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Editors

Aesthetics of Interdisciplinarity: Art and Mathematics

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In memory of Reza Sarhangi (1952–2016)

Foreword

It is hard to believe that 25 years have gone by since I wrote the book, *Connections: The Geometric Bridge between Art and Science* (Kappraff 1990, 2nd edition 2000). At that time, only a few individuals were studying a new discipline that has become known as Design Science. What is Design Science? It is a subject that has advanced from the twin perspectives of the designer and the scientist sometimes in concert with each other and sometimes on their own and may be considered a bridge between art and science. Design Science owes its beginnings to the architect, designer, and inventor Buckminster Fuller (1975).

The chemical physicist Arthur Loeb is one of the individuals most responsible for recognizing Design Science as an independent discipline. He considers it to be the grammar of space and describes it as follows:

> Just as the grammar of music consists of harmony, counterpoint, and form (sonata, rondo, etc.) which describes the structure of a composition and poetry has its rondeau, ballad, virelai and sonnet, so spatial structures whether crystalline, architectural, or choreographic, have their grammar, which consists of such parameters as symmetry, proportion, connectivity, valency, stability. Space is not a passive vacuum; it has properties which constrain as well as enhance the structures which inhabit it. (Loeb 1993, 1)

At first, Prof. Loeb was almost alone in teaching the principles of Design Science to generations of students at Harvard over a 25-year span. At the same time and duration, Mary Blade also developed similar ideas at the Cooper Union and Haresh Lalvani began a Design Science program at Pratt.

Prior to the ideas of Design Science coming together, a person who wished to participate in this academic adventure would have to consult a library of books on various aspects of this discipline. There were few attempts to unify this knowledge and few outlets to communicate the results of any endeavor to an outside audience. There was even the question as to who was the audience. Due to the spirit of community of the early pioneers, this problem was soon remedied. Two conferences arose in the early 1980s initiated by Marjorie Senechal on *Shaping Space and Symmetry* (1988). The vastness of the scope of Design Science was illustrated in Istvan Hargittai's two-volume set *Symmetry: Unifying Human Understanding*

(1986) which demonstrated that art and science were indeed close companions. Other early pioneers were Nat Friedman with his conferences in Albany and his ISAMA conferences. This was followed in 1998 by Reza Sarhangi's Bridges Conferences (just having completed its 18th conference), Kim William's Nexus conferences, the Symmetry Festival Conferences of György Darvas, the series of Symmetry conferences led by Dénes Nagy, and the Ars GEometrica summer workshops led by Dániel Erdély and Kristóf Fenyvesi in Pécs, Hungary. Fast-forwarding to the present, we see Finland beginning to place Design Science at the center of its educational system.

The air was electric with ideas which led to new journals such as *Structural Topology* created by János Baracs and Henry Crapo, the Proceedings of the Bridges Conferences edited by Sarhangi, the electronic journal *Visual Mathematics* edited by Slavik Jablan and Ljiljana Radovic (Vismath), the Nexus Network Journal of Kim Williams (Nexusjournal.com), the *Journal of Mathematics and the Arts* inspired by the Bridges Conference, the journal *Symmetry and Culture*, inspired by the Symmetry Festival directed by György Darvas, and the *International Journal of Space Structures* edited by Tibor Tarnai.

The beauty of these publications and events was that all contributions were welcome and it turned out that artists, sculptors, and musicians, often with little or no formal background in mathematics, mingled at the conferences with mathematicians and scientists and contributed significant geometric ideas. This led the renowned geometer Branko Grunbaum to lament:

> It is a rather unfortunate fact that much of the creative introduction of new geometric ideas is done by non-mathematicians, who encounter geometric problems in the course of their professional activities. Not finding the solution in the mathematical literature, and often not finding even a sympathetic ear among mathematicians, they proceed to develop their solutions as best as they can and publish their results in the journals of their disciplines. (Grunbaum 1983, 166)

Grunbaum felt that mathematics had abandoned the concrete problems of Euclidean geometry for more abstract and distant areas of mathematics remote from the interests of nonmathematicians.

Also up to this time, there occurred a kind of Tower of Babel of academic disciplines in which mathematicians, chemists, crystallographers, artists, architects engineers, and craftspeople each had their own language. In fact, often people from different disciplines were saying the same thing but in their own languages. These interactions enabled emerging ideas and energies the opportunity to merge with each other and celebrate their commonality. Mathematics arose from these diverse domains as the grand unifier due to its abstraction with mathematics rediscovering its roots in the wellspring of geometry.

The Design Science movement also acknowledged the contribution of ancient cultures and ethnicities through the analysis of ancient art and architecture and the study of ethnomathematics. As Slavik Jablan discussed in his article, "Do You Like Paleolithic op-art?" (Kappraff et al. forthcoming), he traced these designs as far back as 23000 BC. Over the years, a constant theme was Islamic patterns celebrating the golden age of Islam. Paulus Gerdes discovered that sand drawings in the Sona tradition of the Tchokwe people of Angola and Zaire resulted in mirror curves and Lunda patterns (Gerdes 1999a, b). At the same time as these folk arts resulted in beautiful patterns, they also had an inner structure based on familiar mathematical principles such as modular arithmetic, abstract algebra, fractals, and the theory of knots.

It was also noted that while the themes of Design Science gave rise to scholarly research, the ideas could also be appreciated not only by artists and crafts people but also by young children so that workshops for children and the uninitiated became a regular part of the conferences. And this has played a role for young children to develop an interest in STEAM programs. These ideas also filtered into places such as the Museum of Mathematics (MoMath, New York). I have been teaching a course to Students from the College of Architecture and now the college of Design in Mathematics of Design at New Jersey Institute of Technology since 1978. Some of the students' work can be found in *Connections*, and a textbook of teaching materials will soon be available (Kappraff et al. forthcoming). Another fine educational resource came about due to the collaboration of Annalisa Crannel, a mathematician, and Marc Frantz, an artist who led a series of Viewpoints workshops, an NSF program, leading to the book *Viewpoints on a Study of the Iterative Function System of Fractals and Perspective Geometry in an Artistic Context* (Frantz and Crannel 2011). Michael Frame and Benoit Mandelbrot also contributed a book on fractals in the classroom (Frame and Mandelbrot 2002).

Always in the background of Design Science were what I would call the "elders," luminaries such as Buckminster Fuller, Arthur Loeb, M.C. Escher, HSM Coxeter, Branko Grunbaum, and Magnus Wenninger whose work has had such a great impact on the field. This brings me to the content of *Aesthetics of Interdisciplinarity: Art and Mathematics*, by Kristóf Fenyvesi and Tuuli Lähdesmäki. At a manifest level, this book presents a set of coherent essays on the theme of art and design. But at a deeper level, the book grapples with the question of what makes a design have an instantaneously felt sense of "rightness." A design, by nature, is abstract, yet we know a great design when we see it, and this is not a random value judgment. There are indeed objective criteria.

Think of what goes into a design. The design must first of all express the skill of its maker. We often refer to this as a design being "elegant," and it is interesting that mathematicians use the same term to express its aesthetics. Although I speak of art, architecture, and design, the same criteria can be shown to hold for mathematics, particularly geometry, although patterns of number have a charm of their own. Although skill is essential, the skill of an artist or designer or, for that matter, a mathematician is not enough. The work must have discernable content. In other words, it must be the product of a system. For example, the great architecture of Le Corbusier was based on his system of the Modulor derived from the golden mean (Kappraff 2000, Kappraff et al. forthcoming). The glorious designs of the Alhambra were based on the 17 wallpaper symmetries of the plane and the geometry of the golden age of Islam. My work on the proportions of the Parthenon, along with the work of Anne Bulckens and Ernest McClain (Kappraff and McClain 2005), showed that the proportions were based on the tones of the pentatonic scale. Although mathematics can survive as a simple manipulation of symbols, it will be vacuous unless it is based on a system, the richer the better. I can say that all great works have some system at its root.

Mathematics has placed great currency in the concept of duality. Often when a design or a mathematical theory is derived, a second result, dual to the first, is derived at the same time. This concept pervades the theory of graphs, mathematical logic, and projective geometry, and M.C. Escher has found it in his art.

It is generally recognized that a great design has a kind of simplicity or lack of clutter, an economy of form. Jablan has shown that even the designs of the Mezin culture in Ukraine from 23000 BC, and other early civilizations from Eastern Europe, were based on a single tile with stripes on it derived from basketry and weaving or clothing design. A great deal of diversity can be derived from even a single module.

In an otherwise chaotic world, the element of symmetry presents the mind with repeating themes. In general, the mind recoils from endless novelty and prefers to see something that it has seen before. In mathematics, it can be said that without symmetry there are no laws or theorems or, for that matter, mathematics, only endless, featureless serendipity. In fact, the theories of science and the theorems of mathematics can be thought to be based on symmetry principles and can be likened to narrow channels lying in the otherwise featureless terrain of art, science, and mathematics. Some would say that a great design should contain the element of surprise often achieved by an admixture of both symmetry and symmetry breaking just as smooth mathematics largely prevails in a mathematical domain only to come upon singularities, which break the symmetry and signify some unusual event or property of the system. This also breaks the monotony of the sameness introduced by symmetry. Instead of surprise, one can talk in terms of stability and instability or balance and imbalance. For example, the golden mean has found its way into the organization of great art because the ratio $1{:}\phi$, where $\phi = 1.618\ldots$ is the symbol of the golden mean, is a way of defining a "middle" that is not exactly in the middle and so introduces the element of tension into the work of art. The golden mean also introduces the element of self-similarity wherever it is found.

It is a further challenge and source of delight to find cultural connections within a design such as remnants of Islamic tilings or the geometry derived from the "flower of life," a component of the subclass of design referred to as sacred geometry, bestowing spiritual content to a design. The folk art of Gerdes mentioned above is another example of design in a cultural context. Ethnomathematics has grown up to follow the cultural aspects of this subject. Perhaps the most advanced application of the use of cultural connections is the work of Crowe and Washburn in their book, *Symmetries of Culture* (1988), which correlated folk design with the symmetry group employed to create the design with each design providing clues as to its fabricator.

The artist or designer should also give as much consideration to the space left over in a design as to the built space itself, i.e., both figure and ground. Finally, a design comes alive when the magical element of color is introduced. I always tell my students to find someone who knows about color theory and study with them.

In summary, the impact and interpretation of a work of art or design can be related to the following concepts which also have importance for mathematics: elegance, content, duality, simplicity, symmetry, symmetry breaking, managing of stability and instability, cultural context, figure and ground, and color.

In large part, this book will explore this terrain through a set of well-chosen essays.

Newark, NJ Jay Kappraff
10 March 2017

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Contents

About the Authors

Mark Daniel Cohen is the Assistant Dean and Chief Loan Officer of the Philosophy, Art and Critical Thought Division of The European Graduate School and one of the founding editors of the academic journal *Hyperion: On the Future of Aesthetics*. He is also a freelance author who writes regularly on art in New York City, with over 400 articles, art reviews, and essays on contemporary art and aesthetics in publication in a variety of art exhibition catalogues and commercial, university, and art school journals. He has recently published *Coarctate: Antigone's Return and Selected Poems*, *Ilan Averbuch: Public Projects*, and *The Judenporzellan of Izhar Patkin* and has contributed chapters to *Chawky Frenn: Art for Life's Sake*, *Abstraction in the Elements*, *The Archeology of the Soul*, and the second edition of *Dictionary of the Avant Gardes*. He is currently working on two philosophical works: *The Power of the Right* and *Treatise on Poetic Reason*. In addition, together with Dr. Friedrich Ulfers, Cohen is preparing a book of Nietzsche's ontology and cosmology and the Eternal Recurrence of the Same viewed from a mathematical perspective.

Donald Crowe was born in Nebraska in 1927 and educated at the University of Nebraska, the ETH in Zurich, Switzerland, and the Universities of Michigan and Toronto with a PhD in geometry directed by H.S.M. Coxeter. After 3 years as a lecturer at University College, Ibadan, Nigeria, he began a 35-year career at the University of Wisconsin, during which time he became interested in continuing the pioneering studies of Andreas Speiser and Edith Müller on the symmetries of real-world decorations, using the geometric tools developed by crystallographers in the nineteenth century. Beginning in 1971, he made short field trips to Mesa Verde (USA), Ghana, Australia, Fiji, and Tonga to record the occurrence of repeated patterns, especially "two-color" patterns. Long collaboration with Dorothy Washburn led to their 1988 book *Symmetries of Culture*. He is currently active in the Bridges organization and the *Journal of Mathematics and the Arts*.

György Darvas (1948), PhD, took his degrees in theoretical physics and in philosophy of science at the Eötvös Loránd University, Budapest. His research concentrates on symmetries in the sciences and arts and interdisciplinary bridges between disciplines promoted by the concept and phenomenon of symmetry. He is the director of the nonprofit institute Symmetrion (1992–), is the founder and CEO of the International Symmetry Association (2003–2013), and has lectured at the R. Eötvös University (1993–2015). He has been a senior research fellow of the Hungarian Academy of Sciences for 40 years; an invited professor at the Tsukuba University, Japan (1996), and the Lomonosov Moscow State University, Russia (2006–2008); and invited lecturer at several universities and conferences from the USA through all of Europe to Russia and from Japan to Israel. He was founder (1990) and editor of the journal *Symmetry: Culture and Science* (1990–), chaired many science-art conferences and festivals, and organized exhibitions on symmetry and related interdisciplinary subjects (1989–). He is an author and editor of over 250 publications, including 14 books.

Kristóf Fenyvesi (1979), PhD, is a researcher of STEAM (Science, Technology, Engineering, Arts and Mathematics) Learning and Contemporary Cultural Studies in Finland, at the University of Jyväskylä's Department of Music, Art and Culture Studies. He is the Vice President of the world's largest mathematics, arts, and education community, the Bridges Organization (www.bridgesmathart.org). In 2013, he was appointed as Chief Executive Officer of International Symmetry Association (www.symmetry.hu), and in 2008, he started Experience Workshop—International Math-Art Movement for Experience-oriented Education of Mathematics (www.experienceworkshop.org). His main areas of research are mathematics and art connections in learning; STEAM education; inquiry-based, cooperative, playful, and experience-oriented approaches in mathematics education; problem solving in mathematics, in science, and in art education; connecting hands-on activities and digital modeling in mathematics, science, art, and design education; science and art connections in learning; phenomenon-based, multi- and transdisciplinary learning and co-teaching; inter-, cross-, multi-, and transdisciplinary management and trans-curricular leadership in education; and interdisciplinary aesthetics and philosophy. Fenyvesi has edited a number of math-art-education handbooks and other mathematics and art resources. He is an associate editor of the Journal of Mathematics and the Arts and has been very active in organizing various international conferences, scientific and education events, exhibitions, and STEAM festivals all around the globe.

Axel Gelfert is an Associate Professor of Philosophy at the National University of Singapore. He has published widely on issues in historical and social epistemology, on the philosophy of art history, as well as on questions in the philosophy of science and technology. He has held visiting fellowships at Collegium Budapest (Institute for Advanced Study) and at the Institute for Advanced Studies in the Humanities (IASH), University of Edinburgh. Before turning to philosophy, he studied Theoretical Physics at the Humboldt University, Berlin, and the University of Oxford.

He is the author of *A Critical Introduction to Testimony* (Bloomsbury 2014) and *How to Do Science With Models: A Philosophical Primer* (Springer 2016).

Paulus Gerdes (1952–2014) was the Professor of Mathematics and Chief Advisor for Research and Quality of Education at the ISTEG-University in Belo Horizonte, Boane, Mozambique. He published on geometry, the culture, and history of mathematics. His specialty was ethnomathematics. Professor Gerdes was a visiting professor at the University of Georgia from 1996 to 1998. He has served the African Mathematical Union as chair of AMUCHMA, the commission on the History of Mathematics in Africa, since 1996, and was the secretary of the Southern African Mathematical Sciences Association (SAMSA, 1991–1995). From 2000 to 2004, he was President of the International Association for Science and Cultural Diversity. In 2000, he was elected to be the President of the International Group of Ethnomathematical Studies. He was member of the International Academy for the History of Science and was elected Vice President of the African Academy of Sciences in 2005. Professor Gerdes published over 60 books and over 100 articles in peer-reviewed journals.

Slavik Jablan (1952–2015) was born in Sarajevo and graduated in mathematics at the University of Belgrade (1971), where also gained his MA (1981) and PhD degree (1984). He has published a few monographs (*Symmetry, Ornament and Modularity*, World Scientific, New Jersey, London, Singapore, Hong Kong, 2002; *LinKnot—Knot theory by computer*, World Scientific, Singapore, 2007); *LinKnot,* webMathematica (http://math.ict.edu.rs) and more than 70 papers on the knot theory, theory of symmetry, antisymmetry, colored symmetry, ornamental art, and ethnomathematics.

Satu Kähkönen PhD, works as a postdoctoral researcher at the Department of Music, Art and Culture Studies, University of Jyväskylä, Jyväskylä, Finland. Kähkönen's major fields of research are the concept of ornament in art theories and the nineteenth- and twentieth-century design and architecture. She has written articles on modern, especially Finnish, design. In her PhD thesis (2011), Kähkönen investigated the concept of ornament in discussions concerning aesthetics of architecture and design during the past 200 years.

Antal Kelle is an artist working in Hungary. He has specialized in creating objects on the border between sculpture, scientific curiosity, and pure playfulness. He has conducted his technical and art studies in Hungary and Germany (Bauhaus Dessau) between 1972 and 1983. Kelle has worked as a teacher of Fine and Applied Art School in Budapest; he has held numerous lectures and workshops at Art Universities in India, Estonia, Armenia, Thailand, Germany, the UK, and Hungary. He has recently held lectures at conferences including Bridges, TEDx, and Ars Geometrica. He has received a number of awards including Aron Kiss lifetime achievement award in 2003, "Pro Ludo" award in 2004, Charles Eames Fellowship in 2008, Noémi Ferenczy Hungarian State Art Award in 2011, and 2015 I. Award

Hungarian Quadriennale of Statues. He has had exhibitions in Museum of Applied Art (Hungary), Gallery of Moholy-Nagy at University of Art and Design (Hungary), Ahmedabad (India), Tartu (Estonia), Bauhaus Meisterhauses Kandinsky and Klee Atelier (Germany), Museum of Fine Arts (Hungary), Pécs 2010 European Capital of Culture (Hungary), Hungarian National Gallery, 2012 Biennale di Venezia (Italy) and Nicolas Schöffer Museum Kalocsa (Hungary), 2013 Vasarely Museum Budapest, 2014 International Mobil MADI Museum Vác (Hungary), Unesco Palace Miro room Paris (France), 2016 Collegium Hungaricum Vienna—Austria, and Center for Hungarian Culture and Science, Helsinki—Finland. He is a member of RegioArt Hungary, National Association of Hungarian Creative Artists (MAOE), Association of Hungarian Fine and Applied Artists (MAKISZ), The International Kepes Society, and International Association of Art (I.A.A. UNESCO).

Tuuli Lähdesmäki PhD in Art History and DSocSci in Sociology, works as Academy of Finland Research Fellow at the Department of Music, Art and Culture Studies, University of Jyväskylä, Finland. Lähdesmäki's major fields of research are reception of art and cultural phenomena, meaning-making processes in contemporary art, cultural identification, cultural politics, and the use of public and urban space. Lähdesmäki has worked in several interdisciplinary research projects, which combine perspectives from art history, cultural studies, reception studies, cultural policy research, sociology, European Studies, and urban studies. She has published various articles on her research topics in scientific journals and edited volumes in Finnish and in English and given scientific presentations on these topics in international conferences. She is a coeditor of a book *Philosophies of Beauty on the Move* (Inter-Disciplinary Press, 2015) that explores the idea of beauty from an interdisciplinary point of view.

Robert Moody has spent the majority of his mathematical life studying symmetry, particularly the study of infinite dimensional Lie groups and algebras, which are analogues of the finite dimensional simple Lie groups. This study revolves around the beautiful internal symmetries of these algebraic objects. They have been used extensively in string theories in the search for the hidden symmetries of nature. More recently, he has been studying long-range aperiodic order, especially its manifestations as point diffraction and the amazing new forms of symmetry arising from aperiodic tilings and metallic quasicrystals. These days his work involves the mathematical foundations pattern as it occurs across a spectrum of artistic and scientific endeavors. He has an abiding interest in the aesthetics of symmetry and pattern and a particular interest in black and white photography (www.labyrinth.zenfolio.com). He is an Officer of the Order of Canada.

István Orosz (b. 1951) is Hungarian poster designer, graphic artist, and an animated filmmaker. He has graduated at the Hungarian University of Arts and Design in Budapest in 1975. His individual graphic works of art are often related to postmodernism by archaic forms, art historical references, stylistic quotations, and playful self-reflection. Themes of the natural sciences, especially of geometry and optics, appear in most of his works. He is also concerned with the theories of vision and sight such as the way the beholder's hypothetical expectations influence the visual and empirical perception of spatial constructions. He is likely to experiment with the extremes and paradoxes of the representation of the perspective to create the illusion of space. In addition, he does experiments to renew the techniques of anamorphosis when he distorts the pictures in such a way that it can only be seen from a particular aspect or that its new layer of meaning only reveals by the interposition of reflective surfaces. Orosz is a regular participant in international art exhibitions, symposiums, and film festivals. He was elected to the Alliance Graphique Internationale and the Hungarian Academy of Arts and Letters. Utisz (pronounced Outis, meaning "Nobody") is his pseudonym used since 1984. It was also Odyssey's feigned name in the well-known affair with the Cyclops that ended in the blinding of the monster's only eye. According to Orosz's symbolic and ironic name, his art is a kind of attack on the eye.

Ljiljana Radović was born in Niš (Serbia) in 1969 and graduated in mathematics at the University of Nis (1993), where she also gained her MA (2000) and PhD degree (2004). She works as the assistant professor of mathematics at the Faculty of Mechanical Engineering, University of Niš. She is interested in the theory of symmetry, antisymmetry, ornamental art, ethnomathematics, math art, and the knot theory.

Charalampos Saitis is currently a Research Fellow of the Alexander von Humboldt Foundation at the Audio Communication Group of the Technical University Berlin. His primary research is focused on quantifying how musicians process and conceptualize timbral qualities of sound, what aspects of the sound experience are essential, and what associations are formed between perception and acoustics. Charalampos also collaborates with the Data Science Group of the ISI Foundation (Turin, Italy) within the Horizon2020-funded Sound of Vision project, developing affective computing methods for stress detection in visually impaired mobility. His research has been published in the *Journal of the Acoustical Society of America* (ASA) and in *Acta Acustica* united with Acustica and presented at major international conferences, where he has won several Best Paper Awards. Charalampos holds a PhD in Music Technology from McGill University (Montreal, Canada), where he also taught undergraduate subjects in acoustics and audio engineering; an MA in Sonic Arts (with Distinction) from the Sonic Arts Research Centre of Queen's University Belfast (Northern Ireland, UK); and a BSc (Hons) in Mathematics from the National and Kapodistrian University of Athens (Greece).

Doris Schattschneider holds a PhD in mathematics from Yale University and is Professor Emerita of Mathematics at Moravian College in Bethlehem, Pennsylvania. Her dual interest in geometry and art led to the study of tiling problems and the work of M.C. Escher. She has lectured and published widely on Escher's work. She is coauthor of a book and collection of geometry models, *M.C. Escher*

Kaleidocycles (Taschen), and author of the book *M.C. Escher: Visions of Symmetry* (Harry Abrams, 2004). She is primary editor of the book with CD Rom, *M.C. Escher's Legacy: A Centennial Celebration* (Springer 2003), which contains 40 contributions from contemporary artists and others whose work has been deeply influenced by Escher. She has been a frequent participant in the Bridges conferences on mathematics and the arts.

Sirkkaliisa Usvamaa-Routila (b. 1955) is Doctor of Philosophy and Social Sciences and Lecturer in Art and Design Education at the University of Jyväskylä (Finland), where she has taught since 1989. She is vice president of the Finnish Society for Philosophy and Phenomenological Research, and she has published books and articles on the history of architecture and design, design theory, book history, and aesthetics.

Angela Vierling-Claassen is an Associate Professor of Mathematics at Lesley University in Cambridge, Massachusetts. Her research interests include using mathematics to understand human relationships, and she has researched how housekeeping duties are split between roommates, how parents of young children divide childcare duties, and how the acceptance of gays and lesbians spreads in social networks. Currently, Angela is collaborating with an educator and a therapist to explore the role of shame in mathematics education. Angela has also developed a course in mathematics designed to engage students of art and design in the "big ideas" of mathematics such as the nature of infinity. She received a PhD in algebraic geometry in 2004 from Boston University and has also been a preceptor of mathematics at Harvard University.

Julian Voss-Andreae is a German sculptor based in Portland, Oregon. Starting out as a painter, he later changed course and studied physics, mathematics, and philosophy at the Universities of Berlin, Edinburgh, and Vienna. Voss-Andreae pursued his graduate research in quantum physics, participating in an experiment considered one of the modern milestones of unifying our everyday intuition with the famously bizarre world of quantum physics. He moved to the USA to study Sculpture at the Pacific Northwest College of Art from where he graduated in 2004. Voss-Andreae's work, often inspired by his background in science, has captured the attention of multiple institutions and collectors in the USA and abroad. Recent institutional commissions include large-scale outdoor monuments for Rutgers University, the University of Minnesota, and Texas Tech University. Voss-Andreae's work has been featured in print and broadcast media worldwide.

Dorothy K. Washburn was born in Ohio in 1945 and educated at Oberlin College (BA 1967, American History) and Columbia University (PhD 1972, Anthropology). She has devoted her career to exploring the kinds of cultural information imbedded in preferences for pattern symmetries by studying archaeological and ethnographic databases from all over the world, but especially from the American Southwest. She has collaborated with Donald Crowe on numerous papers and books, as well as with archaeologists, ethnographers, psychologists, linguists, and native peoples. She held a Miller Fellowship for Research in the Basic Sciences at the University of California, Berkeley, before moving to be the Chairman of the Department of Anthropology at the California Academy of Sciences in San Francisco and later a professor at the Maryland Institute of Art in Baltimore. She is currently a Consulting Scholar in the American Section, University Museum, University of Pennsylvania.

Bridging Art and Mathematics: Introduction

Tuuli Lähdesmäki and Kristóf Fenyvesi

There is a long history of interdisciplinary discussions on the relations between science, mathematics, geometry, art, aesthetics, and artistic praxes. These discussions remain active and pertinent today: the aforementioned relations are explored in various scientific communities, journals, and at conferences. Globally, numerous scholars and artists share a common interest in combining creative thinking, intellectual curiosity, and aesthetic sensibility in their work. Various experts working in different scientific and technological fields are inspired by phenomena that combine mathematical and artistic qualities. Respectively, several contemporary artists, graphic designers, craftsmen, and craftswomen are fascinated by scientific discoveries, mathematics, and geometry and use the formulas and principles of each in different ways in their artistic and creative work.

Since the birth of civilization, mathematics and art have been essential instruments with which human beings have discerned, constituted, and reflected reality and expressed their attempts to explain, dominate, and control nature (Hadot 2006). Mathematics and art form their own modes of communication and are used to reveal the alleged structures of the universe and nature. They have both long contributed to the practices of illustrating and manifesting the reality both intrinsic and extrinsic to human beings. Both mathematics and arts are conceptual and symbolic 'languages' that humans have used in their attempts to depict their empirical perceptions and visions. Both 'languages' refer to worlds outside their respective symbolic spheres—they each provide representations of visible and non-visible phenomena. The 'languages' of mathematics and art are both based on cultural agreements and their interpretation requires a 'reader' who is able to decode their messages. As Bas C. van Fraassen (2008, 9) notes: "The world, the world that our science is of, is the world depicted in science, and what is depicted

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there, is the content of its theoretical representations [—].” The same can be said of the arts—the world of art is the world depicted in the language of art. Arts offer us artistic or emblematic representations of the world. ‘Reading’ the language of mathematics and art requires knowledge of these ‘languages’ and competence to decipher their content. Reading and decoding are cultural and human actions—they always take place within a cultural and subjective context. But how well do these two ‘languages’ communicate with each other?

The history of Western culture from antiquity till the present can primarily be seen as a continuum of epistemological battles and alliances between two interpretations of how to describe the world. According to these views, the world can be grasped through explanations that are either cultural, and thus particular, or scientific, and thus universal. Each discipline argues for who has actually revealed the ‘mysteries’ of the world, manifesting the human mind and reality, containing truths, and explaining beauty. These two views have formed a dualistic context that has made philosophers, artists, and scientists question whether the world and its diverse phenomena can be explained and perceived through the universal laws of mathematics, or rather as culture-bound narratives and symbols; and whether the world can be best represented using the ‘language’ of mathematics or that of the arts.

Contemporary linguists have defined different meaning-making modes and distinct strategies and practices of perceiving reality and its phenomena as discourses. Scholars have used the concept of discourse to refer to specific and restricted ways of producing meanings in and through certain kinds of social and cultural practices and uses of language. The concept has also been applied to explain broader societal structures that have had an impact on various domains in societies and which are manifested, in these domains in the similarity of strivings, values, ways of thinking, and the actions of an era (van Dijk 1997). In this broader sense the concept of discourse approaches the idea of an episteme, as discussed by Michel Foucault. For Foucault (1970), certain kinds of configurations of knowledge, as well as underlying assumptions regarding truth, good, and proper, produce the so-called epistemological unconscious of an era, which encompasses not only science, but a wider range of discourses in culture, education, politics, law, morality, etc. Several epistemes may co-exist simultaneously and it is their interaction that produces complex power hierarchies and various systems of power-knowledge (Foucault 1980, 194–198).

Following Foucault’s conceptualization, the opposing views of describing the world can be conceived as two distinct epistemes—the cultural-emblematic and the mathematical-logical epistemes—in which the nature of knowledge and notions about reality, truth, and beauty are differently comprehended. In both epistemes the use of language—in a broad Barthesian sense (Barthes 1973)—produces its objects. The nature of knowledge, reality, truth, good, and beauty are given meanings in linguistic utterances, textual expressions, and artistic, mathematical, and scientific representations.

Despite their epistemological differences, the epistemes share a common conceptual realm; certain terms, words, and concepts are used within both. This common realm stems from the vocabulary of aesthetics. Mathematicians and scientists often refer to the aesthetic qualities of geometry, mathematical theorems, and scientific discoveries using the terms and expressions artists and art critics employ when evaluating artistic objects and visualizations (Goldstein 2005). The idea of beauty is referred to in the fields of science, mathematics, and arts. The investigation of the uses of language, modes of conceptualization, and discursive meaning-making reveal the epistemological and ontological differences between the cultural-emblematic and the mathematical-logical epistemes. The common objects of interest in science, mathematics, and arts—that is visualizations based on scientific discoveries, mathematical theorems, and geometry—offer an interesting platform for these investigations.

Making Sense of the World in Mathematical-Logical and Cultural-Emblematic Epistemes

The pre-Socratic philosophers already sought to understand the world through searching for a single universal law that determines the world and structures all its qualities and phenomena. In their view, a universal law gives everything a certain form, and form is closely related to the idea of beauty. Pythagoras was the first philosopher in ancient Greece to unify diverse views on cosmology, mathematics, natural science, and aesthetics into a complete theory. According to his thinking, 'everything' (the entire universe) was based on numbers. After antiquity, Pythagorean tradition continued to influence notions about the fundamental basis of the world. These notions regained popularity during the Renaissance. The Greek and Renaissance notions that tie the structure of the universe together with aesthetics, form the basis of the mathematical-logical episteme.

In the discourses of the mathematical-logical episteme, geometrical and mathematical principles of visual phenomena relate their aesthetics to the spheres of reason and logic. In general, reason, logic, and the objectivity of perception are determinants that are often related to science (Darvas 2007, 375) and which can be considered as determining the episteme. In its discourses, the 'language' of geometry and mathematics is deemed universal, and, thus, the images and objects based on them can be considered to 'carry' fundamental universality. In addition, the episteme's discourses relate the universality of geometry and mathematics to the idea of their beauty, and thus come to emphasize the idea of beauty as a universal quality. Since antiquity, mathematicians, scientists, and philosophers have brought to the fore how, for example the facades of architectural masterpieces and the compositions of famous paintings obey the proportions of the Golden Ratio, the Fibonacci sequence, geometrical patterns, or other mathematical sequences. The beauty of such works of art has been located in the geometrical and mathematical principles they follow and which are considered to imply the existence of universal aesthetics.

The fundamental point of departure in the mathematical-logical episteme is in the overall theories and total views of a world based on mathematical principles. This episteme relies on the belief/knowledge of the rationality of the world and its physical phenomena. According to this perspective, the world and its structures are therefore possible to understand, explain, and depict through mathematical theorems and scientific theories. "We live in a universe of patterns", is how Ian Stewart (1995, 1) begins his book on mathematics and numbers, which to him are the elements that underlie everything. Besides mathematics, geometry, and numbers in general, several scholars have perceived symmetry as the key to explain the structure, function, and logic of the world and its diverse physical, social, and cultural phenomena (e.g., Rosen 2008; Darvas 2007; Stewart 2007).

As mathematics, geometry, and numbers are often discussed within the episteme as the laws of nature on which the whole world is based, images and objects which obey the laws of geometry and mathematics can at the same time be considered as obeying the natural laws of beauty. Aside from beauty, mathematics, geometry, and symmetry (see particularly Stewart 2007) have been connected to the idea of truth. In the history of philosophical thinking, the search for beauty and truth has often been complemented by the search for the good (Darvas 2007, 367). Beauty, truth, and the good form the fundamental trinity which, on the one hand, as Darvas (2007) explains, have been perceived to form the final goal of mathematics, while having, on the other hand, the role of an epiphenomenon: beauty, truth, and the good are taken as by-products of the laws of geometry and mathematics—as universal qualities that follow from the mathematical principles. The trinity of beauty, truth, and the good was brought to the fore already in the dialogues of Plato, and the interest in it has continued throughout the philosophical history of the Western science and arts (Darvas 2007).

In the discourses within the mathematical-logical episteme, the history of Western art can be perceived in and explained by discoveries and inventions in the use of geometry and by mathematics in artistic work. Styles, epochs, and works of art have been discussed as a reinvention of, rethinking of, or return to geometry. The history of Western art has been seen as an evolution of the use of geometrical and mathematical rules developing from the invention of the perspective to modern art in which artistic expression was finally reduced and transformed into the composition of pure geometric forms and solids (see e.g., Darvas 2007; Cucker 2013). The development of computing programs and the possibilities they offer for creating graphic representations of algorithmic processes and various types of digital art and music form the latest phase in this evolution, as Charalampos Saitis describes in his article in this volume. Because the discourses within the mathematical-logical episteme emphasize the universality of geometry and mathematics and the aesthetics related to them, the aesthetics of images and objects based on geometry and mathematics are not considered dependent on the cultural, historical, or individual contexts of their receivers. In the discourses of the episteme, the aesthetics of these images and objects is considered to be non-subjective and non-historical. Their aesthetic value remains throughout history and is accessible within different cultures.

Contrary to the views of the mathematical-logical episteme, images and objects based on geometry and mathematics can be conceived of, understood, and explained as cultural representations and artistic emblems that transmit diverse meanings to different receivers. In particular, scholars in the humanities, such as historians, art historians, ethnologists, anthropologists, and also ethnomathematicians, have emphasized the cultural, social, and historical contexts both in the production and reception of geometrical and mathematical images. These notions rely on the world-view of the cultural-emblematic episteme within which universal laws are not considered to explain the meanings of reality and in which reason and logic are believed to be unable to reveal any fundamental truths. In the discourses within the episteme, different truths are perceived as cultural formations and, thus, historically transforming constructions. In this episteme, mathematical and scientific explanations of the world are also seen as cultural, and ideas such as intrinsic beauty and the universal explanatory power of symmetry, geometrical patterns, and mathematical proportions, such as the Golden Ratio, can also be perceived as cultural constructions (Livio 2003).

Within the cultural-emblematic episteme, notions of beauty and aesthetics are considered to be culturally-bound conceptualizations and experiences based on conventions and shared cultural and social habits produced and learned in and through social and cultural reproduction. Particularly in the fields of art history, cultural history, and art philosophy, ideas about beauty, changes in artistic expressions, aesthetics ideals, and styles have been explained by transforming historical and cultural schemas based on learning and on previous experiences (see e.g., Goodman 1976). In the discourses within each episteme, aesthetic experiences do not have a universal or an objective basis. Instead of reason and objectivity, the discourses of each episteme stress the subjects and their subjective emotions and cultural positions in the reception of images. Instead of the non-sensible, the discourses highlight the emotional and affective nature of images. In the views of the cultural-emblematic episteme, each culture and its historical phase has its own system of knowledge, interpretational frame, and aesthetic ideal, which determine the production, interpretation, and use of images.

In the field of art, aesthetic ideals and definitions of good art have been, and still are, a matter of negotiation and contest. In the field of art, the so-called gate-keepers (acknowledged experts, established art critics, workers in art museums and galleries, art historians, etc.) either intentionally or unintentionally determine and define what is considered to be art, and what kinds of expressions are included in the aesthetic sphere. Agents in the field of art do not unanimously agree on these definitions, rather quite the contrary. Art sociologists, such as Pierre Bourdieu (1984), have emphasized how the fields of art and culture are founded on a continuous battle about meanings and the positions out of which these meanings can be produced. In mathematics, the idea of beauty and aesthetics can also be considered as discursively and socially determined. Daniel J. Goldstein (2005, 94) states that mathematical beauty and artistic beauty are both cultural constructions. New mathematical theories and artistic innovations become objects of beauty and aesthetic experience only after new generations of experts and practitioners in the fields are educated in them. In both fields, the sensations of aesthetics require familiarity with the ‘language’ and conventions of the field, which is gained over time through effort and exercise (Goldstein 2005; Rota 2008, 128).

The mathematical-logical and cultural-emblematic epistemes and their different modes of explaining the world and its phenomena can be described as being opposite to each other. Similarly, the fields of science and the arts are often considered to be two incompatible modes of ‘grasping’ the world; they are split into objective and subjective, theoretical and practical, and perceived as appealing to either reason or the emotions. However, throughout history these two modes have been intertwined in various ways. Since antiquity, several scholars and artists have fruitfully sought to merge these modes and create interdisciplinary explanations of the world by combining the views of the two epistemes in their thinking. The interdependence and interaction between the cultural-emblematic and the mathematical-logical world views culminated with Renaissance scholarship, theoretical treatises, and artistic praxes. During the Renaissance, geometry and mathematical proportions were adopted as underlying principles e.g., in perspective theory, architectural concepts, definitions of musical harmonies, and ideals of bodily beauty. Several scholars have discussed and highlighted the continuity of the affinity between the mathematical and artistic ethos in the Western world from the Renaissance to the emergence of modernism in the end of the nineteenth century (Kemp 1990).

However, in academic and scholarly practices the cultural-emblematic and mathematical-logical epistemes can also be interpreted as gradually diverging following the Renaissance. The development of modern science and academia through distinguished scholarship had an impact on the specialization of disciplines and the deepening particularization of each field of inquiry. As a consequence, the core questions, methods, and epistemological and ontological understanding in different disciplines were distinguished and the interaction and dialogue between them narrowed. Similarly, art developed into its own field with its own criteria of evaluation, special value systems, expertise, and connoisseurship. The field of art, its agents, praxes, and knowledge was institutionalized as a hierarchical system. Because of this, acting in the realm of the arts required and still requires a special competence and an acknowledged position within its hierarchy. In addition to emphasizing the affinity of science, mathematics, geometry, arts, and aesthetics in the Western world, the relations of the cultural-emblematic and the mathematical-logical epistemes can be thus presented as a collision or a dis-encounter. On the one hand, the differences between the epistemes have caused disinterest in the world views and modes of thinking of the other episteme. On the other hand, the agents in science and the arts have fostered epistemic thinking in their own disciplines, and, thus, created even stronger differences and confrontations between the epistemes.

Throughout the past decades, multidisciplinary, cross- and transdisciplinary, and interdisciplinary approaches have been emphasized in the natural, social, and human sciences. The recent developments in academia have aimed at producing new bridges between the cultural-emblematic and the mathematical-logical epistemes. Along with the recent development and innovations in science, technology, and the digital world, we are facing a variety of new possibilities strengthening the interaction between the epistemes and producing a deeper understanding and co-operation between the experts interested in the interdisciplinary discussions between science, mathematics, and art.

Opening New Views to Interdisciplinarity Through Aesthetics

Science and technology are not only subjects of functional rationality, but like any other social objects and cultural artefacts, they have hermeneutically interpretable dimensions that can be perceived as their *social meaning* and *cultural horizon*. The social meaning, cultural horizon, and functional rationality are intertwined: the scientific and technological phenomena can be interpreted as embodying complexity, which includes aesthetic characteristics as well. Scientific discoveries, inventions, innovations, experiments, observations, models, systems, and simulations often become the objects of aesthetical interpretation and experience, and their aesthetic dimensions are exposed by multifaceted approaches and interdisciplinary conceptions used in their meaning-making processes. For example, scientific models not only have informative and cognitive values but symbolic, iconic, or even psychological meanings. As creative and 'fictive' representations of reality and its diverse phenomena, scientific models can be examined also as results of complex socio-cultural experiences (Feenberg 2010), including elements that can also be perceived within the aesthetic framework.

Throughout the history of science, the aesthetic dimension has stimulated the development of scientific and technological innovations: artworks, art history, and theories in aesthetics have provided sources of inspiration, conceptual tools, and even research material for science. This reciprocity of disciplines has not occurred in a scientific or artistic vacuum, but it rather is an interdisciplinary outgrowth of the natural interaction between arts, mathematics, and science (Henderson 2013), practiced by the scientists, mathematicians, and artists themselves.

What is the role of aesthetic dimensions in contemporary scientific discoveries, and what is the role of science in contemporary art? While the specific relationship between science and art has recently been studied from various aspects, we still lack a common theoretical ground that we could call the *aesthetics of interdisciplinarity*—an approach that would provide a deeper understanding of the following chiasmatic relations:

- scientific representations, models, and simulations as an artwork <> artworks as a scientific representation, model, or simulation;
- implementation of scientific and technological tools and concepts during the creation of art <> implementation of artistic creativity and inspiration during the scientific work;

- communicating aesthetic experiences with the tools of science and technology <> communicating scientific, technological content through the artistic creation;
- a same object as a modeling kit, a toy, and material for artistic creation;
- a creation of patterns and algorithms as games <> creating art with the implementation of patterns and algorithms;
- useful, playful, and creative dimensions of a scientific object.

Along with the discussions in this volume we aim to offer a conceptual framework that combines the different perspectives of science, mathematics, and art, and to unite these perspectives into a third conception—the *aesthetics of interdisciplinarity*. The aim is to develop a scientifically interesting and artistically inspiring new framework of research, which can function as a fruitful interdisciplinary discourse on the aesthetic aspects of scientific objects and the scientific aspects of aesthetic artefacts. With this conception of the aesthetics of interdisciplinarity we aim to surpass the discursive duality of the mathematical-logical and cultural-emblematic epistemes in grasping the world, and to emphasize instead the significance of perceiving and making sense of the world by accepting the interdependence and interplay of aspects in this interdisciplinary field.

More than half of the articles in this volume evolved from papers presented in the course of the Bridges conference series. The authors of these papers (Schattschneider, Jablan-Radović, Saitis, Washburn-Crowe, Gerdes, Vierling-Claassen, Moody, Orosz, Voss-Andreae, and Kelle) have subsequently adapted and expanded their work for publication in this volume. It is precisely the research discussed in these studies that convinced the editors to rethink how interdisciplinarity is related to the field of aesthetics. To provide even a broader examination of the nexuses joining interdisciplinarity to aesthetics, several other scholars (Cohen, Usvamaa-Routila, Gelfert, Darvas, Kähkönen, and Lähdesmäki) was invited to contribute to the volume. The discussion of the aesthetics of interdisciplinarity in this volume can be seen as a continuation of the lively dialogue between mathematics and art and their connection to culture, art theory, and education that has consistently taken place during the Bridges conferences.

Since 2017 marks the twentieth anniversary of the Bridges conferences, this volume was compiled with the goal of utilizing and selecting articles from the more than 8000 pages of material contained in the Bridges archive.[1] Ultimately, the editors' aim was to take the results stemming from research in mathematics in particular and render them available to experts in cultural studies and art theory, while also providing a selection of international examples of mathematical art—visualizations based on diverse scientific discoveries, mathematical formulas, and geometry—for readers' perusal. For students interested in cultural studies, art theory, and art education, this volume serves as an introduction to the vast common ground stretching between mathematics and art.

[1]See: http://archive.bridgesmathart.org/. Retrieved on 22 December 2016.

As a reflection of the great debt our volume owes Bridges conferences, we consider important to offer a brief overview of the history of the Bridges community. This overview reveals how Bridges was founded on a primarily mathematics-based experiment in establishing an ongoing conversation between mathematics and art. In the course of the past two decades, numerous outcomes originating from this dialogue strongly suggest that examinations in cultural studies or art theory cannot be complete without taking the field of mathematics into consideration. At the same time, the demand to become more familiar with, as well as explore, the cultural and artistic possibilities inherent in mathematics has palpably grown within the mathematics community itself. In this respect, the aesthetics of interdisciplinarity and the development of an interdisciplinary aesthetics provides a means of establishing a platform that joins a) cultural studies and art theory that is steeped in a knowledge and awareness of mathematics; b) mathematical examination of artistic and cultural phenomena; c) educational practices; and d) artistic production itself.

Furthermore, our attempt to look back at the history of the Bridges community is motivated by a personal and intellectual loss the Bridges and international math-art community has recently faced. In the beginning of July, 2016, just a few months before the closing of this volume as well as the Bridges Finland Conference held at the University of Jyväskylä, the founder of the Bridges community, Reza Sarhangi, unexpectedly passed away. This volume is therefore intended as a token of respect to the memory of Reza Sarhangi and the intellectual legacy that this volume will hopefully pass on to a wider audience. With this volume the editors would also like to remember the pioneer in ethnomathematics, Paulus Gerdes, who died in 2014, and Slavik Jablan, the distinguished researcher of visual mathematics, knot theory and inspired mathematic artist, who passed away in 2015. These three researchers were not only internationally recognized scholars, but also extremely influential educators who devoted great energy to attaining exemplary results in teacher education while simultaneously spreading knowledge in science and art to wider audiences. The editors of this volume wish that the touchstones Sarhangi, Jablan, and Gerdes laid out in their work, will offer a source of inspiration for researchers of art history, cultural studies and education. It is with this goal in mind that we hope this volume will also prove useful to instructors in teacher education as well.

History, Organization, and Conferences of the Bridges Community

The Bridges conferences were started by Reza Sarhangi in 1998, and the first few meetings were hosted by Southwestern College in Winfield, Kansas, USA. Sarhangi emigrated to the United States from Iran in 1986. Sarhangi's broad range of personal interests (Shrestha 2016) as well as his research into ancient Persia's mathematical and artistic past helped him to direct attention to mathematics' complex cultural roots. For his integrational approach to mathematics and art, Sarhangi looked far beyond the well-known works of artists in mathematical art, such as M. C. Escher (Sarhangi and Martin 1998) and explored in particular the medieval period of Persian history,[2] when mathematics, arts and crafts coexisted side-by-side. Sarhangi had studied and continuously emphasized as a great example the work of Abul Wafa al-Buzjani (940–997/998), one of the most renowned mathematicians of his time. Al-Buzjani's treatise, *On Those Parts of Geometry Needed by Craftsmen*, was written to educate craftspeople in geometry. For medieval craftspeople, creating the decorative motifs common to the Persian art of that era demanded not only continuous training, but also regular consultation with mathematicians. For example, decorating the inner as well as outer spherical surface of cupolas with tiles featuring highly regular, yet still extremely complex geometric patterns would have required advanced knowledge of geometry. The sophisticated, mathematical nature of Persian decorative arts not only makes them interesting from a historical perspective, but also provides a fascinating area of research for mathematicians today (Lu and Steinhardt 2007).[3]

Before emigrating to the United States, Sarhangi was more than a teacher of mathematics interested in Persian traditions. He was a graphic artist, teacher of drama, playwright, theatre director, and props designer. When added to his background in mathematics and history, his first-hand experience of complex and collective artistic processes—such as creating and performing a play—gave him deep insight into the equally complex processes involved in designing and producing medieval Persian tilings. Sarhangi made great use of his many areas of expertise, first as an innovative, young university professor open to new experiments, then later on as an educator of future mathematics teachers. As department chair of Southwestern College's Department of Mathematics, he already introduced creative study modules and theatrical plays on mathematics to change how mathematicians were educated.

In his new country Sarhangi searched for an academic community capable of supporting his broad range of interests. In the early 1990s, the college implemented a new Integrative Studies Program that drew together faculty from all other traditional and professional programs. As director of the Integrative Studies Program, Daniel F. Daniel, Sarhangi's close friend and mentor at Southwestern College, suggested that Sarhangi would establish a new course for this program. His course on the connections between mathematics and the arts became very popular among students.

Sarhangi also attended the Art and Mathematics (AM) conferences organized by Nat Friedman at the State University of New York at Albany from 1992 to 1998. The AM conferences gathered artists, architects, and other experts applying mathematics creatively. The spirit of cooperation engendered by these AM gatherings

[2]On artistic consequences of the connections between the European Renaissance and the medieval Arabian science's visual investigations, see Belting (2011).

[3]See Reza Sarhangi's numerous articles on Abul Wafa al-Buzjani, e.g. Sarhangi (2006).

led to the publication of several interdisciplinary papers uniting different perspectives to form a kaleidoscope-like vision of the given topic. Sarhangi felt he was witnessing the rebirth of a long-forgotten paradigm from Abul Wafa al-Buzjani's time. He could see first-hand how a new form of art araised from the dialogue between the mathematician creating theories for solving complex artistic and architectural problems, and the master putting theories into practice. This art possesses a unique, aesthetic quality all its own, whose analysis demands a new approach, which we can call interdisciplinary aesthetics, because it is both mathematical and artistic in nature.[4]

Known as ISAMA (*International Society of the Arts, Mathematics, and Architecture*)[5] as of 1998, the AM movement was the direct, American antecedent to the later Bridges conferences. Indeed, three of Bridges' first four directors: George Hart,[6] Carlo Séquin[7], and Reza Sarhangi were ISAMA veterans. The fourth director, Craig S. Kaplan[8], started his career in mathematical art in 1999 after joining the ISAMA and Bridges communities and co-organizing the MOSAIC 2000 conference, which examined connections between computer programming and the arts. Robert W. Fathauer[9] soon assumed responsibility for organizing the Bridges art exhibits.[10]

Many groups and "schools" have connected to form the background for the American, European, and Asian science and art communities currently involved in the Bridges community. These include the *Mathematics and Culture* conferences (Emmer 2004–2012, 2012–2014, 2015), the *Nexus* conferences[11], and the Symmetry Festivals[12] started by György Darvas and Dénes Nagy in 1989 in Budapest, Hungary. The increasingly active presence of the *European Society of Mathematics*

[4]Cf. http://www.isama.org/org/history.html. Retrieved on 24 October 2015.

[5]See the organization's website: http://www.isama.org/. Retrieved on 24 October 2015.

[6]Research professor of Computer Science at Stony Brook University (USA). Hart is also a sculptor whose work is recognized around the world for its mathematical depth and creative use of materials.

[7]Professor of Computer Science at the University of California, Berkeley (USA). His works in computer graphics and in geometric design have provided a bridge to the world of art. In collaboration with several sculptors of abstract geometric art, Séquin has found a new interest and yet another domain where the use of computer-aided tools can be explored and where new frontiers can be opened through the use of such tools.

[8]Professor of University of Waterloo (Canada), and former main editor of *Journal of Mathematics and the Arts*. The focus of Kaplan's research is on the relationships between computer graphics, art, and design, with an emphasis on applications to graphic design, illustration, and architecture.

[9]A former researcher of *Jet Propulsion Laboratory*, and the founder of *Tessellations*, a company that specializes in products that combine art and mathematics. He is an internationally renowned author of activity books on art and mathematics and a mathematical artist, whose work has been shown in numerous exhibits in the US, Canada, and Europe.

[10]The virtual galleries of Bridges Art Exhibit can be accessed here: http://bridgesmathart.org/bridges-galleries/art-exhibits/. Retrieved on 24 October 2015.

[11]Cf. http://www.nexusjournal.com/the-nexus-conferences.html. Retrieved on 24 October 2015.

[12]Cf. http://www.symmetry.hu and http://symmetry-us.com/. Retrieved on 24 October 2015.

and Arts[13] also deserves mention. Bridges also enjoys a close-knit relationship with the *International Mathematics & Design Association* located in Buenos Aires, established by the mathematician Vera W. de Spinadel, in 1998. *The Mathematical Association of America's Special Interest Group on Mathematics and the Arts* (SIGMAA-ARTS) was established by Bridges members and participants, and currently has more than 200 members.[14]

Barely 60 participants attended the first Bridges conferences. Today they attract annually around 300–400 conference participants from around the globe and thousands of audience members. Bridges conferences have been held in Winfield, USA (1998–2001); Towson, USA (2002); Granada, Spain (2003); Winfield, USA (2004); Banff, Canada (2005); London, UK (2006); San Sebastian, Spain (2007); Leeuwarden, The Netherlands (2008); Banff, Canada (2009); Pécs, Hungary (2010); Coimbra, Portugal (2011); Towson, USA (2012); Enschede, The Netherlands (2013); Seoul, Korea (2014); Baltimore, USA (2015); Jyväskylä, Finland (2016); Waterloo, Canada (2017).

Today Bridges conferences enjoy wide recognition from around the globe. Both leading media sources and scientific circles regularly report on Bridges events. The multi-faceted nature of these events offers a likely explanation for this circumstance: the sheer diversity of topics and areas addressed understandably attracts a diverse audience of scholars and artists. In addition to mathematicians, scientists, art scholars, and education experts, Bridges conferences attract painters, musicians, architects, literary scholars, computer programmers, sculptors, dancers, craftspeople, model builders, etc.

In accordance with its highly diverse audience, the aims of each conference reflect many different aspects. Bridges always places great emphasis on providing opportunities for approaches that both innovate and integrate, thereby emphasizing the importance of interdisciplinary cooperation in the research of mathematics and the arts. It presents a platform for scholars, teachers, and artists intent upon surpassing boundaries while also exchanging their own experiences. In addition to supplying professional support, it encourages mathematics teachers to utilize creative, artistic processes and tools in passing on mathematical knowledge, and art teachers to reveal the mathematics involved in diverse artistic processes.

Like musical interludes and concerts included in the Bridges program, intense workshop projects are conducted with the purpose of analyzing the practical application of given topics, and these have formed an integral part of the conference. The educational relevance of math-art approaches has been demonstrated in interactive, experience-oriented workshops since Bridges' early days. The conference emphasizes active participation: mathematical artworks are put on display in Bridges' math-art exhibits, rather than simply being discussed in conference lectures. The Bridges exhibit has since grown to be the largest exhibit of mathematical-art in the world. Since its inception, creative programs have become

[13]Cf. http://www.math-art.eu/. Retrieved on 24 October 2015.

[14]See: http://sigmaa.maa.org/arts/index.html. Retrieved on 24 October 2015.

increasingly structured and have now evolved into separate areas of expertise directed by skilled professionals.

The following key elements form the backbone of Bridges conferences:

- *Plenary lectures*—organized in a joint effort by the members of the Bridges Organization's Board of Directors[15] as well as the local academic coordinators. Visitors to Bridges events can personally meet internationally celebrated members of the scientific world, such as mathematician Harold S. M. Coxeter, physicist John C. Mather, and chemist Sir Harold Kroto, both recipients of the Nobel Prize. Ernő Rubik, inventor of the Rubik's Cube, and László Lovász, chosen for the Wolf Prize, have presented their concepts at Bridges. The Fields Medal-laureate Cédric Villani has contributed his expertise to Bridges. The presence of Ingrid Daubechies, the first female president of the International Mathematics Union, also deserves a mention. John H. Conway, founder of the theory for cell automation, Alan C. Kay, the revolutionary developer of personal computers and programming, Marjorie W. Senechal as many others have numbered among past Bridges plenary speakers. As a part of the plenary addresses, lectures related to the host location or country highlight those significant contributions in math-art achieved locally.
- *Section lectures*—selected and reviewed anonymously by the Bridges Organization's roughly 60-member program committee, out of proposed papers sent in response to the conference's open call. The program committee's work is directed by members responsible for editing the annual conference volume, a project headed by different people each year.
- *Workshops*—selected and reviewed among all workshop papers sent in response to the program committee's open call. The committee for workshop papers as of 2015 to this day has been directed by Kristóf Fenyvesi. The interactive nature of these workshops familiarizes participants with different kinds of math-art applications and content, much of which can be utilized in education. Started by Mara Alagic and Paul Gailiunas, the workshop series, *Bridges for Teachers—Teachers for Bridges*, deserves a mention due to its attention to the development, interactive testing, and joint evaluation of educational applications.
- *Bridges Mathematics and Art Short Movie Festival*—the program of which is compiled by Nathan and Amy Selikoff, as well as by a selection panel led by Robert Bosch and comprising many representatives from various artistic and scientific fields.
- *The Mathematical Art Exhibit*—in recent years displaying works by more than 100 artists from over 30 countries, thereby earning its title as the world's single largest exhibit of math-art. The exhibit's leading curator is the former NASA researcher, tessellation expert and math-artist, Robert W. Fathauer, who leads the interdisciplinary selection committee responsible for choosing exhibit works. Of the many artists whose work has been shown at Bridges exhibits,

[15] About the Bridges Organization's Board of Directors, see: http://bridgesmathart.org/about/. Retrieved on 26 November 2016.

Brent Collins deserves a special mention. Collins not only supported this event from its very inception, he also created *Genesis*, a Möbius-like loop that came to be used as the logo of the Bridges Organization. All works shown at the exhibit are featured in a full color album.

- *The Mathematical Theatre Show*—performance of a play which explores connections between art and science. As of 2009, Steve Abbott not only leads this event, but also stages new works every year, performed by conference participants who volunteer to be a part of his theatrical troupe. Occasionally, these performances also include dancing. Directed by mathematician and dancer Karl Schaffer and dancer Erik Stern, the *Schaffer and Stern Dance Ensemble* has performed multiple times at Bridges, similar to *Mime-Matics*, the mime group created by the mathematicians Tim and Tanya Chartier.
- *Music Night*—a program frequently featuring the debut of new compositions in mathematical-music. A tradition begun by the musician and mathematician Corey Cerovsek, and after him the baton for Bridges music nights has been passed on to Princeton University's musicologist, Dmitri Tymoczko. *Music Night* programs have included Bach interpretations by Harvard's mathematician, pianist and chess master, Noam Elkies, as well as works by contemporary composers such as Fernando Benadon, Clifton Callender, or Adrian Childs. In addition to musical programs featuring professional musicians, the informal music nights performed by music-loving conference participants is at least as important Bridges tradition—begun by internationally renowned math popularizer and musician Vi Hart.
- *Poetry Afternoon*—an event organized by the mathematician and poet Sarah Glaz, who introduces the audience to the international scene of mathematical-poetry.
- *Family/Public Day*—one of the conference's most essential elements which provides the Bridges community with the opportunity to showcase its scientific and artistic work in front of a lay audience. For no entrance fee, workshops for building giant models and other activities are staged for both adults and children alike. As of 2010, this day has been organized by Kristóf Fenyvesi.
- *Public Event*—is open to the entire audience and not only to professional conference participants and provides an opportunity for the introduction of interesting artistic and scientific projects in the form of presentations, videos, and live on-stage demonstrations.

Due to the wide-ranging nature of its programs, Bridges not only promotes new achievements in the area of mathematical art, but it also allows the seemingly distant worlds of academics and the artists to spread their message among a much broader audience. All plenary and section lectures, as well as workshop descriptions are published in a richly illustrated, massive volume of conference materials frequently 600 or more pages in length, yet available free-of-charge on the Bridges website.

Content of the Volume

The aim of this volume is to contribute to the bridge building efforts between cultural and art studies and the mathematical domain of knowledge by exploring interdisciplinary approaches to the aesthetics of mathematical art. The authors of the volume approach their topics from the point of view of aesthetics, anthropology, art history, art theory, artistic practice, ethno-mathematics, geometry, mathematics, philosophy, physics, study of visual illusions, and symmetry studies. The selection of articles in the volume is based on their interdisciplinary nature, their accessibility for a large scientific audience, and their contribution to broaden the views on the role of aesthetics in the confluence of science, mathematics, culture, and art. The volume contains four thematic parts.

Part I, entitled 'Bridging Art and Mathematics: Concepts, Theories, and Philosophies' provides theoretical, conceptual, and philosophical insights on the main topic of the volume. The section includes four articles in which mathematics, geometry, art, design, and visuality are approached as cognitive phenomena and as intertwined to each other in various ways exceeding the borders of separate scientific disciplines. Interfaces of these phenomena form a favorable ground for theoretical and philosophical discussions on sensation, perception, conceptualization, understanding, creativity, imagination, and imaging. Although the articles in the section explore the similarities and differences of mathematical and artistic thinking and production of knowledge from an interdisciplinary point of view, each article takes to the topic a particular approach through which the relations between arts and mathematics are discussed. These approaches include: discussions on dimensions and transformations in the philosophies of science; theorizations of image and percept; exploring the discourses and conceptualizations of beauty and aesthetics; and applying symmetry studies. Philosophical discussions in the articles is strengthened particularly from the point of view of phenomenology.

Mark Daniel Cohen's article discusses the relationship between science and visual art by focusing on transformations in the history of the philosophy of science. The starting point of the article is in the conception of truth. As Cohen notes, both science and art can be considered as pursuits of truth, however, differing in their notions of it. Cohen discusses the epistemological questions in the production of knowledge and understanding of the real, and the contradiction between objectivity and subjectivity in the points of view and practices of science and art. For him, the contradictions between science and art result from the clash between the Newtonian worldview and the empiricist paradigm of the philosophy of science, i.e., the question of the role of human senses, perception, and observation in the production of knowledge of the real. As Cohen writes, mathematics is central to the Newtonian worldview, while anthropomorphic and art-influenced world view form a core in the empiric perceptual science. Cohen takes Goethe's thinking as an example of the perceptual point of view to the real. Since Goethe's time, the ideas of perception of the real and the investigation of the truth in science and art have alienated. However, the ways of conceptualizing the real and the truth in mathematics and arts were approached at a new way in the beginning of the twentieth century. Cohen notes how developments in mathematics (such as the non-Euclidean geometry) shifted the interests of the discipline to the areas in which the investigated situations were impossible to be formed as sensory experiences—they were imperceptible. Similar trajectories can be found in the transformation of art at the beginning of the twentieth century: in various Modern art movements, artists aimed to visually reveal the truth that was, however, considered non-visualizable. Both disciplines shared an interest to non-visible, imperceptible, and even to non-imaginable worlds.

In her article Sirkkaliisa Usvamaa-Routila discusses architecture as a visible phenomenon. If mathematical properties, such as proportions, are assumed to be capable of arousing aesthetic experiences, they are also bound to an assumption that they are in one way or another *seen*, not only imagined. However, Usvamaa-Routila brings to the discussion the role of imagination and the importance of will in the aesthetic experience. Usvamaa-Routila's starting point is the notion of how visual implication aids dealing with specific problems: How can we see properties that cannot be visually apprehended? This problem was discussed, e.g., by Le Corbusier, who used the term *linee occulte* to describe all the lines of a building that cannot be seen, but which are most significant to the successful architectural design. Usvamaa-Routila continues this discussion by suggesting that properties that are exhibited and seen in an architectural work of art "visually implicate" properties that are not seen, although they are experienced. We may assume that for instance the corners of a window visually implicate the diagonals, although these are not exhibited and therefore cannot be seen in the proper sense of perception. We may therefore assume that they are seen only mentally, by implication only. This means that our aesthetic perceptions may be affected by our imagination. In her article, Usvamaa-Routila discusses the dimensions of visual implication through Husserl's phenomenology and Sartre's observations of the interplay of perception and imagination.

Axel Gelfert's article focuses on the confluences of art and mathematics, and introduces ways in which mathematics has attracted the attention of artists since the Renaissance. The notion that mathematical objects, such as proofs or theorems, can have an aesthetic value has long had some currency among mathematicians and philosophers of mathematics. As Gelfert notes, in popular discussions of mathematics, it is common to come across references to the perceived beauty of a theorem, or the superior elegance of one proof as compared with another. However, the invocations of mathematical beauty, its source, and cognitive function are surrounded "an air of mystery", as Gelfert notes, and such aesthetic judgments often require a considerable degree of familiarity with mathematics itself. In the philosophy of science and mathematics, this debate has a parallel in discussions, sparked by Eugene Wigner's article, about "the unreasonable effectiveness of mathematics in the natural sciences". Gelfert's article discusses the crossing of these two debates by exploring the appeal that the aesthetic dimension of mathematics has had on artists and scientists alike. The article contributes to the philosophical discussion on these topics through introducing various empirical examples ranging from the sciences to the arts.

György Darvas discusses in his article interdisciplinary applications of the symmetry phenomena. He introduces the basic terms and geometric appearances of symmetry (e.g., mirror reflection, rotation, translation, and similitude), and how they appear in decorative arts. Symmetry operations in the decorative arts can be presented in one dimension (frieze patterns), in two dimensions (wallpapers or tiling), in three dimensions (crystals), and extended to cover surfaces (spheres and polyhedra). The article indicates how symmetry studies have been used in determining and understanding the world and its basic substance since ancient Greece. In the ancient times, symmetry was connected to the theories on primary elements, cosmological explanations, the movement of planets, and the harmony of numbers and their equivalences, e.g., to music and various phenomena in nature. As an example of the interrelation of diverse symmetry phenomena, Darvas discusses proportion theories, Golden section, Fibonacci sequence, Platonic solids, and their geometric applications. The article provides examples of how the aesthetically pleasing proportions and shapes are embodied not only in artworks, but also applied in recent scientific achievements, such as in quasi-crystals, new molecules (such as the fullerene and the graphene—that are important to nanoscience), and particle physics. In general, the article demonstrates the productivity of interdisciplinarity for science-art relationship.

Part II, entitled 'Studying Mathematical Principles of Composition', includes five articles which explore patterns, designs, and images by explaining their geometric and mathematical principles and logic. Since ancient times, craftsmen, craftswomen, artists, and designers have created patterns and images based on geometry or mathematical sequences. These patterns and images have intrigued various scholars in the fields of mathematics, art history, cultural studies, and ethnology. In different disciplines geometric patterns and images are approached in different ways by emphasizing e.g., their mathematic structures, art historical development, or cultural historical meanings. New points of view to these visual phenomena enrich the understanding of them. Geometry and mathematics based patterns, designs, and images include multilayered information and meanings. Recognizing or investigating only one layer of their character narrows the understanding of their diversity, flexibility, and interrelatedness as interdisciplinary phenomena. Multi- and interdisciplinary points of view are thus needed in the investigation of these patterns and images. The first two articles in the section focus on artists M. C. Escher and Victor Vasarely who are famous for their artworks which contain various geometric puzzles and visual illusions, paradoxes, and problems. The third article introduces the logic of fractals, which are often considered to be a fundamental link between mathematics, art, nature, and aesthetics. The section ends with articles that approach geometrical patterns by combining points of view from ethno-mathematics and history. This kind of interdisciplinary approach to cultural history of ethnic communities or ancient civilizations may renew our understanding of human interaction, communication, and cultural meaning-making processes.

In her article, Doris Schattschneider discusses the works of art of Dutch graphic artist M. C. Escher (1898–1972), who carried out mathematical investigations that led to symmetry drawings of three distinct kinds of tiling with two colors. Escher used several of these drawings as key elements in his prints that further expressed ideas of duality. Schattschneider analyses Escher's art works by investigating in detail their use of color, tiles, forms, symmetry composition, and tiling principles. In addition, she introduces Escher's working methods and sources of inspiration, which led to new artistic discoveries in his duality-symmetry studies. Escher's discoveries are demonstrated in the article with numerous illustrations.

Slavik Jablan and Ljiljana Radović investigate in their article Victor Vasarely's (1906–1997) artworks from the point of view of the theory of visual perception, mathematics, and modularity. The authors introduce art movements, Op Art in particular, which form the artistic context to the Vasarely's works. The article indicates that almost all construction methods of composition, modular elements, optical effects, and visual illusions used in Vasarely's works were mostly (re) discovered by him through intuition, creative visual thinking, and experimenting. Jablan and Radović discuss the artist's altering visual means through rich illustrations from different decades in Vasarely's oeuvre.

Charalampos Saitis discusses in his article the world of fractals and their definition and appearance in the arts and nature. In general, understanding nature has always been a reference point for both arts and science. Several aesthetes have put nature at the forefront of artistic achievement: artworks have been expected to represent and manifest nature. Science has likewise been trying to explain the laws that determine nature. As Saitis notes, technology has provided both fields with the appropriate tools to deal with their common goal. After Benoit Mandelbrot formulated his findings in non-linear dynamical systems into a theory of fractals, a broad artistic interest exfoliated, resulting in a new form of digital art. Fractal images and music, and the application of the principles of fractal theory in the study of various natural phenomena, became popular among the artists and scientists. Saitis indicates how fractals stand right at the heart of the art-science-technology triangle. The article examines the new perspectives brought into art by fractal geometry and chaos theory, and how the study of the fractal character of nature offers possibilities towards theorizing art.

Dorothy K. Washburn and Donald W. Crowe's article focuses on the pattern symmetries of cultural artifacts by providing systematic descriptions of the investigated patterns to allow the comparison of them between different geographical areas and historical periods. In general, culturally produced patterns can be described in many ways, each useful for different purposes. The article discusses how in early pattern studies, designers of textiles and wallpapers created classificatory groupings that were descriptively idiosyncratic and grouped patterns by motif similarities that were very different in their symmetrical arrangements. With a reference to several recent studies, the article illustrates how a grouping by symmetry rather than a motif similarity reveals new insights to the study of cultural objects and activities. As the article indicates, systematic analysis on the continuities, changes, and use of preferential symmetry can enhance the understanding and interpretation of material objects and cultural information that is embodied in their decorations. Washburn and Crowe illustrate this enhancing of understanding through two different cases: Greek Neolithic pottery decoration and basket decoration of Native-American tribes.

In his article, Paulus Gerdes introduces his fieldwork research about a village called Palmeira in Mozambique. Gerdes has investigated for decades mathematical and geometric patterns in mats and baskets weaved by the local people in the region. An important role of ethno-mathematics is to initiate the recognition of the mathematical ideas of peoples who are rarely referred to in books on the history of mathematics. In such scientific studies, the individuality of the local craftsmen and craftswomen is however often forgotten. In them, the weavers and decorators of handicrafts often remain anonymous. Gerdes's article makes an exception to this tradition by giving a face to a local basket weaver master Arlindo Bendzane. In his article, Gerdes introduces various Bendzane's pattern inventions.

Part III, entitled 'Cultural Meanings of Geometric Composition, Structure, and Form', explores geometry-based visual art by discussing the cultural, historical, artistic, and conceptual contexts in which these works of art are created and in which they gain their meanings. Geometry has various functions in the field of visual arts: it has offered a source of inspiration to various periods and stylistic movements; it has served as a tool in the development of designs and ornaments; it is used in the analysis and interpretation of works of art; and the artists use it as an effect and a method in artistic practice. Even though geometry is based on universal mathematical principles and rules, the geometrical patterns as such include various cultural and artistic meanings, which resonate the historical, societal, and cultural conditions of the time of their creation.

In Western cultural history, abstract art has often been situated at the intersection of art, geometry, mathematics, and engineering. The emergence of various abstract art movements in the beginning of the twentieth century have been seen as an attempt to bring artistic and scientific thinking closer to each other in a new way. Besides Western abstract art, this section brings to the discussion geometrical principles in old Indian art and Arabic decorations and ornaments, which are often considered as fundamental examples of junctions of artistic practice and geometry. In general, the section emphasizes the contextual understanding of geometry based art: the points of view from the history of science, art history, cultural history, sociology, and anthropology are utilized in attempts of explaining the cultural meanings of geometrical designs and artworks.

Angela Vierling-Claassen's article focuses on the development and construction of models of algebraic curves and mathematical surfaces for scientific and educational purposes and their influence on artistic movements at the beginning of the twentieth century. The article gives a historical overview of the scientific aims and interests in the field of mathematics at the end of the nineteenth century and at the start of the twentieth century when several scholars started to produce models of mathematical surfaces out of plaster, wire, and other materials. These models were used in university instruction to illustrate research and scientific findings. Gradually, mathematical interest in these models faded, but the models themselves were stored and displayed in universities and museums. Vierling-Claassen indicates how the models were discovered by several artists from the movements of abstract art. Particularly artists in constructivist and surrealist movements drew inspiration from the models of surfaces. The article brings to the fore the concrete paths of influences and inspiration between mathematics, science, and art.

Satu Kähkönen's article discusses the concept of ornament and the ornament and decoration's problematic relation in the Western world during the nineteenth and twentieth centuries. Decoration and ornament have often been considered as counterparts to modernism, which aimed at the simplification of design. Thus several modernist theoreticians have stressed the grid as an emblem of modernity. Kähkönen indicates how discussions on ornament include various cultural and historical meanings which have influenced the design, use, and interpretation of them in the Western world, particularly in Western architecture. However, the idea of ornament has a profoundly different role and meaning in the Arabic world. In contrast to the development of ornament in the Western world, in Arabic ornamental design, geometrical grids function as starting points of decorations. While in Western architecture, ornament has been seen as a secondary element compared to form or structure, in Arabic architecture, ornament has been considered an inseparable part and a fundamental element of buildings. How do the different conceptualizations of ornament encountered in the contemporary globalized world? Kähkönen discusses these encounters through contemporary modernist architecture used in Islamic cultural centers in Europe. In them, the idea of the grid as a starting point and the end of ornament merges.

In his article, Robert V. Moody discusses Swiss artist Alice Boner (1889–1981), who lived and worked in India for 40 years. Boner's passion was the old temple cave sculptures and reliefs dating from the sixth to ninth centuries and which appeared in a number of sites around India. During her decades in India, Boner drew numerous sketches of temple sculptures and wrote a diary about her artistic discoveries. In his article, Moody follows Boner's artistic studies by interpreting her diary remarks, which offer insights into the creative artistic process, Boner's struggles and doubts in her work, and the passions that led her to the discoveries about the geometrical underpinnings of Indian temple cave art. The discovery was unexpected: unlike in Arabic art or various periods in Western art, there was no previous evidence of any formal underlying geometrical principles in old Indian art. At the end of his article, Moody explores Boner's own artworks wherein she varied the traditional motives and geometrical structure of Indian art. The findings of geometrical design principles in ancient temple art influenced the design of composition in Boner's own artistic production.

In Part IV, entitled 'Geometry, Mathematics, and Science in Artistic Practice', contemporary artists introduce in their own words the theoretical and conceptual background of their works in which they utilize geometry, mathematics, and science in different ways. Making a work of art is a creative process, that includes planning, thinking, experimenting, testing, and structuring. In a creative process, subjective sensations, experiences, inspiration, and intuition are taken as working instruments. The very same instruments are crucial in the creation of artworks based on geometry, mathematics, and science, although these disciplines usually invoke impressions of intellectuality, reason, and, knowledge. The section reveals the multifaceted creative process in artistic work, in which the sources of inspiration vary from literature to quantum physics. All the three artists contributing to this section are very familiar with the geometrical, optical, mathematical, technological, and physical phenomena that are part of the focus of their artistic work. The artistic study of these phenomena demands taking into account ideas of perception, imagination, seeing, viewing, and interaction. As the section indicates, the common element in a creative process is the attitude towards experimenting: the creation of art approaches that of play, in which the function of the imagination is crucial. The section ends with an art historical discussion on the reception and interpretation of public artworks using geometry as their basis and as a source of inspiration.

Hungarian artist István Orosz discusses in his article his literary sources of inspiration—Shakespeare, Poe and Verne—and describes how he depicts the world of their texts through double pictures by using techniques usually associated with Renaissance and Mannerist art. Besides interpreting the writer's texts, Orosz's artworks aim to encode hidden anamorphic portraits of the writers revealed only by viewing them in a special way. Orosz have been making experiments with anamorphosis since the 1970s attempting not only to resurrect anamorphic art but to improve and further develop this old method of composing images. In anamorphosis unrecognizably distorted images become visible from a special view point or via an object, which mirrors the surface of the image when placed on it. In his article, Orosz explains through his own works and sketches how the anamorphic images are planned, outlined, and implemented. An understanding of geometry and perspective are the core elements in creating anamorphosis. Orosz emphasizes, however, that intuition, inspiration, and the "inexplicable" will always have a role in a creative work.

Julian Voss-Andreae, a German-born sculptor with a background in physics, introduces in his article his works of art which are inspired by ideas, images, and experiments from quantum physics and its philosophical implications. Voss-Andreae argues that art can indicate aspects of reality that science cannot, and therefore it has the potential to liberate us from the deep impact which the paradigm of classical physics continues to have on our every perception of reality. The ability of art to transcend the literal representation and the function of illustrating only the perceived world enables the artworks to mediate deeper aspects of reality that are hidden to the human eye. In the article, Voss-Andreae describes in detail how his studies and scientific theories and experiments in quantum physics are translated into works of art that invoke both scientific curiosity and sensual experiences.

Hungarian artist Antal Kelle introduces in his article his artworks, which are based on geometrical solids constructed either with modular components or sliced parts of an unbroken solid. Kelle describes his artworks as objects at the border of scientific curiosity, playfulness, and sculpture: in many of his works the components can be moved, and thus the works themselves can be transformed into various new forms by rotating them according to the receiver's curiosity and mood. The basic underlying form of Kelle's works approximates to a regular solid, such as a cone, which may be turned into a random form or an organic statuette. Kelle's works reveal how a geometric solid may mediate various sensations when their form is altered. With modern technology the alteration of the artworks can be even remote-controlled or automatized, as his latest artistic projects indicate.

Tuuli Lähdesmäki's article forms a continuation to Vierling-Claassen's, Orosz's, Voss-Andreae's, Kelle's articles by discussing the development of abstract art and its application to public sculpture. In particular, her article focuses on the problem of representation in geometry-based monuments. Since the 1920s, constructivist and concretist visual art movements have promoted the use of geometric forms and proportions as a basis for artistic expressions and aesthetic experience. After World War II, the ideals of geometric art were transferred to public sculpture in Western countries. Since then, the use of geometrical abstraction in public sculptures and monument art in particular has sparked debates and confrontations. The question of representation has been particularly problematic in the reception of monument art: geometrical abstraction has often been interpreted (or tried to be interpreted) as metonymic, metaphoric, or symbolic depictions of the person or event in honor of whom they are erected. The idea of representation and symbolic meanings are, however, against the principles of constructivist and concretist visual art movements. The article discusses from the discursive and semiotic points of view how the problem of representation has been solved in the reception of monument art based on geometric abstraction. As examples, Lähdesmäki uses two presidential monuments that were erected in Finland in the 1990s.

Future Perspectives: Education Potentials in Aesthetics of Interdisciplinarity

The Bridges community has combined forces in organizing the MoSAIC (*Mathematics of Science, Art, Industry and Culture*)[16] event series, sponsored by the *Mathematical Sciences Research Institute* (MSRI) to spur the STEAM approach (integration of Science, Technology, Engineering, Arts, and Mathematics learning for developing inquiry, critical thinking, and dialogue) among young people. As part of this program, popular aspects of Bridges events are held at university campuses throughout the USA. Thanks to international media, the attention paid to Bridges events—hosted by resident scientific and cultural institutions—strengthens also the local math-art-education communities. The *Experience Workshop International Math-Art Movement*,[17] an independent community of mathematicians, artists, and educators, established in Hungary in 2008 preceding the Bridges Pécs 2010 conference, is still growing. Following Bridges Finland Conference in 2016, the *Experience Workshop* has also started education activities in Finland and is now organizing events in several locations around the world. Through this

[16]See http://bridgesmathart.org/mosaic/. Retrieved on 24 October 2015.

[17]See http://www.experienceworkshop.org. Retrieved on 24 October 2015.

movement, tens of thousands of students and thousands of teachers have been exposed to the Bridges philosophy of interdisciplinary aesthetics and experience-oriented mathematics education through the arts.

Based on the positive experience from the Bridges community and the *Experience Workshop*, we encourage assessing pedagogical potentials of interdisciplinary aesthetics-based approaches and developing further conceptual and methodological tools for the aesthetics of interdisciplinarity in education. As successful international examples have already shown, *Learning Through the Arts* (LTTA) approaches can be beneficial for multiple aspects of the learning process (Elster and Ward 2007). However, these approaches have also faced criticism, as that often the subject being taught, less art itself is the focus of these processes (cf. - Sotiropoulou-Zormpala 2016). To re-establish art's equal role in the interdisciplinary integrated learning process, Marina Sotiropoulou-Zormpala (2012) set up an upgraded model of "aesthetic teaching", in which the aesthetic understanding of the subject; the utilization of multiple literacies; the meaningfulness of the learning space; interaction between logical information and moods, desires, and emotions elicited by the study topic; engaging activities; multiple approaches; and creative play take the lead. James and Marjorie Bullitt Bequette (2012) argue that interdisciplinary work in the arts and sciences can lead to curricular components that combine aesthetic and analytical modes of thinking enhancing both science and art. Respectively, Julia Marshall (2014) proposes an integrative approach to art which enables learning across the curriculum in a transdisciplinary framework by employing "Systems Thinking/New Sciences vision of art integration". Patricia Lynch (2007) notes how this kind of approach can have several positive implications for collaborative learning processes and community building. This viewed is emphasized also by Helene Robinson (2013) who argues that art integration approach may function as a valuable tool for supporting inclusion as well.

Due to the differences between the traditions, contexts, and possible goals of diverse mathematics and art education approaches, their notions of the learning process, learning activities, and collaborative learning and their approaches to teaching, problem-solving, creativity, and understanding of originality and authorship are radically different. However, through meticulous comparison several joint potentials may emerge, which can be re-contextualized and further developed into a mathematics and arts education framework based on the aesthetics of interdisciplinarity (a) to provide motivation and engagement for students and their teachers; (b) to enrich mathematics and arts learning on a meaningful way; (c) to enhance inter- and transdisciplinary STEAM learning frameworks with strong cultural embeddedness and social impact, where art is an integrative and transformative element of the STEAM concept, not just a vehicle for STEM learning.

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Part I
Bridging Art and Mathematics: Concepts, Theories, and Philosophies

The Geometric Expansion of the Aesthetic Sense

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Abstract There is an ambiguity in the relation between science and the arts. They have, in one obvious sense, an intimate connection. Both are fields dedicated to an incisive pursuit of truth, fields of deep inquiry to which traditionally genius flocks, to which the most high-temperature intellects over the centuries devote their energies. But even so, they are also disposed in diametric opposition. The truths they tell are not merely different—their truths have proved to be essentially incompatible. Increasingly, art and science are tending to negate each other.

Mathematics stands at the intersection of the two fields. It bears its own intimate relationship to each. The pertinence of math to art is long standing and well understood. Math is the foundation of harmonics in music, it regulates compositional strategies in visual art, most evidently in perspective rendering, it is the measure of meter in poetry. By the same token, mathematics is the heart of the scientific enterprise. Hard science is not just expressed but conceived in the formulations of mathematics, to such an extent that when science is at its most ambitious, when it is at its purest, math is the only form of conception and expression available to it—verbal phrasing becomes unavailing.

And that speaks to the core of the opposition between their aspirations for truth telling. Art is dedicated to human experience. The intimate details of human experience, and the precise qualities of sensory and subjective experience, are its raw materials. Art is about the human condition—the examination of the qualities of being human is very nearly its defining characteristic. However, increasingly, over the last several centuries science has departed from the intimacies of how human beings directly experience the world. After a series of what Freud called "Copernican Revolutions", not only has the status of the human species been reduced from the assumption that we are the crown of creation to the recognition that we are an inconceivably minor event occurring on an unremarkable planet in a merely typical galaxy, but—far more significant—the truth of the physical universe has proved to be something unavailable to the human senses and incomprehensible to human thought in any manner other than through mathematical formalisms. We are capable of experiencing through our senses at best 4% of what exists, leaving aside the equally

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evident fact that over the course of the history of our species, we will directly encounter nearly none of even that small portion. Science has shown that the truth of things is unvisualizable and conceivable only in equations. It cannot be seen, not even by the mind's eye. And so the intimate specifics of direct personal experience, the very material of art, is, to the search for the truth of things, irrelevant.

Early in the twentieth century, the artists of Modernism, and most particularly the visual artists, confronted the problem by adopting geometric composition in place of observations from nature—in place of the direct personal, sensory experience. But the challenge they began to face remains: if art has no choice but to reflect the truth as science is discovering it, in what sense is art still needed, what can it contribute, and in what sense is it still, recognizably, art? How can art wrestle with our new knowledge of a world that is fundamentally unlike what we understand ourselves to be, and do so in a form that remains legitimately artistic, that reflects the art of our heritage and relates it to the demands of our time?

Introduction

There is a natural kinship between science and the arts, a relationship that seems to be self-evident and undeniable. Both are fields to which traditionally genius flocks, fields of intellectual effort that occupy the minds of those whose mental capabilities are the greatest—both are places to which those who can achieve the most naturally go. Sometimes, it is the same genius: Leonardo is one instance; Einstein is another.

Nevertheless, there is a fundamental difference between the natures of science and art. They are incompatible, in that the truths they tell, as both are truth-seeking enterprises, cannot be reconciled. Art tells us of a human-centered world—its concerns are the concerns of human prospects. Science tells us of the truths of the world, the world as it is, and as it was before we arrived—the world independent of us.

For the past two centuries, the most significant and compelling efforts of the imagination have been in mathematics and science, perhaps to the embarrassment of art. We have learned much of the universe, and relatively little about ourselves, and our fate. Since the beginning of Modernism, art has turned toward the interests of science and has emphasized the use of mathematics, which in essence makes possible abstract art. The challenge confronting visual artists now is to retain the aesthetic integrity of their work, to find the way by which art can recognize what mathematics and science have discovered, but to find the methods by which their art remains, authentically, artistic.

A Janus-Like Division of Temperaments

The soul of mathematics is one divided. It is suspended in its affinities between two seemingly opposite poles. Throughout much of the history of known civilization, mathematics has provided the foundations and organizing principles of directly contending vectors of intellectual life, and thus has represented and supported, actively made possible, diametric tendencies of the imagination. It is as much as to say that one of the essential forms of thought is riven in its allegiances, conflicted in its attitudes and contradictory in its implications, straddling the unavoidable chasm between the two core inflections of mental sentiment, between the argumentative and the artistic, between the thoroughly logical and the deeply lyrical.

On the one hand, mathematics has a long-standing relationship with the arts. Its role in musical composition is both obvious and well studied, harmonics being rooted in simple ratios of wave lengths. In the visual arts, the use of mathematics as a compositional device dates back centuries. The application of the Golden Ratio in the composition of paintings can be found in numerous artists of the Italian Renaissance, including Leonardo, Michelangelo, and Raphael, as well as many since, such as Rembrandt, Dali, and Mondrian. Early in the Italian Renaissance, linear perspective was devised by Brunelleschi as a precise geometric compositional strategy. Shortly afterward, mathematically exacting treatises on perspective were written by Leone Battista Alberti and, later, by Piero della Francesca. In his early notebooks, Leonardo also proposed mathematical laws regarding perspective. And the use in the visual arts of regular geometric forms of varying degrees of complexity stretches from the Egyptian pyramids to the geometric ingenuities of M. C. Escher and the several movements of abstract art in the twentieth century, including Cubism, Suprematism, Constructivism, and Op Art. In the visual arts as well as music, mathematics provides the underlying structural principle for coherent organization and harmonious effects.

On the other hand, mathematics is the heart of the scientific enterprise. Galileo was the first properly to recognize that the universe was organized according to mathematical laws, and physical science has been written in mathematical theories ever since. The alignment of mathematics with the cold intellectual clarity that seems the very antithesis of art has been all the more pronounced since the project, initiated in the early twentieth century, of establishing a formalism for mathematics, of erecting the logical foundation of mathematics and demonstrating that mathematics is a logically coherent system. Although the final goal of demonstrable logical coherence proved unattainable, the project has been sufficiently successful, or at least rewarding, to make plausible the claim that mathematics is logic, such that the logical contradictions that cannot be dispelled from math by logical proof are acknowledged as properties of logic itself.

On the face of it, there could not be two more incompatible modes of thought. And the bifurcation of attitude—opposing proclivities of viewpoint on the way in which to approach and evaluate, essentially, everything—denotes an irreconcilable difference of core concerns, a difference in the field of interest that each of these disciplines in the life of the mind is pointed at. As both the projects of science and art can be considered as pursuits of the truth, it is a difference in the kind of truth they seek, or more precisely the districts of truth they take as their objectives. In short, science and art tell different, and increasingly incompatible, stories.

The purpose of art in all its forms is to tell the human story, its orientation is on the human view of the world, the possibilities of human fate, the inevitability of human downfall. In literature, the issue is the complexities of the human predicament—its promises and reversals of fortune, its triumphs and its betrayals, the comedy of its absurdities and the tragedy of its losses. The very idea of a story that is not populated by human figures—or in the case of fantasy, animal stand-ins for human figures—is virtually inconceivable. In poetry, the vast majority of its images, its metaphors and literary figures, are anthropomorphisms, methods of treating inanimate objects as if they were alive and possessing attributes of human manners or consciousness, as if the morn were in fact "in russet mantle clad". The practice is so rampant, so inescapable, that John Ruskin (1856) identified its over-indulgence as "The Pathetic Fallacy". Music is the most direct artistic presentation of emotion—so much so it often seems music conveys emotion without modality, without intermediary.

And the visual arts, our primary concern in this volume, are almost always devoted to images of human beings and their constructed interior environments or, in the case of landscapes and the occasional animal depictions, images that are results of and evocative of specifically the human gaze. Which is to say that the visual arts do not attempt to show the world as it is but as it appears to us, as it appears through the scrim of associations and evaluations we bring to our experience. The visual images of art are carriers of what the psychologist C. G. Jung (1971, 5) called the "subjective content" of perception, which is projected onto the objects of experience such that, in some cases, the object itself is of secondary importance. In the human gaze, and thus in all forms of art, the objects of experience are comparable to what T. S. Eliot (1975, 48) called, when it occurs in poetry, "objective correlatives": objects, situations or events that are natural symbols for emotions, or, more broadly, inner states or events. Everything that the visual arts depict is overlaid with what we unwittingly bring to it—values, implications, promises, condemnations—all of it seemingly so mixed together that they are inextricable. This fact may be the very heart of the human dilemma: the fact that we have so much trouble distinguishing between the world and ourselves.

Art is the interested view of life, science the disinterested view of the world. The purpose of science is precisely to conceive of and explain the world as it is, without the overlay of human values, judgments, and preferences. And from the time modern science was devised, as a mathematically rooted intellectual discipline, increasingly it has moved human concerns from their central position to the periphery of our vision of the real.

That viewpoint on the status of human interests is now known as the Copernican Principle, which holds that not only is the earth not the center of the solar system, but that human beings have no privileged point of view on the universe. More broadly, the idea—or the sentiment at the heart of this idea—is that the human view of things is not central in any way, not as a yardstick for determining the nature of the real, as a measure of the purpose of the world, as an evaluation of degrees of importance to events, or as a microscope for the determination of the truth of anything other than ourselves, if even that. In brief, the story of the universe, of reality, is not our story, not in any sense that it is more ours than it is the story of anything else. We are not special, and the purpose of reality, if that phrase is meaningful in this context, is not us. We just happen to be here; we are merely something that happened.

The thought that the unimportance of the human race in the broad scope of things has been an increasingly evident implication of science has been taken up by a number of thinkers. The first was Sigmund Freud. In his essay “A Difficulty in the Path of Psycho-Analysis” (Freud 2001, 139–143), he designates three primary historical events that provided decisive blows to human narcissism, to our sense of ourselves as the purpose of all existence and the center of the world, of ourselves as created in the image of god and that for which all else was created, of ourselves as the crown of creation.

The first blow to human narcissism was what Freud called “the cosmological one”: the discovery by Copernicus that the sun, not the earth, is the center of the solar system. The human being’s belief that the earth was the center of all things “was a token to him of the dominating part played by it in the universe and appeared to fit in very well with his inclination to regard himself as lord of the world”.

The second, “the biological blow to human narcissism”, was delivered by Darwin. In Freud’s view, during the development of civilization, the human being acquired a dominating position over the other animals. “He denied the possession of reason to them, and to himself he attributed an immortal soul, and made claims to a divine descent.” Darwin, of course, demonstrated precisely the opposite.

The third blow in Freud’s estimation, one that “is psychological in nature”, was delivered by Freud himself, along with all those who established the foundations of psychology as a science. “Although thus humbled in his external relations, man feels himself to be supreme within his own mind. Somewhere in the core of his ego he has developed an organ of observation to keep a watch on his impulses and actions.” It is consciousness and the will that are the faculties by which the mind exercises control of itself and its activities, through the use of which it is its own master. However, psychoanalysis has realized that the mind is little aware of much of its own actions and has little control therefore over what it does.

Of course, the reduction of the status of the human condition to the ordinary is not the same as the eradication for the human condition of any credible claim of significance. The humbling of our pride as a species, the loss of any defensible claim of uniqueness, does not force us to forsake any sense of importance to the issues of distinct human concern, particular when considered as strictly important from our own point of view. We may not be able to claim any role in the meaning of the universe, but what we are, what we can be, and what will become of us retains legitimate significance to us as the party of concern. The field has been leveled, the bias tilted toward our interests has been corrected in the light of day, but we have not been ejected from the game of existence.

Goethe's Opposition to Newtonian Science

But to some, matters are not so simple or straightforward, and the shift from the projection of human values onto all perception to the mathematically precise study of a self-sustaining physical reality as it exists by its own specifications has subtle and worrying implications. Johann Wolfgang von Goethe, the towering figure of German literature—who also devoted himself to scientific studies in the fields of botany, morphology, meteorology, and optics—placed himself in vigorous dispute with the increasing development of an ontologically oriented science, a science that sought a vision of the world beyond the limits and particulars of human perception and conception. According to Werner Heisenberg, the physicist who created the theory of Quantum Mechanics and who wrote one of the more penetrating analyses of Goethe's position on science, Goethe felt there was a serious risk in the form of imagination represented and practiced by science—he "sensed an injury in the advance of science". (Heisenberg 1979, 76.) There was, for Goethe, a danger in the quality of self-conception science fostered in the human mind, for it had developed as a philosophy of the truth of the world that built its explanations on the foundations of theoretical entities—objects, forces, and events that are unobservable and the result of postulation, thoroughly divorced from direct perception. This was particularly evident to Goethe in Newtonian science, and Goethe directly disputed Newton's theory of optics with his own *Theory of Colors*.

"Newton's starting point appeared strange and unnatural to Goethe" (Heisenberg 1979, 63). What is natural to Goethe as a starting point for science is direct observation uninterpreted, transported unalloyed into conceptual frameworks so as to retain the living quality of the thing science attempts to understand. Human perceptions are both preferable to theoretical entities for their authenticity and are comprehensive, providing an immediate and complete vision of the real. From his *Theory of Colours*:

> Effects we can perceive, and a complete history of those effects would, in fact, sufficiently define the nature of the thing itself. [--] The colours are acts of light; its active and passive modifications: [--] we should think of both as belonging to nature as a whole, for it is nature as a whole which manifests itself by their means in an especial manner to the sense of sight. (Goethe 1840, xvii–xviii.)

Thus for Goethe, the human perceptual apparatus is not an augmentation of nature and is not a particular viewpoint on nature. Rather, it is the ideal mechanism for the investigation of nature.

> In one word, our senses themselves do the real experimenting with phenomena, testing them and proving their validity, in so far as phenomena are what they are only for the respective sense in question. Man himself is the greatest, most universal physical apparatus. (Goethe 1949, 123.)

Goethe's selection of colors as the foundational element for this theory of optics is in preference to Newton's orientation on light, for it is color that is our direct experience, and, as a number of commentators have observed, neither Newton's proposal of the particulate nature of light nor the wave mechanics that were incorporated after Newton's time would have received Goethe's assent, for neither is directly observable—neither is within the realm of our perception. Both, as theoretical constructs, are divorced from reality. Reality is entirely rooted in what our senses report.

The pertinence of these matters to the consideration of the relationship between math and art comes through the clean distinction that Goethe, despite his intent to redirect science, drew between the differing temperaments that characterize science and art. In estimating the human sensory array as the "greatest, most universal physical apparatus" for investigating the world, Goethe specifically rejects the position of science on sensory input, already well established in his day, as described by Heisenberg during his analysis of Goethe's dispute with Newtonian physics: "In a way, science represents the attempt to describe the world to the extent that it is independent of our thought and action. Our senses rank only as more or less imperfect aids enabling us to acquire knowledge about the objective world." (Heisenberg 1979, 67–68.)

On the other hand, Goethe's emphasis on retaining the "living quality" of the thing being perceived and examined through reliance on direct observation rather than hypothetical constructs—along with the inevitable subjective intrusions on those perceptions undisciplined by reference to abstract, and particularly mathematically rigorous, formulations—is the essence of the artistic approach to the world, as Heisenberg points out:

> To this objective reality, proceeding according to definite laws and binding even when appearing accidental and without purpose, there stands opposed that other reality, important and full of meaning for us. In that reality events are not counted but weighed, and past events not explained but interpreted. Useful (*sinnvoll*) interrelations here mean a 'belonging together' within the human mind. True this reality is subjective but it is no less powerful for all that. This is the reality of Goethe's theory of colour. Every type of art is concerned with this reality and every important work of art enriches us with a fresh understanding of its scope. (Heisenberg 1979, 68.)

In this assessment of the world as envisioned by art, of the reality other than the objective one, we see the same "subjective content" that Jung explained is a component of all human perception, a component threaded throughout the world as we perceive it through our senses unrinsed by laboratory apparatus and regimens. It is a world that is more one of human exigencies and fortunes than it is of the ontologically real, "independent of our thought and action"—that is seen through the veil of human interests, that is filled with the mathematical imprecisions of interpretations, associations, judgments, possibilities and futilities, joys and terrors, ultimately, of human sentiments, and sentimentalities. This is the world as it appears to an aesthetician, the world of an artist, the world in which human concerns are the prevailing issues, in which everything is a symbol of our inner dramas.

Although Goethe sensed an injury in the advance of science, it is of course theoretically possible for these two orientations—those of art and of science—to coexist, to be limited to their appropriate fields of application. It is theoretically possible for art and the subjectively tinctured thinking it represents to limit itself to the analysis of human fortunes, without attempting to account for the physical explanations of the world in which its dramas occur, neither by propounding physical theories (which it generally does not) nor by proposing legends, myths, occults, prognostications, inevitabilities or other intrusions upon the proper purview of the scientific. And it is theoretically possible for science to limit itself to theories of physical events without attempting to be reductive regarding human experience and treat consciousness as an object rather than a subject, as something perceived rather than as a, as *the*, perceiver. Theoretically, all this is possible. Theoretically, science and religion are capable of coexisting. But that is generally not how things work out.

The difficulty is that both viewpoints, both fundamentally different temperaments for approaching how one conceives of the world, are totalizing views. Each by its nature possesses the implicit dynamic to explain the other away, to make it subordinate as an intellectual construct, either a subcategory of implication or simply misconceived.

The ambition of science to explain away the human reality, the world of human associations and perceptual constructs, is fairly easy to comprehend. It is the natural business of the scientific outlook to break down everything it perceives into data that can be mathematically mapped, locate regularities of occurrence, and formulate theories to account for the structures of regularity. It is the natural business of science to do the same with consciousness, treating it as an object of study and for theorizing and not as a subject. And there seems no necessary reason this can't be done. Science is capable of putting anything, including consciousness, under its microscope.

Rather more difficult to account for is the urge and method for absorbing science into the human view of the world, or more precisely, the view of the human world, the world of human values and associations, in which all observable things are signs with human import, bearing implications of what is about to happen, of what will become of us. It is the world that is built of our failure to recognize that the face in the astrological charts is merely a reflection in the window through which we otherwise are viewing the stars, that the report of our significance in the larger scale of things is an intellectual optical illusion—that legends are just tales. There is, in all ages it would appear, the urge to account for the world as a function of consciousness, as long as consciousness is allowed into the picture at all—that the world exists because it exists in a mind—in short, that consciousness is a mystical event, and the objective facts of the world, the very objective reality of the world, is a mental construct, something we formulate, or imagine, or dream—or it fashions, or imagines, or dreams.

The core proposition appears to be that since consciousness is inexplicable in physical terms (or for as long as it appears to be), it must, therefore and of necessity and by no further argument, be central to reality. Since consciousness is unaccountable in terms of the world and in accordance with the nature of everything that is not consciousness, it therefore must be the source of everything that is not consciousness. It seems a one-step argument; it seems to compel its conclusion by something other than the force of the argument it appears to be making. Since it is barely a logical event, one has to suspect it is a psychological event, and the compulsion of the conclusion is some form of need other than logical necessity. And so one feels urged to suspect that here is a result of consciousness contemplating its own potential insignificance, maybe its own evisceration. Perhaps for consciousness to consider the question of consciousness is to create an absolutism, an all-or-nothing sense of the alternatives—either it is the foundation of the world, the master of all things, or it is a mere mechanical event, as ordinary as an earthquake, or a pulsar, or a leaf falling on a stream. Perhaps the proposition is the theoretical equivalent of a panic attack.

The mutual totalizing claims of these vying and incompatible views of reality suggest not only the reason for the urgency Goethe clearly felt in the face of the continuing advance of Newtonian science but a risk implicit in the altered form of science he was proposing. Here, a pair of psychological dangers in which it is arguable that Goethe's position constituted a greater peril than the one he feared.

The "injury" Goethe sensed was a wound to the status of the human spirit, a blow to its standing in the world. At the ultimate extension of its logic, the Newtonian world view is a vision of nature comprising nothing but unobservable entities and events, a world in which human consciousness plays no role and the human soul is reduced down to being merely a function of underlying mechanics, at best an emergent property of background conditions invisible to it, at worst nothing but an epiphenomenon. The Newtonian world view is a vision in which the human spirit depatronizes itself, in which it evaporates to nothingness—in which we conceive of ourselves as not existing.

It was viewed as a dangerous vision, an injury to the spirit, a vision in which the living quality of experience becomes lost, and us along with it. But it can be argued that the alternative view Goethe argued in his ideals of a new science harbors a greater danger. A world in which human perception is the ideal data gathering device and in which everything that exists is taken to be what human awareness determines it to be is itself a function of human awareness. In this scenario, we effectively construct the world. This situation places consciousness in a God-like position. We become the source of the world, or at the least, we are functionally omniscient—if not aware of everything simultaneously, clearly capable of such awareness. Nothing is beyond our grasp and we are incapable of necessary error. In such a vision, we have been rescued from insignificance, but we are no longer protected by fallibility, no longer cautioned by our failings, and in the centuries since Goethe's time, we have seen what the arrogant delusion of the perfection of our own ideas has led to. It can be argued that, despite his otherwise wisdom, Goethe's science argument led us from non-existence to that—to the arrogance that inevitably results in tragedy.

The incompatibility of these two world views—the human spirit as mere illusion and the human spirit as the foundation of all truth, perhaps even of reality—leaves no third alternative. There is no room between them; each negates the other. The pertinence here is that, although mathematics has had a significant function in both science and art, between these fundamental alternatives regarding reality, mathematics is embedded and innate within the conceptions of science. It is central to

Newtonian science, and not to the art-influenced, human-centered view of Goethe's perceptual science. And in the years since Goethe's time, the gulf between science and art has become even greater, with science becoming ever less compatible with a human-based conception of reality—with an understanding in which human perception can be accepted as essentially accurate regarding the root elements of the real—and art has faced a fundamental choice regarding its direction, a choice that has directly engaged its relationship with mathematics, a relationship as new as has been the mathematics that has arisen since Goethe made his argument.

Philosophy as Art and Art as Science

It can be claimed, and this author would argue the case in greater detail than can be accommodated here, that Goethe's proposals for a new science constitute one of the first formulations, perhaps the first, of modern Phenomenology. (The core proposition can be traced as far back as Protagoras, for whom, famously, man is the measure of all things.) Although the term was used in the nineteenth century, principally by Hegel, the contemporary philosophical field was formulated in twentieth century, initially by Husserl to practice a scientific study of the world of human impressions, of experience, rather than the world beyond experience, the world as it is on its own terms—"independent of our thought and action". The impulse no doubt is to be found in the dominance of the core Kantian proposal—that our perceptions of the world do not match the nature of the world as it is beyond us, that we template our perceptions with the formal constancies they demonstrate. With this, Kant kicked out one of the legs of Goethe's scientific proposal, the assertion that human perceptions are reliable, are in fact the most reliable data gathering method. Thereby Kant made the reliance on direct perception a necessary capture in a world of our own devising, rather than a royal road to the truth of things.

Even so, it seems likely that the roots of Phenomenology go back to the projects on the part of physicists in the nineteenth century to analyze the workings of the sensations—physicists such as Hermann von Helmholtz and Ernst Mach, among others—for the purpose, it can be said, of reverse engineering the Kantian Transcendental Idealism, in order to infer the nature of our sense impressions before they are transformed into conscious perceptions, and thus to infer the nature of the incoming data, and ultimately thereby the *ding an sich*, the world as it is, unperceived, unconstructed by us. However, in Phenomenology, the ambition to infer the nature of nature *per se* has dropped out, and the objective is purely a study of a world fashioned by our perceptions, mediated by our senses. In this, it can be said that Phenomenology is an attempt for philosophy to take up the orientation and labor of art, to examine and try to understand human fortunes and sentiments and futilities, having some time back lost the capability to practice natural philosophy, which has now become physics and requires the research technology that philosophy does not employ.

Far more intriguing and intellectually compelling, and far more pertinent to the future possibilities for art, have been the developments in mathematics and science that occurred over the past two centuries. Shortly after Kant's time, which is also Goethe's— the end of the eighteenth century and the beginning of the nineteenth—the development of new mathematics began to accelerate, and much of the mathematics that was created dealt with the envisioning of scenarios that not only did not exist in the world of experience but that eluded the capability of being imagined as a hypothetical event, as a vicarious experience. Increasingly, math formulated and examined situations that were purely conjectural and that were implicitly imperceptible—they could be described in mathematical language but they inherently could not be formulated as sensory experience. They could be dreamed up, but they could not be dreamed of.

The mathematics of imperceptibles existed prior to the nineteenth century. Its history is as long as that of pure mathematics, the mathematics of Greek antiquity. Rather obviously, Pythagoras was aware of irrational numbers, such as *Pi*. Two thousand years ago, Hero of Alexandria knew of imaginary numbers, and complex numbers were discovered by Gerolamo Cardano in the sixteenth century. The nineteenth century saw an explosion of such mathematics, including Fourier analysis of wave mechanics and Georg Cantor's transfinite math. But nothing had such implication for science and for art as the developments in alternate geometries: *n*-dimensional geometry and Riemannian geometry.

The mathematician Morris Kline makes the point:

> During the nineteenth century, developments in mathematics came at an ever increasing rate. [--] The most profound in its intellectual significance was the creation of non-Euclidean geometry by Gauss. His discovery had both tantalizing and disturbing implications: [--] disturbing in that it shattered man's firmest conviction, namely that mathematics is a body of truths. [--] So shocking were the implications that even mathematicians refused to take non-Euclidean geometry seriously until the theory of relativity forced them to face the full significance of the creation.
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> [--] the devastation caused by non-Euclidean geometry did not shatter mathematics but released it from bondage to the physical world. The lesson learned from the history of non-Euclidean geometry was that though mathematicians may start with axioms that seem to have little to do with the observable behavior of nature, the axioms and theorems may nevertheless prove applicable. Hence mathematicians felt freer to give reign to their imaginations and to consider abstract concepts such as complex numbers, tensors, matrices, and *n*-dimensional spaces. This development was followed by an even greater advance in mathematics and, surprisingly, an increasing use of mathematics in the sciences. (Kline 1967, 25–26.)

Mathematics increasingly provided and explored the non-visualizable, but once the acknowledgement has been made that the human perceptual apparatus falsifies our conception of reality, this is hardly a drawback. It is moreover the requisite intellectual technology. As Kline explains, non-Euclidean geometry in particular provides the tools necessary to describe reality as science is now discovering it to be, a reality that can be described only with mathematical formalisms, a reality that cannot be visualized. The developments of relativity theory, quantum theory, superstring theory, dark matter, dark energy, and multiverse theories have continued to make clear that only equations can give us a comprehension of what cannot be viewed even by the mind's eye.

Science now tells us that the majority of the universe, of the real, is invisible to us. Estimates of the amount of the universe we are capable of perceiving run from 4% to 0.5%. This fact settles the issue upon which the entirety of Goethe's scientific proposal rests: the human perceptual apparatus is not the best mechanism for gathering scientific data. It is incapable of perceiving the vast majority of the real, and human perceptual logic is essentially no longer pertinent. The very categories of thought we obtained from sensory input are irrelevant.

Our near-complete "blindness" to the real also creates both an opportunity and a dilemma for art. The opportunity was the one taken up by art at the beginning of the twentieth century, with the advent of Modernism. Many of the movements in Modernist Art, and most particularly the development of non-representational abstraction in the visual arts, have been attributed to the developments in science, primarily to Einstein's Theory of Special Relativity. However, Linda Dalrymple Henderson, in her remarkable book *The Fourth Dimension and Non-Euclidean Geometry in Modern Art*, argues that the impulse behind much of the visual art came not from science but from the nineteenth-century work on *n*-dimensional geometry and geometry of curved dimensions. She points out that these subjects were not hermetically isolated under the seals of abstruse mathematical erudition. In the last several decades of the nineteenth century, numerous popular books were published on these subjects, among them a few we continue to read: *The Time Machine*, by H. G. Wells, and *Flatland*, by Edward A. Abbott. The idea of alternate geometries had popular currency through much of Europe and the United States.

That the greater exercise of aesthetic ambition was on the part of the visual arts under Modernism is particularly interesting in the face of an apparent paradox that may well be more an irresistible temptation than a contradiction. The geometric formulations art was being inspired by are implicitly non-visualizable, and were becoming increasingly useful to the scientific investigation of a universe that was becoming increasingly imperceptible. What more inappropriate subject could there be for the visual arts, and what greater temptation: to attempt to reveal visually the truth that cannot be visualized? It was an attempt to turn away from the standard human concerns, to create an art that gave up artistic subjects and did what science did, but by non-scientific means. What it lacked in credibility—its justifications were inevitably mystical—it made up for in brash heroism.

This, however, is not where the dilemma lies. To attempt the impossible, even that which is impossible on the basis of an internal contradiction, is admirably ambitious, and is no doubt virtually every young person's idea of what art is for. The problem is in the identity of the enterprise. If the standard human concerns of traditional art have been laid aside for an investigation of the aspects of nature that science has taken up, then in what sense is the project any longer art?

It is, of course, not enough simply to claim that these abstract exercises, largely of geometric compositions, are the new art, for that is merely a negative definition—a definition with no attributes, a null category—and thus is nothing more than meaningless insistence, and an abuse of language. What is needed is a positive definition that makes clear in what way a geometric composition is not just an exercise in pure mathematics, like an instructional model in a geometry class. What is needed is a definition that specifies the recognizable aesthetic quality in the new art.

The one major artist of the Modernist era who understood the necessity of the question and who was able to propose an answer of his own was Naum Gabo, the Russian sculptor who created the Constructivist movement. One of his sculptures, *Kinetic Construction (Standing Wave)*, 1919–1920, originated as a model he used to demonstrate kinetic energy to a class. So, he should well have been aware of the need for a geometric work of sculpture to have something to it beyond the sheer employment of geometry for it to be a work of sculpture, that mathematical art must both be mathematical and be art.

Gabo conceived of Constructivism as the leading edge of the growth of a new art that matched the developments in scientific understanding of the laws of nature at the beginning of the twentieth century, a clear reference to relativity theory. His ambition for Constructivist sculpture was to open up new knowledge of the inner world of human life just as new knowledge of the outer world of physical reality was being revealed, and by similar means, through the use of new forms of geometric composition.

> The Constructive idea [--] has revealed a universal law that the elements of a visual art such as lines, colours, shapes, possess their own forces of expression independent of any association with the external aspects of the world; that their life and their action are self-contained psychological phenomena rooted in human nature; that those elements are not chosen by convention for any utilitarian or other reason as words and figures are, they are not merely abstract signs, but they are immediately and organically bound up with human emotions. The revelation of this fundamental law has opened up a vast new field in art giving the possibility of expression to those human impulses and emotions which have been neglected. Heretofore these elements have been abused by being used to express all sorts of associative images which might have been expressed otherwise, for instance, in literature and poetry. (Gabo 1996, 366.)

It is a distinctly Platonic turn of thought for Gabo to commit, but then, all of abstract art can be argued to be Platonist. The art that Plato condemned in the tenth book of *The Republic* is specifically imitative art (Plato 2000, 313); it is the imitative in poetry that makes the image a copy of a copy, twice removed from the real. For Gabo, the “associative images” in “literature and poetry”—that which in art is the “imitative”— masked the innate expressive potential of the pure elements of pure art, non-representational art, abstract art: lines, colors, and shapes. Released from their use for the construction of the traditional associative imagery, these pure compositional components are available to take on new forms, impelled by the new geometry to simulate the unvisualizable, to expand the aesthetic sense and the artistic possibilities, and thereby to express new, previously neglected “impulses and emotions”—the recognizable subject matter of art. In short, new geometry makes possible not only new science but also new forms of art, by providing the tools by which areas of the inner life of the human spirit that had previously been unexplored can now be discovered.

This is not a call for a specifically artistic rendering of a quark, or a wormhole, or a hypercube, whatever an artistic version of such things would be. However, it is clear that for Gabo such sculpture is related to the science that the same geometric conceptions make possible, for he adds that it is the role of art to create the ideas that give the meaning to science.

> Science is the vehicle of facts—it is indifferent or at best tolerant, to the ideas which lie behind facts. Art is the vehicle of ideas and its attitude to facts is strictly partial. Science looks and observes, Art sees and foresees. (Gabo 1996, 367.)

It is a tenable idea if by "Art" Gabo also means creativity, including the creativity that is employed by scientists to devise new theories, explain new worlds. If so, then the explanatory power of human ideas, and the mathematics that provides and in many senses *is* their explanation, are human constructs, products of human creativity. Art then becomes a complementary function to science, revealing the core of the ideas that equations can only represent, showing what the equations are trying to convey. The art becomes what the equations mean—a vision of the reasons they work.

Conclusion

The recent expansion of mathematics and of the science whose theoretical formulations it makes possible has become the principal story of humanity. The larger part of the mathematics we know was devised in the past 200 years. Over the last century, the very scale of the universe around us erupted in our understanding. We know this, we have discovered this, because there has been a comparable expansion of the tools of our imagination, and they have opened the capabilities of art as much as science. But there has also been a dwindling of our status, perhaps of our very nature. We once stood at the center of a solar system. We now live at an undistinguished point in a netting of galaxies whose size it seems impossible to comprehend.

The explosion of mathematical meaning has been our meaning as a species, and the expansion has been geometric. But the situation in which we now find ourselves is an exploded version of that of Pascal. We now stand before the universe as he did—contemplating the frightful spaces in which we exist, made sick by the unspeakable scale of the emptiness. We are confronted as he was by the Copernican Principle, by the immensity of our insignificance. And that is also our meaning as a species, for it is also what the mathematics tells us. Only we are aware of the naiveté that Pascal suffered. He could not have suspected the true scale of the void, and perhaps neither do we yet. This is our poignancy.

To contemplate the human situation meaningfully outside of this context is simply obsolete. Like Pascal, we are what our discoveries are making us; we are what our mathematics means. The visual artists of Modernism who early in the twentieth century took up the new geometries as the tools of their art understood this fact. The challenge to artists since then and now is not only to wrestle with our new knowledge, but to do so in a form that is legitimately art, that relates the work of our heritage with the demands of our time. And the material to create human-centered art is available, for the story of our growing insignificance is a human drama. It may be a tragedy, but it is ours; it may be a dirge, but it is for us to sing. It is what Goethe feared, but it is also the truth.

> 'We are poets', said Pythagoras, and in the sense that a mathematician is a creator he is right. (Gabo 1996, 367.)

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On ‘Visual Implication’: Outline of a Theory

Sirkkaliisa Usvamaa-Routila

Abstract Most of us are ready to accept the view that the front elevation of a building is essentially determined by such openings of the wall as windows, doors, bays, and niches. Especially their location and their sizes create a compound of parts and details that appear as an orderly arrangement, as it might be called. Normally we are able to feel when everything seems to be in order, in the right place, thus creating a good and balanced picture of the wall. The lack of such an order can be felt equally easily. One reason of seeing such a balanced order, and/or the lack of it, is the system of rectangles presented by the openings within the parameter rectangle formed by the whole wall. It is easy to see that the main reason for the feeling that everything is in the right place is depending on some of the diagonals of the rectangles and their mutual directions as we experience them in accordance with the movement of our eyes. But then, where do we see all these diagonals when we look at the architectural object or the photo picturing it in its corporeal and factual actuality? Indeed, there are no lines to be seen, no lines present either in the photo or in the front wall of the house! Le Corbusier called the lines of his drawings “les tracés régulateurs” and “linee occulte”. For him they were, though lines not to be presented, extremely important anyway. He wrote that his buildings become visual poetry, cast in reinforced concrete—thanks to these guiding lines.

How is it possible that our eyes can be guided to see the front elevation, according to lines that are not drawn and are nowhere to be seen? Assuming that they are actually so important for our aesthetic experience, the problem comes up urgently: how is it possible that we see the diagonals though they are not made visible at all, how is it possible to see what cannot be seen? I have claimed that in all these cases the diagonals and lines are implicitly given in our perceptions. They are implied by those details we can see because they are there, although unseen and as if hidden, but actually present in some obscure way of absence. I introduced the term visual implication in order to refer to all these cases. I have continued to think whether it would be possible to add to the phenomenological theory of perception a corollary that would solve this problem. The present paper drafts such a corollary. It is based on the role of imagination involved in our capacity of visual perception.

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A Preliminary Remark: Architecture and the Language of Seeing

Architectural objects are normally buildings, and buildings are primarily raised up in order to fulfil some practical purpose.[1] Hence it might be felt that they cannot be examined, in any essential meaning, as visual things, as things primarily to be looked at and thus appreciated as objects of our visual world. In what follows I shall, however, be concerned with architecture in this sense only, as something *seen*. This is a point of view I have chosen entirely because of the problem I am going to discuss, and it does not propose any claims concerning the primary way of studying the aesthetics of architecture in general.

There is of course nothing new in my attempt to think of architecture in a specified way. Most of the traditional talk of architecture has used, and still uses, terms revealing that it takes its roots from the soil of the visual world. In spite of that, one of the prime questions requiring an explanation remains: why has the aesthetics of architecture taken this course of devoting its attention so extensively to the language of seeing? Some of my observations below may contribute to the understanding of this somewhat obscure fact.

Before I begin I would like to remark that I am a philosopher with a background in phenomenology. I have tried, however, to write this essay for the general reader with midget philosophical training. So I have avoided, whenever it appeared possible, the specialized terms typical of phenomenological discussion (such as *noesis, noetic, noematic*). Especially the topic of the latter part of my contribution was difficult to work out, and I had to draw on any resources that seemed to help. Valuable insights and suggestive cues I gained from the writings by as different authors as Schopenhauer, Husserl, Sartre and Wittgenstein, P.F. Strawson, Roger Scruton and Colin McGinn.[2]

The Problem: The Principal Claim of the Classical Theory of Proportions

Many years ago, when writing my doctoral thesis about the classical theory of proportions, I had to frequently face the problem which I am going to deal with in this paper.[3] According to the classical theory certain proportions have the ability to

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[1]Schopenhauer discusses this topic cannily in the second part of his *Die Welt als Wille und Vorstellung*, especially in relation to Kant's definition of what is beautiful (Zweckmässigkeit ohne Zweck), see Schopenhauer (1968, II, ch. 51).

[2]I am indebted also to my husband for encouraging discussions about the topics I have dealt with in this paper, see Routila (1976, 125ff. and 1999, 25ff).

[3]I am referring to my Finnish book on Alberti's architectural tractate: *Kaunis ja sopiva* (The Beautiful and the Suitable). See also my paper 'Partitio and lineamenta in Alberti's *De Re Aedificatoria*' (Opuscula Phaenomenologica Fennica) Turku 2006.

please us so powerfully that we cannot abstain from applause and praise. Since ancient times, many theorists of architecture and even architects themselves have accepted and respected the view that such proportions are, when used in buildings, exceptionally beautiful and capable of creating unusually good-looking and attractive architecture.

It is not my business here to argue for such a view. I simply assume the basic tenet of the classical theory—by reason of my own experiences and similar experiences of many other people who share this view and have given testimony about experiences reminiscent of the ones I have had myself. I agree, however, that there may be quite a lot of people not able to see such things at all, as if they were somehow blinded. I do not discuss here whether it would be possible to teach them to see differently and thus to contest their ignorance. All this must remain outside the present enquiry.

In my profession as a practitioner and theorist of art education, I have nevertheless occasionally discovered that a certain kind of *attitude* may be a prerequisite to the ability to appreciate the visual properties of such things as architectural objects. The notion of attitude announces a highly interesting feature of the problem, for attitudes are dependent on the will. The voluntary character of our seeing certain things has been very little examined and I have to encounter it briefly in the latter part of my paper. It will be my claim that the aesthetic experience of architecture and art in general is dependent on the will. I shall try to show that this dependence is inherently associated with the role of *imagination* in all aesthetic experience.

In order to illustrate the problem involved in the classical theory of proportions, let me now take a very simple example as our starting point. Most of us are ready to accept the view that the front elevation of a building is essentially determined by such openings of the wall as windows, doors, bays and niches. Especially their location and their sizes create a compound of parts and details that appear as an orderly arrangement, as it might be called. Normally we are able to feel when everything seems to be in order, in the right place, thus creating a good and balanced picture of the wall. The lack of such an order can be felt equally easily. We may then think spontaneously that everything is out of order: *look, what an ugly wall they have built there!* One reason of seeing such a balanced order, and/or the lack of it, is the system of rectangles presented by the openings within the parameter rectangle formed by the whole wall. It is easy to see that the main reason for the feeling that everything is in the right place is depending on some of the *diagonals* of the rectangles and their mutual directions as we experience them in accordance with the movement of our eyes.

Le Corbusier and His "Occult Lines"

Let me take a more concrete example. Le Corbusier designed in 1923 for a rich Parisian art collector a fairly large private house, the famous *Maison La Roche*, today the residence of the *Le Corbusier foundation*. Some years later he designed the *Villa Stein*, an equally legendary house in Garches. These two houses became the first manifestations of modern luxury domicile, cast in reinforced concrete according to a modernized version of the classical theory of proportions.

Amongst the original drawings of these houses we find representations of their front elevations. I have reproduced here a copy of the front and rear elevations of *Villa Stein* (Fig. 1).

We see in them several lines the architect has drawn in order to show how the details of the elevations are connected. They appear to be diagonals of the openings and certain other rectangles present in the elevation. When we are shown a photo of

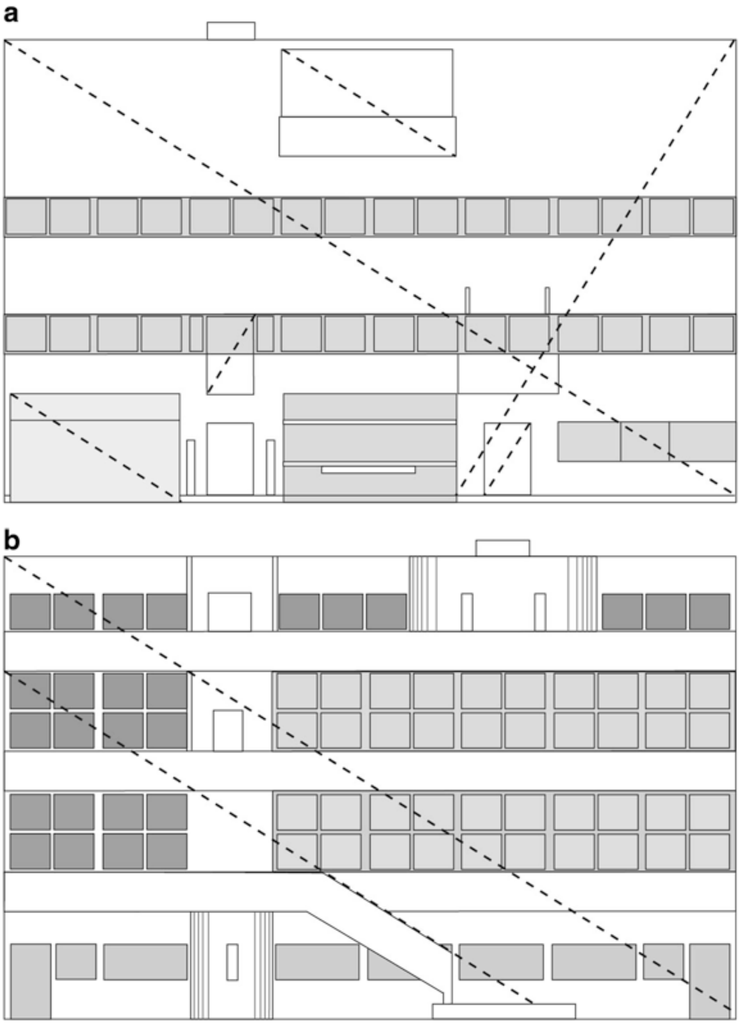

Fig. 1 (**a**) and (**b**) Le Corbusier, *Villa Stein*. The front elevation and the rear elevation. Illustration by the author

the front of the *Maison La Roche* or the *Villa Stein,* or when we, having had enough good luck, have the advantage of looking at the facade face to face with our own eyes, it would be obvious that our eyes would follow similar lines and "read" the elevation as if guided by them.[4]

Discussing the division between inside and outside of a house, Roger Scruton (1979, 253f.) remarks that the façade is the vesting of architectural significance: "The façade is the face of the building: it is what stands before us; it wears the 'expression' of the whole." Referring to Le Corbusier he points out that the classical law of the façade, the *upward movement of its lines*, has been brushed aside "or confusedly ignored" and he quotes, perhaps with some resentment, from Le Corbusier's *Vers une architecture* an illuminating sentence:

> Reinforced concrete has brought a revolution in the aesthetics of construction [--] suppressing the roof and replacing it by terraces [--] these setbacks and recessions will [--] lead to a play with half lights and heavy shades, *with the accent running not from top to bottom, but horizontally, from left to right*. (Le Corbusier, *Vers une architecture* (1928, 61f.), quoted from Scruton (1979, 254).)

So there seems to exist a genuine reason for Le Corbusier to draw these lines and make them explicit in the drawings. I think we are allowed to suppose that he wanted his drawings to be precise and to the point: even they had to be manifestations of the "revolution".

But then, where do we see all these lines when we look at the architectural object or the photo picturing it in its corporeal and factual actuality? Indeed, there are no lines to be seen, no lines present either in the photo or in the front wall of the house! Le Corbusier called the lines of his drawings "les tracés régulateurs" and "linee occulte" and that is what they really seem to be. For him they were, though lines not to be presented, extremely important anyway. He wrote that his buildings become *visual poetry, cast in reinforced concrete*—thanks to these guiding lines (Le Corbusier 1942, 39f.).

How is it possible that our eyes can be guided to see the front elevation, *horizontally, from left to right,* according to lines that are not drawn and are nowhere to be seen? Assuming that they are actually so important for our aesthetic experience, the problem comes up urgently: how is it possible that we see the diagonals though they are not made visible at all, how is it possible to see what cannot be seen?

It is indeed a well-known fact that people tend to scan a design from the top left corner to the bottom right corner. This tendency has been found even in cultures where people are accustomed to reading printed matter from right to left or in vertical columns from top to bottom, so the theory once propagated by scientists, according to which this tendency was based only on our "western" reading habits, does not hold (Eibl-Eibesfeldt and Sütterlin 2007, 159ff.). Many artists have made

[4] I am utterly unsatisfied with the way we have learned in the wake of semiotic approaches to speak of "reading" when we should speak of "seeing" and "looking". I allow the word here only as a convenient simplification.

use of their knowledge of this tendency of the mind and directed the movement of the spectator's eyes. Sometimes they have even tried to prevent the eye to move off the design through using other items as obstacles that are able "to slow down the movement of the eye or to direct it back into the design" in order to increase their attentiveness (Zelanski and Fisher 1984, 32ff)[5]—as Le Corbusier in our example seems to have done.

But even this explanation does not help us, though his knowledge of it may have helped Le Corbusier. We still have to ask: how can we be guided by these "occult" lines? *And why leave them in secrecy if they are so important for the spectator's eyes?*

Another Example: Geometry and Visual Experience

According to the classical theory, the rectangle formed by the fabulous *Golden Section* has peculiar visual features. It is claimed that the proportions inherent in it appear as extraordinarily beautiful and pleasing. Accordingly it has been asserted that the pleasing properties are due to the underlying mathematical paradigm. This in turn has an enigmatic and mysterious property: You can generate an infinite series of golden section rectangles if you form a square of which the side is the shorter side of the rectangle and then remove it from the design. The remaining part is a golden section rectangle, and you can repeat this procedure *ad infinitum*. All golden section rectangles can easily be used in an architectural design to generate a bouquet of right angles and diagonals and to form a proportionate composition just in the same way as the occult and hidden lines in the *Maison La Roche*. Mathematical properties clearly become visually satisfying aesthetic properties of the design, in most cases at least. Besides, the proportions Le Corbusier used both in *Maison La Roche* and in *Villa Stein* were based on golden rectangle; the famous *Modulor* did not yet exist in the twenties.

Yet there remains a puzzling question which the opponents of the classical theory have frequently raised at this point. One of the interesting critical comments comes from Roger Scruton (1979, 63):

> [--] despite the widespread confidence in a concept of proportion which derives from it, the theory provides no general aesthetic of construction. The first difficulty arises when we try to relate the abstractly calculated proportionality of a building to a concrete experience of proportion.

There are two points I want to bring out here. First, we have to be a bit careful in formulating such critical comments. Scruton speaks of the "abstractly calculated proportionality" and accordingly formulates the Golden Section (and other proportions as well) in terms of the ratios, for instance here $(1 + \sqrt{5}) : 2$. Of course, since

[5]*Design. Principles and Problems* by Zelanski and Fisher (1984) is a splendid book about the understanding of visual arts really as something seen.

Descartes and Leibniz transferred geometry to arithmetic, proportions can be expressed mathematically in this way. But in the classical theory the proportions come from *geometry:* they are visual to the extent that geometry is visual! Second, I agree that the mathematical significance of proportions do not change when the way of their expression is correctly changed. Their possibilities as tools for creating the guiding lines for our eyes and to contribute to our aesthetic experiences do change totally. Only geometrical relations can be seen. Scruton seems to be surprised to find that *geometrical relations* “impose themselves in a façade with a certain immediacy and absoluteness”. But isn’t that rather natural? We can actually *see* them.

However, the problem still remains. It is true that what we are seeing in these cases is visually given, but the guiding lines are not seen. We only see the geometrical basis from which the lines are obviously derived. We see only the vehicle, but how can our mind scan what isn’t there?

Visual Implication and Its Alleged Conveniences

In my book about the classical theory mentioned above, I claimed that in all these cases the diagonals and proportions are in one way or another *implicitly given* in our perceptions. They are implied by those details we can see because they are there, although unseen and as if hidden, but actually present in some obscure way of absence. I introduced the term *visual implication* in order to refer to all these cases. I used the term in my lectures and it gained ground as my students were eager to adopt it. As a matter of fact, I didn’t have the faintest idea what kind of an implication could a visual implication be, for usually we speak of implication when we mean thoughts. Such a situation could obviously not be tolerated, and I continued to think whether it would be possible to add to the phenomenological theory of perception a corollary that would solve this problem. The present paper drafts such a corollary.

As a keen student of Husserl’s phenomenology I am used to taking as the starting point of the discussion about experiencing and seeing a discovery originally made by Husserl’s teacher Franz Brentano. It was developed further by Husserl and it has become one of the few doctrines of his that is universally accepted by philosophers of pretty much any tenet and belief. It is the discovery of intentionality of our consciousness. (For brevity, the states of our mind are characterized in the philosophy of mind by *consciousness*.)

Intentionality: The ‘Whereof’ of the Consciousness

According to the theory of intentionality, all our thinking, perceiving, imagining, experiencing etc. is necessarily *of* something, referred to and directed to the object of our thinking and experiencing. “In the very essence of experience lies determined not only *that*, but also *whereof* it is a consciousness,” Husserl (1931, 63) wrote in his archaic, academic style.[6] Thus, there is no thinking, loving, fearing, remembering as such. Every fear is fear of something, every remembering is remembering and every experiencing is experiencing of the object that is experienced and remembered, thought, imagined or feared. Husserl designated this relationship as *intentionality*.

The attraction of this theory is obvious. The first reason is that the intentional character of the consciousness necessarily involves a sharp distinction between the act of experiencing on the one side, and the objects to which the experiencing is directed on the other. Each intentional act grasps an object in a certain way, takes it *as* something while the object may nonetheless have properties not taken into account. Following Husserl's line of thought one can say that we are not only entitled to, but we *have to* distinguish between the object *which* is intended (experienced, thought, seen) and the object *as* it is intended (experienced, thought, seen) (Husserl 1970, 578; cf. Routila 1976, 125f; Carr 1987, 34ff.). The same object can be intended in different ways or, speaking without professional terms, it can be *taken-as* and *seen-as* differently according to our intention. The object is always perceived, for instance, from some angle or another, and we have to distinguish the object as seen from the object with other perspectives. In the language of more recent discussions of intentionality, this means for instance that we cannot substitute such an expression as "evening star" with the expression "morning star" though they refer to the same planet, Venus. The intentional meaning of these two expressions is different. Our descriptions of perceiving are therefore said to be "referentially opaque" (Carr 1987, 37).

The second reason is that in experience it is not only the experienced object that is experienced. With the object of perception, its *perceptual background* is also given to our consciousness. And it in turn presents itself as a mere segment of what *can* be perceived. The object is "surrounded by a horizon of undetermined reality" of which we are also conscious: the spatiotemporal surroundings of the object. Our consciousness is always of more than what is explicitly experienced. I cannot see the room behind my back, but I am conscious of it as a potential focus of my attention.

What I perceive of the object at one moment of time is only one aspect of it, but it is evocative, for it suggests other aspects. The front side of the *Maison La Roche* suggests its back, the façade the interior and the door the place of entrance. All these parts together form what phenomenologists call the inner horizon of the perceived object. All the things within my visual field may be labelled as the outer horizon of the perceived object, such as the houses behind and beneath the *Maison La Roche,* the trees in the gardens, the children playing in the park, the motor cars on the street.

Thus, in the example of the background and surroundings of the perceived object we encounter the *implicitly given*. Of course, the hidden sides of the object cannot be experienced in the manner of the side that faces the perceiver. We experience the surroundings only in the manner of a horizon.

[6]For the German text see Husserliana (III p. 80, see Schutz 1971, 108 ff).

For the eye the surrounding horizon appears as the visual field that can be described as a space which is filled with many other things, in some spatial relation to one another and also in relation to the *body* of the perceiver. We want to emphasize the body of the perceiver because we are dealing with the visual world and we grasp it primarily with our bodily eyes. So, some things are seen in the background, some in the foreground, some far away in the periphery, some in the centre, some bright and illuminated, some remaining in shadowy darkness. In the pre-phenomenological philosophy it was not customary to take much notice of the body. Inspired by Husserl’s analysis of the human body, published for the first time in Paris in 1931, Merleau-Ponty and Jean-Paul Sartre have made pioneering work in this area, Sartre even in his 1938 novel *Nausea*.[7]

Though there are no isolated objects of perception, we have some possibilities of choosing what we perceive. I may merely notice the object of my perception without any further curiosity. It may not stand out anyhow within my visual field and I can even eliminate it by closing my eyes. Otherwise it can also strike out and arouse my interest in it. In this manner most of us would experience the *Maison La Roche* and *Villa Stein*. As an intentional object of our perceiving consciousness, the percept would now impose itself upon our faculty of *attention*. But if I thus focus my attention to a particular area of the visual field, the rest of it does not disappear. It only remains more or less unattended in the horizon of perception. So we may conclude that we consciously see only what we choose to focus on. The rest of our visual perceptions are consigned to a less conscious level of awareness.

This means, however, that we always see much more than we can perceive with full awareness. In our visual field there are constantly objects that we fail to notice. I certainly saw the colour of the eyes of a student who was having a chat with me after my lecture, but I didn’t notice it. Much of what we fail to notice at one time can become attended at another time. We live within a constant spatiotemporal stream of information. In phenomenology this stream appears primarily as temporal, because of the faculties of memory and anticipation.[8] They both affect the way our perception really works, but this topic would take me too far from the problem with which my enquiry began. So let us return to the problem of classical theory and reformulate the principal question: How can we *see* what is only *implied?*

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[7]Husserl lectured at Sorbonne 1929; a French version of the text of his lecture was published 1931 in Paris as *Méditations cartésiennes*. The German text appeared 20 years later in Husserliana I. Husserl’s new venture into the Phenomenology of intersubjectivity with an accent on the human body remained partially unknown in Germany until that. The English translation by Dorion Gairns titled *Cartesian Meditations* appeared 1960.

[8]Husserl’s analysis of time consciousness (*Vorlesungen zur Phänomenologie des inneren Zeitbewußtseins*) was first published 1928 by his pupil and follower Martin Heidegger. Heidegger published his *Sein und Zeit* (trans. *Being and Time*) in the same year presenting a radically new version of Phenomenology that was based on criticism and reception of Husserl. See Routila (1976, 125ff).

The Role of Imagination

Philosophers have recently tended to broach the topic of visual experience by starting anew. There has been considerable interest in a mental capacity that at first seems to be too far away from any serious attempt to explain experiences of the exterior world. I mean the faculty of mind called imagination. I am now going to make use of some promising and, it seems to me, generally accepted discoveries made within the modern theory of imagination. Much of what I have to say will be familiar to the readers of Strawson, Scruton and McGinn. However, there are points which I have to emphasize and cannot leave without exposition.

First, it is worth noting: imagination is the faculty of mind that enables us to conceive of the world in other ways than it is presented in perception. It is true that when we picture the world in such a fashion we only *imagine*. But what does that mean? Let us begin this section by thinking about the distinction between perception and imagination, between a percept and an image.

Perception is the faculty of perceiving, imaging is the faculty of forming images. It might be doubted, however, whether the presence of images is the distinguishing feature of all acts of imagination (see Scruton 1974, 92ff.) but I don't go into that at this point.

So what are images then? Though we are usually well aware of the distinction between what we are seeing in the proper sense of perception and what we are merely imaging, we rather seldom come to think that our "imaginative consciousness" is intentional and always has an intentional object. Neither do we in distinguishing images from perceptions seldom take into account that the creations of our imaginative faculty, the images, are in a peculiar way *like* perceptions and share a component we are inclined to call "sensory". As Scruton (1974, 103ff.) and McGinn (2006, 7ff. 40ff.) remark, we even express images very much the same way that we express perceptual experiences. If I am asked to form an image of my father, I may well concentrate my attention for a while before I catch it and am able to say, "Now I have got it." And my image of my father might be so vivid that I can almost see him standing beside me. This is to say: having an image of X is in some way like *seeing* X.

Note that I am speaking here primarily of visual images. Scruton's assumption (1974, 100ff.) that an image is a kind of a thought of something is, I think, mistaken. It may have its origin in his method that confuses phenomena with our way of speaking of them. When I call up the image of my father, I do not primarily *think* what he was like. My image of him is not my thought of the way he looked. On the contrary, I form an image of him in my imagination from my past memory images and they appear in a sensory form. This is not to say, of course, that no thinking can be involved in my imagination. After I have experienced that I cannot form an image, I usually have to say, "Let me see, I have to think."

True, we often imagine by thinking what it would be like for one to do something. Lewis Carroll, to use an example mentioned by Scruton (1974, 99), imagined what it would be like to live behind the looking glass. Similarly, I may imagine what it would be like for a baron Münchhausen to climb up the ladder in order to visit the Moon. And I don't see any reason to deny our ability to imaginatively transform such thinking into visual images. It is extremely difficult to determine the boundary between thinking and imagination. Scruton puts an end to his discussion by admitting his unwillingness to conclude that "imagery is just a species of thought", but he admits only that the faculty of imagination "lies across the boundary between thought and sensation" (1974, 101f.). Even this might be doubted, however.

Second, I want to reiterate the common sense view that our images *can* be of external physical things—even though they are created by our imagination, they can be exactly of what our perceptions normally are of.[9] Many modern writers have confused the faculty of imagination with pure fantasy and daydreams and other images unrelated to the external reality. That is a serious mistake. This is of course not to say that there are no images of the fantasy kind. On the contrary, although it is important to see that all images are not fantasies and without an intentional correlation to the objects of our apparent reality.

That means first of all that we can see one and the same object both as an image and as a percept. We may have an image of the *Maison La Roche* and a perception of it as well. We can only have the perception if we go to Paris, but we can have an image of it while sitting comfortably in our favourite chair in our home. However, both the perception and the image have the same intentional object, the only difference being that we are able to see the object either with our bodily eyes or—as the saying goes—"with the mind's eye". The image is, however, directed to the physical object of the *Maison La Roche* just like ocular seeing.

Remembering Scruton's warnings one might ask: doesn't such a fusion of imagination and seeing mean that we are ready to change it from seeing to "seeing"?

Indeed, Colin McGinn has recently discussed how literally the common talk of seeing with the so-called mind's eye should be taken (McGinn 2006, 42ff.). Does the mind's eye involve a type of genuinely visual experience that can take external objects as its intentional objects? Suppose that you form an image of the *Maison La Roche:* is the object presented to your mind really in the visual mode? McGinn answers "yes", but he is not trying to postulate an additional organ in the brain—his purpose is to "legitimate the sensory credentials of the imaginative faculty and to show how it is possible for the concept of visual to apply to certain images". This view is certainly intricate and maybe even risky but I think provisionally acceptable. So I will also conclude that seeing with the mind's eye is one kind of sensory seeing.

Besides, images may be more or less vivid. Sometimes images even contain objects we have failed to notice before. Through calling up an image of the student I had a chat with at the university I may see the colour of his eyes as if instead of an image I was looking at a picture of the object. It is told that Kant, teaching geography before he became a professor of philosophy, was able to describe the

[9]For an instructive discussion of the topic see McGinn (2006, 19ff, 128ff).

London bridges so vividly that when his students visited London they could not believe that Kant had actually never left Königsberg. As he could not have any memory images of London he must have been able to carry out his imagining by pure thinking.

There is a feature of imagining which has been much used for distinguishing imagination from perception: the will (see Scruton 1974, 94ff; McGinn 2006, 28ff.). For instance, McGinn writes straightaway: "Perhaps the most obvious difference between percepts and images is that images can be willed but percepts cannot." This is however an intricate topic on which a lot of confusing opinions has been asserted. Imagining is not always voluntary and perception is sometimes voluntary, as I pointed out above: we can often decide not to see an object present in our field of seeing. But we cannot decide to see an object that is not present in it. I suppose we also all have experiences of obsessive images that resist any attempt to eradicate them. When McGinn writes the attractive sentence, "[f]orming images is something I do, while perceiving is something that happens to me", it sounds good but is difficult to agree with. Anyway, many images can be called up at will, and it is to a certain extent "entirely contingent", as Scruton writes (1974, 95), that images are sometimes involuntary and sometimes impossible to banish.

Taking advantage of this admittedly far too sketchy discussion, I am now reaching a conclusion: it is possible that we may see an object both as an image and as a perception. We may have an image of the thing X and we may have a perception of it. But then, it might be asked, what about the difference between an image and a percept? Fortunately, we can point to one and only one feature of imagination that is sufficient to settle down that question.

The Shadowy Power of Absence

One way to learn the difference between percept (what is perceived) and image (what is imagined) is to take notice of the fact that we cannot have them simultaneously, at the same time and of the same object. They exclude each other, at least to a certain extent that can be described in terms of duration.

For this step I have taken my cue from Sartre. I quote two dictums from the *Psychologie phénomenologique de l'imagination* (16, 9f.):

> [--] the characteristic of the intentional object of the imaginative consciousness is that the object is not present.

> No matter how long I may look at an image I shall never find anything in it but what I put there. It is in this fact we find the distinction between an image and a perception.

It is true that Sartre has a complicated view of imagination. It may be odd, but it is by no means absurd. It is penetrated by thinking that appears also in his massive and almost incomprehensible work *Being and Nothingness*. Even his curious theory of art is derived from what these two dictums express; from the idea of the absence and not-being of the intentional object of imagination (cf. 211ff.). For him this is a paradigm case of intentionality in a strict Husserlian sense.

Above, in my discussion of the visual field, I referred to the faculty of attention. As I said, we can pay attention to an object present in our visual field or we can fail to notice it. Usually we fail because we did not pay attention. To a certain extent we can, as I asserted, choose between attending or not attending to the given object, but the object is not affected by our choice. As McGinn remarks in his inspiring discussion of attention, we do not have this choice in the case of images, for images "involve necessarily attentive intentionality" (2006, 26ff.). They are attention-dependent, or to quote McGinn, they are "attention greedy". What Sartre seems to have in mind in the quoted dictums is that we cannot have an image of an object that is present in our visual field and to which we are paying attention. Suppose I have an image of my son and suddenly he appears beside me. My image disappears just as suddenly because my attention is now focusing in full awareness on him as the bodily present person. Image and its object exclude each other when they occur simultaneously.

Why this is so becomes obvious in the light of the latter dictum I quoted from Sartre: Our images are forming in our imagination and we can find in them only what we put in them. For we have them only because there is no real object that regulates its content. Thus, it is the absence and not-being of the intentional object that powers the imagination. It is the absence of X that enables the move of our consciousness "from here to there",[10] from the real world to the world of imagination.

Here Sartre makes an intricate departure. As I remarked above, his view of perception takes notice of the body of the perceiver. That is true also of his view of imagination. When I see my son, my percept of him includes information of his spatiotemporal whereabouts. The image of my son, by contrast, does not specify any particular spatial relation to my, to the imaginer's body. In his clever discussion of Sartre's point of view, McGinn clarifies the idea to which Sartre is pointing at (2006, 29f.):

> We can put this by saying that perceptual consciousness contains a kind of double reference—to the object and to the perceiver's body while the imaginative consciousness makes no such reference. The "absence" of the imagined object is an indication that the body has "transcended" in imagination. [--] The "absence" of which Sartre speaks could as well be described as the absence of body from imaginative intentionality.

So, we cannot be both imagining and perceiving the same object at the same time. The image of an object and its perception exclude each other because our faculty of attention cannot work in two different ways simultaneously. Our perceptive faculty demands that its object is present in the actual visual field. Our images presuppose the absence of the object in order to exist at all.

This chapter had to assert that there is a sharp distinction to be made between what we imagine and what we perceive. Now I must amplify this view in an essential point and retract what I said above.

[10]The expression has a strong association for me: Plotinus uses it in his discussion of imagination, see Plotinus (1984, 237ff).

Imaginative Seeing

There is a special way of seeing, which has been much debated since Wittgenstein discussed it in an interesting and somewhat obscure chapter of his *Philosophical Investigations*. In the wake of Wittgenstein it has been mostly described as *seeing-as* or sometimes as *aspect seeing*. We are already familiar with the phenomenon itself from the aforementioned Husserl's theory of intentionality and the distinction between the object *which* is intended and the same object *as* it is intended or taken.

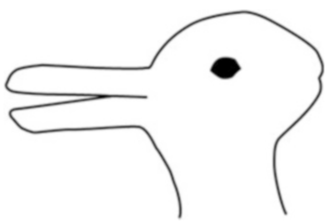
There are several cases in which we can speak of "seeing-as". In his earlier work about imagination Roger Scruton (1974, 107ff.) distinguished (I suppose exhaustively) the different locutions, but I am being concerned with cases that seem to belong to the same category, or, to be sure, to the same type of seeing. It has been called *imaginative perceiving* by Scruton in a later book (1979, 71ff.) and more recently (mutatis mutandis) *imaginative seeing* by McGinn (2006, 47ff.). I will adopt the latter mode of expression. Imaginative seeing is—to quote McGinn—a "hybrid form" of seeing in which we have both an image and a perception of the same object.

Wittgenstein reproduced in his *seeing-as* chapter a drawing of a triangle, a little tilted, as I recall, and made the famous remark according to which it can be seen as a triangular hole, as a solid object, as a geometrical drawing, as hanging from its apex, as a mountain, as a wedge, as an arrow, "and as various other things" (1958, 200f.). The concept of *seeing-as* is thus introduced by enumerating many possible interpretations we can give to the triangle as a result of some exercising of our imaginative faculty. Wittgenstein commented his example by writing (1958, 207):

> The aspects of the triangle: it is as if the image came into contact, and for a time remained in contact, with the visual impression.

Wittgenstein's discussion of *seeing-as* aroused initially out of his attempt to deal with the problem of optical illusions and ambiguous figures such like Jastrow's duck/rabbit drawing. In it the drawing of the duck transforms into a drawing of the rabbit but it is impossible to see the drawing as both a rabbit and a duck simultaneously (Fig. 2). Wittgenstein argued convincingly that in the visual experience of such a drawing the faculty of imagination and the perception are fusing into a compound that remains unstable and cannot be "interpreted" in only one way.

Fig. 2 The duck/rabbit-figure. Illustration by the author



Architecture as Visual Poetry

Using this promising piece of advice I want to suggest that in the duck/rabbit drawing we have a paradigm for what happens when something is visually implied by a perceived object.

Numerous theorists of both philosophical and empirical aesthetics have assured that works of art are inherently ambiguous and "open", not "closed" by one meaning, "reading", or interpretation. A great deal of attention has been devoted to this phenomenon of opacity or indeterminacy, especially since the beginning of the modern movement in art, architecture and design. Much of such indeterminacy is certainly due to the same phenomenon that we experience in the fusion of the components of ambiguous figures.

Discussing the notion of imaginative perceiving Scruton writes about the tracery of the Palazzo Pisani Moretta in Venice (1979, 86, 90ff.):

> [--] imaginative experiences—unlike ordinary perceptions—may be inherently ambiguous [--] *The arrangement may be seen in at least two ways, depending upon how the observer directs his attention* [--] Once aware of the ambiguity he can readily alternate the two interpretations (just as with the famous ambiguous figures investigated by Gestalt psychologists).

So far I would be willing to side with Scruton and I really find his writings about aesthetic experience inspiring. However, I have another idea in mind, an idea that I hope will explain what I have called visual implication.

Thus, unlike Scruton, I do not assert that one can "readily alternate the two interpretations" of an ambiguous figure. On the contrary, I am asserting that visual implication is something that happens to us and cannot be called up at will—or can just as little as the elements of the ambiguous figure. Unlike Scruton, I do not assert either that an ambiguous figure serves as a model of deciding between two or more equally possible interpretations, although I have nothing against this view. But the ambiguity inherent in works of art is not the phenomenon I am dealing with in this paper. I am interested in another topic, the visual implication and its role in the classical theory of proportions. It is in this sense I want to suggest that an architectural *masterpiece* appears to our consciousness with double faces, very much like an ambiguous figure, in the mode of the fusion of a percept and an image of it.

I will now introduce a new concept, the "imaginative counterpart" in order to designate the alternatives of an ambiguous figure. In the duck/rabbit drawing the counterpart of the duck is the rabbit and *vice versa*. In Necker's cube the foreground and the background are counterparts of each other. In an ambiguous figure of the Gestaltists, one imaginative counterpart transforms into another and it is impossible to see them simultaneously. I would like to describe the occurrence of such a fusion of percept and image temporarily as *swinging,* for it *is* some kind of swinging to and fro.

I now attempt to sketch a model of how the visual implication is working in an architectural object. So let us imagine what it would be like to construct a two-layer model consisting of a pair of imaginative counterparts.

Le Corbusier was an artist. He didn't want to design just buildings. He wanted to design buildings that are art, which for him meant objects that are pleasant and exciting to see: visual poetry that is able to get the spectator into a swing while looking at a mass of reinforced concrete. There is nothing mysterious in that, for it means that one has to work a piece of reinforced concrete into something that has the capacity of becoming ambiguous. So the *Maison La Roche* and the *Villa Stein* became the faces of Janus: the two layers work as imaginary counterparts. Both layers appear to the consciousness of the spectator as sensuous pictures:

1. A perceptual layer of picture-like representation (percept) of the minutely detailed sensuous content cast in the concrete shell, the "mass of masonry", related to the spectator's body and seen with the bodily eye.
2. An imaginary layer of a picture-like representation of the geometrical two-dimensional schema, very much like the architectural drawings, but appearing only as an image to the spectator's eye of the mind, and essentially without relatedness to the body of the spectator.

According to this model we experience both layers not simultaneously, but alternating between an image and a percept. The layers are "given" to the spectator's consciousness as imaginative counterparts that *evoke* each other swinging "forwards and backwards",[11] the percept vanishing from the foreground of the visual field into the background, the image vanishing from the imaginary space to the nowhere of "unattendedness". The percept has its object in the building that is real and present, the image has its object in the very mode of an intentional correlate only, as something to which the imaginative seeing is directing itself, but that is present only in the mode of absence and not-being as Sartre suggested in his theory of imagination. If he was right and we can imagine that the image is in no bodily relation to the spectator, it is *as if* it were an idea in the intelligible heaven, or what Russell, in his funny *History of Philosophy*, called the Heaven of the Platonists.

I would, however, like to follow Sartre's advice and speak not of passing from one world to another in the manner of Plotinus. I would like to think, it is "only" a passing from our day-to-day attitude to the aesthetic attitude (cf. Sartre 1966, 213 ff.). But that is another story.

A Note on Geometry

Imagination plays an important role in our experiences of geometric entities. First, we cannot perceive them in the proper sense of perception because they do not exist in the actual world. In the actual world we have only things that approximate geometric entities, round tellers rather than circles. As I said above, imagination is the faculty of mind that enables us to conceive the world in other ways than it is presented in

[11]The imaginative counterparts form together what is analyzed in Phenomenology as an *appresentative pair*. See my paper *Form and Counterform* (Usvamaa-Routila 2003, 6ff.). About appresentation see Schutz (1971, 278ff).

perception. Thus, we may assume that geometry takes roots in the imagination. Second, our experiences of geometric entities clearly contain both sensory and conceptual ingredients and our concern with them may therefore presuppose a distinction between sensory and cognitive imagination and their collaboration for our mind's "flight from the world of perceived reality", to quote Colin McGinn from whom I have adopted the notion of cognitive imagination (cf. McGinn 2006, 128–138)—a captivating topic which I have to leave at that in this paper.

However, we do encounter geometric entities not only as mathematical items of the science of geometry. They are also intimately connected with the expressive power we experience frequently in visual arts, notably within the framework of architectural line-drawings. In the mathematical sense lines and diagonals of rectangles do not have a direction, they do not point to anything, lines do not run from bottom to top nor from top to bottom, nor horizontally from left to right or *vice versa*. However, our mind has both the ability and the inclination to "read" the neutral mathematical properties as having these aesthetic features. There must be a good reason why we are able and also inclined to do so. Unfortunately, we do not know what this reason is. At least some principles has been discovered by the *Gestaltists* and, in a more intuitive way by such masters of modern art as Wassily Kandinsky (*Punkt und Linie zu Fläche)* and Paul Klee (*Pedagogisches Skizzenbuch*, 1925). The theoretical interest in the expressive powers of geometry has a long history that can be traced back at least to the mathematical doctrines of the Pythagorean Brotherhood in ancient Greece. A promising new approach can be found in: Georg Lakoff and Rafael E.Nuñes: *Where mathematics comes from* (New York 2000), 29ff.

The problem of geometry is, however, much more complicated. Although the science of geometry comes to us ready made, it must have appeared once in the history for the first time when it was created. Geometry can thus be seen as a cultural heritage available to us through tradition. In his famous essay on *The Origin of Geometry,* Husserl argued that in order to be available to us in this way of a ready-made, geometry must have an origin in a "primal establishment" (Urstiftung). The science of geometry and our experiences of the expressive powers of geometric entities may thus be equally basic, the prime difference being that they belong to different categories: expressions are emotional and arise feelings, science of geometry is intellectual and arise *discourse*, as might be said, perhaps, for brevity. An excellent analysis of Husserl's cultural theory is still David Carr: *Phenomenology and the Problem of History* (Northwestern University Press 1974), on Husserl's *Origin of Geometry* cf. 202ff.

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The Unreasonable Attractiveness of Mathematics to Artists and Scientists

Axel Gelfert

Abstract Mathematics, not least in popular writings on the topic, is often credited with a special affinity to beauty. For example, mathematical theorems are sometimes described as 'beautiful', or a particular proof may be deemed more 'elegant' than another. But it is far from clear how mathematical objects such as theorems or proofs could function as bearers of aesthetic value. Thus, an air of mystery surrounds these invocations of 'mathematical beauty', its source and cognitive function. The sense of mystery is compounded by the fact that such aesthetic judgments are not exclusively the preserve of mathematical experts, but have also proved attractive to non-experts, including artists. This 'unreasonable attractiveness' of mathematics suggests that, perhaps, there is more to the appreciation of the aesthetic dimension of mathematics than initially meets the eye. The present chapter argues that there are at least three ways in which mathematics, over the past century, has attracted the attention of artists: as tool, as subject matter, and as ideal. It is the latter—mathematics as intellectual ideal and practice—which, it is argued, explains why interest in the relation between mathematics and art reached considerable levels towards the middle of the twentieth century. At the same time, sparked by a growing interest in the role of mathematical models, philosophers of science increasingly turned their attention to mathematics and its cognitive function. Following a review of extant notions of beauty and elegance in mathematics, it is argued that the 'beauty' of an elegant mathematical proof may not reflect any intrinsic feature of a proof, but rather its performative success in establishing the truth of its conclusion. In what has sometimes been called 'aesthetic induction', the very signs of epistemic success (e.g. the mathematical features of empirically successful scientific theories) are being re-interpreted as signs of 'beauty'. Artists, too, have valued mathematics not for its subject matter per se, but for its perceived rigour as an intellectual practice; this is illustrated using examples from a range of artists across the ages. The final three sections of the chapter aim to establish a link between Wigner's puzzle—concerning 'the unreasonable effectiveness of mathematics in the natural sciences'—and the recent philosophical debate about mathematical models. The chapter concludes by reflecting on a convergence (of sorts)

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between art and science, as both have moved away from a simplistic understanding of the goal of representation and have instead looked to mathematics as an anchor of artistic and scientific practice.

Introduction

The notion that mathematical objects—proofs or theorems, say—can be bearers of aesthetic value has long enjoyed some currency among mathematicians and philosophers of mathematics. Not least in popular discussions of mathematics, it is not uncommon to come across references to the perceived 'beauty' of a theorem, or the superior 'elegance' of one proof as compared with another. An air of mystery surrounds these invocations of 'mathematical beauty', its source and cognitive function—not least since one would expect such aesthetic judgments to require a considerable degree of familiarity with mathematics itself. Yet, as the attractiveness of mathematics to many twentieth-century artists shows, even non-experts can appreciate the aesthetic dimension (and, in the case of science: epistemic value) of mathematics. This suggests that, perhaps, there is more to the attractiveness of mathematics to artists and scientists than initially meets the eye. In the present chapter, I argue that there are at least three ways in which mathematics, over the past century, has attracted the attention of artists: as *tool*, as *subject matter*, and as *ideal*. It is the latter—mathematics as intellectual ideal and practice—which, I argue, explains why interest in the relation between mathematics and art reached a new peak towards the middle of the twentieth century. Moreover, I argue that a similar shift in the perspective on mathematics and its cognitive function occurred in philosophy of science, sparked by a growing interest in the role of mathematical models.

The rest of this chapter is organised as follows: The second section reviews notions of beauty and elegance in mathematics and, via a discussion of what has been called 'aesthetic induction', concludes that the 'beauty' of an elegant mathematical proof may not reflect any intrinsic feature of a proof, but rather its *performative success* in establishing the truth of its conclusion. The third section looks at the attractiveness that mathematics has exerted on artists—from Dürer to Bragaglia and El Lissitzky—many of whom have valued mathematics not for its subject matter *per se*, but for its perceived rigour as an intellectual practice. The fourth section develops a parallel strand of the debate, by reviewing what Eugene Wigner in an influential eponymous paper has called 'the unreasonable effectiveness of mathematics in the natural sciences' (Wigner 1960). 'Wigner's puzzle', as it has become known, concerns nature's apparent amenability to mathematical ways of describing and explaining it. In a less 'global', but no less acute way, this question also arises for mathematical models in science. One influential way of thinking about scientific models, to be discussed in the penultimate section, is in terms of positive and negative analogies between the model and the target system which it purports to describe. The chapter concludes by reflecting on a convergence (of sorts) between art and science, as both have moved away from a simplistic understanding of the goal of representation and have instead looked to mathematics as an anchor of artistic and scientific practice.

Beauty and Elegance in Mathematics

In *A Mathematician's Apology*, first published in 1940 and since reissued and translated into various languages, G. H. Hardy provides the classic statement of a view that places aesthetic considerations at the heart of mathematics:

> The mathematician's patterns, like the painter's or the poet's must be *beautiful*; the ideas like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics. (Hardy 1992, 85)

What mathematical beauty is 'may be very hard to define', Hardy continues, 'but that is just as true of beauty of any kind'. Instead of offering a comprehensive analysis of the concept of 'mathematical beauty', Hardy attempts to defend mathematics against its detractors and vindicate it in the eyes of those who are merely ignorant of it. In particular, he explicitly sides with fellow mathematician (and philosopher) Alfred North Whitehead, who, in his book *Science and the Modern World*, lamented the dominance of an 'erroneous literary tradition which represents the love of mathematics as a monomania confined to a few eccentrics in each generation' (1926, 26). It is perhaps not by chance that later editions of Hardy's *Apology* featured an extensive foreword by C.P. Snow, who—apart from being friends with Hardy and, therefore, eminently qualified to contribute a largely biographical introduction—shared a keen awareness of the drifting apart of the 'Two Cultures', viz. the 'traditional' humanities and the sciences. Whereas widespread familiarity with the former was ensured, not least via school curricula with their emphasis on literature and the arts, awareness of the latter was largely confined to professional circles of practising scientists and technologists. Hardy, too, noted that, whereas lack of an appreciation or capacity for the arts was widely regarded as 'mildly discreditable', most educated people would readily admit to, or even exaggerate, 'their own mathematical stupidity' (Hardy 1992, 87). For practicing mathematicians who, ever since Galileo, could claim that the very 'Book of Nature' was written in the language of their discipline, this state of affairs must have naturally seemed unsatisfactory—quite apart from worries about the political dangers of an opening rift between the 'traditional' intelligentsia and the new techno-scientific elites.

Whitehead attempted to re-establish common ground between the 'literary' and the 'mathematical' cultures by pointing out that, after all, both language and mathematics trade in abstraction and analysis. The 'history of the development of language', says Whitehead (1926, 31), is 'a history of the progressive analysis of ideas'. Mathematics, on this account, is simply 'a resolute attempt to go the whole way in the direction of complete analysis, so as to separate the elements of mere matter of fact from the purely abstract conditions which they exemplify'. For Whitehead, the 'direct aesthetic appreciation of the content of experience' (ibid.) is merely the first step in a sequence of abstractions, leading to 'complete abstraction from any particular instance' (1926, 27). It is in this respect that mathematics differs from 'certain forms of literary art', which reject analysis and abstraction in favour of a creative enrichment of its basic terms—for example by 'compact absorption of auxiliary ideas into the main word' (Whitehead 1926, 31). However, any gulf that is perceived to exist between the literary and the mathematical—and, by extension, the scientific—cultures, is not the result of any intrinsic antagonism between the two, but is merely a side effect of the tension between our desires to represent both the world in its generality and each object in its specificity. Whereas Whitehead implied that mathematics and language draw from the same source, Hardy suggested that the perceived gulf between mathematics and 'traditional' culture may be bridged by an appeal to their shared concern with beauty.

What are some of the items that mathematicians regard as (potentially) beautiful? Certain theorems, no doubt. Thus, Gian-Carlo Rota (2005, 4) writes: 'The theorem stating that in three dimensions there are only five regular solids (the Platonic solids) is generally considered to be beautiful.' Other examples cited by Rota include the prime number theorem (regarding the distribution of primes) and the Weierstrass approximation theorem (which states that a continuous function on a closed and bounded interval can always be uniformly approximated by polynomials, to any arbitrary degree of accuracy); to this, one might add—for the benefit of the mathematical layperson—theorems that are easy to grasp, yet difficult to prove, such as Fermat's Last Theorem, which states that no positive integers a, b, and c can satisfy the simple equation $a^n + b^n = c^n$, for any n > 2. Proofs, too, can be beautiful. Indeed, the Hungarian mathematician Paul Erdős used to claim that God had a special book (which Erdős simply referred to as 'The Book') that contained the most elegant proof for each mathematical theorem.[1] Then there are whole mathematical theories, such as Galois's theory of algebraic equations, which used the novel idea of permutation groups to describe how the various roots of a given polynomial equation relate to one another, and why their solutions take the form that they do. (The beauty of Galois's theory is perhaps only surpassed by the drama associated with the fate of its discoverer—Evariste Galois was killed in a duel in 1832, at the tender age of twenty.) What about the beauty of mathematical conventions? Rota, for one, appears to be in two minds about this. On the one hand, he acknowledges that, although 'it is not uncommon for a definition to seem beautiful', mathematicians tend to be 'reluctant to admit' its beauty; on the other hand, he grants that 'axiom systems can be beautiful'. (Rota 2005, 4–5.) What seems clear, though, is that these different kinds of mathematical beauty can come apart. Not every proof of a beautiful theorem is itself beautiful—just think of the convoluted proof, by Andrew Wiles (1995), of Fermat's Last Theorem—and mere elegance in presentation does not necessarily imbue mathematical theories with beauty:

> The mixed blessing of an elegant presentation will endow the theory with an ephemeral beauty that seldom lasts beyond the span of a generation or a school of mathematics. (Rota 2005, 6)

Hardy (1992, 89) had similarly noted that the beauty of a mathematical theorem had to be matched by what he called 'seriousness'—where the latter was to be assessed 'not in its practical consequences, which are usually negligible, but in the significance of the mathematical ideas which it connects'.

[1] Erdős's idea has even given rise to an edited volume titled *Proofs from THE BOOK*, now in its fourth edition (see Aigner and Ziegler 2010).

If mathematical beauty attaches differently to different kinds of mathematics—theorems, proofs, definitions, and theories—it may seem not all that far-fetched to think that there may be no such unitary thing as 'mathematical beauty' in the first place. Indeed, as James McAllister has noted, while 'there is wide agreement among mathematicians at any time about the mathematical entities that merit the predicate "beautiful" [--], the community's aesthetic tastes change in time' (McAllister 2005, 16). The same, of course, is true of the literary and fine arts. Changing standards of taste, one might argue, do not tell against the unity of the concept of beauty, whether in the arts or in mathematics. Against this suggestion, I wish to suggest that, in the case of 'mathematical beauty', we have additional reasons for suspecting a conceptual disunity. A clue is contained in the way in which 'beauty' and 'elegance' tend to be run together in discussions of the aesthetic merits of mathematics. Perhaps some mathematical objects—geometrical patterns, symmetrical shapes, algebraic equations—do indeed have an intrinsic aesthetic appeal, but the 'beauty' of an elegant proof reflects not any intrinsic feature of an object, but its *performative success* in establishing the truth of its conclusion. Similarly, 'beautiful' mathematical theories and axiomatizations do not impress in virtue of any intrinsic aesthetic qualities, but in virtue of their exhibiting other theoretical desiderata, such as generality, explanatory depth, economy, coherence, and—to use Hardy's expression—overall 'seriousness'.

The Artist as Mathematician

The relationship between aesthetics and mathematics is a mutual one. Just as mathematicians regularly invoke aesthetic considerations to justify what it is they are doing, so artists have turned to mathematics for inspiration and vindication. More specifically, I wish to distinguish between three distinct uses of mathematics by artists: mathematics as *tool*, as *subject matter*, and as *ideal*.

A good illustration of the first kind—the *instrumental* use of mathematics—is the development of the perspective technique in European painting in the fifteenth and sixteenth centuries. One of its major proponents and practitioners, Piero della Francesca, in his *De Prospectiva Pingendi* (ca. 1482), makes a case for the use of perspective as follows:

> Painting is nothing but a representation of surfaces and bodies diminished or enlarged [--] It is necessary to use perspective, which detects proportions as real science and determines the increasing or the diminishing of the various quantities by means of straight lines. (Cited after Geatti and Fortunati 1993, 207)

While not the first discussion of the principles of perspective *per se*—detailed discussions were widely available in texts on optics and geometry—Piero's *De Prospectiva Pingendi* was, as Vincent Ilardi puts it, 'the first systematic mathematical treatise *for painters*', resembling a 'shop manual' more than a traditional humanist treatise on the subject of painting: 'It is full of calculations and instructions for perspectivist constructions without the use of mirrors—just mathematics and geometry.' (Ilardi 2007, 197; emphasis added.) Painting, in effect, was portrayed as a form

Fig. 1 William Hogarth, *Satire on False Perspective*, 1753 (Frontispiece to Joshua Kirby's *Perspective of Architecture*, London: R. Francklin, 1761; public domain)

of 'applied mathematics', yet it continued to stand in the service of the artist's representational goals. While the geometrical characteristics of (esp. early) perspectival paintings are often salient, the artist retains considerable control over the composition and content of his work, not least through the choice of vanishing points and depicted scene. Mathematics, like any tool, imposes certain constraints on the artistic process, but it need not determine its outcome and the artwork's overall effect (except on occasion, as demonstrated by Hogarth's famous *Satire on False Perspective*, 1753; see Fig. 1).

When mathematics is made the *subject matter* of artistic representation, it ceases to function as a mere tool and no longer blends in with the overall compositional structure of an otherwise representational image. The depiction of mathematical objects in Albrecht Dürer's famous 1514 engraving *Melencolia*—which shows a sphere, a magic square, a geometrical solid, as well as manifestations of mathematics in nature, such as a rainbow—lends the image an enigmatic and allegorical quality that sets it apart from ordinary representational scenes (see Fig. 2). In an almost ironic twist, even though the picture explicitly represents mathematical objects, its own perspectival composition is not at all easy to grasp.

The proliferation of explicit representations of mathematical objects in modern art correlates with Modernism's growing 'interest in visualizing contemporaneous mathematical ideas' (Corrada 1993, 235). However, such representations of mathematics no longer serve purely allegorical or merely illustrative purposes, but are part of a broader movement that treated mathematics *as an intellectual ideal*. Rather than treat mathematics as a mere instrument for faithful representation, it is its rigour, along with its ability to transcend immediate sense perception, which now makes mathematics attractive to artists in search of new aesthetic principles. The very idea of faithful representation is placed under general suspicion, and with it any artistic techniques that are seen as perpetuating this idea. When Filippo Tommaso Marinetti, in *The Founding and Manifesto of Futurism* (1909), writes that 'sick lamplight through window glass taught us to distrust the deceitful mathematics of our perishing eyes' (Marinetti 2001, 186), this may well be understood as an attack on the idea that our intellectual techniques of construction should always be subordinate to our limited sensory abilities.

The distrust of imagery has an interesting parallel in twentieth-century mathematics itself, notably in the movement, from 1934 onwards, of the group of (mainly French) mathematicians who wrote under the pseudonym Nicolas Bourbaki and who made it their goal to give a rigorous set-theoretical reconstruction of modern mathematics—one that would render mathematics autonomous from both science and society. As James Gleick describes the goals of Bourbaki:

> Mathematics was mathematics—it could not be valued in terms of its application to real physical phenomena. And above all, Bourbaki rejected the use of pictures. A mathematician could always be fooled by his visual apparatus. Geometry was untrustworthy. Mathematics should be pure, formal, and austere. (Gleick 1987, 89)

Jean Dieudonné, one of Bourbaki's first convenors, echoes this attitude in the preface to a textbook on *Linear Algebra and Geometry*, in which he purposefully takes 'the liberty of omitting all diagrams from the text, if only to show that they are unnecessary' (1969, 13)—somewhat facetiously leaving it to his readers to repair this omission for themselves 'if they wish'.

Such distrust of pictorial representation, however, was more than mere iconoclasm; it was paired with an ambitious attempt to free the creativity of inquiring minds from the arbitrary constraints of our sensory apparatus. Scepticism about the traditionally privileged position of sense perception did not—at least not always—give rise to an unbridled embrace of technology and its artifacts. The use of technology and its

Fig. 2 Albrecht Dürer, *Melencolia*, 1514 (Reproduced from Erwin Panofsky, *Albrecht Dürer*, Vol. 2, Princeton: Princeton University Press 1943)

theoretical correlates—such as replicability, precision, and accuracy—required skill and artistic imagination, lest artists repeat the same mistakes as previous generations. A good example is the adoption of photography as a means of artistic expression. While it afforded new ways of representing the world—for example through chronophotography, which captures complex movements in a quick succession of exposures, which may then be superimposed in a single frame—some artists, such as Anton Giulio Bragaglia, were growing impatient with what they saw as a misguided attempt to provide a 'precise, mechanical, icy reproduction of reality' (Bragaglia 2008, 369). What was needed, instead, were new ways of characterising the 'shape of movement'—conceived of not as a succession of time slices, but in terms of how it impinges on human consciousness. Bragaglia's photodynamism attempted just that, by going beyond the traditional 'algebra of movement', identifying the 'anatomy' of each movement—that which underlies its overall character—which in turn was then revealed and represented 'as complex, raising it to the level of an infinitesimal calculation of movement' (Bragaglia 2008, 372). Bragaglia's rhetoric conveys a sense of the degree to which mathematical ideals informed and pervaded modernist practice and self-image.

Mathematics, of course, was but one of many factors that motivated artists of various persuasions in the twentieth century, yet it proved to be a remarkably resilient influence. In particular, its attractiveness cut across political and ideological divides. It inspired Soviet artists, such as El Lissitzky who, in 1925, had gleefully noted that 'in the period between 1918 and 1921 a lot of old rubbish was destroyed'[2] (thus freeing up space for artworks that interpreted the new findings of science and mathematics); however, it also provided a focal point for Futurists like Marinetti, who had at one point lobbied Mussolini to make Futurism the official art form of fascist Italy, and who developed the idea of an *Imaginative and Qualitative Futurist Mathematics* (1941), which was at once 'antistatic antilogical antiphilosophical' (Marinetti 2009, 298), hostile to 'narcotic symmetry', and a strangely atavistic calculus of sorts ('one can multiply health by pride muscles by joy'). After the end of the Second World War, mathematics provided a neutral safe haven for those who, like Max Bill in his influential 1949 essay *The Mathematical Way of Thinking in the Visual Art of Our Time*, argued that 'we have good reason to be skeptical about any "political art" [--] because this is not art at all but simply propaganda' (1993, 6). Instead, Bill proposed 'a new form of art in which the artist's work could be founded to quite a substantial degree on a mathematical line of approach to its content' (1993, 7)—with mathematics fulfilling a dual role as the provider of 'some degree of stability' and the source of 'fresh content', which would 'launch us on astral flights which soar into unknown and still uncharted regions of the imagination' (1993, 9).

Wigner's Puzzle(ment)

The title of the present chapter is inspired by an influential paper by Eugene Wigner, a theoretical physicist who also contributed to the debate about the role of mathematics in science. In his paper 'The Unreasonable Effectiveness of Mathematics in the Natural Sciences', delivered at New York University in 1959 as the

[2]Quoted after Allen (2000, 17).

Richard Courant Lecture in Mathematical Sciences, Wigner developed a line of argument that has since become known as 'Wigner's puzzle', but that should perhaps be called 'Wigner's puzzlement'. The sense of puzzlement arises from the juxtaposition of two observations: first, the realisation 'that the enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious and that there is no rational explanation for it' (Wigner 1960, 2); second, the worry that it may be 'just this uncanny usefulness of mathematical concepts that raises the question of the uniqueness of our physical theories' (ibid.). Wigner does not conclusively resolve this tension, and his interpreters have taken different approaches.[3] It is interesting to note, however, that Wigner's puzzle(ment) also has an *aesthetic* component. For it is the question of which mathematical concepts are to be considered 'interesting', 'fruitful', or 'beautiful' that shapes advanced mathematics—unlike more elementary mathematics, which was still 'formulated to describe entities which are directly suggested by the actual world' (Wigner 1960, 2). Yet, as Sorin Bangu (2009, 245) has noted, it is not clear to Wigner why concepts that were not formulated 'with an eye to the physicist's needs' should still find applicability in physics. Why, one might ask, is mathematics not only beautiful, but also applicable to the messy world around us? And, moreover, why are some of our most explanatorily successful physical theories based on some of the most beautiful mathematics?

In contemporary philosophy of science, this line of thinking has turned towards the explanatory role of mathematics in the natural sciences. In particular, what is being debated is the question of what the specific contribution of mathematical truths is to the explanation of natural phenomena that exhibit salient mathematical features. Consider the following example from evolutionary biology (see Baker 2005): 'Periodical' cicada are well-known for their long developmental cycles, which last for several years. Three such species are known to share the same striking life cycle:

> In each species the nymphal stage remains in the soil for a lengthy period, then the adult cicada emerges after 13 years or 17 years depending on the geographical area. Even more strikingly, this emergence is synchronized among the members of a cicada species in any given area. The adults all emerge within the same few days, they mate, die a few weeks later and then the cycle repeats itself. (Baker 2005, 229)

That biological phenomena display cyclical behaviour—at the level of ontogenetic development as well as at the level of populations (cf. Lotka-Volterra predator-prey dynamics)—is, of course, well-known and has to do with the exigencies of biological reproduction. What sets the current example apart, however, is the salient mathematical fact that the period of the life cycles are prime numbers. There is an intuitive explanation of why life cycles with prime periods confer an evolutionary advantage: they ensure that life cycles between different cicada populations (as well as between cicadas and predators that feed on them) will only rarely overlap—thus reducing competition for limited resources and the threat from

[3] See for example Gelfert (2014).

predator species. Yet this explanation appears to rely crucially on a number-theoretic truth about prime numbers—which is not itself derived from any physical or biological law of nature. This suggests that mathematics is not merely a convenient way of representing law-like regularities and other phenomena in nature, but that it is indispensable: Mathematical truths are both essential to the success of many explanations, and their contribution is of genuinely explanatory value.

The question of whether mathematics can make novel explanatory contributions to the study of empirical phenomena may, at first sight, seem far removed from considerations of mathematical beauty *per se*. Interestingly, however, a case can be made that the two questions are deeply intertwined. Recall that, according to Hardy, mathematical theorems are beautiful if they exhibit the characteristics of generality, depth, economy, coherence, and overall 'seriousness' (see Section "Beauty and Elegance in Mathematics"). Several of these criteria—generality, depth, coherence and economy—have also been invoked by philosophers of science as desiderata of scientific theories and, all else being equal, as offering good heuristic criteria for theory choice. Even if the interpretation of the criteria varies across the two disciplines—what determines the 'economy' of a mathematical theorem may differ significantly from what makes something an 'economical' scientific theory—the parallels between the two sets of criteria are striking. Indeed, it has been argued, notably by McAllister (2005), that in actual scientific practice the two sets of criteria are often run together, for example when physicists characterise their scientific theories in terms of their perceived beauty. This gives rise to what McAllister (2005, 28) calls a process of 'aesthetic induction' whereby 'scientists at a given time attach aesthetic value to an aesthetic property roughly in proportion to the degree of empirical success scored up to that time by the set of all past theories that exhibit the property'. Evidence of such shifts in the aesthetic evaluation of scientific theories may be gleaned from the history of quantum physics. Whereas many mathematical physicists initially resisted the new quantum theory on the grounds that it made essential use of abstract mathematical entities that could not properly be visualised, once the theory moved from one empirical success to another, its very abstractness came to be seen as a perhaps austere, yet undeniable aesthetic quality (McAllister 2005, 29). What this suggests is not so much that the beauty of scientific theories is 'in the eyes of the beholder', but rather that collective attributions of aesthetic value, over time, tend to track the aesthetic properties of empirically successful theories—whatever these may be. While this may, on occasion, license inferences from the (perceived) mathematical beauty of theories to their likely empirical success, such inferences will always have to be tentative: As with simplicity, there is no guarantee that mathematical beauty, in and of itself, is a reliable guide to the truth of empirical theories.[4]

[4]For an extended argument that the truth need not be simple, see Bas van Fraassen's *The Scientific Image* (1980).

Models and Analogies in Art and Science

It might seem natural to conclude from the foregoing discussion that a *general* account of why nature is amenable to mathematical description is too ambitious: perhaps the question of whether the very 'fabric of the world' is fundamentally mathematical in nature is simply one of those that are beyond the grasp of finite beings like ourselves. The general thought that certain aspects of reality are in principle unknowable has, of course, a long ancestry in philosophy; in recent philosophy of science, e.g. in the work of Nancy Cartwright, this thought has sometimes taken the form of the assertion that the very notion of a 'fundamental theory'—that is, a unified scientific account of the physical world—may itself be incoherent and merely a symptom of misguided methodological prejudices.[5] Rather than 'eye-balling' a scientific phenomenon 'to see what can be abstracted from it that has the right form and combination' to serve as input to reputedly 'fundamental' theories (Cartwright 1999, 247), scientists are advised to focus on *phenomenological* models that adequately capture the phenomena in all their empirical fullness, without obsessing about reducing them to fundamental 'first principles'. Thus, phenomenological models will often be formulated in terms of observable (e.g., macroscopic) variables rather than unobservable (e.g., microscopic) mechanisms.[6]

It is important to realise, however, that the move from 'global' fundamental theories to more 'local' phenomenological models does not suffice to sidestep the sorts of questions discussed in the previous section. Even if it renders the question of nature's amenability to mathematics *in general* less pressing, there still remains the puzzle of how mathematics applies to specific cases (or classes of cases). Indeed, this is why there has been considerable recent debate about the role of mathematics in achieving model-based scientific representation.[7] Scientific models, of course, come in many shapes and forms, and mathematical models form but one subclass—albeit an important one—alongside a plethora of material models, theoretical models, model organisms, toy models etc.[8] Few philosophers have done more to rehabilitate the standing of models in the philosophy of science—and of mathematical models, in particular—than Mary Hesse. Best-known for her view of models as analogies (Hesse 1963), Hesse has also contributed to an analysis of the role of mathematics in physics more generally. Thus, in her 1953 paper 'Models in Physics', Hesse sets out to defend the thesis that '*most physicists do not regard*

[5]See especially her 1999 book *The Dappled World: A Study of the Boundaries of Science*.

[6]Cartwright's main example is the case of superconductivity, where phenomenological models (such as the Ginzburg-Landau model, which examines only the macroscopically observable quantities) compete with microscopic models (such as the BCS model, which posits specific mechanisms at a quantum level). See also Gelfert (2016, 45–58).

[7]See especially Hughes (1997) and Gelfert (2011b).

[8]For a survey of the various types of models and the varied functions of scientific modeling, see Gelfert (2016).

models as literal descriptions of nature, but as standing in a relation of analogy to nature' (Hesse 1953, 201; italics original). Her defence of this claim sparked a long-lasting debate over the analogical character of models. Why might one be tempted to regard models as analogies in the first place? Central to the analogical view is the recognition that models function as substitutes for more complex target systems under investigation. Some properties are found to apply to both the model and the target system; others belong to the model but not to the target system that is being investigated. The former set of properties constitutes a 'positive analogy' between model and target system; the latter amounts to a 'negative analogy'. (Swinburne 2007, 57) There will typically be further ('neutral') properties of the model, whose status with respect to the target system is unknown: for all we know, they may or may not be instantiated in the target system—only future investigation can tell. As Hesse puts it, 'the important thing about this kind of model-thinking in science is that there will generally be some properties of the model about which we do not yet know whether they are positive or negative analogies, and these are the interesting properties' (Hesse 1963, 9). In particular, it seems reasonable to assume that the greater the number of positive analogies between model and target, the more we are justified in thinking that the neutral properties of the model also extend to the target system. Richard Swinburne, by way of summarising Hesse's views, discusses the example of the wave equation:

> Thus water waves exhibit properties of diffraction, interference and reflection. So too do sound and light. These constitute the positive analogy of sound and light waves to water waves. Also like water waves, sound is produced by motion of a source [--]. Sound, like water, is transmitted by a medium—air—whereas light (despite initial hypotheses to the contrary) is not. (Swinburne 2007, 57–58)

As the example of the wave equation illustrates, even simple mathematical models—such as the wave equation—need to be understood against the backdrop of both positive analogies (e.g., the like occurrence of diffraction, interference, and reflection, in sound, light, and water) and negative analogies (the dependence, or not, on a medium for propagation of sound and light waves, respectively).

Hesse's early work must be understood against the backdrop of a tradition that largely equated scientific models with either material or mechanical models. On this view, which derives from nineteenth-century scientific usage, a typical scientific model is 'a (real or imagined) concrete, material representation of something' (De Regt 2005, 215). Obvious examples would include the billiard-ball model of an ideal gas, or Maxwell's vortex model of the ether. Yet, by the time Hesse embarked on her analysis of models and analogies, the term 'model' had already proliferated beyond the realm of the mechanical or 'picturable' to also include, for example, quantum-mechanical models. Indeed, it is this proliferation of the use of the term 'model' in actual scientific practice—perhaps best expressed by Hesse's statement that a model can be 'any system, whether buildable, picturable, imaginable, or none of these' (Hesse 1963, 21)—which led to a more thoroughgoing philosophical reassessment of the status of models in science. Subsequent accounts, for example the influential 'models as mediators' view developed by Mary Morgan and

Margaret Morrison (1999), gave an even clearer expression to this sentiment.[9] Thus, Morgan and Morrison emphasise both the constructedness of scientific models and their heterogeneity; model-building as a scientific activity is characterised as 'not only a craft but also an art, and thus not susceptible to rules' (1999, 12), while models themselves are described as 'made up from a mixture of elements, including those from outside the domain of investigation' (1999, 23).

Two significant shifts in philosophical outlook are related to this recognition of the partial autonomy of models from both high-level theories and the sometimes murky world of empirical findings. The first of these concerns the status of scientific models as *epistemic tools*, which are themselves constructed or 'invented' (rather than simply discovered) and which often stand in need of continuous refinement. On this view, models—including mathematical models—are not simply theoretical objects that stand in some timeless abstract relationship to scientific theories, but their primary role is that of a means of inquiry; as such, they can be improved on by scientists and may become obsolete by a change in the goals of inquiry. The second shift is a renewed appreciation of the heterogeneity of the *formats and media* in which models are formulated and expressed. After all, in order for a model to properly function as a tool of inquiry, a model user must be able to access its fundamental properties in a way that makes them salient, for only by doing so can she hope to identify relevant positive and negative analogies. Often, as in the case of the wave equation, the medium will be the language of mathematics (which, of course, a model user must have successfully mastered), but equally often there will be other ways of accessing the content of a model. Consider how James Watson, in *The Double Helix* (1968), describes how hands-on manipulation of various material models played an essential role in eventually uncovering the DNA's geometrical structure:

> The α-helix had not been found by only staring at X-ray pictures; the essential trick, instead, was to ask which atoms like to sit next to each other. In place of pencil and paper, the main working tools were a set of molecular models superficially resembling the toys of preschool children. (Watson 1981, 34)

Tactile interaction with material models of chemical structures thus offered a mode of epistemic access that was qualitatively different from both visual inspection (of X-ray images) and analytical investigation. As Tarja Knuuttila has aptly put it, 'models are intentionally constructed and materially embodied things, *epistemic artefacts*, the constraints of which are characteristically turned into affordances for epistemic purposes' (Knuuttila 2005, 49). Given the ubiquity of models in science, along with their artefactual character, it is only to be expected that one should find instances of fruitful cross-fertilisation between art, mathematics, and science—even if these were not explicitly framed in terms of scientific models. Examples that spring to mind are Ernst Haeckel's idealised depictions of micro-organisms in his

[9]Other recent edited volumes and special issues have explored the role of three-dimensional models (see de Chadarevian and Hopwood 2004) and the relation between scientific practice and model-based representation (see Gelfert 2011a and references therein).

Kunstformen der Natur (1904), which represent exemplary life forms in a highly artistic manner, or the exploration, in art and architecture, of geodesic structures pioneered by R. Buckminster Fuller, which in turn inspired scientific research into the three-dimensional structure of viruses (and, much later, carbon nanostructures).[10]

Conclusion: From Representation to Practice

In this chapter, I have attempted to identify areas of overlap between art, mathematics, and science, both at the level of discourse and at the level of practice. Mathematicians, in talking about mathematics, sometimes explicitly invoke aesthetic notions—for example, when they speak of 'beautiful' theorems or 'elegant' proofs. It might seem, then, that the beauty of scientific theories—for example in physics—is entirely derivative, in the sense that science can at best hope to 'inherit' whatever intrinsic features confer aesthetic qualities on the mathematics employed. However, as the example of McAllister's 'aesthetic induction' shows, it is not at all clear that the criteria of mathematical beauty can be divorced from (epistemically motivated) criteria such as generality, explanatory depth, coherence, and other desiderata of scientific theory choice. Likewise, twentieth-century artists have looked towards mathematics as a source of inspiration not merely out of curiosity, or because of an unmediated attraction to its austere beauty, but because they were looking for new intellectual and artistic ideals. The realisation that faithful representation—where it can be achieved at all—requires considerable work (and, in many cases, may not even be the most worthwhile goal), became a commonplace in the second half of the twentieth century, both in art and science, yet it stopped neither artists nor scientists from searching for fixed points that could anchor artistic and scientific practice, while cutting across political and ideological divides. Mathematics provided such an anchor—not, as is sometimes assumed, in a 'timeless' manner, via the appeal to notions of symmetry or 'mathematical beauty' *simpliciter*—but by showing how one might strike the right balance between stability and dynamic change. As Max Bill put it, with respect to his prospective 'art based on the principles of mathematics':

> The art in question can, perhaps, best be defined as the building up of significant patterns from the ever-changing relations, rhythms and proportions of abstract forms, each one of which, having its own causality, is tantamount to a law unto itself. [--] It is a field in which some degree of stability may be found, but in which, too, unknown quantities, indefinable factors will inevitably be encountered. (Bill 1993, 8–9)

These words, first written in 1949, ring as true in relation to art as they do in relation to science. At the same time, they convey a distinct twentieth-century optimism about our ability to chart those (as yet unknown) regions of the

[10]For a review of the architectural history of geodesic dome structures and their influence on mid-twentieth century biology, see Tarnai (1996).

imagination. How the relationship between art, mathematics and science will unfold in our ever more complex (and infinitely more jaded) twenty-first-century world, remains to be seen.

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Interdisciplinary Application of Symmetry Phenomena

György Darvas

Abstract This chapter gives a short overview of a few possible contributions of interdisciplinarity to aesthetics, as the latter appears in the sciences. Symmetry phenomena play an important role in these considerations. Therefore, the chapter first introduces the most modern interpretation of symmetry. In this course there is discussed what is the common in the different geometric appearances of symmetry (e.g., mirror reflection, rotation, translation, similitude, etc.), and how do they appear in decorative arts. Symmetry, perfection and beauty were considered in close relation to each other since the ancient times. Then the title theme is exemplified by interdisciplinary applications of symmetry phenomena. At first, symmetry operations in decorative arts are presented in one dimension (frieze patterns), in two dimensions (wallpapers or tiling), with just a short reference to the beauty of crystals (three dimensions), then extended to the even coverage of surfaces (sphere and symmetric flat-faced polyhedra).

At second, there is mentioned that symmetry operations appear in algebra and arithmetic as well. The paper presents how the translation of the natural numbers and their sequences appear in aesthetic representations. This leads the reader over to the discussion of the so-called golden section in aesthetic terms. Then, we turn back to the perfect (Platonic) polyhedra as reflected in the so called golden section kaleidoscope. We explore step by step how the perfect proportions appear in the perfect bodies. We give also a short historical overview how the study of beauty of the symmetrically perfect bodies led to interdisciplinary applications in the sciences.

At third, examples are presented, how the discussed aesthetically outstanding proportions, shapes, are embodied not only in artworks, but also in recent scientific achievements. First of all, the discovery of quasicrystals is treated—that represent the so far “missing” golden proportional fivefold symmetry in concrete material structures. Then there are shortly mentioned also such successful new molecules—like the fullerene and the graphene—which is so important in nanoscience. These discoveries cited from the recent decades demonstrate productivity of interdisciplinarity, in terms of science-art relations. They led from aesthetic considerations (based on different appearances of symmetries) to realized scientific ideas.

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The Concept of Symmetry

Connotation of (scientific) truth with (aesthetic) beauty (or *vice versa*) goes back to *Timaeus* (circa 360 BC) from Plato, who repeatedly referred to his teacher, Socrates in this respect too. Copulation was established by the means of (our and nature's efforts to) perfection. This aimed perfection was embodied in good proportion, harmony, golden mean, common measure, consonance, rhythm, in short, what Plato's Greek contemporaries denoted with the common word *symmetry* (let us refer to the five perfect bodies described first by Plato). We know references (e.g., back till Polykleitos) that several books treated symmetry phenomena in the Hellenistic period, but the first, which survived the storms of centuries were *Ten Books of Architecture* by Vitruvius written a few decades BC. The Vitruvian descriptions on proportions and symmetry served as a basis for the *renaissance* of symmetry studies in the *quattrocento*, starting with Alberti (1435, 1436), through Ghiberti (1455) to della Francesca (1474), Pacioli and Leonardo (1494, 1509), Dürer (1525, 1528) (and to the synthesizer of the Renaissance art theory, Ripa later). The concept of symmetry used to be a link between scientific and art concepts in the fifteenth–early sixteenth century. This based its usages in the modern times, when science and arts separated for three centuries again (seventeenth–nineteenth).

Modern notion of symmetry has been elaborated in crystallography during the nineteenth century. We will return to this notion, founded mainly on geometric terms, in the next section. Now we explain how did this crystallographic notion develop into the contemporary concept of symmetry when the main point of its application was removed into, and distributed among other disciplines and the arts in the twentieth century.

What is common in the different geometric appearances of symmetry (e.g. mirror reflection, rotation, translation, similitude and others)?

– In each instance we performed some kind of (geometrical) *operation* (transformation).
– In this process, *one or more* (geometrical) *characteristics of the figure* remained unchanged.
– This characteristic proved to be *invariant under the given transformation* (did not change as a result of the operation performed).

This was the classical geometric concept of symmetry, elaborated till the end of nineteenth century. Science generalised this in such a way that the interpretation be valid not only for geometrical operations and geometric objects, and not just for geometrical characteristics. In a generalised sense, we can speak of symmetry if

– *in the course of any kind of* (not necessarily geometrical) *transformation* (operation)
– *at least one* (not necessarily geometrical) *characteristic of*
– *the affected* (arbitrary and not necessarily geometrical) *object remains invariant* (unchanged).

The generalisation, that is, took place with reference to three things:

– to any transformation,
– to any object,
– to any characteristic.

The first generalisation made it possible for us not just to look for unchanged characteristics during geometrical *operations* we have learned (reflection, rotation, translation, etc.). This made it possible, for instance, for us to understand invariance under charge reflection (called charge conjugation in precise terms) in physics, and invariance under the swapping of colours in art. The second generalisation makes our concept of symmetry capable of making any kind of *object* of science or art the subject of a symmetry operation. This paved the way, among other things, for us to use symmetry operations on the abstract objects of physics. Finally, we allow the constancy of any *characteristic* to be considered as symmetry. Of the examples familiar to every reader, this is true of electric charge, but any physical quantity considered charge-like can be the object of symmetry, just as can the rhythm of a poem or the motif of a piece of music (Darvas 2007).

Related terms to symmetry are asymmetry, dissymmetry and antisymmetry. Asymmetry is used to denote the absence of symmetry. We speak of asymmetry when none of the characteristics of a given object displays symmetry.
There are instances where an object displays symmetry, but this symmetry is broken in one of its characteristics or a not too significant detail. Example of this is e.g., a bubble in a diamond. We refer to this as dissymmetry. These two concepts related to the absence of symmetry were defined in their current use by Pierre Curie (1859–1906). It is to him that the now famous phrase is attributed: "it is dissymmetry that makes the phenomenon" ("*c'est la dissymmétrie qui crée le phénomène*", 1894). We can take this to mean that phenomena which are important for researchers to discover exist at the points where they encounter dissymmetry. This set of concepts were completed by Shubnikov (1940, 1951) in the 1930s with antisymmetry. We talk of antisymmetry when a characteristic is preserved by being transformed into its opposite. A chessboard is antisymmetric, for example: if reflected, the white squares turn black, and vice versa. The shape of the well-known antisymmetric yin-yang displays rotational symmetry, but if rotated 180° around its centre, black turns to white, and white to black; in terms of its colours, that is, it is antisymmetrical.

The mathematical tool to describe symmetries is group theory. In mathematical terms a group is a set of elements, which have an operation applied to them, and which is subject to four simple axioms. Any symmetry, appearing either in a discipline of science or in a kind of arts, can be characterised by a group.

Description of Symmetries in Classical Crystallography

Nature produces many symmetric phenomena. Beauty of symmetry is manifested in the most spectacular way in crystals. Crystals consist of periodically arranged atoms and groups of atoms (molecules). This periodic arrangement resulted in their ability to allow light to let through their body and to reflect light on their plane cut surfaces, not mentioning the perfect geometric shape formed by the covering surfaces of their bodies.

Periodic arrangements can be produced not only by atoms, and not only in three dimensions, like in crystals. Art learned from nature and the beauty of crystals appears in human creativity, for example in decorative arts, since the early periods of culture. Two and one dimensional periodic arrangements appear in decorative motives, called wallpaper motifs (periodic tiling) and friezes, respectively. They are copies of cross-sections and edges of crystals, where the atoms, molecules can be replaced by either man-made or from the nature borrowed motifs with an infinite abundance. The classification of the frieze and wallpaper motifs follows the terminology and methods elaborated in crystallography. This classification is based on the kinds of symmetries manifested in the given arrangements. Historically, this could be mentioned for the first explicit example how scientific exactness coupled with treatise of artistic beauty.

The symmetry of a crystallographic system (extended in the before mentioned sense also to decorative patterns) has two constituents: the point group symmetry of the basic motive placed in a single point of the motive, and the symmetry of the lattice that these latter single points form. In other words, one constituent is the basic element, or design, that is repeated. A planar example is a crocheted tablecloth composed of repetitions of the same motif; a spatial one is the elementary cell. We usually circumscribe the basic design of the motif with the help of a concept borrowed from crystallography, point groups. The other constituent is the order of repetition: the potential operations and transformations with which one of the elementary motifs can be brought into coverage with the others. These potential operations determine a lattice on the plane or in space. The lattices are represented by the angles given by the edges of the cells, and the length of the edges. Together, we term these defining data of the lattice as the lattice parameters. Again we use a specialist term from crystallography, space lattice, to refer to them (even if the lattices in question are only two-dimensional). Together, the point groups and the space lattices determine the space group structure of a pattern.

The basic motif (point group) of a design can be completely asymmetrical—displaying single-fold rotational symmetry, which is transposed onto itself with a 360° rotation—but with translations and reflections a symmetrical pattern can be made out of it. In other instances, the basic motif can display a number of symmetries. A basic plane motif can be a figure both with mirror symmetry and n-fold rotational symmetry.

Notations

Many operations can be applied to a basic motif which can build a pattern. We can place a point group at the vertices of plane and space lattices of various shapes [this situation is marked by *p*, i.e., primitive cell], or at the centre-points of their edges, faces and cells [marked by *c*, i.e., centred cell]. We can make one overlap the other using some kind of symmetry operation: translation (translation with one unit is marked by *1*), mirror reflection (marked by *m*), rotation (marked by the number of rotations, like *2, 3, 4, 6*) to return to the original position) or glide reflection (marked by *g*). The basic motif must satisfy two conditions, however: whether it is a plane or space lattice, we require that it should fill the plane or the space without gaps, in such a way that it be made up of *congruent cells* (of one given type).

These two conditions present certain restrictions. If we want to be very strict, we could also require that we fill the plane in continuous fashion with *regular*, congruent *polygons*. (The edges and angles of regular polygons are all equal.) Only three shapes are capable of covering the plane in this way: the regular triangle, the square, and the regular hexagon. (Moreover, the regular hexagon can be composed of six regular triangles, what means, we cannot make distinction between a triangular and a hexagonal symmetry.) We are not so strict, so we allow five types of cells in tiling the plane: parallelogram, primitive and centered rectangles, square, and 60° rhombus (consisting of two regular triangles).

Placing the point groups in the distinguished points of a lattice, we receive a pattern, which is repeated. The set of transformations that bring the pattern in coverage with itself form a group in the defined mathematical sense. This is called the symmetry group of the given pattern.

Frieze Groups

We can arrange a given object (in crystallography, point groups) along a straight line in seven different ways. Frieze groups are composed of particular point groups and the operations (transformations) applied to them. Whatever basic motif we place on the point group, the basic unit can be transformed in the following seven ways. To put it another way, taking all possible ways into account, the following seven frieze groups can be produced from the applicable group operations (symmetry transformations) used, i.e. translation, reflection, glide reflection and half-turn. These can be seen in Fig. 1. Their notations, in line with the above conventions, are as follows: *11* (translation), *12* (two successive half-turns, i.e., rotations by 180°), *1m* (reflection in the line of the frieze), *1g (*glide reflection, i.e., we reflect a basic motif in the axis of the frieze, then perform a translation by one unit), *m1* (reflection in an axis perpendicular to the longitudinal axis of the frieze (and translate by one unit), *mm* (reflection in two axes perpendicular to one another),

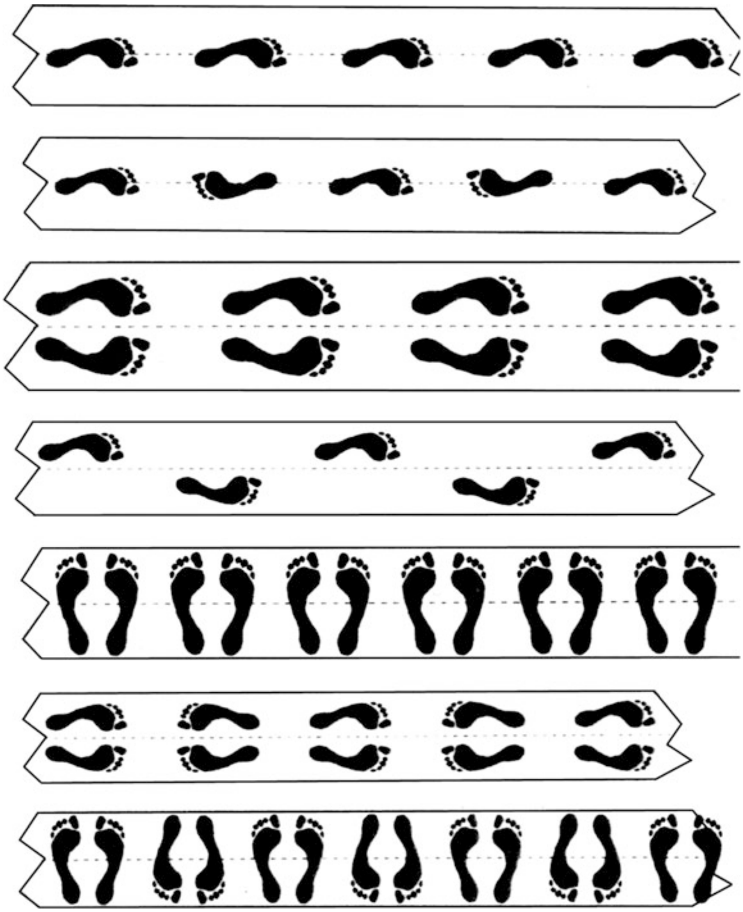

Fig. 1 Footprints demonstrating the seven frieze groups (Kinsey and Moore 2002). Reprinted with the permission of L. Christine Kinsey

mg (combination of reflection in an axis perpendicular to the axis of the frieze and a glide reflection).

One can show that no more frieze groups can be composed, although the low number of the possible frieze patterns seems surprising.

Wallpaper Groups

How many ways can we fill the theoretically infinite planar surface in a continuous fashion with a single (uniform) repeated element? For classification, we use the knowledge gained when we looked at friezes.

We can place an infinitely large number of basic motifs in the place of the individual point groups. There is a limit, however, to how many types of lattice we can use to cover the plane with infinite repetitions, and to the symmetries displayed by the point groups we place in the lattices in order for them to remain, together with the given lattice, invariant under a symmetry operation. The following example illustrates why the last condition represents a genuine limitation. We cannot place a point group with threefold symmetry onto the points of a square lattice, which is symmetrical with regard to reflection in its edges, while satisfying the condition for mirror symmetry, for the mirror symmetry of the pattern would not be preserved during reflection. It would not form a group, as reflection would produce an element which falls outside the group, which contradicts the closure condition formulated among the axioms demanded to fulfil by a group.

On the basis of this, we can interpret the previously introduced concept of *point group* more precisely. In an exact sense, we consider a point group to be the sum of the symmetry operations that can be applied at one point of a planar figure or a spatial body (the term lattice cell applies to both) and under which the given figure or body (cell) is invariant.

We can also make our definitions more precise as concerns lattices. We give the name space lattice (Bravais lattice) to those lattices (even on a plane) which are required by the point group operations applied to the lattice points. What do we mean by required by them? We mean that only those lattices come into consideration that are of a symmetry that the given point group is capable of producing (on the basis of its own symmetries). The only lattices which can be attributed to a given point group (in other words, the only lattices which can be realised in nature) are those which possess the symmetries of that point group. We saw, there are four plane transformations that can be applied on a lattice: translation, rotation, reflection and glide reflection, and we have got five types of possible lattices.

We place point groups in the points of these lattices. One characteristic of point groups is that they can be considered as being fixed on a single point. So we can exclude translation and glide reflection. So, in the case of point groups, only the following symmetry operations can be taken into account:

– 1-, 2-, 3-, 4- or 6-fold rotation around a point, or
– reflection in a straight (symbol: *m*)

Using the above operations both individually and together in two dimensions, the following ten point groups are possible (we can easily see that it is only these):

– 1, 2, 1*m*, 2*mm*, 4, 4*mm*, 3, 3*m*, 6, 6*mm*.

Comparing that said about space lattices and point groups, the ten point groups that can be placed on the lattice points of the five space lattices possible in two dimensions give a total of 17 possibilities for tiling the plane. We call these wallpaper groups. It is such a small number because we saw that the symmetries of particular point groups make it impossible to place them on the point of any lattice. It can be proved that there are 17 and only 17 plane transformation groups that can exist.

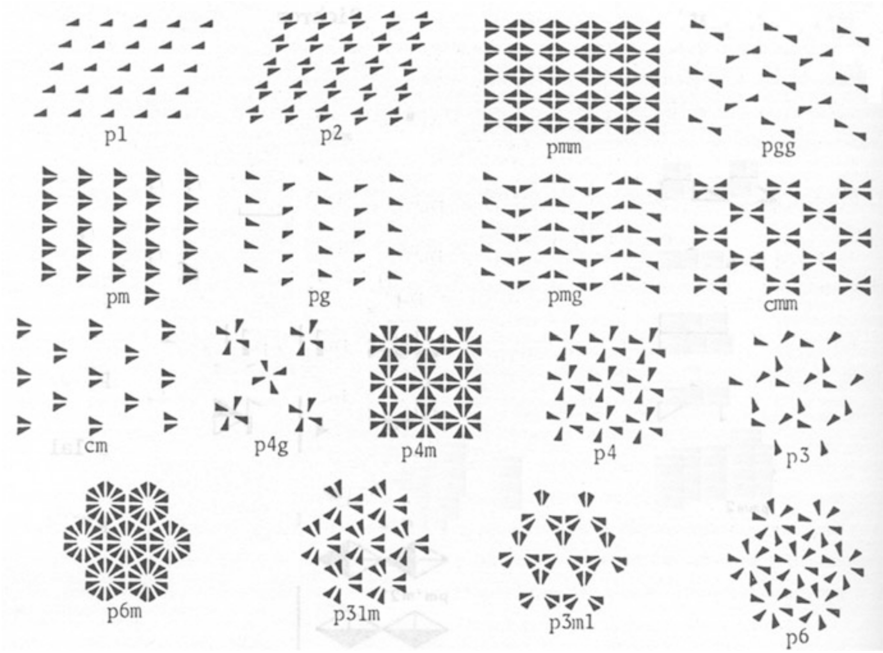

Fig. 2 The 17 wallpaper groups with their crystallographic notations. The image was prepared by Gergely Darvas and extended by György Darvas based on Shubnikov, A. V. and Koptsik, V. A. (1974) *Symmetry in Science and Art*; in: Darvas, György (2007) *Symmetry*. Basel: Birkhauser. Figure 3.17a, p. 90

To conclude, in 2D we have the following possibilities:

- 10: 2D point groups
- 5: 2D space lattices (Bravais lattices)
- 17: 2D plane groups (wallpaper groups).

The wallpaper groups are summarized in Fig. 2. Whatever surprising is it, there are no more than 17 wallpaper groups in one colour. The set of symmetry transformations of any plane pattern can be classified under one of these groups.

The mentioned classification holds for the plane only. More precisely, the surface of a cylinder behaves in the same way like the plane, so one can decorate a cylinder with the same 17 wallpaper groups.

Space Groups

As regards our starting point, the discussion of the beauty of the crystals, we mention only that the continuous filling of space with point groups to be placed in congruent elementary cells is a fundamental task of crystallography. In order to determine the number of ways of possible space-filling, attention has to be paid to: cells of different shapes, point groups that can be fitted in the cells, and the position of the point groups within the cell. On the basis of this, in three dimensions there are a total of: 32 possible 3D point groups, 14 possible space lattices (Bravais lattices), and 230 possible space groups. The various different crystals can be classified on this basis.

Decorating the Sphere, and the Perfect Bodies

The situation is quite different with the surface of the sphere than on the plane or the cylinder. Remember, the plane can be covered without gaps with regular hexagons, but not with regular pentagons. The surface of a sphere can be tiled without gap with regular pentagons, but not with regular hexagons. (The latter would contradict to the well known law of Euler.) Therefore, covering the sphere deserves a few separate paragraphs.

One needs 12, identical edge regular pentagons to cover the sphere without gap. The vertices of these pentagons are distributed equally over the sphere. Each five neighbouring of them determine a plane. One can cut the sphere along such a plane and remove the cap above it. Repeating this procedure over all the 12 plane pentagons, one gets a regular *dodecahedron* (dodeca- means in ancient Greek 12).

Dodecahedron is a perfect body, with equal edges plus congruent faces and vertices. It belongs to the five-member family of the perfect polyhedra, with similar properties. Plato, whom we know as the first who treated them in his *Timaeus*, associated the four others, *tetrahedron, cube, octahedron, icosahedron*, with four primary elements (or substances, identified by Empedocles), namely with *fire, earth, air*, and *water*, respectively. To these four substances, Plato added a fifth, the *universe* (cosmos). Since dodecahedron with its fivefold symmetry appeared to be the most mystical (cf., the theses of the Pythagoreans) among the five perfect bodies, he associated it with the complex universe (Fig. 3). The properties of perfection, which Plato projected onto all areas of existence—including, for example, the social relations between people—were, for him, of symbolic significance. In his interpretation, symmetry was the harmonious order of the universe, that is the very cosmos itself. He writes of the elements of this generalization in *Gorgias* as follows: “And philosophers tell us, Callicles, that communion and friendship and orderliness and temperance and justice bind together heaven and earth and gods and men, and that this universe is therefore called Cosmos or order, not disorder or misrule, my friend.”

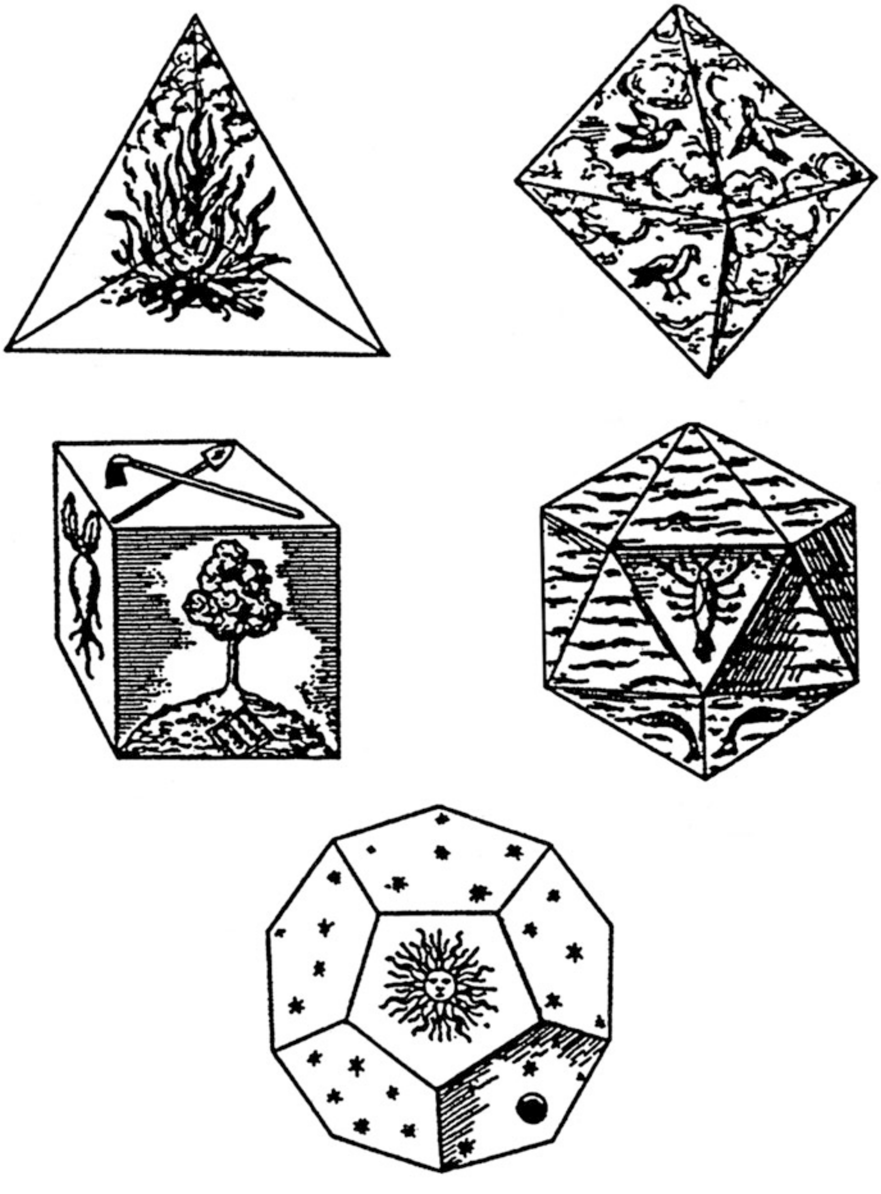

Fig. 3 The five Platonic perfect polyhedra. Source: J. Kepler (1619) *Harmonices mundi*; Prepared by G. Darvas in: (2007) *Symmetry*. Basel: Birkhauser. Figure 2.27, p. 55

Harmony Expressed in Terms of Natural Numbers

Pythagorean "perfection" also included musical harmony in the list of things of artistic beauty. The relationship between the length of strings and the proportions of musical notes was the first law of nature put into mathematical terms. The enumeration of musical sounds and the association of their harmony with the cosmos appear in the so called music of the spheres, which would later have such an intuitive impact on Kepler's discoveries.

The proportions of musical sounds could be written down in the form of the relationships between the smallest natural numbers. Sequences of natural numbers play then an important role in the study of proportions. In the case of the Pythagoreans, numerology meant tracing the material world back to a limited number (1, 2, 3, 4, 5 or many) of primary elements or substances. The schemes of the world based on substances were an invariant element in many different cultures. In the world of Ancient Greece alone (as in Oriental philosophies) we encounter a few schemes of the world based on different numbers of substances. It was perhaps Empedocles (c. 495–435 BC) who had the strongest influence on the Pythagoreans, and whose already mentioned theory was based on four primary elements.

The (displacement) symmetry of the natural numbers seems obvious. However, one can demonstrate that in a stricter sense the set of integers generates a group for the operation of addition, for it satisfies the group axioms:

1. the sum of two integers is an integer,
2. addition is associative: $(a + b) + c = a + (b + c)$,
3. there exists a neutral element, 0, for which $a + 0 = 0 + a = a$, where a is any element of the group,
4. $(-a)$ is the inverse element for a, because $a + (-a) = (-a) + a = 0$.

This was later to become the basis for the harmony in Kepler's picture of the world (Kepler 1596). In India, at around the same time as Empedocles, a similar role was played by *charvaka* (four substances, comparable to his) and *vaisheshika* (five substances: earth, water, light, air, ether). In China, at around this time, the Confucian Hsu Hsing (c. 300–c. 230 BC) associated five *chi* with the five primary elements (metal, wood, water, fire, earth), that could be deduced from the material dichotomies of the yin-yang. (Here, by *chi* we mean what was originally the only substance, primary element, as elaborated in the *Tao* by Lao-Tse (c. sixth–fifth century BC), though *chi* also means knowledge, wisdom, and intellectual essence).

Symmetries of the Perfect Bodies in the World View of the Renaissance and at the Birth of Modern Science

During the *Renaissance* in Europe, the Platonic perfect bodies appeared as the embodiment of the divine proportion. This is well illustrated by the drawings which Leonardo prepared for Luca Pacioli's book *Divina Proportione* (1509). In science, the *Renaissance* brought the triumph of rationality. In contrast to the ancient picture of the world, which, in line with its anthropomorphic attitude, put man at the centre of the universe, and thus preferred the geocentric world-view with its planets tracing complicated cycloid orbits, *Copernicus' revolution* (as Kant referred to it) brought to the fore the heliocentric world-view which described the orbits of the planets with circles. On the basis of the observations provided by astronomical measurements, both models were suitable for correctly describing the movement of the planets. The heliocentric view of the world offered circular planetary orbits, and the Earth-centred one cycloid orbits. The latter gives us a much more complicated description of the world. Yet it was still the more acceptable, because man stood at its centre, and this symmetry proved to be the more powerful. This way of thinking was reflected in ancient philosophy, as well as artistic depiction of nature and of man. As empirical experience was not able to decide between the two models, the choice fell on the simpler, more symmetrical mode of description. In place of apparent symmetry (if we look up at the starry sky above us, we see it as a hemisphere with us standing at its centre), the victor was the path for the Earth given by the symmetry in the measured data (that put the Earth on the same footing as the other planets, tracing a similar orbit around the Sun). Copernicus put the coin on the more symmetric model.

This also brought the rejuvenation of astronomy. Kepler came to determine the laws of the movement of the planets while searching for harmony in the world (Kepler 1609, 1619). His model was based on nested spheres fitting the five Platonic bodies both internally and externally. He wanted to find the perfect picture of the world, the harmony, symmetry embodied in the world. He was himself the most surprised that the orbits of the planets proved not to be symmetric circles, but rather ellipses where the presence of the Sun favoured one focal point over the other.

After Herodotus, Kepler was the first, who preferred empirical experience against the unquestionable belief in symmetry when the two conflicted. This was a greater revolutionary step in scientific thinking than that of Copernicus at the advent of modern science. The destroyed symmetry had to be recovered somehow. Like Plato, Kepler (1619) went back to the music of the spheres of the Pythagoreans. He found the proportions of the angular velocity of the various planets when closest (in perihelion) to and farthest (in aphelion) from the Sun to be as follows: for the Earth 16/15, for Mars 3/2, for Saturn 5/4, which can in order be compared with a half-note, quint, and third. It was these that he composed in his musical motifs.

From Geometry Through Cosmology to Interdisciplinary Applications

Packing of spheres led Kepler to other problems as well, what have got importance later in other sciences. One of the consequences of the problem of drawing regular polyhedra around spheres was Kepler's original formulation of the problem of the densest packing. We know that, on a plane, a circle can be circumscribed by exactly six circles of the same radius, touching it and each other. In space the story is nothing like as simple. We cannot surround a sphere in space with spheres of the same radius in such a way that all the neighbouring spheres touch each other as pairs. There is no solution with integer number of spheres, which means a terrible asymmetry in classical geometry. The classical Kepler problem is thus as follows: how can we arrange similar spheres in a bowl in the densest fashion (i.e. with the most spheres)? (The planar equivalent of this would be how we could cover the surface of the table in the densest fashion with a given denomination of coin, i.e. with the most coins.) The set of problems concerning the densest tilings and closest packings has, over time, become and increasingly independent branch of mathematics. Perhaps it is surprising that centuries had to pass after Kepler before the problem of the densest filling of space was solved. In the nineteenth century, Gauss (1777–1855) provided an estimate that with congruent spheres the space could not be more densely packed than 74.048% (which coincided with the original Kepler conjecture), but this was only proven in 1998 by Thomas Hales, who reported his proof to the fourth symmetry congress in Haifa in that year.

This set of problems is not merely an intellectual game. The areas of application of the densest filling of space and close packing of spheres are very wide. Examples of some such areas are: in crystallography, the determination of the optimal position of atoms; in chemistry, determination of optimal bonding directions; in physics, in atomic nuclear models; in biology, in the division of egg cells; in architecture, the selection of the most stable supporting structures. So did a distortion of symmetry stimulate interdisciplinary applications in different sciences.

Perfect Proportion

We know from Aristotle that ancient Greeks attributed significance to perfect proportions. We learned from Vitruvius that perfect proportion was identified what later was associated with golden mean. Pythagoreans sought the harmony in sequences of natural numbers. They found the arithmetic, geometric and harmonic means between numbers.

The perfect proportion, which was instinctively applied by sculptors and architects since the ancient times, turned to be associated with the so called Fibonacci sequence of natural numbers. Although the perfect proportion is an irrational number, it can be expressed as a limit of rational quotients. The Fibonacci numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, ... etc. are composed as the sum of the preceding two members of the sequence. Where is the symmetry in this sequence? It is in the regularity in the rule of formation. The summing is shifted step by step to the next (pair of) members of the sequence. What is more interesting, the quotient of the consecutive members of the sequence a_n/a_{n-1} approaches to the limit in the infinity what is known as Φ, the Fibonacci number.

The same number Φ can be obtained also in a geometric way, from the proportion of length of sections, which are defined by the division of a section in two, so that the length of the longer section is so to the shorter as the length of the whole section to the longer portion: $a:b = (a + b):a$. It may be surprising that these two quite different (algebraic and geometric) definitions lead to the same result, $\Phi = 1618\ldots$. Nevertheless, the literature treating the peculiar properties of the Fibonacci numbers and the number Φ can fill a library. This Φ can be identified with the—later so called—*golden section*, or golden proportion, known both from artworks (from fine arts to performing arts like music) and scientific results (in diverse disciplines). We can refer here only to a very few examples that may enlighten the interdisciplinary applicability of this proportion.

Golden Section and the Perfect Bodies

In the preceding introductory sections to the interdisciplinary application of symmetry phenomena mention was made of the perfect (or Platonic) solids and the Fibonacci sequence (together with the related golden section—or Fibonacci—number). The two were put side by side intentionally. They are interrelated.

Let us take a square and divide its two neighbouring edges according to the golden section as shown in Fig. 4a, then connect the section points with a straight line to the opposite vertex of the square. Fold the generated triangles in the left and bottom along the inner lines so that the two not divided edges of the square meet. You will get a tetrahedron. If the inner surfaces of this tetrahedron are made of mirrors, the three facets will be reflected into each other. This is the so called golden section kaleidoscope. The possibility of such a kaleidoscope was shown by E. S. Fedorov (1891) by the end of the eighteenth century. He was the person who (parallel with A. M. Schönflies [1891], and almost parallel with W. Barlow [1895]) proved that there exist exactly 230 3-dimensional crystallographic groups. H. S. M. Coxeter (1907–2003), probably the greatest geometer of the twentieth century called the attention that due to the angles of this kaleidoscope, it could be cut from a square, in the 1960s. The implementation, with some adjustments of the order of the angles, was made by Nicolas and Caspar Schwabe (1993) in the 1980s. If allow light inside the kaleidoscope through the arcs marked in Fig. 4a, and cover the arc sections in the individual facets with different colours, the arcs will be reflected in the kaleidoscope to shape polyhedra.

Figure 4c shows what do we see inside. The mirror images of the shortest arc—marked by blue—will shape a regular dodecahedron. The mirror images of the

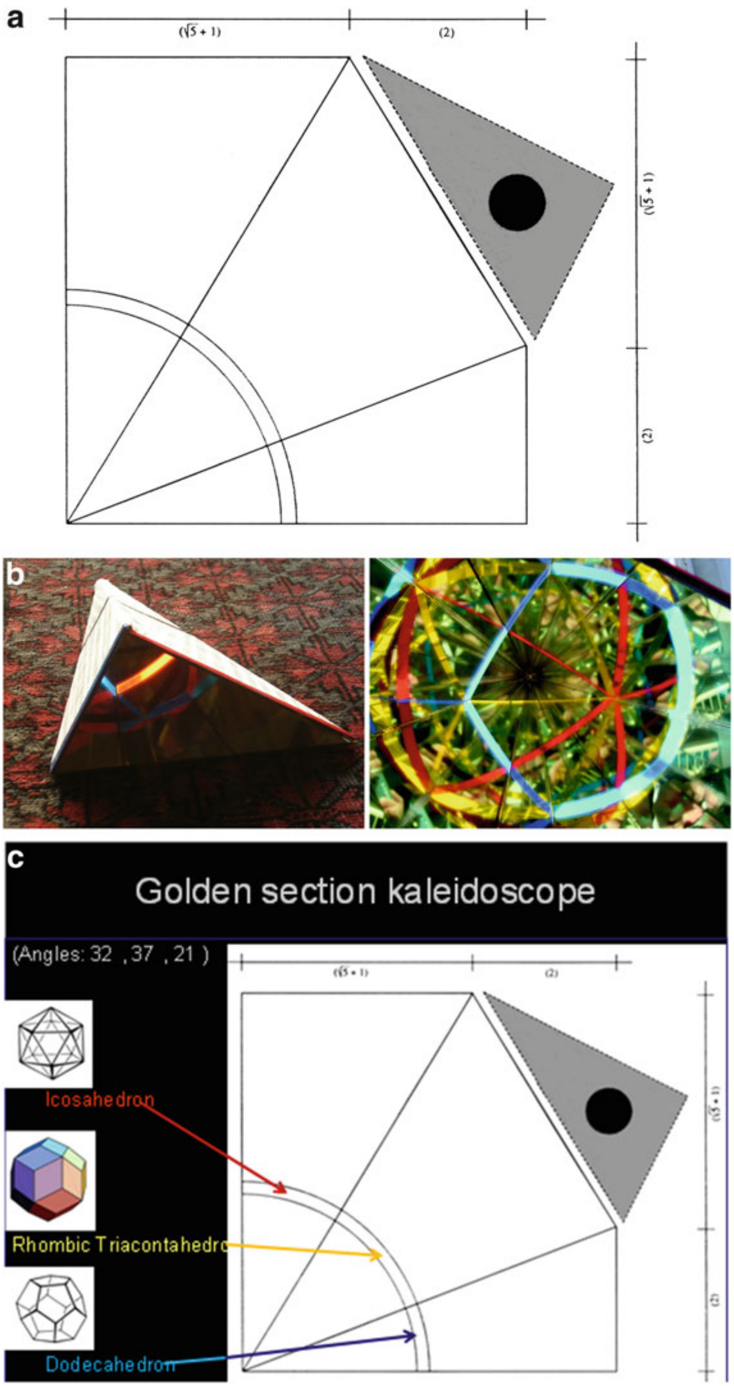

Fig. 4 (**a**) Golden section of a square. Image reproduced by courtesy of the journal *Symmetry: Culture and Science (1990/1992),* based on the paper by C. Schwabe (1986), copyright *Turicum*, Schweizer Kultur und Wissenschaft, April/May 1993, S. 6–7, Reproduced there by courtesy of the

medium length arc—marked by red—will shape a regular icosahedron. The longest arc—which is marked by yellow—and its mirror images will shape a rhombic triacontahedron (Fig. 4c). The rhombic triacontahedron is a polyhedron whose facets are all identical rhombuses. The rhombic triacontahedron is covered just with 30 facets, like the number of edges of either the dodecahedron or the icosahedron. They are called golden rhombuses, because the lengths of their diagonals (which intersect each other perpendicularly) are in proportion to the golden section! We can recognise a miracle in our kaleidoscope: these golden section proportion diagonals coincide with the edges of the dodecahedron (blue) and the icosahedrons (red), as one can check in the right image in Fig. 4b.

We have found mathematical connection between the properties of the regular (Platonic) polyhedra and the golden section

Fivefold Symmetry in the Plane and the Space with the Help of the Golden Section

We knew from the proportion of the length of diagonals to the edge of the regular pentagon that fivefold symmetry is related to the golden section. Now, we learned that golden section is related to regular polyhedra, too. We also knew that the plane cannot be tiled without gap with unique elements showing fivefold symmetry. The surface of the sphere, or a polyhedron can be, this is the dodecahedron. However, the dodecahedron is unable to fill the space without gap. There is only the cube (among the regular polyhedra) that can fill the space without gap. Tetrahedra and octahedra together can fill the space, but that is not considered as a crystal lattice. Crystals—with the condition that only one kind of unit cell be applied—were not allowed to contain two (or more) types of unit cells in their lattice. Thus, there is no classical crystal with fivefold symmetry.

If one is unable to tile the plane with a single regular unit element, without gap showing fivefold symmetry, needs to sacrifice one or more of the conditions: either the congruency of the tiles, or the condition to use only one element. The goal is to solve the problem with the minimum of the disregarded conditions, e.g., to reduce the number of the used elements to two. The same is aimed at the space.

The planar problem was solved by R. Penrose (1974) in 1973. First it was a decorative pattern for artistic purposes, and he patented it as a puzzle. After working with kite and dart shapes, he reduced the two elements of the tiling to two kinds of equal length edges rhombuses. One of them is identical with the above discussed golden rhombus. The tiling displays local fivefold symmetry, that means,

Fig. 4 (continued) editor of the—already ceased—journal *Turicum* (private letter of the former editor to G. Darvas, 2004. (**b**). The golden section kaleidoscope (*left outside*, *right inside* view). Image copyright by the author. (**c**) The shapes seen in the golden section kaleidoscope. Image copyright by the author there is no shift in the tessellation, which brings the pattern into coverage with itself, therefore the pattern is not periodic, but there are similar domains in it that show locally fivefold symmetry. This quasiperiodic tessellation can be continued up to the infinity in the plane. John Conway proved soon (Gardner 1977), inter alia, that we can find a domain tiled similarly to that we happen to have chosen within a very small distance. If the diameter of a chosen (say circular) domain is *d*, then we reach a domain with similar tiling in some direction from the boundary of the original domain, at a distance of at most $s \leq \frac{\Phi^3}{2} d = 2.11 \ldots d$. (As further evidence of the universal connections between symmetries, Φ here is again the golden number we know from the golden section.) This theorem not only gives a certain limit to repetition (even if it is not periodic)—this limit is a surprisingly low, "visually observable" boundary distance. The quasi-periodic arrangement also means that we find certain "local" symmetries in it, which break once past a certain boundary, but which are repeated elsewhere, locally. There are also those that infinitely preserve their symmetry with regard to a certain point, but where the periodic translation of this symmetry centre-point does not make the entire tiling of the plane overlap itself.

One of the other applicable mathematical theorems for the Penrose tiling of the plane with the most interesting symmetry relates to the proportion of the elements used. The proportion of the number of dart and kite shapes is—like the proportion of their area—equal to the golden proportion. 1.618 ... times as many kites are needed as darts. If the tiling is infinite, this number is the precise proportion. The fact that this proportion is not rational is used by Penrose to prove that tiling is not periodic, for if it were periodic, this proportion would have to be a rational number. This theorem—like all those concerning kites and darts—holds true even if the tiling takes place with the Penrose rhombuses (made up of darts and kites) with edges of equal length, and angles of 72° and 108° or 36° and 144° respectively.

One did not need to wait too long to solve the problem of space filling without gap with two elements, namely with two rhombohedra which have edges all of the same length, and which are both delimited by a single congruent rhombus identical with one of those used in the planar Penrose tiling. However, geometric space filling with local fivefold quasiperiodic arrangement does not guarantee that nature produces the same structures with atoms in the vertices of the found arrangement. Remember, two-cell spatial arrangement contradicted to the conditions set up by classical crystallography, even not mentioning quasiperiodicity and (at least local) fivefold symmetry. Events started to accelerate.

The Discovery of Quasicrystals

Ammann's discovery (1976) (Gardner 1977), described below, has paved the way for the possibility of space filling in non-periodic ways displaying fivefold symmetry. He began by constructing from the two mentioned rhombohedra which have edges all of the same length, and which are both delimited by a single congruent "golden" rhombus. The proportion between the diagonals of the faces is the golden section. One looks like a cube flattened along one of the diagonals of its body, while the other looks like a cube stretched along one of its diagonals. H. S. M. Coxeter referred to these as "golden rhombohedra". Apart from these two, no other golden rhombohedra exist. Both were already studied by Kepler. Martin Gardner, the great expert in this field, drew Penrose's attention to Ammann's results. Ammann took a set of two rhombohedra, parallelepipeds with six congruent rhombus faces, and showed that, when face-matching rules are applied, they could tile space non-periodically without gap. As the two rhombohedra are golden rhombohedra, the faces have diagonals in the golden ratio.

Penrose came to the conclusion that it might be a model for certain unexplainable molecular formations, such as viruses. Alongside his congratulations to Ammann, Penrose made the following reply to Gardner:

> [--] some viruses grow in the shapes of regular dodecahedra and icosahedra. [--] But with Ammann's non-periodic solids as basic units, one would arrive at quasi-periodic 'crystals' involving such seemingly impossible (crystallographically) cleavage directions along dodecahedral or icosahedral planes. Is it possible that the viruses might grow in some such way involving non-periodic basic units, or is the idea too fanciful? (R. Penrose to M. Gardner, 4 May 1976, cited in Gardner 1997, 24.)

Nevertheless, interdisciplinary research turned first towards crystalline structures before deepening in the really fascinating applications of the idea in the structures of virus growth. In 1977, Koji Miyazaki at the University of Kyoto reached a discovery similar to but independent of Ammann's (Miyazaki 1986). In addition, he found another method of filling space with two golden rhombohedra in a non-periodic way. In this method, five golden rhombohedra with acute angles and five rhombohedra with obtuse angles fit together to form a rhombic triacontahedron. Two such bodies can each be surrounded by a further 30 golden rhombohedra of each type, which results in a larger rhombic triacontahedron, and this extension can be continued infinitely. This gives us a honeycomb-like filling of the space, the centre of which displays icosahedral symmetry.

Naturally—almost in the minute Penrose had made his planar discovery—the theoretical work began in structural quantum chemistry. The theoretical possibility of the filling of three-dimensional space with atoms with two types of cell stretched by atoms, in quasi-periodic fashion, displaying local fivefold symmetry, was first shown by London crystallographer Alan Mackay in 1978 (Mackay 1982). His solution was rendered more precisely by Tohru Ogawa in 1981—in the knowledge of Miyazaki's result. After this, Mackay and Ogawa joined forces to use calculations to confirm the theoretical possibility of creating stable electron bonds in the space directions determined by their models.[1] The space filling suggested by

[1]The author was invited by the Nobel laureate rector of the University of Tsukuba Leo Esaki, to give a keynote lecture in a panel discussion in an afternoon seminar at the University of Tsukuba, where A. Mackay and T. Ogawa read the discussed lectures, together with K. Miura and D. Weaire

Ogawa was demonstrated in a bamboo shoots model by Akio Hizume, where the cells represent pentagonal space filling and rhombohedral cells.

The dilemma of crystallography at the beginning of the 1980s was as follows: can quasi-periodic space filling be regarded as a crystal? It seemed that, after the theoretical work of Mackay and Ogawa, even if the concepts of classical crystallography did not allow quasi-periodic space filling into the system, their existence could no longer be ignored, although no one had reported to find such structures.

Dan Shechtman and his colleagues in a laboratory at the American National Institute of Standards and Technology near Washington, D.C. examined the material structure of alloys with the help of X-ray diffraction images. Shechtman found X-ray diffraction images displaying tenfold symmetry. As it later transpired, a number of crystallographers investigating the structure of matter had already encountered such images. They had always thrown them away as bad exposures. In 1982, when he looked through many hundred exposures in a single night, and rejected several seemingly "defective" images, he was struck by the idea that they were perhaps not defective, after all. It could not be an accident that there were so many of them. Maybe there was some kind of regularity behind their occurrence? With his colleagues, he set about examining the rejected pictures again, together with the sample of the material they had been taken of, a suddenly cooled aluminium-manganese alloy (Shechtman et al. 1984). This was how he discovered the materials that would later be called quasicrystals.

Their result was an encouragement to all those who had previously not dared to believe in the existence of samples of materials displaying fivefold symmetry. From this point on, "faulty" X-ray images were no longer thrown in the rubbish bin, and one alloy after the next was discovered to have similar symmetry. Quasicrystals, with their fivefold symmetry, opened a new chapter in material science. However, the community of crystallographers resisted to acknowledge the existence of quasiperiodic crystalline arrangements of atoms with fivefold symmetry and multiple golden section proportions, which contradicted to the canon accepted since the Fedorov-Schönflies doctrine. It took years while a journal has accepted to publish their paper.[2] Shechtman was awarded the Nobel prize for his epoch-making discovery of crystalline structures demonstrating fivefold symmetry only in 2011.

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on 25 November 1994 (Darvas 1996; Ogawa et al. 1996). In 1996, when the author spent a semester in Tsukuba for the invitation of T. Ogawa as a visiting professor, he lived in the same house, where the Mackay couple did in 1981.

[2]The author co-organized the ten years anniversary session to remember the publication of the discovery of quasicrystals in Washington, D.C. with the participation of Dan Shechtman and his co-author colleagues in 1995 (Darvas et al. 1995). That time, the quasicrystals were worldwide acknowledged, and many studied them. It was in the same year, when Shechtman organised an interdisciplinary seminar for the faculty members of the Technion in Haifa, and he introduced the author, as the invited speaker of that event, to the audience. Later, Shechtman and the author were co-organizers of the fourth symmetry congress and exhibitions held in Haifa, 1998, and were co-editors of the proceedings volume (Darvas et al. 1998).

Summary

Symmetric properties of material structures led to many discoveries. To remain in the domain of material science, one can mention the discoveries of the fullerenes (Kroto et al. 1985, acknowledged by Nobel prize in 1996) and the production of graphene, which is important to nanoscience (acknowledged by Nobel prize in 2010). Quasicrystals took their place in artworks as well. All they—along with many others in particle physics and life sciences—demonstrate the productivity of interdisciplinary clues and holistic thoughts bridging between arts and sciences.

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Part II
Studying Mathematical Principles of Composition

Lessons in Duality and Symmetry from M. C. Escher

Doris Schattschneider

Abstract The Dutch graphic artist M. C. Escher (1898–1972) carried out mathematical investigations that led to symmetry drawings of three distinct kinds of tilings with two colors, capturing the essence of duality. He used several of these drawings as key elements in his prints that further expressed ideas of duality. One of the most complex of his "duality" tilings was realized in Delft ceramic tile, wrapped around a large column for a school in Baarn, Holland. Recently, a Salish artist in Victoria, BC, Canada, has independently produced a tiling that contains many of the same elements as Escher's complex duality tilings.

Introduction

Duality This term, familiar to mathematicians, is a concept central in the work of M.C. Escher. For mathematicians there are dual spaces, dual functions, sets and their complements, statements and their negations. For Escher there were opposing ideas, states, or perceptions, complementary forms, the juxtaposition of figure and ground (recall a few of his prints: *Day and Night*, *Convex and Concave*, *Encounter*, *Verbum*, *Sun and Moon*) (Bool et al. 1982, cat. nos. 303, 399, 331, 326, 357, respectively).

The essence of duality is two forms mutually defining each other—knowing one is enough to know the other. And yet, it is often impossible to clearly perceive both simultaneously. Think of the iconic test of perception in Fig. 1: focus on the black and two faces mirror each other; focus on the white and a goblet appears. Taking in the whole, you see one interpretation, then the other: the black and white figures shift in and out of focus, each vying to be recognized.

In 1952–1953, Escher corresponded at length with an ophthalmologist, Dr. J. W. Waganaar, on the topic of perception, and raised the question: Is it possible to create a picture of recognizable figures without a background? Waganaar maintained, "In my opinion you do not in fact create pictures without a background. There are compositions in which background and figure change functions alternately; there is

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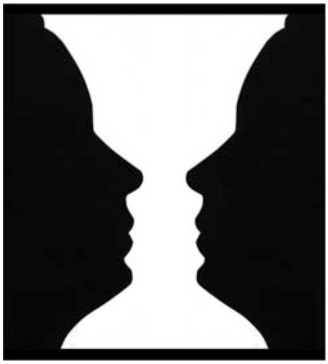

Fig. 1 The classic figure/ground dichotomy investigated by psychologist Edgar Rubin in 1915. Image by Doris Schattschneider

a constant competition between them and it is actually not even possible to go on seeing one of the elements as the figure. ... Your compositions do not have a visual static balance, but a dynamic balance, in which there is, however, a relationship between figure and background at every stage." (Bool et al. 1982, 160.)

In his essay in which he quotes Waganaar, Escher then tells the reader, "Thus it seems that two units [tiles] bordering on each other cannot simultaneously function as 'figure' in our mind; nevertheless, a single dividing line determines the shape and character of both units, serving a double function." He goes on to admit how difficult it is to create his creature tiles, "After a great deal of patience and deliberation, and usually a seemingly endless series of failures, a line is finally drawn [simultaneously delineating the two figures], and it looks so simple that an outsider cannot imagine how difficult it was to obtain." (Bool et al. 1982, 160.) In his hand-done colored drawings of tilings, it is true that Escher's adjacent tiles compete to be figure (or ground). However, in his prints that use fragments of these tilings, Escher emphasizes both their duality and their individuality by having the flat shapes that define each other in the tiling metamorphose into three-dimensional figures, gaining their independence from each other. His print *Sky and Water I* in Fig. 2 perfectly illustrates this important aspect of his work.

Symmetry One definition of symmetry is balance, and this can be interpreted in many ways. Escher was especially sensitive to symmetry in the composition of many of his prints, balancing foreground and distant scenes in his early Italian landscapes (e.g., *Pines of Calvi, Castrovalva* in Bool et al. 1982, cat. nos. 230, 132). He balanced the placement of figures or even scenes in other prints (e.g., *Paradise, Scapegoat, Sky and Water I and II, Day and Night* in Bool et al. 1982, cat. nos. 67, 69, 306, 308, 303, respectively).

However, Escher's greatest obsession was with symmetry as a graceful, regular repetition of forms that cover the plane. What we today call tessellations or tilings, he called "Regular Divisions of the Plane," and he adapted a strict definition that he found in the journal article by F. Haag (1923). For Escher, a *regular division of the plane* was a tiling by congruent tiles (he insisted they be recognizable shapes), with the additional property that every tile be surrounded in the same manner. Today

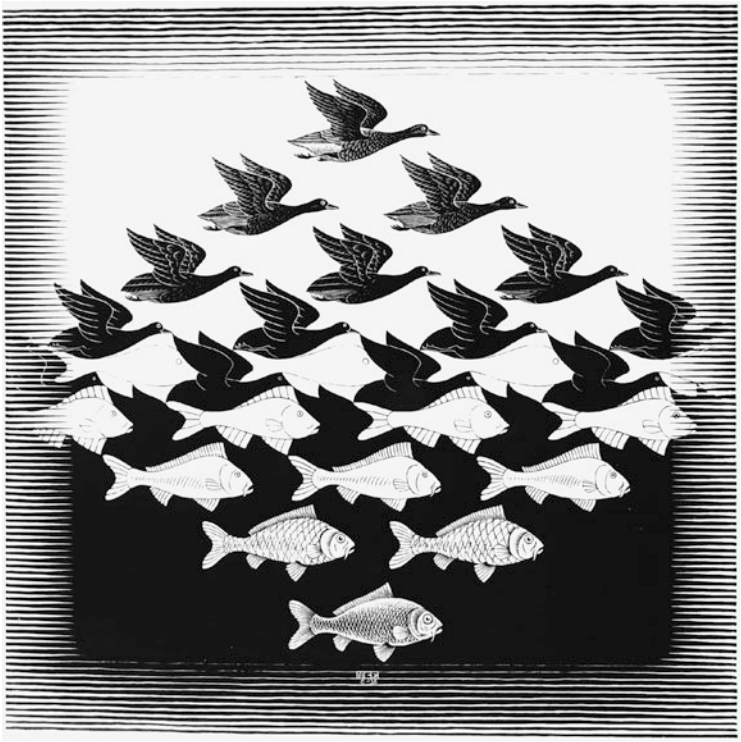

Fig. 2 M. C. Escher. *Sky and Water I*. Woodcut in *black* and *white*, June 1938, 435 × 439

such tilings are termed *isohedral*. They are defined by the property that for any chosen pair of tiles in the tiling, there is an isometry (a translation, rotation, reflection, or glide-reflection) that transforms the first tile in the pair into the second tile, and at the same time, superimposes every tile in the tiling onto another tile. Such an isometry is called a *symmetry* of the tiling. Other ways to express this property is to say the symmetry group acts transitively on the tiles, or the orbit of one tile under the symmetry group is the whole tiling.

Escher traveled to the Alhambra, in Granada, Spain, in 1922 and again in 1936, and carefully copied many of the majolica and incised plaster tilings found there. From these, he gleaned what information he could on their underlying geometry, and employed that knowledge to produce about a dozen original tilings. But it was his careful study in 1937 of a page of illustrations of tilings in a journal article by the mathematician George Pólya (1924) that set Escher on a 4-year intense investigation to determine the characterizations of shapes of tiles that satisfied the conditions to produce a regular division of the plane. In 1941–1942, at the culmination of his study, he summarized the results of his investigations in two notebooks. This information enabled him to produce imaginative and complex tilings, governed by the geometric relationships he had discovered. The keys to his discoveries were the mathematician's tools of symmetry—translations, reflections, rotations, and glide-reflections. For many separate cases, he recorded in terms of these transformations, how a tile was related to each tile that surrounded it. And his investigations were groundbreaking in what today is called color symmetry: his colorings of tilings were compatible with the tilings' symmetries. (For details on 2-color symmetries, see Schattschneider 1986, and Washburn and Crowe 1988.) By the end of his life, he had produced over 150 original tilings, each hand-drawn on squared copybook paper, and then hand-colored. These drawings were his "storehouses" from which he could select ideas for his prints (Schattschneider 2002, 2004).

Escher's Tilings: Duality Through Two-Color Symmetry

By a tiling, we mean a covering of the plane by tiles, without gaps, and no overlaps of their interiors. We use the term *vertex* for a point at which at least three tiles meet, and *edge* for a boundary curve of a tile that connects two consecutive vertices. More than two-thirds of Escher's tilings use two colors: each tile is colored in one of the two colors and tiles that share an edge have contrasting colors. This property (called "map-coloring" by mathematicians) ensures that each tile is surrounded by tiles of the opposite color, so in the infinite jigsaw puzzle of tiles, each individual tile is recognizable. And of course this property satisfied Escher's fascination with duality of figure and ground.

In his notebooks, Escher outlines many cases of tilings in which an even number of tiles meet at each vertex of the tiling, and hence can be map-colored with two colors. He first enumerates classes of isohedral tilings in which the congruent tiles have four edges; these tilings require only two colors. He next shows how a process he called "transition" can produce from a 2-colored isohedral tiling a new isohedral tiling that requires three colors in order to be map-colored. Later in the notebook, he goes on to explore ways in which the tiles in these isohedral tilings can be split into two different shapes to produce new tilings that can be 2-colored.

Escher produced three distinct types of 2-color tilings; each illustrates duality in a different way. In what follows, we discuss separately each of these types of duality tilings and how he created them.

Type (1): Two Colors, One Tile More than a third of Escher's tilings are isohedral (so the tiles are congruent), and the tiling is map-colored with two contrasting colors. The simplest common example of such a tiling is an infinite checkerboard, with its familiar red and black square tiles. Escher encountered many such tilings in the Alhambra with more interesting shapes. One that he copied is shown in Fig. 3 with black and white tiles; he later turned it into a tiling by blue and white butterflies (Schattschneider 2004, 125).

Through the clever use of symmetry, Escher emphasized the dual roles of the congruent dark and light tiles. In these tilings, the pattern formed by all tiles of one color (say, white) was exactly the same as that formed by all tiles of the other color

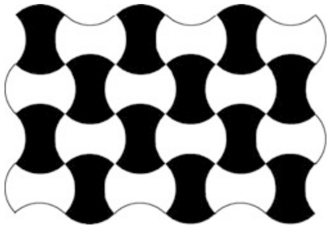

Fig. 3 A tiling from the Alhambra that Escher transformed into butterflies

(say, black). One layer is positive, the other negative, both exactly the same pattern. The Alhambra tiling in Fig. 3 has this property: if the pattern formed by the white tiles was lifted and rotated 90°, this layer could be superimposed exactly on the pattern of black tiles. Either layer can serve as figure, the other ground.

But the dual symmetry employed by Escher is even stronger than that. For any chosen pair of tiles in the tiling, every symmetry of the tiling that sends the first tile in the pair onto the second tile is a *color symmetry*. This means that each symmetry of the tiling either preserves all colors or interchanges all colors: if the two chosen tiles have the same color, then the color symmetry sends every tile onto another tile of the same color, and if the two chosen tiles have opposite colors, the color symmetry sends every tile onto another tile of the opposite color. In mathematical terminology, this kind of isohedral tiling is *perfectly colored*, and there is at least one symmetry that interchanges colors. In the world of textiles and art, a tiling or pattern with these properties is sometimes said to have *counterchange symmetry*.

The Alhambra tiling in Fig. 3 is such a tiling. Here 90° rotation symmetries (with centers at the vertices of tiles) interchange colors, while reflection symmetries (in the vertical and horizontal center lines of the tiles) and 180° rotation symmetries (with centers at the vertices and at the centers of the tiles) preserve colors. Vertical and horizontal glide-reflections preserve colors, while diagonal glide-reflections interchange colors. If all tiles had the same color, the tiling has a *p4g* symmetry group. With tiles colored as in Fig. 3, the symmetry group of the pattern of tiles of a single color is type *cmm*; one notation for the symmetry group of the 2-colored tiling is *p4g*/*cmm*. (This notation varies; two sources are Coxeter 1986, and Washburn and Crowe 1988.)

Escher chose one of his perfectly-colored isohedral tilings as the cover illustration for his book *Grafiek en Tekeningen* (*Prints and Drawings*), first published in 1959 (Fig. 4). The tiles are griffins (winged cats), and were used in his whimsical print *Magic Mirror* (Bool et al. 1982, cat. no. 338). In the infinitely extended tiling, all golden cats face left, outlined by their black mirror-images that face right. Each row of cats is lined up chin-to-tail, and is defined by a single color. The golden cats can serve as figure with the black cats as ground, or vice-versa. In the tiling, all translation symmetries preserve colors, while all (vertical) glide-reflection symmetries interchange colors.

Type (2): Two Colors, Two Different Tiles, Each Occurring in a Single Color The duality of the figure/ground illusion of the faces and goblet in Fig. 1 is achieved through two devices: figures of two different shapes share a boundary and these shapes have contrasting colors. In one-third of Escher's tilings, two distinct tiles with contrasting colors interlock, with all repetitions of a single tile

Fig. 4 Symmetry Drawing E66 with counterchange symmetry, used on the cover of Escher's book *Grafiek en Tekeningen*

the same color. Necessarily, tiles of the same shape touch only at vertices of the tiling; each tile is completely outlined by tiles of the complementary shape. In addition, these tilings are *2-isohedral*, that is, given any pair of congruent tiles, there is a symmetry of the whole tiling that transforms the first tile in the pair to the second. Since each shape occurs in only one color, all symmetries preserve colors.

One of Escher's best-known tilings of this type shows white angels and black devils (Schattschneider 2004, 150); as a result, Andreas Dress named such tilings *Heaven and Hell tilings* (Dress 1986). We will refer to them as HH tilings for short. For an HH tiling, how did Escher manage to get two different motifs (tiles) to perfectly interlock and then fill the plane with their regular repetitions? He achieved this by beginning with an isohedral tiling, splitting one tile to produce two separate shapes, then repeating that same split on every tile in the tiling. In performing the splitting, he had to ensure that the resulting tiling could be map-colored with two colors in such a way that all tiles of a single shape were the same color (Schattschneider 2004, 70–76). Figure 5 recreates the first two cases in his notebook (Schattschneider 2004, 71).

The first case begins with a 2-color tiling in which four tiles meet at every vertex, and edges meet every vertex in the same way. Here, (Fig. 5, left) he splits the tile with a curve that joins opposite vertices; this produces a new tiling with two distinct tiles in which six tiles meet at every vertex, allowing the new tiling to be 2-colored. The second case (Fig. 5, right) begins with a 3-color tiling in which three tiles meet at each vertex, and there are two distinct vertex types. In this case, he splits the tile with a curve that joins opposite vertices of different types; this produces a new tiling with two distinct tiles in which four tiles meet at each vertex, so it can be 2-colored. While the original tilings have translation color-changing symmetries, in the HH tilings produced, those same symmetries now preserve all colors.

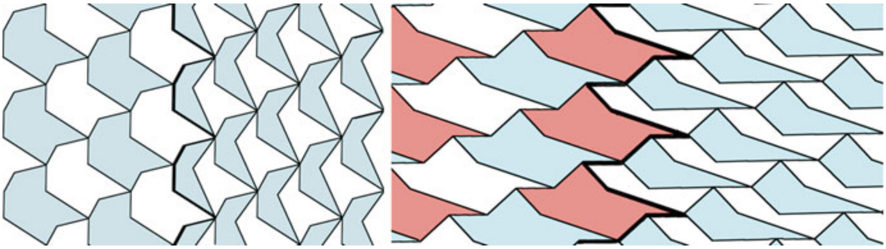

Fig. 5 Splitting each tile in an isohedral tiling can produce a 2-color 2-isohedral HH tiling

Escher investigated splittings of all of the categories of isohedral tilings he had discovered, and considered variations in the choices of splitting curves for a single tiling. In most cases, the splitting produced an HH tiling; in some cases, a 2-colored tiling was produced in which each of the two different tiles occurred in both colors (these type (3) tilings are discussed in our next section). Although the technique may be simple in theory, it was a painstaking process for Escher to derive the shape of a tile and its splitting curve so as to produce two distinct, recognizable, complementary figures.

In most of Escher's HH tilings, the two tiles have very different shapes (e.g., angel/devil, fish/frog, fish/boat, bird/fish, horse/bird). But he produced a few HH tilings in which the two tiles have very subtle differences, and some viewers have believed the tiles to be congruent. The geese in the tiling for the print *Day and Night* fooled the editor of *Scientific American*, who identified the birds as being identical (*Scientific American* 1961, 4 and cover). In fact, as Escher pointed out, they are two different shapes—white geese have tails up, blue geese have tails down. This was Escher's first HH tiling, and has only translation symmetries (Fig. 6).

Type (3): Two Colors, Two Different Tiles, Each Occurring in Both Colors As Escher investigated splittings for each of his ten categories of isohedral types of quadrilateral tiles and several "transitional" tilings derived from them, he found two or even more variations of splitting the tiles. Each variation produced a different 2-color, 2-motif tiling. Several of these new tilings were surprisingly complex—they had two distinct shapes of tiles, but each shape repeated in both the dark and the light color. These tilings, like those of type (2), were 2-isohedral, but also had the property of counterchange symmetry possessed by those of type (1), and were perfectly colored. Although Escher inventoried over 25 of these tilings, he completed only seven numbered colored drawings exhibiting this "double duality."

His first drawing with "recognizable" tiles, made in 1941, shows graceful interlocked birds and fish (Fig. 7a). Although he designed an inlaid door with this pattern, it was not fabricated. In 1952 he used the tiling cleverly in his print *Two Intersecting Planes* (Bool et al. 1982, cat. no. 377). The print shows thickened tiles, each layer (one dark, one light) cut from a slab of wood. The two slabs with fish and bird cut-outs intersect at a sharp angle, showing how they perfectly fit together, one serving as figure, the other ground.

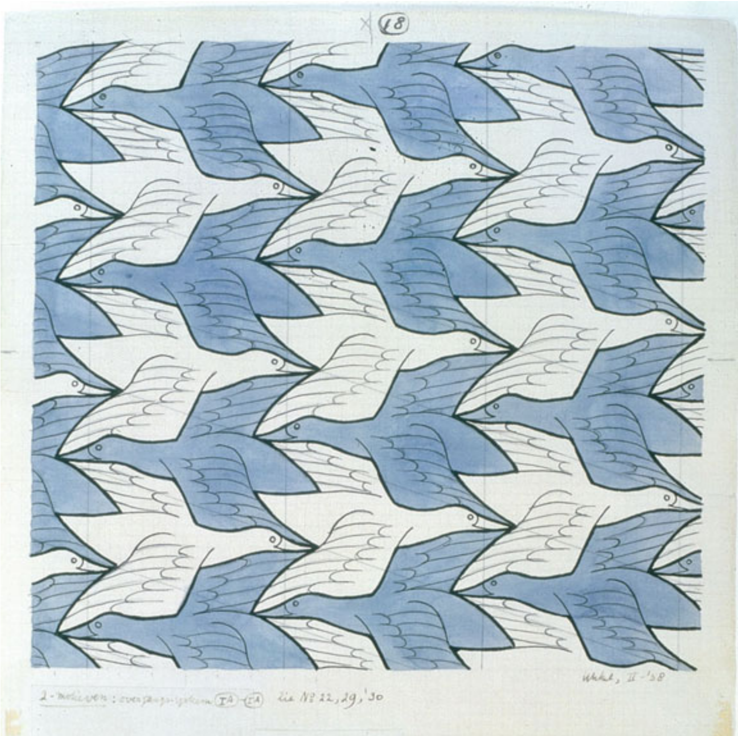

Fig. 6 Escher's Symmetry Drawing E18. 1938. An HH tiling used in the print *Day and Night*

The tiling in Fig. 7a likely began with a 3-color tiling resembling interleaved fish (Fig. 7b); Escher then split each tile into a bird and a fish. Each tile in Fig. 7b has six vertices (identified by dots on one tile) and the edges that connect consecutive vertices on this tile are labeled (in cyclic order, starting at the left) with Heesch notation $TG_1G_2TG_2G_1$. The two edges labeled T are related by a horizontal translation, while the pair labeled G_1 and the pair labeled G_2 are related by glide-reflections (axes of these glide reflections are shown as dotted lines). Escher's splitting curve joins opposite vertices of the original tile so that in the new tiling by birds and fish, one glide-reflection interchanges birds in a column and the other glide-reflection interchanges fish in a column. All glide-reflections interchange colors while all translations preserve colors.

One of the last tilings produced by Escher was another intricate bird-fish tiling, perfectly colored and having counterchange symmetry (Fig. 8a). Created in 1967, it was based on an underlying square grid, so curved square tiles could be fabricated by the Delft tile company. The tiles wrapped around a large column, recreating the creatures' graceful dance. Escher's use of symmetry produced economy: only one mold was needed for the single ceramic square tile containing (in pieces) two birds and two fish. By glazing the tile in two different states—one with white birds and black fish, the other with colors interchanged (Fig. 8b), the repeating design was

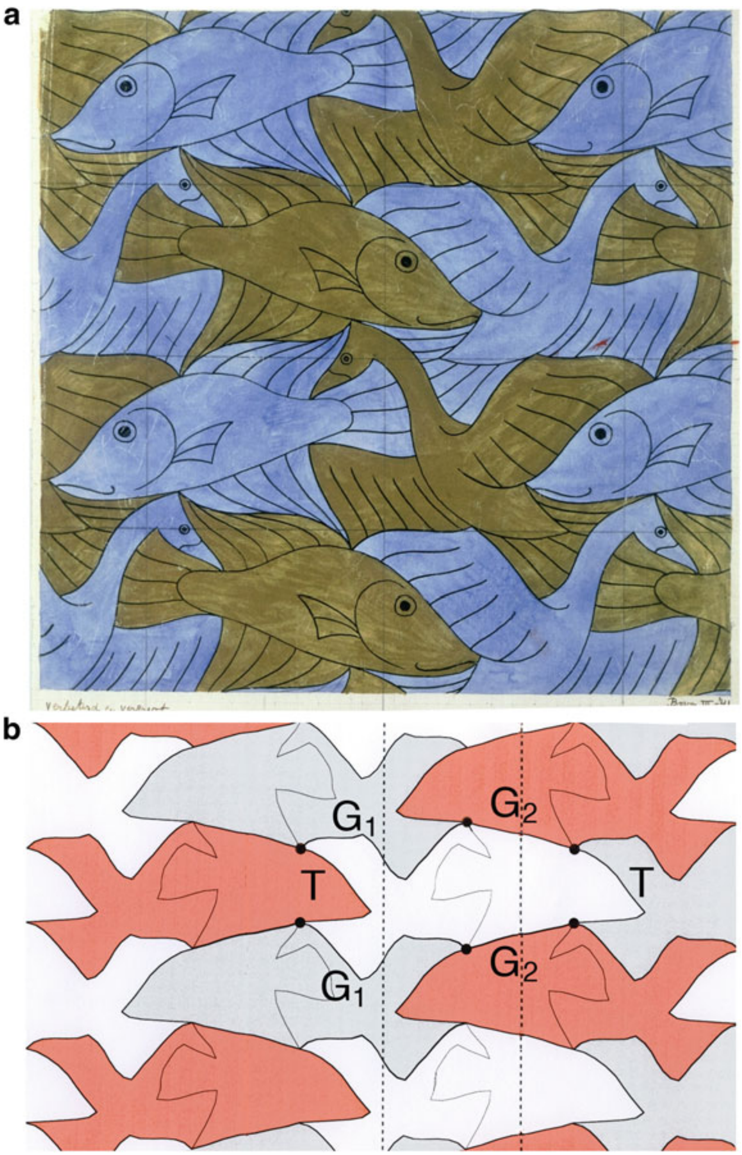

Fig. 7 (**a**) Escher's Symmetry Drawing 34B, 1941. Used in the print *Two Intersecting Planes*. (**b**) The 3-color tiling from which Escher likely derived the 2-motif, 2-color tiling 34B

produced by alternating positive and negative versions of the square tiles in horizontal rows, while matching the same-colored tiles vertically (Fig. 8c).

Although this tiling, like that in Fig. 7a, began with an isohedral tiling having glide-reflection symmetry, the boundary of the original tile and Escher's manner of

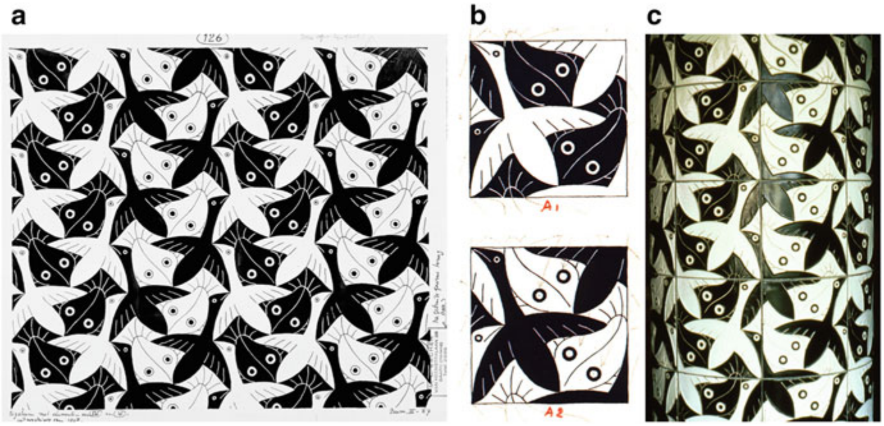

Fig. 8 (**a**) Escher's Symmetry Drawing E126, 1967. (**b**) Design for Delft tile, to be glazed in two ways. (**c**) Tiled column in Baarn school

splitting it into a bird and a fish caused significant differences in the two derived 2-isohedral tilings. In Escher's drawing 34B (Fig. 7a), each fish shares two edges with other fish; similarly, adjacent birds also share two edges. But in drawing 126 (Fig. 8a), no fish share any edges, while each bird shares two edges with adjacent birds. Also, the color symmetries in drawing 126 are different from those in drawing 34B. As you can see in Fig. 8c, there are vertical glide reflections (with axes near the edges of the square tiles) that interchange colors and other vertical glide-reflections (with axes near the centers of the square tiles) that preserve colors. In addition, horizontal translations interchange colors while vertical translations preserve colors.

Escher favored glide-reflection symmetry in his tilings, and so it is not surprising that all but one of his double-duality tilings have glide-reflection symmetry. Those discussed thus far have this symmetry. His drawing 130 of fish and donkeys (Schattschneider 2004, 223) has exactly the same color symmetries as those of drawing 126 in Fig. 8. Three additional double-duality tilings have two fish motifs (not surprising, since viewed from above, these creatures can have curvy outlines, and can swim in any direction). Drawings 57 and 58 (Fig. 9) both show tilings by two species of fish and display new or additional symmetries.

His tiling of two swirling fish in drawing 57 (Fig. 9, left) has only rotation and translation symmetries. Here 60° and 180° rotation symmetries interchange colors, while translation and 120° rotation symmetries preserve colors. In drawing 58 (Fig. 9, right), fat and skinny fish glide past each other in columns, all light fish swimming up, all dark fish swimming down. Here, 180° rotation symmetries interchange dark and light columns of fish and horizontal glide-reflection symmetries also interchange colors. Vertical glide-reflections and all translation symmetries preserve colors. Escher used this design for a commissioned New Year's greeting card. His third drawing of a double-duality tiling of two different fish is numbered 41 (Schattschneider 2004, 147), and has the same color symmetries as his drawing 58.

Fig. 9 Escher's Symmetry Drawing E57 (*left*) and Symmetry Drawing E58 (*right*), 1942

There is one other 2-colored tiling by Escher that might be considered a double-duality tiling, but it depends on interpretation. In his 2-colored drawing 125 (Schattschneider 2004, 218), if you omit the internal detail of the tiles, they are all congruent, resembling flying fish, and so the tiling is type (1). If internal detail is taken into account, then there are two different creature tiles, and the tiling is perfectly colored, with the same color symmetries as drawing 34B in Fig. 7a.

Escher Inspiration: A New Double Duality Tiling

Escher's fascination with duality and symmetry is contagious. Although the types of duality tilings of types (1) and (2) are relatively common, double duality tilings of type (3) are rare. Recently I learned of the work of Dylan Thomas, a Coast Salish artist from the Lyackson First Nation, in British Columbia, Canada. The Salish are known as "The Salmon People," and for centuries, fished the rich waters of the Pacific Northwest for salmon. But in the past decades, the number of salmon has dropped considerably. Thomas's print *Salmon Spirits* (Fig. 10) represents the overcrowding of salmon in the spirit world. The print was inspired by Escher's work in an indirect way.

The artist says that it was Escher's tiling of butterflies with 6-fold rotation symmetry (Schattschneider 2004, 70) that inspired him. He decided to look for a

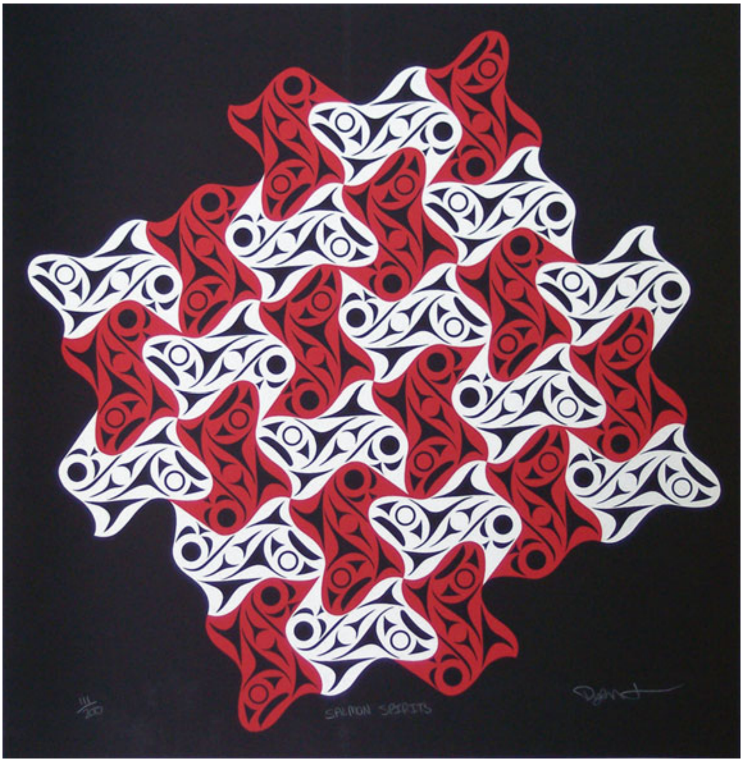

Fig. 10 *Salmon Spirits*. © Dylan Thomas, 2007

way to have 4-fold symmetry with a tiling of salmon, the motifs done in the style of Salish art. He began with a grid of squares, and guided by the concept of "flow," important in their art, developed the edges of the square into sinuous curves. The curvy squares served two roles: they outlined four salmon rotating about the center point where dorsal fins touched, and they fit together in a tiling. Thomas also wanted four salmon to be seen rotating about a point where their pectoral fins touched; in order to fill the curvy square in this manner, he had to change the salmon motif slightly by joining the pectoral fin to the tail with a deeper curve. He then put together the curvy squares, each with four salmon, alternating the two types of squares in rows and columns, like a checkerboard. Just as with Escher's geese in Fig. 6, it is easy to believe (falsely) that the salmon tiles are identical.

Escher discovered three types of 2-isohedral, 2-color tilings with counterchange symmetry by splitting tiles in a tiling with 4-fold (90°) rotation symmetry. His arrangement of color symmetries in one of these (Schattschneider 2004, 75, top left) is the same as those in *Salmon Spirits*, but Thomas's arrangement of tiles is quite different from Escher's. Figure 11 shows schematically how the tiles and the symmetries in *Salmon Spirits* are arranged in an extended tiling.

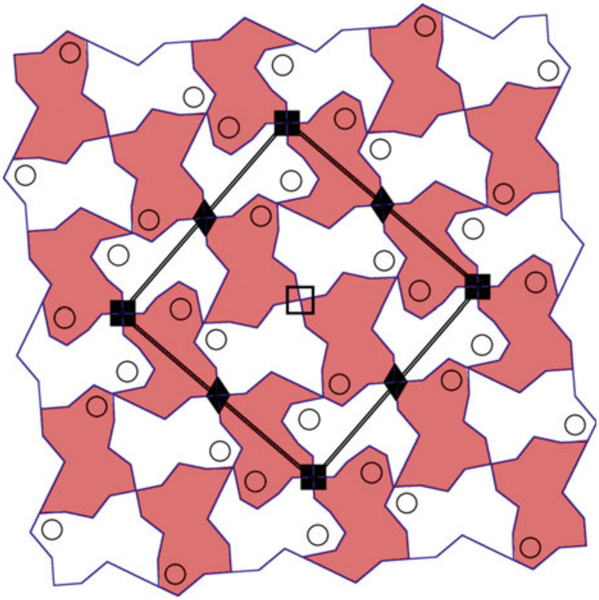

Fig. 11 Schematic showing the symmetries of *Salmon Spirits*

Just as in the print, hollow circles identify eyes of the salmon. Small solid squares identify 4-fold rotation centers where pectoral fins of identical salmon meet; the small open square is a 4-fold rotation center where dorsal fins of the other salmon meet. Small diamonds identify 180° rotation centers for the tiling. The edges of the large superimposed square define minimal translation vectors for symmetries of the extended tiling; this square encloses 8 salmon, four of each type (in pieces). Translation symmetries and 180° rotation symmetries preserve colors, while 90° rotation symmetries interchange colors.

I think that Escher would have been pleased.

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The Vasarely Playhouse: Invitation to a Mathematical and Combinatorial Visual Game

Slavik Jablan and Ljiljana Radović

Abstract The authors analyze Victor Vasarely's works from the viewpoints of the theory of visual perception, mathematics and modularity. The chapter concludes that almost all construction methods, modular elements, optical effects and visual illusions belonging to these fields were (re)discovered by Vasarely, mostly by intuition, creative visual thinking and experimenting, and then used in his artworks.

Introduction

This chapter introduces the mathematical background in the oeuvre of Victor Vasarely. Kepler's cube, Koffka cube, antisymmetry, Moiré effect, layer, modularity, Kufic tiles, random structures, grids, affine transformations, recombinations, impossible figures, ambiguity and visual illusions are some of the terms that originated from mathematics or the theory of visual perception that can be used to describe Vasarely's works. Without the intention to give an exhaustive overview of all the mathematically relevant moments of Vasarely's extremely rich artistic career, we offer a number of starting points which enable readers to explore the mathematical connections of the scientific interest underlying Vasarely's perspective. This is the invitation to a reader to play and experiment with Vasarely's works as well as to reach perceptions of their own. Through this experience of interaction, in accordance with Vasarely's original intentions, everyone has a chance to become an active participant in the creative process and not only to understand, but also to enjoy the artworks of combinatorial and permutational character.

Interdisciplinary discussions on the relations between art and mathematics, aesthetics and artistic practice have a long history. Still today, cultural phenomena under the influence of mathematics and art continue to inspire people working in

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different fields of science. Similarly, many scholars share a common interest in combining creative thinking, intellectual curiosity, and aesthetic sensibility in their work. This paper, based on the collection of the Vasarely Museum in Pécs, Hungary and aims to initiate a dialogue between the artistic and mathematical points of view in the field where artistic and mathematical thinking and practice merge (Jablan and Radović 2011, in Preface by Fenyvesi).

Op Art

Op Art appeared in the late 1950s. The term Op Art (short for *optical art*) was first used in an October 1964 *Time* magazine article (Barrett 1970, 6), which pointed out that one of its distinguishing features is the use of certain optical effects, studied from a scientific point of view by psychologists and physiologists, and unified in the theory of visual perception. However, this term is somewhat misleading, because it suggests that this kind of art produces an optical physiological response only. The real response to Op Art works is psychological, and, according to Josef Albers, "it happens behind the retina, where all optics end" (Barrett 1970). Hence, Op Art mostly uses a few important features of our *visual thinking* (the term introduced by Rudolf Arnheim in 1969): antisymmetry, complementarity and chromatic tension (i.e., a tendency to balance between opposite colors), ambiguity (which our mind does not seem to accept thus logic forces us to make a choice between two equally accurate interpretations of the same object), and perspective illusions.

Despite the fact that Op Art was established as an art movement in the sixties, one of the first Op Art works was done by Marcel Duchamp in 1935. His six black and white *Rotoreliefs* (1935)—systems of almost concentric circles on a turntable—give an illusion of depth while the rotation of this system produces the visual illusion of homothety (Barret 1970, 31). Another is *The Fluttering Hearts* (1936): four hearts alternately colored and enclosed within one another, which produce a pulsating effect—heartbeat, by means of flicker, after-image, advance, and recession of colors (Barrett 1970, 24).

Bauhaus

In a certain sense, Op Art represents a continuation of the constructivist practices of Bauhaus, the German art school founded by Walter Gropius. Bauhaus artists made various experiments with black and white colors, similar to Op Art works. As an example we can take *Black and White Stripes Cut into Concentric Stripes and Displaced*, 1928–1933 (Barrett 1970, 33). Some of the most prominent Op artists of the twentieth century include Bridget Riley, Richard Anuszkiewicz, Jesus Rafael Soto, Jeffrey Steele, François Morellet and, in the present time, Akiyoshi Kitaoka.

The psychology of visual perception was systematically studied and applied in the Op Art movement. Some intuitive attempts to use the principle of modularity, layers, visual illusions and similar methods in art appeared much earlier, but were not so systematically used.

Victor Vasarely, a Hungarian artist born in Pécs (1906–1997), was the originator of Op Art: almost all the methods and techniques used in Op Art can be traced back to his works. During the 1960s and 1970s his optical images became a part of popular culture, having a deep impact on architecture, computer science, and fashion. His innovations in color and optical illusions have had a strong influence on many modern artists and reality-defying pure geometric abstraction has become synonymous with his name. Here we place Vasarely's work in historical context, and discover the earliest sources of his inspiration.

Vasarely and Kepler's Cube

One of Vasarely's favorite geometric illusions was the rhombic tessellation, which he named Kepler's cube. In the theory of visual perception, it is named the Koffka cube (after the Gestalt psychologist Kurt Koffka) (Jablan 2002). Looking at the transparent crystals which form a cube, crystallographer Louis Albert Necker discovered that a single transparent crystal produces two different images: one convex, and the other concave, i.e., it looks either like a cube, or like one corner of a room. This visual illusion is referred to as a Necker's illusion.

The Koffka cube can be viewed as a simplification of Necker's illusion: the image of a cube, when we see the regular hexagon, can be concave or convex. It has two equally valuable interpretations: division of the regular hexagon into three rhombuses, arranged as a regular tessellation {6, 3}, or the plane pattern containing projections of (convex or concave) cubic elements. This is the source of the ambiguity: our eye and brain oscillate between two equally possible interpretations of the same image. The oldest examples of this visual illusion can be found in the mosaics from Antioch, and Vasarely named the illusion with the phrase "Kepler's cube", probably taken from Kepler's works about isohedral tessellations (Fig. 1).

The Koffka cube is multi-ambiguous: it can be interpreted as convex or concave, as three rhombuses with a joint vertex, as a convex or concave trihedron, or as a cube. If we accept its "natural" 3D interpretation—a cube—than for a viewer there

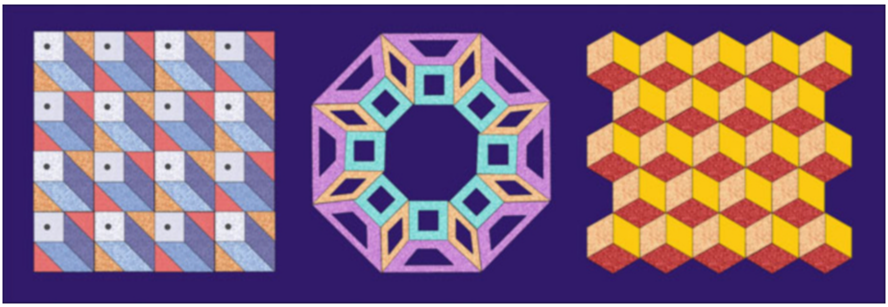

Fig. 1 Mosaics from Antioch. Illustration by Slavik Jablan

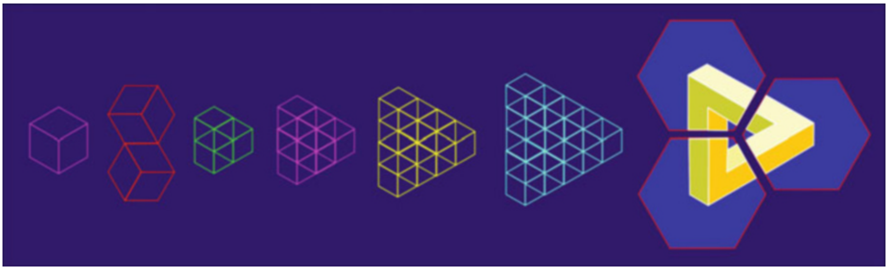

Fig. 2 Koffka cube, Thiery figure, and Penrose tribar Illustration by Slavik Jablan

are three possible positions in space: upper, lower left, and lower right, each having equal right to be a legitimate point of view. So, for the corresponding three directions, a Koffka cube represents a turning point. Having such multiple symmetry, it fully satisfies the conditions to be a suitable basic modular element. Now, we can return to several well-known impossible objects: Thierry figures (proposed at the end of nineteenth century), which consist of two Koffka cubes, the object created by Oscar Reutersvärd in 1934, the Penrose tribar, Vasarely constructions, the alphabet by T. Taniuchi, etc. Almost all impossible figures (Penrose tribar, Escher's infinite staircases, or the geometrical construction of Escher's graphics *Waterfall*) can be constructed using Koffka cubes (Fig. 2).

The Koffka cube and regular hexagons are one of Vasarely's favorite motifs: among 360 graphics from the catalogue of the collection of the Vasarely museum, 86 of them are based on the Koffka cube and its variations. One of the remarkable works of this kind is the paper screen-print from *Bach Album—La-Mi* (1973), with a hexagonal meander composed from Koffka cubes (Fig. 3).

Gestalt Illusion

Vasarely created a series of paintings—*Homage to the Hexagon*—in parallel with the Gestalt series, where he abundantly used Koffka cubes and their combinations, as well as a visual illusion similar to the "Devil's fork" illusion (Figs. 4, 5 and 6).

Black and White

Vasarely's black-white phase was initiated in the figurative Op Art graphics *Zebra* (1938), *Harlequin* (1935), and *Zebra* carpets (1939/1960), based on antisymmetry ("black-white" symmetry, symmetry of opposites), dynamic spiral works with contrasting black and white parts, producing by collision a linear drawing where the parts of lines between two white contours are obtained as a visual illusion

Fig. 3 Victor Vasarely, *La-Mi*, 1973

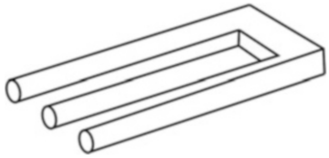

Fig. 4 Devil's fork. Illustration by Ljiljana Radović

similar to Kanitza illusions (Fig. 7). *Zebras* (Fig. 8) is composed of black and white stripes varying in width and formed into a loose spiral containing the interlocking heads and necks of two zebras. Dynamic spiral works with contrasting black and white parts produce by collision the linear drawing, where the parts of lines between two white contours are obtained as a visual illusion. *Payta* (1955/1967) can be viewed as a periodically interrupted system of black and white stripes in such a way that it gives an impression of transparency (Fig. 9). The instability of the composition, which is due to its black-white line breaks, creates the impression of movement. *Novae* (1959/1967) also employs interrupted visual systems. The distortion of the black and white squares results in a bulging or hollowing of certain areas of the picture (Fig. 10). Although this effect is concentrated in certain areas, it affects the entire picture.

All the effects and visual illusions that can be obtained in black and white are present in Vasarely's works from the fifties, beginning with *Transparency* (1953), Vasarely's brilliant homage to Duchamp's *Rotoreliefs*, and in mature works like *Taymir II* and *Supernovae* (1959/1963), which is based on after-images and irradiation.

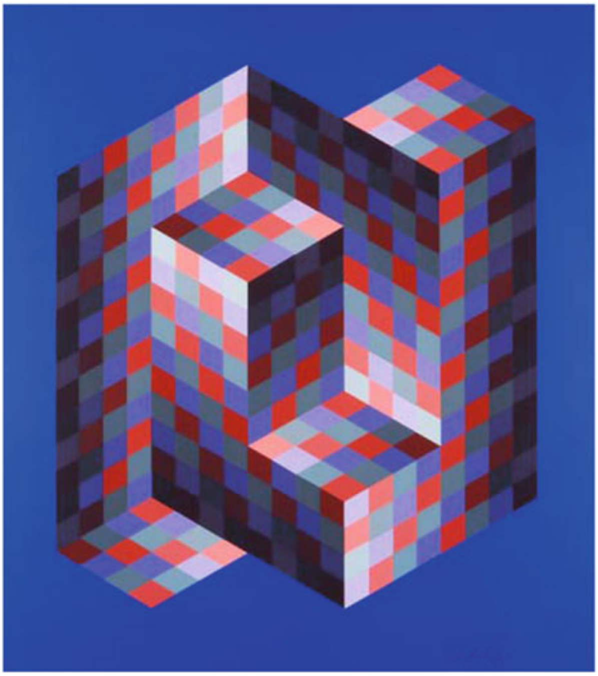

Fig. 5 Victor Vasarely, *Iz-zo*, 1973

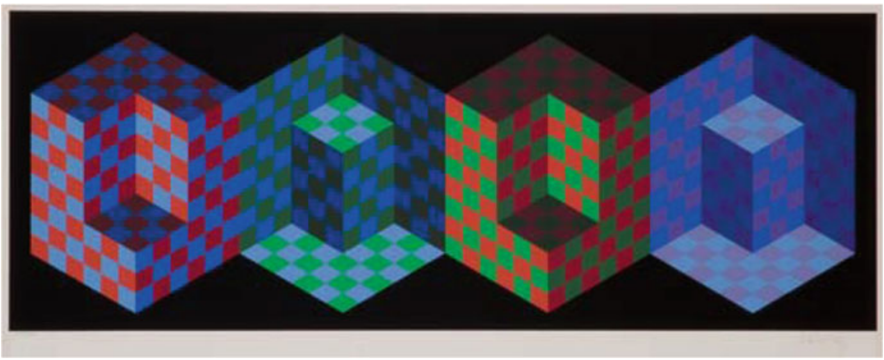

Fig. 6 Victor Vasarely, *Vancouver*

Antisymmetry

Antisymmetry is not just black-white symmetry: in general, it is the symmetry of opposites ("positive-negative", "convex-concave", "light-shadow", "warm-cold", "circle-square", or symmetry between complementary colors) (Radovic and Jablan

Fig. 7 Kanizsa triangle

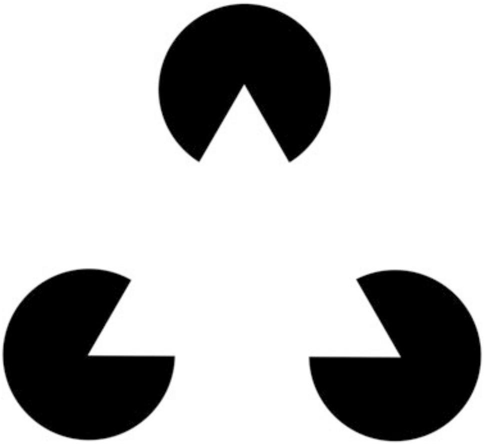

Fig. 8 Victor Vasarely, *Zebras*, 1939/1960

2001). In the perceptual and philosophical/logical sense, antisymmetry can be used to express duality, based on binary codes (0–1).

After the 1960s, black-white compositions were replaced in Vasarely's work by colored structures, which were mostly based on complementary colors, or by a continuous change from light to shadow, and *vice versa*. With this, he introduced a spatial component in his works. Our perception of three dimensional objects

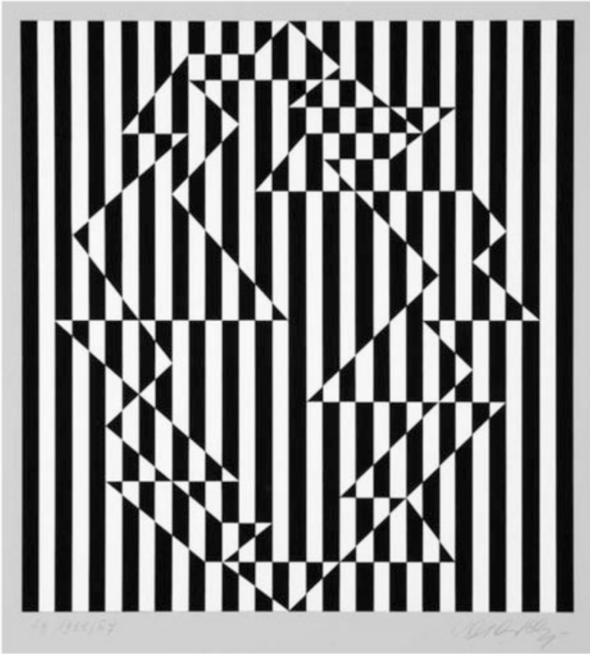

Fig. 9 Victor Vasarely, *Payta*, 1955/1967

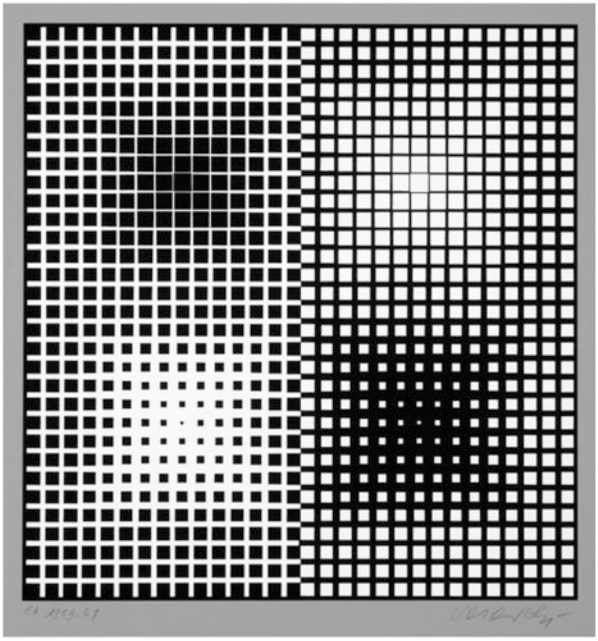

Fig. 10 Victor Vasarely, *Novae*, 1956/1967

strongly depends on light: without light and a changing position of viewpoint, it is not possible to distinguish a concave crater on the Moon from a mountain of the same shape. In general, almost all Op Art works are based on some kind of duality (Figs. 11 and 12).

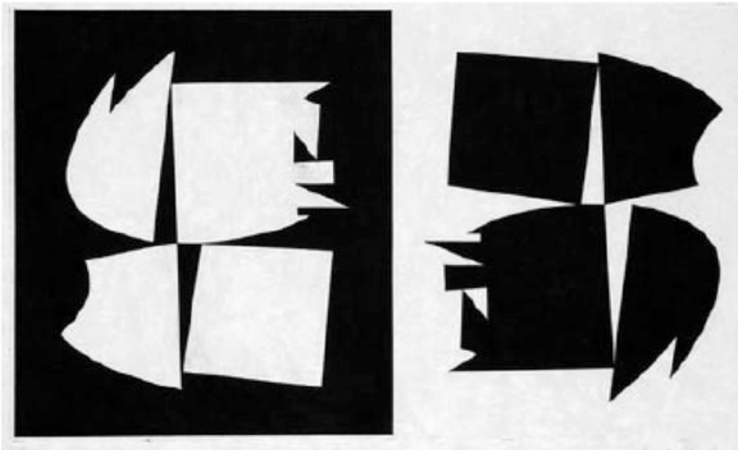

Fig. 11 Victor Vasarely, *Zaira Pos-neg*, 1976

Fig. 12 Victor Vasarely, *II/v.h.c.*, 1973

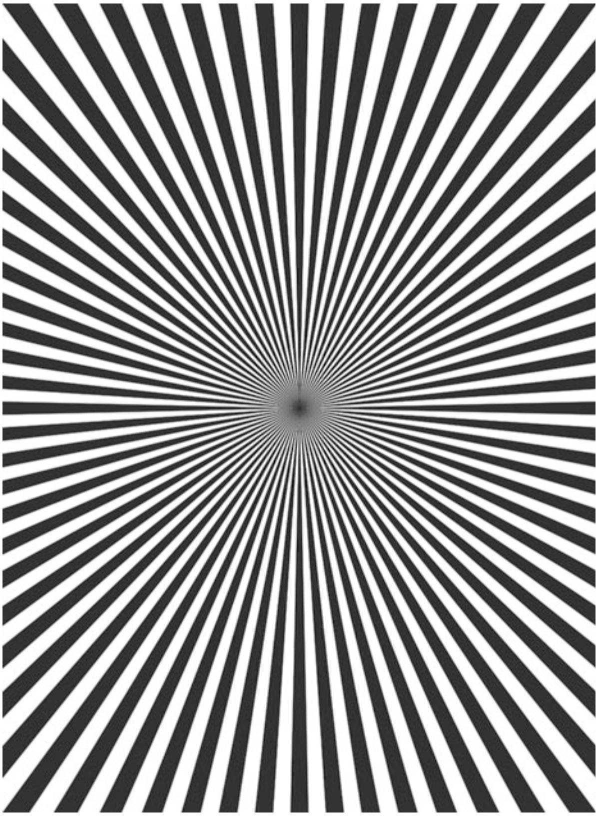

Fig. 13 Homage to the MacKay-illusion. Illustration by Ljiljana Radović

Antisymmetry and Irradiation

A specific form of antisymmetry is the situation where a figure (usually black) and the ground (white) both represent meaningful images, or are even congruent; from this, ambiguity arises, and our eye and brain oscillate between the two possible perceptual interpretations. Moreover, in abstract paintings a figure could be congruent to the ground, and in the case of periodical or quasi-periodical structures or patterns this results in the strange visual effect of flickering and dazzle, or even in very simple periodic black and white patterns (e.g., in the system of concentric periodic black and white circles or a spirals), producing the effect of irradiation, similar to that produced by MacKay's figure.[1] In the case of complementary colors, this results in colored irradiation and the appearance of colored after-images (Fig. 13).

[1]See Donald MacKay's figure at http://popgive.blogspot.rs/2008/12/examples-of-kinetic-illusions-in-op-art.html. Retrieved on 30 January 2017.

The *Moiré* Effect

Moiré is the French word meaning "watered". The *moiré* effect can be produced by superimposing two periodic structures (e.g., grids) which are overlaid at an angle, or when they have slightly different mesh sizes, resulting in an interference pattern. In general, the *moiré* effect is based on the use of interfering layers. Vasarely describes his own (re-)discovery of the layer method in the following way:

> In my native Hungary the windows are double because of the extremes of the continental climate. One winter when I drew a sun-face on the outer pane, shutting the inside window frame, I tried to reproduce the same drawing on the second transparent surface, separated from the first by some six or eight inches... These two sun-faces, which were superimposed when looked at directly, doubled their grimaces when I moved my head to the right or to the left. This crude little cinema left deep traces in my sub-conscious. (Barrett 1970, 146.)

In a similar way, Vasarely (re)discovered the *moiré* effect, which originated from his passion for grids:

> In about 1913, when a child, I injured my forearm while playing... The wound was dressed with gauze, a light white fabric that changed its shape at the slightest touch. I never tired of gazing at this micro-universe, ever the same and yet different. I would play with it, pulling the crowded threads one by one. (Barrett 1970, 147.)

The *Bo-ra* (1973), Vasarely's rectilinear homage to Duchamp's *Rotoreliefs*, consists of identical patterns in black in white which are superimposed and move over one another (Fig. 14).

Afterimages

Centavri (1959/1965/1967) is based on a periodic structure of the simplest kind: a grid of white stripes and a regular system of black squares. In the theory of visual perception, it is well known as a tool for creating after-images: small flickering gray dots in the crossings of white stripes, further elaborated by using affine transformations, after-images, and irradiation (Figs. 15, 16 and 17).

Magnifying Glass Illusions

The circular areas appear as magnified parts of the stripes and grids laying in the back. However, it is pure illusion: only certain parts of the circular areas can be magnifications, and the remaining parts are either exactly the same or totally different from the surrounding area (Figs. 18 and 19).

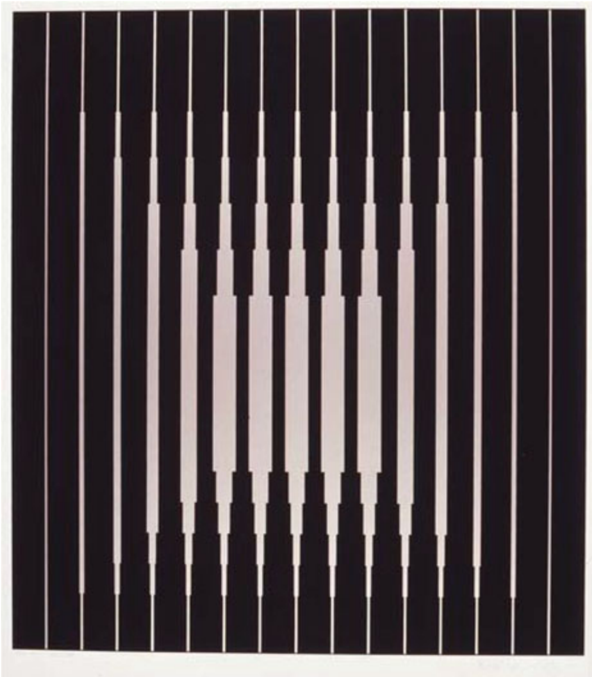

Fig. 14 Victor Vasarely, *Bo-ra*, 1973

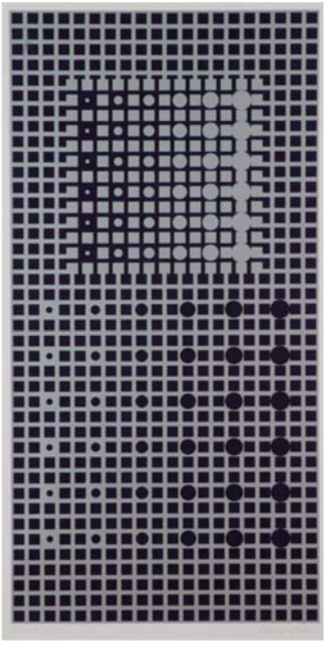

Fig. 15 Victor Vasarely, *Centavri*, 1959/1965/1967

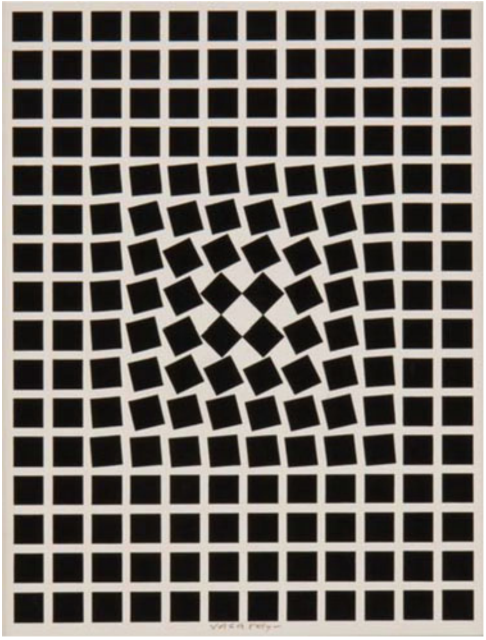

Fig. 16 Victor Vasarely, *Eridan*-os, 1956

The Kinetic Component

Kinetic and time components can be incorporated into static artworks in a similar way. Besides color changes, the layer method, familiar to all users of graphical computer programs, can be used to produce dynamic visual effects. This is achieved through making several layers, usually copies of the same pattern, and placing them one over another. In art history, the layer method of construction is as old as the discovery of transparent materials: waxed paper, or glass. An old example of the application of the layer method in ornamental art is the construction of Islamic patterns from the Mirza Alakbar collection (Fig. 20).

Kinetic objects and graphics are based on layers, in which the objects are static and the viewer moves. Both Vasarely and Jesus Rafael Soto created kinetic objects. Based on the same layer method principle, Vasarely constructed 3D kinetic objects as well (e.g., the aluminum and screen-print construction, *Sir-Ris*, 1968) (Fig. 21).

Principle of Modularity

In a general sense, the modularity principle is a manifestation of the universal principle of economy in nature: the possibility for diversity and variability of structures, resulting from some (finite and very restricted) set of basic elements by their recombination. In all such cases, the most important step is the first choice (recognition

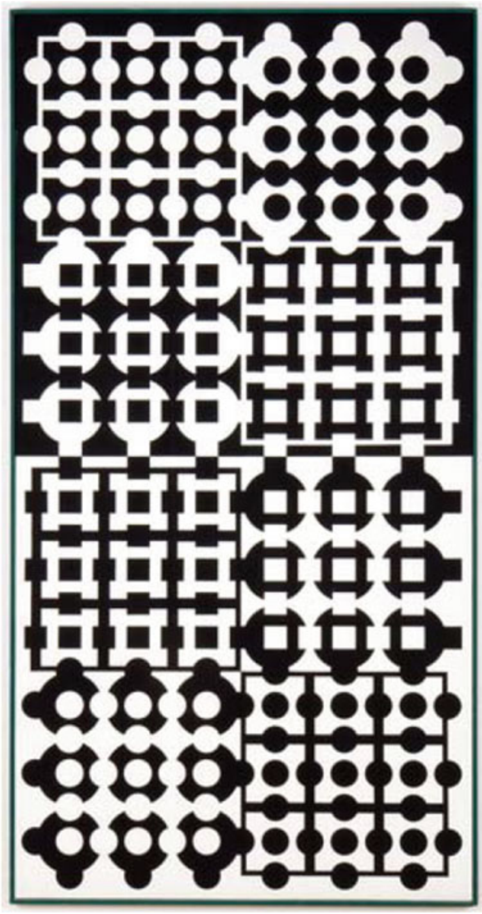

Fig. 17 Victor Vasarely, *Helios VIII*, 1959

or discovery) of basic elements. In science, the modularity principle is represented by the search for basic elements (e.g., elementary particles, 'proto-tiles' for different geometric structures, etc.). In art, different modules (e.g., bricks in architecture or in ornamental brickwork, etc.) occur as the basis of modular structures.

Vasarely used the principle of modularity and supposed that basic compositional elements could be arranged (and rearranged) in millions of variations and would become available to everybody for a reasonable price. In this way, modularity offers unrestricted possibilities for variation and creativity. By using the basic geometric forms of circle, square, ellipse, rhomboid, spiral, meander, etc. (i.e., visual archetypes), a fantastically rich scale of colors and a relatively small set of basic elements (modules) can create an infinite variety of structures.

This can be demonstrated with examples from ornamental art, where some elements, originating from Paleolithic or Neolithic art, remain present even to this day as a kind of "ornamental archetypes". Such modules are, for example, a square with the set of diagonal lines, two antisymmetrical squares, black-white

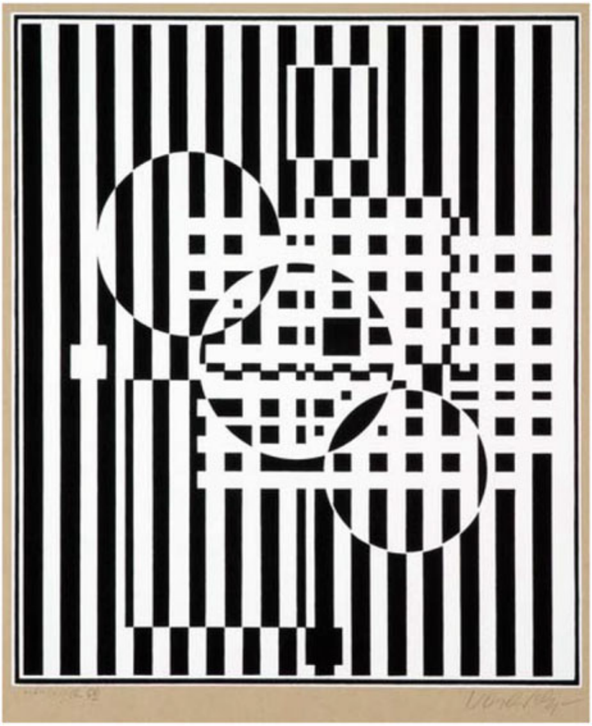

Fig. 18 Victor Vasarely, *Encelade*, 1956/1962

square (abundantly used in the prehistoric or ethnical art, known also as the element of mosaics: Truchet tile). This module, a square or rectangle with a set of parallel diagonal black and white strips, is known as an "Op-tile" (Figs. 22 and 23).

Black and white Op-tile (used as a single element) can be found in the right upper corner of the plexi screen-print *Bach Album- Naissances No. 137* (1973) (Fig. 24).

The paper screen-print *Delocta* is composed from four colored curvilinear Op-tiles (two "positives" and two "negatives", colored in complementary colors) with three diagonal strips, decorated by patterns in complementary colors (Fig. 25).

Kufic Tiles

If in a black and in a white square we construct one diagonal region of opposite color, we obtain two modular elements, which we can call Kufic tiles, because of the similar principle to what is implemented in constructing geometric or square Kufic scripts, which are often used as tilework patterns in Islamic architecture. The Kufic tile is the simplest Op-tile: a white square with one black diagonal strip, or its negative. If you start from the simplest antisymmetrical squares with only one

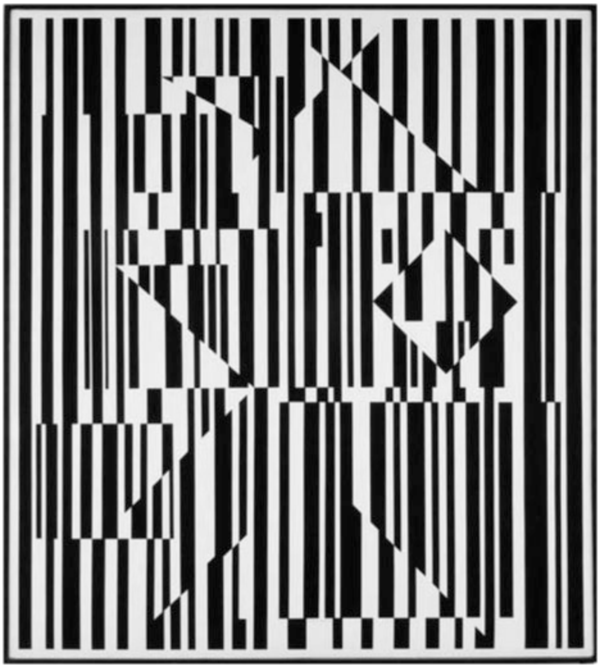

Fig. 19 Victor Vasarely, *Taymir II*, 1956

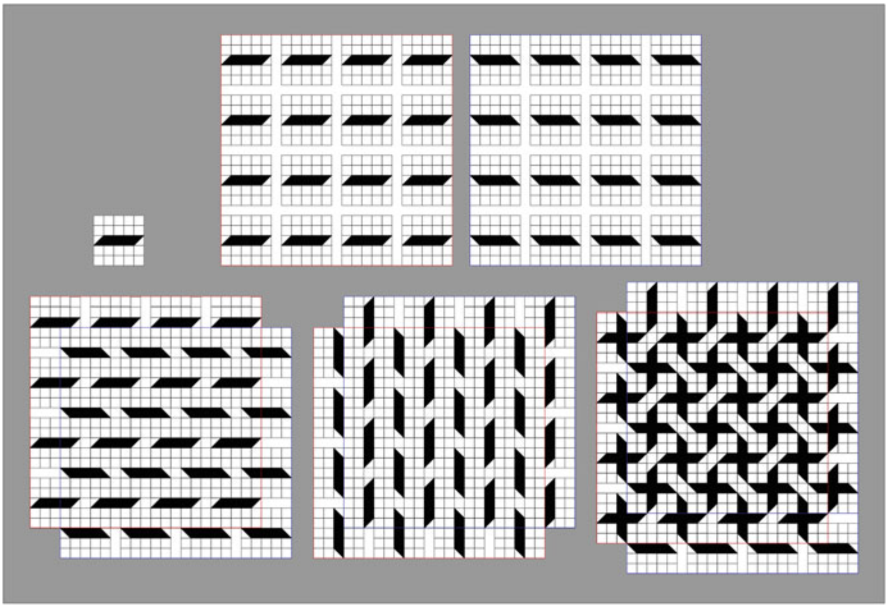

Fig. 20 Construction of Islamic patterns based on the layer method Illustration by Slavik Jablan

Fig. 21 Victor Vasarely, *Sir-Ris*, 1968

diagonal field, you can derive an infinite series of black-and-white ornaments, including in this class many key-patterns, different Neolithic ornaments, as well as Kufic writing. The same proto-tile is well known in the Renaissance and later European ornamental art as a basic element for the Persian and Islamic schemes (Fig. 26).

Similar modules like Kufic tiles are present in Vasarely's works, but in their implicit form: as four decorated rectilinear Kufic tiles composed into a square in the tapestry *Dia-argent* (1969) (Fig. 27).

Curvilinear decorated Kufic tile obtained from two quarter parts of blown-up hemispheres with the centers in the opposite corner of a square, and the remaining part representing a diagonal strip in the Kufic tile (*Énigmes album*) (Fig. 28).

Colors and Light

After the sixties, Vasarely replaced the black-white compositions in his work with colored structures, which were mostly based on complementary colors, or by a continuous change from light to shadow, and *vice versa*, thereby introducing into his works a spatial component.

In *Vonal-Fegn* (1971) (Fig. 29), the feeling of movement and depth is created by Vasarely's use of lines of decreasing scale advancing towards the centre of the canvas—the further in we look to the centre, the further away the field appears to be from us. The use of changing colors across the field also serves to provide the viewer with the feeling of kinetic energy, depth, and space.

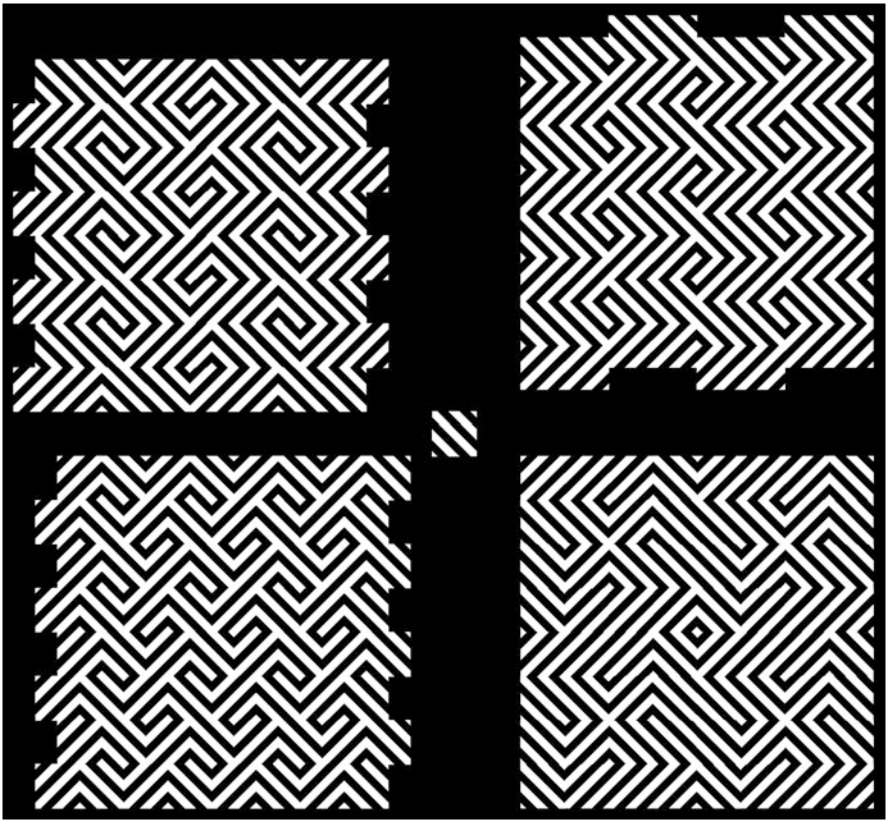

Fig. 22 “Op-tiles”. Illustration by Slavik Jablan

Fig. 23 Bracelet from Mezin, 23,000 BC, on display at the National Museum of History of Ukraine in Kyiv. Used with the permission of Irina B. Vavilova and Tetyana G. Artemenko

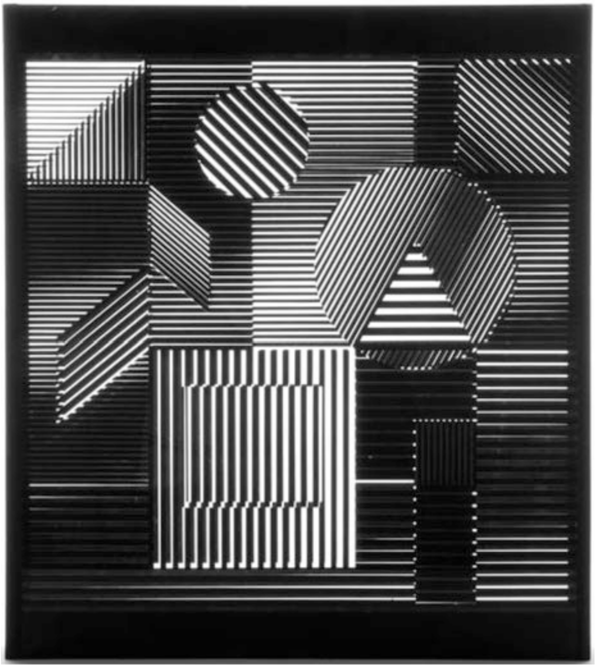

Fig. 24 Victor Vasarely, *Bach Album-Naissances No. 137*, 1973

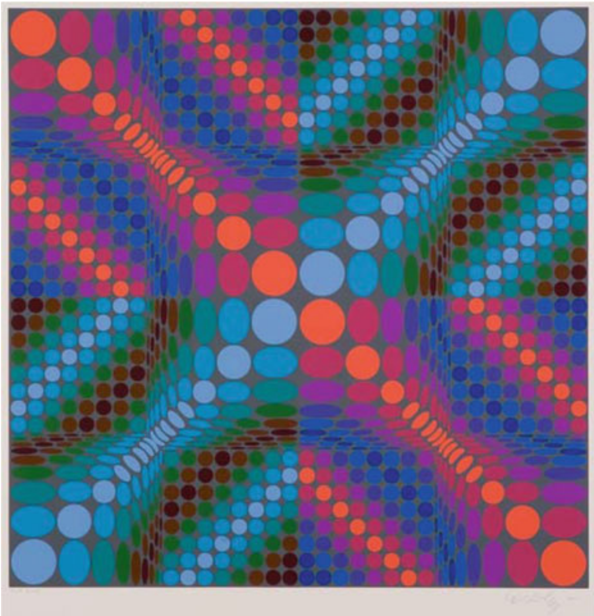

Fig. 25 Victor Vasarely, *Delocta*, 1979

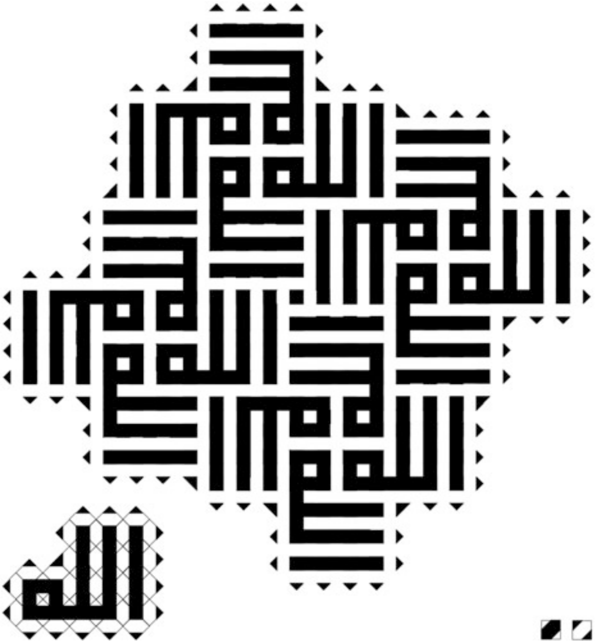

Fig. 26 Kufic writing. Illustration by Slavik Jablan

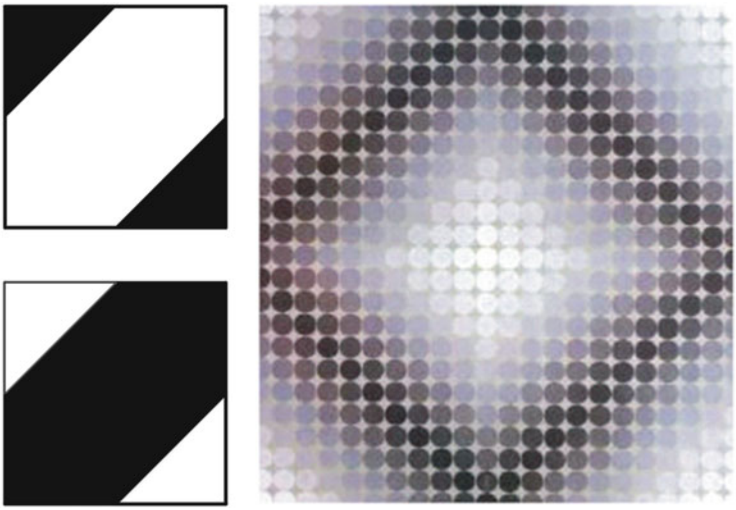

Fig. 27 *Left*: Illustration for rectilinear 'Kufic tile' design principle. *Right*: Victor Vasarely, *Dia-argent*, 1969

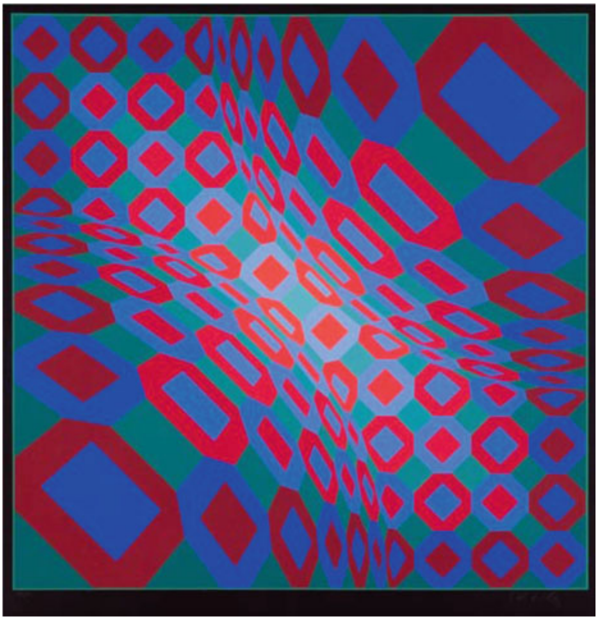

Fig. 28 Victor Vasarely, *Énigmes album*

Fig. 29 Victor Vasarely, *Vonal-Fegn*, 1971

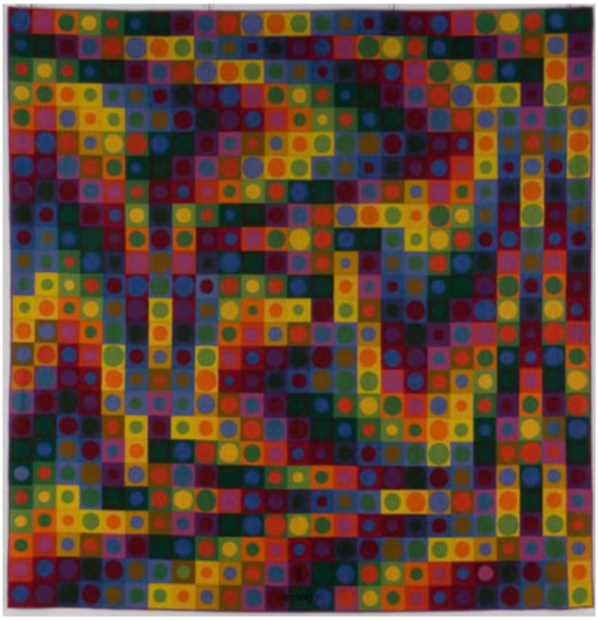

Fig. 30 Victor Vasarely, *Majus MC*, 1967

Random Structures

Random and stochastic structures, used by many Op artists (e.g., by F. Morellet), found their place in Vasarely's works—for example *Majus MC* (1967) (Fig. 30)—which reminds us of Mondrian's *Broadway Boogie Woogie* (1942/1943).

Some of Vasarely's visionary graphics, for example *Tuz* (1966/1970) (Fig. 31), presaged the artworks of one of the contemporary masters of kinetic art based on visual illusions, Akiyoshi Kitaoka.[2]

Zigzag Lines

Arrangements of zigzag lines, representing examples of interrupted systems, were used by Vasarely to create very subtle and refined suggestions of 3D space structures (e.g., the paper screen-prints *Ilile*, or *Zint*) (Figs. 32 and 33).

[2]See Akiyoshi Kitaoka's works at www.ritsumei.ac.jp/~akitaoka/index-e.html. Retrieved on 1 August 2016.

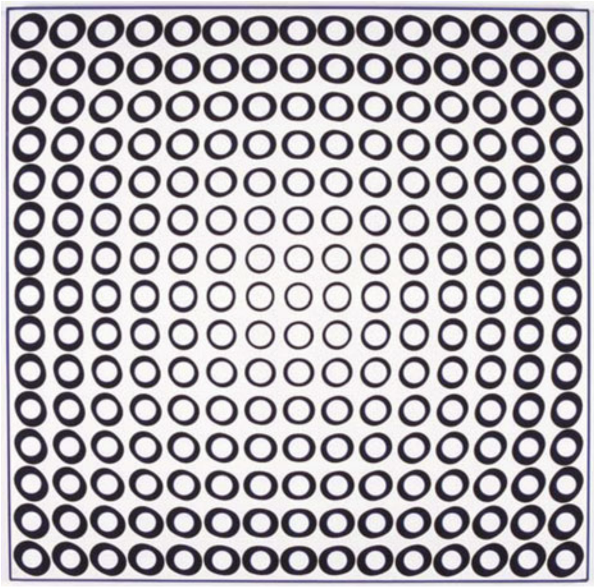

Fig. 31 Victor Vasarely, *Tuz*, 1966–1970

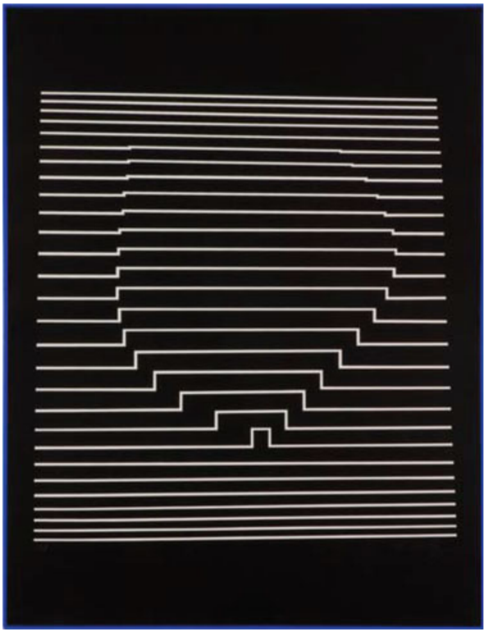

Fig. 32 Victor Vasarely, *Ilile*

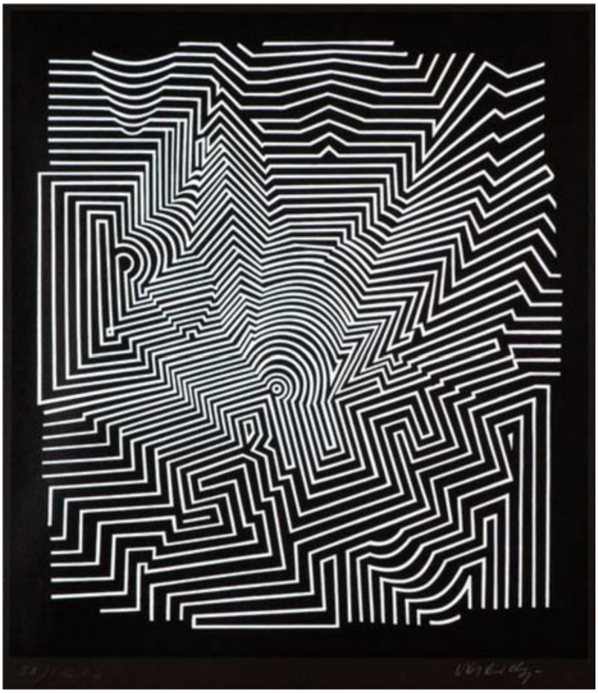

Fig. 33 Victor Vasarely, *Zint*, 1952/1964

Passion for Grids

Vasarely developed a passion for grids, and in his works used affine deformations of the regular square grid in order to produce visual illusions of abstract 3D surfaces. A representative example of this method is Vasarely's acrylic painting *Biadan* (1959), which uses 3D computer graphics based on grids: triangulations of surfaces, where the size of the elements depends on from the local curvature of the surface (Fig. 34).

Vasarely's fascination with geographic cards and isohypses are evident in his paper screen-print *Bi Rhomb* (Fig. 35).

Many of Vasarely's works suggest a continuous affine transformation from squares to rhombuses or *vice versa*, followed by a color change which is in perfect accordance with the geometrical changes (e.g., *Quasar-Dia-2*, 1965, Fig. 36). A more complex topological transformation of a black square grid into its curvilinear equivalent is shown in the tempera painting *Yapoura* and *Yapoura-2* (1951/1956) (Fig. 37 and 38).

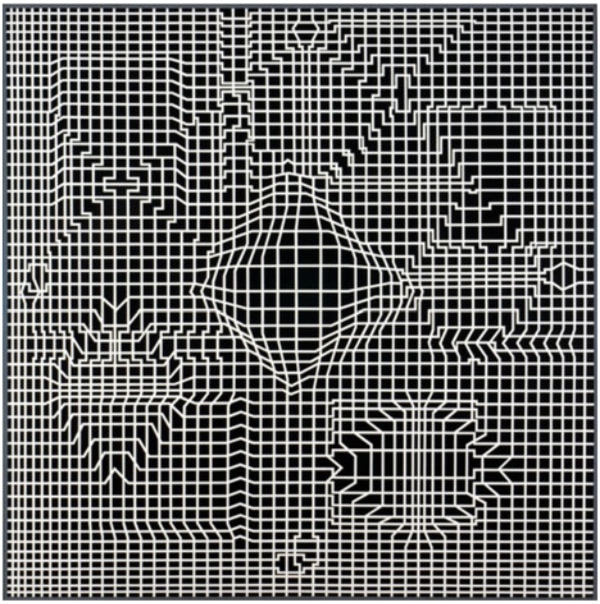

Fig. 34 Victor Vasarely, *Biadan*, 1959

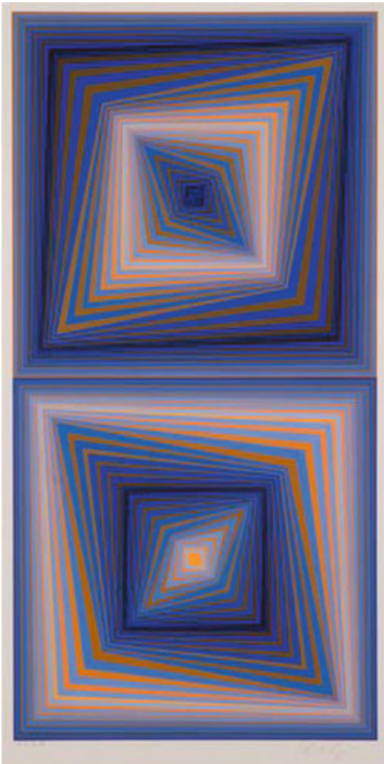

Fig. 35 Victor Vasarely, *Bi Rhomb*

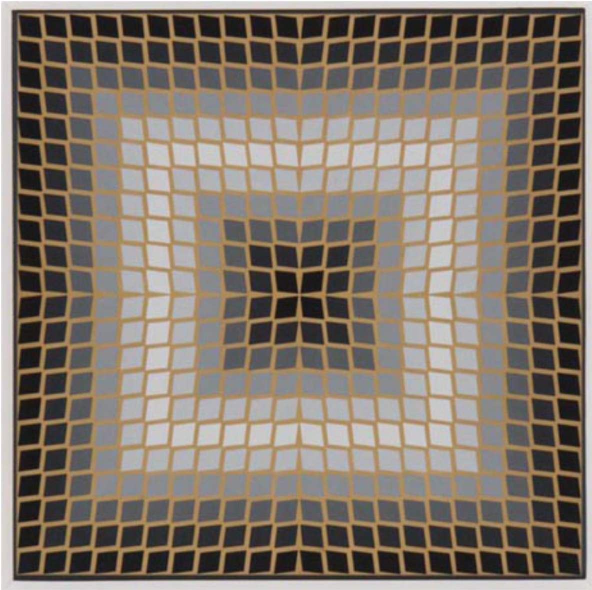

Fig. 36 Victor Vasarely, *Quasar-Dia-2*, 1965

Fig. 37 Victor Vasarely, *Yapura*, 1951/1964

Fig. 38 Victor Vasarely, *Yapoura-2*, 1951/1956

Affine Transformations

According to the frequency of occurrence, first place is occupied in Vasarely's works by his favorite form—the affine transformation of a blown-up hemisphere, or hemicube (topologically equivalent to hemisphere), which is from his "galactic" phase (beginning with tempera works *Vega-Arl*, and *Vega Blue*, 1968). In the Vega series we see some of the most advanced applications of Vasarely's systematic approach to form and colour. The paintings are based on spherical distortions of a polychromatic grid (Figs. 39, 40, 41, 42 and 43). The surface appears to have been warped, giving the feeling of something trying either to break out or to recede back into the depths of the surface. *Vega-Nor* (1969) is one of the best known of these works.

Following *Vega-Nor*, Vasarely painted more than 110 works from the same collection, sometimes combined with Koffka cubes (e.g., in the acrylic painting *Cheit-Pyr* from 1970/1971, Fig. 39) and hexagons. A similar example of the hemisphere coming from figurative art is M. C. Escher's lithograph *Balcony* (1945) (Fig. 44 and 45).

While Kazimir Malevich with his white square on a white background (*White on White*, 1918) reached the boundaries of geometrical abstraction, Vasarely created colorful artworks full of joy and visual surprises. He offered viewers the chance to participate in the artistic process itself by recomposing modular elements from his *Combinatoire "Planetary"* (Fig. 46) metal and fiberglass kit, and become an artist and creator. In this way, Vasarely was a visionary of contemporary art, promoting its democratization, multiplicability, and the interactivity of artworks. In order to make his art more available, Vasarely developed his own printing techniques, silk-screen reproductions, and computer programs for modeling his works.

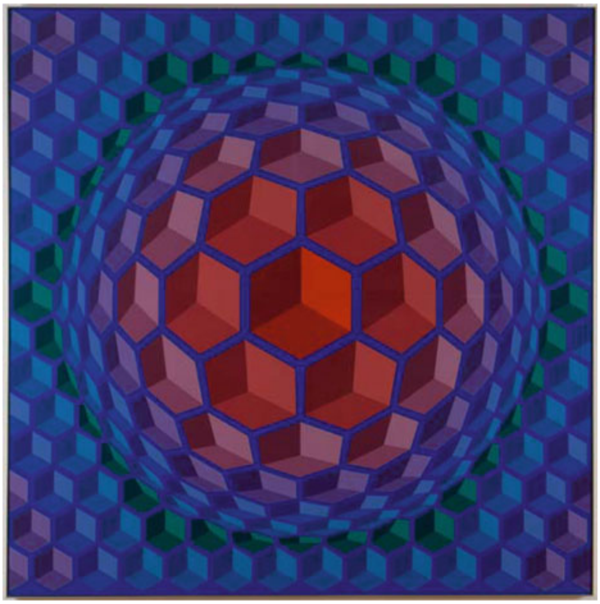

Fig. 39 Victor Vasarely, *Cheyt-Pyr*, 1970/1971

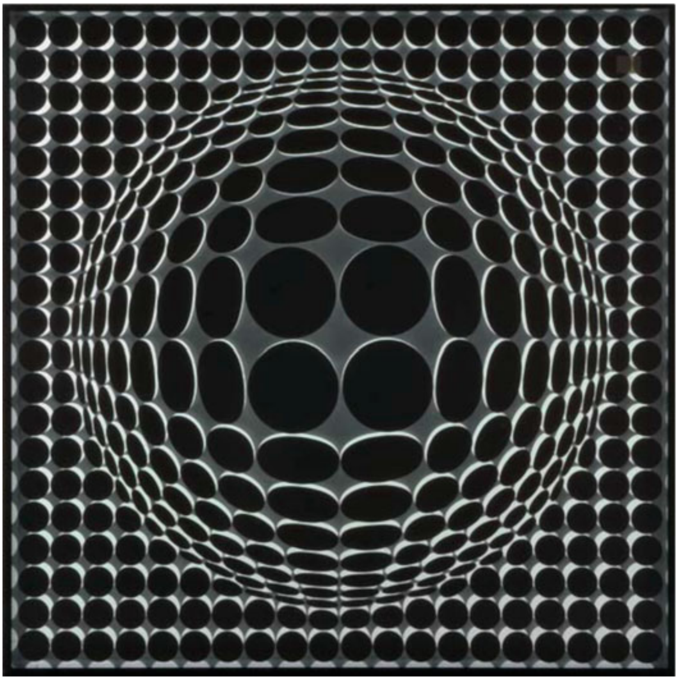

Fig. 40 Victor Vasarely, *Vega-Mir No. 137*, 1973

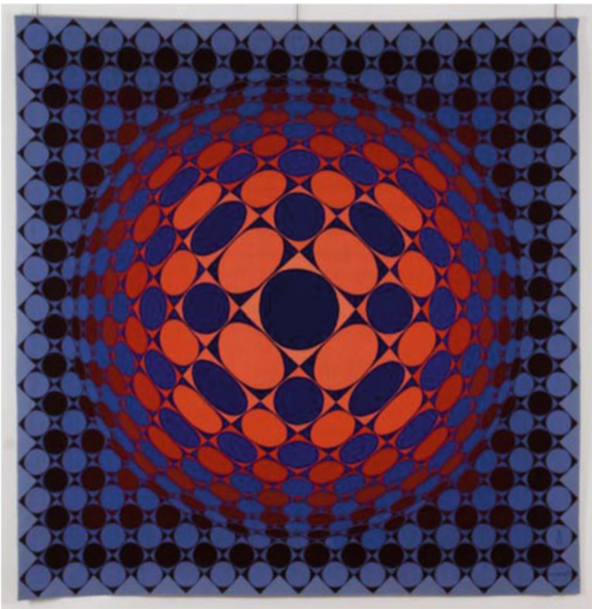

Fig. 41 Victor Vasarely, *Vega-Sakk*, 1958/1969

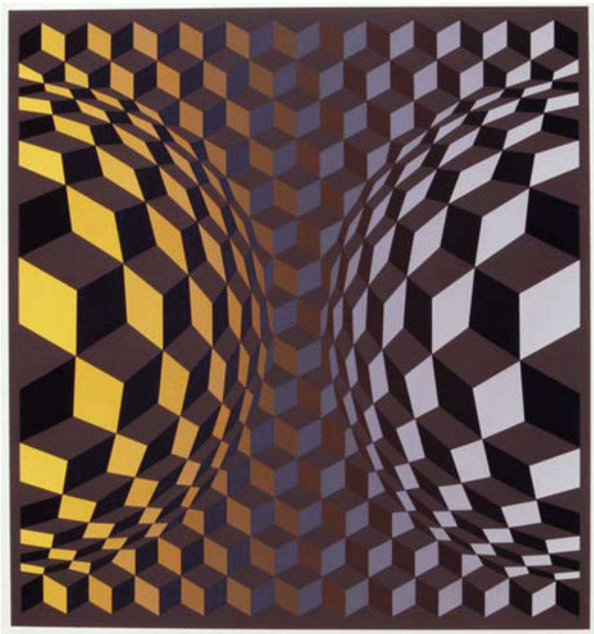

Fig. 42 Victor Vasarely, *G.A.Cheyt Ond*, 1971

Vasarely's interest in the democratization of art is closely connected with his design practice, beginning from the time when he worked as a commercial artist and fabric designer, and continued in his tapestry works, architectural projects, print series, multiplies–small sculptures reproduced in large series, the wonderful

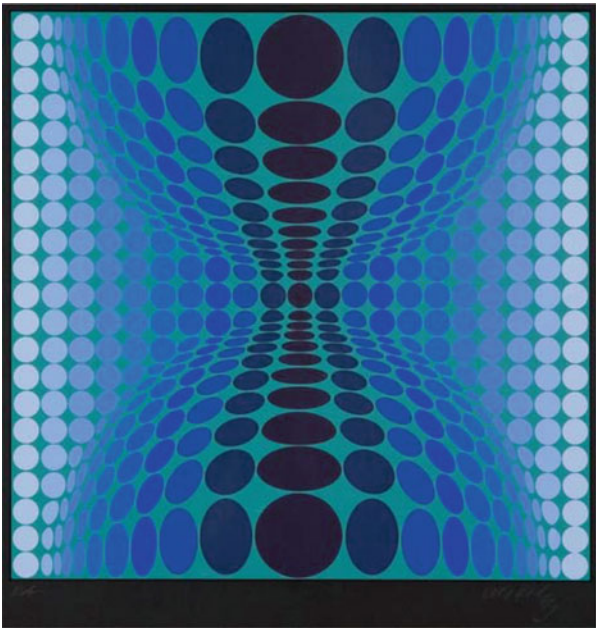

Fig. 43 Victor Vasarely, *Ond All*

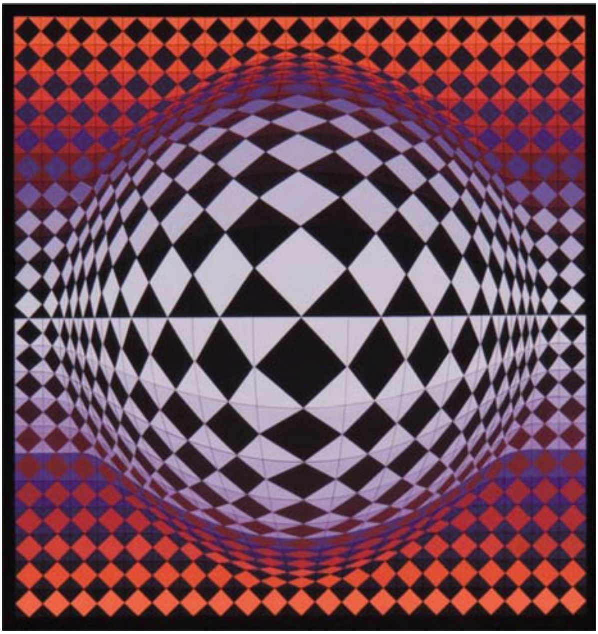

Fig. 44 Victor Vasarely, *Vega-Arl*, 1968

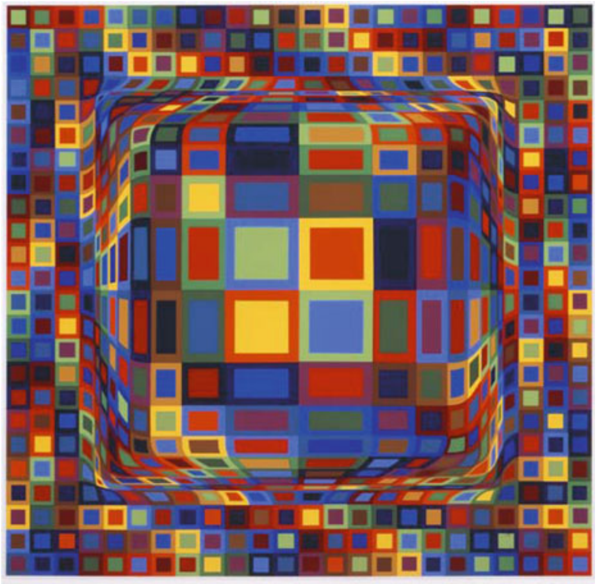

Fig. 45 Victor Vasarely, *Koska MC*, 1970

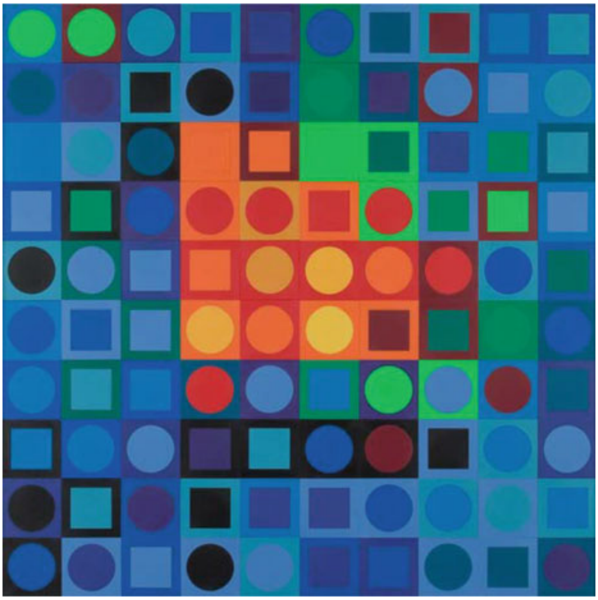

Fig. 46 Victor Vasarely, *Combinatoire "Planetary"*, 1971

chess-set *Sakk* (1979) (Fig. 47), and works from his final period based on infinite variations generated by his computer graphic program. He used the principle of modularity and supposed that basic compositional elements could be arranged (and rearranged) in millions of variations and would become available to everybody for a

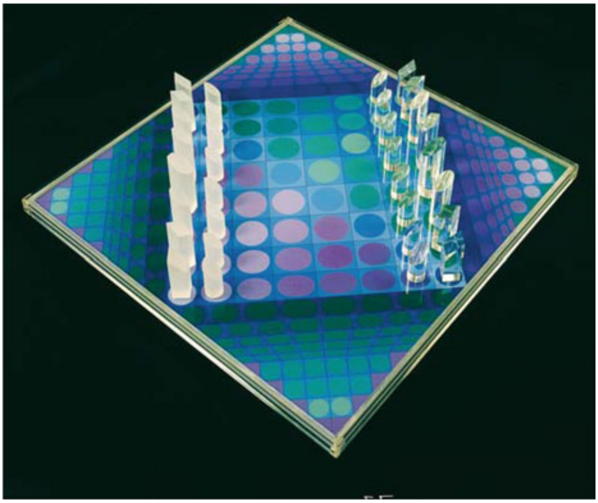

Fig. 47 Victor Vasarely, *Sakk*, chess-set, 1979

reasonable price. In this way, modularity offers unrestricted possibilities for variation and creativity, by using the basic geometric forms of circle, square, ellipse, rhomboid, spiral, meander, *etc*. (i.e., visual archetypes), fantastically rich scale of colors (the RGB color palette, used today for producing images with 16 million colors on the screens of our monitors), and a relatively small set of basic elements (modules) in order to create an infinite variety of structures. Vasarely created art of particular originality while thinking rationally, using pictorial systems defined by mathematical formulas, based on periodic and semi-periodic structures, or interrupted systems (using “symmetry breaking”), accompanied by visual illusions.

Vasarely’s work is not only his own art experiment: it is an invitation to the viewer to play, experiment, and do research with it.

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Fractal Art: Closer to Heaven? Modern Mathematics, the Art of Nature, and the Nature of Art

Charalampos Saitis

Abstract Understanding nature has always been a reference point for both art and science. Aesthetics have put nature at the forefront of artistic achievement. Artworks are expected to represent nature, to work like it. Science has likewise been trying to explain the very laws that determine nature. Technology has provided both sides with the appropriate tools towards their common goal. Fractal art stands right at the heart of the art-science-technology triangle. This chapter examines the new perspectives brought to art by fractal geometry and chaos theory and how the study of the fractal character of nature offers promising possibilities towards art's mission.

Introduction

"When I judge art, I take my painting and put it next to a God-made object like a tree or flower. If it clashes, it is not art." Paul Cézanne

Mathematics has always affected art. Mathematical concepts such as the golden ratio, the platonic solids and the projective geometry have been widely used by painters and sculptors whereas the Pythagorean arithmetic perception of harmony dominates western music to this very day. Infinity and probability theory inspired artists such as M. C. Escher and composers such as Iannis Xenakis forming trends in contemporary art and music respectively. The evolution of technology in the twentieth century created new areas of intersection between mathematics and art or music: digital art, computer music and new media.

The founding of chaos theory introduced mathematics to a fascinating and intriguing new reality: nature itself. Until then, scientists were only capable of observing the nonlinear dynamical character of the natural structures and processes. Now they had mathematical tools, which described, explained and proved the chaotic properties of nature, like the weather systems and the butterfly effect. Computers became an integral part of this scientific revolution, being capable of calculating nonlinear dynamical systems and visualizing the results over time;

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scientists were able to watch the chaotic evolution of a naturally-occurring phenomenon on a computer's screen. Eccentric shapes, irregular geometrical objects and extraordinary figures emerged, revealing a world where the unreal (graphic representations of algorithmic processes) implied the real (nature). Benoit Mandelbrot was the first mathematician to shape this new area into an individual self-standing theory, which instantly became the most popular of all. He introduced the neologism *fractal* to unite all these strange objects under one term.

> I coined *fractal* from the Latin adjective *fractus*. The corresponding Latin verb *frangere* means 'to break'; to create irregular fragments. It is therefore sensible – and how appropriate for our needs! – that, in addition to 'fragmented' (as in *fraction* or *refraction*), *fractus* should also mean 'irregular', both meanings being preserved in *fragment*. (Mandelbrot 1982, p. 4)

Artists were instantly attracted to fractals. Artistic interest burst out, resulting in a new form of digital art, which rapidly became popular both inside and outside the artistic and scientific communities. Musical interest followed almost simultaneously, focused mainly on the area of algorithmic composition. Consequently, challenging questions arose: Is fractal art the next big thing? Is fractal art the fulfilment of art's mission? Are we getting *closer to Heaven*?

The following sections of the chapter will discuss some initial concepts in a mathematical and aesthetical approach. Section "What Is a Fractal Anyway?" will introduce the general mathematical notions of fractal geometry and chaos theory. Section "Welcome to Fractaland!" will introduce fractal art, its areas and sub-areas and their mathematical origins. Section "Fractals Have Always Been There!" will outline a brief background on the pre-existing use of fractal structures in artworks. Section "It Is a Fractal World After All, Isn't It?" will discuss the view that the geometry of nature is neither Euclidean nor projective but mainly fractal and chaotic, and how this implies that fractal art is (promised to be) the art of nature.

What Is a Fractal Anyway?

Fractal Geometry (and Some Topology) The *topological dimension* of a set is defined as the number of independent parameters needed to describe a point in the set. For example, a point in the plane is described by two independent parameters (also known as the Cartesian coordinates of the point), so in this sense the plane is two-dimensional. By definition topological dimension is always a natural number. It essentially refers to the dimensions of the space in which the set exists. For example, the topological dimension of a solid sphere is 3, whereas that of a hollow sphere is 2. However, topological dimension behaves in unexpected ways on certain highly irregular sets such as fractals. *Hausdorff dimension* provides an alternative way to define the dimension of such sets.

Let F be a fractal and $\mathrm{N}(r)$ the minimum number of balls of radius less than or equal to r required to cover F completely. Clearly, as r gets smaller $\mathrm{N}(r)$ gets larger, that is the smaller the radius of the balls the more balls are needed to cover F. Very roughly speaking, as $r \to 0$, then N(r) is found to be proportional to $1/r^d$, where d is a real number. The number d is called the Hausdorff dimension of F. The Hausdorff dimension essentially measures the space-filling ability of the covered surface (or fractal in our case); moreover, it refines the concept of topological dimension by relating it to other properties of the space such as volume. Interestingly enough, the Hausdorff dimension is more often a fraction and not an integer. For example, if we consider a solid sphere, then its Hausdorff dimension equals its topological dimension. However, if we consider instead a hollow sphere, the resulting Hausdorff dimension will eventually be a number between 2 and 3. Therefore fractals and other highly irregular sets also share a fractional dimension that is usually greater than their topological dimension. There are various closely related notions of possibly fractional dimensions. These are usually referred to as *fractal dimensions*. Note that 'fractal' is a neologism and as such is not semantically related to 'fractional'. Mandelbrot formally defined a fractal to be a set with Hausdorff dimension strictly greater than its topological dimension. However, such a definition proved to be unsatisfactory in that it excluded a number of sets that clearly ought to be regarded as fractals. As Falconer (2002, pp. xx–xxi) notes:

> the definition of a 'fractal' should be regarded in the same way as the biologist regards the definition of 'life' [--] just a list of properties characteristic of a living thing [--] most living things have most of the characteristics on the list, though there are living objects that are exceptions to each of them. In the same way, it seems best to regard a fractal as a set that has (a list of) properties, rather than to look for a precise definition, which will almost certainly exclude some interesting cases.

Therefore, a fractal F is (defined as) a geometrical object that generally has the following features:

1. F has a fine structure, i.e., detail on arbitrarily small scales.
2. F is too irregular to be described in traditional geometrical language, both locally and globally.
3. F has some form of self-similarity, at least approximately or stochastically.
4. F has a 'fractal dimension' (defined in some way) that is greater than its topological dimension.
5. F has a simple and recursive definition in most cases of interest.

Fractals are generally generated following three techniques:

1. Escape-time fractals: defined by a recurrence relation at each point in a space.
2. Iterated function systems (IFS): a fixed geometric replacement rule exists.
3. Random fractals: these are generated by stochastic rather than deterministic processes.

Fractals can also be classified according to their self-similarity. Three types of self-similarity are identified in fractals, listed below in a direct correspondence to the generation techniques presented above:

1. Quasi-self-similarity: the fractal appears approximately (but not exactly) identical at different scales. Quasi-self-similar fractals contain small copies of the entire fractal in distorted and degenerate forms. This is a loose form of self-similarity.

2. Exact self-similarity: the fractal appears identical at different scales. This is the strongest type of self-similarity.
3. Statistical self-similarity: the fractal has numerical or statistical measures that are preserved across scales. This is the weakest type of self-similarity.

Chaos Theory Chaos theory describes the behaviour of systems of nonlinear dynamical equations when iterated. Iteration refers to the process, whereby an initial value is input to a system of equations and the output is fed back into the system as a new input value. The same process is repeated in infinitely many steps. Each step provides a value, which represents a point in *n*-dimensional space (*n* being determined by the number of variables in the equations). The *orbit* of the system is defined as the set of these points over time. An *attractor* of the system is a set to which the orbit of the system converges. There are three categories of behaviour into which the system can fall upon iteration: constant, where all points in the orbit tend towards a stable value; oscillatory, where all points in the orbit belong to a repeating set; and chaotic, where no point in the orbit is visited twice in a finite time period. The last category of behaviour is the most interesting. The system wanders around a range of points, often returning to close, but never identical points and its attractor, being seen from a geometrical point of view, is a complicated set with fractal characteristics called a *strange attractor* or a *fractal attractor*. Strange attractors are actually fractals, which many people consider as the most interesting and beautiful ones. The most essential feature of a strange attractor is its sensitivity to the initial conditions, i.e., minor differences in the initial input values of the system can produce drastic, unexpected results after a certain number of iterations (O'Brien 2004, pp. 22–23). Figure 1 shows a digital image of a strange attractor that looks like a butterfly, referring to the infamous *butterfly effect* in weather prediction systems, first observed by American mathematician and meteorologist Edward Lorenz in 1961.

Scaling self-similarity is the most critical property of a fractal; it is the 'strange attractor' of the dynamical system defined by mathematics, art and nature. Historically, mathematical 'strange' structures existed before fractals. They were characterized as 'pathological' since they did not fit the patterns of Euclid. However:

> the mathematicians who created (them) regarded them as important in showing that the world of pure mathematics contains a richness of possibilities going far beyond the simple structures that they saw in Nature. [--] Now [--] [n]ature has played a joke on mathematicians. The nineteenth-century mathematicians may have been lacking in imagination, but Nature was not. The same pathological structures that the mathematicians invented to break loose from nineteenth-century naturalism turn out to be inherent in familiar objects all around us. (Mandelbrot 1982, pp. 3–4)

Fig. 1 *Butterfly effect* by Nathan Selikoff (2007). The term was coined by Lorenz to describe how very small changes–the flap of a butterfly's wings–in the initial conditions of a chaotic system such as the weather may drastically affect the system's behaviour in later stages (iterations). Courtesy of the artist, available online at http://nathanselikoff.com/236/strange-attractors/butterfly-effect

Welcome to Fractaland!

"Just as the creation of a fractal structure involves the process of iteration, so the production of artistic works involves iteration. The creative process is a system wherein the output eventually becomes part of the input. In this way, the process of making art becomes self-similar, self-referential and an iteration of itself." Edward Berko

Very broadly speaking, *fractal art* refers to calculating fractal mathematical functions (algorithms) and then converting the results into digital still or animated images and music. Fractal images are visual representations of the algorithmic data, while fractal music maps the calculation outputs to music pitches or other sounds. Although the initiative of the artist/programmer is what defines the resulted artwork, fractal art, as with any computer-based artistic movement, has been strongly criticized as to whether it should be considered a form of art because of its computational origins.

Fractal images are digital images that either consist of a single fractal or are composed of several fractal and non-fractal objects. Software tools have become widely available, whereby the artist is able to algorithmically generate a fractal, to apply colour patterns, and to compose digital images. The algorithms used fall under the three generating techniques presented in the previous section, thus allowing the artist to decide on the type of self-similarity and the level of complexity. One of the most popular fractal generating programmes is *Ultra Fractal*. Its innovating feature is the built-in layering capability; in addition, it was the first programme to place total control of the image in the hands of the user. It should be noted here that colouring a fractal image is an essential part of the process. Geometrically, a fractal object consists of 'neighbourhoods' of different importance. Computers generate a fractal by applying different colours to different levels of importance, the result being outstanding colourful images that challenge imagination. Figure 2 shows a digital *fractal ocotillo* created by iterating parts of photographic images of two different ocotillos. An example of a spatial fractal image can be seen in Fig. 3.

Two sub-areas of fractal images are fractal flames and fractal landscapes. Fractal flames are mainly generated with IFS, whereas stochastic algorithms are used to generate fractal landscapes. *Apophysis* is the most popular fractal flame editor. Kenton "Doc Mojo" Musgrave is considered a leading authority on fractal landscapes and his most recent computer program, *MojoWorld*, is one of the more convenient ways to investigate them. Finally, sequencing fractal images creates so-called fractal animations. An example of a fractal landscape can be seen in Fig. 4.

Fractal music is primarily an area of algorithmic composition. Fractal generating algorithms are applied to pitch, dynamics, duration, time and other audio parameters to determine compositional processes. Most common is to apply the results to MIDI parameters, though application to wav is not unheard of. There are several types of fractal music according to the methods and software used. Programmes like *MusiNum* and *Gingerbread* generate music using solely fractal mathematics, and transformations to fit a chosen musical scale. However, the majority of the software uses fractal as well as a great many other algorithmic techniques combined. In other cases, composers use fractal music as a starting point for further development using conventional instruments, samples, or other sounds.

Another way of composing fractal music is using iterated systems of nonlinear dynamical equations as note generating algorithms. The result is directly related to the behaviour of the system. Obviously, constant behaviour will produce no interesting musical results. Oscillatory behaviour has the possibility of producing interesting repetitions, if the period (the number of distinct points in the repeating set) is large enough, but in practice the set is usually rather small, resulting in melodies that circle between only a few pitches. It is the chaotic behaviour that holds the most musical interest. When the strange attractor is interpreted musically the result is very similar to variations on a theme. The material produced has a high degree of correlation with its past, but is always

Fig. 2 *Ocotillo in Bloom, Iterated* by Robert Fathauer (2011). Courtesy of the artist, available online at http://members.cox.net/fathauerart/Ocotillo.html

producing something new at the same time. *Chaosmuse* is a programme in pure ANSI C. It allows the composer to specify what equation systems, parameters, and musical parameters to use through a text-based, interactive interface. The generated music is output as several standard file formats, including MIDI Types 1 and 0, Adagio, CSound score file format, and mtr2, a simple format for input into MAX/MSP.

Fractals Have Always Been There!

"Since a long time I am interested in patterns with 'motives' getting smaller and smaller till they reach the limit of infinite smallness." M. C. Escher

Well before the mathematical establishment of fractals and chaos, fractal patterns, self-similarity, chaotic structures and infinity have been used by painters, sculptors and composers as both motivation and quantitative tools.

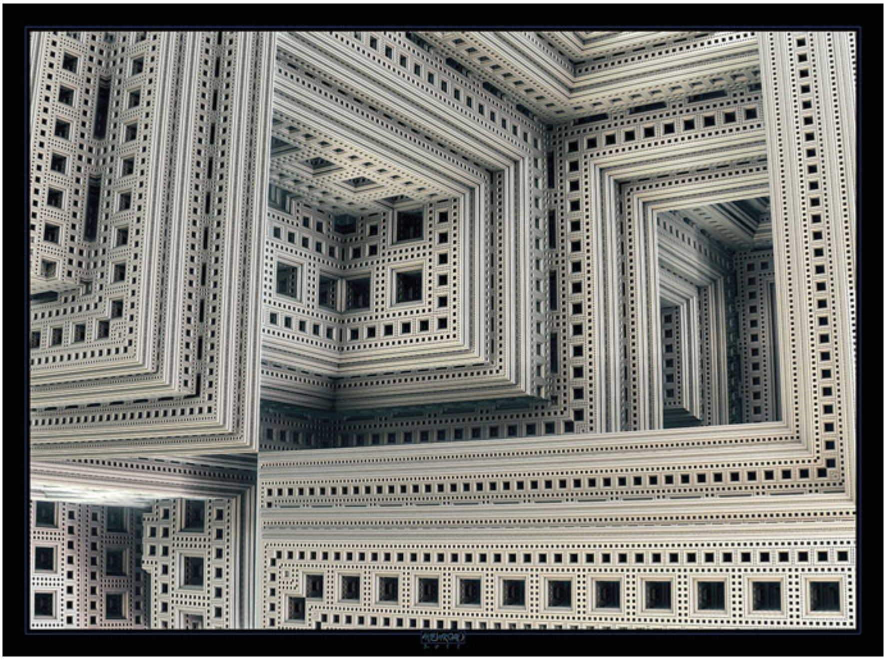

Fig. 3 *Obsession* by Mehrdad Garousi (2011). Example of a spatial fractal image. Courtesy of the artist, available online at http://mehrdadart.deviantart.com/gallery/#/d41x45d

Jackson Pollock created some abstract paintings in the late 1940s, in which he used two revolutionary methods both implying chaotic patterns. In the first he used his whole body to introduce a wide range of length scales into his painting motion. In the second he applied paint by letting it drip on the canvas. Furthermore, electronic de-construction of Pollock's paintings into their constituent coloured layers showed that each of the individual layers consist of a uniform, fractal pattern (Taylor et al. 1999). Escher created his famous tessellation drawings inspired by the notions of infinite repetition and scaling self-similarity, often producing optic illusions. Salvador Dali's *La Visage de la Guerre* depicts a fractal progression of ever smaller death masks.

Self-similarity abounds in canonical works of western music, present in different forms at all time periods. Canons and fugues are the main examples. A canon is a contrapuntal composition that employs a melody with one or more imitations of itself played after a given duration (usually one measure). The initial melody is called the leader, while the imitative the follower. The follower is either an exact replica of the leader or a transformation, thus there are several different types of canons. A fugue is a more complex form of contrapuntal composition where one main theme, the subject, sounds in successive imitation in each voice. Studies on J. S. Bach's famous canons and fugues prove the masterly use of self-similarity and scaling. The music of Beethoven and Mozart also consists of similar elements. As

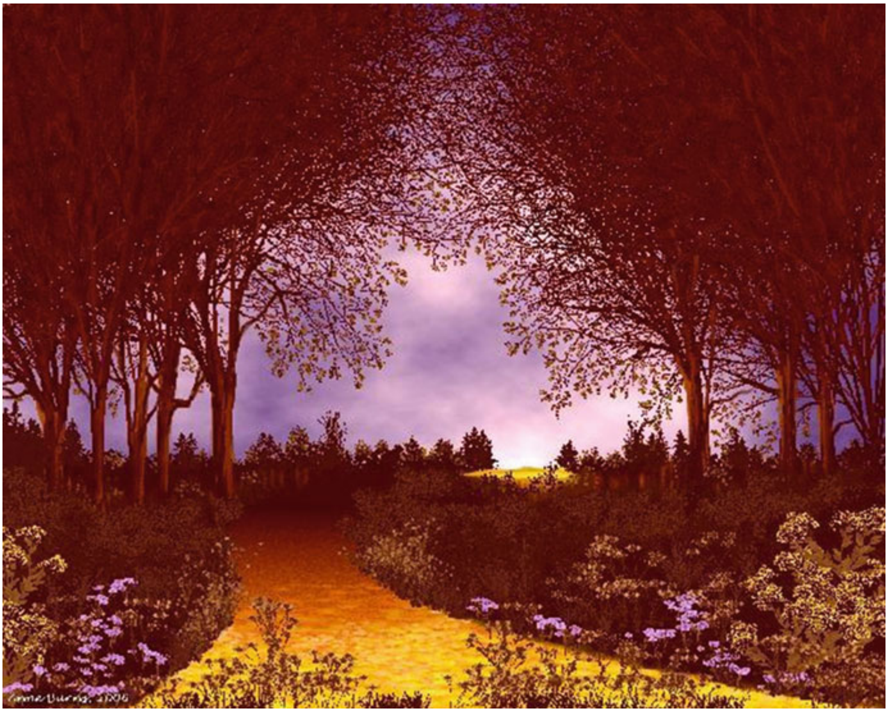

Fig. 4 *Fractal Scene II* by Anne Burns (2007). Example of a fractal landscape. Courtesy of the artist

Mandelbrot pointed out, “music displays fractal characteristics because of its inherently hierarchical nature” (O’Brien 2004, p. 26).

It Is a Fractal World After All, Isn’t It?

“The observation by Mandelbrot of the existence of a ‘Geometry of Nature’ has led us to think in a new scientific way about the edges of clouds, the profiles of the tops of forests on the horizons, and the intricate moving arrangement on the wings of a bird as it flies.” Michael F. Barnsley

With the establishment of fractal geometry and chaos theory it soon became clear that these are useful mathematical tools for describing nature. Scientists went on examining natural patterns and objects from a completely new angle, the results being more than just interesting. Irregularity, chaos, abrupt changes, discontinuity, self-similarity, scaling: all rule both the inner and outer beauty and harmony of nature and life. Trees, branches, leaves, the roots of a plant, cauliflowers, snow-flakes, diamonds, coastlines, mountains, clouds, stars, the sky, galaxy clusters:

fractal attractors describe visible natural shapes. The weather, the solar system, plate tectonics, turbulent fluids, population growth, economy: examples of chaotic dynamical systems. The brain and bronchial lobes are also examples of bodily structures with elements of self-similarity and scaling.

In 1982 Benoit Mandelbrot wrote his fundamental essay *The Fractal Geometry of Nature*, in which he introduced his revolutionary ideas in a mathematical, philosophical and artistic way (Mandelbrot 1982). The essay has had a strong influence on many scientists and artists. The innovative concept of Mandelbrot was to usefully represent a natural pattern by a fractal set. Mandelbrot (1982, pp. 193–194) notes that

> many facets of Nature can only be described with the help of fractals [--] Nature's patterns are irregular and fragmented [--] self-similarity is [--] the fabric of Nature.

Chaos theory brought out a whole new aesthetic in science. Traditionally, science has considered the relation between the observer and the observed to be purely 'objective'. However, chaos dramatically negated this assumption by showing that this relation is chaotic dynamical and thus 'subjective'. Artists on the other hand have always understood that a change in one small part of a painting or a musical composition may destroy or transform the work. Chaos united the two worlds of art and science towards a further understanding of nature (Briggs 1992, pp. 31–33).

Conclusion

"Mathematics is on the artistic side a creation of new rhythms, orders, designs, harmonies, and on the knowledge side, is a systematic study of various rhythms, orders, designs and harmonies." William L. Schaaff

Continuity and infinity characterize nature itself. Therefore, image and sound, as integral parts of nature, and consequently art and music, could be redefined by taking on a completely new vision and mission. Modern Mathematics provides the necessary tools, both scientific and artistic, through Fractal Geometry and Chaos Theory. Fractal Art is located at the very point where the nature of the Art meets the art of the Nature. Both scientists and artists should embrace Fractal Art and explore its infinite (but not self-similar!) possibilities.

The American Heritage Directory defines art as:

1. Human effort to imitate, supplement, alter or counteract the work of nature; and
2. The conscious production or arrangement of sounds, colors, forms, movements, or other elements in a manner that affects the sense of beauty, specifically the production of the beautiful in a graphic or plastic medium.

Fractal art seems able to follow this definition more closely than other forms of artistic expression because of its strong natural relevance. Nature, on the one hand, forms patterns somewhere between order and chaos, some of them being orderly in time but disorderly in space, others orderly in space but disorderly in time: fractal patterns (Burns 1994, p. 7). We meet these patterns everywhere in the real world we live in. Even the way humans react to natural phenomena is in most cases unpredictable. Art, on the other hand, has been either totally orderly (classicism, cubism) or totally disorderly (dada, surrealism); with fractal art the in between gap is likely to be filled in.

In the field of music, fractal algorithms are expected to produce more realistic natural sounds and subtle melodies than conventional algorithmic approaches. Music produced by chaotic nonlinear dynamical systems is highly acclaimed as more aesthetically interesting than music obtained with other stochastic algorithms such as Markov chains (O'Brien 2004, pp. 26–27). Fractal music yields patterns that originate from natural procedures, thus being harmonious and beautifully complex. Further processing of these patterns, either as succession of musical events or as complex frequency spectra, promises to produce musical material, which will bring new horizons in composition. As fractal structures are identified in nature, in the nature of sound itself, and in our perception of musical beauty, fractal music is about to be the next musical revolution (Scrivener 2000).

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Cultural Insights from Pattern Symmetries

Dorothy K. Washburn and Donald W. Crowe

Abstract The authors have published texts and studies documenting the procedure for and application of the use of plane pattern symmetries to classify cultural patterns. This chapter contrasts the difference in cultural insights gained between pattern studies that simply describe patterns by motif type and shape and those that describe the way motifs are repeated by plane pattern symmetries. Culturally produced patterns can be described in many ways, each useful for different purposes. We describe how early pattern studies aimed at designers of textiles and wallpapers created classificatory groupings that were descriptively idiosyncratic, grouping patterns by motif similarities that are arranged by very different symmetries. We then cite several recent studies that illustrate how a symmetry rather than a motif similarity grouping reveals new insights from continuities, changes and preferential symmetry use that can enhance our understanding and interpretation of the material.

Introduction

The use of plane pattern symmetries to analyze geometric designs that cover the surfaces of textiles, ceramics, tiling, wood, basketry, floor coverings and other media created by cultures around the world has resulted in extremely productive collaborations between the objective world of mathematical geometry and the subjective world of human decorative activity. A wide array of studies has shown how plane pattern symmetry classes not only enable researchers to systematically define the artistic output of a given culture, but also to study temporal and spatial continuities and changes in patterns (e.g., Gerdes 1998; Hargittai 1986; Hargittai and Laurent 2002; Washburn 2004; Washburn and Crowe 2004; as well as the journal *Symmetry: Culture and Science* edited by György Darvas and Dénes Nagy). In this article we first clarify

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how this analytical approach differs from most studies of design oriented around pattern as decorative art and then we describe a few examples of the kinds of useful insights on cultural activity that we have discovered from our collaborative work.

The best analytical approaches for cultural artifacts are those that target features containing information pertinent to the problem under discussion. As obvious as this may seem, in fact, the standard typological/stylistic approach bundles co-occurring features, each of which relate to different aspects of production, function and meaning. For example, art historians often focus on modes of presentation and subject matter that, considered together as a style, typify a particular historical period in which a work was produced.

Anthropologists concerned with cultural activity need analytical approaches tailored to extract the meaning, if any, of patterns present on material culture related to particular activity. A primary concern is to use an analytical approach that not only enables systematic description of a given body of data but also one that allows comparison of design data from different areas and periods. In addition, optimally, the approach would focus on features in the designs that reflect the thinking of the people who created the patterns, rather than the analysts who seek to interpret them. Finally, we need an approach that can address non-representational pattern.

Clearly there are many features that can be analyzed in a realistic image that are clues to its meaning. But when confronted with designs composed of purely geometric shapes arranged systematically into a pattern, as is the case on many textiles, ceramics, tile, floor coverings, etc., we have found that Euclidean plane pattern symmetries, that is, the finite (point), one-dimensional (band) and two-dimensional (wallpaper) symmetries that describe patterns in the flat plane, have been immensely useful for systematic descriptions and comparisons.

The cultural meanings imbedded in patterns composed of purely geometric shapes are particularly difficult to "deconstruct." Happily, from the beginnings of our collaborative efforts, we discovered that the pattern output from a given culture seemed to be consistently structured by a limited number of symmetries. We then began exploring why particular symmetries seemed to dominate the patterns in a given culture. We asked whether the symmetrical structure itself carried cultural meaning, what that meaning might be, and how it changed over time and space. Results from a number of studies suggest that it is possible to "read" symmetrical arrangements of design elements not only as records of constancy and change within cultural decorative systems, but also as metaphors of imbedded cultural information (Washburn 1999).

Principles of Plane Pattern Symmetries

Before we examine how analysis of pattern by their geometrical symmetries can provide insights into cultural meaning, we describe the plane symmetry classes and their nomenclature. Euclidean geometries characterize the generation of repeated patterns in the flat plane. These geometries, known for centuries, were systematically enumerated in the nineteenth century and are described in our work *Symmetries of Culture* (Washburn and Crowe 1988). We emphasize that this use of mathematics is simply a tool which we use to uncover consistencies in the way people in different cultures make patterns. Indeed, because the human visual system has evolved to use the property of symmetry to recognize and identify form, it is a particularly appropriate feature to study, although individuals may not be aware of the symmetries they are using to create pattern in terms of the principles of Euclidean geometry.

Plane pattern symmetries, or more properly, distance-preserving transformations, are created by four rigid motions: translation, mirror reflection, rotation, and glide reflection. These four motions move identically shaped pattern parts around a point axis (finite), along a single line axis (one-dimensional), and along multiple line axes (two-dimensional). Although an infinite number of rotations and mirror reflections may pass through a single point axis, there are only seven permutations of the four motions possible along a single line axis, and only 17 permutations along axes arranged in 30°, 60°, 90° and 180° two-dimensional grids. Of course, if the colors of the motifs are systematically alternated, then the number of different classes increases. But, we have found that in culturally produced designs, most are either one-color or two-color designs. Two-color one-dimensional designs have 17 color classes and two-color two-dimensional patterns have 46 color classes.

Each of the classes so generated is described by a nomenclature. Here we use the one described in Washburn and Crowe (1988) that accounts for the presence or absence of each of the four motions in the design. For finite designs, the format *cn* or *dn* is used for cyclic or dihedral designs respectively; the *n* refers to the number of rotations or reflections. For one-dimensional designs the four-symbol notation *pxyz* is used. *P* begins each notation, followed by successive symbols for vertical reflection, horizontal or glide reflection and bifold rotation; the *m* indicates mirror reflections, the *a* indicates glide reflection, the 2 indicates bifold rotation, and 1 indicates no motion in that position. For the 17 two-dimensional patterns we use the short form from the International Tables for X-Ray Crystallography (Henry and Lonsdale 1952).

Descriptive Analyses of Pattern

We contrast our approach with the descriptive classifications of pattern created by designers of textiles and wallpapers for these treat both representational and non-representational pattern. One of the best-known pattern studies is that of Archibald Christie's wide ranging historical survey easily available as *Pattern Design*, a Dover reprint (1969) of his original 1910 study. Christie focused on the rhythmic movement of element repetition, both naturalistic and geometric, that he found pervasive in patterns from the earliest times. For example, he illustrates two sixteenth century borders, one from Egypt and one from Persia (Fig. 1). We would classify them both as having class *pma2* symmetry. Christie, however, used

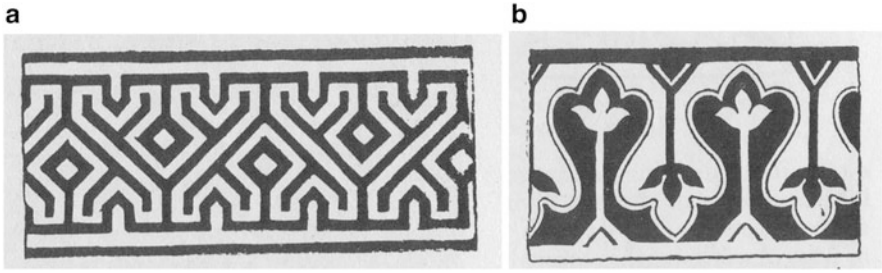

Fig. 1 Border designs with *pma2* symmetry. From Christie, *Pattern Design*, 2nd ed., 1929. With permission from Dover Pub., 1969. (**a**) Fig 183, (**b**) Fig 185

idiosyncratic descriptive terms and classes to describe how isolated units (spots) and continuous units (stripes) are combined into different kinds of repeated patterns by processes he described as "powdering, striping, interlocking, interlacing, branching, and counterchanging" (reversing colors).

Improvements in the technology of pattern production, such as mechanized looms for woven textiles, enabled the rapid generation of an endless succession of different patterns. It also fostered the appearance of pattern books such as Amor Fenn's *Abstract Design: A Practical Manual on the Making of Patterns for the Use of Students Teachers Designers and Craftsmen* (1920) that provided line diagrams of numerous border and textile patterns that showed how simple units can be recombined and elaborated into very complex, decorative patterns. Fenn does not differentiate patterns by their generating symmetries but rather by the angles by which "enclosed shapes," such as squares, circles, polygons, are juxtaposed and repeated to create borders and textiles. His borders generally correspond to one-dimensional band designs, and his textiles correspond to two-dimensional overall patterns.

However, Fenn describes his units, such as "frets," in such a way that border bands composed of these units can be generated by a number of different symmetries. For example, Fig. 2a is a fret generated by bifold rotation and vertical mirror reflections and glide reflections, *pma2*, Fig. 2b is a fret generated only by bifold rotation, *p112*, and Fig. 2c is a fret generated by vertical and horizontal mirror reflections and bifold rotation, *pmm2*. In another example, his "interlacing" units can be arranged by several of the seven one-dimensional symmetries, such as by simple translation, *p111* (Fig. 3a) as well as by bifold rotation, *p112* (Fig. 3b). For overall patterns Fenn describes how the repeated unit, such as squares, hexagons, and undulate lines, are arranged to cover space as in "drop" patterns (Fig. 4a, b). Although quite different decorative effects are achieved, both patterns are organized and repeated by the two-dimensional symmetry *cmm*.

It may well be asked: In what way(s) is a symmetry classification of pattern superior to the descriptive classifications based on motif shape and/or shared repetition configurations as exemplified above? Such purely descriptive classifications, such as those by design historians or pattern designers, group pattern by

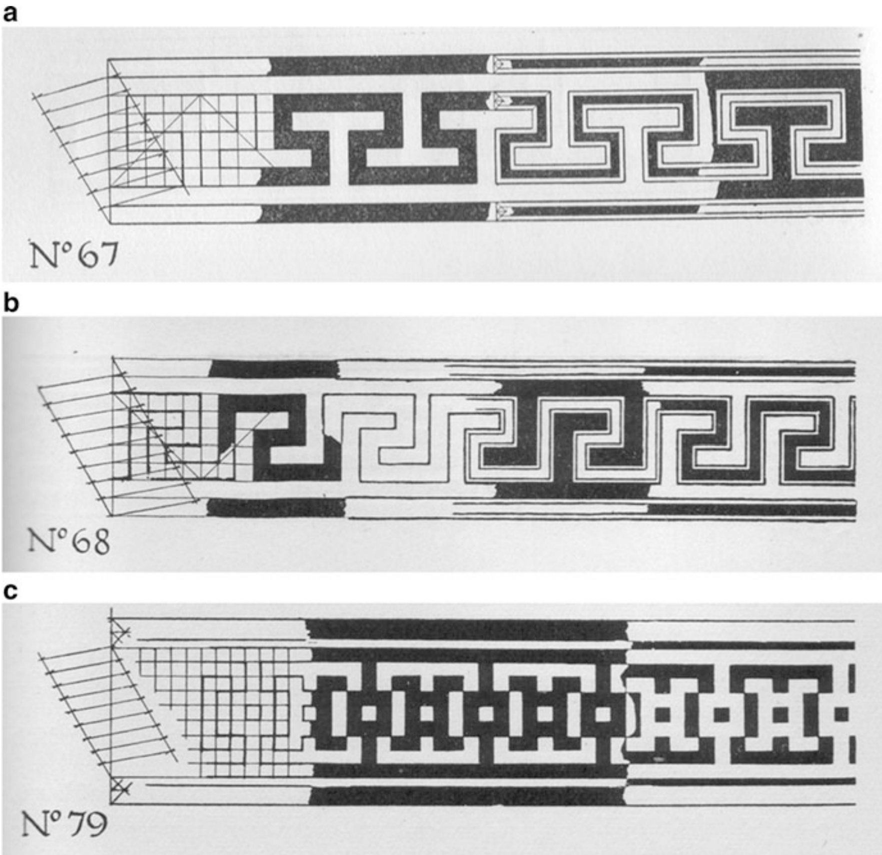

Fig. 2 (**a**) *pma2*, (**b**) *p112*, (**c**) *pmm2*. From Fenn. (**a**) #67, (**b**) #68, (**c**) #79

similarity in motif shape or subject matter regardless of underlying differences in symmetrical structure. While these may be appropriate for the description of a specific body of data or perhaps for the production of designs within the strictures of a given technology, we argue that they do not facilitate studies concerned with cultural aspects related to their function and meaning.

Thus, we have set aside interest in the way pattern may stylistically decorate, an issue that may well be a Western preoccupation, and instead we have queried how pattern informs. We have found that a focus on symmetrical structure rather than motif enables us to explore how cultures without writing systems use pattern in different kinds of information transmitting capacities. We present here several examples of studies that demonstrate how symmetry consistencies and differences in pattern over time and space correspond to important ethnic differences and geographical interaction patterns as well as to environmental factors that stimulated major social adaptations.

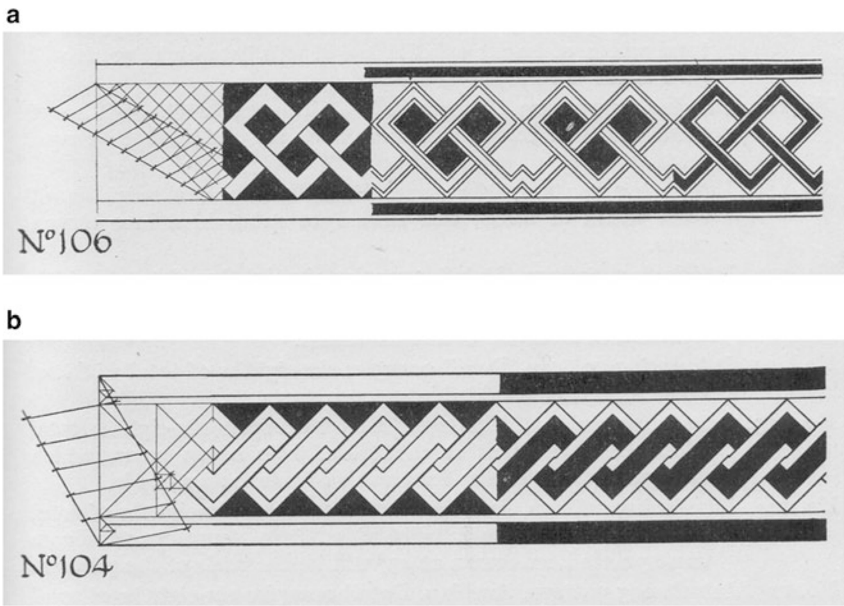

Fig. 3 (**a**) *p111* and (**b**) *p112*. From Fenn. (**a**) #106, (**b**) #104

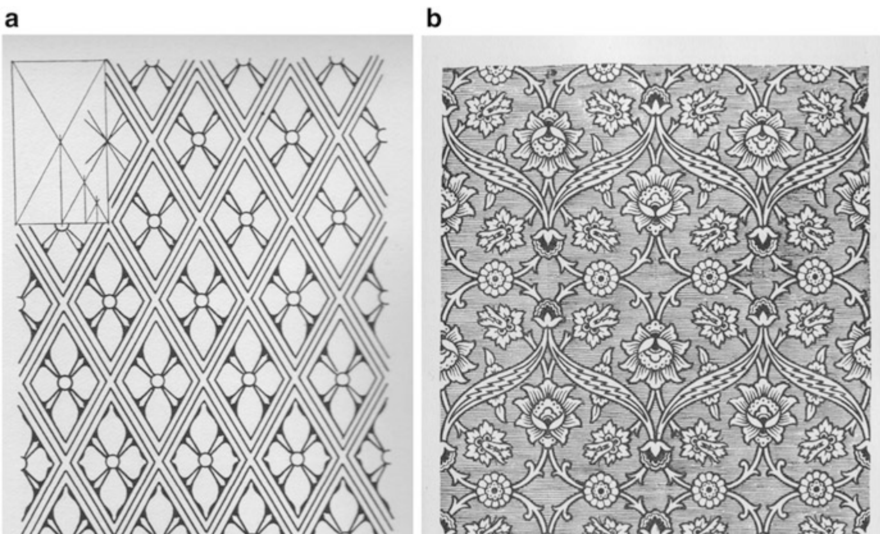

Fig. 4 (**a**) *cmm*, (**b**) *cmm*. From Fenn, (**a**) #296, (**b**) #358

Spatial and Temporal Distinctions Revealed by Symmetry in the Greek Neolithic

We begin with an archaeological study of decorated pottery made during the Neolithic from sites throughout mainland Greece (Washburn 1983). Art historians have described the designs in terms of the mode of manufacture, such as incision, their color, and by idiosyncratic terms that describe the constituent motifs, such as flames, triangles, zigzags, or nets (Wace and Thompson 1912; Zervos 1962) (Fig. 5).

A distributional study of the motifs described by these idiosyncratic terms shows that they occurred on red/cream pottery from every area where Greece was occupied during the Early Neolithic. For example, all patterns composed of triangles described as "flame" patterns are found on sites throughout occupied Greece (Fig. 6). However, if these same motifs are described by the symmetries used to configure them into patterns, they appear grouped in geographically separate enclaves, each characterized by a different symmetry arrangement of each motif (Fig. 7). Notably these enclaves are separated by mountain ranges or bodies of water, suggesting that geographic factors impeded free movement and interchange during this early period, indicating that pottery was produced and used within each area.

However, if we apply this same methodology to incised patterns from the Late Neolithic, we find a new distribution of motifs arranged by the same symmetry. Instead of mutually exclusive areas, each characterized by a different symmetrical arrangement of the motifs, one centrally located site, Orchomenos, appears to have a number of motif arrangements (i.e., pattern symmetries), while surrounding sites such as Lerna, Nemea and Corinth display only one or two patterns and symmetries.

The shift to this new distribution pattern correlates precisely with the beginnings of trade in the Aegean, suggesting that the central site with the greatest variety of patterns composed by different symmetries was a market place as well as a bulking center where goods from outlying sites were brought in to be sold and/or bulked and shipped throughout the Aegean. This analysis is an excellent example of the power of symmetry classifications of pattern to highlight cultural interaction spheres and changes in interaction routes over time and space.

The Confluence of Native Exegesis and Design Symmetry

We next examine a study of designs on historic period twined baskets from three Indian tribes in northern California, the Yurok, Karok, and Hupa (Washburn 1986). While these peoples are all salmon fishers living in large villages along the swiftly flowing northern rivers of the Sierras, they speak mutually exclusive languages. Nevertheless, their common lifestyle has resulted in basket forms and designs that are difficult to differentiate in technique and pattern motif. In the 1930s Lila O'Neale (1932) of the University of California, Berkeley, visited a number of weavers to study the aesthetic principles that guided the creation of their basket designs. She discovered

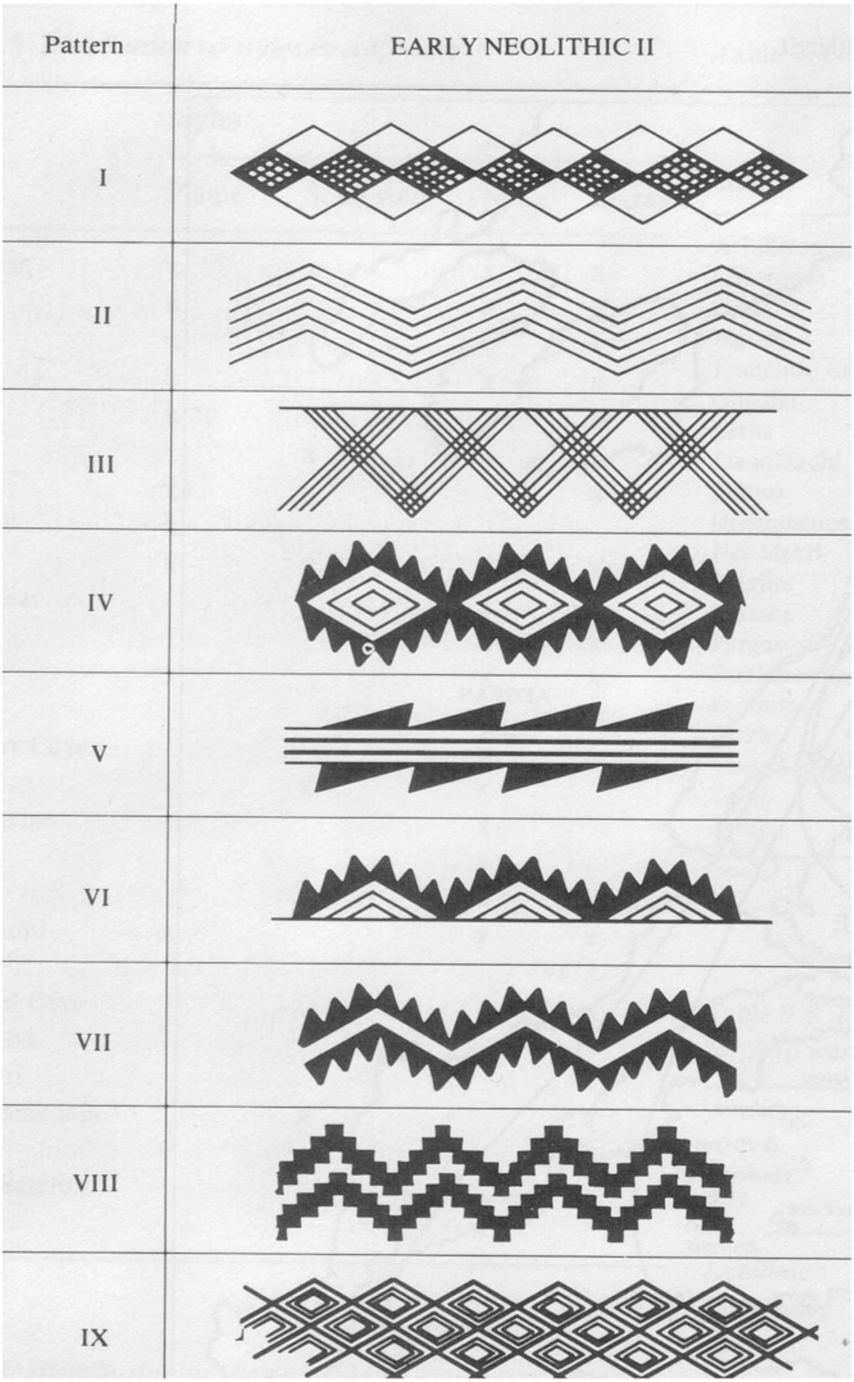

Fig. 5 Four motifs in nine patterns from the Early Neolithic, Greece. Source: (Washburn 1983, Fig. 9.10). Image reprinted with permission, Cambridge University Press

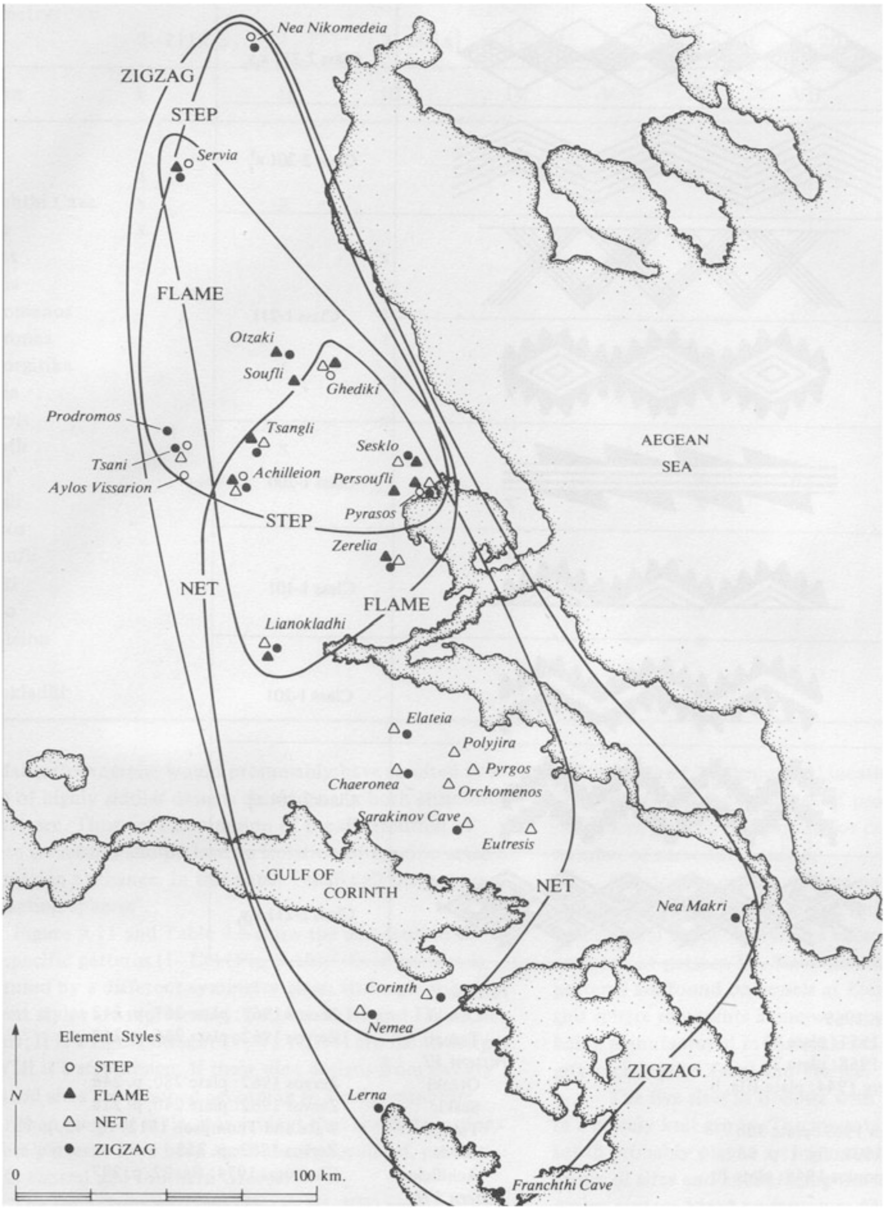

Fig. 6 Distribution of the four motif styles in the Early Neolithic, Greece. Source: (Washburn 1983, Fig. 9.9). Image reprinted with permission, Cambridge University Press

a dichotomy between baskets that were said to be aesthetically "good" and thus would be worn for tribal ceremonies and those that were "bad" that were made for sale to non-Indians, whether dealers, tourists or anthropologists.

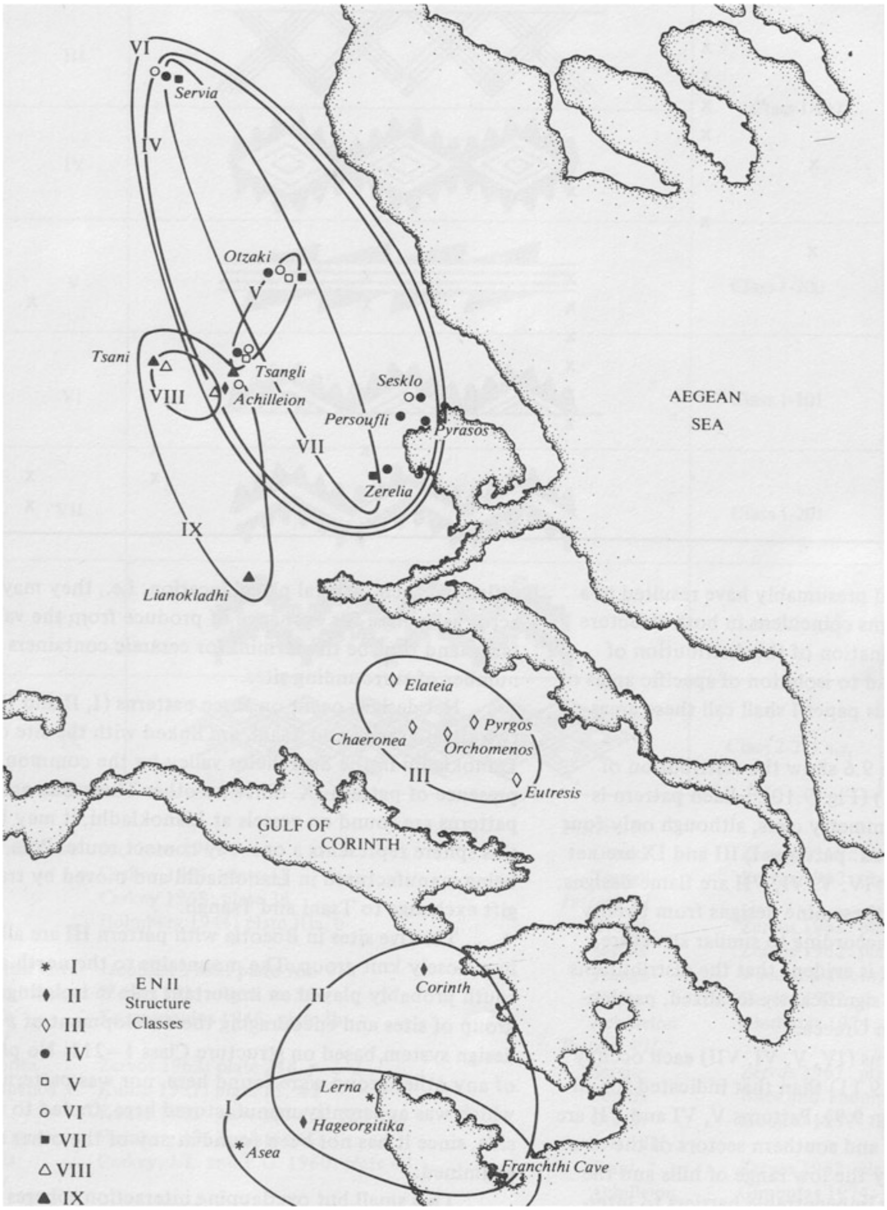

Fig. 7 Distribution of the nine symmetry configurations of the four motifs, Greece. Source: (Washburn 1983, Fig. 9.11). Image reprinted with permission, Cambridge University Press

Notably, both the good and bad baskets were made with the same twined technology, raw materials, care in execution, and design motifs. The ONLY feature that distinguished the good and bad baskets was the difference in the symmetries that they used to create

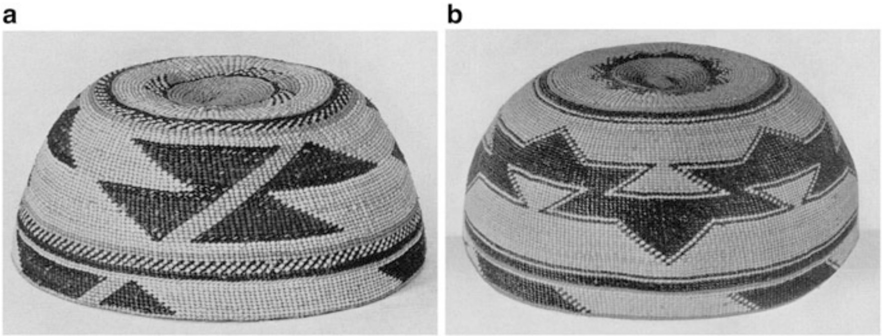

Fig. 8 (**a**) Hat with “good” *p112* design. (**b**) Hat with “good” *pma2* design. From O’Neale, (**a**) Fig. 24a. (**b**) Fig. 21a

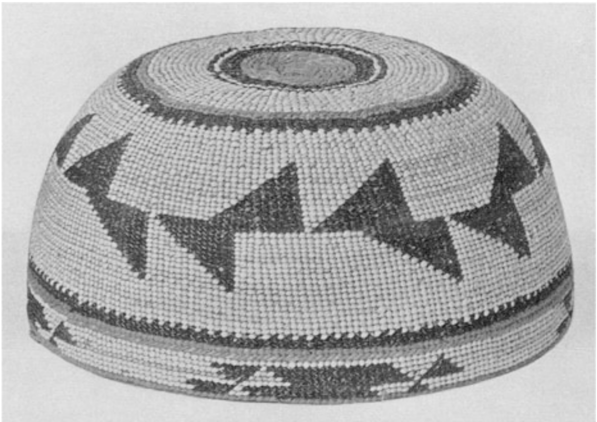

Fig. 9 Hat made for sale with “bad” *p1a1* design. From O’Neale, Fig. 22a

designs on the good and bad baskets. The good designs were constructed exclusively by *p112* and *pma2* symmetries (Fig. 8) while the bad designs organized the same motifs by other symmetries (Fig. 9). A blind sort of basket hat images by the symmetries used to repeat the designs on the hats resulted in a perfect separation of the good basket hats made for traditional home use from the bad basket hats made to sell outside the tribal sphere.

This case appears to exemplify a deliberate decision by basket makers to use pattern structure, rather than motif, as a way to differentiate objects that carry designs appropriate for internal traditional purposes versus those made to satisfy the desires of outside collectors. Non-Indian buyers, unfamiliar with the structural requirements for appropriate pattern, willingly bought baskets that appeared traditional from visible features such as technique, materials, design elements, even though the configuration of the designs elements had no ethnic authenticity. Early anthropologists as well as collectors filled museums with these tourist baskets, long thought to be representative of traditional material culture (Washburn 1984).

Temporal Continuity and Change in Design Symmetries in the Pueblo Southwest

The final example illustrates how an analysis of pattern structure change reflects the kinds of social configurations and changes to those configurations that organized communities of different sizes under changing environmental conditions. The data comes from study of a very large data base of 17,000+ geometric designs on ceramics made by the prehistoric puebloan peoples living in the northern American Southwest between AD 600 and 1600 (Washburn et al. 2010). Figure 10 charts the changing use frequencies of the most prominent symmetries over this 1000-year period, revealing clear shifts in the AD 800–900 period from *C2* and *D2* to *p112* and then again in the post-AD 1175 period from *p112* band designs to finite designs that are asymmetric or that have *C1* or *D2* symmetries as well or simple translational *p111* or mirror reflection arrangements, especially in paneled *pm11* vertical reflection configurations.

Both the shifts in Periods II and VII correspond to periods of environmental change severe enough to require changes in subsistence practices, habitation type and location, and social organization. During the first shift, cooler temperatures and lower rainfall forced people to abandon large pithouse villages on mesa tops whose inhabitants decorated pottery with *C2* and *D2* designs (Figs. 11 and 12). They moved into smaller masonry pueblo units scattered adjacent to valley bottoms, floodplains or arroyo fans that could capture the rainfall from the summer thunderstorm rainfall regime, a farming technique known as dry farming.

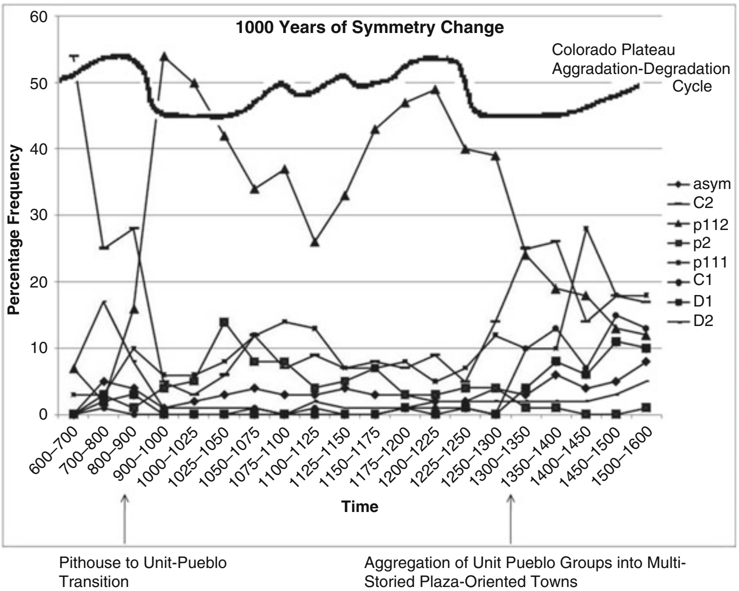

Fig. 10 Prevalence of five symmetries on Anasazi ceramics during nine periods. Source: (Washburn et al. 2010, Fig. 4)

Fig. 11 *C2* ceramic design. From Lister and Lister, *Earl H. Morris Memorial Pottery Collection*. Fig. 17, #9622. Copyright University of Colorado Museum of Natural History

Fig. 12 *D2* Ceramic design. From Lister and Lister *Earl H. Morris Memorial Pottery Collection*, Fig. 14, #9551. Copyright University of Colorado Museum of Natural History

A lengthy 400-year period of conditions generally favorable for this dry farming corn agriculture lifeway ensued that enabled the spread of these small unit pueblo villages throughout the Four Corners area. The prevailing *p112* symmetry on the ceramic designs of this period (Fig. 13) is a structural metaphor of the simple reciprocities between husbands and wives, ritual societies, and people and the deities that brought the rain. Such cooperative relationships in performing daily and ritual work are visually indicated by the interlocking of the motifs in the bands of *p112* designs on ceramics produced in these small farming villages.

However, by the late twelfth century successively longer droughts defeated even the best food production strategies. Almost the entire Four Corners area was depopulated. In New Mexico people moved to the permanently watered tributaries of the northern Rio Grande where many of their descendents continue to live in pueblos today. In Arizona people moved to the Little Colorado River and its tributaries as well as adjacent to permanent springs, such as on the southern edge

Fig. 13 *p112* ceramic design. From Lister and Lister, *Earl H. Morris Memorial Pottery Collection*, Fig 18, #9449. Copyright University of Colorado Museum of Natural History

Fig. 14 *pm11* ceramic design. From Mera, plate XXVII. Ogapoge Polychrome Olla. Style Trends of Pueblo Pottery, H.P. Mera. Library Collections, Museum of Indian Arts & Culture, Labaratory of Anthropology, Santa Fe

of Black Mesa where the Hopi live today in 12 villages. These site locational shifts to take advantage of different topographical conditions and water resources were accompanied by the development of new agricultural techniques. Along the Rio Grande use of mulched gravel fields and irrigated plots enabled the growth of large towns and thus necessitated the development of new forms of social organization to organize these larger groups. We suspect that the rise of *D2* and other mirror reflection symmetries, such as the *pm11* design on the jar in Fig. 14, reflects the development of the dual moiety divisions of the pueblos. In this organizational format the yearly ritual and political activities of the pueblo are shared by 2 moieties; each is in charge for one half of the year.

Conclusion

These few examples illustrate the kinds of information that can be gleaned from studying patterns on cultural materials by the plane pattern symmetries that generate them. This approach can be applied to patterns produced by cultures throughout the world, from the prehistoric to the present. The studies cited here exemplify how analysis of pattern by the plane pattern symmetries that arrange the pattern motifs can reveal continuities and changes in cultural activity that correlate with varying patterns of activity such as political interaction, trade, ethnic identity, environmental change and social organization. They suggest that the structural symmetries underlying designs and patterns may be more than compositional vehicles for creating pleasing decoration. Not only do the consistencies and changes in design symmetries appear to correlate with key factors in the environmental and social domains, but also the symmetries themselves appear to function for their makers and users as visual displays of socially important information.

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Interweaving Geometry and Art: Examples from Africa

Paulus Gerdes

Abstract One role of ethnomathematics as a research area resides in contributing to studies that permit to initiate the recognition of mathematical ideas of peoples 'forgotten' in books on the history of mathematics, valuing their knowledge in diverse manners, including stimulating the view that this knowledge may be incorporated into mathematics education. A fertile area of exploration of symmetry and geometry among these peoples has been the conception and fabrication of twill-plaited mats and baskets. Ethnomathematical study of plane patterns of mats and baskets contributes to the maintaining the collective and national cultural memory and prevents the permanent loss of several original, creative ideas and their resulting patterns. The basket and mat weaving presents a concrete example of how tradition and innovation may be interwoven. The article introduces an exceptional basket weaver, Arlindo Bendzane, whom the author met at the market in Palmeira, Mozambique. Bendzane has invented and explored a great variety of designs by changing the regular weaving texture through the introduction of lines of discontinuity or by using colored strips.

Introduction

Many peoples do not appear referred to in books on the history of mathematics. This does not mean that these peoples have not produced mathematical ideas. It only means that their ideas have not (yet) been recognised, understood or analysed by professional mathematicians and by historians of mathematical knowledge. One role of ethnomathematics as a research area resides in contributing to studies that permit to initiate the recognition of mathematical ideas of these peoples, valuing their knowledge in diverse manners, including stimulating the view that this knowledge may be incorporated into mathematics education.

A fertile area of exploration of symmetry and geometry among many cultures 'forgotten' in books on the history of mathematics has been the conception and fabrication of twill-plaited mats and baskets. Twill-plaited basketry designs (see the example in Fig. 1) embody and express the creativity, imagination, knowledge and reflections of the women and men who weave mats, make hats and baskets. Design elements may be symmetrical or not, and they may be combined in a symmetrical way or not. The patterns and designs may transmit various meanings.

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Fig. 1 Twill-plaited wall decoration in a Bamileke house in Cameroon

The invention of the plane patterns responds to the intellectual, geometric and artistic capacities of their creator(s), revealing, demonstrating and underscoring the mathematical inventiveness of the weavers. Ethnomathematical study of plane patterns of mats and baskets contributes to the maintaining the collective and national cultural memory and prevents the permanent loss of several original, creative ideas and their resulting patterns. Publishing studies and catalogues of weaving patterns and ideas may contribute to a creation of a cultural pyramid of personal and communal memory, intellectual capacity, and creative innovation.

Bringing 'Music' into the Texture

Designs and patterns result from the particular arrangement or twill of the strands interwoven by the mat or basket maker. Twilling is a form of plaiting whereby the strands in one direction may go over or under more than one strand in the opposite direction; in other words, the weaving is not always "over one, under one". The 2/2 twill indicates the "over two, under two" weave. The 3/3 twill represents the "over three, under three" weave.

When the regular 2/2 or 3/3 twills are used, diagonal zigzag lines become visible that make angles of 45 degrees with the (horizontal and vertical) weaving directions. When the basket weaver interrupts the regular, balanced twill at certain places of the fabric, then (s)he brings *ndzimo*—song and music—into it, as Tonga women say in Inhambane (Southeast Mozambique): zigzag lines change direction (see the example in Fig. 2); designs and patterns are invented.

Often *discontinuity lines* are interwoven in the fabric (see the examples in Fig. 3), and at the intersection of discontinuity lines appear 'toothed squares' or 'multiple spirals' (see the examples in Figs. 4 and 5). Toothed squares may be combined in various ways (see the example in Fig. 6). When plant strands are coloured with different dyes, more complex looking designs can be produced.

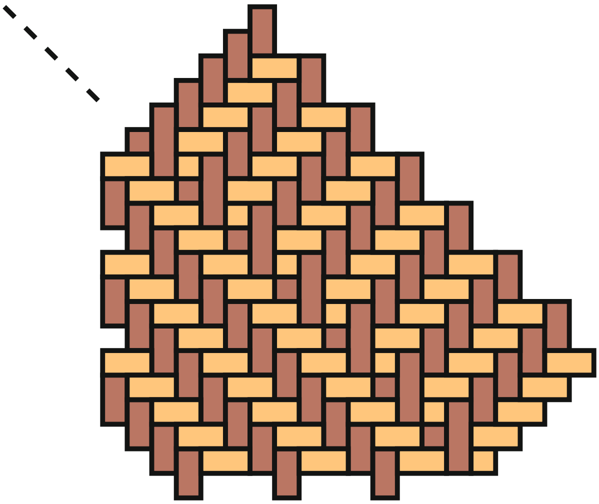

Fig. 2 An example of changing zigzag directions by Tonga basket weavers (Basic twill 2/2)

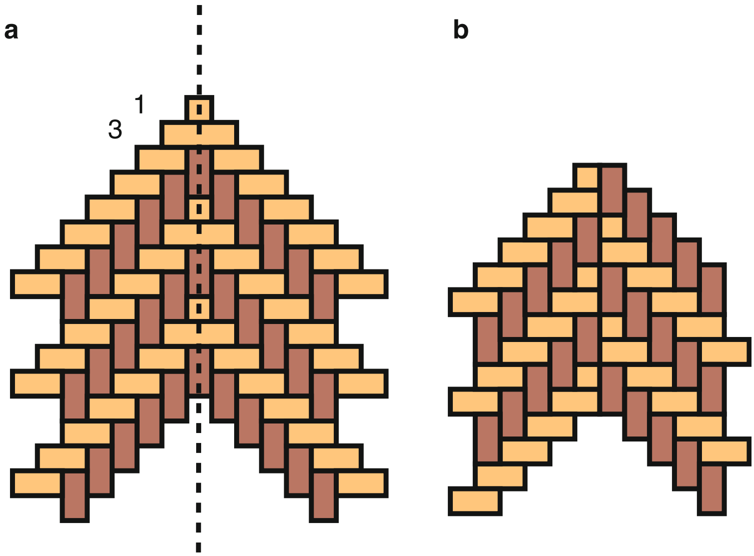

Fig. 3 Examples of 'discontinuity lines'. (**a**) (1,3) Discontinuity line, basic twill 2/2 for instance used by Makhuwa basket weavers (Mozambique). (**b**) Alternating discontinuity line, basic twill 2/2 for instance used by Kuba basket weavers (Congo)

The cultural phenomenon of basket and mat weaving presents a concrete example of how tradition and innovation may be interwoven. Experimentation, exploring possibilities and systematic variation are characteristics of the mathematic-artistic weaving activity. The weavers themselves say that they 'weave music' into the

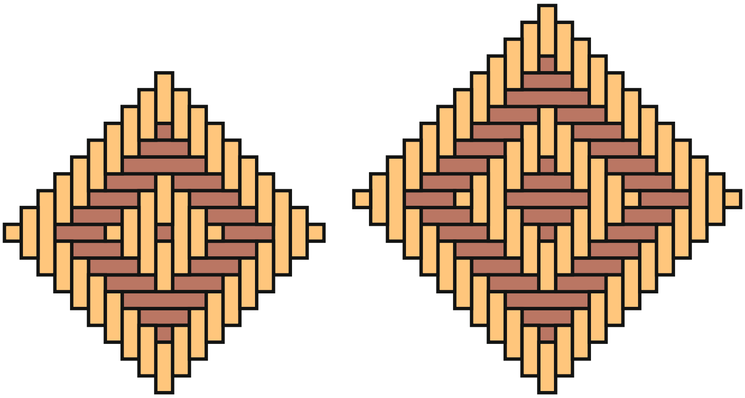

Fig. 4 Examples of toothed squares (Basic twill 3/3)

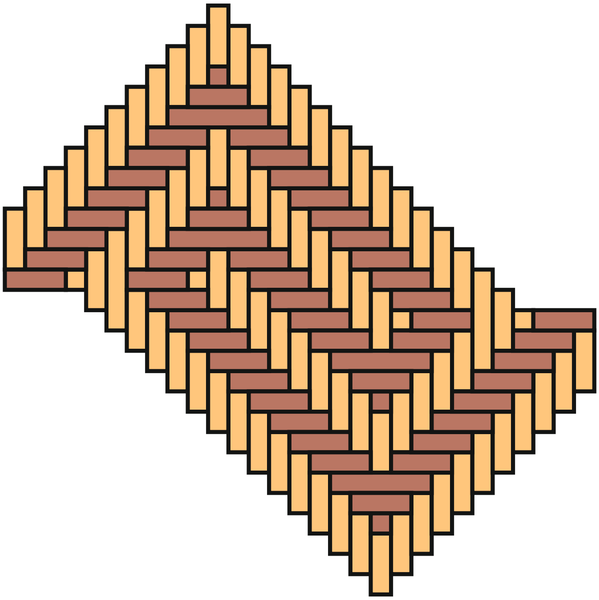

Fig. 5 Example of triple spirals in Makhuwa basketry (Basic twill 3/3) (Mozambique)

designs and patterns. By bringing ‘music’ into their textures, basket and mat weavers invent designs and patterns, expressing their emotions, ideas and reflections.

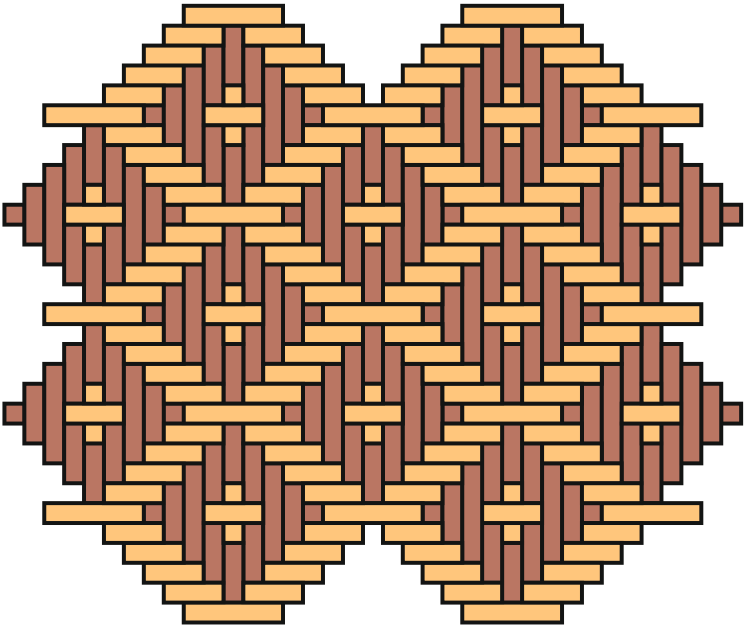

Fig. 6 Example of a combination of toothed squares from the Obamba (Gabon)

Earlier Studies and Collections

Probably the first studies both in the international and in the Brazilian context, in which the possible reasons for the appearance of concentric toothed squares in twill-plaited basketry are discussed, are those of Max Schmidt (1874–1950). In his books (Schmidt 1905, 330–403 [Brazilian translation 1942]) he introduces the concept of a twill-plaiting quadrilateral [German: *Geflechts-viereck*], talks about the mathematical properties of those quadrilaterals and studies the transposition of plaiting patterns to other contexts of ornamentation (cf. Schmidt 1904, 1926). In his autobiography Schmidt underlines that he proved that the origin of the majority of the geometric ornaments of the South-American indigenous peoples derives from the twill-plaiting technique (Schmidt 1955, 120). Schmidt's studies of Brazilian baskets at the beginning of the twentieth century may be considered ethnomathematical studies 'avant la lettre', many years before Ubiratan D'Ambrosio coined the term ethnomathematics. Schmidt's studies inspired the ideas of Georges-Henri Luquet (1876–1965) on the role of twill-plaited basketry in the 'origin of mathematical notions' (Luquet 1929).

Beautiful collections of twill-plaited baskets from around the world are included in the studies by LaPlantz (1993) and Sentance (2001). In the case of Brazil, Ribeiro (1985) presents a taxonomy of indigenous baskets and the ethnography of the Wayana baskets (Velthem 1998) presents material that invites to a historical-ethnomathematical reflection and to field research. In the study by Ricardo (2000) information is given about Baniwa basketry, including about some plaiting patterns and about the types of twill-plaited circular trays, *waláya* and *dopítsi*. For the Baniwa, the plaiting design formed by concentric toothed squares is the first that every child learns. In one of my previous papers (Gerdes 1989), some aspects of arithmetic and geometrical ornamentation of indigenous baskets from Brazil are analysed. In various studies the author has tried to contribute to the understanding of forms of geometrical thinking involved in the production of twill-plaited objects. In my book *Awakening of Geometrical Thought in Early Culture* (Gerdes 1990, 1991, 1992, 2003), some relationships between twill-plaited basketry and the early history of geometrical thinking and symmetry are explored.

Encounter with a Weaver

For years I used to pass, once or twice a month through a village called Palmeira in Portuguese, supposedly after an old, tall palm tree visible from far away on the top of a hill. The village lies about 70 miles North of Maputo, the capital of Mozambique on the southeast coast of Africa. At the village market and along the main street transport baskets and sleeping mats are sold, along with smaller quantities of circular winnowing trays, called *rihlèlò* (Plural: *tinhlèlò*) in the local Changana language.

Most basket trays in this region of the Mozambique are plaited without any design. The basket weaver, Arlindo Bendzane, who lives about 25 miles east of Palmeira, however, is very exceptional: he has invented and explored a great variety of designs by changing the regular weaving texture through the introduction of lines of discontinuity or by using colored strips.

On the left hand side of the vertical line of discontinuity in Fig. 7 the diagonals go up from left to right: ╱. On its right hand side, the diagonals go down from left to right: ╲. In the regular 2/2 twill, the horizontal strips always go over and under two vertical strips. When they pass a vertical line of discontinuity in the plaiting texture, however, they pass once over one or three vertical strips.

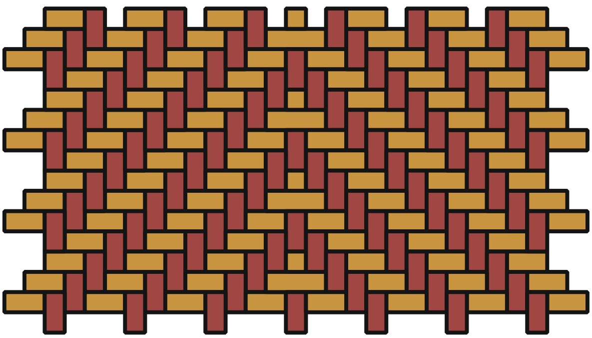

Fig. 7 Vertical line of discontinuity

Fig. 8 Example of a basket made by Arlindo Bendzane

Bendzane had learned from his father how to make *tinhlèlò*. It seems that the period 2002–2004, when Bendzane was in his early 60s, was a high point in his creative life. During that period, I acquired 65 basket trays made by Bendzane that display distinctive designs. Becoming older, he stopped making his colorful, design-rich basket trays (see an example in Fig. 8). During the last years only design- and color-less basket trays, made by other weavers, were for sale in Palmeira.

Over a period of several years, I had selected and collected Bendzane's winnowing trays, each with a distinctive design. Then suddenly and unexpectedly, I had an opportunity to meet Bendzane on Sunday May 2, 2004. The old master weaver had become curious and wanted to see that 'strange' person who was so interested in his trays and designs, and who always selected trays with a notebook in his hands. On this occasion, I passed with my family through Palmeira on a Friday and was returning on Sunday. When the news spread among the basket sellers—one of whom was Bendzane's son—that I would most probably be on my way back on Sunday, Bendzane was waiting to meet me. Figure 9 shows Bendzane with one of his *tinhlèlò* together with me, with my notebook in my hands. He was very eager to look through my notebook. Surely my notebook was more interesting than me.

The notebook displays black-and-white photographs and computer drawings of basket trays in my collection. This collection contains basket trays from different parts from Mozambique (cf. Gerdes 2010a, b), from other parts of Africa (like Senegal in West Africa, cf. Gerdes 2000), from South America (in particular from the Bora the Peruvian Amazon, cf. Gerdes 2009a), from North America (in particular twelfth century winnowing baskets from the Anasazi in today's Arizona, cf. Gerdes 2000) and several from Asia. The order of the photographs and drawings in the notebook follows my own, mathematical criteria and is not by country or culture. We sat down on a mat and during more than one hour the master weaver was looking into my notebook, page by page. Bendzane was very glad to recognize each of his baskets immediately as his personal product and invention. About other designs he made comments like "This is not my design, but I could have made it", "This is not my design, I could not have thought about it, but now I will try to reproduce it", "This is a very intricate design...". At the end, he stood up and proudly he addressed the crowd that had gathered. He spoke about his knowledge, about how he had often tried to achieve 'equal parts' and how he had sometimes deliberately broken the symmetry. I found it most significant, that

Fig. 9 Basket weaver Arlindo Bendzane with the author (May 2, 2004)

speaking in Changana, he used the Portuguese word for science—'ciência'—when he referred to his knowledge.

From a technical point of view, when making a good winnowing tray a challenge is how to weave a square mat and how fasten to it to a circular hoop. Normally, a rectangular piece of wood is cut and bent into a circular hoop, and then the basket weaver binds the two ends together. The loosely plaited square mat may be dampened with water, and, after being tied to the hoop initially in four places (see the example in Fig. 10), it is then molded into shape. The corners of the square mat are cut off and the remaining section is pushed through the rim.

The initial fastening of the edges of a square mat, at their midpoints, to the hoop may be easier if these four midpoints or the corresponding two midlines are visible in one way or another. The midlines of the square become two perpendicular diameters of the circular base of the winnowing tray (see Fig. 11).

Creative basket weavers in Mozambique, like Bendzane, and in other parts of the world have invented and used two different methods to achieve this visibility of the midpoints and midlines: either they introduce some differently colored strips,

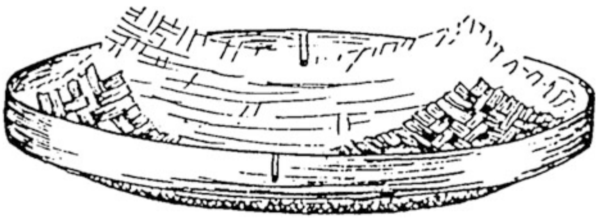

Fig. 10 Fastening the edges of a square mat at their midpoints

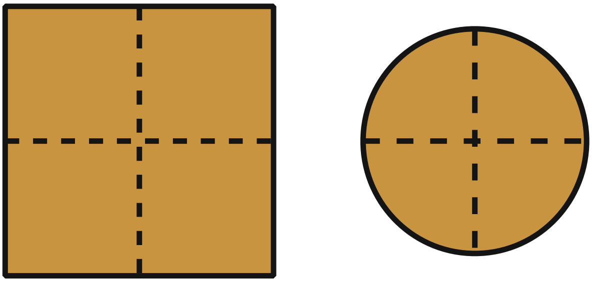

Fig. 11 A square and its two midlines; a circle with two perpendicular diameters

breaking the monotone of the base, or they change, in a systematic manner, the plaiting texture, breaking thus the regularity of the 2/2 or 3/3 twill by introducing lines of discontinuity.

The most widespread solution is to weave the mat in such a way that the midlines are lines of discontinuity. Two types of designs may be generated in this way as Fig. 12 illustrates: either a design with concentric squares or a X-design.

Sometimes the master weaver Bendzane opted for this type of 'simple' solution, as Fig. 13 illustrates.

More often, however, Bendzane invents more laborious, symmetric structures like the ones exhibited in Fig. 14, where the red lines represent lines of weaving discontinuity and the blue squares represent the groups of concentric toothed squares generated by the lines of discontinuity.

Figures 15 and 16 present two winnowing trays made by Bendzane that correspond to the weaving structure in Fig. 14a. On the winnowing tray in Fig. 17 there are five lines of discontinuity in each of the two weaving directions, leading to a design structure with twelve sets of concentric toothed squares. The resulting beautiful design has not only a vertical and a horizontal axis of symmetry, but also two diagonal axes if one does not take into account the colors of the strips, as Fig. 18 underscores.

The examples presented may give an idea of the creativity of virtuosity of the master weaver Bendzane from Palmeira. For an analysis and colorful exhibition of winnowing trays made by Bendzane and other basket weavers in the South of Mozambique, see the book "*Tinhlèlò*, Interweaving Art and Mathematics: Colorful Circular Basket Trays from the South of Mozambique" (Gerdes 2010b). The reader, who likes to know more about geometrical ideas in African cultures in general, may consult "Geometry from Africa" published by the MAA (Gerdes 1999). The books "Sipatsi" (Gerdes 2009b) and "Otthava" (Gerdes 2010a) present an analysis of

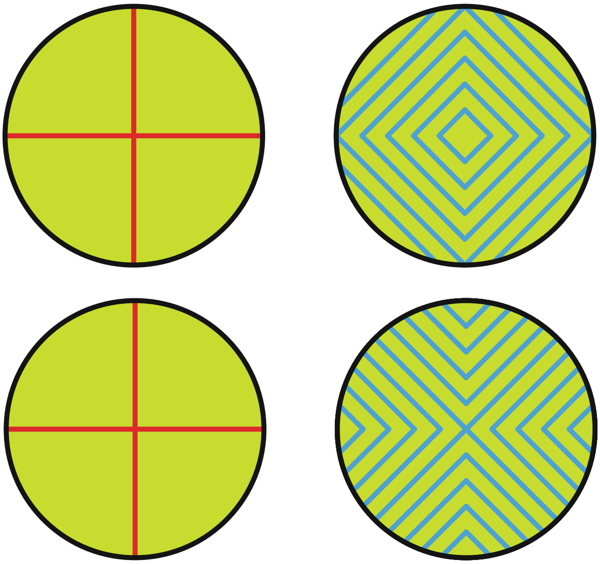

Fig. 12 Generation of designs with concentric squares or with a central X

Fig. 13 A tray with a X-design made by Bendzane

basket weaving and geometry among the Tonga and Makhuwa populations in Mozambique. The book "Sona Geometry" (Gerdes 2006) presents an introduction to and analysis of the geometry of sand-drawings from Angola. For a bibliography and research overview of mathematical ideas in African cultures, see the books by Gerdes and Djebbar (2007, 2011).

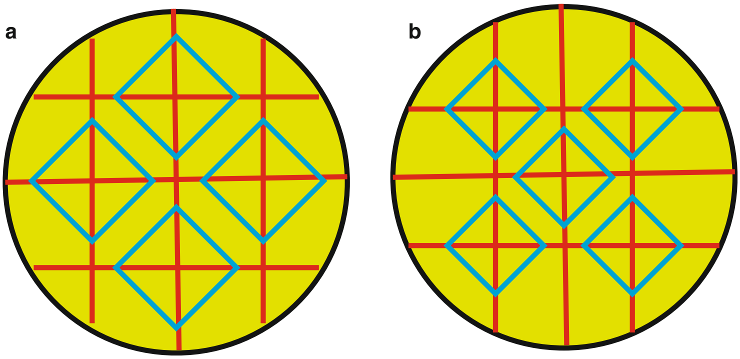

Fig. 14 (**a**) X-centre; Four sets of concentric toothed squares. (**b**) ◊-centre; Five sets of concentric toothed squares

Fig. 15 A winnowing tray made by Arlindo Bendzane

Conclusion and Perspectives

On the one hand, it seems necessary to continue in Mozambique and elsewhere with longitudinal studies to expand the understanding of the mathematical reasoning and creativity of basket weavers in diverse cultural contexts. On the other hand, continuing the collection of data from cultures all over the world that have explored twill-plating, may contribute to the further comparative understanding of the development of human thinking, including mathematical reasoning, reacting to similar societal problems to be solved, like in the case of making circular basket trays.

Fig. 16 A winnowing tray made by Arlindo Bendzane

Fig. 17 A winnowing tray made by Arlindo Bendzane

For instance, in my book of woven hats from Mozambique a possible starting point for an international comparative study is presented (Gerdes 2010c). In the case of Brazil, it would be interesting to make an analysis of twill-plaited basket weaving among indigenous peoples, like the Guarani, Tupi, Krenak, Terena, and Kaingang in the State of São Paulo, and in other states.

Fig. 18 Four axes of symmetry

Further educational experimentation with twill-plaited basket weaving may be explored. The work of the late Maurice Bazin may serve as inspiration: a reflection on a consequent praxis of mathematics education, in the context of the struggle for survival of an indigenous people from Brazil, may be found in the manuscript "Teaching mathematics and indigenous science or how I learned with the Tuyuka people" (Bazin 2001). He has made an experiment with the educational use of basket weaving (cf. Bazin and Tamez 1995). The analysis of mathematical ideas involved in twill-plaited basketry will surely continue ranging from the cycle matrices and specific number friezes to new mathematical ideas.

Acknowledgements The article is a combination of two papers by the author: "Analyzing and Exploring Mathematical Ideas Involved in Twill-Plaited Basketry" and "Interweaving art and geometry: Encounter with an African master weaver" (*Math Horizons* 19(4), 25–27, published by Mathematical Association of America, Washington).

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Part III
Cultural Meanings of Geometric Composition, Structure, and Form

Models of Surfaces and Abstract Art in the Early Twentieth Century

Angela Vierling-Claassen

Abstract In the late 1800s and early 1900s mathematicians were producing models of mathematical surfaces out of plaster, wire, and other materials. These models were used to illustrate research and for university instruction. Gradually, mathematical interest in these models faded, but the models themselves were still on display in universities and museums. There they were found by several artists from the Constructivist and Surrealist movements, two movements of abstract art that were active in the early twentieth century. Artists from each of these movements drew some inspiration from these models of surfaces.

Models of Surfaces

Humans have been constructing approximations of polyhedra and simple geometric solids for a long time. Spherical forms that show all five regular polyhedra have been found in Scotland dating as far back as 2000 BCE (Artmann 1994). However, models of more complicated three dimensional surfaces do not appear until the beginning of the nineteenth century after more sophisticated mathematical theories led mathematicians to attempt to visualize such forms.

In the 1700s, Gaspard Monge developed descriptive geometry, which involves representing three dimensional objects in two dimensions (for instance, by sketching a projection of the object). His ideas were developed at a school for military engineering and were a military secret until after the French Revolution. Monge made models of ruled surfaces for instructional purposes; his are the earliest known models of this type (Kidwell 1996).

Monge's student, Thèodore Olivier, also built models. Olivier taught descriptive geometry at the École Centrale des Artes et Manufactures and the Conservatoire National des Arts et Métiers and designed models of ruled surfaces some of which were manufactured by the firm of Pixii, Père, and Sons. Their successor, Fabre de Lagrange, continued to manufacture the models (Kidwell 1996), selling models to

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locations such as the South Kensington Museum in London, now known as the Science Museum (South Kensington Museum 1877).

Around the mid-1800s, the "golden age" of model building began. Many mathematicians began to build models out of a variety of materials, including plaster, cardboard, metal, and string. Many of the model-builders were German, and the names of those involved include influential mathematicians such as Eduard Kummer, Felix Klein, and Alexander Brill. Felix Klein was a major proponent of both visual intuition and model-building. After the 1893 World's Columbian Exposition (World's Fair) in Chicago, Klein gave a series of lectures at Northwestern University in which he said, "I wish to insist in particular on what I regard as the principal characteristic of the geometrical methods that I have discussed today: these methods give us an actual mental image of the configuration under discussion, and this I consider as most essential in all true geometry" (Klein 1894, 32). The German exhibit at the Chicago World's Fair featured a display of models which were available for purchase by universities (Kidwell 1996).

In the fourth lecture at Northwestern, Klein discussed the shape of algebraic curves and surfaces, making reference to models that were displayed during the World's Fair. One such model is diagonal cubic created by Alfred Clebsch (see Fig. 1). "In 1872 we considered, in Göttingen, the question as to the shape of surfaces of the third order. As a particular case, Clebsch at this time constructed his beautiful model of the diagonal surface, with 27 real lines" (Klein 1894, 26). Any smooth surface which is the zero locus of a polynomial of degree three has exactly 27 straight lines lying on the surface. In the diagonal cubic all 27 of these lines are real. In the plaster model in Fig. 1, these lines have been etched on the surface. In this model, several "passages" (as they were called in the classic literature) can be seen. Klein had the idea to generate all of the possible types of cubic surfaces by collapsing and deforming these passages (for some details, see Fischer 1986) and a series of models illustrating this was executed by Klein and his student Carl Rodenburg. For example, collapsing three of the passages results in a surface with three ordinary double points which contains nine lines (see Fig. 1).

There are many other gems to be found in a collection of mathematical models, including the hyperboloid of one sheet and the helicoid (see Fig. 2) as well as Kuen's Surface and the generalized helicoid of constant negative curvature (see Fig. 3).

Many of the models built were reproduced and sold by publishing houses (especially the German publishing house of Ludwig Brill which was later taken over by Martin Schilling) to schools and museums around the world. In 1911, the catalogue of Schilling models contained 40 series which included 4000 models and devices (Fischer 1986). By 1932, Martin Schilling informed the mathematics institute at Göttingen that "in the last years, no new models appeared" (Fischer 1986, Volume 1, IX). The most productive phase of model-building was over, perhaps due in part to the shift in the mathematical culture away from intuition and visualization and towards formality and rigor. Many of the models, however, survived in universities and museums, sitting in dusty cases when they were no longer brought out as part of mathematics instruction. There they awaited discovery by the non-mathematical world.

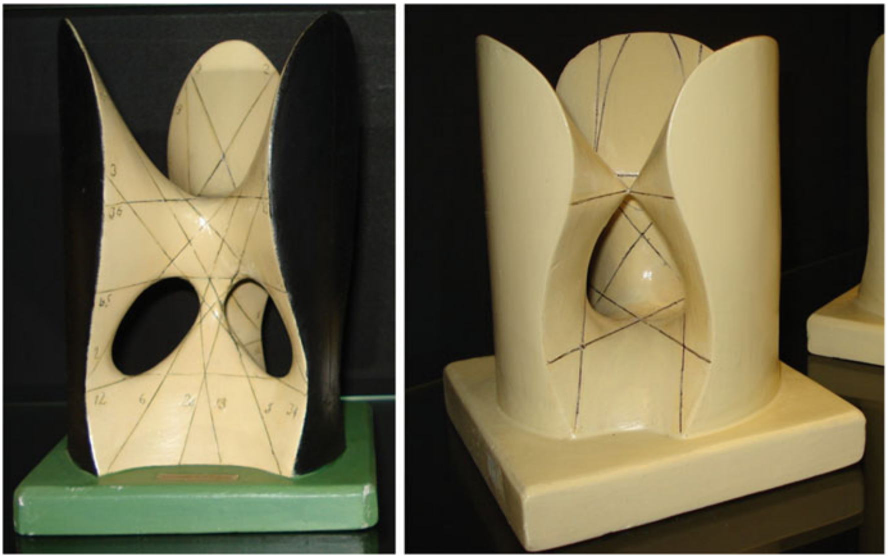

Fig. 1 *On left*: Clebsch diagonal cubic, showing the 27 real lines lying on the surface. Brill Series 7, Number 1. *On right*: Cubic with three A_1 double points. Brill Series 7, Number 8. From the MIT collection, photograph by the author

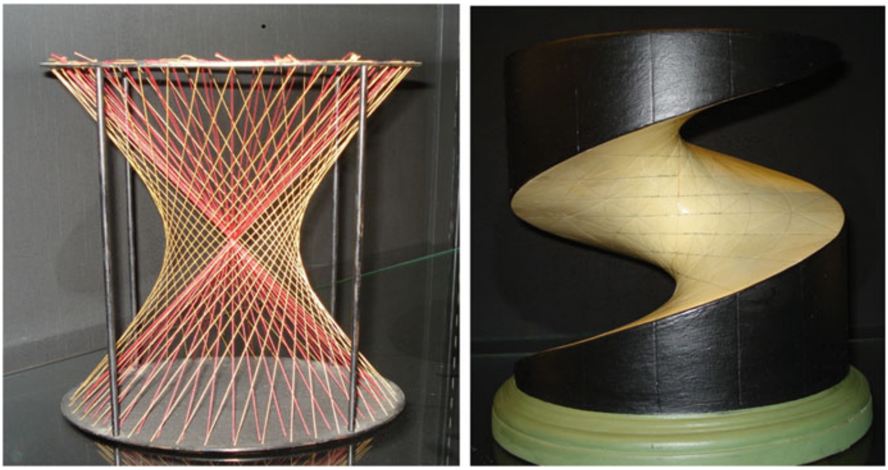

Fig. 2 *On left*: Hyperboloid of one sheet with asymptotic cone. Unknown origin, but possibly Brill Series IV, Number 1. *On right*: Helicoid, showing the generating line. Brill Series 8, Number 6a. From the MIT collection, photograph by the author

Fig. 3 *On left*: Two views of Kuen's Surface. Brill Series 8, Number 1. *On right*: Generalized Helicoid. Brill Series 5, Number 4. Both of these surfaces have constant negative curvature. From the MIT collection, photograph by the author

Birth of Modern Art

Until the late nineteenth century, Western art was mostly representational. That is, art consisted of drawings, paintings, and sculptures that were intended to represent some aspect of the physical world-people, animals, landscapes, and common objects. In the years preceding the First World War, a new kind of artistic movement began. This was a movement of abstract art, or art that was not intended by be representational. Wassily Kandinsky is generally regarded as having been the first western artist to paint purely abstract pictures. This first wave of abstraction was widened by movements such as Dada, de Stijl, and Constructivism.

Another kind of modern art was being developed during the time in between the world wars, the Surrealist movement. The Surrealists were not trying to remove representation from art (like Constructivists) nor to destroy traditional art (like Dada) but instead to give a voice to the unconscious and irrational.

Both the Surrealists and Constructivists seem to have had some exposure to the mathematical models of surfaces. These models of geometric surfaces have an eerie beauty; it is not surprising that artists would be intrigued by them. It is even less surprising when you put their interest in context. The mid- to late- nineteenth century was a turbulent time in mathematics. Old ideas, such as Euclidean geometry, were being turned on their heads. Many revolutionary mathematical ideas made their way into the public sphere and sparked the imagination of writers and artists alike. Writers such as H. G. Wells and artists like Marcel Duchamp were fascinated with non-Euclidean geometry and the idea of a spatial fourth dimension.

The surrealists found that "non-Euclidean geometry signified a new freedom from the tyranny of established laws" (Henderson 1983, 339). Culturally, mathematics represented both scientific progress and the potential for chaos.

Naum Gabo and the Constructivists

Constructivism was an artistic and architectural movement that began in Russia in the early twentieth century. Naum Gabo and his brother Antoine Pevsner, were, to a large extent, responsible for popularizing and spreading the movement outside of Russia, notably to Paris and England. Constructivists also had an influence on the Abstraction-Creation group, the de Stijl movement, and the Bauhaus.

It is likely that Gabo saw mathematical models on display while he was a student in Munich. Gabo had some interest in art as a teen, but went to the University of Munich to study medicine. He took other scientific coursework while he was there, particularly physics and engineering (Hammer and Lodder 2000). Notably, he also followed some courses at the Technical University in Munich, where there were certainly mathematical models on display, as Felix Klein and Alexander Brill produced and studied models in problem sessions with students (Fischer 1986). In his 1936 Museum of Modern Art exhibition catalog Cubism and Abstract Art, Alfred Barr made even more of Gabo's connection with models, stating that Gabo "had been studying mathematics in Munich and had made mathematical models" (Barr 1936, 133). This statement is probably inaccurate, but it does show that the topic of Gabo's connection with mathematical models had surfaced during Barr's research.

Gabo's early cubist-influenced sculptures such has Head No. 2[1] are reminiscent of cardboard models of surfaces made with interlocking cross-sections, such as would have been on display at the university. Pictures of such models could also be found in encyclopedias of the time (Baynes and Robertson Smith 1875).

This early similarity of Gabo's work to certain mathematical models may simply be coincidence, but later influence of models on Gabo's work in the 1930s is clear. In their book, Constructing Modernity: The Art and Career of Naum Gabo, Hammer and Lodder (2000) note that a 1936 drawing Study for Construction in Space: Crystal appears to be a tracing of a figure from the article "Mathematical Models" in the 14th edition of the Encyclopedia Britannica. Indeed, the Encyclopedia drawing has many similarities to the final sculpture Construction in Space: Crystal (1937–1939, cellulose acetate, Tate Modern, London).

Others sculptures and drawings show evidence of Gabo's encounter with mathematical models. For instance, Gabo's 1933 Sketch for a Stone Carving (crayon on paper, Gabo family collection) is reminiscent of a ruled surface or helicoid (see

[1]Many versions of Head No. 2 exist, one version can be seen at http://www.tate.org.uk/art/artworks/gabo-head-no-2-t01520

Fig. 2) which would have been on display at the Poincaré Institute in Paris. Gabo was living in Paris at the time this sketch was done.

Antoine Pevsner was the brother of Naum Gabo. Constructions of Pevsner's from the mid-1930s show evidence of a possible influence of mathematical models. Pevsner began his artistic career as a painter, and during the 1920s Gabo encouraged his brother to pursue sculpture and taught him constructive techniques. Pevsner always denied any direct influence of mathematics on his work. However, his Developable Surface series was perhaps inspired by models of ruled surfaces (see Fig. 2). Mathematically, the term developable refers to a surface which can be constructed from a flat plane without stretching or tearing (at least locally). Such a surface can thus be swept out by lines precisely in the manner of Pevsner's sculptures.[2]

British sculptor Barbara Hepworth had contact with Naum Gabo while Gabo was in England—from 1936 to 1946. Hepworth may have seen mathematical models before she encountered Gabo. In December 1935, Hepworth sent a letter in which she said that architect John Summerson had told her that there were "some marvelous things in a mathematical school in Oxford—sculptural working out of mathematical equations—hidden away in a cupboard" and that she intended to go and look at them soon (Hammer and Lodder 1996). Some of Hepworth's work exhibits mathematical influence. The sculpture Helicoids in Sphere (1938, lignum vitae on original base, private collection), for instance, has similarities to a mathematical model of a surface known as Steiner's Roman Surface. Other sculptures, such as Pelagos (1946, painted wood and strings, Tate Modern, London) and Winged Figure (1963, on the side of John Lewish store at Oxford and Holles Streets in London) echo mathematical models in their form and in their use of string. During the 1930s, Hepworth was producing sculptures using string and plaster as in Sculpture with Colour (Deep Blue and Red) (1940, painted plaster and string, Tate Modern, London). Both of these materials were materials widely used in the mathematical models.

Henry Moore is another British sculptor who is sometimes associated with constructivism. Moore stated more than once that his use of string in his sculpture, which began in 1937, was influenced by seeing models at the Science Museum in London.

> I was fascinated by the mathematical models I saw there, which had been made to illustrate the difference of the form that is halfway between a square and a circle. One model had a square at one end with 20 holes along each side [--] Through these holes rings were threaded and lead to a circle with the same number of holes at the other end. A plane interposed through the middle shows the form that is halfway between a square and a circle [--] It wasn't the scientific study of these models but the ability to look through the strings as with a bird cage and see one form within the other which excited me. (Hedgecoe and Moore 1968, 105)

[2]For instance, see Construction in Space and in the Third and Fourth Dimensions, which can be found at the University of Chicago. See https://arts.uchicago.edu/public-art/by-work/construction-in-space

The influence of such models can be seen in pieces such as Stringed Figure No. 1 (1937, cherry wood and string, Hirshhorn Museum and Sculpture Garden, Washington, DC).

Man Ray and the Surrealists

In The History of Surrealist Painting, Marcel Jean suggests that it may have been Max Ernst who brought the mathematical models into the surrealist consciousness. "Max Ernst had originally come across these constructions in the Institute Henri Poincaré and had mentioned them to the director of Cahiers d'Art, Christian Zervos, who in his turn had asked Man Ray to photograph them" (Jean and Mezei 1960, 251). This is backed up by Baldwin's (1988, 199) account in his book on Man Ray, in which he says this was a series

> [--] of photographs of items created in the 1880s by a physicist attempting to render algebraic formulae correctly. Max Ernst had taken Man Ray to see the objects on display at the Poincaré Institute in Paris and had photographed them in a deliberately impressionistic style.

And in Man Ray's own words in the film A Life in the Day of Man Ray he says that he "was told about some mathematical objects at the Institute Poincaré in Paris," although he does not mention who told him. Man Ray then executed a series of photographs, entitled Mathematical Objects, which were studies on models at the Poincaré Institute. Some of these photographs appeared in a 1936 issue of Cahiers d'Art along with an essay on mathematics and abstract art by Christian Zervos.

Man Ray's photographs, as well as his later series of paintings, Shakespearean Equations, based on them[3], gave mathematical models a lot of exposure. The surrealists displayed mathematical objects in their May 1936 show "Exposition Surraliste d'Objets" at the Galerie Charles Ratton in Paris. In his famous "Crisis of the Object", André Breton (1972, 274) writes,

> The laboratories of mathematical institutions throughout the world already display side by side objects constructed according to both Euclidean and non-Euclidean principles, equally mystifying in appearance to the layman, but which nevertheless bear a fascinating and equivocal relationship to each other in space as we generally conceive it.

Man Ray's interest in the mathematical objects also may have had an influence on the 1936 International Surrealist Exhibition at the New Burlington Galleries in London. This show was supported financially and organized by Roland Penrose, who was befriended by Man Ray and André Breton in Paris. Man Ray went back and forth between London and Paris, carrying art in preparation for the show (Baldwin 1988). His photographs of mathematical objects were on display at the show, which was open from June 11 through July 4, 1936, with attendance of 1500

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[3]See images at http://www.manray.net/shakespearean-equation.jsp

people per day. Even the catalogue from this exhibition reveals a fascination with mathematical models—the front cover of the catalog is a collage by Max Ernst featuring a statue with a reptilian head holding and standing near a number of mathematical models, including Kuen's surface (see Fig. 3). Man Ray's photographs showed up at yet another big exhibition of the time, the 1936–1937 Fantastic Art, Dada, and Surrealism exhibition at New York's Museum of Modern Art.

Other surrealists may have been inspired by seeing the models and their photographs. Max Ernst, who did the cover of the London exhibition catalog, did several other collages and paintings that seem to relate to mathematical models. Examples include The Feast of the Gods (1948, oil on canvas, Museum of the twentieth Century, Vienna, Austria), Chemical Nuptials (1948), and Young Man Intrigued by the Flight of a Non-Euclidean Fly (1942–1947) which contains several forms reminiscent of mathematical models such as the Spindle Cyclide and the Horn Cyclide.

Conclusion

As exciting as it is to trace the influence of mathematics on a group of artists, in the episode described in this paper, there is still a large disconnect between mathematicians and artists themselves. There was mutual influence, but never direct connection. The creators of mathematical models must have been influenced by the arts. After all, when creating a model of a mathematical surface, there are an infinite number of choices to be made: What view of the surface will be used? What perspective? What are the important features to be displayed? What materials and methods will be used in the construction? All of these choices are informed by aesthetics, visual culture, and visual literacy, which are developed and refined in the world of art and design.

And it is clear that through the models, mathematicians influenced artists, even without having any direct contact. The surrealists saw in the mathematical models the contradiction between the rationality of mathematics and the fantastical nature of the forms. The constructionists saw the mathematical models as a pure form of sculpture, removed from representation. Artists from both movements were able to use the models as a source of forms and visual ideas.

As artists, designers, and mathematicians learn to communicate about their works, appreciate the interconnections between their fields, and engage in joint projects, we will continue to see the kind of influence represented by the models turn into true collaboration between artists and mathematicians.

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The Grid: As a Starting Point and an End Point of Ornament

Satu Kähkönen

Abstract Ornament was one of the key concepts in the nineteenth century discussions on architecture and decorative arts. In the beginning of the twentieth century, modernism abandoned ornament from its vocabulary. In the last decades of the twentieth century, the heritage of Modernism and its rejection of ornament were critically analyzed and ornament became a topical subject once more. Geometry played an important role in the search for rational and intellectual basis for the architecture and decorative arts in the beginning of industrial age. Arabic ornamental practice with its geometrical foundation served as one precept in searching and formulating appropriate design principles for architecture and decorative arts in the nineteenth century. In Arabic ornamental design different geometrical grids function as a starting point which offers unlimited possibilities for designs. In the twentieth century, the Western aesthetics limited to the simple grid formed by horizontal and vertical lines. This chapter reflects on the history and the heritage of the ornament discussions focusing especially on the role of geometrics in the conceptions of ornament.

Introduction

Generally speaking, ornament has been seen as an inseparable part and a fundamental element of the Arabic architecture, whereas in Western architecture it has been seen as secondary, additional, or even unnecessary. This dissimilarity does not concern only the aesthetics but it implies a profoundly different way of thinking. Conventionally, it is stated that modern art and architecture abandoned ornament in the beginning of the twentieth century. In its ambition to be modern and progressive, Western art and architecture no longer considered ornament as a valuable element. Instead, ornament became to represent the past, the feminine and other cultures. The rejection of ornament is typically related to the writings of Adolf Loos (1870–1933) and Le Corbusier (1887–1965).

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However, unlike in the twentieth century, ornament was of great interest to diverse fields of study and one of the central concepts of art theory in the nineteenth century: ornamental practices were studied from architectural, artistic, anthropological, and psychological perspectives. Interest in ornament was not purely historical or theoretical. In many cases both historical and theoretical questions were closely related to the contemporary design practices. The concept of ornament and ornamental practices returned into focus in the Western architecture and design in the end of the twentieth century. Topics like the ornament revival and reassessing the theory as well as the functions of ornament were once more regarded to be up-to-date.

In this chapter I will start with the Western conceptions of ornament. Then I will illustrate how the examples of Arabic ornamental tradition served as one exemplary practice when the nineteenth century reformers tried to find appropriate design principles for decorative arts in the industrial age. In general, geometry played an important role in the search for rational and intellectual basis for the decorative arts in the mid-nineteenth century England. Geometry was not understood just as a compositional design matrix but also as a tool for control or restrain of pattern design. One of the main objectives was to separate decorative arts and pattern design, both in theory and in practice, from fine arts and direct imitation of nature. Towards the end of the nineteenth century, ornament was seen to an increasing extent as essentially abstract or linear. In the beginning of the twentieth century concepts like structure, form and space came into focus in architectural theory, and ornament as a word was cut out of theory and discussions. But did ornament disappear in practice? I will address two interrelated questions: first, if the grid, a proportional system or a pattern of regularly spaced horizontal and vertical lines, can be seen as modernist ornament, and second, how the Arabic ornamental tradition is modernised in modern European architecture. In conclusion, I will ask if the concept of ornament is still relevant and can it be related to contemporary design practices?[1]

Western Conceptions on Ornament

Ornament became a topical subject in the last decades of the twentieth century. This is explained partly by the ideas of the end of modernism and the beginning of postmodernism. Thus, most of the studies on ornament from the 1980s until today have shared an objective to critically analyze the heritage of Modernism and its rejection of ornament (see, for example, Schafter 2003). In these studies the *fin de siècle* Vienna with figures like architect Adolf Loos (1870–1933) and the artists of the Secession movement have played a central role. The ornament debate has been

[1]The chapter is based on the author's PhD study on the concept of ornament in discussions concerning aesthetics of architecture and design during the past 200 years.

seen as for example a battle between Jugendstil architects and those who advocated simplicity and rationality. In some studies the question has been seen in a wider context and the focus has been on tracing the origin of the Western thinking of ornament as a mere additional element (Raulet and Schmidt 1993; Franke and Paetzold 1996). According to the German art historian Frank-Lothar Kroll (1987, p. 2), it is possible to trace the interpretation of ornament as a secondary, additional element of the Western architecture at least since Vitruvius. For some others, it was the Renaissance architect Alberti who made the definitive division between structure and ornament in his famous treatise on architecture (Eck 1994, p. 45). Nevertheless, most of the writers agree that before the twentieth century ornament could be separated from the structure only in theory, but not in practice.

In his summary of the interpretations of ornament in the Western theory of art in the nineteenth century, Kroll (1987, pp. 153–156) makes the following observations: The question of the origin of the ornament (simultaneously a question of the origin of the arts) was an important and controversial topic for the nineteenth century writers. For example the German architect Gottfried Semper (1803–1879) related the origin of ornaments to different materials and techniques. Semper emphasized the role of textiles and for example the technique of weaving in the genesis of the first ornaments. On the other hand, in his study of the history of the development of ornamental motives, the Austrian art historian Alois Riegl (1858–1905) claimed that ornament was first of all a result of a creative impulse, "Kunstwollen". However, most of the nineteenth century writers agreed on the formal and material essence of ornament, but they used different terms to describe it. For example, for the architect Karl Bötticher (1806–1889) and the philosopher Theodor Lipps (1851–1914) ornament was art added to the core form (Kernform), and for Semper it was the decorative clothing or dress of the object or architecture. The writers also agreed that ornament was always dependent on the form, object or background and it could not exist alone. Thus, in contrast to architecture or fine arts, ornament was not an independent art form. The writers used different art forms in their illustrations of the topic; they found ornamental practices in architecture, painting and sculpture. According to Kroll (1987, p. 154), if one wants to try to explain the ontology of ornament, one has to ask which are the unchanging elements of ornament in different art forms. He highlights that the specificity of ornament lies in its dual role as the connective element between arts and as something that cannot exist alone.

Ornament was one of the central concepts in the nineteenth century discussions on architecture and decorative arts. It had its role in discussions on both beauty and practice. On their writings on architecture, architects, designers and critics such as Augustus Welby Northmore Pugin (1812–1852), Owen Jones (1809–1874), Gottfried Semper (1803–1879) and John Ruskin (1819–1900) focused on finding the basis and the sound principles for the architecture of the industrial age. Pugin and Ruskin both emphasized the superiority of Gothic architecture, but in their attempt to codify its lessons to the modern architecture, they arrived at different conclusions. In Gothic architecture Pugin found "the true principles of art". One of these admirable principles was the adjustment of the beauty of nature to geometrical forms. He gave examples of this "adoption of nature for decorative purposes" in his book *Floriated Ornament* (1849). Ruskin, on the contrary, admired the unique touch of human hand in the sculptural decoration of Gothic architecture and detested the fascination for geometry of his contemporary architects and designers. In contrast to Pugin and Ruskin, Jones and Semper did not limit themselves to study just one style or only European and Christian architecture. Instead, they studied and compared ornamental practices from different eras and geographical locations.[2]

The Arabic Ornament Tradition as a Source of Principles

One of the most important nineteenth century writers on ornament, the Welsh architect Owen Jones (1809–1874), was a great admirer of the Arabic ornament. He wrote in his *Grammar of Ornament* in 1856 that "Mohammedans" managed very early in history to form and perfect a style of art, which was truly unique. Jones regarded, for example, the mosques of Cairo as being among the most beautiful buildings in the world. He asserted that "they are remarkable at the same time for grandeur and simplicity of their general forms, and for the refinement and elegance which the decoration of these forms displays" (Jones 1972, pp. 56–57). Above all the other examples, Jones valued the Alhambra in Granada, Spain (Fig. 1a–e). He studied the Alhambra in detail in the 1830s and published studies and illustrations from the 1840s onwards. The example of the Alhambra also influenced his own designs, for example the ones he made for the 1851 Great Exhibition in London. For Jones, the Alhambra served as an example in which the Arabic "system of decoration reached its culminating point". In his *Grammar of Ornament*, where he had collected "illustrated examples" of ornaments from different times and cultures, it was the Moorish ornament of the Alhambra that was considered exemplary. He stated: "We can find no work so fitted to illustrate a Grammar of Ornament as that in which every ornament contains a grammar in itself. Every principle which we can derive from the study of the ornamental art of any other people is not only ever present here, but was by the Moors more universally and truly obeyed." Jones admired Egyptian ornamental art especially because of its specific kind of symbolism, and this was the only thing he could think that was missing in the ornamental

[2]In all of these studies of architecture and its decoration, ornament was related to both spirituality or religion and different design or execution techniques and materials. The influence of Pugin, Ruskin, Jones and Semper is noticeable in writings of the next generation of designers such as Christopher Dresser (1834–1904), William Morris (1834–1896), Lewis Foreman Day (1845–1910) and Walter Crane (1845–1915), but also in the seminal studies of art historians Alois Riegl (1858–1905) and Wilhelm Worringer (1881–1965). Both the designers and the art historians formulated the concept of ornament suitable to their own use. At the turn of the century, ornament was increasingly seen as essentially abstract or linear, for example in the design theory of Henry van de Velde (1863–1957).

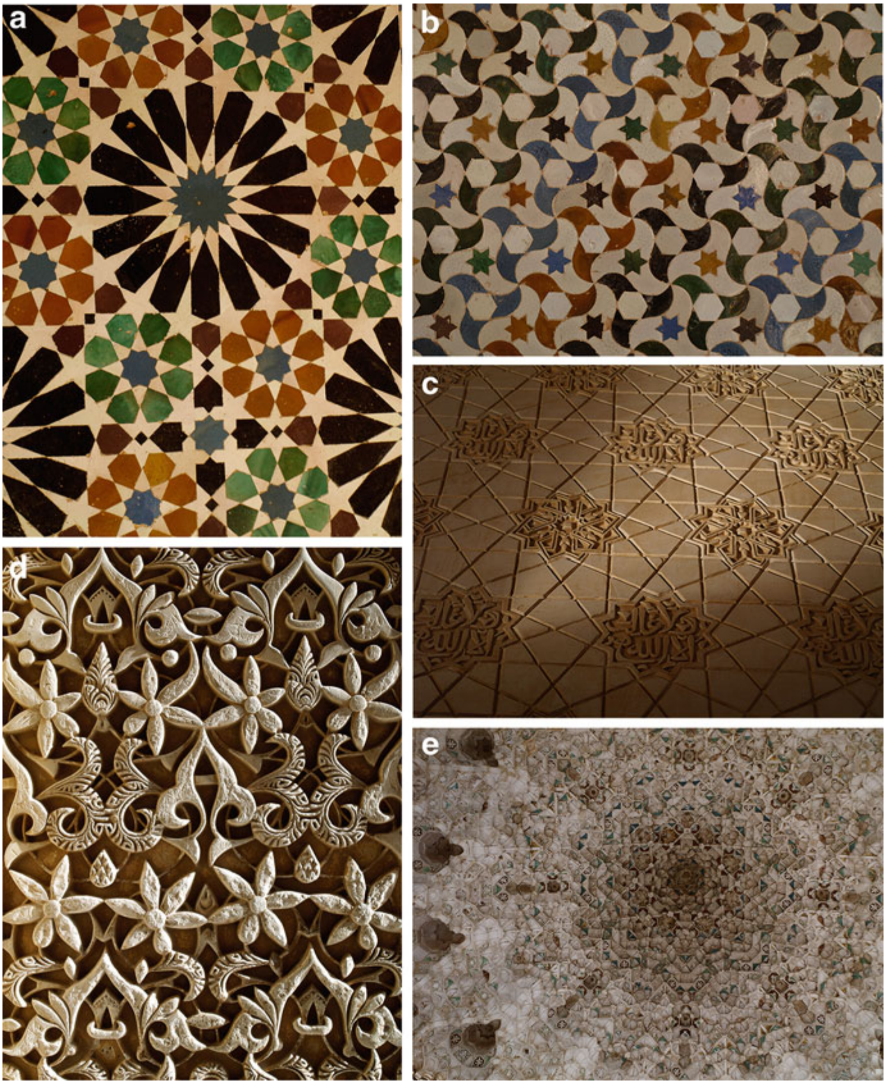

Fig. 1 (**a**–**e**) Decoration based on geometry from Alhambra. Photos: Satu Kähkönen. Copyright: Satu Kähkönen

art of the Moors. This kind of symbolism was missing because of the religious prohibition of visual depiction of living beings. But according to Jones, the Moors chose instead to use inscriptions which "were addressing themselves to the eye by their outward beauty, at once excited the intellect by the difficulties of deciphering their curious and complex involutions, and delight the imagination when read, by the beauty of the sentiments they expressed and the music of their composition" (Jones 1972, pp. 66–70).

For Jones the ornamental art of the Moors praised the glory of God and proclaimed the power and majesty of the King, but in his *Grammar of Ornament* it served as an example and lesson of beauty for the artists. From the examples of the Alhambra he conducted some of the most important principles which formed his “grammar of ornament”. Jones believed that it was possible to find general laws from different ornamental styles, and the search for the new styles, or like Jones put it “the future progress of Ornamental Art”, should be based on these universal principles and not on the imitation of past styles. He listed 37 propositions in *Grammar of Ornament*, calling them the “general principles in the arrangement of form and colour in architecture and the decorative arts.” Maybe the most famous of these was the proposition number 5: “Construction should be decorated. Decoration should never be purposely constructed” (Jones 1972, pp. 5–8). In his ornamental design Jones emphasized the role of geometry. Proposition number 8 stated: “All ornament should be based upon a geometrical construction”. He also discovered the mathematical principle of the stalactite domes that mystified Western visitors (Flores 2006, p. 20). In his writings Jones underlined that the architecture was the material expression of the culture that created it. Jones blamed the Reformation for severing the connection between religion and art and causing fragmentation in the Western societies. As stated by Jones, in earlier societies architecture reflected religious belief but because of the decline of faith in the nineteenth century, the Western architecture turned from art to a science of building and engineering (Fig. 1a–e).

From the middle of the nineteenth century until the beginning of the twentieth century, designers and architects in Europe and the United States studied Owen Jones’ principles and examples. Jones played an important role in the mid-nineteenth century English design reform circles and even though he was not the only one emphasizing the role of geometry in pattern design, his influence was notable. The reformers started to emphasize the role of geometry more widely in their design principles in the beginning of the 1850s (Bury 1988, pp. 45–46; Lubbock 1995, pp. 261–270; Brett 2000, pp. 6–15). Geometry was seen as a compositional tool that made pattern design intellectual, rational and scientific. Reformers wanted to get rid of the undesirable imitative practices, both the imitation of nature and the imitation of historical styles. Instead of being imitations, the new designs were to be based on understanding the profound “natural laws”. Geometry also had a role in controlling ornament, keeping it within the limits of “good taste”, opposite to what was seen as its bourgeois, extravagant and uncontrolled use.

The Grid: As a Starting Point and an End Point of Ornament

Brent Brolin (1985, p. 121) has argued that even if the modernists detested the nineteenth century designers’ fascination with ornament, they nevertheless adopted to their own means the design principles formulated for the pattern design by the nineteenth century reformers. In the twentieth century grid was seen as controlling the layout of the surface and even the structure of the material world. It often signified the rational and impersonal. But the decorative role or the symbolic functions of the grid as such, were not generally emphasized or articulated. At the same time, the notion of ornament and the practice of decorating became loaded with negative connotations, and they were seen above all as manifestations of (the nineteenth century) “bad taste”.

Can grid, a proportional system or a pattern of regularly spaced horizontal and vertical lines, be seen as a modernist ornament even if it claims to be quite the opposite? On the one hand, it has been explained that ornamental or decorative practices arise from our psychological need for visual order. On the other hand, ornamentation can be regarded as something by which we can recognize the culture in question. As stated by Rosalind Krauss (1985, pp. 9–15), the grid is an emblem of modernity. In other words, it is a form through which we can recognize something as being Modern and Western. According to Krauss, the early employers of the grid, the artists of the early modern art movements who “discovered” the grid, incorporated the spiritual meanings in it. But the split between spirit and matter was also declared through it.

In Arabic ornamental design different geometrical grids function as a starting point, and geometry in general offers unlimited possibilities for designs. In the twentieth century the Western architecture seemed to limit itself to the basic geometrical forms, like the simple grid formed by horizontal and vertical lines and the other emblem of modernism, the “white cube”. Writing in the 1980s Krauss (1985, p. 9) considered that “the grid has increasingly become a ghetto”. Has the situation changed during the past decades?

The French architect Jean Nouvel, the designer of the Arab World Institute (1981–1987) in Paris, has stated that “a cultural position in architecture is a necessity”.[3] Nouvel explained that in the Arab World Institute he selected an approach which is both global and specific. He considered the southern side of the Arab World Institute with its symbolic “moucharabiehs”, polygons of varying shapes and sizes creating a geometric effect recalling the Alhambra, as “a contemporary expression of Eastern culture”, and the northern side as “a literal mirror of Western culture”. A more contemporary example is the Royal Embassy of Saudi Arabia (2008) in Berlin. The building is a combination of a glass cube and a cylinder that is covered with an ornamental metal grid. Also here the ornamental grid functions as an indication of an Arab culture, but it is subordinate to the formal language of the Western architecture (Fig. 2a, b).

[3]See: www.jeannouvel.com (Arab World Institute Paris 1981–1987).

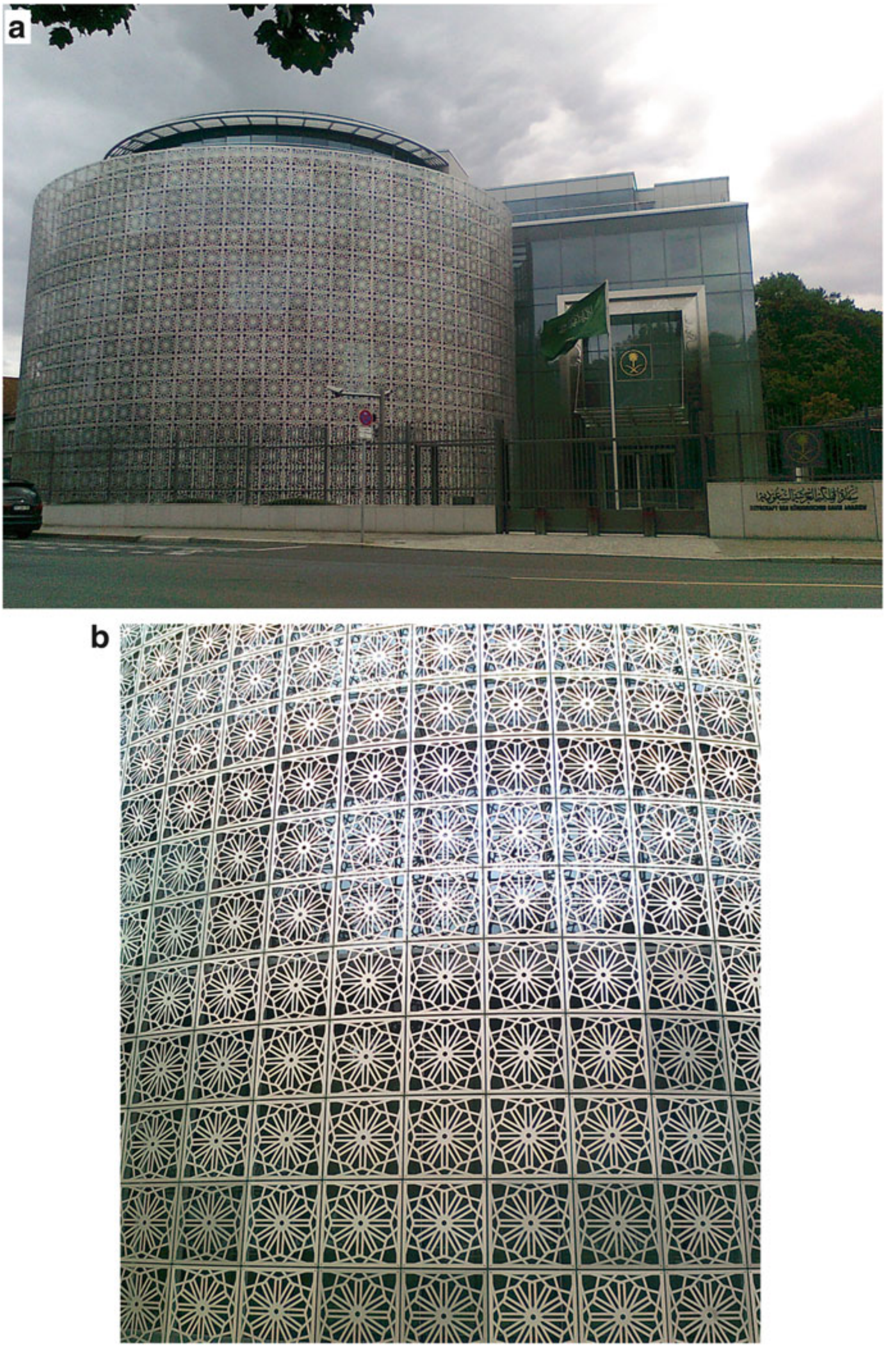

Fig. 2 (**a**, **b**) Royal Embassy of Saudi Arabia, Berlin. Gerhard Bartels, Nabil Fanous and Braun Schlockermann & Partner. Photos: Tuuli Lähdesmäki. Used with permission

Conclusion

The historical question of the meaning of the ornament is dependent on the cultural environment it is observed in. Ornament has been seen as a purely formal way to animate art. It has also been seen as a symbolic expression of the function, as an expression of a worldview, and as a metaphorical indication of divinity. In general, it is understood that ornament is the element through which the personal or the worldview is expressed.

In many cases it seems that ornament still represents something in contrary to legitimate modern Western aesthetics. It is considered as an element of the past, or a thing by which we can recognize another culture. Whereas in the Western architecture ornament is seen as decoration (symbolic or non-symbolic) added to structure, or form, and geometry (for example linear perspective) as a technique, the Arabic tradition sees geometrical ornament as a legitimate art form on its own with specific spiritual and intellectual meanings. Due to these deep-rooted differences in the understanding of ornament, there is something that makes the comparison of these two traditions unfruitful. In the light of the Western architecture and in its general attitudes towards ornament, changing the practice requires changing the way of thinking. If we continue to see ornament as an element added to structure, we cannot expect to see real change in attitudes towards ornament or ornamental practice. And if we are willing to change the way we think, we might learn something from the Arabic art.

Advances in technology and material innovations had made it possible to design and produce decorative elements and structures, which were previously impossible or expensive to realize. For example algorithmic design has the potential to make us think differently about design and architecture. But how are we going to discuss these forms and decorative elements made possible by the new technology? Does ornament belong to the vocabulary of algorithmic architecture? Is ornament going to be liberated from its negative connotations and given a new meaning? Does it mean that ornament and structure are one and the same? Does it make architecture ornamental by nature? This is up to the current generation of practitioners to decide.

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Alice Boner and the Geometry of Temple Cave Art of India

Robert V. Moody

Abstract Alice Boner (1889–1981) was a Swiss-trained sculptor and artist who lived in the ancient Indian city of Varanasi from 1936 until 1978. She made significant contributions to the culture of India through her realization and explication of the geometrical principles underlying India's ancient temple cave art, principles that are unlike anything with which we are familiar in western art. The question we raise here is to what extent these principles entered into her own art, and in particular into her great triptych based on the Indian trinity of gods, Shiva, Vishnu, and Brahma. We know from her diaries (Boner 1993) that the creation of this triptych was extraordinarily important to her and occupied a number of years of her life. Although it seems to have passed by unnoticed, we argue here that an analysis of the triptych makes it abundantly clear that indeed the underlying structure of these three paintings is based precisely on these geometrical ideas, and so reveals hidden layers of meaning within them. In this paper, based on her diary, her book on Hindu sculptures, and a detailed look at the triptych, we discuss the remarkable geometric principles which lie at the root of Indian temple cave art and the way in which Alice Boner implemented them in her great paintings.

Introduction

Alice Boner (1889–1981) was a Swiss-trained sculptor and artist who lived in the ancient Indian city of Varanasi (a.k.a. Benares) from 1936 until 1978. Her passion was oriental art, particularly the art of India. She made a significant contribution to the culture of India through her detailed analysis of India's ancient temple cave art, particularly by elucidating the very striking geometric/mathematical elements that underlie its coherence and dynamic flow. In addition, she painted numerous pictures based on Indian subjects, and one outcome of this was a triptych based on the Indian trinity of gods. We know from her diaries (Boner 1993) that the creation of this triptych was very important to her and occupied quite a number of years of her life. When I saw

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these pictures in a specially designed exhibit dedicated to her in the Bharat Kala Bhavan (museum of fine art) on the campus of Benares Hindu University in Varanasi, and when later I read her geometric analysis of Indian cave art (Boner 1962), it seemed natural to ask whether or not she consciously based these pictures on this same geometric/mathematical formalism. I have seen no mention of this anywhere, but, as I hope to demonstrate here, these paintings were very much created on these principles.

India's rich cultural history goes back at least three millennia, although sadly much of its art is lost: in India the climate rapidly destroys anything remotely perishable, and over the course of centuries much of what did not succumb to climate was intentionally destroyed in the various Mogul invasions and endless strife between local contending kingdoms. Notable exceptions are massive sculptural reliefs in stone that date from the sixth to ninth centuries and appear in a number of sites around India. It is to these that Alice Boner was drawn over and over again. Fortunately for us she kept a diary (Boner 1993), and though she wrote into it rather infrequently, what she did write was deeply personal and offers a fascinating insight into her creative artistic life, her struggles and doubts, and the passions that led to her discoveries about the geometrical underpinnings of Indian temple cave art.

In August 1940 she had an epiphany at the famous temple caves at Ellura. To quote from her diary of August 20:

> I had been here twice before and each time I had been completely taken out of myself... So I went again this summer... In order to approach the images I started to draw them. It was stiflingly hot and I was on the point of fainting, so I had to lie down on the ground to regain my senses. The drawings were awfully dull and inartistic. But in the peace of the guest-house I started analyzing them in their geometrical scheme and to build up the diagrams in terms of lines of energy. From such an analysis, all of a sudden, a revealing light broke forth. I grasped with my inner intuition of form that there I was really touching the hidden meaning, and that I was approaching the mystery of their unique and incomparable power of suggestion and expression... And where previously I had only seen the magnificent composition, the powerful movement, the supremely alive modeling of form, all these more or less aesthetic considerations gave way to a symbolic, underlying conception to which they were only humble accessories. (Boner 1993)

What she discovered is one half of the story that we tell here. The other relates to her personal creative artistic expression. Although trained as a sculptor, she decided early on that working in stone in the sweltering heat of India would be too debilitating, and she chose to paint instead. Sometime in 1946 she decided to create a triptych—three large paintings—representing each of the three primary manifestations of God in the Hindu theology: the Creative aspect, the Preserving aspect, and the Destroying aspect. This project was to consume much of her time, thought, and energy over the course of the next 10 years and there are many entries in her diary relating to her struggles in depicting in line and form what she intuitively felt about metaphysical concepts involved. Although nothing explicit appears in the diaries, it seems implicit that geometric form was an important ingredient in her conception of these works, and it is natural to wonder if she may not have used the same geometrical principles that she had discovered in the Indian sculptures as a guide to her own paintings. Here we will see explicitly that she did.

The role of geometry in art is nothing new. We know of it in the theory of proportion in Greek art, in the richness of the tilings in Islamic art, and in the use of perspective in Renaissance art. Furthermore these things are more or less rather apparent to the viewer.

In Indian sculptural art the primary entities are celestial and other mythological beings depicted in significant scenes from Indian folklore and legend. In fact, given the injunction against depiction of the divinity in Islamic art, it is remarkable to see how totally opposite the Indian Hindu experience has been, with its kaleidoscopic and often very sensuous depictions of gods and goddesses. In fact these two very different outcomes probably stem from the same original idea: if no conception of God can be complete, then one either makes no images or many.

Although often full of life and energy, there is little evidence of any formal underlying geometrical principles Indian temple cave art, even when one consciously looks for it. In fact several of the serious scholars of Indian art were to expressly declare that there are none. To quote the great Heinrich Zimmer 1955:

> [Each separate form element] seems to be floating in the air all by itself. They all enjoy a life of their own and do not depend on any free artistic economy that would govern the detail of their appearance and build them up together, as supporting parts, into a structured whole. The meaning of whatever might appear individual in them does not depend on their being related to an artistic total that would control them throughout, turning them into mere members of an artistic organism. (Quoted in the Preface of Boner 1962)

Given this backdrop, let us consider her geometrical analysis (Boner 1962) of a panel called Narasimha Avatara, which is inside one of the temple caves at Ellura. The panel (Fig. 1) dates from the eighth century. The figures are roughly life size. Ellura is a phenomenon that has to seen to be fully appreciated. It is a complex of monumental architecture built, not by assembling stone, but by the simple process of removing it. The panel shown here is small part of an elaborate cave temple that was hewn out of solid rock: images, columns, freestanding statues, and all. It depicts the destruction by the god Vishnu of the asura (=demon), King Hiranyakasipu, who, based on some powers that he had won, decided to make personal claims to divinity and insist that he be worshipped. The panel is unfortunately rather damaged, but it still conveys the energy and tension of the moment, as Vishnu, transformed into the form of a man-lion called Narasimha, prepares a death-dealing blow to the asura.

Alice Boner's suggestion is that although the panels are square or rectangular, the primary geometrical feature is a single circle[1], inscribed in the square or having two opposing sides as tangents in the rectangular case. The circle is divided by diameters, here 8, and generally 6, 8, or 12, into equal sectors, the vertical and horizontal diameters being the fundamental.

[1]When the panel shape is more rectangular than square, two or even three circular fields may be used. In this case the circle centres are in a line, one radius apart. Boner's triptych has one example of each type.

Fig. 1 Narashima Avatara, Ellura (Source: Boner 1962)

The intersections of the diameters with the circle form the basis of a rectangular grid (see Fig. 2, left side). Note that the grid spacing varies vertically and horizontally. This forms what she calls the **space-division** of image. It is static in nature and serves to locate the important elements of the composition. In our panel we see how the vertical neatly divides the space for the two opposing forces. The two central grid interspaces are taken with the bodies of Narasimha and the asura, while the horizontal interspaces divide the bodies into the heads, torsos, and thigh regions.

Boner's real insight is that there is a further division into an oblique grid obtained by selecting two or three families of parallel chords, also based on the cutting points of the diameters and the circle. In our case there are two families (see the right hand side of Fig. 2). She calls this the **time division**, and it is this that determines the direction of the movement.

> Narasimha's body from his foot on the ground, his leg, his trunk, his face and left upper arm with the sankha is channelled into the space between C'A to B'B (the first oblique interspace). The drapery falling behind the leg is doubling and reinforcing his movement. (Boner 1962)

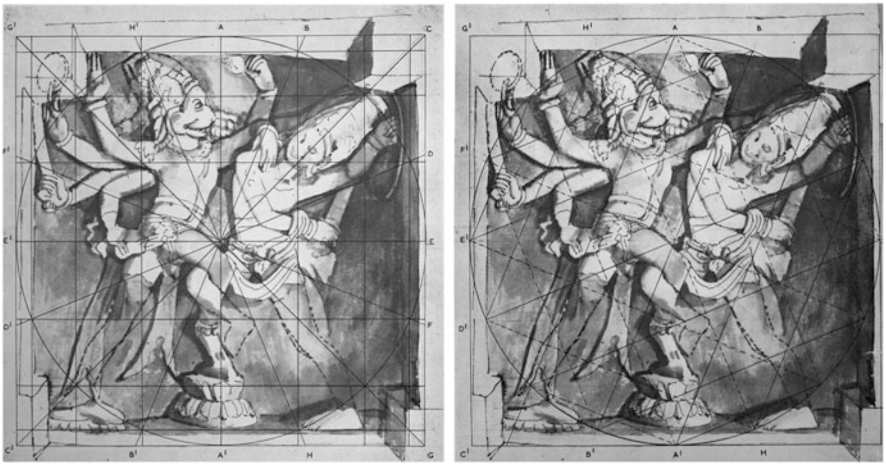

Fig. 2 Space and time divisions for Narashima Avatara (Source: Boner 1962)

To these two main features, the static space division and the dynamic time division, Boner adds the further notion of the **integration** or artistic synthesis of the two, which gives the composition its specific character and meaning. Here she writes:

> A fierce clash between two opposing forces is brought into a simple but eloquent formula: From the lower corners of the panel the bodies of the contending enemies are rising towards one another in converging oblique lines. . . .Further up the slant of their bodies reaches the point at which they should crash into one another. . . One movement is carried victoriously across the middle line dividing the two figures, while the other is broken by this thrust and has to recoil upon itself at right angles. (Boner 1962)

Boner spells all this out in more detail and more axiomatically, but the main points are the circular field, the division by diameters, the space and time divisions, and finally the integration. Panels that were more elongated were based on two or three circles spaced at one radius apart. In her book she analyzes twenty-one panels in this manner and makes a convincing case that these principles occur again and again in this genre of work.

Boner discovered these hidden laws of composition for herself. However, initially she was very reluctant to publish them. First of all such leading figures as Zimmer had said they did not exist. Second she felt sure that if there had been well-understood principles of composition such as she was suggesting, then, even given the ravages of India's climate and history, some knowledge and textual evidence of them would have survived. Fortunately, Boner was well connected and eventually she was led to a scholar of ancient architecture, Sadasiva Rath Sharma, who not only immediately understood what she was talking about, but was even able to point to other examples. More importantly he pointed out to her a 10 –11 c. work in Sanskrit, the Silpa Prakasa, which though primarily a manual on the construction of a temple, also gives detailed instructions on how the images are to be carved on the walls. Together they were to publish a translation (Boner/Sharma) of this work along with an annotated analysis of it.

The book is by no means an earlier version of what Boner was suggesting. However, it makes certain things very clear. The circle and its vertical and horizontal diameters were the primary principles behind temple images. Images, which by their nature were for worship, were to be constructed with extreme attention to appropriate ritual and detail. Thus one can assume that the compositions of the panels were guided by very strict principles and, given the wide spread distribution of such works of art across India, these principles were fairly universally understood by the artists of those times. Boner considered this book as confirmation of her general thesis and it gave her the necessary courage to publish her findings. Her book was published in 1962. It is still in print.

The Triptych

> Jan. 1, 1946: I woke up with a sense of alertness and greeted the sun when it had just risen over the horizon. I went up and greeted my Krishnaji [an image of Krishna to which she was devoted] and my Vishvarupa Krishna… I thought of what might be left over in my life and surrendered it in thought to Krishna. I felt that I had no great desires left, except the one that I may be granted to accomplish my three great paintings, my trilogy Shrishti, Sthiti, Samhara [also called Prakriti, Vishvarupa, and Kali] and that painting them was not only painting them, but realizing their intrinsic contents. (Boner 1993, 126)

This moving statement is in itself an indication of how seriously she treated the creation of the triptych and the sort of spiritual context in which she immersed herself. Given the importance of her discoveries of the geometric underpinnings of Indian cave art (her contributions to the art of India were recognized by the prestigious award of the Padmabhushan by the President of India in 1974 and with an honorary doctorate from the University of Zurich), her devotion to Indian spirituality, the subject of her triptych, and the number of years over which she worked and agonized over it, it is reasonable to assume that the two aspects of her creative life would be melded in this project. Items in her diaries also point to the importance of the underlying geometry: April 2, 1946:

> …suddenly I saw the entire picture as it should be when determinedly put into geometrical patterns. I saw those patterns underlying it, even as it is today, but they must be brought out fully, which will raise the whole vision onto another plane. It will then express the inherent thought more deeply, the unity of the principle in contrast to the division of manifestation… it must be started afresh, on a bigger scale… No work of mine has ever fascinated me so much or tired me less. (Boner 1993, 128)

Specifically, then, we would expect to find strong evidence of one or more circular fields divided into 6, 8, or 12 equal sectors by diameters and a strong relationship between this geometry and the main features of the subject. We would expect this relationship to be not only visually significant, but also contextually with the metaphysical meaning attached to each image by Indian culture. Let us see how this works out by detailed analysis of each of the three paintings.

In order to understand the context of these three paintings it is necessary to recall a basic idea in Hinduism—or sanatana dharma (the eternal way), as it is more properly called. According to this there are three fundamental qualities or tendencies in Nature: a centripetal pure ascending tendency, called *sattva* which is associated with order, knowledge, and light; a centrifugal descending tendency, called *tamas*, towards darkness, inertia, dissolution which is associated with time and destruction; and finally a third tendency, called *rajas*, arising from the tension between the first two, which is associated with motion, activity (mental and physical), expansion, and creation in all its myriads of aspects and forms. The constant interplay of the three tendencies is the play of the physical and mental universe. Each is personified in the form of a god: Vishnu, preserver; Shiva, destroyer, and Brahma, creator. Yet each is only a manifestation of one power of Brahman (the Immensity), which is beyond all categories of conception. Each of the three panels of Boner's triptych is devoted to one of these three tendencies: sattva in the form of Vishnu, tamas in the form of a manifestation of Shiva called Kali, and finally rajas in a scene, which though highly metaphysical is primarily naturalistic in content. I will cover these in the order of increasing sophistication as far as the underlying geometry goes.

The canvases are large, about a meter in width, and very colourful. They are also full of a lot of local detail as one can see from Figs. 3–7.

Destruction: Kali

Time (Kala) is that which disassociates all things. Conceived as a divinity, the Power of Time is represented as the goddess Maha-Kali... *Kali is represented as supreme night, which swallows all that exists* (Danielou 1985, 271). She dances [amazingly!] on the dead body of Shiva, the corpse of the ruined universe. The canvas is largely made up of warm brown through orange tones.

The geometrical foundation of the painting is a very simple, a circle centred exactly on the navel of Kali, see Fig. 4. There are 6 diameters and the space and time divisions (superimposed here) are straightforward. The iconography of the four hands representing the power of destruction [the sword], the fate awaiting each of us [the severed head], the open hand removing fear, and the bowl of blood [which is in the so-called giving hand] is quite standard, as is the necklace of skulls. Perhaps the minimization of structure here (much more severe than in the other two paintings) represents the tendency towards lack of creativity, lack of form, and ultimately dissolution.

This image was actually the last of the three that she painted. Geometrically it is the most straightforward. Nonetheless, it consumed vast amounts of Boner's thought and time, from 1953 to 1956, and she was never fully satisfied with it (Fig. 5).

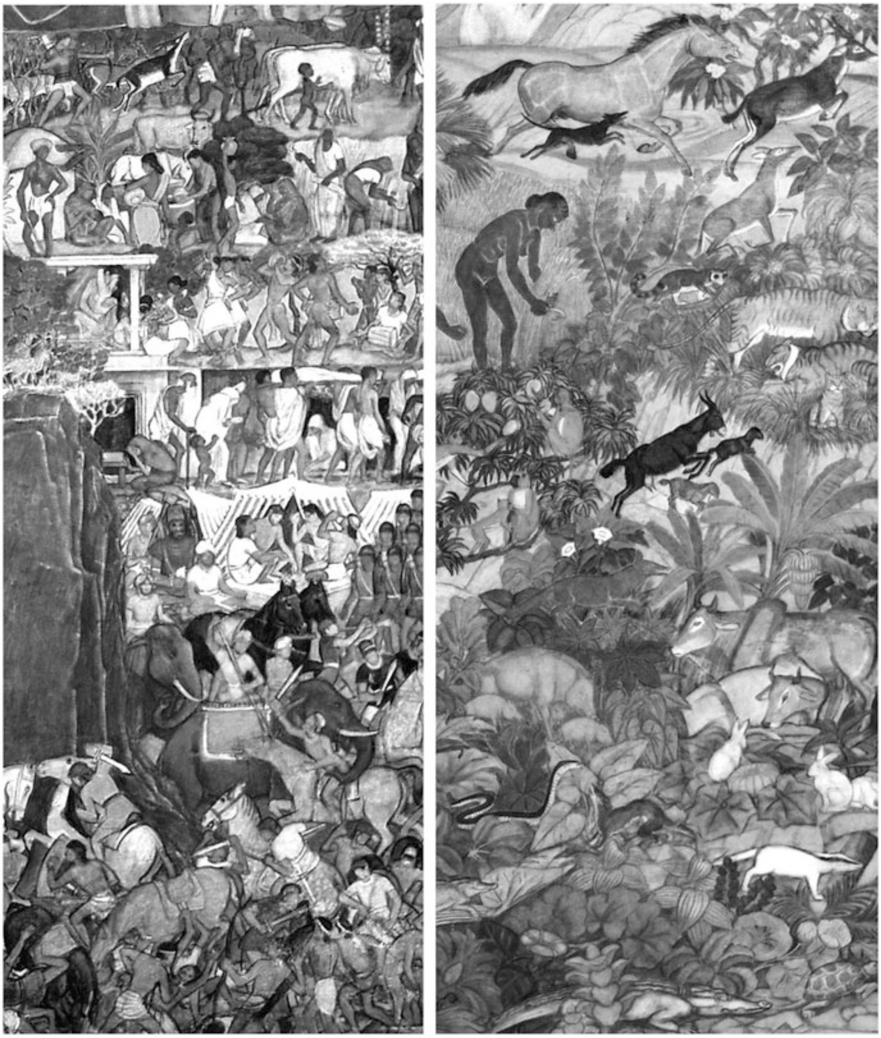

Fig. 3 Alice Boner, details from *Vishvarupa* and *Prakriti*

Preservation: Visvarupa

This painting represents one of the great moments in the Indian epic *Mahabharata*. Arjuna, the great warrior, stands on his chariot between the two contending armies of what is about to be an epic internecine battle. With him is his chariot driver who is no less than Krishna, one of the manifestations of Vishnu. Arjuna, mighty warrior though he is, seeing the carnage about to ensue, with his family and friends pitted against each other, loses heart and feels his spirit fading. He wants no part of this battle. Turning to Krishna, he begins to question him. The text of their dialogue (the *Bhagavad Gita*), which is something akin to the sermon on the mount for millions of Hindus, is a

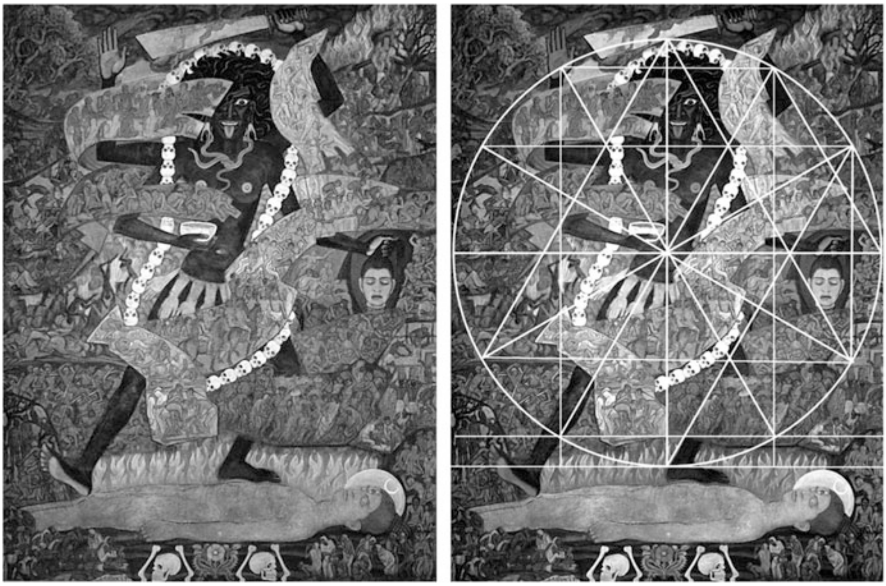

Fig. 4 Alice Boner, *Sumhara Kali (Destruction)*

theological and philosophical discussion of the nature of reality and divinity, as well as a practical spiritual manual for daily living. Arjuna knows that Krishna is not only a personal friend but also nonetheless a manifestation of the supreme godhead itself. Still, when he asks Krishna to show his real form he is completely unprepared for what he sees: *suppose a thousand suns should rise together into the sky: such is the glory of the Shape of Infinite God* (Bhagavad Gita) (Fig. 6).

The image, here is based on three overlapping circles. It is a form that occurs in some of the more elongated rectangular panels in the temple cave art, and which Boner discusses in her book. No doubt she used the three circles here in reference to the story of the famous three strides of Vishnu: the Asura King Bali has acquired sovereignty over heaven and earth. Vishnu appears as a dwarf and makes of King Bali the seemingly innocuous request to have as much land as he can stride in three steps. Being granted this, Vishnu assumes his divine form and with one stride covers all of earth, with a second all of heaven, and with a third, the underworld, crushing King Bali in the process. In the painting Vishnu appears in his universal form, covering all of the cosmos, *brilliant like the sun; like fire blazing, boundless... infinite of arms, eyes, mouths, and bellies. . . the sun and moon his eyeballs. . . shouldering the sky in hues of rainbow. . ., come as Time, the waster of all peoples* (Bhagavad Gita).

Geometrically we see three equal circles, centred one radius apart, the centres being directly between the eyes of Vishnu, on his navel, and on a multi-headed serpent (*When he is asleep and creation is withdrawn, Vishnu is represented resting on a thousand-headed serpent called Endless* [Danielou]). The circles cover heaven, the earth, and the underworld. The division is by 12 diameters. The geometrical formalism is striking, with the space division used, particularly horizontally, to indicate repetition and hence order and form (and even sterility), while the time division emphasizes its

Fig. 5 Sketch from Boner 1993, supporting our hypothesis for the strong geometrical elements underlying her final image of Kali

vastness. Note the great vertical sweeps up and down suggested by the two elongated triangles running from top to bottom that frame the main figure.

Creation: Prakriti

This panel is geometrically and also, I think, artistically the most interesting of the three. Here we see the multitudinous forms of the natural world in a sort of evolutionary sequence from bottom to top. As we move up through the plant and animal kingdoms we see the emergence of man, the beginning of agriculture, the

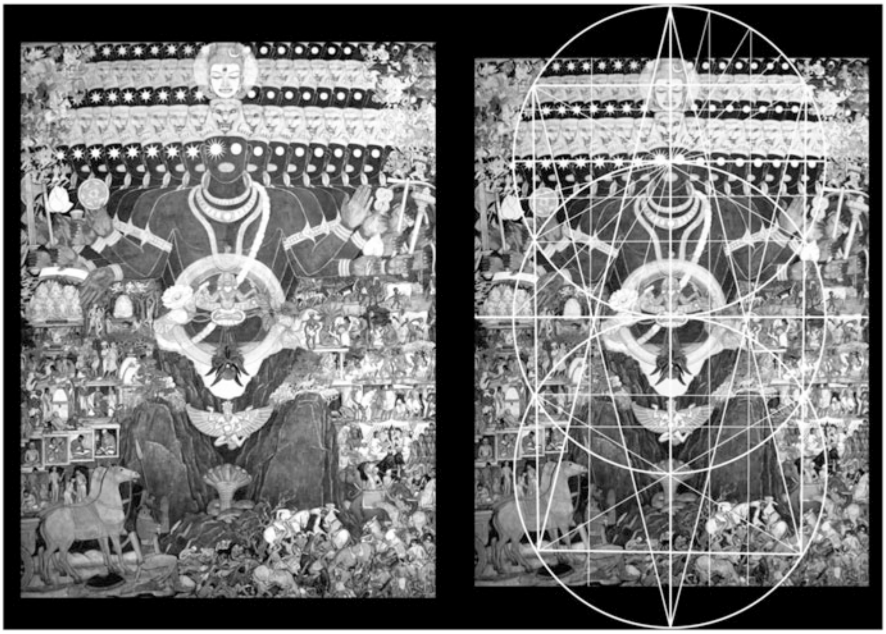

Fig. 6 Alice Boner, *Vishvarupa* (*Preservation*)

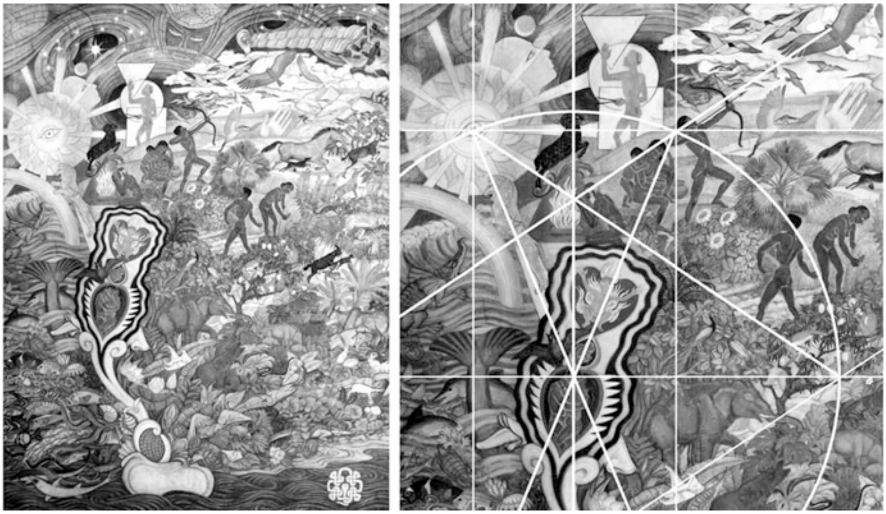

Fig. 7 Alice Boner, *Prakriti (Creation)*

discovery of fire and its use in worship, the aspiration to higher things, and the evolution of the human mind, mathematics, and culture (Fig. 7).

The geometrical underpinnings to this work are subtler than in the others. In keeping with its subject, Boner has created something new, going beyond the strict formalism that she had established in her theory. The key feature is the womb-like structure that emerges like some exotic plant form the cosmic ocean out of which all creation seems to explode. Within this womb is a couple embracing, in the very act of creation. The centre of the circular field is their kiss (... *the relation which the Hindu religion establishes between God and the soul takes the poignancy and ecstasy of human relationship, of love between man and woman...*) (Boner 1993, 103). The division is into eight diameters, but in fact it is the two diameters nearest the vertical that carry the main symbolism. The one diameter passes along the axis of the womb through the unborn fetus, through its fiery transformation by spirit, to the family grouping with the mother and her two children, and then to the father aiming to the sky about to release an arrow from his bow. The second diameter passes directly to the great source of all earthly life, the sun, which is of course a symbol for the Immensity itself. The secant created by the points of intersection of this diameter and principal horizontal diameter at the circle's circumference passes directly through the fingertips of the devotee worshipping in front of his fire. The secant vertically symmetric to this gives rise to a major dynamic thrust to the upper right, originating at the centre of the rainbow arc (just off the picture), passing again through the fingertips of the devotee, as well as his head, and the heads of the mother, father, and older child and finally up through the head of the soaring eagle that is the target of the arrow.

There is a second circle of the same size situated laterally one radius to the right of the one that I have shown. Its interior seems to correspond to the terrestrial part of the image and its circumference traces the arc of the womb-family grouping. However its importance seems subordinate to the remarkable symbolic logic of the first circle, so I have omitted it for the sake of clarity.

There are numerous symbolic references in this painting, for example the representations of the five senses, the sacrificial ram above the fire, the lion lying along the diameter to the sun whose meaning I do not know, and no doubt many of which I am simply unaware. It would be worthwhile for an expert to analyze this painting in more detail for all of its symbolic references.

Conclusion

Alice Boner discovered a uniform geometric foundation for a certain genre of sculptural relief that appears in Indian art of the sixth–ninth centuries. Her theory has proven convincing in its ability to give intellectual coherence to this art form, and she was honoured on a national level in India for this work. Although she does not say it explicitly in so many words, and though we can probably never prove it for sure, it is manifest once one sits down and analyzes the pictures that these same principles were used to guide her own work. We have seen from the extreme importance she gave to her triptych that it would be natural for her to have done so. The triptych illustrates the geometric form that she discovered in temple art in three ways—simple, complex, and imaginative. In keeping with its three subjects, it uses the one, two, and three circular field types, and uses each of the three types of circle division appearing in the temple panels. These three paintings are perhaps unique in the world of art in having this particular geometric basis.

These paintings, along with a number other pieces of her art, are to be found in a specially designed exhibit dedicated to her in the Bharat Kala Bhavan (museum of fine art) on the campus of Benares Hindu University in Varanasi.

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Part IV
Geometry, Mathematics, and Science in Artistic Practice

Art with a Double Meaning

István Orosz

Abstract Artists have many sources of ideas, just as they have a favorite medium. Here I describe how I was inspired by the work of poets William Shakespeare and Edgar Allan Poe and writer Jules Verne to create etchings with double meaning. My artworks interpret their writings and also encode hidden anamorphic portraits of the writers, revealed only through viewing in a special way. For Poe, I not only used his poem *The Raven* but also his essay *The Philosophy of Composition* to guide my creation process, just as he did for the poem.

Introduction

I would like to present some of my typical work, as well as sharing some thoughts on them. All of them have several ties to literature. Although they conjure up many similarities with the latter, these references are hidden and you need a special, so-called anamorphic view to grasp them. Concealment and anamorphosis, these words will most frequently encounter in this essay. And of course geometry as well, which inspired me with its sober rules and innate transcendent poetry. Naturally I will not only describe my artworks, but also try to explain, or rather to show some technique, which I came across during my experiments with the creation of anamorphosis.

I studied graphic design at the University of Art and Design in Budapest where my diploma works were poster designs; later I started to make animated films, as well. The strange duality of posters—they mean different things if seen from a distance and from close up—and the miracle of the moving image led me to experiment with illusion in fine arts. All this happened in the seventies and eighties in the previous century in that part of the world where speaking enigmatically had a strange political piquancy. Enigmas and illusion lead to anamorphosis, a design technique well-known in the Baroque era and later forgotten. This two-dimensional technique uses a special point of view or a mirroring object to reveal the secret of the distorted image. I have been making experiments with anamorphoses for quite

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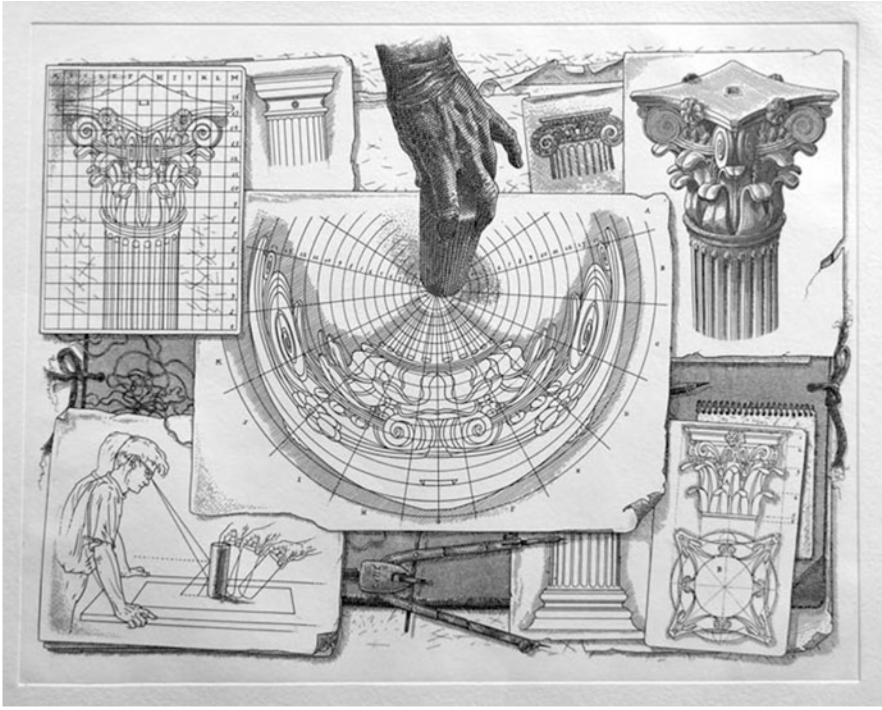

Fig. 1 *Anamorphosis with Column 1*, 1994, etching, 550 × 395 mm. Copyright by István Orosz

some time: I drew the first one in the late 1970s. My interest was not only in the resurrection of anamorphoses but to improve and further develop this old-fashioned genre.

To coin the commonplace, at least for the persons dealing with the intersections between art and geometry, the anamorphosis is a technical word describing the unrecognizably distorted images, which became visible again with the help of a special viewpoint or an object with mirrored surface placed on them. Of course, if you want to draw anamorphoses, you have to understand geometry and need to study the science of perspective. But once you take it up, sooner or later you will also fall in love with it. As for me, I want to get others to fall in love with anamorphoses, and maybe this is the reason for this short note.

Figure 1 gives a visual summary of a captropic, or mirror anamorphosis of a Greek column (Fig. 2). To construct one, I use a grid of squares that was already in use by the early Middle Ages to enlarge or to reduce an image. It might have been Alberti to be the first one to use such grids to distort an image in perspective; however, to distort the grid of squares into circle arcs and radii, that is, to transform it into a cylindrical anamorphosis was a Parisian monk's, Jean-Francois Niceron's (1613–1646) invention.

Fig. 2 *Anamorphosis with Column 2*, 1994, etching, 406 × 550 mm. Copyright by István Orosz

Shakespeare

For sorrow's eye, glazed with blinding tears,
Divides one thing entire to many objects;
Like perspectives which, rightly gaz'd upon,
Show ***nothing but confusion*** - *eyed awry,*
Distinguish form.
(William Shakespeare 1623, *Richard II*, Act II, Scene 2, 16–20.)

This short poem occurs in Shakespeare's *Richard II*, and shows that Shakespeare was well aware of the technique of anamorphosis. Also the theatre-goers of the time in London must have known it, although it was not yet named "anamorphosis" but "the perspective". "There is nothing but confusion" reads Shakespeare's text. In representing this visually, that means that there should be only the chaos of a confused, almost unrecognizable base image. But, instead of having a totally confused image, I intend to bring some sense to the base anamorphosis, and to give it meaning on its own. "Eyed awry, Distinguish form"—a second reading of the etching, eyeing it from a different viewpoint, will reveal a hidden form. The ambiguous layers revealed by this approach make use of the analogy or contrast between the two images within the same picture, which are, in the meantime, independent from each other. This approach also can bring a philosophical reading to these anamorphoses.

Fig. 3 *The Theatre of Shakespeare*, 1998. Copperplate etching, aquarelle colouring, 1000 × 360 mm. *Front view*. Copyright by István Orosz

In my etching of *The Theatre of Shakespeare*, the two images complete each other in a thematic way. If we look at it from the front (Fig. 3), as we usually do in the case of a picture, we will see a theatre (Shakespeare's Globe Theatre) from the sixteenth century, with a pointed roof over the stage. There are actors, the audience, and people gazing around. In the foreground at the right, two members of the audience are actively engaged, and a mustached artist (guess who?) is trying to capture a scene. If we, the viewers of the etching, step to the right side of the exceptionally wide panoramic picture and look at it from a narrow acute angle from which the picture is seen as a narrow strip, a second picture is revealed. The elements of the theatre not only disappear but also transform into a portrayal of Shakespeare. The picture of a busy theatre becomes a portrait (Fig. 4).

Jules Verne

"The Pole's [--] so special, my good Johnson, because it's the only point of the globe that is motionless, while every other point turns at great speed" (Verne 2005 [1866], 326). This special spot is where Captain Hatteras in *The Adventures of Captain Hatteras* by Jules Verne is heading. He is a character with the typical duality of Verne's figures, since he passes through romantically exaggerated extreme adventures while he is obsessed to solve one of the relevant scientific problems of his time. For Captain Hatteras, this problem is the exploration of the North Pole. At the time, the book was published in 1866, the expeditions already were relatively close to the Pole, but the Norwegian explorer Amundsen could only reach it in 1926. Verne's protagonist maniacally, and in the end as a true madman, is after the Pole. There is a similar special point that you as the spectator of an anamorphosis should discover or, if you like, that is the spot where you should place a mirror cylinder in order to solve the riddle and to realize the meaning of the elements in the picture (Fig. 5).

Fig. 4 *The Theatre of Shakespeare*, 1998. *Side view*. Copyright by István Orosz

Fig. 5 *Jules Verne Anamorphosis*, 1983, offset print, 500 × 700 mm. Copyright by István Orosz

In the illustration for *The Adventures of Captain Hatteras* I used a mirror anamorphosis to encode double meaning in my etching. If you place the etching on a flat surface, and place a mirror cylinder with its base on the sun-disk, it will reflect the face of the French writer (Fig. 6). The polar landscape is transformed into a portrait.

Edgar Allan Poe

Leave my loneliness unbroken! — quit the bust above my door!
Take thy beak from out my heart, and take thy form from off my door!"
Quoth the Raven, "Nevermore".
And the Raven, never flitting, still is sitting, still is sitting
On the pallid bust of Pallas just above my chamber door;
And his eyes have all the seeming of a demon's that is dreaming,
And the lamplight o'er him streaming throws his shadow on the floor;
And my soul from out that shadow that lies floating on the floor
Shall be lifted — nevermore!
(Edgar Allan Poe, The Raven. *New York Evening Mirror*, January 29, 1845.)

Thus ends the poem *The Raven* by Edgar Allan Poe. Poe wrote an essay, *The Philosophy of Composition* (1846), in which he offers a radical theory on the

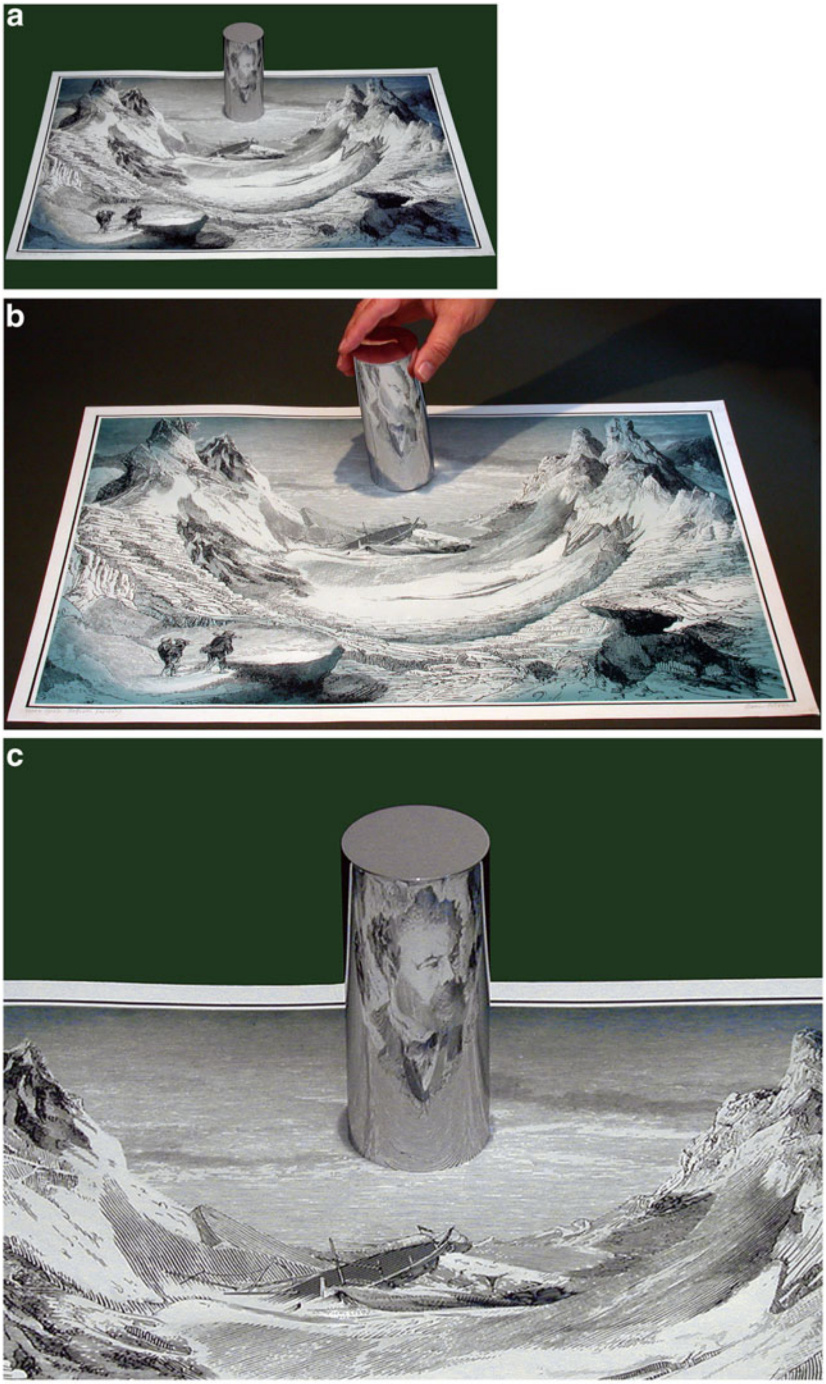

Fig. 6 (**a**, **b**, **c**) *Jules Verne Anamorphosis* with mirror cylinder. *Mirror size*: 60 × 100 mm. Copyright by István Orosz

creative process as he describes what lies behind his poem. Inspiration or design? When speaking about a work of art, you may want to know which of these predominated in its creation. Poe also asked this question and his answer was that a work of art can be created consciously "with the precision and rigid consequence of a mathematical problem."

The Raven remains one of the most widely recognized and respected poems in literature. But the scene it describes is astonishing, embarrassing and even faintly comic. A talking raven lands in a room: at a first reading (or listening), you can hardly take it seriously. However, with Poe, you can never tell what he takes seriously and how seriously he takes it. An idea making no sense could end in a question of life and death, and sentences that sound straightforward may be intended as parodies. Therefore his *Philosophy of Composition* too has to be read with certain reservations. Who would believe Poe's claim that intuition is not needed to compose poems, that inspiration does not exist? That all you need to compose a poetical work is a logical, step-by-step design and that effects have to be cautiously and precisely calculated?

My first step translating the poem visually was to show the time and the place, or at least what we know of it from the text. Clearly, it is the home of the narrator, easily depicted through "homely" disorder (shoes kicked off, books all over the place, etc.). Rather than refer to the fact that the poem was written in 1845, I depicted these objects as of today. Also, I "left" some of the devices there that I needed for designing my anamorphosis (Fig. 7).

Picturing the precise time of day was more complex—the poem says it is midnight. The geometrical equivalent of midnight in the image is symmetry, and so I used a sphere within a square as the basic structure of the composition and distinctly highlighted its centre. The symmetry, present in Poe's poem as well, is the result of viewing from above, a bird's-eye perspective. But to picture a top-view, with the raven's eye view of the milieu would definitely not be enough to fully interpret the situation. To depict the raven itself would be perhaps overdoing it.

This problem also goes back to the question of whether in the poem a raven really does make its appearance, or is only the product of the narrator's imagination. The greatest virtue of the poem is that it does not come down decisively on either side. If a real raven is depicted, then a side is taken. Is it possible for the illustrator, as it was for Poe, to keep an open mind? Yes, it is, if a depiction of the bird is avoided, and only its shadow, its image in a mirror, is represented. My illustration presents all three possibilities. The dark shade in the middle of the picture can be taken as the shadow of the bird hovering above; in the wine cup, the beat of a wing is reflected; the illustrations in a natural history book lying open on the table also depict a raven.

The next issue is the identity of the narrator. The poem begins,

Once upon a midnight dreary, while I pondered, weak and weary,
Over many a quaint and curious volume of forgotten lore,
While I nodded, nearly napping, suddenly there came a tapping,
As of someone gently rapping, rapping at my chamber door. (Poe 1845)

Fig. 7 *The Raven (Edgar Allan Poe Anamorphosis)*, 2006. Copperplate etching coloured with aquarelle, 360 × 500 mm. Copyright by István Orosz

He hovers between sleep and wakefulness, hence he is shown as slumped over the table. This also circumvents the question of whether Poe and the narrator are the same. The empty armchair opposite this figure and the drapery on it refer to the loss of the lady in the poem. To symbolize love, two books are intertwined.

In *The Philosophy of Composition* Poe argues at length that the monotony of the frequently-repeated refrain lends the poem its melancholy. In the illustration, there is a monotony in the repetitive pattern of the parquet floor, and in the books and the sheets of paper scattered everywhere. The books and their illustrations bear out Poe's notion that consciousness and calculation, intellect instead of "fine frenzy" lead the creative process. None of the important details should be omitted. The open books allowed me to include the bust of Pallas Athene, highlighted by Poe in his poem.

On re-reading the poem or Poe's comments on its creation, one senses that he is intentionally hiding something. The blurred mystical-metaphysicality of the poem and the provocative brainstorming of *The Philosophy of Composition* seem to distract the reader. It this so that you wouldn't recognize a soul torn by fear and doubt, so that you wouldn't take the first-person narrator seriously, so that you wouldn't identify him with Poe. The poet did not have a dead lover called Lenore, his room did not contain a bust of Pallas Athene; yet there is no doubt that the shadow of the raven hovers over Poe's soul, destiny and life. The fifth line of the penultimate stanza ("Take thy beak from out my heart") confirms that here it is the poet speaking and not his narrator, slumped over his books. This is the first metaphor that re-interprets the whole poem as it has developed and clarifies the symbolic character of the bird.

Someone viewing my illustration when it lies flat will place a cylindrical mirror over the reflection of the raven's wing in the wine glass; in so doing, they emphasize the metaphoric interpretation of the poem and of the picture. In the mirror is reflected Edgar Allan Poe's virtual face, made up of the objects that lie in the etching horizontally, the requisites of the illustration for *The Raven*. Once the cylinder is raised, the face disappears and what is left are these scattered objects, the shades, the man lying on his face and the empty room (Figs. 8 and 9).

Poe claims in his essay that the most important effect to be created in a work of art is that which allows it to be interpreted backwards. His conclusion explains all the parts of the composition and their role in the whole. In fact, the same backwards

Fig. 8 Placing a mirror cylinder on the coloured engraving, *mirror size*: 60 × 100 mm. Copyright by István Orosz

Fig. 9 Looking at the image in the mirror reveals Edgar Allan Poe. Copyright by István Orosz

interpretation is at work in an anamorphosis that has a double meaning, since by placing a cylindrical mirror onto the centre of the paper, the viewer will realize why certain objects have been placed in the picture.

How can you distort an image so that it only becomes visible and recognizable in a mirror of the right size and from a certain angle? Here is how I created *The Raven* etching. I selected a photograph of Poe and made a line drawing from it (Fig. 10, left). Then I produced a reflection of the picture (so that when its reflection appears in the mirror, the original image can be seen). I drew a grid of 11 columns and 9 rows upon the picture, adding numbers and letters for the sake of clarity. The drawing was first reflected vertically, and then horizontally to be readied to turn into the anamorphosis (Fig. 10, right).

Next, I drew eleven equally-spaced concentric circles from the centre of the illustration, and split the area into nine pie slices. In other words, I made a distorted form of the grid of squares. I then had to redraw the image in each small square exactly deformed to fit into the corresponding annular region (Fig. 11). I placed the

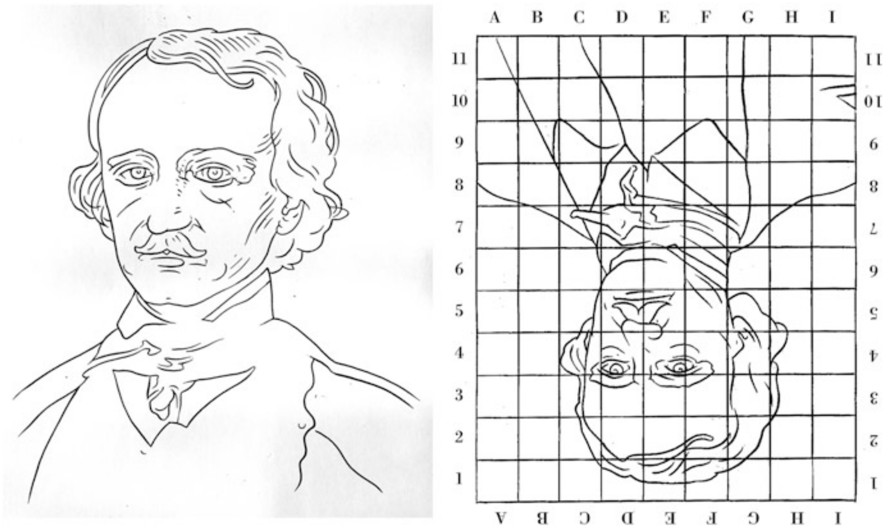

Fig. 10 Poe's photograph converted to a line drawing, then inverted and transferred to the grid. Copyright by István Orosz

cylindrical mirror from time to time onto the centre to check the work. Once the squares were filled in correctly, Poe's image appeared in the mirror. Now came the most interesting part: I had to figure out how the elements of the distorted image could be replaced by other objects that would be in the picture. I had to see a round desk-top as the curve of the forehead, a pen or a pencil as an eyebrow, a watch as a necktie, the shadow of the raven as the writer's waistcoat, and so on. I drew these objects conventionally then arranged them in the composition (Fig. 11).

I find it interesting that Poe expressed great interest in the characteristics of optics and visual perception. In his short story, *The Murders in the Rue Morgue* (1841), the detective Auguste Dupin explains how the details and the different viewpoints should be used to examine something as a whole—and then, with surprising precision, he describes the technique of distorted perception (Fig. 12).

Reflections

Truth is not always in a deep well. In fact, as to most important knowledge, I do believe that it is invariably superficial. The depth lies in the valleys where we seek it, and not upon the mountaintops where it is found. This is well typified in the contemplation of the heavenly bodies. To look at a star by successive glances—to view it in a side-long way, by turning the exterior portions of the retina toward it (these are more susceptible to feeble impressions of light than the interior) to behold

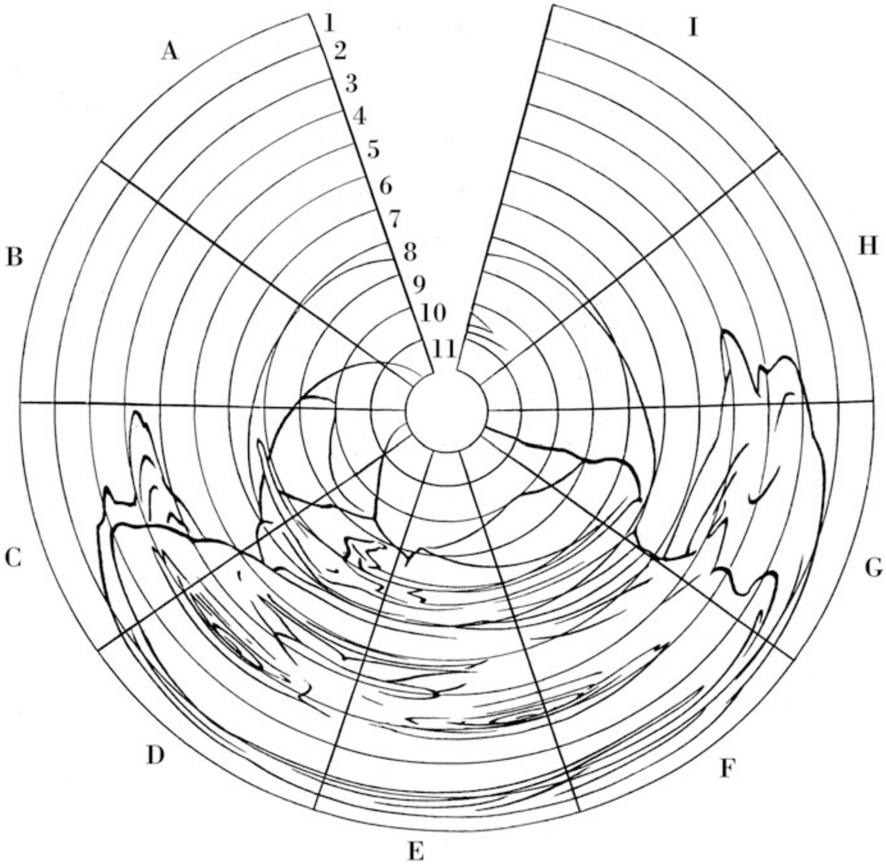

Fig. 11 The drawing of Poe transferred to the *circular* grid. Copyright by István Orosz

the star distinctly—is to have the best appreciation of its luster. This lustre grows dim exactly in proportion to the degree we turn our vision fully upon it. A greater number of rays actually fall upon the eye in the latter case, but in the former, there is a more refined capacity for comprehension.

When designing my anamorphoses, I attempted to work with a conscious and calculating mind, but I was also aware of traps that such childish logic might lead me into. All I could hope for was that the "inexplicable" too, will always have a role in all kinds of creative work.

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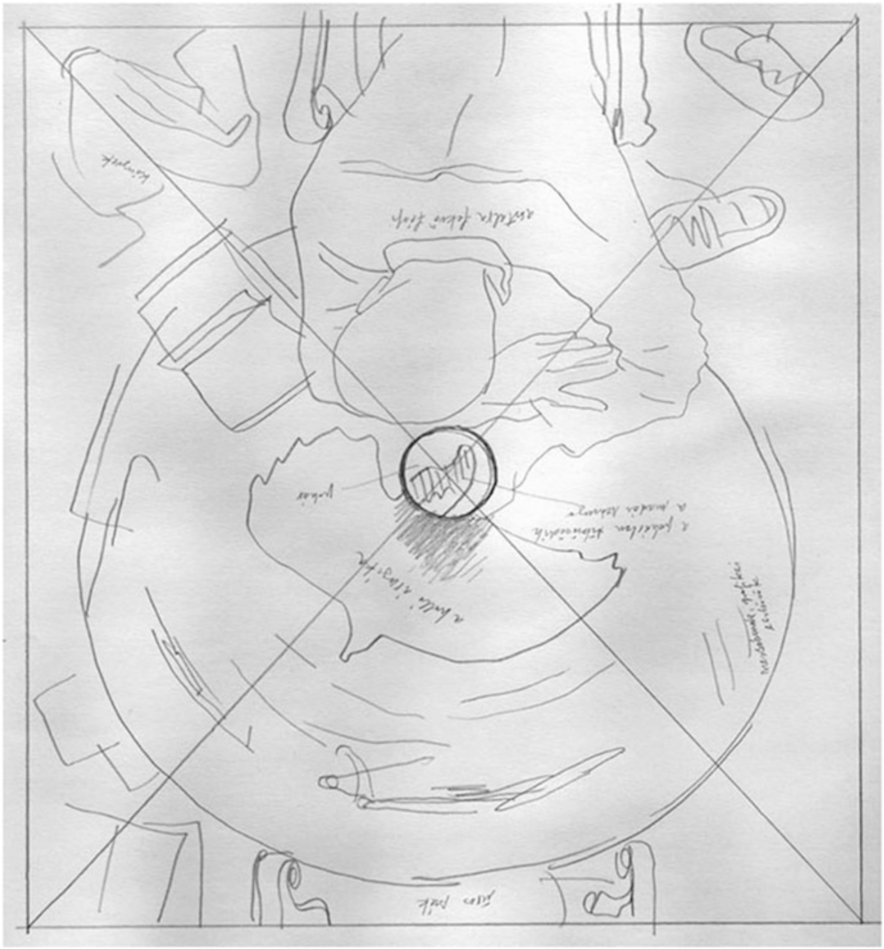

Fig. 12 The objects arranged in the positions needed on the distorted image. Copyright by István Orosz

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Quantum Sculpture: Art Inspired by the Deeper Nature of Reality

Julian Voss-Andreae

Abstract The author, a sculptor with a background in physics, describes sculptures he creates inspired by quantum physics. He argues that art such as the presented sculptures can indicate aspects of reality that science cannot and therefore has the potential to help liberate us from the deep impact the paradigm of classical physics continues to have on our every perception of reality.

Introduction

After receiving a physics degree in 2000 from the Free University Berlin and the University of Vienna, I moved to the U.S. and studied fine art. Science has inspired my work since my days as an art student when I began to create sculptures based on the form and function of *proteins*, the molecular building blocks of life (Voss-Andreae 2005, 41–45). Throughout my art studies I also retained a strong interest in the field that had most fascinated me as a scientist: quantum physics and its philosophical implications. Quantum physics is the scientific foundation of practically everything we encounter in the world, ranging from virtually every aspect of current high-technology to the miracle of life itself. Despite its overwhelming importance and its fundamental status in science, quantum theory remains philosophically extraordinarily problematic. I will begin this chapter by describing the challenges in attempting to create a consistent mental image of a world ruled by quantum physics. I will then briefly outline a seminal experiment at the boundary between physics and philosophy (Arndt et al. 1999, 680–682) and describe how this experiment, which I was fortunate to participate in as a graduate student, has had a deep influence on my art. Finally, in the main part of this chapter, I will present select sculptures inspired by ideas, images and experiments from quantum theory.

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On Visualizing Quantum Physics

With our intuitions schooled within the paradigm of *classical physics*[1] we tend to assume that reality has definite properties, regardless of whether or not there is anyone around to observe them. This view, called “objective realism”, turns out to be incompatible with quantum theory (Einstein et al. 1935, 777–780; Bohr 1935, 696–702). For example, there is no accurate space-time representation of, say, an electron: It is neither a particle nor a wave nor any other “thing”. So when attempting to visualize concepts from quantum physics there is a danger in presenting artificially concrete representations without making sure they are correctly understood as only a facet of something more complex or as something altogether different. A well-known example of such a misunderstanding is the ubiquitous hydrogen atom model. In earlier models, now widely recognized as grossly false, electrons are displayed as particles circling the nucleus in discrete orbits. Then there are the representations of electrons as wave-functions, the orbitals pictured in physics textbooks. Even if the three-dimensional shape of the probability density is pictured correctly[2] it is still a potentially misleading abstraction because this shape merely represents *tendencies* for results of possible electron position measurements, whereas the phenomenal reality it refers to are the discrete and apparently random positions at which the electron is actually measured when an experiment is carried out. The problem is the very notion that a hydrogen atom, or any quantum “object” for that matter, is an object and has a particular appearance or properties *independent* of the means used to observe it. Consequently, it seems impossible to assign a “quantum object” any objective existence at all. And by extension, the same is true for everything material we encounter in this world.

There is always a danger of taking any image or model too literally. Using images in science or philosophy to illustrate states of affair is generally a two-edged sword because it is essential that the audience knows the limits of a picture and uses it with discrimination and intelligence. With that caution, I believe that art in general, especially after having shed the requirement to visually represent reality accurately, is uniquely capable of instilling an intuition for the deeper aspects of reality that are hidden to the naked eye. The ability of art to transcend the confines of logic and literal representation and to offer glimpses of something beyond, can help us open up to a deeper understanding of the world and to wean ourselves from the powerful grip that the world view of classical physics has had over the last centuries on our every perception of reality.

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[1] “Classical physics” refers to the physics before the twentieth century advent of quantum physics.

[2] These models often contain an additional imprecision in that they illustrate only the angular dependence of the wave-function without including the radial one. I am sure many if not most scientists would draw these spherical harmonics if asked to depict ‘what a hydrogen atom looks like’.

First Sculptures

For my graduate research in Anton Zeilinger's experimental physics group in Vienna I participated in an experiment that successfully demonstrated quantum behavior for very large particles by sending them—as quantum mechanical matter waves—through a double-slit experiment.[3] The particles probed were C_{60} *buckminsterfullerenes*, at that time by a large margin the most massive particles ever probed in such a setup.[4] Affectionately called *buckyballs*, these unusual molecules consist of 60 carbon atoms located at the vertices of a truncated icosahedron, the classic soccer ball. In 1999 we saw the first *interference pattern*, the telltale sign for quantum behavior. The only way to explain the experimental results in classical terms is to conclude that a single buckyball (or, more accurately, the entity that is later detected as a single buckyball) goes through two openings at once—two openings that are a hundred times farther apart than the diameter of one buckyball.

Buckyball Sculptures (2004–2007)

Inspired by Leonardo da Vinci's illustration of a truncated icosahedron for a Renaissance mathematics book (Pacioli 1509), I created my first buckyball sculpture in 2004. I noticed that the cut-outs on each face provide the exact amount of material for another, smaller buckyball. After cutting openings into the smaller buckyball's faces, the same is true again for the next buckyball and, taking advantage of this reiterative procedure, I fabricated a succession of four buckyballs from bronze sheet. I placed the buckyballs inside each other and attached them in place by running thin rods radially through the 60 vertices (Fig. 1). It is appealing to me that the nested structure of *Quantum Buckyball* echoes the mathematical structure of the wave-function associated with the buckyball in our experiment: a spherical wave, emanating from a central source.

A sculptural object occupying a considerable volume of space while consisting of comparatively little material is an apt metaphor for the ephemeral nature of the quantum object. I started making larger buckyballs from steel consisting only of the edges, culminating in a large piece with a diameter of 30′ (9 m) that was first installed in 2006. Now permanently sited in a picturesque private park in Oregon, the buckyball hovers above arm's reach over a sloped terrain with a small creek running under it. Suspended by three majestic Douglas firs that grow through the structure, the

[3]The double-slit experiment is a beautifully simple experimental setup that consists of a beam of particles or light that is sent through two neighboring openings, the slits. The detected pattern behind the slits ("interference pattern") reveals whether or not the beam has traveled as a wave and passed through both openings at once.

[4]Our experiment was technically not a double-slit experiment since we used a grating with more than two slits. But the difference is not significant because the wave-function of one buckyball extends coherently only over about two slits in width.

Fig. 1 *Quantum Buckyball*, bronze, diameter 24″ (60 cm), 2004. © Julian Voss-Andreae. Photo: Dan Kvitka. Four buckyballs are nested inside each other, attached in place by thin rods going radially through the 60 vertices

buckyball's orientation was chosen such that two opposing hexagons, one at the bottom and one on the top, are lying between the trees on horizontal planes. Figure 2 shows a view up from below the buckyball. The reason that such a basic shape succeeds as a piece of art is its placement within nature. Despite its considerable size, the buckyball's visual impact is quite subtle due to the relatively thin 2″ (5 cm) tubing and the natural color of the corroding steel. The trees intersecting the buckyball dissolve the mathematical shape, symbolizing quantum physics' revelation that matter has no clear-cut boundary. On a more general level, the installation speaks of the dichotomy between nature and culture, symbolized by the trees and the mathematical shape respectively. Reading the sculpture and its environment this way, culture is poised between the two poles of embracing nature and caging her.

Quantum Man (2006–2007)

In quantum theory, matter is mathematically described as a wave and therefore each portion of moving matter is associated with a specific wavelength, the distance between two consecutive waves. My former group leader Anton Zeilinger once remarked jokingly that the fact that the wavelength associated with a typical walking person happens to be approximately the Planck length[5] cannot possibly

[5] A very small distance (1.6×10^{-35} m) that is thought to be of fundamental meaning in physics.

Fig. 2 *Quantum reality (Large Buckyball around trees)* (view from *below*), steel and trees, diameter of the steel structure 30′ (9 m), 2007. © Julian Voss-Andreae. Located in a park-like setting in Portland (Ore.), a 30′ (9 m) diameter buckyball is suspended in the air by large Douglas firs intersecting with the buckyball

be a coincidence. This comment made me think about what such a wave-function might look like and a few years later I created a series of sculptures inspired by this idea. Modeled in the shape of a stylized human walker, *Quantum Man* consists of numerous vertically oriented parallel steel slabs with constant spacing to represent the wave fronts (Service 2006, 913). The slabs are connected with short pieces of steel. These irregularly positioned connectors between the regularly spaced slices evoke associations with stochastic events and, more concretely, with the formulation of quantum mechanics in terms of *path integrals*.[6] When approached from the front or back, the sculpture seems to consist of solid steel, but when seen from the side it visually disappears almost completely (Fig. 3). This fascinating visual effect[7] offers a range of possible interpretations. In one such interpretation *Quantum Man* serves as a metaphor for the *wave-particle duality*—depending on the experimental

[6]Richard Feynman's path integral formalism is a tool for calculating quantum mechanical probabilities by adding up all possible paths ("sum over histories"). This is done by "slicing up time" to parameterize arbitrary paths. The slabs suggest the time slices and the irregularly placed rods the random path points. See also the description of *Night Path* (2009) in the last section.

[7]For footage of the sculpture displaying this effect see V. Patton, "Quantum Sculptures with Julian Voss-Andreae," *Oregon Art Beat (Oregon Public Broadcasting TV)* Episode 1012 (2008); URL http://www.youtube.com/watch?v=LqsQYVFAgPo (link checked September 2011).

Fig. 3 *Quantum Man*, stainless steel, 100″ × 44″ × 20″ (2.50 m × 1.10 m × 0.50 m), 2007. © Julian Voss-Andreae. The images show three views of the same sculpture, located at the Maryhill Museum of Art in Goldendale (Wash)

question we ask, the same physical system reveals either properties of waves or of particles:

> A feeling of intangibility and the subjectivity of points of view pervades *Quantum Man*, a walking figure created from parallel slices of steel in which the particle-like concreteness seen from the front shifts to wave-like near-invisibility when the piece is viewed from the side (Ball 2009, 416).

Quantum Woman (2008–2009)

Quantum Man's slices are oriented vertically, corresponding to horizontal motion. To create a female counterpart I rotated the slices into a horizontal orientation, quantum mechanically associated with motion in the up-down direction. The initial idea was that *Quantum Woman* would symbolize a connection between earth and the heavens, as opposed to her male counterpart symbolizing involvement in the orthogonal direction, the worldly realm. I made two versions of *Quantum Woman*, both based on a traditional life-size figure I created after a live model. For the first version, later titled *Science (Quantum Woman)*, I cut 175 slices out of a virtual model of the figure and had them laser-cut from stainless steel sheet to faithfully recreate the body's shape. The relationship between the fertile, female figure and its image in the shape of a stack of cold stainless steel slabs evokes the relationship between nature and the natural sciences; a complex reality is represented as a set of simplified maps. Both versions of *Quantum Woman* have four "seams", tension elements made from bent steel rod that vertically connect all slices. Those seams divide the figure neatly into the four Cartesian quadrants, further playing off science's insistence of imposing a grid onto the world in order

Fig. 4 *Left panel*: *Science (Quantum Woman)*, mirror-polished stainless steel, 69″ × 19″ × 16″ (1.75 m × 0.50 m × 0.40 m), 2008. *Middle and right panel*: *Quantum Woman II*, stainless steel, 69″ × 19″ × 16″ (1.75 m × 0.50 m × 0.40 m), 2009. © Julian Voss-Andreae. The second version of *Quantum Woman* (*middle and right panel*) consists of slices that contain additional irregular fluctuations to dissolve the smooth surface that is a feature of the first version (*left panel*)

to make it mathematically ascertainable. For the second version I decided to lighten the materiality of the piece and to dissolve the accuracy of the outline by using fewer and thinner slabs and adding "quantum fluctuations", random oscillations to the outlines of each slice's original shape. Both versions of *Quantum Woman* are depicted in Fig. 4.

The "Quantum Objects" Exhibition

When I was offered to exhibit my work at the *American Center for Physics*, I decided to show about thirty smaller-scale sculptures, all inspired by quantum physics. Titled "Quantum Objects",[8] the exhibition contained small versions of *Quantum Man* (see Frontispiece) and *Quantum Buckyball* as well as a head study for *Quantum Woman II*. Most of the sculptures were created specifically for this exhibition, ranging from translations of quantum physical concepts that many scientists would recognize as such, to more abstracted works. Common to all pieces is a well-defined conceptual origin. The complete collection of sculptures can be viewed on my website.[9]

[8]"Quantum Objects" was part of the three-person exhibition "Worlds Within Worlds" (Fall 2009–Spring 2010).

[9]URL http://www.julianvossandreae.com/under "Work" and "Archive" (link checked September 2011).

Concerning the title of the exhibition, I should stress that the term "quantum object", although regularly used in physics, is really an oxymoron. An "object" is something that lives completely in the paradigm of classical physics: Its reality is one that is independent of the observer, it behaves deterministically, and it has definite physical properties, such as occupying a well-defined volume in space and time. For the "quantum object" all those seemingly self-evident truths become false. As we saw in the introduction, it has a reality that is relative to the observer. In addition, the principle of causality is violated and other features of materiality, such as clear boundaries in space and time and being objectively located or even possessing identity, do not pertain.

The Well (Quantum Corral) (2009)

One of the objects in the exhibition, *The Well*, was created by directly utilizing data from a landmark experiment performed by Mike Crommie, Chris Lutz, and Don Eigler at the IBM Almaden Research Center (Crommie et al. 1993, 218–220). The researchers prepared a very clean copper surface with a few iron atoms scattered on it and used a *scanning tunneling microscope*, a device that "feels" a surface with subatomic resolution, to produce data that represent the shape of this tiny landscape. This same device was then used to push the iron atoms into a neat circle, termed "quantum corral", after which the surface was scanned again. The single iron atoms show up as peaks and the experiment reveals the concentric circles of a standing matter wave inside the corral, analogous to the standing sound waves in musical instruments. This is a rare example of directly visualizing quantum mechanical matter waves.[10] The researchers kindly provided their experimental data which I then converted into code that was used to mill the shape out of a block of wood. In order to see the peaks and waves, the height of the contour had to be greatly exaggerated compared to its width and depth.[11] After milling, the object was traditionally gilded with gold leaf (Fig. 5). My motivation for making such an object goes beyond showcasing the data which is fascinating in itself. I want to evoke a sense of wonder in the audience and convey the feeling of witnessing something extraordinary. Philip Ball writes about *The Well* in his review of the "Quantum Objects" exhibition in the journal *Nature*:

> The gilded surface reminds physicists that it is the mobility of surface electrons in the metal which accounts for its reflectivity (and the coloration of gold is itself a relativistic effect of the metal's massive nuclei). But for art historians, this gilding not only invokes the crown-like haloes of medieval altarpieces but could also allude to the way gold was reserved in the Renaissance for the intangible: the other-worldly light of heaven (Ball 2009, 416).

[10]What is imaged in this experiment is essentially the square of the surface state electrons' wave-function.

[11]The same had been done to prepare all the published images of the quantum corral including the images in the original publication by Crommie et al. (1993, 218–220).

Fig. 5 *The Well (Quantum Corral)*, gilded wood, 3″ × 13″ × 12″ (6 cm × 34 cm × 31 cm), 2009. © Julian Voss-Andreae. Photo: Dan Kvitka. Original experimental data were used to turn a man-made subatomic "quantum landscape" into an art piece

Night Path (2009)

Night Path was inspired by Richard Feynman's *path integral* approach to quantum mechanics. Feynman calculated quantum mechanical probabilities by adding up all possible paths between a start point and an end point. This quantum mechanical concept of a *path* only makes sense as long as it cannot be observed[12]; it is really a tendency for a path and not an actual path. Feynman handled the continuum of paths mathematically by dividing time into "slices" and filling each slice with a continuum of path points. When this approach is modeled on the computer, only some random paths in the neighborhood of the classical trajectory are calculated since they contribute most to the result.[13] Guided by this image, I started with a parabola, the classical trajectory of a falling object, and computer-generated a distribution of random paths around it. The paths get successively closer to the parabola and eventually merge into one point (Fig. 6). The image of a point expanding into a curved, fuzzy tail resembles historical depictions of comets. I wanted to connect the idea of the quantum mechanical path to the image of this celestial body that is often portrayed in art and literature as portending important events.

Spin Family (Bosons and Fermions) (2009)

Spin Family playfully equates the two fundamental kinds of matter in the universe, bosons and fermions, with the two human genders, female and male. Due to their difference in a quantum physical property called *spin*, bosons tend to attract each

[12]"Cannot be observed" in this context does not mean "when nobody looks", but rather that an observation is *in principle* impossible because there is no physical carrier of information (e.g. a photon that could get detected by an observer's eye).

[13]For a more detailed discussion about the relationship between *Night Path* and the physics that inspired it, see Ball (2009, 416) and the Q&A section in Philip Ball's blog "homunculus", URL http://www.philipball.blogspot.com/2009/11/quantum-objects.html (link checked September 2011).

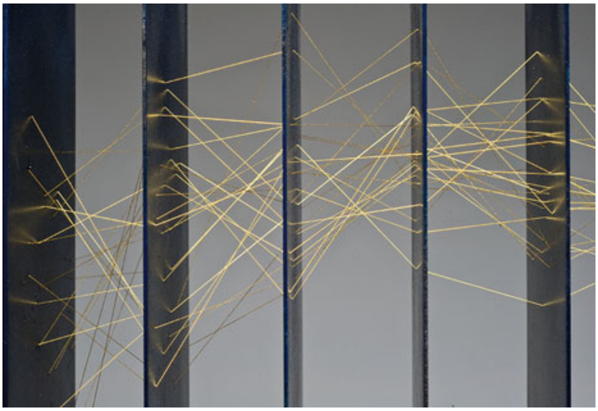

Fig. 6 *Night Path* (detail), painted steel and *golden* thread, 18″ × 19″ × 6″ (46 cm × 48 cm × 15 cm), 2009. © Julian Voss-Andreae. Photo: Dan Kvitka. Held in place by a frame of *dark-blue* painted steel plates, *golden* threads fluctuate randomly around the trajectory of a falling object to meet in one point

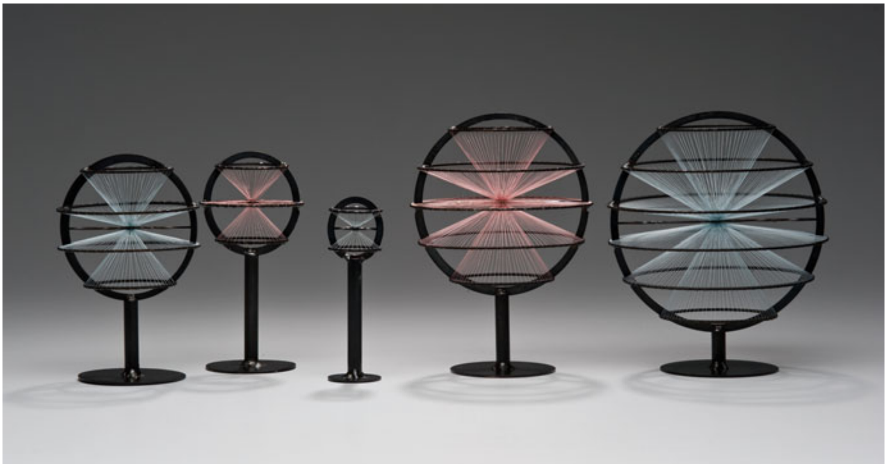

Fig. 7 *Spin Family (Bosons and Fermions)*, steel and colored silk, largest object 7″ × 6″ × 6″ (18 cm × 15 cm × 15 cm), 2009. © Julian Voss-Andreae. Photo: Dan Kvitka. A continuous silk thread representing quantum mechanical spin is woven in and out of *circular* metal frames. The 'bosons' in *pink* portray the 'female' members and the 'fermions' in *light blue* the 'male' members of a family

other whereas fermions have a tendency to stay isolated. *Spin Family* is a series of five objects displaying the three-dimensional structure of the spin essentially as it follows from the rules of quantum mechanics. A continuous silk thread representing the spin is woven in and out of circular metal frames expanding the single, well-defined direction of the spin in classical physics into quantum physics' continuum of possibilities, giving a diaphanous quality to the overall forms (Fig. 7).

Self-Portrait on the Brink of Detection (2009)

Unable to perceive the world on the quantum level without sophisticated technology, our intuition about the nature of reality is shaped by the comparative crudeness of our unaided senses. If we, for example, observe an apple falling from a tree, we naturally assume that the apple has an identity and is one and the same thing before, during, and after the fall. Quantum physics, however, teaches us that there is no real continuity of "objects" around us. The image we perceive as "the apple" is actually the rapid accumulation of an astronomical number of single, indivisible quanta of experience, or *events*. These quanta of experience are individual little flashes of light that our brain automatically connects into familiar objects that then appear to us as constant. *Self-Portrait* imagines this process of experiencing slowed down and captures the moment where the successive accumulation of events has just lead to a first recognition of the familiar. I created an image made up of events represented by small holes in a backlit steel plate. To that end I wrote a computer program that transforms an image, in this case a photograph of my face, into a distribution of spots. The lighter a particular area of the image is, the higher the density of random spots, or "events", the algorithm generates in this area. Figure 8 shows the program's output used to create the piece on display, a free-standing darkened steel sheet with 1500 small holes. Lit from the back, the holes resemble shimmering stars on the night sky.

Quantum Field (Profiles) (2009)

Quantum Field was born out of my interest in giving material representation to what it is that connects people. In physics the space between two interacting objects contains a *field*. Guided by this analogy, I utilized an old shipbuilders' technique to draw smooth lines by clamping long, thin, flexible strips of wood, so-called *splines*, between nails. Splines generally bend into curves that are perceived as elegant because the mechanics of the system, with the splines moving freely along the nails, allows the total bending energy of the spline to settle down at its minimum.[14] I marked the contours of two identical human profiles facing each other with two sets of nails. Extrapolating between the two contours, I placed additional sets of nails in between the faces and wove wooden strips through them to represent something reminiscent of a field between the two human profiles (Fig. 9). This work also evokes an association with the phenomenon of *entanglement*, another puzzling but ubiquitous aspect of reality revealed through quantum physics. In the most basic manifestation of entanglement, two twin-like particles share a connection that is

[14]For quantum mechanical wave-functions the situation is actually similar since the kinetic energy operator in the Schrödinger equation involves a second derivative, i.e. the curvature, of the wave-function.

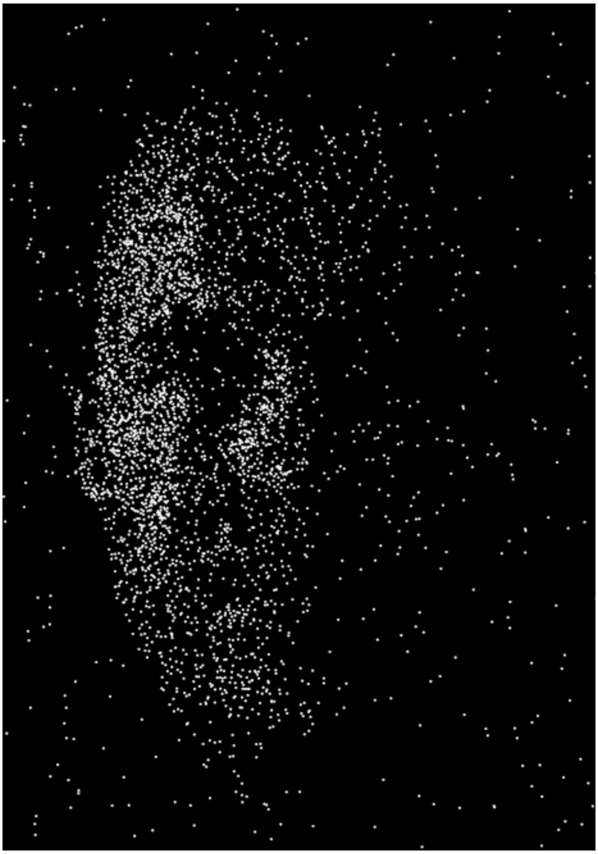

Fig. 8 *Computer sketch for 'Self-Portrait on the Brink of Detection'*, 2009. © Julian Voss-Andreae. The art work made after this sketch is a back-lit steel plate with 1500 small holes. The image resembles what our retina would detect during a very short moment with only very few photons available to build up an image of what we see. At this point, the stochastic nature of reality is still visible

deeper than anything possible in classical physics.[15] The two particles' states are tied together as if they were located at the same spot, even though they might be separated by light-years.

The Universe (The Cellular Structure of Space and Time) (2009)

It is often believed that space-time itself is made up of smallest indivisible units, analogous to the quanta of experience or the atoms of matter that reveal themselves to us only with sufficient magnification. But how would those presumed quanta of space-time be arranged? *The Universe* portrays the cells of space-time as arranged

[15]Einstein famously called this fascinating phenomenon "spukhafte Fernwirkung" (*spooky action at a distance*).

Fig. 9 *Sketch for 'Quantum Field (Profiles)'*, plywood with *pencil marks*, wooden splines, and nails, 32″ × 24″ × 2″ (80 cm × 61 cm × 5 cm), 2009. © Julian Voss-Andreae. Two facing profiles are formed with wooden strips woven in between nails. "*Field lines*" were added in between the faces using the same technique in order to give material representation to the connection between the two figures

in an organic fashion, namely as cells in a foam, the ubiquitous natural system that is comprised of irregularly shaped polyhedral bubbles with, nevertheless, well-defined properties. To make this piece, I created an artificial foam by squeezing water-filled balloons, the foam's bubbles, into a spherical mold and filling the gaps in between the balloons with hot wax. After the wax had hardened, I popped the balloons to produce an open network of polyhedral cells with rounded edges. I cast the structure in bronze, gold-plated the interior and applied a dark patina to the exterior (Fig. 10).

Conclusion

The simultaneous advent of quantum physics in the sciences and the rise of modernism in the arts in the early twentieth century marks a profound shift in the cultural evolution of humankind. The uneasiness many of us experience when dealing with either illustrates how little we have grappled yet with the consequences of this paradigm shift. The sculptural work presented in this chapter aims at exploring the character of this shift by transforming ideas that emerged in the isolated intellectual realm of quantum physics into art that evokes a sensual

Fig. 10 *The Universe (The Cellular Structure of Space and Time)*, bronze, diameter 8″ (20 cm), 2009. © Julian Voss-Andreae. Photo: Dan Kvitka. An irregular network of *golden* cells conspires to make up a regular *sphere*. Could this be the way the smallest units of space-time arrange themselves to make up the universe?

experience. My hope is that my work will help to lift those ideas into the sphere of our collective consciousness and aid us in intuiting the unfathomable deeper nature of reality.

Acknowledgments I would like to express my thanks to the following people: Sarah Tanguy invited me to show new sculptural works at the American Center for Physics which resulted in the "Quantum Objects" exhibition; Philip Ball asked me tough questions for his review in "Nature" which forced me to think more deeply about the topic; I had many discussions with George Weissmann about quantum physics and the implications it has on our world view and I am especially indebted to him for his central contributions to this manuscript. I am also very thankful to all the people who read the manuscript and gave me valuable feedback: Leo Gross, Jack R. Leibowitz, Michael May, Arthur I. Miller, and Kenneth Snelson as well as the six anonymous reviewers from *Leonardo* and the *Bridges 2010* conference. They all provided excellent comments which benefitted the paper greatly. And last, but by no means least, I would like to thank my wife Adriana for always inspiring me, carefully reading and editing this manuscript and, most importantly, her love.

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Artformer Geometry

Antal Kelle

Abstract In this chapter I describe some of my artworks, which are reconfigurable geometrical objects—somewhere on the border between scientific curiosity, pure playfulness and sculpture. After a brief description of each item, I included some of the thoughts I tried to express through them. All of my sculptures described here are interactive: they can be transformed eider manually or remotely. My aim is to show that contemporary problems can be expressed through simple geometric forms.

Theory and Practice

The elements of my objects can move, and they can be transformed into various forms by rotating them according to our curiosity and mood. The basic underlying form always approximates to a regular solid, which may be turned into a random form or a pleasing, organic statuette. There is an intimate interplay between the sculptures and anyone who chooses to investigate them; there is no set goal and no right or wrong answer. With modern technology they can be automatic, and could be remote-controlled.

As an artist I use geometrical terminology in its freer, everyday sense, rather than with its precise abstract meaning. This artistic approach often does not match the strict mathematical usage. We say ice-cube for instance, refer to block buildings as cubic, and use the word pyramid for ancient buildings. In fact, in this we are saying which form they resemble the most. Pyramids most closely resemble the mathematical solids of the same name, however their surface is not what we would expect in geometry because of the technical steps of the construction. That is, they are not the planar faces of a polyhedron. I was inspired by this fact, and tried to turn the process the other way around, and examined to what extent I could depart from the physical forms while keeping to the original geometrical form in a philosophical and artistic sense. For instance, if I cut a piece from the cube, to what extent can the seven sided cube (heptahedron) still be called a cube for art and by non-specialists? My first object, named KREABAU (Fig. 1), is a set of basic geometrical solids that are cut by other regular geometrical forms. These deformed spheres, cylinders and cubes are not

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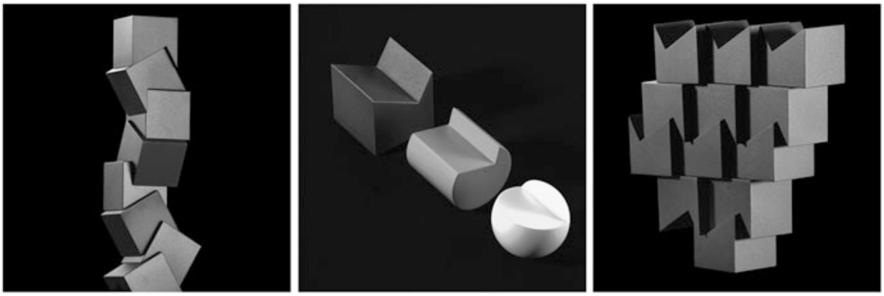

Fig. 1 KREABAU, 1999, the basic elements 6 × 6 × 6 cm painted beech. Photo: István Oravecz. Copyright: Antal Kelle

imperfect, but have become different from regular forms. They have direction and individuality, and provide many possibilities for building Cubist sculptures.

Axioms, Parameters and Preconceptions Mathematical axioms are simple, strict and were programmed into us a long time ago, however we all are aware of their limitations. The great explorers and mapmakers had to deal with the problem of the difference between plane and spherical geometry, however, for many people, it is still a surprise that the sum of the angles of a spherical triangle is not fixed but varies within a certain range. New geometries can be born with the help of new axioms and parameters. I am proud to mention János Bolyai, who created the non-Euclidean geometry with his work *Appendix* (1831). His geometry is an essential basis for twentieth century theoretical physics.

Deformations and Projections We don't need to wander so far away because even within Euclidean geometry we have countless possibilities yet to be utilised. For instance, if we distort a circle in a regular way we get an ellipse. If we build a prism or a cone on this ellipse, the resulting solids maintain some of the defining features of their circular equivalents while replacing some other features with new ones, but there are some surprises; we can find strange characteristic features in them. Distorting a circle into an ellipse can be regarded as projection.

Sections and Variability Geometrical solids can be divided into parts by slicing. We know that elliptical cylinders can be cut at angles so as to have exactly circular cross sections, and the joints that can be seamlessly assembled with arbitrary angular rotations. ELLIPSO, a classical wooden toy designed by Xavier de Clippeleir, is a good example of this. With my HELIX object (Fig. 2), the non-parallel sections are also circles! It is movable, having 11 rotatable segments. The basic form is a simple (but deformed) cone cut by planes to create a variable object consisting of segments. The segments are arranged consecutively, meeting each other at the intersecting planes that divide the object into segments. They can rotate with respect to each other along the section planes. At each of the sections a profile—defined by the bounding surface—is formed belonging to that particular section. There is complete freedom of rotation between connecting parts.

Fig. 2 HELIX, 1999, 14 × 14 × 25 cm mixed beech varnished wood, plastic joint parts. Photo: István Oravecz. Copyright: Antal Kelle

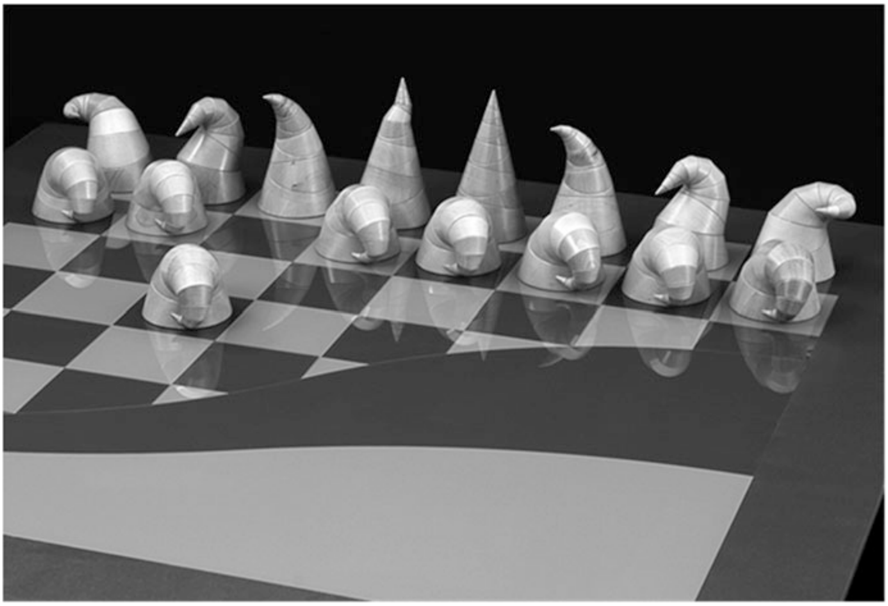

Fig. 3 *MODERN FOLK TALES*, 2004, Helix characters 160 × 160 × 75 cm mixed acrylic fiberglass, wooden. Photo: István Oravecz. Copyright: Antal Kelle

Thus one can produce an endless variety of forms by rotating the segments. The different shapes may express different personality or emotions. Thus we end up with not pure geometrical forms, but characters with personality, such as chess pieces (Fig. 3). In my version of 'chess', MODERN FOLK TALES, there is only a single set of chess pieces, and the fight takes place between the figures of this one team. Stand up, and you become a king!

Fig. 4 INFLEXIO, 2000, 40 × 20 × 2 cm mixed, varnished wood, plastic joint parts. Photo: István Oravecz. Copyright: Antal Kelle

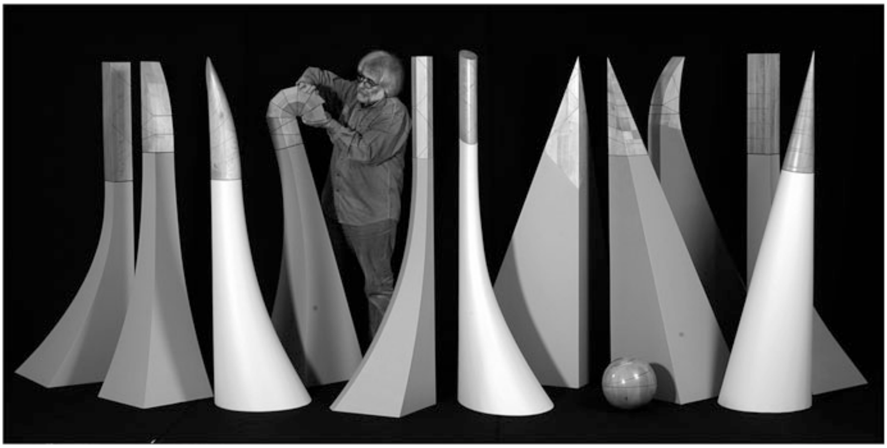

Fig. 5 GEOMETRICAL PANTHEON, I-XII 2003–2004, mixed, varnished wood, plastic joint parts, painted base. Photo: István Oravecz. Copyright: Antal Kelle

Ranges, Sequences and Isomorphism We can create a series of three dimensional pieces ordered like numbers. In this way, organic looking mobile plastic art can be produced by building simple, abstract geometrical elements. One example is my INFLEXIO (Fig. 4), which has identical articulated elements connected to form a Möbius strip.

In more complex cases the elements that are built next to each other are not identical, but partially similar, forming parts of a systematic series. The segments belonging to each particular sculpture are joined by a special connection, allowing an unlimited number of rotations, just like with the HELIX. This is how my object group GEOMETRICAL PANTHEON (Fig. 5) was made. Here, circles are not the only cutting planes: some of the sculptures have triangles and squares, too. SPHERE (Fig. 6) as a part of the GEOMETRICAL PANTHEON, has unlimited number of rotations and positions (Fig. 6a). Figure 6b shows the SPHERE's most opened and excentric position.

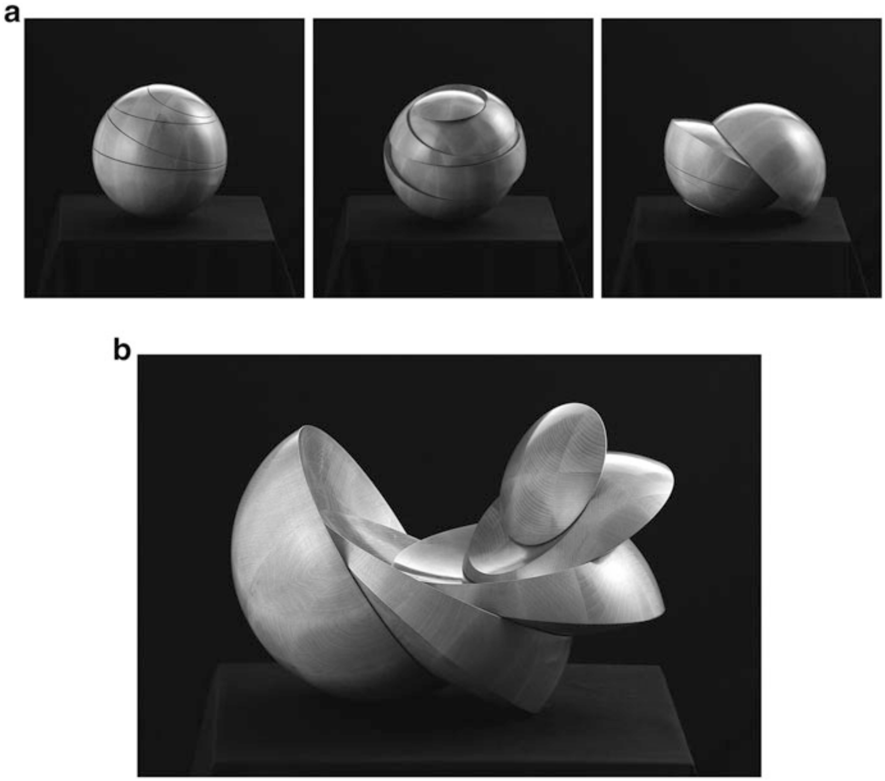

Fig. 6 (**a**) SPHERE, 2004, 28 × 28 × 28 cm open 40 × 32 × 28 cm, mixed, varnished wood, plastic joint parts. Photo: István Oravecz. Copyright: Antal Kelle; (**b**) *SPHERE*, 2004. Photo: István Oravecz. Copyright: Antal Kelle

Mobility and Interactivity

The Geometrical Pantheon consists of 12 individual sculptures, the elements of which can be rotated and moved manually. By rotating them, they can be transformed into varied forms according to our curiosity and mood. All the basic underlying forms approximate to a regular solid, which may be turned into a random form or a pleasing, organic statuette. There is an intimate interplay between the sculptures and anyone who chooses to investigate them. There is no set goal and no right or wrong answer. With modern technology they can be automatic, and could be remote-controlled. They can come alive; their movements and dance are beyond simple demonstration or variations. Analogies can be found and symbolic messages can be imagined. The short film LIBRETTO available on my website at www.artformer.com gives an idea of these possibilities.

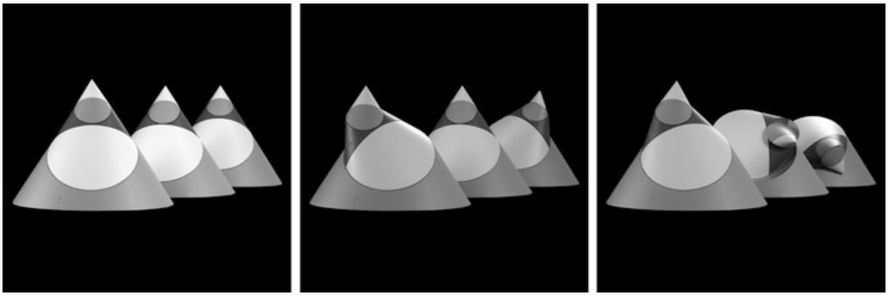

Fig. 7 UNIFORMS, 2005, 160 × 160 × 100 cm mixed wood, resin, metal mechanism. Photo: István Oravecz. Copyright: Antal Kelle

The mobile sculpture group named UNIFORM (Fig. 7) is an interpretation of being different and the feeling of not being the same. This piece of work has also been exhibited in Dessau, at the Studio-Gallery of Kandinsky and Klee, two of the Bauhaus's famous teachers. The three cones with exactly the same shape, size and decoration are made of different components. The sections of the first figure are parallel. The second one has periodically varied angles while the third one has a spherical surface. Visitors tried to set and move their mechanism in the same way unsuccessfully, because the results were always different except for the base form.

My NEXUS project (Fig. 8a, b) was exhibited in Hungary in the Renaissance Hall of the Museum of Fine Arts in 2009. It is a platform for the interaction of several individuals, and uses the most up-to-date technologies. The statue of a couple, almost 6-meter-tall, that uses the entire space is exceptional in the history of art from the point of view of interactivity. The sculptures are equipped with remotely controlled motors at their joints. The change in the joint angles is controlled by the combination of input signals coming from different sources, which can be local to the exhibit or may come from spectators in remote locations who view the sculpture over the internet or even on cell phones. Visitors to the Hungarian Museum of Applied Arts, at the Moholy Nagy University of Art and Design in Budapest, could watch the sculptures move according to the input they have given at the local terminal. The indirect behaviour and reactions revealing themselves in the movements of the statues developed into play, co-operative acceptance, or even denial.

My newest project is both the biggest and the most time consuming to build. It will be an interactive moving monument, a more than 25-meter-high moving steel tower, named INDIAN DESIRE (Fig. 9), that is going to be realized in Ahmedabad, India. Its structure is similar to the sculptures of the GEOMETRICAL PANTHEON, with square sectional faces. The shape of the column came from the traditional Hindu temples and Muslim mosques (like Qutab Minar). I abstracted and combined these forms into a non-regular prism, and cut them with planes into six pieces. The moving steel sculpture will twist, change shape and colour, and will be controlled by the visitors with the help of sophisticated, robotic motors and special-purpose software.

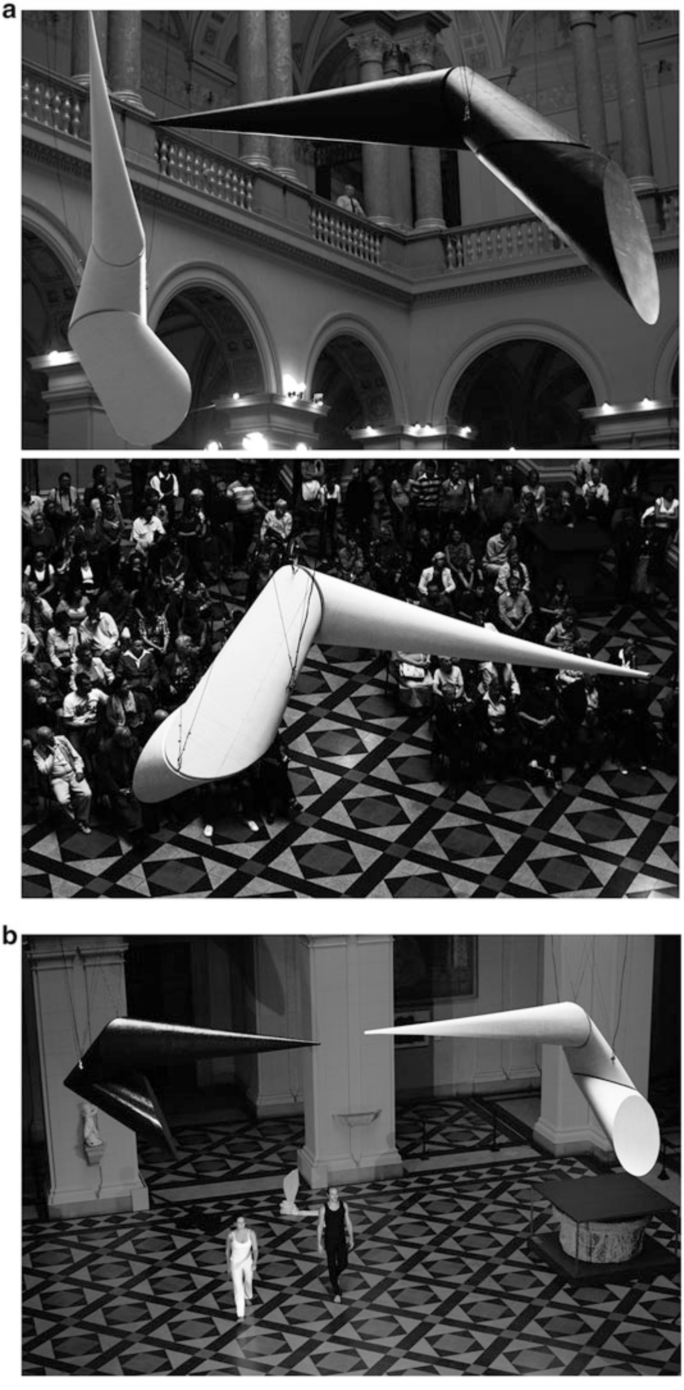

Fig. 8 (**a**) NEXUS at the opening ceremony with dancers, 2009, 6 × 12 × 8 m mixed: carbon fiber, metal mechanism, computerised internet/wifi control. Photo: Emese Kelle. Copyright: Antal Kelle. (**b**) NEXUS at the opening ceremony with dancers, 2009. Photo: Emese Kelle. Copyright: Antal Kelle

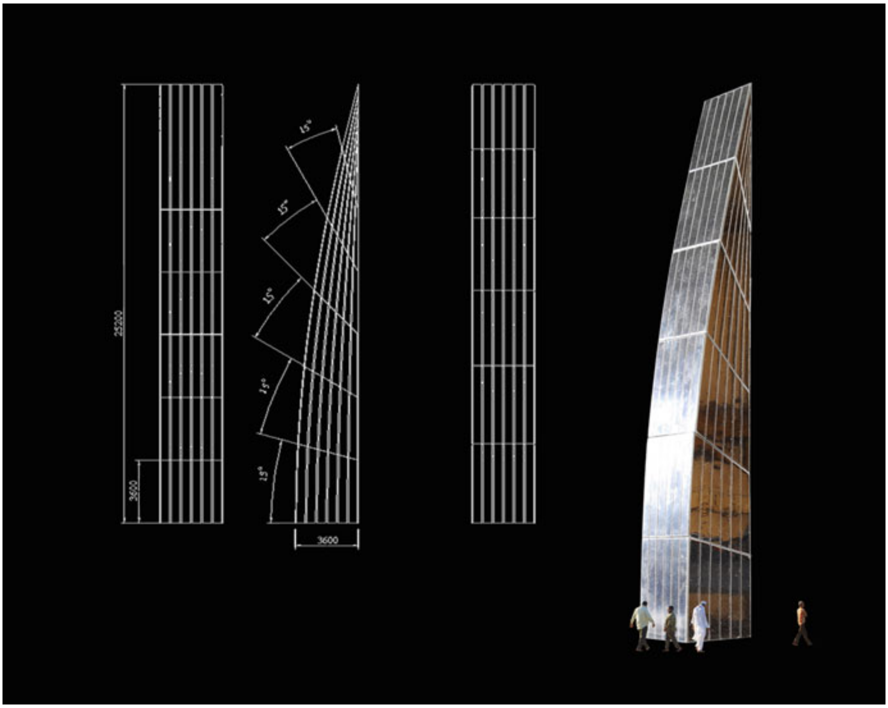

Fig. 9 *INDIAN* DESIRE, 2007 25.2 × 3.6 × 3.6 m mixed: stainless steel, metal mechanism, computer controlled. Photo: Antal Kelle. Copyright: Antal Kelle

Conclusion

I have always been interested in objects made with geometrical oddities, especially ones made from identical or similar modules, or ones based on similar principles. When preparing the present chapter, I was pleased to discover that several contemporary artists are exploring similar ideas. Tom Verhoeff and Koos Verhoeff (2008)—with the systematic of the mathematician—examined the possibilities of miter joints and variants. Rinus Roelofs (2009) builds—in real and in virtual—module-base sculptures. His Non-Flat tilings resemble to my objects built of identical modules. Roland de Jong Orlando (2008) and George Hart build objects of modular kits. In the beginning I made toys with the concept of two-dimensional modular building blocks which were not joined, and were like puzzles or tangrams. The three-dimensional sculptures described in this article are cut and joined in a special way, and the shapes, principles and concepts are patent pending. I believe that these objects deal with contemporary problems, approached more from an artistic than a mathematical point of view.

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Aesthetics of Geometry and the Problem of Representation in Monument Sculpture

Tuuli Lähdesmäki

Abstract Since the 1920s and 1930s, constructivist and concretist visual art movements have stressed geometric forms, proportions, and orders as a base for artistic expression and aesthetic experience. After the World War II geometric form was adopted to the public sculpture. Abstract, geometrically constructed sculpture was also used in commemorative functions in modern monument art. The combination of the commemoration of a significant historical event or a national hero, and the aesthetic ideas based on constructivist or concretist art movements caused a lot of debates and confrontations in many Western countries. In particular, the interpretation of abstract monuments was problematized: abstract monuments were often interpreted (or tried to be interpreted) as metonymic or metaphoric depictions or more or less symbolic images of the person or event for whom they were erected. The idea of representation and the symbolic meanings, however, contradict the principles of constructivist and concretist visual art movements. The chapter discusses two contemporary constructivist and concretist monuments in Finland and illustrates how the problem on representation has been solved in the public reception of them.

Representation and Reception as a Discourse

Erecting a public monument for the commemoration of a person or an event is an act, which pertains to several areas of social life: art and aesthetics, reminiscence and memory, communality and identity, understanding and narration of history, spatial experience, social norms and politics. The monument projects often cause active public discussions or even severe debates on different meanings and possible interpretations of the monument. The question on representation is particularly controversial regarding modernist monument sculpture, and it has evoked a varying kind of confrontation and dispute. This question became particularly problematic when forms of art based on geometric abstraction were applied to monument sculpture: the idea of representation and the symbolic meanings contradict the principles of constructivist

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and concretist visual art movements. The combination of the commemoration of a significant historical event or a national hero, and the aesthetic ideas based on constructivist or concretist art movements caused a lot of debates and confrontations in many Western countries after the World War II. In particular, the interpretation was problematized in the reception of the abstract monuments.

In this chapter I will discuss the problematics of representation in the reception of constructivist and concretist monuments, and explore how the geometric abstraction is perceived and conceptualized in the interpretation process of the work of art attached to various cultural and social meanings due to its commemorative function and the tradition related to the practice of erecting monuments. The empirical focus of the chapter is on two Finnish presidential monument projects that illustrate different aspects of the problematics of representation. The reception of these monuments was profoundly controversial and both projects caused an intense public debate in Finland in the beginning of the 1990s. Despite the polemics, petitions, and negative criticism, the constructivist or concretist monuments of the former Finnish presidents Risto Ryti and Lauri Kristian Relander were unveiled in the capital city of Helsinki in 1994 and 1996.

I will approach the problematics of representation and the aesthetics of geometrical forms from a discursive point of view. The discursive approach to the reception of monuments enables the analysis of the notion of representation and meaning-making processes. Understanding the aesthetics of geometrical forms from the discursive point of view rejects the universalistic explanations of beauty in geometry, and rather emphasises the transforming character and historicity of the principles in aesthetics of geometrical forms. The perception, reception and description of the geometric abstraction are related to various cultural and linguistic practices and conventions. The theoretical background of the chapter arises from the approaches of social constructionism which emphasize the reality and the 'truth' as constructions produced in language, interaction and social practices (Shotter 1993, pp. 6–10, 99–101; Gergen 1999). In this chapter I define discourse as a particular way of representing reality (Fairclough 1992, pp. 3–4). These representations, expressed in the reception of geometric abstraction, construct the monuments and the aspects related to them in a complex way. These representations also indicate the power positions and hierarchies intertwined in the use of language and the meaning-making processes. The discursive approach is combined in the chapter with the use of semiotic concepts. Both points of view share a similar kind of understanding of the interpretive nature of reality and the constructive character of language and concepts in the production of meanings. Therefore semiotic concepts, which explain and theorize the formation of meanings, are often used as analytic tools in discourse studies (van Dijk 1995; Wodak and Meyer 2009; Jaipal-Jamani 2011).

The empirical data of the chapter consists of various types of texts which were a part of the debates of the two monuments in question. The data related to the Ryti monument project consists of 271, and the Relander monument project of 72 texts. The empirical material has been collected from libraries and Finnish archives. The largest part of the material is formed by news texts and letters to the editors of local, regional and national newspapers. The speakers in the texts represent several positions such as journalists, art critics, artists, officials, and the so-called lay people, who do not hold a public position in cultural or societal fields.

Aesthetic Aims of the Art Movements in Geometric Abstraction

Since the 1920s and 1930s, constructivist and concretist visual art movements have stressed geometric forms, proportions and orders as a base for artistic expressions and aesthetic experiences. The constructivist art movement has its origins in Russia, where artists were inspired to create works of art that—instead of representative qualities—stressed abstraction composed of structural elements: lines, planes and geometric solids with an aesthetics based on the rhythm, proportions and balance of structures. These artistic ideals originated from the earlier pioneers of abstract art, such as the suprematist paintings of Kazimir Malevich. In 1920 a group of Russian artists and art theorists—who called themselves the First Working Group of Constructivists—formulated the aims of the movement. They thought that the constructivists should turn away from any experimental activity 'removed from life' and towards 'a real experimentation' and the mastering of the creation of structures in a scientific and disciplined way (Harrison and Wood 2003, p. 342). Among the group the three-dimensional constructivist works of art were paralleled to non-representational and non-symbolic works of engineering. Thus, like industrial products, constructivist works of art were often given a serial number instead of a title (Hohl 2002, pp. 985–986). Since the revolution the new social and political circumstances in Soviet Russia influenced artistic creation. Besides the artistic and aesthetics dimension, the constructivists considered art and design as instruments in social and ideological transformation of the society. The constructivists actively participated in public life creating propaganda posters, decorations and urban and architectural plans stressing the political and ideological aims of the Bolshevik government, which in the beginning of the 1920s still supported constructivist aesthetics.

As a part of the ideological strivings, constructivist principles were also applied to public sculpture and monument art. A well-known example of a constructivist monumental work is Vladimir Tatlin's model for the Monument to the Third International. The model—known from photographs and later reconstructions—was made in 1920 but never executed. The model reveals the utopian nature of Russian constructivism: in Tatlin's visions the monument would have been 400 metres high and included three geometrical building blocks rotating at different speeds (Hohl 2002, p. 993; Harrison and Wood 2003, p. 336).

The constructivist ideas in art spread quickly to Western Europe and had an influence on other modern art movements. The constructivist ideas were adapted particularly in Germany and The Netherlands, where artists, architects and designers of the Bauhaus and De Stijl further elaborated the aesthetic principles of geometric abstraction. Geometric abstraction was applied to all kinds of designs by both movements, including monument art. For example Walter Gropius—the leading Bauhaus architect—used a constructivist composition in his monument for the memory of the workers shot during the Kapp Putsch in 1920. The monument, made of concrete, was erected in a Weimar cemetery in 1922.

Like constructivism, concretism—or Art Concrete—stressed geometric abstractions and structural composition of forms. Art Concrete was an heir of the constructivist movement in the West. It was originally formulated and introduced by the Dutch artist and architect Theo van Doesburg in 1930. His Manifesto of Concrete Art (1930), which stemmed from the De Stijl movement, emphasized colour, line and surface, and their geometrical composition as the main elements in art. These were considered as concrete elements which did not refer to any subject outside the work itself or symbolise anything. Thus, concretists saw their art as being rational and pure, that is free from the outside elements (Strietman 1988; Harrison and Wood 2003, pp. 282–284; Frampton 1982, pp. 112–113). According to the concretists, the reception of Art Concrete was also 'pure' and 'direct'—concretist art was not understood as requiring any complex interpretation process, because it was seen as referring to nothing else but itself.

The aesthetic ideas of constructivism and concretism had an influence on post-World War II art movements in Western countries. The continuity of the geometry based aesthetics can be perceived in minimalism, which developed in the USA in the 1960s as a counter reaction to abstract expressionism and its stress on colour, energy, emotions, the process of creation, and subjectivity. Again, artists found their artistic inspiration from geometry, structural compositions, surfaces, spatial dimensions, proportions, series and industrial materials. The aim of the movement was to exclude the pictorial, illusionistic and fictive subjects from the works of art, and instead focus on the mode of expression in which the presence of the reduced work itself was seen as creating the content of the work. For example, Donald Judd, the leading figure in minimalist art, stressed an anti-illusionist attitude in the production of art and demanded that the artists had to make their material and spatial qualities 'literally real' (Harrison and Wood 2003, pp. 824–828).

The idea of art that does not refer to anything else but itself seems to follow the modernist art movement from the first decades of the twentieth century to the postmodernist break in culture at the end of the century. In spite of the postmodernist criticism of modernist art, the principles and ideals of 'pure form' and 'direct reception' still influence the discourses of art. Works of art based on geometric abstraction are still produced, and they are perceived and discussed in terms of discourses related to constructivism, concretism and minimalism.

Debates on Form in Monument Sculpture

Although the geometric abstraction was actively elaborated and theorized in Western art during the first half of the twentieth century, it had only a slight impact on the renewal of urban design. Most of the constructivist and concretist sculptures remained private and thus hidden from the public eye (Daval 2002, p. 1037). However, after the World War II constructivism and concretism were little by little established as mainstream movements in art. Another mainstream art movement of the time was free abstraction, which is often discussed as informalism or vitalism in Western art history.

In the Western art history, these different discourses of modernism were intensively brought to the fore during the 1950s and 1960s. The diverse discourses produced a confrontation in the practices and the notions of art—artists and critics positioned themselves either for or against free or geometric abstraction, informalism or constructivism, or vitalism or concretism (Ojanperä 1998, p. 113; Lindgren 1996, p. 158, 2001, p. 139). The discourse, which relied on geometric abstraction, stressed theoretical points of view, rationality, spirit and intelligence as the main dimensions of art. Respectively, the discourse bound to free abstraction emphasized emotion, intuition and emotional and experiential reception (Lindgren 1996, pp. 158–160; Huusko 2001). In addition, the discourse of free abstraction highlighted organity on the levels of creative process, form and function of art. Following the principles of nature, organic (non-mathematical) growth was seen as eligible in the form of art. This discourse also included principles of materiality in sculpture: a sculpture had to be 'honest' to its material. Organity in the creative process was stressed by art talk and criticism which highlighted the ideals of handiwork in the production of sculptures, the process of sculpturing rather than the final artistic product, and the imprints of artistic work on the surface of the work of art (Lindgren 1996, pp. 19–23, p. 157, 2001, p. 135). The discourse of geometric abstraction ignored the physical work and unique imprints of artists in the artistic process and instead emphasized the works' universalistic aesthetics and the systematic, serial and technical realization reminiscent of industrial production.

The adaption of the abstract form into the commemorative function of monument sculpture has caused a lot of dispute among the receivers in all the Western countries since the 1950s. Monument debates and dispute on public art became an international phenomenon which started to characterise the Western modernist public art in general (Burstow 1989, p. 472; Gamboni 1997, pp. 132–133, 155, 170). Although the pre-modernist monument art was often controversial as well, the adaption of abstract form into the monument art aggravated the discussion in a new way. A form, which the receivers could have accepted in the so-called gallery art, did not fulfil the expectations laid on communal and commemorative conventions of the monument genre. In the core of these debates was the question of representation of abstract art. The audience, who was not used to abstraction in art, or at least not in public space, expected the traditional expression and figurative depiction of nationally significant persons and events. The fierce comments in the debates indicated notions of art according to which the fundamental principle in art was to represent something and somehow refer to reality. Therefore, the receivers often interpreted, or tried to interpret, abstract art as a simplified and reduced depiction of some real objects, scenes or phenomena. In the reception of free abstraction this was not necessarily a problem, because the works were often based on organic forms or reduced visualizations of phenomena in reality and nature (Lindgren 2000, pp. 240–242).

Besides the discussions on representation, the expected nationalistic and patriotic nature of monument genre kept the monument projects in the focus of the public interest. Among the receivers, monuments were expected to express a common world view of the commemorating community and an established narration of the nation. On the one hand, breaking the conventions of the form of monuments was interpreted as rupturing the communal and national bases of the commemorative practice and the honouring of national heroes. On the other hand, this was exactly the aim of several modernist artists (Berggren 1999, p. 564; Lindgren 2000, p. 230).

In general, the debates and disputes of the monument art in the 1950s and 1960s in Western countries reflected the post-war transformation of the society. From the 1940s till 1960s, all Western countries witnessed several art, literature and culture debates which encapsulated the confrontations caused by the changes in culture and society. These debates often dealt with expressions of national identity, interpretations of the recent past, depictions of religion, and expressions of moral codes. The cultural and social impulse to the debates was above all in the rupture of pre-war patriotic-nationalistic values and in the need to redefine the old value systems (Sevänen 1998, p. 334). Several modernist writers and artists aimed to deconstruct the practices, myths and imageries of national representations and narrations. In monument art this deconstruction occurred both on the levels of content and of form. The abstraction of monument art was a striking mean to renew the nationalistic and conservative values of monument sculpture (Berggren 1999, p. 564). However, the debates and disputes on representation weakened the relationship between the ordinary audience and modern public sculpture, and brought the ordinary audience apart from the discourses of abstract art (Burstow 1989, p. 472; Gamboni 1997, pp. 132–133, 155, 170). The relationship of the ordinary audience to modernist monuments was mainly narrowed to the rejection of them, or general disinterest in the monuments which were considered as poor or difficult to understand (Lähdesmäki 2007).

In spite of the continuous debates, the abstract form became common in the Western countries during the 1960s and 1970s. In Finland the major monuments erected in the 1970s and 1980s obeyed the ideas of free abstraction and the aesthetics of informalism. At the same time the geometric abstraction was used in several public sculptures, which—however—were not dedicated to any commemorative function. Finally, during the 1990s the geometric abstraction was applied to major monument projects as well. The adaption of constructivist and concretist aesthetics to monument art burst out the discussions on representation and reception of 'literal reality'. In different discourses of the reception the problem of representation was solved in different ways.

Interpreting Geometric Abstraction

In 1989, the Finnish Prime Minister's Office set up a committee to prepare monument projects for three former Finnish presidents, who were not yet commemorated with a monument in the capital city. The first two of the projects were launched for Presidents Risto Ryti (1889–1956) and Lauri Kristian Relander (1883–1942). Both were organized as a public sculpture competition in which all inhabitants of Finland could participate. In addition, eight merited Finnish sculptors were invited to participate in Relander's monument competition. The winner of the competition for Ryti was published in 1991. The winning proposal was made by the sculptor Veikko Myller, whose proposal—titled *Years of Responsibility*—consisted of a constructivist composition of rectangular beams with the height of 6 metres. The announcement of the winning proposal caused an intense debate, in which the proposal was disputed in newspapers, radio programs, the city council and government meetings, and in several petitions of political and cultural groups. However, the monument, which followed the original proposal, was unveiled by the sitting President Martti Ahtisaari in 1994 (Fig. 1). The monument is a composition of two parts in which a rectangular beam, executed in bronze, changes its direction in different angles. The competition for the Relander monument was launched while the Ryti debate was still active. The winner of the competition was published in 1993. Matti Peltokangas's winning proposal, titled *From bottom to top, from inside to outside*, included four massive cubes, whose surfaces were enlivened by straight tracks which diagonally cut the surfaces of the cubes. The cubes were installed in composition that left a cross-shaped space between them. Like the Ryti monument project, the Peltokangas' proposal was actively discussed in the media. Finally, the 2-meter high monument made of granite was unveiled by the sitting President in 1996 in Helsinki in the same park the Ryti monument is located (Fig. 2). Both monuments include the name and the years of birth and death of the commemorated president attached either on the monument itself (Ryti) or on the base of the monument (Relander).

As the debates on the Ryti and Relander monuments indicate, geometric abstraction causes various challenges to the reception of monument art. Particularly, the combination of commemorative and honoring function of monuments and aesthetic ideas based on constructivism and concretism can be considered as profoundly problematic. Although the ideas of reference, allusion, representation or symbolic meaning contradict the principles of constructivist and concretist aesthetics, the receivers often aspire to solve the problem of representation in the reception of geometric abstraction through them. In different reception discourses the problem of representation is approached and explained in different ways. These ways can be analyzed the help of semiotic concepts: metonym, metaphor and symbol, and seeing them not as features in the work of art, but as features of the reception (Palin 2004, p. 67; Lähdesmäki 2007, p. 103). In this sense the reception can be defined as a metonymic interpretation, metaphoric interpretation or symbolic interpretation. Different reception discourses include particular modes of interpretation on both the executed monument and the ideas of an ideal monument; what a proper monument should be like and how a monument should be received (Lähdesmäki 2007, pp. 136–137).

Fig. 1 Risto Ryti monument *The Years of Responsibility*, Veikko Myller, 1994 Helsinki, bronze. Photo: Tuuli Lähdesmäki

Metonymic Interpretation

The most polemic arguments in the Ryti and Relander debates were expressed by people who wanted the monuments to depict the appearance of the presidents. Because this kind of iconic interpretation was impossible in the case of the Ryti and Relander monuments, those receivers tried to interpret the monuments as figurative representations of some objects in reality. In this discourse the reference to the presidents was produced by interpreting the monuments as metonyms, which offered a concrete link between the geometric form and the commemorated person. According to this discourse, the presidents should have been depicted through

Fig 2 Lauri Kristian Relander Monument *From bottom to top, from inside to outside*, Matti Peltokangas, 1996 Helsinki, granite. Photo: Tuuli Lähdesmäki

representations of objects which would have produced an understandable material link to the president, his deeds or phases of life. These objects were given a function of representing the missing person in the monument. When monuments are received as metonyms, they are expected to function as a kind of a signpost which directs the approach or angle to the commemorated person or event rather than exposing the person or event as such (Ankersmit 1999, p. 95).

In the metonymic interpretation the Ryti monument was perceived as representing different constructions related to the World War II. The geometric form of the monument offered various possibilities to link concrete objects to Ryti, whose presidency in Finland from 1940 to 1944 was tied to the war events and sceneries. The form of the monument was interpreted for example as "bombed railway tracks" (Olin 1991, Translation TL) or "tank barriers in the fields of Luumäki, which we dug in a rush" (Hyvönen 1991, Translation TL). Respectively, the Relander monument was interpreted representing various objects in reality, as the following quotation from a letter to the editor indicates.

> Again the Finnish nation has to marvel at a monument—the statue of Relander. The jury has worked hard and chosen to the first place a stack of four stone cubes. Are they dice, which depict the surprising election of Relander as a black horse? What do the tracks on the surface of the stone depict? Are they imprints on the suitcases of a man, who travelled a lot? (Pseudonym Tarkkailija Pispalasta 1993, Translation TL)

In the text, the writer is offering interpretations in which the president is referred through objects which have a connection to him or which can be seen as parts of events related to him. Relander, whose presidency took place from 1925 to 1931, is in Finnish popular history called Travelling-Lasse, which refers to his way of doing foreign policy by travelling the neighbouring countries much more than his predecessor. Relander was chosen as the president as a so-called black horse due to a game of party politics: his name came up just in the end of the election in the parliament and surprised many.

In the discourse stressing metonymic interpretation of monuments, the proper and the most unambiguous objects through which the monument should refer to the person or event in question are often disputed. Some objects are often seen as natural and unquestionable metonymic signs. The formation of these kinds of 'automatic' metonyms includes the presence of power: some meanings of the persons, their deeds or some areas in history in general are seen more natural or focal than others. 'Clear' or 'unambiguous' metonymic objects are produced in history writing and media through repeating certain depictions and images of persons. Recurrent images and depictions naturalize metonyms.

The metonymic interpretation does not transfer the meanings from one level to another but sees the signs and their referents on the same conceptual plane (Fiske 1990, pp. 96–97). Metonymic interpretations operate in the framework of one conceptual level. Therefore, metonyms are less abstract and easier to explain than metaphors or symbols (Palin 2004, p. 44). Due to this feature, the metonymic interpretation was often used in the monument debates to bring to the fore sarcastic or humoristic interpretations. In the causeries and caricatures, monuments were presented as objects whose reference to the presidents produced negative or hilarious connotations.

Metaphoric Interpretation

In the reception discourse, which stressed the artistic expression and creativity in the artistic work, receivers often interpreted the monuments as metaphoric signs. Metaphoric interpretation demands association and seeking similarities between different conceptual levels. Thus, metaphoric interpretation appeals to imagination (Fiske 1990, pp. 97–98). Through metaphors it is possible to see some sense and logic in works of art, which would make no sense, were they interpreted literally (Ricoeur 1976, p. 51). In metaphoric interpretation the representation of the monuments was approached through parallels and analogies. Geometric forms were not seen as representing any objects in reality, but they were perceived as associative expressions. Emotions, atmospheres, features and thoughts associated with the form were perceived as abstractions related to the commemorated person, his deeds and decades of presidency, and their significance. Metaphoric meanings in the interpretation of monuments were often attached to abstract qualities, values and virtues. In this discourse, abstraction as such was not concerned as a difficult mode of expression, but often even more expressive and revealing than figuration. Paul Ricoeur has discussed the theory of tension connected to metaphors—metaphors are often understood as solving puzzles rather than just an association based on likeness (Ricoeur 1976). This kind of striving to solve 'the puzzle of geometry' characterized the metaphoric interpretation.

In the metaphoric interpretation the abstract form of the Relander monument was interpreted by using an analogy between different conceptual levels. The cubes were interpreted as associative expressions referring to particular emotions, features and ideas characterising the president, his deeds and presidential term. The rising tracks on the cubes and the open structure of the monument were interpreted as metaphors of Relander's interests and actions in opening Finland towards Europe. In an interview in a newspaper, the artist himself explained: "With lines I wanted to depict the periods of economic booms and depressions that the state underwent. In a way, the state opens from inside to outside, and new things from outside get inside" (Kuisma 1993, Translation TL). The geometric form and the material of the monument—heavy and hard Finnish granite—were interpreted referring to the actions of the president in stabilising the democracy and constitution in Finland, as the following quotation from an art critic indicates.

> I consider the final sculpture composed of four geometric pieces as a modern monument made in the spirit of its subject. Finland's second president Lauri Kristian Relander seems now to be a modern reformist standing out of his time, a reformist, who aimed to open Finland towards Europe. He aimed to stabilise the constitution and democracy. Therefore, strong Finnish granite fits him. (Kivirinta 1996, Translation TL)

In the Ryti monument, the metaphoric interpretations stressed the reference of the angular form and the dark surface of the monument to the difficult years and decisions during Ryti's presidential term, the horrors of war and his tragic destiny. During the war Ryti signed, in a hard-pressed political situation, a personal contract of alliance with Nazi Germany in 1944. After the war, due to the pressure by the Soviet Union, he was sentenced in Finland to 10 years of imprisonment as being 'responsible for the war'. In the following art criticism, the abstract form is explained in reference to the complex decisions related to the war. In addition, the art critic associates the form of the monument proposal to a falling soldier.

> The *Years of responsibility* is art historically linked to a new wave of constructivism and minimalism of the 1960s. However, Myller's work does not emphasize the autonomy of angular forms, but rather an organic growth from the ground. The artist has planned to cast the final work into bronze, even though the style of the proposal belongs obviously to the tradition of steel sculptures.
>
> The name of the work gives some hints for the interpretation and two quick impressions can be suggested. The movement of the beam represents the complexity of wartime decisions. It generates an association on events, which at times dive into the ground to the realm of Demeter, and rise up again. In addition, a falling soldier can be perceived from the Myller's work, if one takes as a mediator for example Henry Moore's vision on this heroic theme that has its roots in the antique. (Valkonen 1991, Translation TL)

As the text indicates, a vitalist discourse of the free abstraction could be applied to the reception of the constructivist monuments. The reference to organic growth intertwines with the vitalist ideas of being ‘honest’ to the materials and traditions and the conventions related to them.

Metonymic and metaphoric interpretations are sometimes intertwined in the same texts. However, in both reception discourses a monument is understood as a mnemotechnic sign, whose function is to remind either on particular or more extensive characteristics and narrations related to the person or event in question.

Symbolic Interpretation

In the previous reception discourses, the geometric form of the monuments was interpreted as a reference to certain objects, qualities or events. However, the monuments could also be interpreted as including no representations of the presidents. In this kind of discourse, a monument was understood as a symbolic sign whose reference to the commemorated person was based on cultural agreement—not on the form of representation (see Fiske 1990, p. 48). Metaphoric and symbolic interpretations differ according to their relation to the reference—metaphors are based on a transposition to another level of meaning or reality, whereas the connection of symbols with its objects is a matter of arbitrary convention, agreement or rule (Fiske 1990, pp. 48, 96–97). In the symbolic interpretation the monuments were seen as symbolising the presidents and their eras without trying to depict or represent them.

In this discourse, the symbolic function of the monument was based on a convention of dedicating a work of art to a person. Detailed explanations of the representation of form or the symbolic character of the monument were seen as unnecessary or pointless, because they were considered as simplifying the interpretational complexity of a work of art and locking certain interpretations of ambiguous works.

In the discourse characterized by the symbolic interpretation, receivers stressed the reception based in the aesthetic ideas of the constructivist and concretist art. In this kind of reception, the receivers paid attention to the composition of masses, surface structure and its effects, light-shadow variations, and space-mass rhythm in the monuments. Besides the geometric form, the material and its qualities were being taken into account—form and material were considered to be important to fit together. These impressions and features of the form and material were understood as the aesthetic core of the monuments. In addition, the aesthetic experience produced by the form and material was considered as the main meaning of the monuments. Formalistic descriptions were recurring particularly in the texts of art critics and in some interviews of the artists. For example, Myller described his Ryti monument as “a two-piece installation, in which he had looked for tension between the parts” (Möttölä 1999, Translation TL). Even though in some interviews Peltokangas had mentioned several metaphoric interpretations of his work, in some other interviews he used a formalistic approach. “I aim to express the movement inside the masses", he described his proposal (Kivirinta 1993, Translation TL).

In the debates, art critics and artists considered monuments first of all as works of art and aesthetic objects, while many of the non-expert receivers stressed the commemorative and honorary function as the main motive for erecting monuments. Views of the art critics and artists reflected their competence and position of expertise in the field of art—phenomena, which were considered as belonging to the field of art, were discussed from the point of view of constructivist, concretist, or vitalist aesthetics.

Conclusions: Social Dimension of Interpretation

In Pierre Bourdieu's art sociological theories, art and culture, as well as other social activities, are basically seen as a struggle for symbolic power and 'capital'—the right and the authority to define 'proper' and 'true' meanings and the ability to distinguish 'good' and 'bad' art (Bourdieu 1984; Bourdieu and Darbel 1991). Bourdieu's sociological field theory and theory about the hierarchies of tastes partly explain the formation of conflicts in the reception of abstract monuments. Monuments can be approached and interpreted through the semiosis of different fields of society, such as art, politics and memory, but the existence of other fields (other than the interpreter's own) is not always recognised or accepted. In Bourdieu's theory the art field forms a hierarchical structure, in which only some tastes and notions of art are appreciated and taken seriously. The struggle over 'good' and 'proper' tastes and notions of art also characterised the monument discussions.

Formalist reception and symbolic interpretation of monuments follow principles, which Bourdieu has characterised as legitimate taste. According to it, works of art are not appreciated because of their representational content but because of their form, which is distinguished from the interest in the subject of the work (Bourdieu 1984, p. 44). This kind of approach is the principle point of departure in ideas of modernist art—form becomes the main content of art, and the mode of expression is merged with the expressed subject. In the monument debates, the art critics and artists used both the constructivist, concretist and vitalist points of view in their comments and articles. These discourses of modernism can be perceived as intertwined and mixed in texts, which combined different modernist perspectives at the same time. The points of view appeared unified particularly when they were confronted with points of view of non-expert receivers. Their critical comments towards abstract monuments in general influenced the texts produced inside the art field—rather than arguing with other art experts, criticism was directed at the arguments of the outsiders. Will to defend the art field, its discourses and its agents against the criticism of the non-expert receivers unified the comments and opinions of the art field (Lähdesmäki 2007, pp. 255–256).

As the monument cases of this chapter indicate, reception and interpretation of geometric abstraction is profoundly complex and includes several discursive and social dimensions. Interpretations of constructivist or concretist monuments are made from different stand points, which reflect the receivers' needs or expectations of representative and illustrative images. In all discussed modes of reception—metonymic, metaphoric and symbolic—interpretations were based on certain traditions or conventions of sculpture. Metonymic interpretations reflected the ideas of traditional figurative sculpture and its ways to represent reality. Metaphoric interpretation leaned on a concept of sculpture in which perceiving the representation requires imagination and a subjective interpretation process—and thus sensitivity for artistic expression. Symbolic interpretation obeyed the aesthetic ideas of constructivism or concretism and followed the modernist discourse, which still partly characterises the discussions of art in the contemporary art field.

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