Relation Algebras, Idempotent Semirings and Generalized Bunched Implication Algebras

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Abstract. This paper investigates connections between algebraic structures that are common in theoretical computer science and algebraic logic. Idempotent semirings are the basis of Kleene algebras, relation algebras, residuated lattices and bunched implication algebras. Extending a result of Chajda and Länger, we show that involutive residuated lattices are determined by a pair of dually isomorphic idempotent semirings on the same set, and this result also applies to relation algebras. Generalized bunched implication algebras (GBI-algebras for short) are residuated lattices expanded with a Heyting implication. We construct bounded cyclic involutive GBI-algebras from so-called weakening relations, and prove that the class of weakening relation algebras is not finitely axiomatizable. These algebras play a role similar to representable relation algebras, and we identify a finitely-based variety of cyclic involutive GBI-algebras that includes all weakening relation algebras. We also show that algebras of down-closed sets of partially-ordered groupoids are bounded cyclic involutive GBI-algebras.

1 Introduction

Idempotent semirings, also known as dioids, play an important role in many applications in computer science, ranging from regular languages and Kleene algebras to shortest path algorithms using tropical semirings such as the max-plus semiring. They are also generalizations of distributive lattices, quantales, residuated lattices and relation algebras, each of which have been studied extensively in mathematics and logic. While it has been known for a long time that Boolean algebras, relation algebras and involutive residuated lattices have two isomorphic semiring reducts that are connected by an anti-isomorphism, the characterization of these algebras by coupled semirings has only recently been formalized in a result by Di Nola and Gerla [2] for MV-algebras and by Chajda and Länger [1] for bounded commutative integral involutive residuated lattices. In Sect. 2 we show that a more general result holds for arbitrary involutive residuated lattices, hence also for relation algebras. This leads to a shorter axiomatization for involutive residuated lattices using only two binary operations, two unary operations and a constant, which is useful for working with relation algebras and their generalizations in automated theorem provers and finite model finders.

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Residuated lattices are generalizations of relation algebras and of many other mathematical structures, including Heyting algebras, MV-algebras, basic logic algebras, lattice-ordered groups and quantales. They are also the algebraic semantics of many logical systems, such as intuitionistic logic, relevance logic, linear logic and other substructural logics. Even though they span so many algebraic and logical systems, residuated lattices have a simple definition and a surprisingly deep and elegant algebraic theory that is shared by all the special cases. In this paper we concentrate mostly on involutive residuated lattices expanded with a Heyting arrow. In the bounded and commutative case these algebras are known as bunched implication algebras, or BI-algebras, and have found significant applications in separation logic, a Hoare logic developed by Reynolds, O'Hearn, Pym and others for the verification of pointer data-structures, memory management algorithms and concurrent software. Most of the algebraic properties of BI-algebras hold also in the non-commutative setting of generalized bunched implication algebras, or GBI-algebras for short, so we take this more general approach. By definition a GBI-algebra is a residuated lattice with a Brouwerian algebra defined on the same lattice. In Sect. 3 we observe that relation algebras are a subvariety of bounded involutive GBI-algebras, so this provides an interesting connection between these classes of algebras. We investigate weakening relation algebras that are intuitionistic versions of representable relation algebras, and show that they are not finitely axiomatizable. In Sect. 4 we give partial-order semantics for these algebras and show that they are based on groupoids, i.e. small categories in which all morphisms are isomorphisms.

2 Coupled Semirings

A *semilattice* is of the form $(A, \vee)$ such that $\vee$ is a binary operation on the set A that is associative, commutative and idempotent ($x \vee x = x$), and in such an algebra $x \leq y \iff x \vee y = y$ defines a partial order. A *monoid* $(A, \cdot, 1)$ has an associative operation $\cdot$ such that $1x = x1 = x$. An *idempotent semiring* is an algebra $\mathbf{A} = (A, \vee, \cdot, 1)$ where $(A, \vee)$ is a semilattice, $(A, \cdot, 1)$ is a monoid and $\cdot$ distributes over $\vee$ in both arguments (i.e., $x(y \vee z) = xy \vee xz$ and $(x \vee y)z = xz \vee yz$). A *lattice* $(A, \wedge, \vee)$ is a pair of semilattices $(A, \vee)$ and $(A, \wedge)$ linked by the absorption laws $x \wedge (x \vee y) = x = x \vee (x \wedge y)$. A *residuated lattice* is of the form $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \backslash, /)$ where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and $\backslash, /$ are the left and right residuals of $\cdot$, i.e., for all $x, y, z \in A$

$$xy \leq z \iff y \leq x\backslash z \iff x \leq z/y.$$

These equivalences imply that $(A, \vee, \cdot, 1)$ is an idempotent semiring since, e.g., $x(y \vee z) \leq w \iff y \vee z \leq x\backslash w \iff y, z \leq x\backslash w \iff xy \vee xz \leq w$. A residuated lattice is *bounded* if it has a bottom element $\bot$, *integral* if 1 is the top element and *commutative* if the identity $xy = yx$ holds. For an arbitrary constant 0 in a residuated lattice define the *linear negations* ${\sim}x = x\backslash 0$ and $-x = 0/x$. The constant 0 is *involutive* if ${\sim}{-x} = x = -{\sim}x$ for all elements x, and an *involutive residuated lattice* (also called an involutive FL-algebra) is a residuated lattice with an involutive 0. Such an algebra is *cyclic* if $\sim x = -x$. Note that a commutative involutive residuated lattice satisfies $x\backslash y = y/x$ and hence is always cyclic.

For example, a relation algebra $(A, \wedge, \vee, \neg, ;, ^{\smile}, 1)$ is a cyclic involutive residuated lattice if one defines $x\backslash y = \neg(x^{\smile}; \neg y)$, $x/y = \neg(\neg x; y^{\smile})$ and $0 = \neg 1$, and omits the operations $\neg, ^{\smile}$ from the signature. The cyclic linear negation is given by $\sim x = \neg(x^{\smile}) = (\neg x)^{\smile}$. An example that is a bounded commutative integral involutive residuated lattice is provided by the *standard MV-algebra* $([0,1], \min, \max, \cdot, 1, \backslash, /)$ where $xy = \max(x+y-1, 0)$ and $x\backslash y = y/x = \min(1-x+y, 1)$. The class of all MV-algebras is the variety generated by this unit-interval algebra (i.e. the smallest class closed under products, subalgebras and homomorphic images). We note that the variety of involutive residuated lattices has a decidable equational theory [3,17] while this is not the case for relation algebras.

In [2] Di Nola and Gerla showed that every MV-algebra is determined by a pair of coupled commutative semirings, and Chajda and Länger [1] generalized this construction to bounded commutative integral involutive residuated lattices. We show here that the result is actually valid in the general setting of involutive residuated lattices and hence includes all (reducts of) relation algebras.

For two algebras $\mathbf{A}, \mathbf{B}$ with the same signature, an *anti-isomorphism* $\alpha\colon \mathbf{A} \to \mathbf{B}$ is like an isomorphism, except that for all binary operations $*$ we have

$$\alpha(x *^{\mathbf{A}} y) = \alpha(y) *^{\mathbf{B}} \alpha(x)$$

(instead of $\alpha(x) *^{\mathbf{B}} \alpha(y)$).

A *generalized coupled semiring* is a triple $((A, \vee, \cdot, 1), (A, \wedge, +, 0), \alpha)$ such that

(i) $(A, \vee, \cdot, 1)$ and $(A, \wedge, +, 0)$ are idempotent semirings
(ii) $(A, \wedge, \vee)$ is a lattice (with order denoted by $\leq$)
(iii) α is an anti-isomorphism from $(A, \vee, \cdot, 1)$ to $(A, \wedge, +, 0)$
(iv) $x \leq y$ if and only if $1 \leq \alpha(x) + y$

Theorem 1. *Let* $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \backslash, /, 0)$ *be an involutive residuated lattice with linear negations* $\sim x = x\backslash 0$, $-x = 0/x$ *and define* $x + y = \sim((-y) \cdot (-x))$. *Then* $((A, \vee, \cdot, 1), (A, \wedge, +, 0), \sim)$ *is a generalized coupled semiring.*

Proof. In any residuated lattice $\cdot$ distributes over $\vee$ since $x(y \vee z) \leq w \iff y \vee z \leq x\backslash w \iff y, z \leq x\backslash w \iff xy, xz \leq w \iff xy \vee xz \leq w$, and likewise for $(x \vee y)z = xz \vee yz$. In an involutive residuated lattice the linear negations are order-reversing bijections, hence $\sim(x \vee y) = \sim y \wedge \sim x$. Replacing x, y by $\sim x, \sim y$ in the definition of $+$ shows that $\sim(xy) = \sim y + \sim x$, and $\sim 1 = 1\backslash 0 = 0$ since $x \leq 1\backslash 0 \iff x = 1x \leq 0$. Therefore (iii) is satisfied, and (i) follows since anti-isomorphisms preserve the structure of idempotent semirings. Obviously (ii) holds, so it remains to check (iv): $1 \leq \sim x + y \iff 1 \leq \sim((-y)x) \iff (-y)x \leq 0 \iff x \leq \sim - y = y$, where the middle equivalence holds because the linear negations are order-reversing and $-1 = 0$.

We now show that the converse also holds.

Theorem 2. *Let* $((A, \vee, \cdot, 1), (A, \wedge, +, 0), \alpha)$ *be a generalized coupled semiring and define* $x\backslash y = \alpha(\alpha^{-1}(y) \cdot x)$, $x/y = \alpha^{-1}(y \cdot \alpha(x))$. *Then* $\mathbf{A} = (A, \wedge, \vee, \cdot, 1, \backslash, /, 0)$ *is an involutive residuated lattice and* $\alpha(x) = {\sim}x$. *If* $\alpha = \alpha^{-1}$ *then* $\mathbf{A}$ *is cyclic, and if* 1 *is the top element of the first semiring then* $\mathbf{A}$ *is bounded and integral.*

Proof. By (i) and (ii) $(A, \wedge, \vee)$ is a lattice and $(A, \cdot, 1)$ is a monoid, so we need to show that $\backslash, /$ are residuals with 0 as involutive element. By (iv) we have

$$\begin{aligned} xy \le z &\iff 1 \le \alpha(xy) + z \\ &\iff 1 \le \alpha(y) + \alpha(x) + z \\ &\iff y \le \alpha(x) + z = \alpha(x) + \alpha(\alpha^{-1}(z)) = \alpha(\alpha^{-1}(z) \cdot x) = x\backslash z. \end{aligned}$$

To see that $xy \le z \iff x \le z/y$, first observe that (iv) is equivalent to $x \le y \iff 1 \le \alpha(\alpha^{-1}(y) \cdot x)$ and after replacing y by $\alpha(y)$ one obtains $x \le \alpha(y) \iff 1 \le \alpha(y \cdot x)$. From (iii) it follows that α and α^{-1} are order-reversing, so we compute

$$\begin{aligned} xy \le z &\iff \alpha(z) \le \alpha(xy) \\ &\iff 1 \le \alpha(x \cdot y \cdot \alpha(z)) \\ &\iff y \cdot \alpha(z) \le \alpha(x) \\ &\iff x \le \alpha^{-1}(y \cdot \alpha(z)) = z/y. \end{aligned}$$

Condition (iv) also implies that $\alpha(0) = 1$ since $1 \le 1 \implies 1 \le \alpha(\alpha^{-1}(1) \cdot 1) = \alpha(1 \cdot \alpha^{-1}(1)) \implies \alpha^{-1}(1) \le \alpha(1) = 0 \implies \alpha(0) \le 1$ and $0 \le 0 \implies 1 \le \alpha(1 \cdot 0) = \alpha(0)$. The element 0 is involutive since, ${\sim}x = x\backslash 0 = \alpha(\alpha^{-1}(0) \cdot x) = \alpha(1x) = \alpha(x)$, and $-x = 0/x = \alpha^{-1}(x \cdot \alpha(0)) = \alpha^{-1}(x1) = \alpha^{-1}(x)$. □

The preceding theorems show that all involutive residuated lattices are completely determined by their $\vee, \cdot$ structure and by an order-reversing bijection that satisfies property (iv). It also follows that the residuals are term-definable $x\backslash y = {\sim}((-y) \cdot x)$ and $x/y = -(y \cdot ({\sim}x))$, though this is a well-known result [6, p. 153].

As an application of the above result we obtain a fairly concise equational basis for the variety of involutive residuated lattices using the signature $\vee, \cdot, 1, {\sim}, -$ since the remaining operations are defined by $x \wedge y = {\sim}(-x \vee -y)$, $x\backslash y = {\sim}((-y) \cdot x)$, $x/y = -(y \cdot ({\sim}x))$ and $0 = {\sim}1$.

Theorem 3. *An algebra* $(A, \vee, \cdot, 1, {\sim}, -)$ *is (term equivalent to) an involutive residuated lattice if and only if the following 12 identities hold:*

- $(x \vee y) \vee z = x \vee (y \vee z)$, $x \vee y = y \vee x$ *(associativity and commutativity)*
- $x(y \vee z) = xy \vee xz$, $(x \vee y)z = xz \vee yz$ *(distributivity of* $\cdot$ *over* $\vee$*)*
- $(xy)z = x(yz)$, $x1 = x$ *(associativity and right identity of* $\cdot$*)*
- ${\sim}{-x} = x = -{\sim}x$ *(involution of linear negations)*

- $\sim(-(x \vee y) \vee -x) = x = \sim((-x) \vee -y) \vee x$ *(absorption laws)*
- $1 \leq \sim(x(\sim x))$, $x(\sim(yx)) \leq \sim y$ *(equivalent to* $x \leq \sim y \iff 1 \leq \sim(yx)$*).*

Proof. From Theorem 1 it follows that an involutive residuated lattice satisfies the above identities, where the last two are derived from condition (iv) of coupled semirings by $\sim y \leq \sim y \Rightarrow 1 \leq \sim(y(\sim y))$ and

$$\sim(yx) \leq \sim(yx) \Rightarrow 1 \leq \sim(yx(\sim(yx))) = \sim(y(x\sim(yx))) \Rightarrow x(\sim(yx)) \leq \sim y.$$

Conversely, assume $(A, \vee, \cdot, 1, \sim, -)$ is an algebra that satisfies the identities, and define $x \wedge y = \sim(-y \vee -x)$, $x + y = \sim((-y)(-x))$ and $0 = \sim 1$. It remains to show that $((A, \vee, \cdot, 1), (A, \wedge, +, 0), \sim)$ is a generalized coupled semiring. The absorption laws translate to the usual form $(x \vee y) \wedge x = x = (x \wedge y) \vee x$, and it is easy to see that $\wedge$ is associative and commutative. Since idempotence of $\wedge, \vee$ follow from the absorption laws, $(A, \wedge, \vee)$ is a lattice. The definition of $\wedge, +, 0$ and the involution identities show that $\sim$ is an anti-isomorphism from $(A, \vee, \cdot, 1)$ to $(A, \wedge, +, 0)$. Note that condition (iv) of coupled semirings is equivalent to $x \leq \sim y \iff 1 \leq \sim(yx)$. To see this holds we compute: $x \leq \sim y \Rightarrow y \leq -x$, hence $1 \leq \sim(y(\sim y)) \leq \sim(y(\sim -x)) = \sim(yx)$, and by distributivity over $\vee$, the operation $\cdot$ is order-preserving in each argument, so

$$1 \leq \sim(yx) \Rightarrow x \leq x(\sim(yx)) \leq \sim y.$$

Finally, since $-y = (-y)1$, we deduce $1x = x$ from the following equivalences: $x \leq y \iff 1 \leq \sim((-y)x) = \sim((-y)1x) \iff 1x \leq y$. Therefore $(A, \vee, \cdot, 1)$ is an idempotent semiring, and the anti-isomorphism $\sim$ shows the same holds for $(A, \wedge, +, 0)$. □

The standard equational basis for involutive residuated lattices has 15 identities and a signature with 5 binary operations. A short equational basis can be useful when searching for finite counterexamples or using automated theorem provers. It is not known if the given basis is irredundant, but it is interesting to note that it suffices to assume that 1 is a right-identity element.

It is easy to extend the preceding theorems to a categorical equivalence between the categories of involutive residuated lattices and generalized coupled semirings.

There is only one 2-element residuated lattice, namely the two-element lattice $\mathbf{2} = \{0, 1\}$ with $x \cdot y = x \wedge y$. Clearly this is a bounded commutative involutive lattice, and is in fact a Boolean algebra. There are three 3-element residuated lattices, the 3-element Gödel algebra $\mathbf{G}_3 = \{\bot, a, 1\}$ with $aa = a$ is integral but not involutive, the 3-element Łukasiewicz algebra $\mathbf{L}_3 = \{0, a, 1\}$ with $aa = 0$ which is both integral and cyclic involutive, and the Sugihara algebra $\mathbf{S}_3 = \{\bot, 1, \top\}$ which is cyclic involutive but not integral.

3 Distributive Residuated Lattices and Generalized Bunched Implication algebras

A *distributive residuated lattice* is a residuated lattice that satisfies the distributive law $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. A typical example of a distributive residuated lattice is given by a collection $\mathcal{R}$ of binary relations on a set X such that $\mathcal{R}$ is closed under intersection $\cap$, union $\cup$, composition $\circ$, residuals $\backslash, /$ and contains a relation E such that $E \circ R = R \circ E = R$ for all $R \in \mathcal{R}$. Here the operations $\backslash, /$ are defined by the usual expressions for residuals on binary relations: $R\backslash S = (R^{\smile} \circ S')'$ and $R/S = (R' \circ S^{\smile})'$, where $^{\smile}$ denotes the converse operation and $'$ is the operation of complementation with respect to the total relation $X^2 = X \times X$. Note that we are not assuming $\mathcal{R}$ is closed under $'$ or that it includes X^2.

A *Brouwerian algebra* is a residuated lattice that satisfies the identity $x \wedge y = xy$. Residuated operations always distribute over lattice joins, hence Brouwerian algebras satisfy the distributive law. Moreover, since $x \wedge y \leq x$ is equivalent to $y \leq x\backslash x$, it follows that Brouwerian algebras have a top element, denoted by the constant $\top$, which is also the identity element for $\wedge$. A *Heyting algebra* is a bounded Brouwerian algebra, i.e., it also has a bottom element $\bot$. The meet operation is commutative, hence $x\backslash y = y/x$ and this operation is usually called a Heyting implication and denoted $x \to y$. We now consider algebras that combine the signatures of residuated lattices and Brouwerian algebras.

A *generalized bunched implication algebra* (or GBI-algebra for short) $\mathbf{B} = (B, \wedge, \vee, \to, \top, \cdot, 1, \backslash, /)$ is a residuated lattice with an additional binary operation $\to$ that is a Heyting implication, i.e., for all $x, y, z \in B$

$$x \wedge y \leq z \iff y \leq x \to z.$$

A GBI-algebra is *bounded* if it also contains a bottom element $\bot$, and we consider $\bot$ to be a constant operation of the algebra. Hence bounded GBI-algebras have Heyting algebra reducts, while GBI-algebras have Brouwerian lattice reducts. In a bounded GBI-algebra, the *intuitionistic negation* is defined by $\neg x = x \to \bot$. An example of a GBI-algebra that is not bounded is given by the nonpositive integers $\mathbb{Z}^-$ with the operations $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$, $xy = x + y$, $x\backslash y = y/x = \min(y - x, \top)$, $1 = \top = 0$ and

$$x \to y = \begin{cases} y & \text{if } x > y \\ \top & \text{otherwise.} \end{cases}$$

An *involutive* GBI-algebra is a (necessarily bounded) GBI-algebra that has an involutive constant 0, while a *Boolean* GBI-algebra is bounded GBI-algebra that satisfies the double negation identity $\neg\neg x = x$. For example, sequential algebras [9,10] are (term-equivalent to) a subvariety of Boolean GBI-algebras and relation algebras are (term-equivalent to) a subvariety of Boolean involutive GBI-algebras (see Theorem 6 below). Boolean GBI-algebras are also known as *residuated Boolean monoids* or *rm-algebras* [8,12].

A *bunched implication algebra* (or BI-algebra) is a bounded GBI-algebra that satisfies the identity $xy = yx$. These algebras are the algebraic semantics of separation logic, a programming logic for modeling mutable data structures and concurrent processes [15,16]. An advantage of the varieties of GBI-algebras and BI-algebras is that they have decidable equational theories [5,7], whereas the subvarieties of Boolean GBI-algebras and Boolean BI-algebras have undecidable equational theories [13].

In this section we study the algebraic structure of GBI-algebras and their connections with relation algebras and residuated lattices. Table 1 summarizes how many residuated lattices there are up to isomorphism on a set with n elements, and provides the same information for some of the subclasses introduced above.

Table 1. Number of algebras up to isomorphism on a set with n elements

Number of elements: $n =$	1	2	3	4	5	6	7	8
Residuated lattices	1	1	3	20	149	1488	18554	295292
GBI-algebras	1	1	3	20	115	899	7782	80468
Bunched implication algebras	1	1	3	16	70	399	2261	14358
Involutive residuated lattices	1	1	2	9	21	101	284	1464
Cyclic involutive resid. lattices	1	1	2	9	21	101	279	1433
Involutive GBI-algebras	1	1	2	9	8	43	49	282
Cyclic involutive GBI-algebras	1	1	2	9	8	43	48	281
Involutive BI-algebras	1	1	2	9	8	42	46	263
Boolean involutive BI-algebras	1	1	0	5	0	0	0	25
Relation algebras	1	1	0	3	0	0	0	13

Since (bounded) GBI-algebras have Brouwerian algebra reducts, they also satisfy the distributive law. A *relational GBI-algebra* is of the form $(\mathcal{R}, \cap, \cup, \to, \top, \circ, E, \backslash, /)$, where $\mathcal{R}$ is a collection of binary relations on a set X, and $\mathcal{R}$ is closed under these operations. Note that $\cup, \cap, \circ$ are the usual set-theoretic operations on binary operations, but $\top$ need only be transitive, $R \circ E = R = E \circ R$, and $\to, \backslash, /$ need only satisfy

$$R \cap S \subseteq T \iff S \subseteq R \to T \quad R \circ S \subseteq T \iff S \subseteq R \backslash T \iff R \subseteq T/S$$

for all $R, S, T \in \mathcal{R}$.

Natural examples of relational GBI-algebras are constructed as follows: Let $\mathbf{P} = (P, \sqsubseteq)$ be a partially ordered set, $Q \subseteq P^2$ an equivalence relation that contains $\sqsubseteq$, and define the set of *weakening relations* on $\mathbf{P}$ by $\mathrm{Wk}(\mathbf{P}, Q) = \{\sqsubseteq \circ R \circ \sqsubseteq \,:\, R \subseteq Q\}$. Since $\sqsubseteq$ is transitive and reflexive, this set can also be defined by $\{R \subseteq Q : \sqsubseteq \circ R \circ \sqsubseteq = R\}$. Theorem 5 below shows that $\mathrm{Wk}(\mathbf{P}, Q)$ is a bounded cyclic involutive relational GBI-algebra with Q as top element. If $Q = P \times P$, then we write $\mathrm{Wk}(\mathbf{P})$ instead of $\mathrm{Wk}(\mathbf{P}, Q)$ and call this algebra the *full weakening relation algebra*.

Weakening relations are the natural analogue of binary relations when the category **Set** of sets and functions is replaced by the category **Pos** of partially ordered sets and order-preserving functions. Since sets can be considered as discrete posets (i.e. ordered by the identity relation), **Pos** contains **Set** as a full subcategory, which implies that weakening relations are a substantial generalization of binary relations. They have applications in sequent calculi, proximity lattices/spaces, order-enriched categories, cartesian bicategories, bi-intuitionistic modal logic, mathematical morphology and program semantics, e.g. via separation logic.

Lemma 4. *Let* $\mathbf{P} = (P, \sqsubseteq)$ *be a poset,* Q *an equivalence relation that contains* $\sqsubseteq$*,* R *any binary relation on* P *and let* $R' = Q - R$*. Then*

1. $\sqsubseteq \circ R \circ \sqsubseteq = R$ *is equivalent to* $\sqsupseteq \circ R' \circ \sqsupseteq = R'$*, and*
2. $(\sqsupseteq \circ R \circ \sqsupseteq)'$ *is a weakening relation.*

Proof. 1. Assume $\sqsubseteq \circ R \circ \sqsubseteq = R$ and $(x, y) \in \sqsupseteq \circ R' \circ \sqsupseteq$. Then there exist $(u, v) \in Q$ such that $x \sqsupseteq u$, $(u, v) \notin R$ and $v \sqsupseteq y$. If $(x, y) \in R$ then $u \sqsubseteq x$ and $y \sqsubseteq v$ imply $(u, v) \in R$, which is a contradiction. Hence $(x, y) \in R'$ and therefore $\sqsupseteq \circ R' \circ \sqsupseteq = R'$. The converse is proved by a dual argument.
2. Let $(x, y) \in \sqsubseteq \circ (\sqsupseteq \circ R \circ \sqsupseteq)' \circ \sqsubseteq$. Then there exist $(u, v) \in Q$ such that $x \leq u$, $v \leq y$ and $(u, v) \notin \sqsupseteq \circ R \circ \sqsupseteq$. If $(x, y) \in \sqsupseteq \circ R \circ \sqsupseteq$ then there exist $(r, s) \in R$ such that $r \sqsubseteq x$ and $y \sqsubseteq s$. However, now transitivity implies $r \sqsubseteq u$ and $v \sqsubseteq s$, hence $(u, v) \in \sqsupseteq \circ R \circ \sqsupseteq$, a contradiction. Therefore $(x, y) \in (\sqsupseteq \circ R \circ \sqsupseteq)'$, and it follows that $\sqsubseteq \circ (\sqsupseteq \circ R \circ \sqsupseteq)' \circ \sqsubseteq \subseteq (\sqsupseteq \circ R \circ \sqsupseteq)'$. The reverse inclusion always holds by reflexivity. □

Theorem 5. *Let* $\mathbf{P} = (P, \sqsubseteq)$ *be a poset,* Q *an equivalence relation that contains* $\sqsubseteq$*, and for* $R, S \in \mathrm{Wk}(\mathbf{P}, Q)$ *define*

- $\top = Q$, $\bot = \emptyset$, $1 = \sqsubseteq$, $0 = \sqsupseteq'$,
- $R \to S = (\sqsupseteq \circ (R \cap S') \circ \sqsupseteq)'$ *where* $S' = Q - S$,
- $R \backslash S = (\sqsupseteq \circ R^{\smile} \circ S' \circ \sqsupseteq)'$ *and* $R/S = (\sqsupseteq \circ R' \circ S^{\smile} \circ \sqsupseteq)'$.

Then $\mathbf{Wk}(\mathbf{P}, Q) = (\mathrm{Wk}(\mathbf{P}, Q), \cap, \cup, \to, \top, \bot, \circ, 1, \backslash, /, 0)$ *is a bounded cyclic involutive relational GBI-algebra with involutive constant* 0*, and the linear negation is* ${\sim}R = R\backslash 0 = 0/R = R^{\smile\prime} = R^{\prime\smile}$.

Proof. Note that $\mathbf{Wk}(\mathbf{P}, Q)$ contains the empty set and is closed under $\circ$ and under (arbitrary) meets and joins. The operation $'$ is complementation with respect to Q, but it is not an operation on $\mathrm{Wk}(\mathbf{P}, Q)$. The relation $\sqsubseteq$ is an identity element for weakening relations since $\sqsubseteq \circ \sqsubseteq = \sqsubseteq$. The formula for $R \to S$ is justified by the lemma above and the following equivalences:

$$\begin{aligned}
R \cap S \subseteq T &\iff R \cap T' \cap S = \emptyset \\
&\iff R \cap T' \subseteq S' \\
&\iff R \cap T' \subseteq (\sqsupseteq \circ (R \cap T') \circ \sqsupseteq) \subseteq (\sqsupseteq \circ S' \circ \sqsupseteq) = S' \\
&\iff S \subseteq (\sqsupseteq \circ (R \cap T') \circ \sqsupseteq)' \\
&\iff S \subseteq R \to T.
\end{aligned}$$

For $R\backslash S$ the calculation is similar:

$$\begin{aligned} R \circ S \subseteq T &\iff R^{\smile} \circ T' \subseteq S' \qquad \text{(by relation algebra)} \\ &\iff R^{\smile} \circ T' \subseteq (\sqsupseteq \circ R^{\smile} \circ T' \circ \sqsupseteq) \subseteq (\sqsupseteq \circ S' \circ \sqsupseteq) = S' \\ &\iff S \subseteq (\sqsupseteq \circ R^{\smile} \circ T' \circ \sqsupseteq)' \\ &\iff S \subseteq R\backslash T \end{aligned}$$

and the argument for R/S is a mirror image.

Lemma 4 shows that $0 = \sqsupseteq'$ is a weakening relation and the linear negations agree since

$$\sim x = x\backslash 0 = (\sqsupseteq \circ x^{\smile} \circ \sqsupseteq'' \circ \sqsupseteq)' = (\sqsupseteq \circ x^{\smile} \circ \sqsupseteq)' = (\sqsupseteq \circ \sqsupseteq'' \circ x^{\smile} \circ \sqsupseteq)' = 0/x = -x.$$

Hence $\sim R = (\sqsubseteq \circ R \circ \sqsubseteq)^{\smile\prime} = R^{\smile\prime} = R'^{\smile}$ for any weakening relation x, so $\sim\sim R = R^{\smile\prime\smile\prime} = R^{\smile\smile\prime\prime} = x$. □

In the previous proof we used the notation $^{\smile}$ for the converse operation on binary relations. In an abstract relation algebra, this operation is simply an order-preserving permutation that satisfies $x^{\smile\smile} = x$ and $(xy)^{\smile} = y^{\smile}x^{\smile}$, and it is definable by the composition of (cyclic) linear negation and complement: $x^{\smile} = \sim\neg x$ (where $\neg x = x \to \bot$). We extend this notation to bounded cyclic involutive GBI-algebras, but note that $x^{\smile\smile} = x$ only holds in the Boolean case, and adding $(xy)^{\smile} = y^{\smile}x^{\smile}$ gives an alternative definition of abstract relation algebras.

Theorem 6. *Boolean cyclic involutive GBI-algebras satisfy the identities* $\sim\neg x = \neg\sim x$, $(x \vee y)^{\smile} = x^{\smile} \vee y^{\smile}$, $(x \wedge y)^{\smile} = x^{\smile} \wedge y^{\smile}$ *and* $x^{\smile\smile} = x$. *They are relation algebras if and only if they also satisfy the identity* $(xy)^{\smile} = y^{\smile}x^{\smile}$.

Proof. By definition, Boolean GBI-algebras satisfy $\neg\neg x = x$, where $\neg x = x \to \bot$. The linear negations $\sim, -$ are anti-isomorphisms of the lattice structure, and since the complement $\neg$ is uniquely determined by the lattice structure, both $\sim$ and $-$ preserve complementation. Therefore $\sim\neg x = \neg\sim x$ and $-\neg x = \neg - x$ hold in any Boolean involutive residuated lattice. With $x^{\smile}$ defined as $\sim\neg x$ it follows that $x^{\smile\smile} = \sim\neg\sim\neg x = \sim\sim\neg\neg x = \sim\sim x$, and if the algebra is cyclic, then $x^{\smile\smile} = x$.

Now assume a Boolean cyclic involutive GBI-algebra satisfies the identity $(xy)^{\smile} = y^{\smile}x^{\smile}$. To see that it is a relation algebra, it suffices to show that De Morgan's Theorem K holds, i.e., $xy \wedge z = \bot \iff x^{\smile}z \wedge y = \bot$. This follows from the calculation below:

$$\begin{aligned} xy \wedge z = \bot &\iff xy \le \neg z \iff x \le (\neg z)/y = -(y \cdot (\sim\neg z)) \\ &\iff x^{\smile} \le (-(y \cdot z^{\smile}))^{\smile} = -(z \cdot y^{\smile}) = \neg y/z \\ &\iff x^{\smile}z \le \neg y \iff x^{\smile}z \wedge y = \bot. \end{aligned}$$

The converse is obvious since relation algebras are Boolean cyclic involutive GBI-algebras and satisfy $x^{\smile\smile} = x$. □

We also note that the identities $\sim\neg x = \neg\sim x$ and $x^{\smile\smile} = x$ both fail in weakening relation algebras and in cyclic involutive GBI-algebras, e.g. in the 3-element Łukasiewicz algebra.

The smallest Boolean cyclic involutive GBI-algebra that fails the converse identity has 8 elements and was originally found in the context of residuated lattices with a De Morgan operation [4]. This algebra has atoms $1, a, b$ and satisfies $aa = a$, $bb = a \vee b$ and $ab = \top = ba$. The involutive element $0 = a \vee b = \neg 1$, and the linear negation satisfies $\sim 1 = 0$, $\sim a = 1 \vee a$ and $\sim b = 1 \vee b$. Hence $a^\smile = b$ and $b^\smile = a$, which implies that $(aa)^\smile = a^\smile = b \neq a^\smile a^\smile = bb = a \vee b$.

If $\mathbf{P}$ is a discrete poset then $\mathbf{Wk}(\mathbf{P})$ is the full representable relation algebra on the set P, so algebras of weakening relations play a role similar to representable relation algebras. Therefore we define the class WGBI of *weakening GBI-algebras* as all algebras that are embedded in a weakening relation algebra $\mathbf{Wk}(\mathbf{P}, Q)$ for some poset $\mathbf{P}$ and equivalence relation Q that contains $\sqsubseteq$. In fact the variety RRA of representable relation algebras is finitely axiomatized over WGBI.

Theorem 7. *1.* WGBI *is closed under subalgebras and products.*
2. RRA *is the subclass of algebras in* WGBI *that satisfy* $\neg\neg x = x$*, i.e., have Boolean algebra reducts.*
3. The class WGBI *is not finitely axiomatizable relative to the variety of all bounded cyclic involutive GBI-algebras.*

Proof. 1. Let $\{\mathbf{A}_i : i \in I\}$ be a family of algebras from WGBI. Then there exists a family of posets $\{\mathbf{P}_i : i \in I\}$ and equivalence relations $\{Q_i : i \in I\}$ such that $\mathbf{A}_i$ is embedded in $\mathbf{Wk}(\mathbf{P}_i, Q_i)$ for each $i \in I$. We can assume that the posets are disjoint, and define $\mathbf{P} = \bigcup_{i\in I} \mathbf{P}_i$, $Q = \bigcup_{i\in I} Q_i$. Then $\prod_{i\in I} \mathbf{Wk}(\mathbf{P}_i, Q_i) \cong \mathbf{Wk}(\mathbf{P}, Q)$ via the map that sends a tuple of disjoint weakening relations $(R_i : i \in I)$ to $\bigcup_{i\in I} R_i$. Since $\prod_{i\in I} \mathbf{A}_i$ is embedded in $\prod_{i\in I} \mathbf{Wk}(\mathbf{P}_i, Q_i)$, it follows that WGBI is closed under products. The closure under subalgebras holds because a composition of embeddings is again an embedding.
2. Let $\mathbf{A}$ be a member of WGBI that satisfies $\neg\neg x = x$. Then $\mathbf{A}$ is embedded in a weakening relation algebra $\mathbf{Wk}(\mathbf{P}, Q)$, so the identity element of $\mathbf{A}$ maps to the partial order $\sqsubseteq$ of the poset $\mathbf{P}$. Assume that $\sqsubseteq$ is not the identity relation on P, so there exist $p \neq q$ such that $p \sqsubseteq q$. Then $(q, p) \in \sqsupseteq\circ \sqsubseteq \circ\sqsupseteq$, hence it is not a member of $\neg\sqsubseteq = \sqsubseteq \to \bot = Q - \sqsupseteq\circ \sqsubseteq \circ\sqsupseteq$. It follows that $(q, p) \in \neg\neg\sqsubseteq$, which means that the identity $\neg\neg x = x$ fails for the identity element of $\mathbf{A}$, a contradiction. Therefore $\sqsubseteq$ is the identity relation, so $\mathbf{P}$ is a discrete poset, and $\mathbf{A}$ is a subalgebra of a representable relation algebra.
3. This is an immediate consequence of 2., since if WGBI were finitely axiomatizable, adding one more identity would give a finite axiomatization of RRA. However, Monk [14] proved that RRA is not finitely axiomatizable. □

Currently it has not been established whether WGBI is closed under homomorphic images, hence a variety, and whether it is a discriminator variety. Another interesting problem that arises is how to define a natural finitely-based variety that contains WGBI similar to Tarski's variety RA of (abstract) relation algebras relative to the variety RRA of all representable relation algebras. Clearly such a basis would include the axioms of bounded cyclic involutive GBI-algebras, but there are other simple identities that are satisfied by all weakening relations. In particular, one can define *domain* and *range* of a relation by the terms $d(x) = x\top \wedge 1$ and $r(x) = \top x \wedge 1$. In any lattice-ordered monoid with top element $\top$, $d(d(x)) = d(x)\top \wedge 1 \le x\top\top = x\top$ and $d(x) \le 1$, hence $d(d(x)) \le d(x)$. Also $d(x) = d(x)1 \le d(x)\top$, so $d(x) \le d(d(x)$, and similarly $r(r(x)) = r(x)$.

Lemma 8. WGBI *satisfies the identities* $d(x)x = x$, $xr(x) = x$ *and* $\top x \top x \top = \top x \top$.

Proof. It suffices to show that the identities hold in any $\mathbf{Wk}(\mathbf{P}, Q)$. From $d(x) \le 1$ it follows that $d(x)x \le x$. For the reverse inclusion, let $(p, q) \in x$. Since $(q, p) \in Q$ and $(p, p) \in 1$, we have $(p, p) \in d(x)$, hence $(p, q) \in d(x)x$.

Clearly $\top x \top \le \top$, so $\top x \top x \top \le \top x \top = \top d(x) x \top \le \top x \top x \top$. □

The smallest cyclic involutive GBI-algebra (or residuated lattice) where these identities fail is the 3-element Łukasiewicz algebra, with $0 < a < 1$ and satisfying $aa = 0$. Since $\top = 1$, we have $d(a) = a = r(a)$, but $aa \ne a$.

4 Partially-Ordered Groupoid Semantics for Some Cyclic Involutive GBI-Algebras

For an element a in a lattice $\mathbf{A} = (A, \wedge, \vee)$ the set $\{x \in A : x < a\}$ always has a least upper bound, which is either a or the largest element below a. In the latter case a is called *completely join-irreducible*, and a lattice is *join-perfect* if every element is a join of completely join-irreducible elements. *Completely meet-irreducible* elements and *meet-perfect* lattices are defined dually. A *perfect* lattice is defined to be both meet and join-perfect. Birkhoff showed that a finite distributive lattice $\mathbf{A}$ is determined by its poset $J(\mathbf{A})$ of completely join-irreducible elements (with the order induced by $\mathbf{A}$). The result also holds for complete perfect distributive lattices. Conversely, if $\mathbf{Q} = (Q, \le)$ is a poset, then the set of downward closed subsets $D(\mathbf{Q})$ of $\mathbf{Q}$ forms a complete perfect distributive lattice under intersection and union. Moreover, $D(\mathbf{Q})$ is a Heyting algebra, with $U \to V = Q - {\uparrow}(U - V)$ for any $U, V \in D(\mathbf{Q})$.

For a poset $\mathbf{P}$ the weakening relation algebra $\mathrm{Wk}(\mathbf{P})$ is a complete and perfect GBI-algebra, and in this case the poset of completely join-irreducible elements is isomorphic to $\mathbf{Q} = \mathbf{P} \times \mathbf{P}^\partial$. The composition $\circ$ of $\mathbf{Wk}(\mathbf{P})$ is determined by its restriction to pairs of $\mathbf{Q}$, where it is a partial operation given by

$$(t, u) \circ (v, w) = \begin{cases} (t, w) & \text{if } u = v \\ \text{undefined} & \text{otherwise.} \end{cases}$$

For an arbitrary complete perfect GBI-algebra $\mathbf{A}$, the operation $\cdot$ is also determined by restricting to $J(\mathbf{A})$, but in general this requires a ternary relation to represent $\circ$. Here we consider the special case when the restriction of $\cdot$ to

$J(\mathbf{A})$ gives a partial operation on $J(\mathbf{A})$. The aim is to characterize the partially-ordered partial algebras that are the result of restricting from certain complete perfect bounded cyclic involutive GBI-algebras to their partially-ordered set of join-irreducibles.

For comparison, we first consider the classical case of relation algebras. A complete perfect relation algebra has a complete atomic Boolean algebra as reduct, and the set of join-irreducibles is the set of atoms. The operation of composition, restricted to atoms, is a partial operation precisely when the atoms form a (Brandt) groupoid [11, Sect. 5], or equivalently a small category with all morphism being invertible. In this case the relation algebra is in fact representable using the Cayley representation of the groupoid.

In the more general setting of cyclic involutive GBI-algebras we have a similar situation using partially-ordered groupoids. We first recall the definitions. A *groupoid* is defined as a partial algebra $\mathbf{G} = (G, \circ, ^{-1})$ such that $\circ$ is a partial binary operation and $^{-1}$ is a (total) unary operation on G that satisfy the following axioms:

1. $x \circ y, y \circ z \in G \implies (x \circ y) \circ z = x \circ (y \circ z)$,
2. $x \circ y \in G \iff x^{-1} \circ x = y \circ y^{-1}$,
3. $x \circ x^{-1} \circ x = x$ and $x^{-1-1} = x$.

These axioms imply $x \circ x^{-1} \in G$, as well as $x \circ y \in G \implies x \circ y \circ y^{-1} = x$ and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$. Typical examples of groupoids are disjoint unions of groups and the *pair-groupoid* $(X \times X, \circ, ^{\smile})$, where $(x, y)^{\smile} = (y, x)$ and $(x, y) \circ (z, w) = (x, w)$ if $y = z$ (undefined otherwise). A *partially-ordered groupoid* $(G, \leq, \circ, ^{-1})$, or *po-groupoid* for short, is a groupoid $(G, \circ, ^{-1})$ such that $(G, \leq)$ is a poset and

4. $x \leq y$ and $x \circ z, y \circ z \in G \implies x \circ z \leq y \circ z$,
5. $x \leq y \implies y^{-1} \leq x^{-1}$,
6. $x \circ y \leq z \circ z^{-1} \implies x \leq y^{-1}$.

Note that the implication $x \leq y$ and $z \circ x, z \circ y \in G \implies z \circ x \leq z \circ y$ follows from these axioms. If $\mathbf{P} = (P, \sqsubseteq)$ a poset with dual poset $\mathbf{P}^\partial = (P, \sqsupseteq)$ then $\mathbf{P} \times \mathbf{P}^\partial = (P \times P, \leq, \circ, ^{\smile})$ is a po-groupoid, called a *po-pair-groupoid*, with $(a, b) \leq (c, d) \iff a \sqsubseteq c$ and $b \sqsupseteq d$.

Theorem 9. *Let $\mathbf{G} = (G, \leq, \circ, ^{-1})$ be a partially-ordered groupoid. Then $D(\mathbf{G})$ is a bounded cyclic involutive GBI-algebra.*

Proof. The downsets of any poset form a complete perfect Heyting algebra under intersection and union. For downsets s, t the operation $\cdot$ is defined by $s \cdot t = {\downarrow}\{x \circ y : x \in s, y \in t\}$, and it is associative by Axiom 1. The identity of $D(\mathbf{G})$ is $1 = {\downarrow}\{x \circ x^{-1} : x \in G\}$, and cyclic involution is defined by ${\sim}s = G - \{x^{-1} : x \in s\}$. Hence $x \in {\sim}s \iff x^{-1} \notin s$. Axiom 5 ensures that ${\sim}s$ is again a downset, and since $x^{-1-1} = x$, it follows that ${\sim}{\sim}s = s$. It remains to check a version of the coupled semirings axiom: $s \subseteq {\sim}t \iff t \cdot s \subseteq 0 = {\sim}1$. Since every downset is a union of principal downsets, it suffices to consider $s = {\downarrow}x$ and $t = {\downarrow}y$ where $x, y \in G$. Now ${\downarrow}x \subseteq {\sim}{\downarrow}y \iff x^{-1} \notin {\downarrow}y \iff x^{-1} \not\leq y \iff x^{-1} \circ y^{-1} \not\leq z \circ z^{-1}$ for all

$z \in G$ using Axiom 6 in the forward direction, and using right-multiplication by $z^{-1} = y^{-1}$ in the reverse direction. This is equivalent to $(y \circ x)^{-1} \notin 1$, $\downarrow(y \circ x) \subseteq 0$ and finally $\downarrow y \cdot \downarrow x \subseteq 0$. □

In fact for a poset $\mathbf{P} = (P, \sqsubseteq)$ the weakening relation algebra $\mathbf{Wk}(\mathbf{P})$ is obtained from the po-pair-groupoid $\mathbf{G} = \mathbf{P} \times \mathbf{P}^\partial$, and for an equivalence relation $Q \subseteq P^2$, $\mathbf{Wk}(\mathbf{P}, Q)$ is obtained from the sub-po-groupoid $(Q, \leq, \circ, ^\smile)$. Hence every weakening relation algebra has po-groupoid semantics. For example, if one takes the 2-element chain $\mathbf{P} = \mathbf{C}_2 = (\{0,1\}, \sqsubseteq)$ with the usual order $0 \sqsubseteq 1$, then $P^2 = \{(0,0), (0,1), (1,0), (1,1)\}$ and

$$\mathrm{Wk}(\mathbf{C}_2) = \{\emptyset, \{(0,1)\}, \{(0,0),(0,1)\}, \{(0,1),(1,1)\}, \{(0,0),(0,1),(1,1)\}, P^2\}.$$

The linear negation $\sim$ dualizes this 6-element lattice and interchanges a, b. The 3-element chain $\mathbf{C}_3$ gives a 9-element po-groupoid, and $\mathbf{Wk}(\mathbf{C}_3)$ has 20 elements (see Fig. 1).

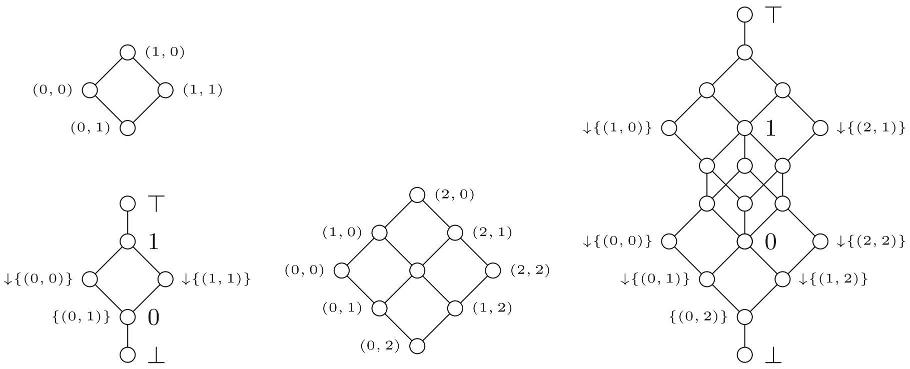

Fig. 1. Weakening relation algebras $\mathbf{Wk}(\mathbf{C}_2)$ and $\mathbf{Wk}(\mathbf{C}_3)$ and their po-pair-groupoids

However, there exist po-groupoids $\mathbf{G}$ such that $D(\mathbf{G})$ is not a weakening relation algebra. The smallest such po-groupoid is based on the pair-groupoid $\mathbf{G} = (\{0,1\}^2, \circ, ^\smile)$, but has only two pairs that are comparable: $(0,1) \leq (1,0)$, so $(0,0)$ and $(1,1)$ are not comparable to any other pairs. The cyclic involutive GBI-algebra $D(\mathbf{G})$ has 12 elements, which does not agree with the cardinality of any of the algebras $\mathbf{Wk}(\mathbf{P}, Q)$.

The last result shows that the cardinality of weakening relation algebras determined by a finite linear order is given by the central binomial series.

Theorem 10. *For an n-element chain $\mathbf{C}_n$ the weakening relation algebra $\mathbf{Wk}(\mathbf{C}_n)$ has cardinality $\binom{2n}{n}$.*

Proof. This follows from the observation that $D(\mathbf{C}_m \times \mathbf{C}_n)$ has cardinality $\binom{m+n}{n}$. For $n = 1$ this is immediate, since an m-element chain has $m + 1$ down-closed sets. Assuming the result holds for n, note that $\mathbf{P} = \mathbf{C}_m \times \mathbf{C}_{n+1}$ is the disjoint union of $\mathbf{C}_m \times \mathbf{C}_n$ and $\mathbf{C}_m$, where we assume the additional m elements are not below any of the elements of $\mathbf{C}_m \times \mathbf{C}_n$. The number of downsets of $\mathbf{P}$ that contain an element a from the extra chain $\mathbf{C}_m$ as a maximal element is given by $\binom{k+n}{n}$ where k is the number of elements above a. Hence the total number of downsets of $\mathbf{P}$ is $\sum_{k=0}^{m} \binom{k+n}{n} = \binom{m+n+1}{n+1}$. □

5 Conclusion

The results in this paper provide connections between idempotent semirings, involutive residuated lattices, generalized bunched implication algebras and relation algebras. These ordered algebras have been extensively studied in algebraic logic and theoretical computer science, and they share many common features that allow techniques to transfer from one theory to the other. Weakening relation algebras extend representable relation algebras to nonclassical logic and are worthy of further investigation.

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