Interdisciplinary Application of Symmetry Phenomena

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Abstract This chapter gives a short overview of a few possible contributions of interdisciplinarity to aesthetics, as the latter appears in the sciences. Symmetry phenomena play an important role in these considerations. Therefore, the chapter first introduces the most modern interpretation of symmetry. In this course there is discussed what is the common in the different geometric appearances of symmetry (e.g., mirror reflection, rotation, translation, similitude, etc.), and how do they appear in decorative arts. Symmetry, perfection and beauty were considered in close relation to each other since the ancient times. Then the title theme is exemplified by interdisciplinary applications of symmetry phenomena. At first, symmetry operations in decorative arts are presented in one dimension (frieze patterns), in two dimensions (wallpapers or tiling), with just a short reference to the beauty of crystals (three dimensions), then extended to the even coverage of surfaces (sphere and symmetric flat-faced polyhedra).

At second, there is mentioned that symmetry operations appear in algebra and arithmetic as well. The paper presents how the translation of the natural numbers and their sequences appear in aesthetic representations. This leads the reader over to the discussion of the so-called golden section in aesthetic terms. Then, we turn back to the perfect (Platonic) polyhedra as reflected in the so called golden section kaleidoscope. We explore step by step how the perfect proportions appear in the perfect bodies. We give also a short historical overview how the study of beauty of the symmetrically perfect bodies led to interdisciplinary applications in the sciences.

At third, examples are presented, how the discussed aesthetically outstanding proportions, shapes, are embodied not only in artworks, but also in recent scientific achievements. First of all, the discovery of quasicrystals is treated—that represent the so far “missing” golden proportional fivefold symmetry in concrete material structures. Then there are shortly mentioned also such successful new molecules—like the fullerene and the graphene—which is so important in nanoscience. These discoveries cited from the recent decades demonstrate productivity of interdisciplinarity, in terms of science-art relations. They led from aesthetic considerations (based on different appearances of symmetries) to realized scientific ideas.

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The Concept of Symmetry

Connotation of (scientific) truth with (aesthetic) beauty (or *vice versa*) goes back to *Timaeus* (circa 360 BC) from Plato, who repeatedly referred to his teacher, Socrates in this respect too. Copulation was established by the means of (our and nature's efforts to) perfection. This aimed perfection was embodied in good proportion, harmony, golden mean, common measure, consonance, rhythm, in short, what Plato's Greek contemporaries denoted with the common word *symmetry* (let us refer to the five perfect bodies described first by Plato). We know references (e.g., back till Polykleitos) that several books treated symmetry phenomena in the Hellenistic period, but the first, which survived the storms of centuries were *Ten Books of Architecture* by Vitruvius written a few decades BC. The Vitruvian descriptions on proportions and symmetry served as a basis for the *renaissance* of symmetry studies in the *quattrocento*, starting with Alberti (1435, 1436), through Ghiberti (1455) to della Francesca (1474), Pacioli and Leonardo (1494, 1509), Dürer (1525, 1528) (and to the synthesizer of the Renaissance art theory, Ripa later). The concept of symmetry used to be a link between scientific and art concepts in the fifteenth–early sixteenth century. This based its usages in the modern times, when science and arts separated for three centuries again (seventeenth–nineteenth).

Modern notion of symmetry has been elaborated in crystallography during the nineteenth century. We will return to this notion, founded mainly on geometric terms, in the next section. Now we explain how did this crystallographic notion develop into the contemporary concept of symmetry when the main point of its application was removed into, and distributed among other disciplines and the arts in the twentieth century.

What is common in the different geometric appearances of symmetry (e.g. mirror reflection, rotation, translation, similitude and others)?

– In each instance we performed some kind of (geometrical) *operation* (transformation).
– In this process, *one or more* (geometrical) *characteristics of the figure* remained unchanged.
– This characteristic proved to be *invariant under the given transformation* (did not change as a result of the operation performed).

This was the classical geometric concept of symmetry, elaborated till the end of nineteenth century. Science generalised this in such a way that the interpretation be valid not only for geometrical operations and geometric objects, and not just for geometrical characteristics. In a generalised sense, we can speak of symmetry if

– *in the course of any kind of* (not necessarily geometrical) *transformation* (operation)
– *at least one* (not necessarily geometrical) *characteristic of*
– *the affected* (arbitrary and not necessarily geometrical) *object remains invariant* (unchanged).

The generalisation, that is, took place with reference to three things:

– to any transformation,
– to any object,
– to any characteristic.

The first generalisation made it possible for us not just to look for unchanged characteristics during geometrical *operations* we have learned (reflection, rotation, translation, etc.). This made it possible, for instance, for us to understand invariance under charge reflection (called charge conjugation in precise terms) in physics, and invariance under the swapping of colours in art. The second generalisation makes our concept of symmetry capable of making any kind of *object* of science or art the subject of a symmetry operation. This paved the way, among other things, for us to use symmetry operations on the abstract objects of physics. Finally, we allow the constancy of any *characteristic* to be considered as symmetry. Of the examples familiar to every reader, this is true of electric charge, but any physical quantity considered charge-like can be the object of symmetry, just as can the rhythm of a poem or the motif of a piece of music (Darvas 2007).

Related terms to symmetry are asymmetry, dissymmetry and antisymmetry. Asymmetry is used to denote the absence of symmetry. We speak of asymmetry when none of the characteristics of a given object displays symmetry.
There are instances where an object displays symmetry, but this symmetry is broken in one of its characteristics or a not too significant detail. Example of this is e.g., a bubble in a diamond. We refer to this as dissymmetry. These two concepts related to the absence of symmetry were defined in their current use by Pierre Curie (1859–1906). It is to him that the now famous phrase is attributed: "it is dissymmetry that makes the phenomenon" ("*c'est la dissymmétrie qui crée le phénomène*", 1894). We can take this to mean that phenomena which are important for researchers to discover exist at the points where they encounter dissymmetry. This set of concepts were completed by Shubnikov (1940, 1951) in the 1930s with antisymmetry. We talk of antisymmetry when a characteristic is preserved by being transformed into its opposite. A chessboard is antisymmetric, for example: if reflected, the white squares turn black, and vice versa. The shape of the well-known antisymmetric yin-yang displays rotational symmetry, but if rotated 180° around its centre, black turns to white, and white to black; in terms of its colours, that is, it is antisymmetrical.

The mathematical tool to describe symmetries is group theory. In mathematical terms a group is a set of elements, which have an operation applied to them, and which is subject to four simple axioms. Any symmetry, appearing either in a discipline of science or in a kind of arts, can be characterised by a group.

Description of Symmetries in Classical Crystallography

Nature produces many symmetric phenomena. Beauty of symmetry is manifested in the most spectacular way in crystals. Crystals consist of periodically arranged atoms and groups of atoms (molecules). This periodic arrangement resulted in their ability to allow light to let through their body and to reflect light on their plane cut surfaces, not mentioning the perfect geometric shape formed by the covering surfaces of their bodies.

Periodic arrangements can be produced not only by atoms, and not only in three dimensions, like in crystals. Art learned from nature and the beauty of crystals appears in human creativity, for example in decorative arts, since the early periods of culture. Two and one dimensional periodic arrangements appear in decorative motives, called wallpaper motifs (periodic tiling) and friezes, respectively. They are copies of cross-sections and edges of crystals, where the atoms, molecules can be replaced by either man-made or from the nature borrowed motifs with an infinite abundance. The classification of the frieze and wallpaper motifs follows the terminology and methods elaborated in crystallography. This classification is based on the kinds of symmetries manifested in the given arrangements. Historically, this could be mentioned for the first explicit example how scientific exactness coupled with treatise of artistic beauty.

The symmetry of a crystallographic system (extended in the before mentioned sense also to decorative patterns) has two constituents: the point group symmetry of the basic motive placed in a single point of the motive, and the symmetry of the lattice that these latter single points form. In other words, one constituent is the basic element, or design, that is repeated. A planar example is a crocheted tablecloth composed of repetitions of the same motif; a spatial one is the elementary cell. We usually circumscribe the basic design of the motif with the help of a concept borrowed from crystallography, point groups. The other constituent is the order of repetition: the potential operations and transformations with which one of the elementary motifs can be brought into coverage with the others. These potential operations determine a lattice on the plane or in space. The lattices are represented by the angles given by the edges of the cells, and the length of the edges. Together, we term these defining data of the lattice as the lattice parameters. Again we use a specialist term from crystallography, space lattice, to refer to them (even if the lattices in question are only two-dimensional). Together, the point groups and the space lattices determine the space group structure of a pattern.

The basic motif (point group) of a design can be completely asymmetrical—displaying single-fold rotational symmetry, which is transposed onto itself with a 360° rotation—but with translations and reflections a symmetrical pattern can be made out of it. In other instances, the basic motif can display a number of symmetries. A basic plane motif can be a figure both with mirror symmetry and n-fold rotational symmetry.

Notations

Many operations can be applied to a basic motif which can build a pattern. We can place a point group at the vertices of plane and space lattices of various shapes [this situation is marked by *p*, i.e., primitive cell], or at the centre-points of their edges, faces and cells [marked by *c*, i.e., centred cell]. We can make one overlap the other using some kind of symmetry operation: translation (translation with one unit is marked by *1*), mirror reflection (marked by *m*), rotation (marked by the number of rotations, like *2, 3, 4, 6*) to return to the original position) or glide reflection (marked by *g*). The basic motif must satisfy two conditions, however: whether it is a plane or space lattice, we require that it should fill the plane or the space without gaps, in such a way that it be made up of *congruent cells* (of one given type).

These two conditions present certain restrictions. If we want to be very strict, we could also require that we fill the plane in continuous fashion with *regular*, congruent *polygons*. (The edges and angles of regular polygons are all equal.) Only three shapes are capable of covering the plane in this way: the regular triangle, the square, and the regular hexagon. (Moreover, the regular hexagon can be composed of six regular triangles, what means, we cannot make distinction between a triangular and a hexagonal symmetry.) We are not so strict, so we allow five types of cells in tiling the plane: parallelogram, primitive and centered rectangles, square, and 60° rhombus (consisting of two regular triangles).

Placing the point groups in the distinguished points of a lattice, we receive a pattern, which is repeated. The set of transformations that bring the pattern in coverage with itself form a group in the defined mathematical sense. This is called the symmetry group of the given pattern.

Frieze Groups

We can arrange a given object (in crystallography, point groups) along a straight line in seven different ways. Frieze groups are composed of particular point groups and the operations (transformations) applied to them. Whatever basic motif we place on the point group, the basic unit can be transformed in the following seven ways. To put it another way, taking all possible ways into account, the following seven frieze groups can be produced from the applicable group operations (symmetry transformations) used, i.e. translation, reflection, glide reflection and half-turn. These can be seen in Fig. 1. Their notations, in line with the above conventions, are as follows: *11* (translation), *12* (two successive half-turns, i.e., rotations by 180°), *1m* (reflection in the line of the frieze), *1g* (glide reflection, i.e., we reflect a basic motif in the axis of the frieze, then perform a translation by one unit), *m1* (reflection in an axis perpendicular to the longitudinal axis of the frieze (and translate by one unit), *mm* (reflection in two axes perpendicular to one another),

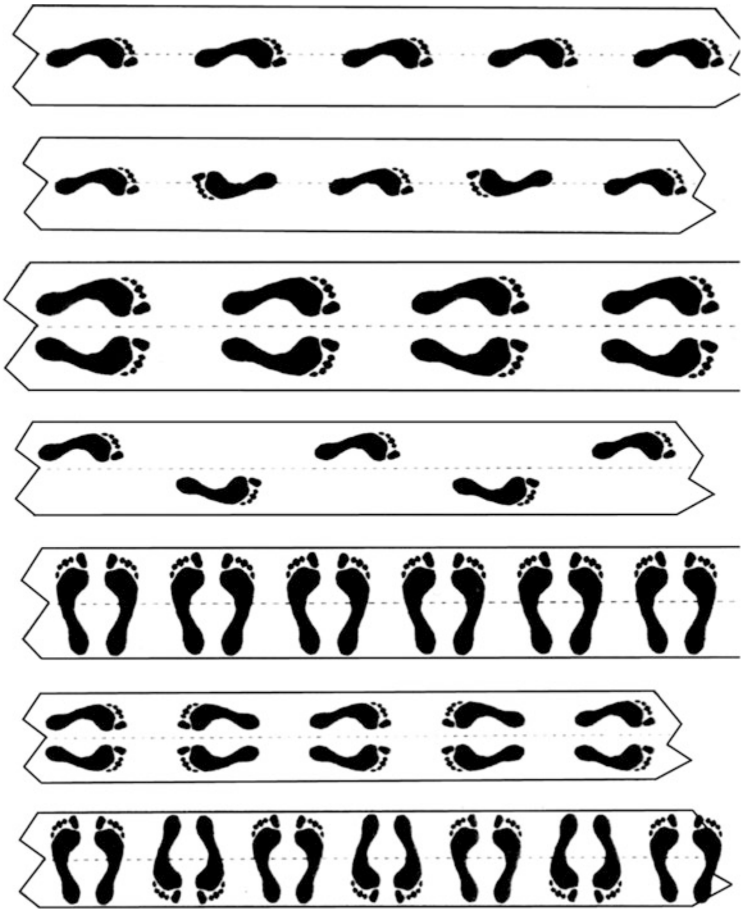

Fig. 1 Footprints demonstrating the seven frieze groups (Kinsey and Moore 2002). Reprinted with the permission of L. Christine Kinsey

mg (combination of reflection in an axis perpendicular to the axis of the frieze and a glide reflection).

One can show that no more frieze groups can be composed, although the low number of the possible frieze patterns seems surprising.

Wallpaper Groups

How many ways can we fill the theoretically infinite planar surface in a continuous fashion with a single (uniform) repeated element? For classification, we use the knowledge gained when we looked at friezes.

We can place an infinitely large number of basic motifs in the place of the individual point groups. There is a limit, however, to how many types of lattice we can use to cover the plane with infinite repetitions, and to the symmetries displayed by the point groups we place in the lattices in order for them to remain, together with the given lattice, invariant under a symmetry operation. The following example illustrates why the last condition represents a genuine limitation. We cannot place a point group with threefold symmetry onto the points of a square lattice, which is symmetrical with regard to reflection in its edges, while satisfying the condition for mirror symmetry, for the mirror symmetry of the pattern would not be preserved during reflection. It would not form a group, as reflection would produce an element which falls outside the group, which contradicts the closure condition formulated among the axioms demanded to fulfil by a group.

On the basis of this, we can interpret the previously introduced concept of *point group* more precisely. In an exact sense, we consider a point group to be the sum of the symmetry operations that can be applied at one point of a planar figure or a spatial body (the term lattice cell applies to both) and under which the given figure or body (cell) is invariant.

We can also make our definitions more precise as concerns lattices. We give the name space lattice (Bravais lattice) to those lattices (even on a plane) which are required by the point group operations applied to the lattice points. What do we mean by required by them? We mean that only those lattices come into consideration that are of a symmetry that the given point group is capable of producing (on the basis of its own symmetries). The only lattices which can be attributed to a given point group (in other words, the only lattices which can be realised in nature) are those which possess the symmetries of that point group. We saw, there are four plane transformations that can be applied on a lattice: translation, rotation, reflection and glide reflection, and we have got five types of possible lattices.

We place point groups in the points of these lattices. One characteristic of point groups is that they can be considered as being fixed on a single point. So we can exclude translation and glide reflection. So, in the case of point groups, only the following symmetry operations can be taken into account:

– 1-, 2-, 3-, 4- or 6-fold rotation around a point, or
– reflection in a straight (symbol: *m*)

Using the above operations both individually and together in two dimensions, the following ten point groups are possible (we can easily see that it is only these):

– 1, 2, 1*m*, 2*mm*, 4, 4*mm*, 3, 3*m*, 6, 6*mm*.

Comparing that said about space lattices and point groups, the ten point groups that can be placed on the lattice points of the five space lattices possible in two dimensions give a total of 17 possibilities for tiling the plane. We call these wallpaper groups. It is such a small number because we saw that the symmetries of particular point groups make it impossible to place them on the point of any lattice. It can be proved that there are 17 and only 17 plane transformation groups that can exist.

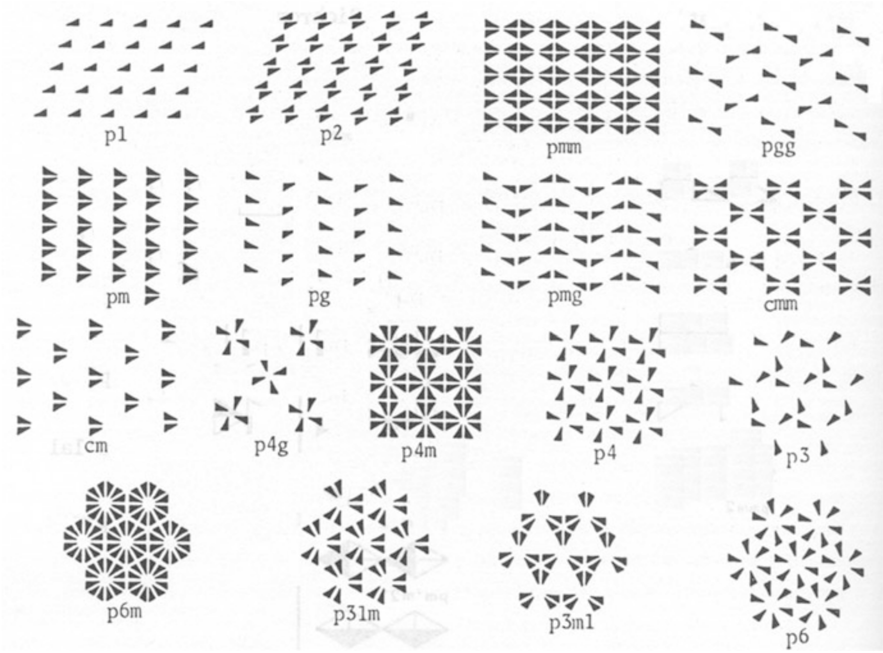

Fig. 2 The 17 wallpaper groups with their crystallographic notations. The image was prepared by Gergely Darvas and extended by György Darvas based on Shubnikov, A. V. and Koptsik, V. A. (1974) *Symmetry in Science and Art*; in: Darvas, György (2007) *Symmetry*. Basel: Birkhauser. Figure 3.17a, p. 90

To conclude, in 2D we have the following possibilities:

– 10: 2D point groups
– 5: 2D space lattices (Bravais lattices)
– 17: 2D plane groups (wallpaper groups).

The wallpaper groups are summarized in Fig. 2. Whatever surprising is it, there are no more than 17 wallpaper groups in one colour. The set of symmetry transformations of any plane pattern can be classified under one of these groups.

The mentioned classification holds for the plane only. More precisely, the surface of a cylinder behaves in the same way like the plane, so one can decorate a cylinder with the same 17 wallpaper groups.

Space Groups

As regards our starting point, the discussion of the beauty of the crystals, we mention only that the continuous filling of space with point groups to be placed in congruent elementary cells is a fundamental task of crystallography. In order to determine the number of ways of possible space-filling, attention has to be paid to: cells of different shapes, point groups that can be fitted in the cells, and the position of the point groups within the cell. On the basis of this, in three dimensions there are a total of: 32 possible 3D point groups, 14 possible space lattices (Bravais lattices), and 230 possible space groups. The various different crystals can be classified on this basis.

Decorating the Sphere, and the Perfect Bodies

The situation is quite different with the surface of the sphere than on the plane or the cylinder. Remember, the plane can be covered without gaps with regular hexagons, but not with regular pentagons. The surface of a sphere can be tiled without gap with regular pentagons, but not with regular hexagons. (The latter would contradict to the well known law of Euler.) Therefore, covering the sphere deserves a few separate paragraphs.

One needs 12, identical edge regular pentagons to cover the sphere without gap. The vertices of these pentagons are distributed equally over the sphere. Each five neighbouring of them determine a plane. One can cut the sphere along such a plane and remove the cap above it. Repeating this procedure over all the 12 plane pentagons, one gets a regular *dodecahedron* (dodeca- means in ancient Greek 12).

Dodecahedron is a perfect body, with equal edges plus congruent faces and vertices. It belongs to the five-member family of the perfect polyhedra, with similar properties. Plato, whom we know as the first who treated them in his *Timaeus*, associated the four others, *tetrahedron, cube, octahedron, icosahedron*, with four primary elements (or substances, identified by Empedocles), namely with *fire, earth, air*, and *water*, respectively. To these four substances, Plato added a fifth, the *universe* (cosmos). Since dodecahedron with its fivefold symmetry appeared to be the most mystical (cf., the theses of the Pythagoreans) among the five perfect bodies, he associated it with the complex universe (Fig. 3). The properties of perfection, which Plato projected onto all areas of existence—including, for example, the social relations between people—were, for him, of symbolic significance. In his interpretation, symmetry was the harmonious order of the universe, that is the very cosmos itself. He writes of the elements of this generalization in *Gorgias* as follows: “And philosophers tell us, Callicles, that communion and friendship and orderliness and temperance and justice bind together heaven and earth and gods and men, and that this universe is therefore called Cosmos or order, not disorder or misrule, my friend.”

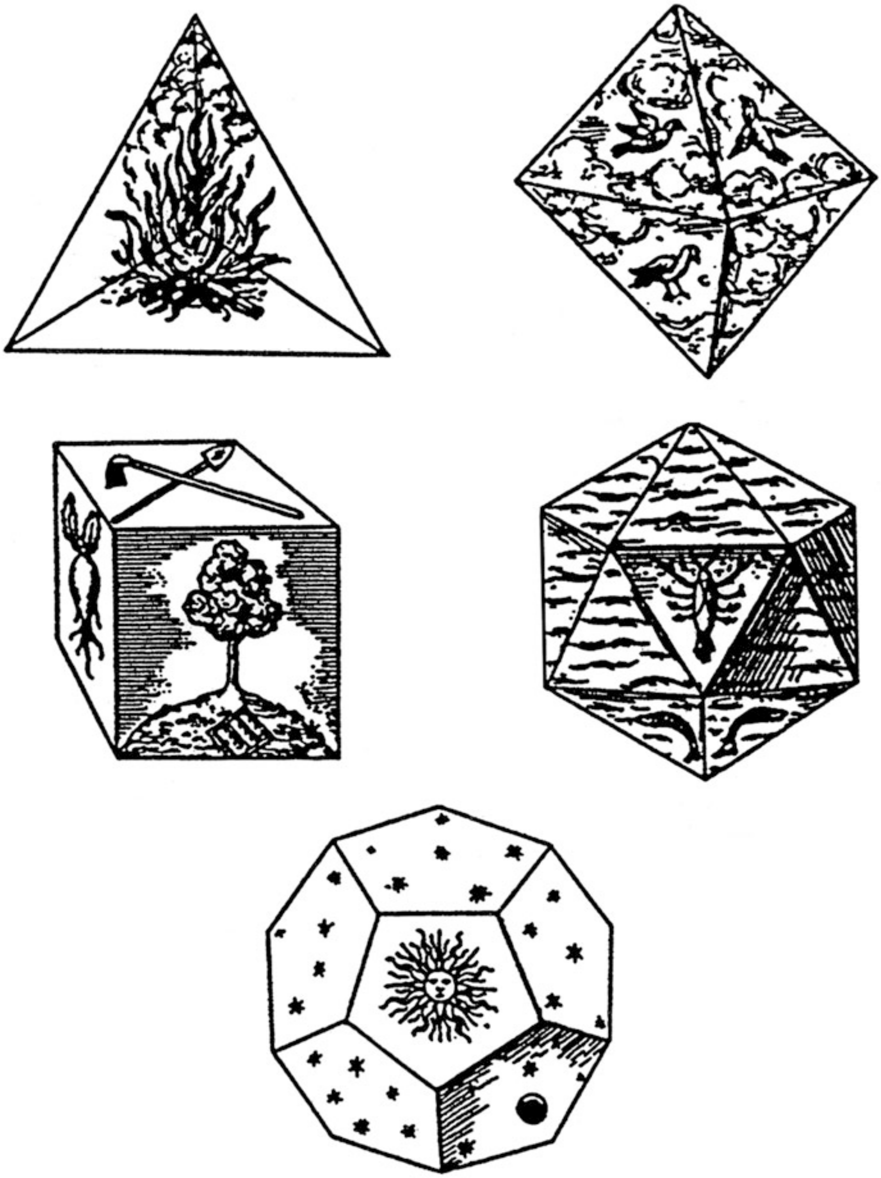

Fig. 3 The five Platonic perfect polyhedra. Source: J. Kepler (1619) *Harmonices mundi*; Prepared by G. Darvas in: (2007) *Symmetry*. Basel: Birkhauser. Figure 2.27, p. 55

Harmony Expressed in Terms of Natural Numbers

Pythagorean "perfection" also included musical harmony in the list of things of artistic beauty. The relationship between the length of strings and the proportions of musical notes was the first law of nature put into mathematical terms. The enumeration of musical sounds and the association of their harmony with the cosmos appear in the so called music of the spheres, which would later have such an intuitive impact on Kepler's discoveries.

The proportions of musical sounds could be written down in the form of the relationships between the smallest natural numbers. Sequences of natural numbers play then an important role in the study of proportions. In the case of the Pythagoreans, numerology meant tracing the material world back to a limited number (1, 2, 3, 4, 5 or many) of primary elements or substances. The schemes of the world based on substances were an invariant element in many different cultures. In the world of Ancient Greece alone (as in Oriental philosophies) we encounter a few schemes of the world based on different numbers of substances. It was perhaps Empedocles (c. 495–435 BC) who had the strongest influence on the Pythagoreans, and whose already mentioned theory was based on four primary elements.

The (displacement) symmetry of the natural numbers seems obvious. However, one can demonstrate that in a stricter sense the set of integers generates a group for the operation of addition, for it satisfies the group axioms:

1. the sum of two integers is an integer,
2. addition is associative: $(a + b) + c = a + (b + c)$,
3. there exists a neutral element, 0, for which $a + 0 = 0 + a = a$, where a is any element of the group,
4. $(-a)$ is the inverse element for a, because $a + (-a) = (-a) + a = 0$.

This was later to become the basis for the harmony in Kepler's picture of the world (Kepler 1596). In India, at around the same time as Empedocles, a similar role was played by *charvaka* (four substances, comparable to his) and *vaisheshika* (five substances: earth, water, light, air, ether). In China, at around this time, the Confucian Hsu Hsing (c. 300–c. 230 BC) associated five *chi* with the five primary elements (metal, wood, water, fire, earth), that could be deduced from the material dichotomies of the yin-yang. (Here, by *chi* we mean what was originally the only substance, primary element, as elaborated in the *Tao* by Lao-Tse (c. sixth–fifth century BC), though *chi* also means knowledge, wisdom, and intellectual essence).

Symmetries of the Perfect Bodies in the World View of the Renaissance and at the Birth of Modern Science

During the *Renaissance* in Europe, the Platonic perfect bodies appeared as the embodiment of the divine proportion. This is well illustrated by the drawings which Leonardo prepared for Luca Pacioli's book *Divina Proportione* (1509). In science, the *Renaissance* brought the triumph of rationality. In contrast to the ancient picture of the world, which, in line with its anthropomorphic attitude, put man at the centre of the universe, and thus preferred the geocentric world-view with its planets tracing complicated cycloid orbits, *Copernicus' revolution* (as Kant referred to it) brought to the fore the heliocentric world-view which described the orbits of the planets with circles. On the basis of the observations provided by astronomical measurements, both models were suitable for correctly describing the movement of the planets. The heliocentric view of the world offered circular planetary orbits, and the Earth-centred one cycloid orbits. The latter gives us a much more complicated description of the world. Yet it was still the more acceptable, because man stood at its centre, and this symmetry proved to be the more powerful. This way of thinking was reflected in ancient philosophy, as well as artistic depiction of nature and of man. As empirical experience was not able to decide between the two models, the choice fell on the simpler, more symmetrical mode of description. In place of apparent symmetry (if we look up at the starry sky above us, we see it as a hemisphere with us standing at its centre), the victor was the path for the Earth given by the symmetry in the measured data (that put the Earth on the same footing as the other planets, tracing a similar orbit around the Sun). Copernicus put the coin on the more symmetric model.

This also brought the rejuvenation of astronomy. Kepler came to determine the laws of the movement of the planets while searching for harmony in the world (Kepler 1609, 1619). His model was based on nested spheres fitting the five Platonic bodies both internally and externally. He wanted to find the perfect picture of the world, the harmony, symmetry embodied in the world. He was himself the most surprised that the orbits of the planets proved not to be symmetric circles, but rather ellipses where the presence of the Sun favoured one focal point over the other.

After Herodotus, Kepler was the first, who preferred empirical experience against the unquestionable belief in symmetry when the two conflicted. This was a greater revolutionary step in scientific thinking than that of Copernicus at the advent of modern science. The destroyed symmetry had to be recovered somehow. Like Plato, Kepler (1619) went back to the music of the spheres of the Pythagoreans. He found the proportions of the angular velocity of the various planets when closest (in perihelion) to and farthest (in aphelion) from the Sun to be as follows: for the Earth 16/15, for Mars 3/2, for Saturn 5/4, which can in order be compared with a half-note, quint, and third. It was these that he composed in his musical motifs.

From Geometry Through Cosmology to Interdisciplinary Applications

Packing of spheres led Kepler to other problems as well, what have got importance later in other sciences. One of the consequences of the problem of drawing regular polyhedra around spheres was Kepler's original formulation of the problem of the densest packing. We know that, on a plane, a circle can be circumscribed by exactly six circles of the same radius, touching it and each other. In space the story is nothing like as simple. We cannot surround a sphere in space with spheres of the same radius in such a way that all the neighbouring spheres touch each other as pairs. There is no solution with integer number of spheres, which means a terrible asymmetry in classical geometry. The classical Kepler problem is thus as follows: how can we arrange similar spheres in a bowl in the densest fashion (i.e. with the most spheres)? (The planar equivalent of this would be how we could cover the surface of the table in the densest fashion with a given denomination of coin, i.e. with the most coins.) The set of problems concerning the densest tilings and closest packings has, over time, become and increasingly independent branch of mathematics. Perhaps it is surprising that centuries had to pass after Kepler before the problem of the densest filling of space was solved. In the nineteenth century, Gauss (1777–1855) provided an estimate that with congruent spheres the space could not be more densely packed than 74.048% (which coincided with the original Kepler conjecture), but this was only proven in 1998 by Thomas Hales, who reported his proof to the fourth symmetry congress in Haifa in that year.

This set of problems is not merely an intellectual game. The areas of application of the densest filling of space and close packing of spheres are very wide. Examples of some such areas are: in crystallography, the determination of the optimal position of atoms; in chemistry, determination of optimal bonding directions; in physics, in atomic nuclear models; in biology, in the division of egg cells; in architecture, the selection of the most stable supporting structures. So did a distortion of symmetry stimulate interdisciplinary applications in different sciences.

Perfect Proportion

We know from Aristotle that ancient Greeks attributed significance to perfect proportions. We learned from Vitruvius that perfect proportion was identified what later was associated with golden mean. Pythagoreans sought the harmony in sequences of natural numbers. They found the arithmetic, geometric and harmonic means between numbers.

The perfect proportion, which was instinctively applied by sculptors and architects since the ancient times, turned to be associated with the so called Fibonacci sequence of natural numbers. Although the perfect proportion is an irrational number, it can be expressed as a limit of rational quotients. The Fibonacci numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, ... etc. are composed as the sum of the preceding two members of the sequence. Where is the symmetry in this sequence? It is in the regularity in the rule of formation. The summing is shifted step by step to the next (pair of) members of the sequence. What is more interesting, the quotient of the consecutive members of the sequence a_n/a_{n-1} approaches to the limit in the infinity what is known as Φ, the Fibonacci number.

The same number Φ can be obtained also in a geometric way, from the proportion of length of sections, which are defined by the division of a section in two, so that the length of the longer section is so to the shorter as the length of the whole section to the longer portion: $a:b = (a + b):a$. It may be surprising that these two quite different (algebraic and geometric) definitions lead to the same result, $\Phi = 1618\ldots$. Nevertheless, the literature treating the peculiar properties of the Fibonacci numbers and the number Φ can fill a library. This Φ can be identified with the—later so called—*golden section*, or golden proportion, known both from artworks (from fine arts to performing arts like music) and scientific results (in diverse disciplines). We can refer here only to a very few examples that may enlighten the interdisciplinary applicability of this proportion.

Golden Section and the Perfect Bodies

In the preceding introductory sections to the interdisciplinary application of symmetry phenomena mention was made of the perfect (or Platonic) solids and the Fibonacci sequence (together with the related golden section—or Fibonacci—number). The two were put side by side intentionally. They are interrelated.

Let us take a square and divide its two neighbouring edges according to the golden section as shown in Fig. 4a, then connect the section points with a straight line to the opposite vertex of the square. Fold the generated triangles in the left and bottom along the inner lines so that the two not divided edges of the square meet. You will get a tetrahedron. If the inner surfaces of this tetrahedron are made of mirrors, the three facets will be reflected into each other. This is the so called golden section kaleidoscope. The possibility of such a kaleidoscope was shown by E. S. Fedorov (1891) by the end of the eighteenth century. He was the person who (parallel with A. M. Schönflies [1891], and almost parallel with W. Barlow [1895]) proved that there exist exactly 230 3-dimensional crystallographic groups. H. S. M. Coxeter (1907–2003), probably the greatest geometer of the twentieth century called the attention that due to the angles of this kaleidoscope, it could be cut from a square, in the 1960s. The implementation, with some adjustments of the order of the angles, was made by Nicolas and Caspar Schwabe (1993) in the 1980s. If allow light inside the kaleidoscope through the arcs marked in Fig. 4a, and cover the arc sections in the individual facets with different colours, the arcs will be reflected in the kaleidoscope to shape polyhedra.

Figure 4c shows what do we see inside. The mirror images of the shortest arc—marked by blue—will shape a regular dodecahedron. The mirror images of the

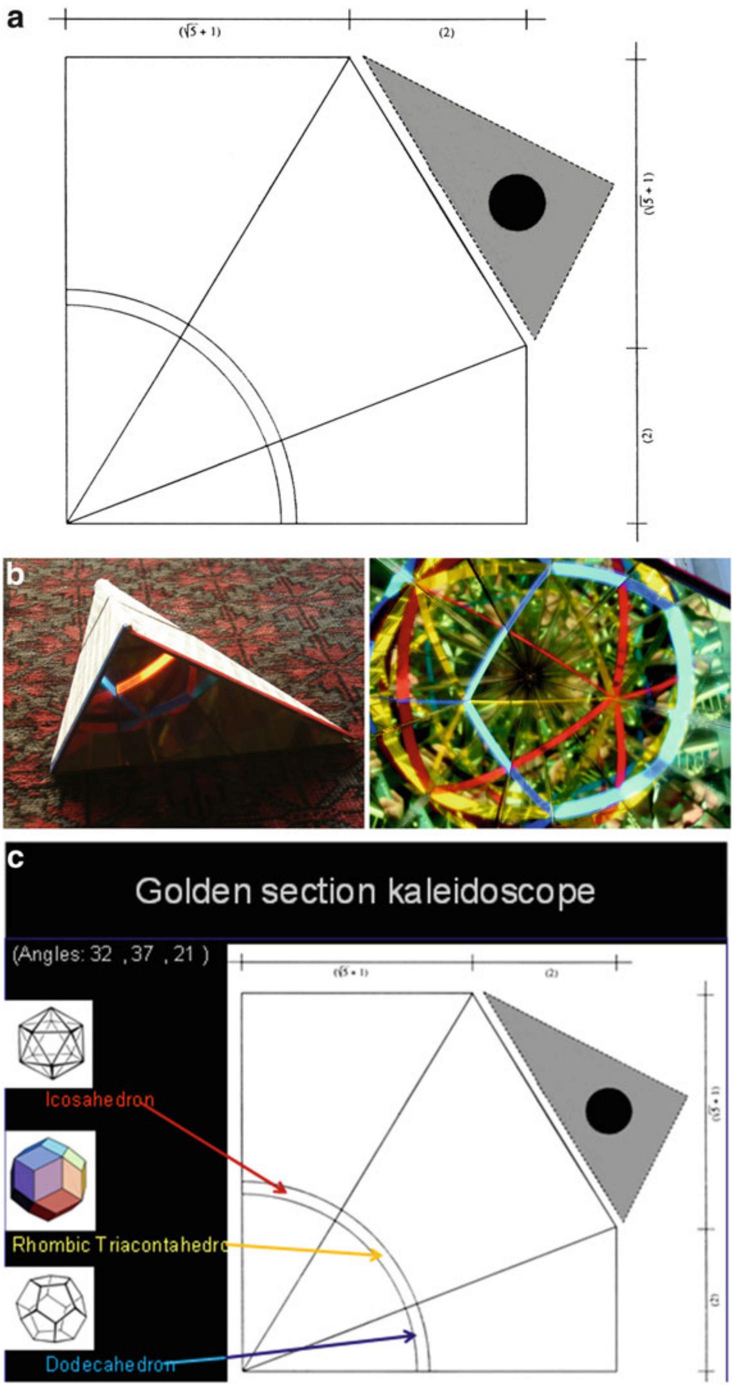

Fig. 4 (**a**) Golden section of a square. Image reproduced by courtesy of the journal *Symmetry: Culture and Science (1990/1992),* based on the paper by C. Schwabe (1986), copyright *Turicum*, Schweizer Kultur und Wissenschaft, April/May 1993, S. 6–7, Reproduced there by courtesy of the

medium length arc—marked by red—will shape a regular icosahedron. The longest arc—which is marked by yellow—and its mirror images will shape a rhombic triacontahedron (Fig. 4c). The rhombic triacontahedron is a polyhedron whose facets are all identical rhombuses. The rhombic triacontahedron is covered just with 30 facets, like the number of edges of either the dodecahedron or the icosahedron. They are called golden rhombuses, because the lengths of their diagonals (which intersect each other perpendicularly) are in proportion to the golden section! We can recognise a miracle in our kaleidoscope: these golden section proportion diagonals coincide with the edges of the dodecahedron (blue) and the icosahedrons (red), as one can check in the right image in Fig. 4b.

We have found mathematical connection between the properties of the regular (Platonic) polyhedra and the golden section

Fivefold Symmetry in the Plane and the Space with the Help of the Golden Section

We knew from the proportion of the length of diagonals to the edge of the regular pentagon that fivefold symmetry is related to the golden section. Now, we learned that golden section is related to regular polyhedra, too. We also knew that the plane cannot be tiled without gap with unique elements showing fivefold symmetry. The surface of the sphere, or a polyhedron can be, this is the dodecahedron. However, the dodecahedron is unable to fill the space without gap. There is only the cube (among the regular polyhedra) that can fill the space without gap. Tetrahedra and octahedra together can fill the space, but that is not considered as a crystal lattice. Crystals—with the condition that only one kind of unit cell be applied—were not allowed to contain two (or more) types of unit cells in their lattice. Thus, there is no classical crystal with fivefold symmetry.

If one is unable to tile the plane with a single regular unit element, without gap showing fivefold symmetry, needs to sacrifice one or more of the conditions: either the congruency of the tiles, or the condition to use only one element. The goal is to solve the problem with the minimum of the disregarded conditions, e.g., to reduce the number of the used elements to two. The same is aimed at the space.

The planar problem was solved by R. Penrose (1974) in 1973. First it was a decorative pattern for artistic purposes, and he patented it as a puzzle. After working with kite and dart shapes, he reduced the two elements of the tiling to two kinds of equal length edges rhombuses. One of them is identical with the above discussed golden rhombus. The tiling displays local fivefold symmetry, that means,

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Fig. 4 (continued) editor of the—already ceased—journal *Turicum* (private letter of the former editor to G. Darvas, 2004. (**b**). The golden section kaleidoscope (*left outside*, *right inside* view). Image copyright by the author. (**c**) The shapes seen in the golden section kaleidoscope. Image copyright by the author there is no shift in the tessellation, which brings the pattern into coverage with itself, therefore the pattern is not periodic, but there are similar domains in it that show locally fivefold symmetry. This quasiperiodic tessellation can be continued up to the infinity in the plane. John Conway proved soon (Gardner 1977), inter alia, that we can find a domain tiled similarly to that we happen to have chosen within a very small distance. If the diameter of a chosen (say circular) domain is d, then we reach a domain with similar tiling in some direction from the boundary of the original domain, at a distance of at most $s \leq \frac{\Phi^3}{2} d = 2.11\ldots d$. (As further evidence of the universal connections between symmetries, Φ here is again the golden number we know from the golden section.) This theorem not only gives a certain limit to repetition (even if it is not periodic)—this limit is a surprisingly low, "visually observable" boundary distance. The quasi-periodic arrangement also means that we find certain "local" symmetries in it, which break once past a certain boundary, but which are repeated elsewhere, locally. There are also those that infinitely preserve their symmetry with regard to a certain point, but where the periodic translation of this symmetry centre-point does not make the entire tiling of the plane overlap itself.

One of the other applicable mathematical theorems for the Penrose tiling of the plane with the most interesting symmetry relates to the proportion of the elements used. The proportion of the number of dart and kite shapes is—like the proportion of their area—equal to the golden proportion. 1.618 ... times as many kites are needed as darts. If the tiling is infinite, this number is the precise proportion. The fact that this proportion is not rational is used by Penrose to prove that tiling is not periodic, for if it were periodic, this proportion would have to be a rational number. This theorem—like all those concerning kites and darts—holds true even if the tiling takes place with the Penrose rhombuses (made up of darts and kites) with edges of equal length, and angles of 72° and 108° or 36° and 144° respectively.

One did not need to wait too long to solve the problem of space filling without gap with two elements, namely with two rhombohedra which have edges all of the same length, and which are both delimited by a single congruent rhombus identical with one of those used in the planar Penrose tiling. However, geometric space filling with local fivefold quasiperiodic arrangement does not guarantee that nature produces the same structures with atoms in the vertices of the found arrangement. Remember, two-cell spatial arrangement contradicted to the conditions set up by classical crystallography, even not mentioning quasiperiodicity and (at least local) fivefold symmetry. Events started to accelerate.

The Discovery of Quasicrystals

Ammann's discovery (1976) (Gardner 1977), described below, has paved the way for the possibility of space filling in non-periodic ways displaying fivefold symmetry. He began by constructing from the two mentioned rhombohedra which have edges all of the same length, and which are both delimited by a single congruent "golden" rhombus. The proportion between the diagonals of the faces is the golden section. One looks like a cube flattened along one of the diagonals of its body, while the other looks like a cube stretched along one of its diagonals. H. S. M. Coxeter referred to these as "golden rhombohedra". Apart from these two, no other golden rhombohedra exist. Both were already studied by Kepler. Martin Gardner, the great expert in this field, drew Penrose's attention to Ammann's results. Ammann took a set of two rhombohedra, parallelepipeds with six congruent rhombus faces, and showed that, when face-matching rules are applied, they could tile space non-periodically without gap. As the two rhombohedra are golden rhombohedra, the faces have diagonals in the golden ratio.

Penrose came to the conclusion that it might be a model for certain unexplainable molecular formations, such as viruses. Alongside his congratulations to Ammann, Penrose made the following reply to Gardner:

> [--] some viruses grow in the shapes of regular dodecahedra and icosahedra. [--] But with Ammann's non-periodic solids as basic units, one would arrive at quasi-periodic 'crystals' involving such seemingly impossible (crystallographically) cleavage directions along dodecahedral or icosahedral planes. Is it possible that the viruses might grow in some such way involving non-periodic basic units, or is the idea too fanciful? (R. Penrose to M. Gardner, 4 May 1976, cited in Gardner 1997, 24.)

Nevertheless, interdisciplinary research turned first towards crystalline structures before deepening in the really fascinating applications of the idea in the structures of virus growth. In 1977, Koji Miyazaki at the University of Kyoto reached a discovery similar to but independent of Ammann's (Miyazaki 1986). In addition, he found another method of filling space with two golden rhombohedra in a non-periodic way. In this method, five golden rhombohedra with acute angles and five rhombohedra with obtuse angles fit together to form a rhombic triacontahedron. Two such bodies can each be surrounded by a further 30 golden rhombohedra of each type, which results in a larger rhombic triacontahedron, and this extension can be continued infinitely. This gives us a honeycomb-like filling of the space, the centre of which displays icosahedral symmetry.

Naturally—almost in the minute Penrose had made his planar discovery—the theoretical work began in structural quantum chemistry. The theoretical possibility of the filling of three-dimensional space with atoms with two types of cell stretched by atoms, in quasi-periodic fashion, displaying local fivefold symmetry, was first shown by London crystallographer Alan Mackay in 1978 (Mackay 1982). His solution was rendered more precisely by Tohru Ogawa in 1981—in the knowledge of Miyazaki's result. After this, Mackay and Ogawa joined forces to use calculations to confirm the theoretical possibility of creating stable electron bonds in the space directions determined by their models.[1] The space filling suggested by

[1]The author was invited by the Nobel laureate rector of the University of Tsukuba Leo Esaki, to give a keynote lecture in a panel discussion in an afternoon seminar at the University of Tsukuba, where A. Mackay and T. Ogawa read the discussed lectures, together with K. Miura and D. Weaire

Ogawa was demonstrated in a bamboo shoots model by Akio Hizume, where the cells represent pentagonal space filling and rhombohedral cells.

The dilemma of crystallography at the beginning of the 1980s was as follows: can quasi-periodic space filling be regarded as a crystal? It seemed that, after the theoretical work of Mackay and Ogawa, even if the concepts of classical crystallography did not allow quasi-periodic space filling into the system, their existence could no longer be ignored, although no one had reported to find such structures.

Dan Shechtman and his colleagues in a laboratory at the American National Institute of Standards and Technology near Washington, D.C. examined the material structure of alloys with the help of X-ray diffraction images. Shechtman found X-ray diffraction images displaying tenfold symmetry. As it later transpired, a number of crystallographers investigating the structure of matter had already encountered such images. They had always thrown them away as bad exposures. In 1982, when he looked through many hundred exposures in a single night, and rejected several seemingly "defective" images, he was struck by the idea that they were perhaps not defective, after all. It could not be an accident that there were so many of them. Maybe there was some kind of regularity behind their occurrence? With his colleagues, he set about examining the rejected pictures again, together with the sample of the material they had been taken of, a suddenly cooled aluminium-manganese alloy (Shechtman et al. 1984). This was how he discovered the materials that would later be called quasicrystals.

Their result was an encouragement to all those who had previously not dared to believe in the existence of samples of materials displaying fivefold symmetry. From this point on, "faulty" X-ray images were no longer thrown in the rubbish bin, and one alloy after the next was discovered to have similar symmetry. Quasicrystals, with their fivefold symmetry, opened a new chapter in material science. However, the community of crystallographers resisted to acknowledge the existence of quasiperiodic crystalline arrangements of atoms with fivefold symmetry and multiple golden section proportions, which contradicted to the canon accepted since the Fedorov-Schönflies doctrine. It took years while a journal has accepted to publish their paper.[2] Shechtman was awarded the Nobel prize for his epoch-making discovery of crystalline structures demonstrating fivefold symmetry only in 2011.

on 25 November 1994 (Darvas 1996; Ogawa et al. 1996). In 1996, when the author spent a semester in Tsukuba for the invitation of T. Ogawa as a visiting professor, he lived in the same house, where the Mackay couple did in 1981.

[2]The author co-organized the ten years anniversary session to remember the publication of the discovery of quasicrystals in Washington, D.C. with the participation of Dan Shechtman and his co-author colleagues in 1995 (Darvas et al. 1995). That time, the quasicrystals were worldwide acknowledged, and many studied them. It was in the same year, when Shechtman organised an interdisciplinary seminar for the faculty members of the Technion in Haifa, and he introduced the author, as the invited speaker of that event, to the audience. Later, Shechtman and the author were co-organizers of the fourth symmetry congress and exhibitions held in Haifa, 1998, and were co-editors of the proceedings volume (Darvas et al. 1998).

Summary

Symmetric properties of material structures led to many discoveries. To remain in the domain of material science, one can mention the discoveries of the fullerenes (Kroto et al. 1985, acknowledged by Nobel prize in 1996) and the production of graphene, which is important to nanoscience (acknowledged by Nobel prize in 2010). Quasicrystals took their place in artworks as well. All they—along with many others in particle physics and life sciences—demonstrate the productivity of interdisciplinary clues and holistic thoughts bridging between arts and sciences.

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