1

Numbers and Mathematical Symbols

Natasha Maurits

After reading this chapter you know:

- what numbers are and why they are used,
- what number classes are and how they are related to each other,
- what numeral systems are,
- the metric prefixes,
- how to do arithmetic with fractions,
- what complex numbers are, how they can be represented and how to do arithmetic with them,
- the most common mathematical symbols and
- how to get an understanding of mathematical formulas.

1.1 What Are Numbers and Mathematical Symbols and Why Are They Used?

A refresher course on mathematics can not start without an introduction to numbers. Firstly, because one of the first study topics for mathematicians were numbers and secondly, because mathematics becomes really hard without a thorough understanding of numbers. The branch of mathematics that studies numbers is called number theory and *arithmetic* forms a part of that. We have all learned arithmetic starting from kindergarten throughout primary school and beyond. This suggests that an introduction to numbers is not even necessary; we use numbers on a day-to-day basis when we count and measure and you might think that numbers hold no mysteries for you. Yet, arithmetic can be as difficult to learn as reading and some people never master it, leading to *dyscalculia*.

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So, what is a number? You might say: 'well, five is a number and 243, as well as 1963443295765'. This is all true, but what is the essence of a number? You can think of a number as an abstract representation of a quantity that we can use to measure and count. It is represented by a symbol or numeral, e.g., the number five can be represented by the Arabic numeral 5, by the Roman numeral V, by five fingers, by five dots on a dice, by 𝍸, by five abstract symbols such as ••••• and in many other different ways. *Synesthetes* even associate numbers with colors. But, importantly, independent of how a number is represented, the abstract notion of this number does not change.

Most likely, (abstract) numbers were introduced after people had developed a need to count. Counting can be done without numbers, by using fingers, sticks or pebbles to represent single or groups of objects. It allows keeping track of stock and simple communication, but when quantities become larger, this becomes more difficult, even when abstract words for small quantities are available. A more compact way of counting is to put a mark—like a scratch or a line—on a stick or a rock for each counted object. We still use this approach when tally marking. However, marking does not allow dealing with large numbers either. Also, these methods do not allow dealing with negative numbers (as e.g., encountered as debts in accounting), fractions (to indicate a part of a whole) or other even more complex types of numbers.

The reason that we can deal with these more abstract types of numbers, that no longer relate to countable series of objects, is that numeral systems have developed over centuries. In a numeral system a systematic method is used to create number words, so that it is not necessary to remember separate words for all numbers, which would be sheer impossible. Depending on the base that is used, this systematic system differs between languages and cultures. In many current languages and cultures base 10 is used for the numeral system, probably as a result of initially using the 10 digits (fingers and thumbs) to count. In this system, enumeration and numbering is done by tens, hundreds, thousands etcetera. But remnants of older counting systems are still visible, e.g. in the words twelve (which is not ten-two) or quatre-vingts (80 in French; four twenties). For a very interesting, easy to read and thorough treatise on numbers please see Posamenter and Thaller (2015).

We now introduce the first mathematical symbols in this book; for numbers. In the base 10 numeral system the Arabic numerals 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9 are used. In general, mathematical symbols are useful because they help communicating about abstract mathematical structures, and allow presenting such structures in a concise way. In addition, the use of symbols speeds up doing mathematics and communicating about it considerably, also because every symbol in its context only has one single meaning. Interestingly, mathematical symbols do not differ between languages and thus provide a universal language of mathematics. For non-mathematicians, the abstract symbols can pose a problem though, because it is not easy to remember their meaning if they are not used on a daily basis. Later in this chapter, we will therefore introduce and explain often used mathematical symbols and some conventions in writing mathematics. In this and the next chapters, we will also introduce symbols that are specific to the topic discussed in each chapter. They will be summarized at the end of each chapter.

1.2 Classes of Numbers

When you learn to count, you often do so by enumerating a set of objects. There are numerous children (picture) books aiding in this process by showing one ball, two socks, three dolls, four cars etcetera. The first numbers we encounter are thus 1, 2, 3, ... Note that '...' is a mathematical symbol that indicates that the pattern continues. Next comes zero. This is a rather peculiar number, because it is a number that signifies the absence of something. It also has its own few rules regarding arithmetic:

$$a + 0 = a$$
$$a \times 0 = 0$$
$$\frac{a}{0} = \infty$$

Here, a is any number and ∞ is the symbol for infinity, the number that is larger than any countable number.

Together, 0, 1, 2, 3, ... are referred to as the *natural numbers* with the symbol $\mathbb{N}$. A special class of natural numbers is formed by the *prime numbers* or primes; natural numbers >1 that only have 1 and themselves as positive divisors. The first prime numbers are 2, 3, 5, 7, 11, 13, 17, 19 etcetera. An important application of prime numbers is in *cryptography*, where they make use of the fact that it is very difficult to factor very large numbers into their primes. Because of their use for cryptography and because prime numbers become rarer as numbers get larger, special computer algorithms are nowadays used to find previously unknown primes.

The basis set of natural numbers can be extended to include negative numbers: ..., -3, -2, -1, 0, 1, 2, 3, ... Negative numbers arise when larger numbers are subtracted from smaller numbers, as happens e.g. in accounting, or when indicating freezing temperatures indicated in °C (degrees Centigrade). These numbers are referred to as the integer numbers with symbol $\mathbb{Z}$ (for 'zahl', the German word for number). Thus $\mathbb{N}$ is a subset of $\mathbb{Z}$.

By dividing integer numbers by each other or taking their ratio, we get fractions or rational numbers, which are symbolized by $\mathbb{Q}$ (for quotient). Any rational number can be written as a fraction, i.e. a ratio of an integer, the *numerator*, and a positive integer, the *denominator*. As any integer can be written as a fraction, namely the integer itself divided by 1, $\mathbb{Z}$ is a subset of $\mathbb{Q}$. Arithmetic with fractions is difficult to learn for many; to refresh your memory the main rules are therefore repeated in Sect. 1.2.1.

Numbers that can be measured but that can not (always) be expressed as fractions are referred to as *real numbers* with the symbol $\mathbb{R}$. Real numbers are typically represented by decimal numbers, in which the decimal point separates the 'ones' digit from the 'tenths' digit (see also Sect. 1.2.3 on numeral systems) as in 4.23 which is equal to $\frac{423}{100}$. There are *finite decimal numbers* and *infinite decimal numbers*. The latter are often indicated by providing a finite number of the decimals and then the '...' symbol to indicate that the sequence continues. For example, $\pi = 3.1415\ldots$ Real numbers such as π that are not rational are called *irrational*. Any rational number is real, however, and therefore $\mathbb{Q}$ is a subset of $\mathbb{R}$.

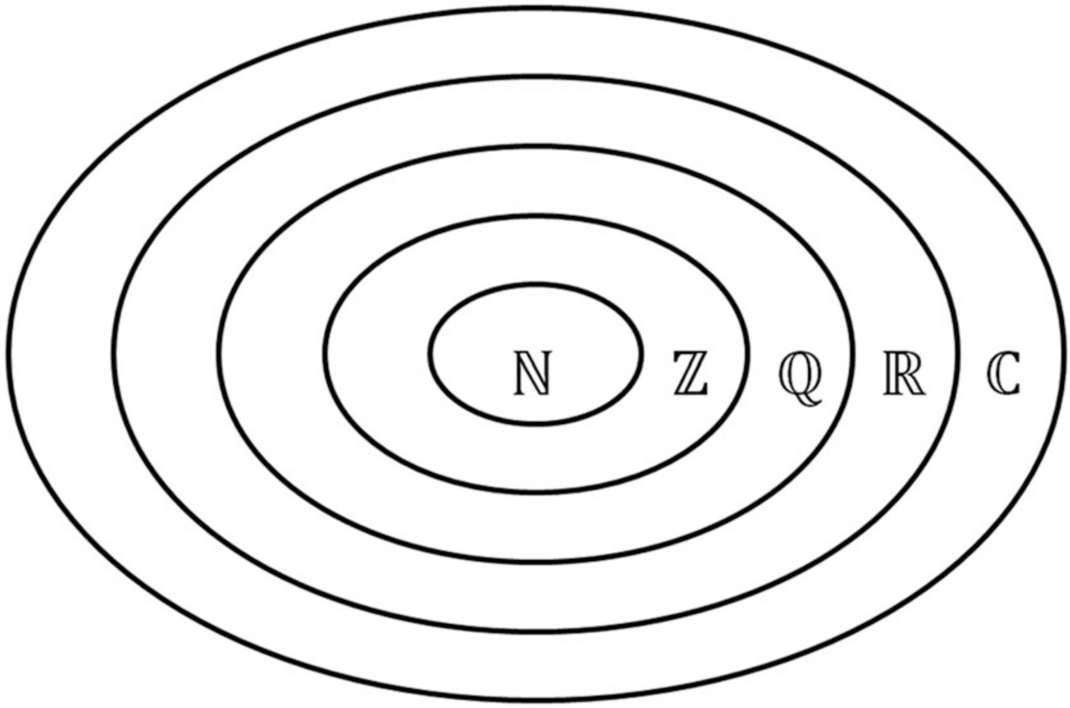


Fig. 1.1 The relationship between the different classes of numbers: $\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}\subset\mathbb{C}$, where $\subset$ is the symbol for 'is a subset of'.

The last extension of the number sets to be discussed here is the set of *complex numbers* indicated by $\mathbb{C}$. Complex numbers were invented to have solutions for equations such as $x^2+1=0$. The solution to this equation was defined to be $x=i$. As the complex numbers are abstract, no longer measurable quantities that have their own arithmetic rules and are very useful in scientific applications, they deserve their own section and are discussed in Sect. 1.2.4.

The relationship between the different classes of numbers is summarized in Fig. 1.1.

Exercise

1.1. What is the smallest class of numbers that the following numbers belong to?

a) -7
b) e (Euler's number, approximately equal to 2.71828)
c) $\sqrt{3}$
d) 0.342
e) 543725
f) π
g) $\sqrt{-3}$

1.2.1 Arithmetic with Fractions

For many, there is something confusing about fractional arithmetic, which is the reason we spend a section on explaining it. To add or subtract fractions with unlike denominators you first need to know how to find the smallest *common denominator*. This is the least common multiple, i.e. the smallest number that can be divided by both denominators. Let's illustrate this by some examples.

Suppose you want to add $\frac{2}{3}$ and $\frac{4}{9}$. The denominators are thus 3 and 9. Here, the common denominator is simply 9, because it is the smallest number divisible by both 3 and 9. Thus, if one denominator is divisible by the other, the largest denominator is the common denominator. Let's make it a bit more difficult. When adding $\frac{1}{3}$ and $\frac{3}{4}$ the common denominator is 12; the product of 3 and 4. There is no smaller number that is divisible by both 3 and 4. Note that to find a common denominator, you can <u>always</u> multiply the two denominators. However, this will not always give you the <u>least</u> common multiple and may make working with the fractions unnecessarily complicated. For example, $\frac{7}{9}$ and $\frac{5}{12}$ do have $9 \times 12 = 108$ as a common denominator, but the least common multiple is actually 36. So, how do you find this least common multiple? The straightforward way that always works is to take one of the denominators and look at its table of multiplication. Take the one you know the table of best. For 9 and 12, start looking at the multiples of 9 until you have found a number that is also divisible by 12. Thus, try $2 \times 9 = 18$ (not divisible by 12), $3 \times 9 = 27$ (not divisible by 12) and $4 \times 9 = 36$ (yes, divisible by 12!). Hence, 36 is the least common multiple of 9 and 12. Once you have found a common denominator, you have to rewrite both fractions such that they get this denominator by multiplying the numerator with the same number you needed to multiply the denominator with to get the common denominator. Then you can do the addition. Let's work this out for $\frac{7}{9} + \frac{5}{12}$:

$$\frac{7}{9} + \frac{5}{12} = \frac{7 \times 4}{9 \times 4} + \frac{5 \times 3}{12 \times 3} = \frac{28}{36} + \frac{15}{36} = \frac{43}{36} = 1\frac{7}{36}$$

Note that we have here made use of an important rule for fraction manipulation: whatever number you multiply the denominator with (positive, negative or fractional itself), you also have to multiply the numerator with and vice versa! Subtracting fractions takes the exact same preparatory procedure of finding a common denominator. And adding or subtracting more than two fractions also works the same way; you just have to find a common multiple for all denominators. There is also an unmentioned rule to always provide the simplest form of the result of arithmetic with fractions. The simplest form is obtained by 1) taking out the wholes and then 2) simplifying the resulting fraction by dividing both numerator and denominator by common factors.

Exercise

1.2. Simplify

a) $\frac{24}{21}$

b) $\frac{60}{48}$

c) $\frac{20}{7}$

d) $\frac{20}{6}$

1.3. Find the answer (and simplify whenever possible)

a) $\frac{1}{3}+\frac{2}{5}$

b) $\frac{3}{14}+\frac{7}{28}$

c) $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}$

d) $\frac{3}{4}+\frac{7}{8}+\frac{9}{20}$

e) $\frac{1}{4}-\frac{5}{6}+\frac{3}{8}$

f) $-\frac{1}{3}+\frac{1}{6}-\frac{1}{7}$

For multiplying and dividing fractions there are two important rules to remember:

1) when multiplying fractions the numerators have to be multiplied to find the new numerator and the denominators have to be multiplied to find the new denominator:

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

2) dividing by a fraction is the same as multiplying by the inverse:

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$$

For the latter, we actually make use of the rule that when multiplying the numerator/denominator with a number, the denominator/numerator has to be multiplied with the same number and that to get rid of the fraction in the denominator we have to multiply it by its inverse:

$$\frac{a}{b}\div\frac{c}{d}=\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{\frac{a}{b}\times\frac{d}{c}}{\frac{c}{d}\times\frac{d}{c}}=\frac{\frac{a}{b}\times\frac{d}{c}}{1}=\frac{a}{b}\times\frac{d}{c}$$

Exercise

1.4. Find the answer (and simplify whenever possible)

a) $\frac{2}{3} \times \frac{6}{7}$

b) $1\frac{2}{5} \times 1\frac{3}{7}$

c) $\frac{5}{6} \div \frac{6}{5}$

d) $\frac{11}{13} \times \frac{2}{3} \div \frac{6}{13}$

e) $\frac{2}{4} \div 2 \times \frac{12}{48}$

Finally, for this section, it is important to note that arithmetic operations have to be applied in a certain order, because the answer depends on this order. For example, when $3 + 4 \times 2$ is seen as $(3 + 4) \times 2$ the answer is 14, whereas when it is seen as $3 + (4 \times 2)$ the answer is 11. The order of arithmetic operations is the following:

1) brackets (or parentheses)
2) exponents and roots
3) multiplication and division
4) addition and subtraction

There are several mnemonics around to remember this order, such as BEDMAS, which stands for Brackets-Exponent-Division-Multiplication-Addition-Subtraction. The simplest mnemonic is PEMA for Parentheses-Exponent-Multiplication-Addition; it assumes that you know that exponents and roots are at the same level, as are multiplication and division and addition and subtraction. Think a little bit about this. When you realize that subtracting a number is the same as adding a negative number, dividing by a number is the same as multiplying by its inverse and taking the *n*th root is the same as raising the number to the power 1/n, this makes perfect sense (see also Sect. 1.2.2).

Exercise

1.5. Calculate the answer to

a) $8 \div 4 - 1 \times 3^2 + 3 \times 4$

b) $(8 \div 4 - 1) \times 3^2 + 3 \times 4$

c) $(8 \div 4 - 1) \times (3^2 + 3) \times 4$

d) $(8 \div 4 - 1) \times (3^2 + 3 \times 4)$

1.2.2 Arithmetic with Exponents and Logarithms

Other topics in arithmetic that often cause people problems are exponentials and their inverse, logarithms. Exponentiation is a mathematical operation in which a *base a* is multiplied by itself *n* times and is expressed as:

$$a^n = a \times \cdots \times a$$

Here, *n* is referred to as the *exponent*. Exponentiation is encountered often in daily life, such as in models for population growth or calculations of compound interest. For example, when you have a savings account that yields 2% interest per year, your starting capital of €100,00 will have increased to $100 + \frac{2}{100} \cdot 100 = 1,02 \cdot 100 = 102$. Another year later, you will have $102 + \frac{2}{100} \cdot 102 = 1,02 \cdot 102 = 1,02 \cdot 1,02 \cdot 100 = (1,02)^2 \cdot 100 = 104,04$. Thus, after *n* years, your capital will have grown to $(1{,}02)^n \cdot 100$. In general, when your bank gives you *p*% interest per year, your starting capital of *C* will have increased to $\left(1 + \frac{p}{100}\right)^n \cdot C$ after *n* years. Here, we clearly see exponentiation at work. Let me here remind you of some arithmetic rules for exponentiation that will come in very handy when continuing with the next chapters (*a* and *b* should be non-zero):

$$\begin{aligned} a^0 &= 1 \\ a^{-n} &= \frac{1}{a^n} \\ a^n a^m &= a^{n+m} \\ \frac{a^n}{a^m} &= a^{n-m} \\ (a^n)^m &= a^{nm} \\ (ab)^n &= a^n b^n \end{aligned} \tag{1.1}$$

Exercise

1.6. Simplify to one power of 2:

a) $\frac{2^3 2^4}{2^2}$

b) $\frac{(2^2)^{\frac{1}{2}} 2^3}{2^{-4} 2^2}$

This is also the perfect point to relate *roots* to exponentials, because it makes arithmetic with roots so much easier. Mostly, when people think of roots, they think of the square root, designated with the symbol $\sqrt{}$. A square root of a number *x* is the number *y* such that $y^2 = x$. For example, the square roots of 16 are 4 and −4, because both 4^2 and $(-4)^2$ are 16. More generally, the *n*th root of a number *x* is the number *y* such that $y^n = x$. An example is given by the cube root of 8 which is 2 ($2^3 = 8$). The symbol used for the *n*th root is $\sqrt[n]{}$, as in $\sqrt[3]{8} = 2$.

And here comes the link between roots and exponents: $\sqrt[n]{x} = x^{\frac{1}{n}}$. Knowing this relationship, and all arithmetic rules for exponentiation (Eq. 1.1), allows for easy manipulation of roots. For example,

$$\frac{\sqrt[4]{9}}{\sqrt[8]{3}} = \frac{9^{\frac{1}{4}}}{3^{\frac{1}{8}}} = 9^{\frac{1}{4}} \cdot 3^{-\frac{1}{8}} = \left(3^2\right)^{\frac{1}{4}} \cdot 3^{-\frac{1}{8}} = 3^{\frac{1}{2}} \cdot 3^{-\frac{1}{8}} = 3^{\frac{1}{2}-\frac{1}{8}} = 3^{\frac{3}{8}} = \left(3^3\right)^{\frac{1}{8}} = \sqrt[8]{27}$$

Exercise

1.7. Simplify as much as possible:

a) $\frac{\sqrt[3]{1000}}{\sqrt[4]{16}}$

b) $\sqrt[4]{25}\sqrt{5}$

c) $\sqrt{3y^8}$

d) $\frac{\sqrt[4]{9}}{\sqrt[8]{3}}$ (this is the same fraction as in the example above; now try to simplify by rewriting the fourth root to an eighth root right away)

e) $\sqrt[3]{x^{15}}$

f) $\sqrt[7]{p^{49}}$

g) $\sqrt[3]{\frac{a^6}{b^{27}}}$

h) $\sqrt[3]{\frac{-27x^6y^9}{64}}$

Finally, I will briefly review the arithmetics of the *logarithm*, the inverse operation of exponentiation. The base n logarithm of a number y is the exponent to which n must be raised to produce y; i.e. $\log_n y = x$ when $n^x = y$. Thus, for example, $\log_{10}1000 = 3$, $\log_2 16 = 4$ and $\log_7 49 = 2$. A special logarithm is the *natural logarithm*, with base e, referred to as *ln*. The number e has a special status in mathematics, just like π, and is encountered in many applications (see e.g., Sect. 3.2.1). It also naturally arises when calculating compound interest, as it is equal to $\left(1 + \frac{1}{n}\right)^n$ when n goes to infinity (see Sect. 6.7; verify that this expression gets close to e for a few increasing values for n). The basic arithmetic rules for logarithms are:

$$\log_b y^a = a\log_b y$$

$$\log_b \sqrt[a]{y} = \frac{\log_b y}{a}$$

$$\log_b xy = \log_b x + \log_b y$$

$$\log_b \frac{x}{y} = \log_b x - \log_b y$$

$$\log_b y = \frac{\log_k y}{\log_k b}$$

The third arithmetic rule above shows that logarithms turn multiplication into addition, which is generally much easier. This was the reason that, before the age of calculators and computers (until approximately 1980), logarithms were used to simplify the multiplication of large numbers by means of slide rules and logarithm tables.

Exercise

1.8. Simplify as much as possible:

a) $\frac{\log_b\left(\left[x^2+1\right]^4\right)}{\log_b\sqrt{x}}$

b) $\log_2(8 \cdot 2^x)$

c) $\frac{1}{\log_{27}(3)}$

d) $\log_2(8 \cdot \sqrt[3]{8})$

1.9. Rewrite to one logarithm:

a) $\log_2 x^2 + \log_2 5 + \log_2 \frac{1}{3}$

b) $\log_3 \sqrt{a} + \log_3(10) - \log_3 a^2$

c) $\log_a a^2 - \log_a 3 + \log_a \frac{1}{3}$

d) $\log_x \sqrt{x} + \log_x x^2 + \log_x \frac{1}{\sqrt{x}}$

1.2.3 Numeral Systems

In the Roman numeral system, the value of a numeral is independent of its position: I is always 1, V is always 5, X is always 10 and C is always 100, although the value of the numeral has to be subtracted from the next numeral if that is larger (e.g., IV = 4 and XL = 40). Hence, XXV is 25 and CXXIV is 124. This way of noting numbers becomes inconvenient for larger numbers (e.g., 858 in Roman numerals is DCCCLVIII, although because of the subtraction rule 958 in Roman numerals is CMLVIII). In the most widely used numeral system today, the decimal system, the value of a numeral does depend on its position. For example, the 1 in 1 means one, while it means ten in 12 and 100 in 175. Such a positional or place-value notation allows for a very compact way of denoting numbers in which only as many symbols are needed as the base size, i.e., 10 (0,1,2,3,4,5,6,7,8,9) for the decimal system which is a base-10 system. Furthermore, arithmetic in the decimal system is much easier than in the Roman numeral system. You are probably pleased not to be a Roman child having to do additions! The now commonly used base-10 system probably is a heritage of using ten fingers to count. Since not all counting systems used ten fingers, but also e.g., the three phalanges of the four fingers on one hand or the ten fingers and ten toes, other numerical bases have also been around for a long time and several are still in use today, such as the duodecimal or base-12 system for counting hours and months. In addition, new numerical bases have been introduced because of their convenience for certain specific purposes, like the binary (base-2) system for digital computing. To understand which number numerals indicate, it is important to know the base that is used, e.g. 11 means eleven in the decimal system but 3 in the binary system.

To understand the systematics of the different numeral systems it is important to realize that the position of the numeral indicates the power of the base it has to be multiplied with to give its value. This may sound complicated, so let's work it out for some examples in the decimal system first:

$$\begin{aligned} 154 &= 1 \times 10^2 + 5 \times 10^1 + 4 \times 10^0 \\ &= 1 \times 100 + 5 \times 10 + 4 \times 1 \\ 3076 &= 3 \times 10^3 + 0 \times 10^2 + 7 \times 10^1 + 6 \times 10^0 \\ &= 3 \times 1000 + 0 \times 100 + 7 \times 10 + 6 \times 1 \end{aligned}$$

Hence, from right to left, the power of the base increases from $base^0$ to $base^1$, $base^2$ etcetera. Note that the 0 numeral is very important here, because it indicates that a power is absent in the number. This concept works just the same for binary systems, only the base is different and just two digits, 0 and 1 are used:

$$\begin{aligned} 101 &= 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 \\ &= 1 \times 4 + 0 \times 2 + 1 \times 1 \\ 110011 &= 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \\ &= 1 \times 32 + 1 \times 16 + 0 \times 8 + 0 \times 4 + 1 \times 2 + 1 \times 1 \end{aligned}$$

Thus 101 and 110011 in the binary system are equal to 5 and 51 in the decimal system. An overview of these two numeral systems is provided in Table 1.1.

In the binary system a one-positional number is indicated as a *bit* and an eight-positional number (consisting of 8 bits) is indicated as a *byte*.

Exercise

1.10. Convert these binary numbers to their decimal counterparts

a) 10
b) 111
c) 1011
d) 10101
e) 111111
f) 1001001

Table 1.1 The first seven powers used for the place values in the decimal (base 10) and the binary (base 2) systems

Power	7	6	5	4	3	2	1	0
Value in decimal system	10,000,000	1,000,000	100,000	10,000	1000	100	10	1
Value in binary system	128	64	32	16	8	4	2	1

There are some special notations for numbers in the decimal system, that are easy to know and use. To easily handle very large or very small numbers with few *significant digits*, i.e., numbers with many trailing or leading zeroes, the *scientific notation* is used in which the insignificant zeroes are more or less replaced by their related power of 10. Consider these examples:

$$10000 = 1 \times 10^4$$
$$0.0001 = 1 \times 10^{-4}$$
$$5340000 = 5.34 \times 10^6$$
$$0.00372 = 3.72 \times 10^{-3}$$
$$696352000000000 = 6.96352 \times 10^{14}$$

To get the scientific notation of a number, you thus have to count the number of digits the comma has to be shifted to the right (positive powers) or to the left (negative powers) to arrive at the original representation of the number. Calculators will use 'E' instead of the 10 base, e.g., 10000 = 1E4.

Exercise

1.11. Write in scientific notation

a) 54000
b) 0.0036
c) 100
d) 0.00001
e) 654300
f) 0.000000000742

To finalize this section on numeral systems I would like to remind you of the metric prefixes, that are used to indicate a multiple or a fraction of a unit and precede the unit. This may sound cryptic, but what I mean are the 'milli-' in millimeter and the 'kilo-' in kilogram, for example. The reason to introduce them here is that nowadays, all metric prefixes are related to the decimal system. Table 1.2 presents the most commonly used prefixes.

1.2.4 Complex Numbers

In general, complex numbers extend the one-dimensional world of real numbers to two dimensions by including a second, *imaginary* number. The complex number i, which indicates the imaginary unit, is defined as the (positive) solution to the equation $x^2+1=0$, or, in other words, i is the square root of -1. Every complex number is characterized by a pair of numbers (a,b), where a is the real part and b the imaginary part of the number. In this sense a complex number can also be seen geometrically as a *vector* (see also Chap. 4) in the complex plane (see Fig. 1.2). This complex plane is a 2-dimensional coordinate system where

Table 1.2 The most commonly used metric prefixes, their symbols, associated multiplication factors and powers of 10

Prefix	Symbol	Factor	Power of 10
Exa	E	1000 000 000 000 000 000	18
Peta	P	1000 000 000 000 000	15
Tera	T	1000 000 000 000	12
Giga	G	1000 000 000	9
Mega	M	1000 000	6
Kilo	k	1000	3
Hecto	h	100	2
Deca	da	10	1
Deci	d	0.1	−1
Centi	c	0.01	−2
Milli	m	0.001	−3
Micro	µ	0.000 001	−6
Nano	n	0.000 000 001	−9
Pico	p	0.000 000 000 001	−12
Femto	f	0.000 000 000 000 001	−15

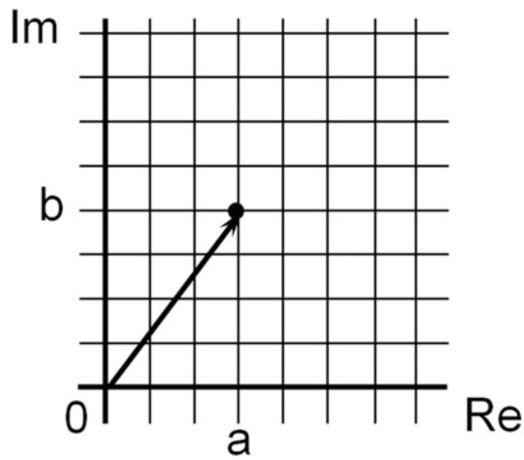


Fig. 1.2 Illustration of the complex number *a* + *bi* as a pair or vector in the complex plane. *Re* real axis, *Im* imaginary axis.

the real part of the complex number indicates the distance to the vertical *axis* (or reference line) and the imaginary part of the complex number indicates the distance to the horizontal axis. Both axes meet in the origin. The horizontal axis is also referred to as the real axis and the vertical axis as the imaginary axis. A complex number z is also written as $z=a+bi$. For manipulating complex numbers and working with them, it helps to remember that a complex number has these two representations, i.e. as a pair or vector (a,b) in the two-dimensional complex plane and as a number $a+bi$.

Exercise

1.12. Draw/position the following complex numbers in the complex plane

a) 1 + i
b) −2 − 2.5i
c) −3 + 2i
d) $4\sqrt{-1}$

1.2.4.1 Arithmetic with Complex Numbers

Let's start simple, by adding complex numbers. This is done by adding the real and imaginary parts separately:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

Similarly, subtracting two complex numbers is done by subtracting the real and imaginary parts separately:

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

Alternatively, adding or subtracting two complex numbers can be viewed of geometrically as adding or subtracting the associated vectors in the complex plane by constructing a parallelogram (see Fig. 1.3).

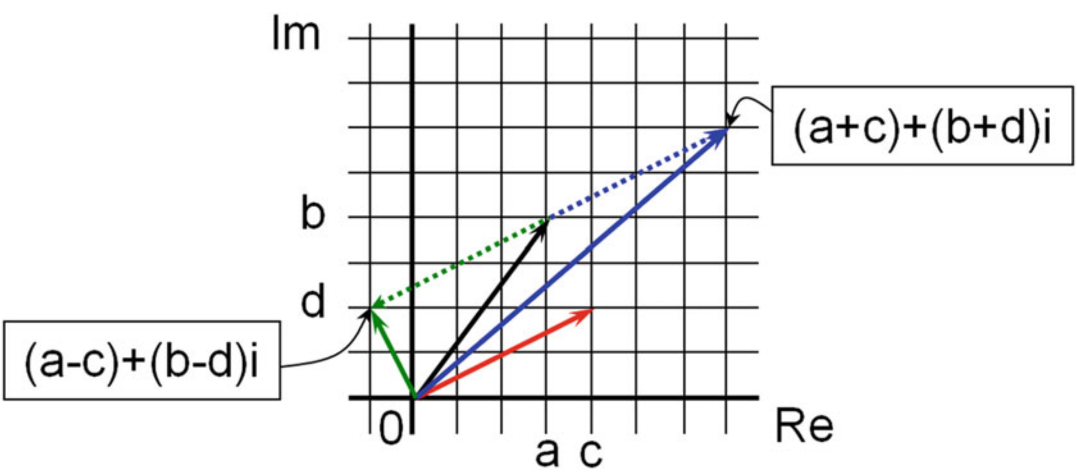


Fig. 1.3 Illustration of adding (*blue*) and subtracting (*green*) the complex numbers *a* + *bi* (*black*) and *c* + *di* (*red*) in the complex plane. The *dashed arrows* indicate how *c* + *di* is added to (*blue dashed*) or subtracted from (*green dashed*) *a* + *bi*.

Multiplying two complex numbers is done by using the *distributive law* (multiplying the two elements of the first complex number with each of the two elements of the second complex number and adding them):

$$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i \tag{1.2}$$

Here, we make use of the fact that $i^2 = -1$. Finally, division of two complex numbers is done by first multiplying numerator and denominator by the *complex conjugate* of the denominator (and then applying the distributive law again) to make the denominator real:

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{(a+bi)}{(c+di)}\frac{(c-di)}{(c-di)} = \frac{ac - adi + bci - bdi^2}{c^2 - cdi + cdi - d^2i^2} \\ &= \frac{(ac+bd) + (bc-ad)i}{c^2+d^2} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i\end{aligned}$$

The complex conjugate of a complex number is indicated by an overbar and is calculated as:

$$\overline{a + bi} = a - bi$$

Hence, for a complex number $z = a + bi$:

$$z\bar{z} = a^2 + b^2$$

Exercise

1.13. Calculate:

a) (1 + i) + (−2 + 3i)
b) (1.1 − 3.7i) + (−0.6 + 2.2i)
c) (2 + 3i) − (2 − 5i)
d) (4 − 6i) − (6 + 4i)
e) (2 + 2i) × (3 − 3i)
f) (5 − 4i) × (1 − i)
g) $\frac{5 - 10i}{1 - 2i}$
h) $\frac{18 + 9i}{\sqrt{5} - 2i}$

1.2.4.2 The Polar Form of Complex Numbers

An alternative and often very convenient way of representing complex numbers is by using their polar form. In this form, the distance r from the point associated with the complex number in the complex plane to the origin (the point (0,0)), and the angle φ between the vector associated with the complex number and the positive real axis are used. The distance r can be calculated as follows (please refer to Fig. 1.4):

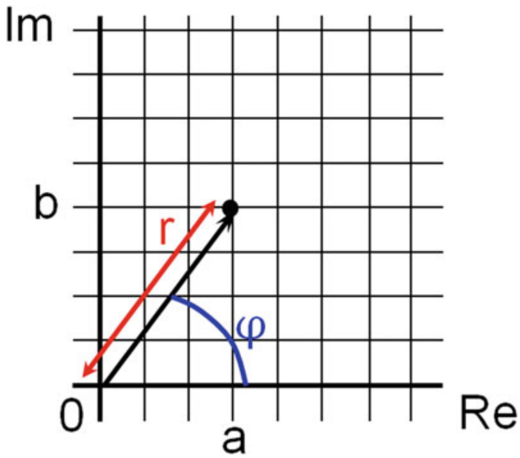


Fig. 1.4 Illustration of the polar form of the complex number *a* + *bi* in the complex plane. *Re* real axis, *Im* imaginary axis, *r* absolute value or modulus, φ argument.

$$r = \sqrt{a^2 + b^2} = \sqrt{z\bar{z}} \equiv |z|$$

Here, the symbol '≡' stands for 'is defined as' and we use the complex conjugate of *z* again. The symbol '|.|' stands for modulus or absolute value. The angle, or argument φ can be calculated by employing the trigonometric tangent function (see Chap. 3).

The polar expression of the complex number *z* is then (according to Euler's formula, see Sect. 3.3.1) given by:

$$z = re^{i\phi}$$

At this point, this may seem like a curious, abstract form of an exponential power and may seem not very useful. However, this polar form of complex numbers does allow to e.g., find all 3 complex roots of the equation $z^3 = 1$ and not just the one obvious real root $z = 1$ (see also Chap. 2 on equation solving and Sect. 3.3.1).

1.3 Mathematical Symbols and Formulas

The easiest way to learn the language of mathematics is to practice it, just like for any foreign language. For that reason we explain most symbols in this book in the context of how they are used. However, since mathematics is a very extensive field and since practicing mathematics takes time, we here also provide a more general introduction to and reminder of often used mathematical symbols and some conventions related to using the symbolic language of mathematics.

1.3.1 Conventions for Writing Mathematics

There are a few conventions when writing mathematical texts, that are also helpful to know when reading such texts. In principle, all mathematical symbols are written in Italics when they are part of the main text to discern them from non-mathematical text. Second, vectors and matrices (see Chaps. 4 and 5) are indicated in bold, except when writing them by hand. Since bold font can then not be used, (half) arrows or overbars are used above the symbol used for the vector or matrix. Some common mathematical symbols are provided in Table 1.3.

1.3.2 Latin and Greek Letters in Mathematics

To symbolize numbers that have no specific value (yet), both Latin and Greek letters are typically used in mathematics. In principle, any letter can be used for any purpose, but for quicker understanding there are some conventions on when to use which letters. Some of these conventions are provided in Table 1.4.

1.3.3 Reading Mathematical Formulas

To the less experienced, reading mathematical formulas can be daunting. Although practice also makes perfect here, it is possible to give some general advice on how to approach a mathematical formula and I will do so by means of an example. Suppose you are reading an article (Ünlü et al. 2006) and you stumble upon this rather impressive looking formula (slightly adapted for better understanding):

$$C_i^m(\varepsilon) = \frac{|\{(j,k)|(|r(i+k-1)-r(j+k-1)| \leq \varepsilon) \; for \; k=1\ldots m, j=i\ldots N-m+1\}|}{N-m+1}$$

The first thing to do when encountering a formula, is to make sure that you know what each of the symbols means in the context of the formula. In this case, I read the text to find out what C is (I already know that it will depend on m, i and ε from the left hand side of the

Table 1.3 Meaning of some common mathematical symbols with examples

Symbol	Meaning	Example
$\Rightarrow$	implies	$z = i \Rightarrow z^2 = -1$
$\Leftrightarrow$	if and only if	$x + 3 = 2x - 2 \Leftrightarrow x = 5$
$\approx$	approximately equal to	$\pi \approx 3.14$
$\propto$	proportional to	$y = 3x \Rightarrow y \propto x$
!	factorial	$3! = 3 \times 2 \times 1 = 6$
$<$	less than	$3 < 4$
$>$	greater than	$4 > 3$
$\ll$	much less than	$1 \ll 100{,}000{,}000$
$\gg$	much greater than	$100{,}000{,}000 \gg 1$

Table 1.4 Conventions on the use of Latin and Greek letters in mathematics

Latin letter	Application	Example
$a, b, c, \ldots$	as *parameter* in equations, or functions	$y = ax + b$ $y = ax^2 + bx + c$ $z = a + bi$
	vectors	**a** or $\vec{a}$ or $\overrightarrow{a}$
e	base of natural logarithm, approximately equal to 2.71828...	
x, y, z	*Cartesian coordinates*	$(x,y) = (1,3)$ $(x,y,z) = (-1,2,-4)$
	axes in 2D- or 3D space	x-axis
d, D	diameter	
	derivative (see Chap. 6)	$\frac{d}{dt}, \frac{d^2}{dx^2}$
i, j, k	counters	$i = 1, \ldots, n$ $\sum_{i=1}^{n} x_i$ $\sum_{i=1}^{n} \sum_{j=1}^{m} x_{i,j}$
	vector element	x_i
	matrix element	$x_{i,j}$
	complex unity	$z = a + bi$
n, m, N	quantity	$i = 1, \ldots, n$ $j = 1, \ldots, m$
	number of participants/animals in experimental science	N
P, Q, R	point in space	$P = (1,2)$ $Q = (-1,1,3)$
r	radius	circle or sphere radius
	modulus in polar coordinates or polar form of complex numbers	$z = re^{i\phi}$
t	time (counter)	
T	time (window), period	

Greek letter	Application	Example
α (alpha)	angle	
	significance level (in statistics)	
β (beta)	*power* (in statistics)	
δ (delta)	Dirac delta	$\delta(x) = \begin{cases} \infty & if\ x = 0 \\ 0 & if\ x \neq 0 \end{cases}$
	Kronecker delta	$\delta_{ij} = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$
Δ (delta)	small increment	Δt
ε (epsilon)	(very) small (positive) number	for every $\delta < \varepsilon$
φ (phi)	angle (in polar coordinates)	
	argument (in polar form of complex numbers)	$z = re^{i\phi}$
ζ (zeta), θ (theta), ξ (ksi), ψ (psi)	angles	
π	relation between circumference and radius r of a circle	circumference $= 2\pi r$

formula), what i, j, and k count (I already know they will most likely be counters because of the mathematical convention explained in Table 1.4), what m and N are and what r is. I already know that ε will be a small number (again, because of the mathematical convention explained in Table 1.4). Then what remains to be known are the symbols .| (here: for every pair (j,k) such that), |.| (here: *cardinality* (the number of pairs (j,k); outer symbols) and distance (inner symbols)) and {.} (the set of).

What the article tells me is that $r(i)$ is a collection of m consecutive data points (an m-tuple) taken from a total number of N data points, starting at the ith data point and that $C_i^m(\varepsilon)$ is the relative number of pairs of these m-tuples which are not so different, i.e. which differ less than a small number ε for each pair of entries. $N - m + 1$ is the total number of different $m -$ tuples that can be taken from the total data set. The largest value that i can take on is thus also $N - m + 1$. The first thing to notice in the formula is that by letting k run from 1 to m and j from i to $N - m + 1$, all possible pairs of m-tuples are indeed considered. This can be understood by assuming a value for m (e.g., 2), taking into account that i runs from 1 to $N - m + 1$ and then writing out the indices of the first and last pairs of m-tuples with indices $i + k - 1$ and $j + k - 1$. The part between the round brackets makes sure that from all possible pairs, only the pairs of m-tuples that have a distance smaller than ε are counted. So this formula indeed calculates what it is supposed to do.

So, what general lessons about formula reading can be learned from this example?

First, you need to know what all symbols mean in the context of the formula. Second, you need to understand what the more general mathematical symbols mean in the formula. Third, you break the formula into pieces and build up your understanding from there. More examples will be given in each of the following chapters.

Glossary

Arithmetic Operations between numbers, such as addition, subtraction, multiplication and division

Axis Reference line for a coordinate system

Base The number b in the exponentiation operation b^n

Cardinality The number of elements in a set, also indicated by #

Cartesian coordinates Uniquely specify a point in 2D space as a pair of numbers that indicates the signed distances to the point from two fixed perpendicular directed lines (axes), measured in the same unit of Length

Common denominator The least common multiple, i.e. the smallest number that can be divided by both denominators

Complex conjugate For a complex number $a+bi$ the complex conjugate is $a-bi$

Complex numbers Pairs of numbers (a,b) where a is the real part and b the imaginary part, also indicated as $a+bi$, where $i = \sqrt{-1}$ is the imaginary unit

Cryptography Discipline at the intersection of mathematics and computing science that deals with various aspects of information security

Denominator Lower part in a fraction $\frac{a}{b}$

Derivative Measure of change in a function

Distributive law To multiply a sum (or difference) by a factor, each element is multiplied by this factor and the resulting products are added (or subtracted)

Dyscalculia Difficulty learning or comprehending numbers or arithmetic, often considered as a developmental disorder
Exponent The number n in the exponentiation operation b^n
Finite decimal number Decimal number with a finite number of decimals
Imaginary Here: the imaginary part of a complex number
Infinite decimal number Decimal number with an infinite numbers of decimals
Infinity The number larger than any countable number
Integer numbers The numbers . . ., −3, −2, −1, 0, 1, 2, 3, . . . collectively referred to as $\mathbb{Z}$
Irrational numbers Real numbers such as π that are not rational
Logarithm The base n logarithm of a number y is the exponent to which n must be raised to produce y
Natural logarithm Logarithm with base e
Natural numbers The numbers 0, 1, 2, 3, . . . collectively referred to as $\mathbb{N}$
Numerator Upper part in a fraction $\frac{a}{b}$
Parameter Consider a function $f(x) = ax^2 + bx + c$; here, x is a variable and a, b, and c are parameters, indicating that the function represents a whole class of functions for different values of its parameters
Power Power is used in the context of exponentiation, e.g. as in b^n, where b is the base and n the exponent. One can also describe this as 'b is raised to the power n' or 'the nth power of b'
Prime numbers Natural numbers >1 that only have 1 and themselves as positive divisors
Rational numbers Numbers that can be written as fractions; a ratio of two integers, collectively referred to as $\mathbb{Q}$
Real numbers Numbers that can be measured but that cannot (always) be expressed as fractions, collectively referred to as $\mathbb{R}$
Root The nth root of a number x is the number y such that $y^n = x$.
Scientific notation Used to write numbers that are too large to be written in decimal form. In this notation all numbers are written as $a \times 10^b$, where a can be any real number and b is an integer
Significance level In statistics: probability of rejecting the null hypothesis given that it is true. Typically, the significance level is set to 0.05, meaning that a 1 in 20 chance of falsely rejecting the null hypothesis is accepted
Significant digit A digit with meaning for the number
Synesthete A person who, when one sense is stimulated, has experiences in another. In the context of this chapter: a person who identifies colors and shapes with numbers.
Vector Entity with a magnitude and direction, often indicated by an arrow, see Chap. 4

Symbols Used in This Chapter (in Order of Their Appearance)

The symbols presented in Tables 1.3 and 1.4 are not repeated here.

0,1,2,3,4,5,6,7,8,9	Arabic numerals used in the base 10 numeral system
. . .	the pattern continues
+	addition
=	equal to
×	multiplication
∞	infinity
°	degree
⊂	is a subset of
$\mathbb{N}$	natural numbers

$\mathbb{Z}$	integer numbers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{C}$	complex numbers
$\div$	division
$\sqrt{}$	square root
e	Euler's number
I	Roman numeral for 1
V	Roman numeral for 5
X	Roman numeral for 10
L	Roman numeral for 50
C	Roman numeral for 100
D	Roman numeral for 500
E	scientific notation for a base 10 exponent on calculators, e.g. 1E3 $= 10^3$
i	complex unity (positive solution of $x^2 + 1 = 0$)
$\bar{\cdot}$	complex conjugate (overbar)
r	here: modulus of complex number
φ	here: argument of complex number
$\lvert\cdot\rvert$	absolute value, modulus, cardinality or distance
$\equiv$	defined as
{.}	the set of
.|	such that
#	cardinality

Overview of Equations, Rules and Theorems for Easy Reference

Relationship between numeral systems

$\mathbb{N}\subset\mathbb{Z}\subset\mathbb{Q}\subset\mathbb{R}\subset\mathbb{C}$

Order of arithmetic operations

1) brackets (or parentheses)
2) exponents and roots
3) multiplication and division
4) addition and subtraction

Arithmetic with fractions

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$$

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$$

Arithmetic with exponentials

$$a^0 = 1$$
$$a^{-n} = \frac{1}{a^n}$$
$$a^n a^m = a^{n+m}$$
$$\frac{a^n}{a^m} = a^{n-m}$$
$$(a^n)^m = a^{nm}$$
$$(ab)^n = a^n b^n$$

Arithmetic with logarithms

$$\log_b y^a = a\log_b y$$
$$\log_b \sqrt[a]{y} = \frac{\log_b y}{a}$$
$$\log_b xy = \log_b x + \log_b y$$
$$\log_b \frac{x}{y} = \log_b x - \log_b y$$
$$\log_b y = \frac{\log_k y}{\log_k b}$$

Arithmetic with complex numbers

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$
$$(a+bi)-(c+di) = (a-c)+(b-d)i$$
$$(a+bi)(c+di) = (ac-bd)+(ad+bc)i$$
$$\frac{a+bi}{c+di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i \tag{1.3}$$
$$\overline{a+bi} = a-bi$$

$z\bar{z} = a^2 + b^2$ for a complex number $z = a + bi$

Answers to Exercises

1.1. a) $\mathbb{Z}$ integer numbers
b) $\mathbb{R}$ real numbers
c) $\mathbb{R}$ real numbers
d) $\mathbb{Q}$ rational numbers

e) $\mathbb{N}$ natural numbers
f) $\mathbb{R}$ real numbers
g) $\mathbb{C}$ complex numbers

1.2. a) $\frac{24}{21} = \frac{8}{7} = 1\frac{1}{7}$
b) $\frac{60}{48} = 1\frac{12}{48} = 1\frac{1}{4}$
c) $\frac{20}{7} = 2\frac{6}{7}$
d) $\frac{20}{6} = 3\frac{2}{6} = 3\frac{1}{3}$

1.3. a) $\frac{1}{3} + \frac{2}{5} = \frac{5}{15} + \frac{6}{15} = \frac{11}{15}$
b) $\frac{3}{14} + \frac{7}{28} = \frac{6}{28} + \frac{7}{28} = \frac{13}{28}$
c) $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{3}{6} + \frac{2}{6} + \frac{1}{6} = \frac{6}{6} = 1$
d) $\frac{3}{4} + \frac{7}{8} + \frac{9}{20} = \frac{30}{40} + \frac{35}{40} + \frac{18}{40} = \frac{83}{40} = 2\frac{3}{40}$
e) $\frac{1}{4} - \frac{5}{6} + \frac{3}{8} = \frac{6}{24} - \frac{20}{24} + \frac{9}{24} = -\frac{5}{24}$
f) $-\frac{1}{3} + \frac{1}{6} - \frac{1}{7} = -\frac{14}{42} + \frac{7}{42} - \frac{6}{42} = -\frac{13}{42}$

1.4. a) $\frac{2}{3} \times \frac{6}{7} = \frac{12}{21} = \frac{4}{7}$
b) $1\frac{2}{5} \times 1\frac{3}{7} = \frac{7}{5} \times \frac{10}{7} = \frac{70}{35} = 2$
c) $\frac{5}{6} \div \frac{6}{5} = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}$
d) $\frac{11}{13} \times \frac{2}{3} \div \frac{6}{13} = \frac{11}{13} \times \frac{2}{3} \times \frac{13}{6} = \frac{22}{18} = 1\frac{4}{18} = 1\frac{2}{9}$
e) $\frac{2}{4} \div 2 \times \frac{12}{48} = \frac{2}{4} \times \frac{1}{2} \times \frac{12}{48} = \frac{3}{48} = \frac{1}{16}$

1.5. a) $8 \div 4 - 1 \times 3^2 + 3 \times 4 = 2 - 9 + 12 = 5$
b) $(8 \div 4 - 1) \times 3^2 + 3 \times 4 = 9 + 12 = 21$
c) $(8 \div 4 - 1) \times (3^2 + 3) \times 4 = 1 \times 12 \times 4 = 48$
d) $(8 \div 4 - 1) \times (3^2 + 3 \times 4) = 1 \times (9 + 12) = 21$

1.6. a) $\frac{2^3 2^4}{2^2} = 2^{3+4-2} = 2^5$
b) $\frac{(2^2)^{\frac{1}{2}} 2^3}{2^{-4} 2^2} = 2^{1+3+4-2} = 2^6$

1.7. a) $\frac{\sqrt[3]{1000}}{\sqrt[4]{16}} = \frac{10}{2} = 5$

b) $\sqrt[4]{25}\sqrt{5} = 5^{2\cdot\frac{1}{4}}5^{\frac{1}{2}} = 5^{\frac{1}{2}+\frac{1}{2}} = 5$

c) $\sqrt{3y^8} = y^4\sqrt{3}$

d) $\frac{\sqrt[4]{9}}{\sqrt[8]{3}} = \frac{\sqrt[8]{81}}{\sqrt[8]{3}} = \sqrt[8]{\frac{81}{3}} = \sqrt[8]{27}$

e) $\sqrt[3]{x^{15}} = \mathrm{x}^{\frac{1}{3}\cdot 15} = \mathrm{x}^5$

f) $\sqrt[7]{p^{49}} = \mathrm{p}^{\frac{1}{7}\cdot 49} = \mathrm{p}^7$

g) $\sqrt[3]{\frac{a^6}{b^{27}}} = \frac{\mathrm{a}^{\frac{1}{3}\cdot 6}}{\mathrm{b}^{\frac{1}{3}\cdot 27}} = \frac{\mathrm{a}^2}{\mathrm{b}^9}$

h) $\sqrt[3]{\frac{-27x^6y^9}{64}} = \frac{-3\mathrm{x}^2\mathrm{y}^3}{4}$

1.8. a) $\frac{\log_b\left([x^2+1]^4\right)}{\log_b\sqrt{x}} = \log_b\left([x^2+1]^4\right) - \log_b\sqrt{x} = 4\log_b(x^2+1) - \frac{1}{2}\log_b x$

b) $\log_2(8\cdot 2^x) = \log_2 8 + \log_2 2^x = 3 + x$

c) $\frac{1}{\log_{27}3} = \frac{\log_{27}27}{\log_{27}3} = \log_3 27 = 3$

d) $\log_2\left(8\cdot\sqrt[3]{8}\right) = \log_2 8 + \log_2 8^{\frac{1}{3}} = 3 + \frac{1}{3}\log_2 8 = 3 + \frac{1}{3}\cdot 3 = 4$ or $\log_2\left(8\cdot\sqrt[3]{8}\right) = \log_2(8\cdot 2)$ $= \log_2 16 = 4$

1.9. a) $\log_2 x^2 + \log_2 5 + \log_2\frac{1}{3} = \log_2\left(x^2\cdot 5\cdot\frac{1}{3}\right) = \log_2\left(\frac{5}{3}x^2\right)$

b) $\log_3\sqrt{a} + \log_3(10) - \log_3 a^2 = \log_3\left(\frac{\sqrt{a}\cdot 10}{a^2}\right) = \log_3\frac{10}{a\sqrt{a}}$

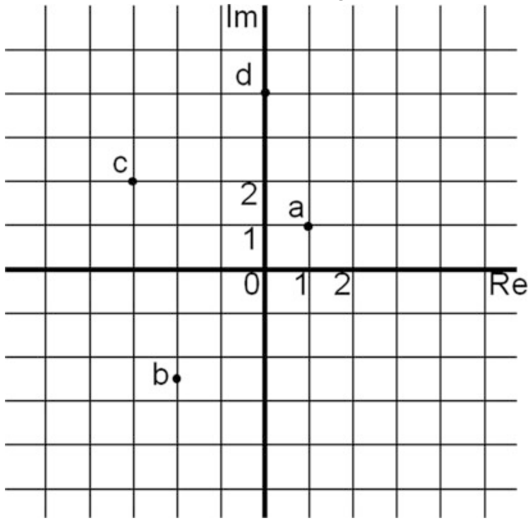
c) $\log_a a^2 - \log_a 3 + \log_a\frac{1}{3} = \log_a\frac{a^2}{3\cdot 3} = \log_a\frac{a^2}{9}$

d) $\log_x\sqrt{x} + \log_x x^2 + \log_x\frac{1}{\sqrt{x}} = \log_x\frac{x^2\sqrt{x}}{\sqrt{x}} = \log_x x^2 = 2$

1.10. a) 10 = 2 + 0 = 2
b) 111 = 4 + 2 + 1 = 7
c) 1011 = 8 + 0 + 2 + 1 = 11
d) 10101 = 16 + 0 + 4 + 0 + 1 = 21
e) 111111 = 32 + 16 + 8 + 4 + 2 + 1 = 63
f) 1001001 = 64 + 0 + 0 + 8 + 0 + 0 + 1 = 73

1.11. a) 5.4×10^4
b) 3.6×10^{-3}
c) 1×10^2
d) 1×10^{-5}
e) 6.543×10^5
f) 7.42×10^{-10}

1.12. The numbers a) to d) have been drawn in this figure



.

1.13. a) $(1 + i) + (-2 + 3i) = -1 + 4i$

b) $(1.1 - 3.7i) + (-0.6 + 2.2i) = 0.5 - 1.5i$

c) $(2 + 3i) - (2 - 5i) = 8i$

d) $(4 - 6i) - (6 + 4i) = -2 - 10i$

e) $(2 + 2i) \times (3 - 3i) = 6 + 6i - 6i - 6i2 = 12$

f) $(5 - 4i) \times (1 - i) = 5 - 4i - 5i + 4i2 = 1 - 9i$

g) $$\frac{5-10i}{1-2i} = \frac{(5-10i)(1+2i)}{1^2+2^2} = \frac{5-10i+10i-20i^2}{5} = 5$$

h) $$\frac{18+9i}{\sqrt{5}-2i} = \frac{(18+9i)\left(\sqrt{5}+2i\right)}{5+2^2} = \frac{18\sqrt{5}+9\sqrt{5}i+18i+18i^2}{9}$$
$$= \left(2\sqrt{5}-2\right) + \left(\sqrt{5}+2\right)i$$

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