

# Bound-Preserving High Order Finite Volume Schemes for Conservation Laws and Convection-Diffusion Equations

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**Abstract** Finite volume schemes evolve cell averages based on high order reconstructions to solve hyperbolic conservation laws and convection-diffusion equations. The design of the reconstruction procedure is crucial for the stability of the finite volume schemes. Various reconstruction procedures, such as total variation diminishing (TVD), total variation bounded (TVB), essentially non-oscillatory (ENO) and weighted essentially non-oscillatory (WENO) reconstructions have been developed in the literature to obtain non-oscillatory and high order finite volume schemes. However, it is a challenge to design strictly bound-preserving finite volume schemes which are genuinely high order accurate, including at smooth extrema. These include maximum-principle-preserving schemes for scalar conservation laws and convection-diffusion equations, and positivity-preserving (for relevant physical quantities such as density, pressure or water height) for systems. In this presentation we survey strategies in the recent literature to design high order bound-preserving finite volume schemes, including a general framework in constructing high order bound-preserving finite volume schemes for scalar and systems of hyperbolic conservation laws through a simple scaling limiter and a convex combination argument based on first order bound-preserving building blocks, and a non-standard finite volume scheme which evolves the so-called “double cell averages” for solving convection-diffusion equations which can maintain the bound-preserving property and high order accuracy simultaneously.

**Keywords** Finite volume schemes · Bound-preserving · High order accuracy · Conservation laws · Convection-diffusion equations

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## 1 Introduction

In this presentation we are interested in solving hyperbolic conservation laws

$$u_t + \sum_{i=1}^d f_i(u)_{x_i} = 0, \quad u(x_1, \dots, x_d, 0) = u_0(x_1, \dots, x_d) \quad (1)$$

or convection-diffusion equations

$$u_t + \sum_{i=1}^d f_i(u)_{x_i} = \sum_{i=1}^d \sum_{j=1}^d (a_{ij}(u)u_{x_j})_{x_i}, \quad u(x_1, \dots, x_d, 0) = u_0(x_1, \dots, x_d), \quad (2)$$

with suitable restrictions on the functions  $f_i(u)$  and  $a_{ij}(u)$  to ensure well-posedness, by finite volume methods. The type of finite volume methods we will discuss involves the evolution of cell averages of the solution only, through suitable polynomial reconstructions to achieve high order accuracy and stability [8].

For example, for the one-dimensional scalar conservation law

$$u_t + f(u)_x = 0, \quad (3)$$

a semi-discrete finite volume scheme takes the form

$$\frac{d}{dt} \bar{u}_j = \frac{1}{\Delta x} \left( h \left( u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+ \right) - h \left( u_{j-\frac{1}{2}}^-, u_{j-\frac{1}{2}}^+ \right) \right) \quad (4)$$

where  $\bar{u}_j$  is the cell average of  $u$  in the cell  $I_j = (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}})$ , and

$$u_{j-\frac{1}{2}}^+ = p_j(x_{j-\frac{1}{2}}), \quad u_{j+\frac{1}{2}}^- = p_j(x_{j+\frac{1}{2}}),$$

with  $p_j(x)$  being a reconstructed polynomial satisfying

$$\frac{1}{\Delta x} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} p_j(x) dx = \bar{u}_j,$$

and  $h(u^-, u^+)$  is a monotone flux, namely it is increasing (non-decreasing) in its first argument and decreasing in its second argument, symbolically  $h(\uparrow, \downarrow)$ .

An important property of the entropy solution (which may be discontinuous) of a scalar conservation law (1) or a scalar convection-diffusion (2) is that it satisfies a strict maximum principle. If

$$M = \max_{(x_1, \dots, x_d)} u_0(x_1, \dots, x_d), \quad m = \min_{(x_1, \dots, x_d)} u_0(x_1, \dots, x_d), \quad (5)$$

then  $u(x_1, \dots, x_d, t) \in [m, M]$  for any  $(x_1, \dots, x_d)$  and  $t$ .

For scalar conservation laws (1), first order monotone finite volume schemes can easily maintain the maximum principle. For example, for the one-dimensional scalar conservation law (3), the first order fully discrete monotone scheme, namely the scheme (4) with zeroth degree polynomial reconstruction and with Euler forward time discretization, becomes

$$\begin{aligned}\bar{u}_j^{n+1} &= H_\lambda(\bar{u}_{j-1}^n, \bar{u}_j^n, \bar{u}_{j+1}^n) \\ &= \bar{u}_j^n - \lambda[h(\bar{u}_j^n, \bar{u}_{j+1}^n) - h(\bar{u}_{j-1}^n, \bar{u}_j^n)]\end{aligned}$$

where  $\lambda = \frac{\Delta t}{\Delta x}$ . It clearly satisfies the monotonically increasing property in all of its arguments

$$H_\lambda(\uparrow, \uparrow, \uparrow)$$

under a suitable CFL condition

$$\lambda \leq \lambda_0. \quad (6)$$

Also, for any constant  $c$ , we have

$$H_\lambda(c, c, c) = c - \lambda[h(c, c) - h(c, c)] = c.$$

Therefore, if

$$m \leq \bar{u}_{j-1}^n, \bar{u}_j^n, \bar{u}_{j+1}^n \leq M$$

then

$$\bar{u}_j^{n+1} = H_\lambda(\bar{u}_{j-1}^n, \bar{u}_j^n, \bar{u}_{j+1}^n) \geq H_\lambda(m, m, m) = m,$$

and

$$\bar{u}_j^{n+1} = H_\lambda(\bar{u}_{j-1}^n, \bar{u}_j^n, \bar{u}_{j+1}^n) \leq H_\lambda(M, M, M) = M.$$

Thus the scheme satisfies the maximum principle under the CFL condition (6).

However, for higher order *linear* schemes, i.e. schemes which are linear for a linear partial differential equation (PDE)

$$u_t + au_x = 0, \quad (7)$$

for example the second order accurate Lax–Wendroff scheme

$$u_j^{n+1} = \frac{a\lambda}{2}(1 + a\lambda)u_{j-1}^n + (1 - a^2\lambda^2)u_j^n - \frac{a\lambda}{2}(1 - a\lambda)u_{j+1}^n$$

where again  $\lambda = \frac{\Delta t}{\Delta x}$  and  $|a|\lambda \leq 1$ , the maximum principle is *not* satisfied. In fact, no linear schemes with order of accuracy higher than one can satisfy the maximum principle (Godunov Theorem).

Therefore, nonlinear schemes, namely schemes which are nonlinear even for the linear PDE (7), have been designed in the literature to overcome this difficulty. These include roughly two classes of schemes.

The first class is the well-known total variation diminishing (TVD) schemes [3]. Most TVD schemes also satisfy strict maximum principle, even in multi-dimensions. TVD schemes can be designed for any formal order of accuracy for solutions in smooth, monotone regions. However, all TVD schemes will degenerate to first order accuracy at smooth extrema [11].

The second class includes the total variation bounded (TVB) schemes [15], essentially non-oscillatory (ENO) schemes [4, 16], and weighted ENO (WENO) schemes [5, 10]. These schemes do not insist on strict TVD properties, and they do *not* satisfy strict maximum principles, although they are “essentially” non-oscillatory (that is, the overshoots and undershoots are tiny and can usually only be observed in amplified graphs). These schemes can be designed to be arbitrarily high order accurate for smooth solutions.

## 2 Bound-Preserving High Order Finite Volume Schemes for Hyperbolic Conservation Laws

As mentioned before, a high order finite volume scheme for solving the one-dimensional conservation law (3) has the following algorithm flowchart:

1. Starting from the cell averages  $\{\bar{u}_j^n\}$  at time level  $n$ ;
2. Reconstruct a piecewise polynomial function  $u^n(x)$ , which in cell  $I_j$  has a cell average agreeing with  $\bar{u}_j^n$ . This reconstruction should ensure both accuracy and stability (non-oscillatory property);
3. Evolve by, e.g., a Runge–Kutta time discretization, to get the cell averages  $\{\bar{u}_j^{n+1}\}$  at the next time level  $n + 1$ ;
4. Return to step 1 above.

We now discuss the design of high order maximum-principle-preserving finite volume schemes. First, it is important to give a correct definition. Let us again take the one-dimensional scalar conservation law (3) as an example. We will call a finite volume scheme to be maximum-principle-preserving, if we have

$$m \leq u^{n+1}(x) \leq M, \quad \forall x$$

provided

$$m \leq u^n(x) \leq M, \quad \forall x.$$

A modified definition allowing the evaluation of the bounds only at certain quadrature points will be given later to facilitate easy implementation.

Following [22], we have the following steps in designing high order maximum-principle-preserving finite volume schemes:

1. Start with  $u^n(x)$  which is a high order accurate reconstruction

$$|u(x, t^n) - u^n(x)| \leq C \Delta x^p$$

and it satisfies

$$m \leq u^n(x) \leq M, \quad \forall x$$

therefore of course we also have

$$m \leq \bar{u}_j^n \leq M, \quad \forall j;$$

2. Evolve for one time step to ensure

$$m \leq \bar{u}_j^{n+1} \leq M, \quad \forall j. \tag{8}$$

3. Given (8) above, obtain the reconstruction  $u^{n+1}(x)$  which

- satisfies the maximum principle

$$m \leq u^{n+1}(x) \leq M, \quad \forall x;$$

- is high order accurate

$$|u(x, t^{n+1}) - u^{n+1}(x)| \leq C \Delta x^p.$$

There are several difficulties in this procedure.

The major difficulty is how to evolve in time for one time step in a high order fashion yet still guarantee the bound for the new cell averages at the next time level (8). This must be achieved by the finite volume scheme with a high order reconstruction, in order to assure that these new cell averages are both high order accurate and satisfy the boundedness (8). Previous works to achieve this goal mainly used one of the following two approaches. The first approach is to use exact time evolution. This can certainly guarantee the bound for the new cell averages at the next time level (8), as the exact solution of the PDE has this property and the cell averaging operator does not affect the bounds. However, exact time evolution can be implemented with reasonable cost only for linear PDEs, or for scalar nonlinear PDEs in one dimension. This approach was used in, e.g., Jiang and Tadmor [6], Liu and Osher [9], Sanders [14], Qiu and Shu [13], and Zhang and Shu [21], to obtain TVD schemes or maximum-principle-preserving schemes for linear and nonlinear PDEs in one dimension or for linear PDEs in multi-dimensions, for second, third or higher order accurate schemes. The second approach is to use simple time evolution such as the TVD or strong stability preserving (SSP) Runge–Kutta or multi-step methods [2, 16]. However, additional limiting such as the minmod or MUSCL type limiting [7] will be needed on  $u^n(x)$  which may destroy accuracy near smooth extrema.

In Zhang and Shu [22], a procedure is designed to prove the bound for the new cell averages at the next time level (8), with simple Euler forward or SSP Runge–Kutta or multi-step methods without losing accuracy on the limited  $u^n(x)$ , as described below.

The evolution of the cell average for a high order finite volume scheme (4) with Euler forward time discretization satisfies

$$\begin{aligned}\bar{u}_j^{n+1} &= G(\bar{u}_j^n, u_{j-\frac{1}{2}}^-, u_{j-\frac{1}{2}}^+, u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+) \\ &= \bar{u}_j^n - \lambda[h(u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+) - h(u_{j-\frac{1}{2}}^-, u_{j-\frac{1}{2}}^+)],\end{aligned}$$

where we can easily verify that  $G$  is increasing with its first, second and fifth arguments but decreasing with its third and fourth arguments, symbolically

$$G(\uparrow, \uparrow, \downarrow, \downarrow, \uparrow).$$

Therefore, even if we insist that all five arguments of the function  $G$  are in the range  $[m, M]$ , the cell average at the next time level  $\bar{u}_j^{n+1}$  may still be outside this range, regardless of how small one takes the CFL number  $\lambda > 0$ . The problem is with the two arguments  $u_{j-\frac{1}{2}}^+$  and  $u_{j+\frac{1}{2}}^-$  which are values at points *inside* the cell  $I_j$ . We would therefore hope to have the first term  $\bar{u}_j^n$  to help. In order to do this, we would need to link the cell average  $\bar{u}_j^n$  with the two point values  $u_{j-\frac{1}{2}}^+$  and  $u_{j+\frac{1}{2}}^-$ . If we take a Legendre Gauss–Lobatto quadrature rule which is exact for polynomials of degree  $k$ , then

$$\bar{u}_j^n = \sum_{\ell=0}^m \omega_\ell p_j(y_\ell),$$

where  $y_\ell$  are the Legendre Gauss–Lobatto quadrature points and  $y_0 = x_{j-\frac{1}{2}}$ ,  $y_m = x_{j+\frac{1}{2}}$ . The scheme for the cell average is then rewritten as

$$\begin{aligned}\bar{u}_j^{n+1} &= \omega_m \left[ u_{j+\frac{1}{2}}^- - \frac{\lambda}{\omega_m} \left( h(u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+) - h(u_{j-\frac{1}{2}}^+, u_{j+\frac{1}{2}}^-) \right) \right] \\ &\quad + \omega_0 \left[ u_{j-\frac{1}{2}}^+ - \frac{\lambda}{\omega_0} \left( h(u_{j-\frac{1}{2}}^+, u_{j+\frac{1}{2}}^-) - h(u_{j-\frac{1}{2}}^-, u_{j-\frac{1}{2}}^+) \right) \right] + \sum_{\ell=1}^{m-1} \omega_\ell p_j(y_\ell) \\ &= \omega_m H_{\lambda/\omega_m}(u_{j-\frac{1}{2}}^+, u_{j+\frac{1}{2}}^-, u_{j+\frac{1}{2}}^+) + \omega_0 H_{\lambda/\omega_0}(u_{j-\frac{1}{2}}^-, u_{j-\frac{1}{2}}^+, u_{j+\frac{1}{2}}^-) + \sum_{\ell=1}^{m-1} \omega_\ell p_j(y_\ell).\end{aligned}$$

Therefore, if

$$m \leq p_j(y_\ell) \leq M$$

at all Legendre Gauss–Lobatto quadrature points and a reduced CFL condition

$$\lambda/\omega_m = \lambda/\omega_0 \leq \lambda_0,$$

where  $\lambda_0$  is the bound for the CFL condition of the first order monotone scheme in (6), is satisfied, then

$$m \leq \bar{u}_j^{n+1} \leq M.$$

The second difficulty is: given

$$m \leq \bar{u}_j^{n+1} \leq M, \quad \forall j$$

how to obtain an *accurate* reconstruction  $u^{n+1}(x)$  which satisfies

$$m \leq u^{n+1}(x) \leq M, \quad \forall x.$$

We would like to avoid the evaluation of the extrema of the polynomial solution  $u^{n+1}(x)$  before limiting, which, for a piecewise polynomial of higher degree, especially in high-dimension, could be quite costly.

Again in Zhang and Shu [22], a procedure is designed to obtain such  $u^{n+1}(x)$  with a very simple scaling limiter, which only requires the evaluation of the unlimited  $u^{n+1}(x)$  at certain pre-determined quadrature points and does not destroy accuracy. The procedure involves replacing  $p_j(x)$  by the limited polynomial  $\tilde{p}_j(x)$  defined by

$$\tilde{p}_j(x) = \theta_j(p_j(x) - \bar{u}_j^n) + \bar{u}_j^n \quad (9)$$

where

$$\theta_j = \min \left\{ \left| \frac{M - \bar{u}_j^n}{M_j - \bar{u}_j^n} \right|, \left| \frac{m - \bar{u}_j^n}{m_j - \bar{u}_j^n} \right|, 1 \right\}, \quad (10)$$

with

$$M_j = \max_{x \in S_j} p_j(x), \quad m_j = \min_{x \in S_j} p_j(x) \quad (11)$$

where  $S_j$  is the set of Legendre Gauss–Lobatto quadrature points of cell  $I_j$ .

Clearly, this limiter is just a simple scaling of the original polynomial around its average. The computational cost is minimal, since it involves only the computation of  $\theta_j$  by (10), which in turn only involves the computation of the local bounds  $m_j$  and  $M_j$  by (11), via evaluating the unlimited polynomial at the pre-determined Legendre Gauss–Lobatto quadrature points of cell  $I_j$ .

The following lemma, guaranteeing the maintenance of accuracy of this simple limiter, is proved in Zhang and Shu [22].

**Lemma** Assume  $\bar{u}_j^n \in [m, M]$  and  $p_j(x)$  is an  $O(\Delta x^p)$  approximation, then  $\tilde{p}_j(x)$  is also an  $O(\Delta x^p)$  approximation.

We have thus obtained a high order accurate scheme satisfying the following maximum principle: If

$$m \leq u^n(x) \leq M, \quad \forall x \in S_j,$$

then

$$m \leq u^{n+1}(x) \leq M, \quad \forall x \in S_j.$$

Recall that  $S_j$  is the set of Legendre Gauss–Lobatto quadrature points of cell  $I_j$ .

We remark that the algorithm can be further simplified without affecting its performance on bound-preserving and high order accuracy, following [25]. Notice that, since the middle quantity in the equalities below is a convex combination of the point values of a continuous function  $p_j(x)$  at the  $m - 1$  points  $y_1, y_2, \dots, y_{m-1}$ , by the mean value theorem, we have

$$p_j(\xi) = \frac{1}{1 - 2\omega_0} \left( \sum_{\ell=1}^{m-1} \omega_\ell p_j(y_\ell) \right) = \frac{1}{1 - 2\omega_0} \left( \bar{u}_j - \omega_0 \left( u_{j-\frac{1}{2}}^+ + u_{j+\frac{1}{2}}^- \right) \right) \quad (12)$$

where  $\xi$  is an unknown location in  $I_j$ . The important thing to notice is that, even though we do not know the location  $\xi$ ,  $p_j(\xi)$  as defined above can be explicitly computed from just the knowledge of  $\bar{u}_j$ ,  $u_{j-\frac{1}{2}}^+$  and  $u_{j+\frac{1}{2}}^-$ . Therefore, we can change the definitions of  $M_j$  and  $m_j$  from (11) to

$$M_j = \max\{u_{j-\frac{1}{2}}^+, u_{j+\frac{1}{2}}^-, p_j(\xi)\}, \quad m_j = \min\{u_{j-\frac{1}{2}}^+, u_{j+\frac{1}{2}}^-, p_j(\xi)\}, \quad (13)$$

where  $p_j(\xi)$  is defined in (12). With this modification, we do not need to compute  $p_j(y_\ell)$  for the internal Legendre Gauss–Lobatto quadrature points  $y_\ell$  with  $\ell = 1, \dots, m - 1$ , resulting in a saving of the computational cost, which could be a significant saving for multi-dimensional unstructured meshes. The resulting limiter (9) is even milder than before, since the scaling factor  $\theta_j$  computed in (10) is closer to one when (13) is used than when (11) is used. This milder limiter was used in, e.g., [18] to compute shallow water equations with positive water heights.

Finally, the last difficulty is how to generalize the algorithm and result to two and higher dimensions. Algorithms which would require an evaluation of the extrema of the reconstructed polynomials  $u^{n+1}(x, y)$  would not be easy to generalize at all. On the other hand, our algorithm uses only explicit Euler forward or SSP (also called TVD) Runge–Kutta or multi-step time discretizations, and a simple scaling limiter involving just an evaluation of the polynomial at certain quadrature points, hence it easily generalizes to two and higher dimensions on structured or unstructured meshes, with strict maximum-principle-satisfying property and provable high order accuracy [26]. This method can be easily generalized to two-dimensional incompressible Euler equations in the vorticity-streamfunction formulation (with strict maximum principle for the vorticity), and two or multi-dimensional passive convections in a divergence-free velocity field, i.e.

$$\omega_t + (u\omega)_x + (v\omega)_y = 0,$$



with a given divergence-free velocity field  $(u, v)$ , again with strict maximum principle. See [22, 26].

The framework of establishing maximum-principle-satisfying schemes for scalar equations can be generalized to hyperbolic systems to preserve the positivity of certain physical quantities, such as density and pressure of compressible gas dynamics.

The main ingredients for designing positivity-preserving schemes for systems are:

- A first order explicit scheme which can keep the positivity of the desired quantities (e.g., density and pressure) under a suitable CFL condition. Examples include the Godunov scheme, Lax–Friedrichs scheme, kinetic scheme, HLLC scheme, etc.
- The quantity for which positivity is desired is one of the components of the conserved variable  $u$  (for example the density  $\rho$ ), or is a *concave* function of the conserved variable  $u$  (for example the pressure  $p$  or the internal energy  $e$ ). Under this assumption, the region of positivity of the desired quantities is a *convex region* in the  $u$  space.

With these ingredients, the technique to enforce maximum-principle for scalar equations can be directly generalized to enforce positivity of the desired quantities without affecting the high order accuracy of the finite volume schemes. That is, we have a high order scheme satisfying positivity-preserving in the following sense: If  $u^n(x)$  has positive density and pressure for all  $x \in S_j$ , then  $u^{n+1}(x)$  also has positive density and pressure for all  $x \in S_j$ . Recall that  $S_j$  is the set of Legendre Gauss–Lobatto quadrature points of cell  $I_j$ . The simplified version mentioned above can again be used. For the discussion of positivity-preserving schemes for Euler equations of compressible gas dynamics, ideal special relativistic hydrodynamics (RHD), shallow water equations, a hierarchical size-structured population model, and cosmological hydrodynamical turbulence, see [12, 17–20, 23, 24, 26, 29]. Some of these papers discuss the procedure using discontinuous Galerkin methods, however the same procedure can be applied to finite volume schemes directly.

### 3 Bound-Preserving High Order Finite Volume Schemes for Convection-Diffusion Equations

A generalization of the bound-preserving technique in the previous section to the convection-diffusion equations (2) is not straightforward. If we follow the approach in the previous section to deal with standard finite volume schemes or discontinuous Galerkin schemes for solving the convection-diffusion equations (2), we can only obtain second order bound-preserving schemes [28], or at most third order bound-preserving schemes for special discontinuous Galerkin schemes [1]. However, in [27], we have designed a non-standard finite volume scheme for solving the general nonlinear convection-diffusion equations (2) on rectangular meshes which can maintain both high order accuracy and bound-preserving property.

Let us again use the following one-dimensional equation

$$u_t + f(u)_x = (a(u)u_x)_x, \quad a(u) \geq 0 \quad (14)$$

to show the ideas. The non-standard finite volume scheme in [27] is based on evolving the *double cell averages*

$$\bar{\bar{u}}_i = \frac{1}{\Delta x^2} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \left( \int_{x-\frac{\Delta x}{2}}^{x+\frac{\Delta x}{2}} u(\xi) d\xi \right) dx.$$

If we take a double integration on the diffusion equation

$$u_t = (a(u)u_x)_x = (A(u))_{xx}, \quad (15)$$

we obtain the identity

$$\frac{d}{dt} \bar{\bar{u}}_i = \frac{1}{\Delta x^2} [A(u(x_{i+1})) - 2A(u(x_i)) + A(u(x_{i-1}))] \quad (16)$$

which is satisfied by the exact solution of the PDE (15). There, taking an Euler forward time discretization of (16), we obtain the non-standard finite volume scheme

$$\bar{\bar{u}}_i^{n+1} = \bar{\bar{u}}_i^n + \frac{\Delta t}{\Delta x^2} [A(u_{i+1}) - 2A(u_i) + A(u_{i-1})]. \quad (17)$$

Equation (17) would become a scheme to evolve the double cell averages  $\bar{\bar{u}}_i$  as long as we have a reconstruction procedure to obtain accurate approximations to the point values  $u_j$  from these double cell averages. Such reconstruction is technically different from the standard reconstruction from standard cell averages to point values, but it can be designed along similar lines [27].

Notice that the right hand side of the scheme (17) is a linear combination of the point values  $A(u_j)$ , and the coefficients are positive for points outside the cell  $I_i$  (namely at  $x_{i-1}$  and  $x_{i+1}$ ). Therefore, in order to obtain a bound-preserving scheme following the lines of the previous section, we now just need the following two ingredients:

1. Exact quadrature rule for the double cell averages with positive weights:

$$\bar{\bar{p}}_i = \sum_{\ell=0}^m \omega_\ell p(y_\ell)$$

with suitable quadrature points  $y_\ell$  and quadrature weights  $\omega_\ell > 0$ , for polynomials  $p$  of degree  $k$ .

2. WENO type reconstruction from the double cell averages  $\{\bar{\bar{u}}_i\}$  to the point values of the function  $\{u_j\}$ .

The procedure for the convection-diffusion equation (14) is similar to the one outlined above for the pure diffusion equation (15). The details have been worked out in [27]. We thus have a high order, non-standard finite volume scheme which is bound-preserving. Unfortunately, it appears that the scheme can only be designed on rectangular meshes, as it is not easy to define “double cell averages” on unstructured meshes.

## 4 Concluding Remarks

We have surveyed recent developments of the design and analysis of uniformly high order accurate, bound-preserving finite volume schemes for multi-dimensional non-linear conservation laws and convection-diffusion equations. For hyperbolic conservation laws, the result is quite general, with schemes designed for arbitrary triangulations and for arbitrary high order of accuracy. For convection-diffusion equations, only a non-standard finite volume scheme on rectangular meshes can be designed to have the bound-preserving property for arbitrary high order of accuracy. For standard finite volume and discontinuous Galerkin schemes, only second order accurate (or for a special discontinuous Galerkin method, third order accurate) bound-preserving schemes can be obtained.

In the future we will investigate whether it is possible to obtain high order bound-preserving property for classical finite volume schemes evolving only cell averages for convection-diffusion equations on general triangulations.

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