A Survey on the Dimension Theory in Dynamical Diophantine Approximation

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Abstract Dynamical Diophantine approximation studies the quantitative properties of the distribution of the orbits in a dynamical system. More precisely, it focuses on the size of dynamically defined limsup sets in the sense of measure and dimension. This quantitative study is motivated by the qualitative nature of the density of the orbits and the connections with the classic Diophantine approximation. In this survey, we collect some recent progress on the dimension theory in dynamical Diophantine approximation. This includes the systems of rational maps on its Julia set, linear map on the torus, beta dynamical system, continued fractions as well as conformal iterated function systems.

1 Introduction

Classic Diophantine approximation concerns how well an irrational number can be approximated by rational numbers. This is motivated by the density of rational numbers. Since the density property is only of qualitative nature, one is led to study the quantitative properties of the distribution of rational numbers. More importantly, this constitutes the main theme of the metric Diophantine approximation [61].

Analogously, there are also many evidences saying that in a dynamical system, the orbit of a generic point is dense. Let's cite two well-known results [71].

Theorem 1.1 (Poincaré's Recurrence Theorem) *Let* $(X, \mathscr{B}, \mu, T)$ *be a measure theoretical dynamical system with* μ *a finite Borel measure. For any measurable set* $B \in \mathscr{B}$ *with positive measure, for almost all* $x \in B$, $T^n x \in B$ *for infinitely many* $n \in \mathbb{N}$. *If there is a compatible metric d, then for almost all* $x \in X$,

$$\liminf_{n\to\infty} d(x, T^n x) = 0.$$

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Theorem 1.2 (Corollary of Birkhoff's Ergodic Theorem) *Let* $(X, \mathscr{B}, \mu, T)$ *be an ergodic dynamical system with a compatible metric d. For any y in the support of* μ*, for almost all* $x \in X$*,*

$$\liminf_{n\to\infty} d(T^n x, y) = 0.$$

Similar to the density of rational numbers, these are also only of qualitative nature. Then it is desirable to know the quantitative properties and leads to the study on the quantitative properties of the distribution of the orbits. More precisely, one is interested in the size of the following limsup sets:

$$\Big\{\star \in X : T^n(x) \in B(y, r_n), \ \ \text{i.o. } n \in \mathbb{N}\Big\}$$

where $\{r_n\}_{n\geq 1}$ is a sequence of decreasing real numbers and *i.o.* denotes *infinitely often*. Here $\star$ can refer to x or y or even the pair (x, y). So, in general, there are three types of questions.

In many cases, instead of considering a general form, one usually focuses on the following more concrete questions:

- Prob1. Let $\{z_n\}_{n\geq 1}$ be a sequence of elements in X and $\psi : \mathbb{N} \to \mathbb{R}^+$. One cares about the points whose orbit can be well approximated by the given sequence $\{z_n\}$ with the given speed. Namely, the size of the set

$$\Big\{x \in X : |T^n x - z_n| < \psi(n), \ \text{i.o. } n \in \mathbb{N}\Big\}.$$

  We call it the shrinking target problems with given targets or shrinking target problems by following Hill and Velani [28].
- Prob2. Let $\psi : \mathbb{N} \to \mathbb{R}^+$. One cares about the point whose orbit will come back to shrinking neighbors of the initial point infinitely often. Namely the size of the set

$$\Big\{x \in X : |T^n x - x| < \psi(n), \text{i.o. } n \in \mathbb{N}\Big\}.$$

  We call it the quantitative Poincaré recurrence properties.
- Prob3. Let $y_0 \in X$ be given in advance. One cares about which points can be well approximated by the orbit of y_0. Namely the size of the set

$$\Big\{x \in X : |T^n y_0 - x| < \psi(n), \ \text{i.o. } n \in \mathbb{N}\Big\}.$$

  We call it the dynamical covering problems as its analogy with the random covering problem [35].

We call the studies on these dynamically defined limsup sets as *Dynamical Diophantine approximation*.

Another type of questions designed for studying the quantitative properties of the distribution of the orbits is called as recurrence time and waiting time. For any $x, y \in X$ and $r > 0$, define

$$\gamma_r(x, y) = \inf\{n \geq 1 : T^n(x) \in B(y, r)\},$$

i.e., the first time needed for the orbit of x entering the ball $B(y, r)$ with radius r and center y. When $x = y$, γ_r is called the recurrence time and when $x \neq y$, it is called waiting time. One concerns the scaling properties of γ_r with respect to r. One is referred to the series works of Saussol, Galatolo, Kim and Galatolo, etc. and the references therein (see, for example, [4, 20, 21, 23–26, 33, 53–56]). This is not included in this short survey.

2 Relationship with the Classic Diophantine Approximation

There are close connections between dynamical Diophantine approximation and the classic Diophantine approximation. Let us present two examples to illustrate this.

2.1 Irrational Rotation and Inhomogeneous Diophantine Approximation

Inhomogeneous Diophantine approximation concerns the Diophantine inequality

$$\|n\alpha - y\| < \psi(n)$$

with $\alpha \in [0, 1]$ an irrational, $y \in [0, 1]$ a real number and $\|\cdot\|$ denotes the distance to the nearest integer.

Naturally there are two types of questions by fixing one parameter and letting the other vary. More precisely, one concerns the following two sets:

$$C(\alpha, \psi) := \Big\{y \in \mathbb{R} : \|n\alpha - y\| < \psi(n), \text{ i.o. } n \in \mathbb{N}\Big\};$$

and

$$W(y, \psi) := \Big\{\alpha \in [0, 1] : \|n\alpha - y\| < \psi(n), \text{ i.o. } n \in \mathbb{N}\Big\}.$$

Let $R_\alpha(x) = x + \alpha \pmod 1$ be the irrational rotation. Then the set $C(\alpha, \psi)$ concerns just the covering problem of the orbit of 0 while the set $W(y, \psi)$ is another type of dynamical Diophantine approximation defined on the parameter space $\{\alpha : \alpha \in Q^c\}$.

2.2 Continued Fractions and Homogeneous Diophantine Approximation

At first, let's recall the Gauss map which induces the continued fraction expansion. The Gauss transformation $T : [0, 1) \to [0, 1)$ is given by

$$T(0) := 0, \ T(x) = 1/x \pmod 1, \ \text{for } x \in (0, 1).$$

Let $a_1(x) = \lfloor x^{-1} \rfloor$ ($\lfloor \cdot \rfloor$ stands for the integer part) and $a_n(x) = a_1(T^{n-1}(x))$ for $n \geq 2$. Each irrational number $x \in [0, 1)$ admits a unique infinite continued fraction expansion of the form

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{a_3(x) + \ddots}}}. \tag{1}$$

The integers a_n are called the partial quotients of x. The nth convergent $p_n(x)/q_n(x)$ of x is given by $p_n(x)/q_n(x) = [a_1, \ldots, a_n]$.

It is already well known that continued fractions are attached great importance to homogeneous Diophantine approximation. This is due to two old theorems [37]:

Theorem 2.1 (Lagrange) *The convergents of a real number $x \in [0, 1]$ are its best rational approximants. More precisely, for any $q < q_n(x)$ and $0 \leq p \leq q$,*

$$|x - p/q| > |x - p_n(x)/q_n(x)|.$$

Theorem 2.2 (Legendre) *Let p/q be a rational number. Then*

$$\left|x - \frac{p}{q}\right| < \frac{1}{2q^2} \Longrightarrow \frac{p}{q} = \frac{p_n(x)}{q_n(x)}, \text{ for some } n \geq 1.$$

Legendre's theorem tells us that if a real number x can be well approximated by some rational, this rational must be a convergent of x. So to find good rational approximations of an irrational, we only need focus on its convergents.

Due to these tight connections of continued fractions with homogeneous Diophantine approximation, the two fundamental results in metric number theory, i.e. Khintchine's theorem [36] and Jarník's theorem [34], were originally proved by using continued fractions.

Let's recall a simple form of the Jarník set: for any $v > 2$, define

$$W_v = \left\{x \in [0, 1] : |x - p/q| < q^{-v}, \ \text{i.o. } n \in \mathbb{N}\right\}.$$

Noting that

$$\left|x - \frac{p_n(x)}{q_n(x)}\right| = \frac{T^n(x)}{q_n(q_n + T^n(x)q_{n-1}(x))} \sim \frac{T^n(x)}{q_n^2(x)}.$$

and $q_n^2(x) \sim e^{(\log T'(x)+\cdots+\log|T'(T^{n-1}(x))|)}$, the set W_v can be reformulated as (almost)

$$W_v = \Big\{x \in [0,1] : |T^n(x) - 0| < e^{-\frac{v-2}{2}(\log T'(x)+\cdots+\log|T'(T^{n-1}(x))|)}, \text{ i.o. } n \in \mathbb{N}\Big\}.$$

So, Jarník set can be viewed as a special case of the shrinking target problem in the dynamical system of continued fractions.

3 Partial Results in Measure

In this section, we give a short review on partial results on measure of the dynamical Diophantine approximation. For more results, one can be referred to subsequent works of those cited below.

3.1 Shrinking Target Problems

Recall that shrinking target problems concern the size of the set

$$\Big\{x \in X : |T^n x - z_n| < \psi(n), \text{ i.o. } n \in \mathbb{N}\Big\}$$

or more generally the set

$$W := \Big\{x \in X : T^n x \in B_n, \text{ i.o. } n \in \mathbb{N}\Big\}$$

where $\{B_n\}$ is a sequence of measurable sets decreasing in measure.

Clearly W is a limsup set, so Borel-Cantelli Lemma is used naturally to quantify its measure. The convergence part of the Borel-Cantelli Lemma works well, while the divergence part may not, since the events $\{T^{-n}B_n\}$ may no longer be independent. But this can be compensated by some strong mixing properties of the system (X, T, μ).

Philipp [51] considered this in the systems of b-adic expansion, β-expansion as well as continued fractions, while a first general result is due to Chernov and Kleinbock [12].

Theorem 3.1 ([12]) *Let $\{B_n\}$ be a sequence of μ-measurable sets. Then for μ-almost all $x \in X$, the iterates $T^n x \in B_n$ infinitely often if*

$$\sum_{n\geq 1} \mu(B_n) = \infty \ \ \text{and} \sum_{1\leq n\leq m\leq N} R_{n,m} \leq C \sum_{n=1}^{N} \mu(B_n), \tag{2}$$

where $R_{n,m}$ stands for the decay of correlations $R_{n,m} := |\mu(T^{-n}B_n \cap T^{-m}B_m) - \mu(B_n)\mu(B_m)|$.

For the special case when $\{B_n\}$ is a sequence of balls with a common center, C. Bonanno, S. Isola, and S. Galatolo proved that

Theorem 3.2 ([7]) *Let $(X, \mathscr{B}, T, \mu)$ be a measure theoretic dynamical system with μ a finite Borel measure. Then for any y, for μ-almost all x one has*

$$\liminf_{n\to\infty} n^{\alpha} d(T^n(x), y) = \infty, \quad \alpha > \underline{d}_{\mu}(y),$$

where $\underline{d}_{\mu}(y)$ is the lower local dimension of y with respect to the measure μ:

$$\underline{d}_{\mu}(y) = \liminf_{r\to 0} \frac{\log \mu(B(y, r))}{\log r}.$$

For a piecewise expanding map on an interval [38] or some hyperbolic maps [12, 15], it is known that given y for μ-almost all x one has

$$\liminf_{n\to\infty} n^{\alpha} d(T^n(x), y) = \infty, \quad \alpha = \underline{d}_{\mu}(y).$$

3.2 Quantitative Recurrence Properties

For the quantitative recurrence properties, M.D. Boshernitzan presented the following outstanding result for general systems.

Theorem 3.3 (Boshernitzan [8]) *Let (X, T, μ, d) be a measure dynamical system with a metric d. Assume that, for some $\alpha > 0$, the α-dimensional Hausdorff measure $\mathscr{H}^{\alpha}$ of the space X is σ-finite. Then for μ-almost all $x \in X$,*

$$\liminf_{n\to\infty} n^{\frac{1}{\alpha}} d(T^n x, x) < \infty.$$

If, moreover, $\mathscr{H}^{\alpha}(X) = 0$, then for μ-almost all $x \in X$,

$$\liminf_{n\to\infty} n^{\frac{1}{\alpha}} d(T^n x, x) = 0.$$

Later, L. Barreira and B. Saussol showed that the above convergence exponent α may relate to the local dimension of the point in the sense that

Theorem 3.4 (Barreira and Saussol [3]) *Let $T : X \to X$ be a Borel measurable transformation on a measurable set $X \subset \mathbb{R}^m$ for some $m \in \mathbb{N}$, and μ be a T-invariant probability measure on X. Then μ-almost surely, for any $\alpha > \underline{d}_\mu(x)$,*

$$\liminf_{n\to\infty} n^{1/\alpha} d(T^n x, x) < \infty.$$

3.3 Dynamical Covering Problems

For covering problems, the system of irrational rotation is paid constant attention to (see [19, 39, 40, 66]). Recently, Fuchs and Kim [22] gave a complete characterization of the size of the set

$$W_1(\psi) := \Big\{ y \in [0,1] : \|n\alpha - y\| < \psi(n), \text{ i.o. } n \in \mathbb{N} \Big\}.$$

Theorem 3.5 (Fuchs and Kim [22]) *Let $\psi(n)$ be a positive, non-increasing sequence and α be an irrational number with convergents p_k/q_k in its continued fraction expansion. Then, for almost all $y \in \mathbb{R}$,*

$$\|n\alpha - y\| < \psi(n) \text{ i.o. } n \in \mathbb{N}$$

if and only if

$$\sum_{k=1}^{\infty} \sum_{n=q_k}^{q_{k+1}-1} \min\left\{\psi(n), \|q_k\alpha\|\right\} = \infty.$$

When T is an expanding Markov map, Fan et al. [18] and Liao and Seuret [45] made excellent contributions to this topic. We will introduce their work in Sect. 13.

4 Hausdorff Dimension and Hausdorff Measure

From this section on, we focus our attention on the dimensional theory of the three types questions presented above. In this short section, we give briefly the definition of Hausdorff measure and Hausdorff dimension. Mainly, we cite the *Mass distribution principle* which is a classic tool to determine the Hausdorff dimension of a set from below.

The Hausdorff measure and dimension have been a widely used tool to discriminate null sets in a measure space. They can be defined in any space endowed with a metric. Before recall the definitions, we fix some notation.

Let (X, d) be a metric space and F be a subset of X. The diameter $\sup\{|x-y| : x, y \in U\}$ of a non-empty subset U of X will be denoted by $d(U)$. A collection $\{U_n\}_{n\geq 1}$ is called a ρ-cover of F if

$$F \subset \bigcup_{n\geq 1} U_n, \ \text{ and } 0 < d(U_n) < \rho, \ \text{ for all } n \geq 1.$$

A dimension function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous, non-decreasing function such that $f(r) \to 0$ as $r \to 0$.

The Hausdorff f-measure of the set F with respect to the dimension function f will be denoted throughout by $\mathscr{H}^f$ and is defined as

$$\mathscr{H}^f(F) = \lim_{\rho\to 0} \inf \left\{ \sum_{n\geq 1} f(d(U_n)) : \{U_n\}_{n\geq 1} \text{ is a } \rho \text{ cover of } F \right\}.$$

In the case that $f(r) = r^s$ $(s \geq 0)$, the measure $\mathscr{H}^f$ is the usual s-dimensional Hausdorff measure $\mathscr{H}^s$ and the Hausdorff dimension $\dim_{\mathrm{H}} F$ of a set F is defined by

$$\dim_{\mathrm{H}} F := \inf \left\{ s : \mathscr{H}^s(F) = 0 \right\} = \sup \left\{ s : \mathscr{H}^s(F) = \infty \right\}.$$

For further details see [16, 47].

A general and classical method for obtaining a lower bound for the Hausdorff f-measure of an arbitrary set F is the following *mass distribution principle* [16, Proposition 4.2].

Lemma 4.1 *Let μ be a probability measure supported on a subset F. Suppose there are positive constants c and r_o such that for any ball $B(x, r)$ with $r < r_o$,*

$$\mu(B) \leq cf(r),$$

then

$$\mathscr{H}^f(F) \geq \mu(F)/c.$$

At the end, we introduce the pressure function, which are tightly related to the dimension of the shrinking target problems. Let (X, d) be a compact metric space with a transformation $T : X \to X$. Call $\mathscr{F}_n(\varepsilon)$ an (n, ε)-separated set of X, if for any $x \neq y \in \mathscr{F}_n(\varepsilon)$,

$$|T^k(x) - T^k(y)| \geq \varepsilon, \ \text{ for some } 0 \leq k \leq n.$$

Let $\psi : X \to \mathbb{R}$ be a function on X. The pressure function $\mathtt{P}$ with respect to the potential ψ is defined as

$$\mathtt{P}(T,\psi) = \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\ \log \sup_{\mathscr{F}_n(\varepsilon):(n,\varepsilon)\ \text{separated set}} \sum_{x\in\mathscr{F}_n(\varepsilon)} e^{S_n\psi(x)},$$

where we use $S_n\psi(x)$ to denote the ergodic sum $\psi(x)+\psi(Tx)+\cdots+\psi(T^{n-1}x)$.

When the system (X,T) is identified with a full shift symbolic space $(\Lambda^{\mathbb{N}},\sigma)$, another form of the pressure function can be given as

$$\mathtt{P}(T,\psi) = \lim_{n\to\infty}\frac{1}{n}\ \log \sum_{w_1,\ldots,w_n\in\Lambda^n}\ \sup_{x\in I_n(w_1,\ldots,w_n)} e^{S_n\psi(x)}, \tag{3}$$

where $I_n(w_1,\ldots,w_n)$ is the set of points whose symbolic representations begin with $(w_1,\ldots,w_n)$.

5 Shrinking Target Problems: *b*-adic Expansion

In a dynamical system (X,T), the shrinking target problems mainly study the size of the following set:

$$\Big\{x\in X : |T^n x - y| < \psi(n,x),\ \text{i.o.}\ n\in\mathbb{N}\Big\}$$

where $\psi : X\times\mathbb{N}\to\mathbb{R}^+$ is a positive function and may depend on x.

The shrinking target problems were studied for the first time by Hill and Velani [28] in the system when T is an expanding rational map and X its corresponding Julia set. But to illustrate the ideas in attacking the shrinking target problems, in this section, we consider a most simple case, namely when T is the b-adic expansion with $b\ge 2$ being an integer.

Fix an integer $b\ge 2$ and define the b-adic transformation T as $Tx = bx \pmod 1$. Then every $x\in[0,1]$ can be expanded into a finite or infinite series

$$x = \frac{\varepsilon_1(x)}{b}+\cdots+\frac{\varepsilon_n(x)+T^n x}{b^n} = \frac{\varepsilon_1(x)}{b}+\frac{\varepsilon_2(x)}{b^2}+\cdots. \tag{4}$$

where

$$\varepsilon_1(x) = \lfloor bx\rfloor,\ \varepsilon_n(x) = \varepsilon_1(T^{n-1}x),\ \text{for}\ n\ge 2$$

are called the digit sequence of x.

Let $\Lambda = \{0, 1, \ldots, b-1\}$. For any integers $(\varepsilon_1, \ldots, \varepsilon_n) \in \Lambda^n$, we use $I_n(\varepsilon_1, \ldots, \varepsilon_n)$ to denote an nth order cylinder in the b-adic expansion, namely,

$$I_n(\varepsilon_1, \ldots, \varepsilon_n) = \Big\{x \in [0, 1] : \varepsilon_k(x) = \varepsilon_k, 1 \le k \le n\Big\},$$

which is an interval of length b^{-n}.

In such a symbolic space, the pressure function $\mathtt{P}$ with a potential ψ is expressed as

$$\mathtt{P}(\psi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{0\le\varepsilon_1,\ldots,\varepsilon_n<b} \sup_{x\in I_n(\varepsilon_1,\ldots,\varepsilon_n)} e^{S_n\psi(x)}.$$

Theorem 5.1 *Let $b \ge 2$ be an integer and T the b-adic transformation. Let $f : [0, 1] \to \mathbb{R}^+$ be a continuous function. Then for any $y_0 \in [0, 1]$, the dimension of the set*

$$D_{z_0}(f) := \Big\{x \in [0, 1] : |T^n x - y_0| < e^{-S_n f(x)},\ i.o.\ n \in \mathbb{N}\Big\}$$

is given by the solution to the pressure function

$$P(-s(\log|T'| + f)) = 0.$$

In the definition of W_f, the shrinking rate $e^{-S_n f(x)}$ depends on x, we try to release a little bit on this dependence. For any $x \in [0, 1]$, let $I_n(x)$ be the nth cylinder containing x. We choose arbitrarily a point x_n in $I_n(x)$. Then by the continuity of f, for any $\delta > 0$, when n is large, for any $x \in [0, 1]$,

$$|S_n f(x) - S_n f(x_n)| < n\delta.$$

Thus, it follows that

$$D_{z_0}(f + \delta) \subset D_{z_0}(f) \subset D_{z_0}(f - \delta).$$

Then by the continuity of the pressure function, we need only pay attention to dimension of the set

$$D'_{z_0}(f) = \Big\{x \in [0, 1] : |T^n x - y_0| < e^{-S_n f(x_n)},\ \text{i.o. } n \in \mathbb{N}\Big\}$$

where x_n can be chosen as any point in $I_n(x)$.

In the following, when we need take a point in a cylinder $I_n(\varepsilon_1, \ldots, \varepsilon_n)$, we write it as $[\varepsilon_1, \ldots, \varepsilon_n]$.

5.1 *Upper Bound of* $\dim_{\mathrm{H}} D'_{z_0}(f)$

The upper bound is established by using a natural cover of $D'_{z_0}(f)$. So at first, we give an expression to reflect the limsup nature of $D'_{z_0}(f)$:

$$D'_{z_0}(f) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{(\varepsilon_1,\ldots,\varepsilon_n)\in\Lambda^n} \Big\{x \in I_n(\varepsilon_1,\ldots,\varepsilon_n) : |T^n x - y_0| < e^{-S_n f([\varepsilon_1,\ldots,\varepsilon_n])}\Big\}.$$

This gives a collection of natural covers of $D'_{z_0}(f)$.

Now we estimate the length of

$$J_n(\varepsilon_1,\ldots,\varepsilon_n) := \Big\{x \in I_n(\varepsilon_1,\ldots,\varepsilon_n) : |T^n x - y_0| < e^{-S_n f([\varepsilon_1,\ldots,\varepsilon_n])}\Big\}.$$

By (4), it follows that

$$T^n x = b^n\Big(x - \frac{\varepsilon_1}{b} - \cdots - \frac{\varepsilon_n}{b^n}\Big).$$

Substituting it in the inequality in $J_n(\varepsilon_1,\ldots,\varepsilon_n)$, it follows that $J_n(\varepsilon_1,\ldots,\varepsilon_n)$ is an interval with length

$$\Big|J_n(\varepsilon_1,\ldots,\varepsilon_n)\Big| \le \frac{1}{b^n} e^{-S_n f([\varepsilon_1,\ldots,\varepsilon_n])}.$$

As a result, the s-dimensional Hausdorff measure of $D'_{z_0}(f)$ can be estimated as

$$\mathscr{H}^s(D'_{z_0}(f)) \le \liminf_{N\to\infty} \sum_{n=N}^{\infty} \sum_{0\le\varepsilon_1,\ldots,\varepsilon_n<b} \left(\frac{e^{-S_n f([\varepsilon_1,\ldots,\varepsilon_n])}}{b^n}\right)^s.$$

So, for any s larger than the solution to $\mathrm{P}(-s(\log|T'| + f)) = 0$, one has $\mathscr{H}^s(D'_{z_0}(f)) = 0$. This gives the upper bound of $\dim_{\mathrm{H}} D'_{z_0}(f)$.

5.2 *Lower Bound of* $\dim_{\mathrm{H}} D'_{z_0}(f)$

The lower bound is obtained by a classic way: at first construct a Cantor subset of $D'_{z_0}(f)$; then define a suitable mass distribution sitting on such Cantor subset; at last the mass distribution is applied. Here we only give the first two steps, while omit the technical estimation of the last step.

Cantor Subset

Bearing in mind that $D'_{z_0}(f)$ is a limsup set, the events

$$W_n := \{x \in [0,1] : |T^n x - y_0| < e^{-S_n f(x_n)}\} \tag{5}$$

should be realized infinitely often.

In a symbolic space, it would be quite convenient to locate a point by determining its symbolic representation. Thus to realize the event (5), we transfer the ball $B(z_0, r)$ to a family of cylinders $\mathscr{G}$. Since we are interested in the dimension, it does not need a strict equality:

$$B(z_0, r) = \cup_{I_n \in \mathscr{G}} I_n.$$

We only need that there is a family of "good" cylinders $\mathscr{G}$ such that

$$\bigcup_{I_n \in \mathscr{G}} I_n \subset B(z_0, r), \ \frac{\sum_{I_n \in \mathscr{G}} |I_n|}{r} \geq c,$$

for an absolute constant $c > 0$.

For b-adic expansion, this is realized in the following lemma. Write the b-adic digit sequence of z_0 as $(b_1, b_2, \ldots)$. It is possible that $b_n = 0$ ultimately.

Lemma 5.2 *Fix a word $(\varepsilon_1, \ldots, \varepsilon_n)$ in Λ^n and $x_n \in I_n(\varepsilon_1, \ldots, \varepsilon_n)$. Let t be the integer such that*

$$|I_t(b_1, \ldots, b_t)| < e^{-S_n f(x_n)} \leq |I_{t-1}(b_1, \ldots, b_{t-1})|.$$

Choose $\mathscr{G} = \{I_t(b_1, \ldots, b_t)\}$. Then clearly one has

$$I_{n+t}(\varepsilon_1, \ldots, \varepsilon_n, b_1, \ldots, b_t) \subset \left\{x \in I_n(\varepsilon_1, \ldots, \varepsilon_n) : |T^n x - z_0| < e^{-S_n f(x_n)}\right\}.$$

The Cantor subset is constructed level by level in the following way.

- The first level of the Cantor set.

Let $t_0 = 0$. Fix an integer $m_1 \gg t_0$. Let $n_1 = m_1 + t_0$. We construct a subset of W_{n_1}, i.e. realizing the event for the first time at $n = n_1$. For each word $(\varepsilon_1, \ldots, \varepsilon_{m_1}) \in \Lambda^{n_1}$, let t_1 be the integer given in Lemma 5.2. Then we have a collection of intervals

$$I_{n_1+t_1}(\varepsilon_1, \ldots, \varepsilon_{n_1}, b_1, \ldots, b_{t_1}), \ (\varepsilon_1, \ldots, \varepsilon_{m_1}) \in \Lambda^{m_1},$$

which is a subset of W_{n_1}.

It should be mentioned that t_1 depends on the word $(\varepsilon_1, \ldots, \varepsilon_{m_1})$. This dependence will not play a role in the argument, thus will not be explicitly addressed.

The first level of the Cantor set is then defined as

$$\mathbb{F}_1 = \bigcup_{(\varepsilon_1,\ldots,\varepsilon_{n_1})\in\Lambda^{n_1}} I_{n_1+t_1}(\varepsilon_1,\ldots,\varepsilon_{n_1},b_1,\ldots,b_{t_1}).$$

The intervals in $\mathbb{F}_1$ are called fundamental cylinders of the first level.

- The second level of the Cantor set.

For each fundamental interval $I_{n_1+t_1}(w_1)$ in the first level $\mathbb{F}_1$, we select a collection of its sub-cylinders to constitute a subfamily of the second level of the Cantor set which will realize the event W_n for the second time.

Fix an integer m_2 which is much larger than

$$\sup\{n_1+t_1 : (\varepsilon_1,\ldots,\varepsilon_{n_1})\in\Lambda^{n_1}\}.$$

Fix a fundamental cylinder $I_{n_1+t_1}(w_1)$ in $\mathbb{F}_1$. For each $(\varepsilon_1,\ldots,\varepsilon_{m_2})\in\Lambda^{m_2}$, let t_2 be the integer given in Lemma 5.2 with respect to the word $(w_1,\varepsilon_1,\ldots,\varepsilon_{m_2})$.

Let $n_2 = n_1+t_1+m_2$. Then we have a collection of intervals

$$I_{n_2+t_2}(w_1,\varepsilon_1,\ldots,\varepsilon_{m_2},b_1,\ldots,b_{t_2}),\quad (\varepsilon_1,\ldots,\varepsilon_{m_2})\in\Lambda^{m_2},$$

which are subsets of W_{n_2}.

The second level of the Cantor set is then defined as

$$\mathbb{F}_2 = \bigcup_{I_{n_1+t_1}(w_1)\in\mathbb{F}_1}\ \bigcup_{(\varepsilon_1,\ldots,\varepsilon_{m_2})\in\Lambda^{m_2}} I_{n_2+t_2}(w_1,\varepsilon_1,\ldots,\varepsilon_{m_2},b_1,\ldots,b_{t_2}).$$

The other levels can be constructed similarly. Then the desired Cantor set is defined as

$$\mathbb{F}_\infty = \bigcap_{n=1}^{\infty}\mathbb{F}_n.$$

It is clear that

$$\mathbb{F}_\infty \subset D'_{z_0}(f).$$

Mass Distribution

We will define a sequence of real numbers which are tightly related to the dimension of $\mathbb{F}_\infty$. For each integer $m\ge 1$, define s_m being the solution to the equation

$$\sum_{(\varepsilon_1,\ldots,\varepsilon_m)\in\Lambda^m}\left(|I_m(\varepsilon_1,\ldots,\varepsilon_m)|\cdot e^{-S_m f(x'_m)}\right)^{s_m} = 1,$$

where $x'_m \in I_m(\varepsilon_1, \ldots, \varepsilon_m)$. By the definition of the pressure function (3), it is clear that

Lemma 5.3 *Let s_∞ be the solution to the pressure function $\mathrm{P}(-s(\log|T'|+f)) = 0$. Then*

$$\lim_{m\to\infty} s_m = s_\infty.$$

Now we define a mass distribution supported on $\mathbb{F}_\infty$. This is given by distributing a suitable mass on each fundamental cylinders defining $\mathbb{F}_\infty$. We express a general fundamental cylinder constituting $\mathbb{F}_\infty$ as

$$I_{n_k+t_k}(w_k) = I_{n_k+t_k}\big(w_{k-1}, \varepsilon_1^{(k)}, \ldots, \varepsilon_{m_k}^{(k)}, b_1, \ldots, b_{t_k}\big) \in \mathbb{F}_k$$

where $I_{n_{k-1}+t_{k-1}}(w_{k-1}) \in \mathbb{F}_{k-1}$. Then we define a measure μ as

$$\mu\Big(I_{n_k+t_k}\big(w_{k-1}, \varepsilon_1^{(k)}, \ldots, \varepsilon_{m_k}^{(k)}, b_1, \ldots, b_{t_k}\big)\Big)$$
$$= \mu\Big(I_{n_{k-1}+t_{k1}}(w_{k-1})\Big) \cdot \Big(|I_{m_k}\big(\varepsilon_1^{(k)}, \ldots, \varepsilon_{m_k}^{(k)}\big)|e^{-S_{m_k}f(x'_{m_k})}\Big)^{s_{m_k}},$$

where $x'_{m_k} \in I_{m_k}(\varepsilon_1^{(k)}, \ldots, \varepsilon_{m_k}^{(k)})$.

To apply the mass distribution principle (Lemma 4.1), we need compare the length of $I_{n_k+t_k}(w_k)$ and its μ-measure. It should be noticed that its length satisfies

$$|I_{n_k+t_k}(w_k)| = |I_{n_{k-1}+t_{k-1}}(w_{k-1})| \cdot |I_{m_k}\big(\varepsilon_1^{(k)}, \ldots, \varepsilon_{m_k}^{(k)}\big)|e^{-S_{n_k}f(x_{n_k})}$$

where $x_{n_k} \in I_{n_k}$ instead of in $I_{m_k}(\varepsilon_1^{(k)}, \ldots, \varepsilon_{m_k}^{(k)})$. But this will not cause much complexity, since

$$|S_{n_k}f(x_{n_k}) - S_{m_k}f(x'_{m_k})| \le (n_{k-1} + t_{k-1})\|f\|_\infty = o(m_k).$$

One can conclude that the dimension of $\mathbb{F}_\infty$ is s_∞. The left task is to verify the condition in Lemma 4.1 is satisfied, so the detailed estimation is omitted.

6 Shrinking Target Problem: Expanding Rational Maps

Let T be an expanding rational map on the Riemann sphere acting on its Julia set J. Let $f : J \to \mathbb{R}^+$ be a Hölder continuous map with $f'(x) \ge \log|T'|$ for all $x \in J$. Define

$$D_{z_0}(f) := \Big\{x \in J : x \in B\big(y, e^{-S_nf(x)}\big), y \in I_n(z_0),\ \text{i.o. } n \in \mathbb{N}\Big\},$$

where $I_n = \{y : T^n y = z_0\}$ is the n-th inverse of z_0.

Theorem 6.1 ([28, 29]) *The Hausdorff dimension of $D_{z_0}(f)$ is given by the unique solution $s(f)$ to the pressure function*

$$P(T, -sf) = 0.$$

Theorem 6.2 ([32]) *Let $s(f)$ be Hausdorff dimension of $D_{z_0}(f)$. Then the $s(f)$-dimensional Hausdorff measure of $D_{z_0}(f)$ is either zero or infinity.*

6.1 Key Properties

Three key properties of the system (J, T) are used in the proof of these theorems. The first one is the bounded distortion property; the second one concerns the distribution of the inverse of z_0, while the third says that there exists a good measure supported on J.

Lemma 6.3 (Köbe Distortion Theorem) *Let $\Delta \subset \overline{\mathbb{C}}$ be a topological disc with boundary containing at least two points and let $V \subset \Delta$ be compact. Then there exists a constant $K(\Delta, V)$ such that for any univalent holomorphic function $F : \Delta \to \mathbb{C}$ it holds that*

$$\sup_{x,y\in V} \frac{|F'(x)|}{|F'(y)|} \le K(\Delta, V).$$

As a consequence, the bounded distortion property holds, namely, there exists a constant K such that if f is a univalent holomorphic function defined on $B(x, 2r)$ in $\mathbb{C}$, then

$$B\big(F(z), K^{-1}r|F'(z)|\big) \subset F\big(B(z, r)\big) \subset B\big(f(z), Kr|F'(z)|\big).$$

In the proof, F is taken to be the inverse branches of T^n for any $n \ge 1$.

Lemma 6.4 *Let T be an expanding rational map with Julia set J. Then there is a neighborhood U of J such that $T^{-1}(U) \subset U$ and for any ball $B \subset U$, all inverse branches of iterates of T are defined on B.*

Let $z_0 \in J$. There exist constants C_1, C_2 and an integer n_0 such that for all $n \ge n_0$,

$$J \subset \bigcup_{y:T^n y = z_0} B\big(y, C_1|(T^n)'(y)|^{-1}\big);$$

and the following union

$$\bigcup_{y:T^n y = z_0} B\big(y, C_2|(T^n)'(y)|^{-1}\big)$$

are disjoint.

Lemma 6.5 *When f is Hölder continuous, there exists an $-s(f)f$-conformal measure.*

7 Shrinking Target Problem: Finite Kleinian Group

Diophantine approximation of real numbers by rationals can be seen geometrically in terms of the orbit of infinity under the Möbius action of the modular group $\mathrm{SL}(2,\mathbb{Z})$. So, the classic Jarník-Besicovitch theorem on the size of τ-well approximable points can be seen as a special case of the shrinking target problems in the system of group actions.

Let G be a non-elementary, geometrically finite Kleinian group acting on the unit ball model B^{k+1} of $(k+1)$-dimensional hyperbolic space with Poincaré metric ρ derived from the differential

$$d\rho = |d\mathrm{x}|/(1-|\mathrm{x}^2|).$$

Thus G is a discrete subgroup of $\mathrm{Möb}(B^{k+1})$, the group of orientation-preserving Möius transformations preserving B^{k+1}. By assumption, there is some finitely sided convex fundamental polyhedron for the action of G on B^{k+1}. Since G is non-elementary, the limit set J of G (the set of limit points in the unit sphere S^k of any orbit of G in B^{k+1}) is uncountable.

The analogue of the set of τ-well approximable points in hyperbolic space (B^{k+1},ρ) is the set of points in the limit set of a Kleinian group G which are "very close" to infinitely many images of a "distinguished" point y in the limit set J. More precisely, for any $\tau \geq 1$, define

$$W_y(\tau) = \Big\{x \in J : |x-g(y)| < g'(0)^{\tau},\ \text{i.o. } g \in G\Big\}.$$

The classical set of τ-well approximable points corresponds to the case when y is the parabolic fixed point at infinity of the modular group $\mathrm{SL}(2,\mathbb{Z})$ and its images are the rationals.

The dimension of $W_y(\tau)$ was treated according to when the geometrically finite group G is without or with parabolic elements. The result for the first case is due to Dodson et al. [14] and [70], while the second case is due to Hill and Velani [30].

Theorem 7.1 ([14, 70]) *Assume that the geometrically finite group G is without parabolic elements, and let the "distinguished" point y be a hyperbolic fixed point of G. Then*

$$\dim_{\mathrm{H}} W_y(\tau) = \frac{\delta}{\tau},$$

where δ is the Hausdorff dimension of the limit set J.

Now assume the geometrically finite group G has parabolic elements and let the "distinguished" point y be a parabolic fixed point p of G. The stabilizer $G_p = \{g \in G : g(p) = p\}$ of p is an infinite group which contains a free abelian subgroup of finite index and of rank $\mathrm{rk}(p) \in [1,k]$. Refer to $\mathrm{rk}(p)$ as the rank of the parabolic fixed point p. Then it was proved that

Theorem 7.2 ([30]) *Let G be a geometrically finite group with parabolic elements and let* $rk(p)$ *denote the rank of the parabolic fixed point p. Then for* $\tau \geq 1$,

$$\dim_{\mathrm{H}} W_p(\tau) = \min\left\{\frac{\delta + rk(p)(\tau - 1)}{2\tau - 1}, \frac{\delta}{\tau}\right\}.$$

For partial results, one is referred to Stratmann [62], Velani [68, 69], and the references therein.

8 Shrinking Target Problems: Parabolic Rational Maps

Let T be a parabolic rational maps $T : \overline{C} \to \overline{C}$ and $J(T)$ be its Julia set. Recall that for parabolic rational maps it is well known that

$$J(T) = J_r(T) \cup J_p(T),$$

i.e., the Julia set $J(T)$ admits a disjoint decomposition into the radial Julia set $J_r(T)$ and the countable set of pre-parabolic points

$$J_p(T) := \bigcup_{\omega \in \Omega} \bigcup_{n \in \mathbb{N}} T^{-n}(\omega),$$

where Ω denotes the set of rationally indifferent periodic points.

For each $\omega \in \Omega$, define the canonical balls $B(c(\omega), r_{c(\omega)})$ associated to ω as follows. Let $I(\omega) := T^{-1}(\omega) \setminus \{\omega\}$. Then, for each integer $n \geq 0$, define the canonical radius r_ξ at $\xi \in T^{-n}(I(\omega))$ by

$$r_\xi := |(T^n)'(\xi)|^{-1},$$

and call the ball $B(\xi, r_\xi)$ the canonical ball at ξ. Roughly speaking, canonical balls are all the holomorphic inverse iterates of $B(\omega, r_\omega)$ which is a standard neighborhood of ω.

Then the shrinking target problem in this setting can be formulated as a Jarník-Julia sets. For $\omega \in \Omega$ and $\sigma > 0$, define

$$\mathscr{J}_\sigma^\omega(T) := \bigcap_{n \in \mathbb{N}} \bigcup_{r_{c(\omega)} < 1/n} B(c(\omega), r_{c(\omega)}), \quad \mathscr{J}_\sigma = \bigcup_{\omega \in \Omega} \mathscr{J}_\sigma^\omega.$$

Call $\mathscr{J}_\sigma(T)$ the σ-Jarník-Julia set and $\mathscr{J}_\sigma^\omega(T)$ the (σ, ω)-Jarník-Julia set.

It was proved by Stratmann and Urbański that

Theorem 8.1 ([63]) *Let T be a parabolic rational map with Julia set* $J(T)$ *of Hausdorff dimension h. For* $\omega \in \Omega$ *and* $\sigma > 0$, *the Hausdorff dimension of*

σ-Jarník-Julia set and the (σ, ω)-Jarník-Julia set are determined by the following, where $p(\omega)$ denotes the number of attracting petals associated with ω, and $p_{\min} := \min_{\eta\in\Omega} p(\eta)$.

- *If $h < 1$, $\dim_H \mathscr{J}_\sigma(T) = h/(1+\sigma)$,*
- *If $h \geq 1$,*

$$\dim_H \mathscr{J}_\sigma^\omega(T) = \begin{cases} \frac{h}{1+\sigma}, & \text{for } \sigma \geq h-1; \\ \frac{h+\sigma p(\omega)}{1+\sigma(1+p(\omega))}, & \text{for } \sigma < h-1. \end{cases}$$

An essential ingredient in the proof of this theorem is to show that, much as for Kleinian groups, for parabolic rational maps there exists a generalization of Dirichlet's Theorem in number theory. Roughly speaking, this result shows that the Julia set admits economical, arbitrarily fine coverings and packings by finitely many canonical balls whose radii are diminished in a dynamically controlled way.

A similar results hold for the so-called tame parabolic iterated function systems [64].

9 Shrinking Target Problem: Markov Expanding Systems

Definition 9.1 (Expanding Markov Map) Let $\mathscr{V} = \{V_i\}_{i\in\Lambda}$ be a countable family of disjoint subintervals of the unit interval with non-empty interior. Let T be a map from $\cup_{i\in\Lambda} V_i$ to $[0,1]$. Given $w = (w_1, \ldots, w_n) \in \Lambda^n$ for some $n \in \mathbb{N}$, let $V_w = \cap_{i=1}^n T^{-i} V_{w_i}$.

Call $T : \cup_{i\in\Lambda} V_i \to [0,1]$ is an expanding Markov map if T satisfies the following conditions.

- For each $i \in \Lambda$, $T|_{V_i}$ is a C^1 map which maps the interior of V_i onto open unit interval (0; 1),
- There exists $\xi > 1$ and $N \in \mathbb{N}$ such that for all $n \geq N$ and all $x \in \cup_{w\in\Lambda^n} V_w$, we have $|(T^n)'(x)| > \xi^n$,
- There exists some sequence ρ_n with $\lim_{n\to\infty} \rho_n = 0$ such that for all $n \geq N$, $w \in \Lambda^n$, and all $x, y \in V_w$,

$$e^{-n\rho} \leq \frac{|(T^n)'(x)|}{|(T^n)'(y)|} \leq e^{n\rho}.$$

The repeller J of an expanding Markov map is the set of points for which every iterate of T is well defined,

$$J := \Big\{x \in [0,1] : T^n(x) \in \cup_{i\in\Lambda} V_i, \text{for all } n \in \mathbb{N}\Big\}.$$

Clearly this system includes the b-adic expansion and continued fraction expansion as special cases.

The shrinking target problem in this system is formulated as

$$D_{z_0}(f) = \Big\{x \in J : |T^n x - z_0| < e^{-S_n f(x)},\ \text{i.o. } n \in \mathbb{N}\Big\}.$$

Theorem 9.2 ([52]) *Let T be a expanding Markov map with J its attractor. Let f be a continuous map on J and $z_0 \in J$. Then the Hausdorff dimension of $D_{z_0}(f)$ is the unique solution s to the pressure function*

$$P\big(T, -s(\log|T'| + f)\big) \le 0.$$

Urbański [67] considered this system for the first time with some restriction on z_0, namely, the orbit of z_0 under T falls into only finitely many generating intervals $\{V_i\}$. The case for the system of continued fraction expansion is solved by Li et al. [43]. A complete result is achieved by Reeve [52].

Let's give words to compare the argument in proving the above theorem with that for the case of b-adic expansion.

Similar to b-adic expansion, there is a symbolic space corresponding to this Markov expanding system, so one would like to work it in the symbolic space. In such a sense, one can define cylinders as usual. Compared with the b-adic, a main difference for Markov expanding system lies in Lemma 5.2. In other words, the ball $B(z_0, e^{-S_n f(x)})$ may not be well packed by merely one cylinder, so more cylinders should be taken into account.

For the case of continued fractions, this is conquered by the following observation:

Lemma 9.3 ([43]) *Let $B(z,r)$ be a ball with center $z \in [0,1]$ and radius $0 < r < e^{-4}$. Then there exist integers $t \le -4\log r$, $b_1, \ldots, b_{t-1}$ and $\underline{b}_t, \overline{b}_t$ such that $3 \le \underline{b}_t < \overline{b}_t$ and the family*

$$\mathcal{G} = \Big\{I_t(b_1, \ldots, b_{t-1}, b_t) : \underline{b}_t < b_t \le \overline{b}_t\Big\}$$

satisfies the following three conditions.

(1) *All the cylinders in $\mathcal{G}$ are of comparable length:*

$$1/24 \le \frac{|I_t(b_1, \ldots, b_{t-1}, b_t)|}{|I_t(b_1, \ldots, b_{t-1}, b_t')|} \le 24, \quad \text{for all } \underline{b}_t < b_t, b_t' \le \overline{b}_t. \tag{6}$$

(2) *All the cylinders I_t in $\mathcal{G}$ are contained in the ball $B(z,r)$.*

(3) *The cylinders in $\mathcal{G}$ pack the ball $B(z,r)$ sufficiently; that is*

$$2r \ge \sum_{\underline{b}_t < b_t \le \overline{b}_t} \big|I_t(b_1, \ldots, b_{t-1}, b_t)\big| \ge \frac{r}{4^6}. \tag{7}$$

For the general case, Reeve proved the following property:

Lemma 9.4 ([52]) *Let $B(z_0, r)$ be a ball with $z_0 \in J$ and $r > 0$. Define*

$$U(z_0, n, r) := \Big\{V_w : w \in \Lambda^n, V_w \subset B(z_0, r)\Big\}.$$

There exists a sequence of natural numbers n_r with $\lim_{r\to 0} n_r = \infty$ and $\limsup_{r\to 0} n_r^{-1} \log r < 0$ such that

$$\limsup_{r\to 0} \frac{1}{r} \sum_{V_w \in U(z_0, n_r, r)} |V_w| > 0.$$

Both of the above two lemmas serve as the same role that a ball can be well packed a collection of cylinders.

10 Shrinking Target Problems: β-Expansions

Fix a real number $\beta > 1$. The β-transformation is defined as

$$T_\beta(x) = \beta x \ (\mathrm{mod}\ 1),\ x \in [0, 1].$$

Then every $x \in [0, 1]$ be expanded as a series expansion

$$x = \frac{\varepsilon_1}{\beta} + \frac{\varepsilon_2}{\beta^2} + \cdots,$$

where $\varepsilon_1(x) = \lfloor \beta x \rfloor$, $\varepsilon_n = \varepsilon_1(T^{n-1}x)$ are called the digit sequence of x (with respect to the base β).

The digit sequence of 1 plays an important role in β-expansions. If the β-expansion of 1 terminates, i.e. there exists $m \geq 1$ such that $\varepsilon_m(1, \beta) \geq 1$ but $\varepsilon_n(1, \beta) = 0$ for $n > m$, β is called a simple number. Whence, we put

$$\big(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \varepsilon_3^*(\beta), \ldots\big) = (\varepsilon_1(1, \beta), \ldots, \varepsilon_{m-1}(1, \beta), \varepsilon_m(1, \beta) - 1)^\infty,$$

where $(\varepsilon)^\infty$ denotes the periodic sequence $(\varepsilon, \varepsilon, \varepsilon, \ldots)$. If β is not a simple number, we also denote by $(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \varepsilon_3^*(\beta), \ldots)$ the β-expansion of 1. In both cases, we say that the sequence $(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \varepsilon_3^*(\beta), \ldots)$ is the β-expansion of unity.

Definition 10.1 A finite or an infinite sequence $(\varepsilon_1, \ldots, \varepsilon_n, \ldots)$ is called β-admissible, if there exists an $x \in [0, 1]$ such that the β-expansion of x begins with $\varepsilon_1, \ldots, \varepsilon_n, \ldots$.

Theorem 10.2 (Parry [49]) *Let $\beta > 1$ be given. A non-negative integer sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ is β-admissible if and only if, for any $k \geq 1$,*

$$(\varepsilon_k, \varepsilon_{k+1}, \ldots) <_{\mathrm{lex}} \big(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots\big),$$

where $(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots)$ is the β-expansion of unity.

When β is a Parry number, the corresponding system is a finite Markov system. But as far as a general β is concerned, this is no longer the case.

For any admissible sequence $(\varepsilon_1, \ldots, \varepsilon_n)$, we define the cylinder set as

$$I_n(\varepsilon_1, \ldots, \varepsilon_n) := \Big\{x \in [0, 1] : \varepsilon_1(x) = \varepsilon_1, \ldots, \varepsilon_n(x) = \varepsilon_n\Big\}.$$

When β is a Parry number, every cylinder has a regular lengths:

$$c\beta^{-n} \leq |I_n(\varepsilon_1, \ldots, \varepsilon_n)| \leq \beta^{-n}.$$

for an absolute constant $c > 0$. But for a general β, it may happen that

$$|I_n(\varepsilon_1, \ldots, \varepsilon_n)| \leq \beta^{-n^2} \ll \beta^{-n}.$$

So, one has to find an alternate of Lemma 5.2. This is done by the following property. Call a cylinder $I_n(\varepsilon_1, \ldots, \varepsilon_n)$ full if

$$|I_n(\varepsilon_1, \ldots, \varepsilon_n)| = \beta^{-n}.$$

Lemma 10.3 ([10]) *Among* $(n + 1)$ *consecutive cylinders of order n, there exists at least one full cylinder.*

Thus for any $\eta > 0$, there exists r_η such that for any ball $B(z, r)$ with $r < r_\eta$, one can find a full cylinder I_n such that

$$I_n \subset B(z, r), \quad |I_n| \geq r^{1+\eta}.$$

The above structure appears for the first time in [60].

Theorem 10.4 ([10]) *Let* $\beta > 1$ *and f a positive continuous function on* $[0, 1]$. *Then the Hausdorff dimension of* $D_{z_0}(f)$ *is the unique solution s to the pressure function*

$$P(T, -s(\log \beta + f)) = 0.$$

11 Shrinking Target: Matrix Transformations on Torus

Let T be a $d \times d$ matrix with integral coefficients. Then T determines a self-map of the d-dimensional torus $X = \mathbb{R}^d/\mathbb{Z}^d$. Let $\{B(n)\}_{n\geq 1}$ be a sequence of cubes in X with the diameters $\{r_n\}$ decreasing. Define

$$W = \Big\{x \in X : x \in T^{-n}B(n), \ \text{i.o.}, n \in \mathbb{N}\Big\}.$$

Hill and Velani [31] proved the following results. Let

$$\tau = \liminf_{n\to\infty} \frac{-\log |r_n|}{n}.$$

Theorem 11.1 *Let $T : X \to X$ be a matrix transformation of the torus $X = \mathbb{R}^d/\mathbb{Z}^d$. Let $e_1, \dots, e_d$ be the absolute values of the eigenvalues of T (with multiplicity). Suppose these are ordered: $e_1 \le \dots \le e_d$. Then for $\tau \ge \log e_d/e_1$, one has*

$$\dim_{\mathrm{H}} W = \min_{i=1,\dots,d} \left\{ \frac{i + \log e_i + \sum_{j=i+1}^{d} \log e_j}{\tau + e_i}. \right\}$$

Theorem 11.2 *Let $T : X \to X$ be diagonalizable over $\mathbb{Q}$, and let $e_1, \dots, e_d \in \mathbb{Z}$ be the eigenvalues of T arranged in increasing order. Then one has*

$$\dim_{\mathrm{H}} W = \min_{i=1,\dots,d} \left\{ \frac{i + \log e_i - \sum_{j:e_j > e_i e^{\tau}} (\log e_j - \log e_i - \tau) + \sum_{j=i+1}^{d} \log e_j}{\tau + e_i}. \right\}$$

Let's give some words on this setting. The main difficulty is that W is the limsup of a collection of subsets of X which are far from being circular since T may expand in one direction and contract in others.

To make the difficulty more clear, we assume that T is a diagonalizable matrix even expanding in every direction. Then

$$T^{-n}\left([0,1]^d\right), \text{ and } T^{-n}(B(n))$$

are collections of rectangles with sidelengths $e_1^{-n}, \dots, e_d^{-n}$ and $e_1^{-n} r_n, \dots, e_d^{-n} r_n$, respectively, instead of balls.

In the definition of Hausdorff measure, we use *balls* to cover a fractal set. So, for the limsup set W defined above, there is no natural covers. A general idea is to partition the rectangles into small balls. Even this, one need also pay attention to the relative positions of the rectangles. It means that if these rectangles are close enough, when one covers one rectangle by balls, it is possible that these balls may also cover the other rectangles in part. The extra condition in the first result that $\tau \ge \log e_d/e_1$ excludes this possibility. Without this extra condition, as one sees in the second result, there is an extra term in dimension W and the dimension drops.

12 Shrinking Target Problem on the Parameter Space

Let $\{T_\alpha : \alpha \in \Omega\}$ be a family of transformations defined on a metric space X where Ω is a subset of another metric space. Instead of considering the Diophantine properties of the orbits under one fixed transformation, one can also consider the set of parameters where the orbit of some point satisfies some Diophantine properties. More precisely, fix $x_0, z_0 \in X$. One considers the set

$$W(\phi) := \Big\{\alpha \in \Omega : |T_\alpha^n(x_0) - z_0| < \phi(n),\ \text{i.o.}, n \in \mathbb{N}\Big\}.$$

Such a setting fits well for irrational rotations and beta expansions.

12.1 Irrational Rotation

Let $\Omega = [0, 1]$ and $X = [0, 1]$. For each $\alpha \in [0, 1]$, T_α is the irrational rotation:

$$T_\alpha : [0, 1] \to [0, 1],\ T_\alpha(x) = x + \alpha \pmod 1.$$

Then the set $W(\phi)$ can be rewritten as

$$W(\phi) = \Big\{\alpha \in [0, 1] : \|n\alpha - y\| < \phi(n),\ \text{i.o. } n \in \mathbb{N}\Big\}$$

where $\|\cdot\|$ denotes the distance to the integers and y is a given point in $[0, 1]$. This is nothing but the inhomogeneous Diophantine approximation.

The dimension of $W(\phi)$ was obtained by Lebesley [41].

Theorem 12.1 ([41]) *Let ϕ be a decreasing function on* $[0, 1]$. *Then*

$$\dim_{\mathrm{H}} W(\phi) = \frac{2}{1 + \tau},\ \tau = \liminf_{n\to\infty} \frac{-\log \phi(n)}{\log n}.$$

Y. Bugeaud, S. Harrap, S. Kristensen and S. Velani studied the set of points y which are badly approximated by the orbit of α (in high dimensional case). Namely, the dimension of the set

$$\mathrm{Bad}_A := \Big\{y \in [0, 1]^n : \exists\, c(x) > 0, \|Aq - y\| > \frac{c(x)}{q^{m/n}},\ \text{for all } q \in \mathbb{Z}^m \setminus \{0\}\Big\}$$

where A is an $n \times m$ real matrix. It was proved that

Theorem 12.2 ([11]) *For any $n \times m$ real matrix A,* $\dim_{\mathrm{H}} \mathit{Bad}_A = n$.

12.2 β-Expansions

Schmeling [57] proved that for any $x_0, y \in [0, 1]$,

$$\liminf_{n\to\infty} |T_\beta^n(x_0) - y| = 0$$

for Lebesgue almost all $\beta > 1$. This is a beginning of the study β-expansions on the parameter space $\{\beta : \beta > 1\}$.

Now we are interested in the dimension of the following set

$$E\big(\{\ell_n\}_{n\geq 1}, x_0, y\big) = \Big\{\beta > 1 : |T_\beta^n x_0 - y| < \beta^{-\ell_n}, \text{ i.o. } n \in \mathbb{N}\Big\}.$$

One has

Theorem 12.3 *For any* $x_0, y \in [0, 1]$,

$$\dim_{\mathrm{H}} E\big(\{\ell_n\}_{n\geq 1}, x_0, y\big) = \frac{1}{1+b}, \text{ where } b = \liminf_{n\to\infty} \frac{\ell_n}{n}.$$

Schmeling and Persson [50] proved the case when $x_0 = 1$ and $y = 0$; for the case of a general y, it was obtained by Li et al. [42]. The full general result is proved by Lü and Wu recently [46].

12.3 Two Parameters

As mentioned in the introduction, one can also consider the case that two parameters are both involved.

Dodson [13] considered the case of irrational rotations and got the following result.

Theorem 12.4 *Let* ϕ *be a decreasing positive function defined on* $\mathbb{N}$. *Then*

$$\Big\{(\alpha, y) \in [0, 1]^2 : \|n\alpha - y\| < \phi(n), \text{ i.o. } n \in \mathbb{N}\Big\}$$

is of Hausdorff dimension

$$1 + \frac{2}{t+1}, \text{ where } t = \liminf_{n\to\infty} \frac{-\log\phi(n)}{n}.$$

For the case of β-expansions, Ge and Lü [27] obtained that

Theorem 12.5 *Let* ϕ *be a decreasing positive function defined on* $\mathbb{N}$. *Then*

$$\Big\{(x, y) \in [0, 1]^2 : |T_\beta^n(x) - y| < \phi(n), \text{ i.o. } n \in \mathbb{N}\Big\}$$

is of Hausdorff dimension

$$1 + \frac{2}{t}, \text{ where } t = \liminf_{n\to\infty} \frac{-\log\phi(n)}{n\log\beta}.$$

13 Dynamical Covering Problem

Let (X, d) be a metric space with a transformation $T : X \to X$. Fix a point $x_0 \in X$. One considers the set of points which can be well approximated by the orbit of x_0, i.e.

$$C(\phi) := \Big\{y \in X : |T^n x_0 - y| < \phi(n), \text{ i.o. } n \in \mathbb{N}\Big\}.$$

The covering problem is closely related to the classical random covering problem. Namely, consider an independent and identically distributed (i.i.d.) sequence $\{x_n\}$ uniformly distributed on the unit circle with respect to Lebesgue measure, a decreasing sequence of positive numbers $\{\ell_n\}$ and the associated random intervals $(x_n - \ell_n/2 \pmod 1, x_n + \ell_n/2 \pmod 1)$. Then one concerns how many or which points can be covered by these random intervals infinitely often [35].

Instead of a uniformly distribution sequence $\{x_n\}$, in our setting, x_n is driven by the orbit of a given point. So we call the setting here a dynamical covering problem.

13.1 Irrational Rotation

When T is the irrational rotation $x \to x + \alpha \pmod 1$ with α irrational, the set $C(\phi)$ can be written as

$$C(\phi) := \Big\{y \in [0, 1] : \|n\alpha - y\| < \phi(n), \text{ i.o. } n \in \mathbb{N}\Big\}.$$

The Hausdorff dimension of $C(\phi)$ was considered for the first time by Bernik and Dodson [6] with partial results. Bugeaud [9] and Schmeling and Troubetzkoy [58] independently proved the following result.

Theorem 13.1 ([9, 58]) *Let $\phi(n) = n^{-t}$ for some $t > 1$, the dimension of $C(\phi)$ is $1/t$.*

Schmeling and Troubetzkoy proved it by using the Three Gap Theorem of the distribution of $\{n\alpha : n \in \mathbb{N}\}$, while Bugeaud proved it by introducing the weak regular system (Regular system was introduced by Baker and Schmidt [1]). However at present, this is a consequence of the Minkowski's theorem by using the powerful mass transference principle established by Beresnevich and Velani [5].

Let's first recall the Minkowski's theorem.

Theorem 13.2 ([48] Minkowski's Theorem) *Let $\alpha \in [0, 1]$ be an irrational number. For any $y \neq k\alpha + m$ with $k, m \in \mathbb{Z}$, one has*

$$\|n\alpha - y\| < 1/4n, \ \textit{i.o. } n \in \mathbb{N}.$$

Mass transference principle discloses a deep phenomenon that Lebesgue measure theoretical statements for limsup sets can imply Hausdorff measure theoretical statements. Let $B(x,r)$ be a ball in $\mathbb{R}^k$. Denote B^f for the ball $B(x, f(x)^{1/k})$.

Theorem 13.3 ([5] Mass Transference Principle) *Let $\{B_i\}_{i\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^k$ with $r(B_i) \to 0$ as $i \to \infty$. Let f be a dimension function such that $x^k f(x)$ is monotonic and suppose that for any ball B in $\mathbb{R}^k$.*

$$\mathscr{H}^k\Big(B \cap \limsup_{i\to\infty} B_i^f\Big) = \mathscr{H}^k(B)$$

Then, for any ball B in $\mathbb{R}^k$

$$\mathscr{H}^f\Big(B \cap \limsup_{i\to\infty} B_i\Big) = \mathscr{H}^f(B).$$

The above results or methods work well when $\phi(n) = n^{-t}$. And in this special case, the dimension is independent of the irrational number α. But this is not the case as far as a general error function ϕ is concerned [17]. For an optimal bound estimations on the dimension of $W(\phi)$, one is referred to a result by Liao and Rams [44].

Theorem 13.4 ([44]) *For any α with Diophantine type β, one has*

$$\min\left\{u_\phi, \max\left\{\ell_\phi, \frac{1+u_\phi}{1+\beta}\right\}\right\} \le \dim_{\mathrm{H}} C(\phi) \le u_\phi,$$

where

$$u_\phi = \limsup_{n\to\infty} \frac{\log n}{-\log\phi(n)}, \quad \ell_\phi = \liminf_{n\to\infty} \frac{\log n}{-\log\phi(n)}.$$

13.2 Doubling Map

Let T be the doubling map $x \to 2x \pmod 1$. Fan et al. [18] considered the problem that how well $2^n x \pmod 1$ approximates a point y. More precisely, the set

$$C(\phi) := \Big\{y \in [0,1] : \|2^n x - y\| < \phi(n), \text{ i.o. } n \in \mathbb{N}\Big\}.$$

This set depends on the point x, since it is clear that when x is rational, $C(\phi)$ contains only finitely many points. Thus, instead of considering every x, the authors considered $C(\phi)$ as a random set of x with respect to an invariant Gibbs measure as the probability measure.

Let ν_φ, ν_ψ be two T-invariant probability Gibbs measures on $[0, 1]$ associated with normalized Hölder potentials φ and ψ. The measure ν_φ is used to describe the randomness of the set $C(\phi)$ with respect to x and the measure ν_ψ to describe sizes of sets.

Let the error function $\phi(n) = n^{-\kappa}$. Write $e_{\max} = \int -\varphi(x)dx$, and h_{ν_φ} for the measure theoretic entropy of ν_φ.

The first result concerns the ν_ψ-measure of $C(\phi)$ for a ν_φ-generic point x.

Theorem 13.5 ([18])

$$\sup\Big\{\kappa : \nu_\psi(C(\phi)) = 1, \nu_\varphi\text{-}a.e.\ x\Big\} = \frac{1}{\int \varphi d\nu_\psi}.$$

The second result concerns the dimension of $C(\phi)$.

Theorem 13.6 ([18]) *For ν_φ-almost all x,*

$$\dim_{\rm H} C(\phi) = \begin{cases} 1/\kappa, & \text{when } 1/\kappa \le h_{\nu_\varphi}; \\ E(1/\kappa), & \text{when } h_{\nu_\varphi} < 1/\kappa < e_{\max}; \\ 1, & \text{when } 1/\kappa \ge e_{\max}, \end{cases}$$

where $E(t)$ is the dimension spectrum of ν_φ, which is defined by

$$E(t) := \dim_{\rm H} \Big\{y : \lim_{r\to 0} \frac{\log \nu_\varphi(y-r, y+r)}{\log r} = t.\Big\}.$$

13.3 Expanding Markov Maps

Liao and Seuret [45] got the corresponding result successfully in the setting of finite Markov expanding systems. Let's first recall the definition of finite Markov expanding maps.

Definition 13.7 A transformation $T : [0, 1] \to [0, 1]$ is an expanding Markov map with finite partitions if there is a subdivision $\{a_i\}_{0\le i\le m}$ of $[0, 1]$ (denoted by $I(k) =]a_k, a_{k+1}[$ for $0 \le k \le Q - 1$) such that:

- (Expanding property) there is a positive integer n and a real number $\rho > 1$ such that
$$|(T^n)'(x)| \ge \rho > 1;$$
- (Piecewise monotonicity) T is strictly monotonic and can be extended to a C^2 function on each $\overline{I(i)}$;
- (Markov property) if $I(j) \cap T(I(k)) \neq \emptyset$, then $I(j) \subset T(I(k))$;

- (Mixing) there is an integer R such that $I(j) \subset \cup_{n=1}^{R} T^n(I(k))$ for every k and j;
- (Rényi's condition) For every $0 \le k < m$,

$$\sup_{x,y,z \in I(k)} \frac{|T''(x)|}{|T'(y)||T'(z)|} < \infty.$$

Let $\mu_{\max}$ be the Gibbs measure associated with the potential $\psi = -\log|T'|$, which is known to be equivalent to Lebesgue measure. Define

$$\alpha_{\max} = \frac{\int -\varphi d\mu_{\max}}{\int -\log|T'| d\mu_{\max}}.$$

Theorem 13.8 ([45]) *Let $T : [0,1] \to [0,1]$ be an expanding Markov map. Let ν_φ be the Gibbs measure with a Hölder potential φ and the error function $\phi(n) = n^{-\kappa}$.*

1. For ν_φ-almost all x,

$$\dim_{\rm H} C(\phi) = \begin{cases} 1/\kappa, & \text{when } 1/\kappa \le \dim_{\rm H} \nu_\varphi; \\ E(1/\kappa), & \text{when } \dim_{\rm H} \nu_\varphi < 1/\kappa < \alpha_{\max}; \\ 1, & \text{when } 1/\kappa \ge \alpha_{\max}, \end{cases}$$

where $E(t)$ is the dimension spectrum of ν_φ, which is defined by

$$E(t) := \dim_{\rm H} \Big\{ y : \lim_{r \to 0} \frac{\log \nu_\varphi(y-r, y+r)}{\log r} = t. \Big\}.$$

2. For ν_φ-almost all x, the Lebesgue measure of $C(\phi)$ is 0 if $1/\kappa < \alpha_{\max}$ and is full if $1/\kappa > \alpha_{\max}$.

It should be emphasized that there is much difference between the general Markov expanding system and the doubling map. For example, for the doubling map, since the Lyapunov exponents are constant, the intervals of generation n have same lengths. While for the Markov maps their lengths may be of very different order. The non-constant Lyapunov exponents bring many difficulties. Also there are essential differences in illustrating the dimension of $C(\phi)$ from below (for a general result, see [2]).

14 Quantitative Recurrence Properties

Quantitative recurrence properties concerns the Hausdorff dimension of the following sets in a metric dynamical system (X, T):

$$R(f) := \Big\{ x \in X : |T^n x - x| < e^{-S_n f(x)}, \text{ i.o. } n \in \mathbb{N} \Big\}.$$

14.1 β-Expansions

A general idea in tackling the dimensional theory in β expansion is that one focuses on the points for which the cylinders containing them have regular lengths. This is called an *approximating method*. But the risk is that, since one neglects some points, one may not get the right result by such a method. In [65], Tan and Wang observed a fact for β expansion which can be used to show that in many cases the approximating method works.

Write the β-expansion of 1 as

$$1 = \frac{\varepsilon_1^*}{\beta} + \frac{\varepsilon_2^*}{\beta^2} + \cdots .$$

Define a sequence of β_N approximating β from below: let $\beta_N > 1$ be the solution to

$$1 = \frac{\varepsilon_1^*}{x} + \cdots + \frac{\varepsilon_N^*}{x^N}.$$

Given a β-admissible block $\omega = (\omega_1, \ldots, \omega_n)$ with length n, one can obtain a β_N-admissible sequence $\overline{\omega}$ by changing the blocks $(\omega_1^*(\beta), \ldots, \omega_N^*(\beta))$ in w from the left to the right with non-overlaps to $(\omega_1^*(\beta), \ldots, \omega_N^*(\beta) - 1)$. Denote the resulting sequence by $\overline{\omega}$.

Proposition 14.1 $\overline{\omega} \in \Sigma_{\beta_N}^n$.

Define the map $\pi_N : \Sigma_\beta^n \to \Sigma_{\beta_N}^n$ as $\pi_N(\omega) = \overline{\omega}$.

Proposition 14.2 *For any* $\overline{\omega} \in \Sigma_{\beta_N}^n$,

$$\sharp \pi_N^{-1}(\overline{\omega}) \le 2^{\frac{n}{N}},$$

i.e., the number of the inverse of $\overline{\omega} \in \Sigma_{\beta_N}^n$ *is at most* $2^{\frac{n}{N}}$.

Corollary 14.3 *Let g be a continuous function on* $[0, 1]$. *The pressure function* $P(g, T_\beta)$ *is continuous with respect to* β.

This enables one to show that

Theorem 14.4 ([65]) *Let* $\beta > 1$ *and f a positive continuous function on* $[0, 1]$. *Then the Hausdorff dimension of* $R(f)$ *is the unique solution s to the pressure function*

$$P\big(T, -s(\log \beta + f)\big) = 0.$$

14.2 Conformal Iterated Function Systems

Let $\Phi = \{\phi_i : i \in \Lambda\}$ be a conformal iterated function system on $[0, 1]^d$ with Λ a countable index set. Denote by J the attractor of Φ.

It would be clear that there is natural dynamical system on J, but since the points in J may have multiple coding representations, the transformation may not be well defined at those points. So instead of using a transformation, we use *the inverse* of ϕ^{-1}.

Let $f : [0, 1]^d \to \mathbb{R}^+$ be a positive function, $S_nf(x)$ be the sum $f(x)+f(\phi_{w_1}^{-1}(x))+\cdots+f((\phi_{w_1}\circ\cdots\circ\phi_{w_{n-1}})^{-1}(x))$ (analogous to an ergodic sum).

In this conformal system, the set $R(f)$ can be formulated as

$$\Big\{x \in J : \big|x - (\phi_{w_1}\circ\cdots\circ\phi_{w_n})^{-1}(x)\big| < e^{-S_nf(x)}, w_i \in \Lambda, 1 \le i \le n, \text{ i.o. } n \in \mathbb{N}\Big\}.$$

Theorem 14.5 ([59]) *Let Φ be a conformal IFS on $[0, 1]^d$ with open set condition, and let $f : [0, 1]^d \to \mathbb{R}^+$ be a continuous function. Then*

$$\dim_{\mathrm{H}} R(f) = \inf\big\{t \ge 0 : \mathrm{P}(-t(\log|(\Phi^{-1})'| + f)) \le 0\big\}. \tag{8}$$

15 Remarks on Shrinking Target Problem

In this last section, we give a possible conjecture on the size of the shrinking target problems:

$$W(\phi) := \Big\{x \in X : T^n x \in B(z, \phi(n)), \text{ i.o. } n \in \mathbb{N}\Big\}.$$

Or we can consider another form

$$W(f) := \Big\{x \in X : x \in B\big(y, e^{-S_nf(y)}\big), y \in \mathscr{I}_n, \text{ i.o. } n \in \mathbb{N}\Big\},$$

where $\mathscr{I}_n := \{y : T^n y = z\}$. These two sets may not be equal but closely related.

In most of these concrete systems cited above, the dimension of $W(f)$ is usually given by a unified formula: [28, 29, 43, 52, 67],

$$\dim_{\mathrm{H}} W(f) = \inf\Big\{s \ge 0 : \mathtt{P}(T, -sf) \le 0\Big\}, \tag{9}$$

where $\mathtt{P}$ is the pressure function.

Recall that in those cases, the dimension of the phase space X is given by the Bowen-Manning-McCluskey formula:

$$\dim_{\mathrm{H}} X = \inf\Big\{s \geq 0 : \mathtt{P}(T, -s\log|T'|) \leq 0\Big\}, \tag{10}$$

Now we pose some conditions on (X, T): Assume there exist $c_1 > c_2 > 0$ such that for every $n \geq 1$,

- Covering: $X \subset \bigcup_{z \in \mathscr{I}_n} B(z, c_1|(T^n)'(z)|^{-1})$,
- Disjointness: $\{B(z, c_2|(T^n)'(z)|^{-1}), z \in \mathscr{I}_n\}$ are pairwise disjoint.
- T is expanding.

We pose the following conjecture for a general system as far as possible.

Conjecture 15.1 Under the conditions given above on the system (X, T), if

$$\dim_{\mathrm{H}} X = \inf\{s \geq 0 : \mathtt{P}(T, -s\log|T'|) \leq 0\}$$

then one would have

$$\dim_{\mathrm{H}} W(f) = \inf\{s \geq 0 : \mathtt{P}(T, -sf) \leq 0\}.$$

One can also compare the situation here (the third item below) with the mass transference principle in the classic Diophantine approximation developed by Beresnevich and Velani [5]. So we call the formula (10) a dimension transference principle.

Let's give some evidences supporting the conjecture:

- It is clear that (10) is a natural upper bound of $\dim_{\mathrm{H}} W(f)$.
- Recall the definition of the pressure function:

$$\mathtt{P}(T, -sf) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x:T^n x = y} e^{-sS_n f(x)},$$

  which concerns also about the distribution of the pre-images. With suitable normalization, the quantity $e^{-sS_nf(x)}$ can be used to define a μ-measure of the ball $B(x, |(T^n)'(x)|)$. So the solution s to $\mathtt{P}(T, -sf) = 0$ is tightly related to a Hölder exponent of the measure μ in average. This leads to the dimension from below of the support of μ by the classic mass distribution principle [16].
- Notice that $|(T^n)'(z)|^{-1} = -S_n(\log|T'|)(z)$. Comparing the first condition on X with the definition of $W(f)$, it looks like that in defining $W(f)$, one shrinks the ball $B(z, e^{-S_n \log|T'|(z)})$ in defining X to the ball $B(z, e^{-S_nf(z)})$.

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