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Geodetic Observables and Their Mathematical Treatment in Multiscale Framework

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Abstract. For the determination of the Earth's gravitational field various types of observations are available nowadays, e.g., from terrestrial gravimetry, airborne gravimetry, satellite-to-satellite tracking, satellite gravity gradiometry, etc. The mathematical relation between these observables on the one hand and the gravitational field and the shape of the Earth on the other hand is called the *integrated concept of physical geodesy*. In this paper, an integrated concept of physical geodesy in terms of harmonic wavelets is presented. Essential tools for approximation are Runge–Walsh type integration formulas relating an integral over an internal sphere to suitable linear combinations of observational functionals, i.e., linear functionals representing the geodetic observables in terms of gravitational quantities on and outside the Earth. A scale discrete version of multiresolution is described for approximating the gravitational potential on and outside the Earth's surface. Furthermore, an exact fully discrete wavelet approximation is developed for the case of bandlimited wavelets. A method for combined global outer harmonic and local harmonic wavelet modeling is proposed corresponding to realistic Earth's models.

Keywords. Integrated wavelet concept, scaling function, Runge–Walsh approximation, geodetic observables, Meissl schemata.

1. Introduction

Gravity as observed on the Earth's surface is the combined effect of the gravitational mass attraction and the centrifugal force due to the Earth's rotation. Under the assumption that the centrifugal force is explicitly known, the determination of the gravity mainly reduces to getting knowledge of the gravitation. According to the classical Newton Law of Gravitation (1687), knowing the density distribution of a body such as the Earth, the gravitational potential can be computed everywhere in the Euclidean space $\mathbb{R}^3$.

Although Earth's gravitational field modeling is always governed by the same physical laws, it changes its nature when it is seen from different spatial and time scales. To be more concrete, if one looks at gravitational field determination on the basis of an increasing spatial magnification and accuracy, we have to go from something that is suitably characterized by a simple mass point, on astronomical scale, to what is described by a global truncated multipole (i.e., outer harmonic) model, at scales corresponding to satellite altimetry, down to wavelengths of about 100 km. By further zooming in we can reach a spatial resolution of about 1 km showing a very complicated pattern, strongly related to the shape of the Earth and to irregular masses inside the Earth's crust. Simultaneously, the error in the knowledge of the gravitational field models goes from 5 Gal, the flattening effect, down to 10 mGal in a today's global model, down to about 10^{-1} mGal at the regional 1 km resolution or even better. There is also a change of the gravitational field in the time scale depending on the time interval under consideration, for instance, gravitational changes due to geotectonic displacements of masses inside the Earth on very long time scales. It changes because of motions of the rotational axis inside the Earth's body and it shows a periodic change because of the continent and ocean reactions to the torques generated by the moon and the sun. Finally, gravitation shows a change because of human activities, for instance, because of the presence of artificial lakes, height's variations in the water-bearing stratum under cities, etc. It is also worth mentioning that there are certain relations between different scales in the time-like behaviour and in the space-like behaviour of the gravitational field. In any way, it may be assumed for global up to regional modeling purposes that the time-like variations of the field are either well predictable (like tides etc.) or so slow as to be neglected, e.g., on the scale of a decade, or so small and local as to be beyond the scope of interest. Thus, global gravitational field modeling as scientific issue is by definition based on the assumption of a stationary gravitational field with a spatial resolution ranging from a worldwide scale down to about 1 km and from about 1000 Gal of the full field down to, at least, 10^{-1} mGal, or even better in some regional areas.

What we would like to present in this contribution are mathematical structures in straightforward continuation to the monograph [19] by which the gravitational part of the gravity field can be approximated progressively better and better, reflecting an increasing flow of observations of terrestrial, airborne and/or satellite type, e.g., terrestrial gravimetry, airborne gravimetry, satellite altimetry, satellite-to-satellite tracking (SST), satellite gravity gradiometry (SGG), etc. More precisely, we shall try to outline the canonical bridge of gravitational field determination from the well-established global outer harmonic approximation corresponding to a spherical Earth to modern multiscale methods involving the actual geometry of the Earth's surface (thereby neglecting, e.g., the small effect of the atmosphere in the outer space).

The so-called disturbing potential is probably the most crucial quantity in gravity field modeling. The disturbing potential is a scalar quantity which is obtained as the difference between the gravity potential of the Earth and the normal gravity potential of a reference surface, usually an ellipsoid. The deviations of the gravity potential from the normal potential are relatively small. Note that both the gravity potential and the normal gravity potential contain the same centrifugal potential. Thus, the disturbing potential is harmonic in the outer space.

At this stage some remarks should be made in order to clarify our approach in more detail:

1. The mathematical connection between the observables, the gravity field and the shape of the Earth is called the *integrated concept of physical geodesy*.
2. The foundation of the integrated geodesy approach is the fact that every geodetic measurement is a functional which may assumed to be suitably linearizable by introducing, e.g., normal potentials associated to a reference surface such as an ellipsoid. In other words, the relation between the object function, i.e., the geopotential and the data, may be supposed to be linear.
3. More and more measurements refer to satellites and cannot be modeled as functionals of the gravitational potential on the boundary. Although these observations show a denser observational distribution, they are much more difficult to handle, since they show an exponentially spectral smoothing while moving to the outer space. As a consequence, essential knowledge of the gravitational potential should be based on ground observations, but gravitational field modeling cannot be treated only within a boundary-value formulation because of spaceborne observations. This fact is the reason why we do not speak of the "geodetic boundary-value problem (GBVP)" but of the "integrated concept".
4. Concerning the layout of this contribution a particular interest is focussed on the satellite methods SST und SGG, which are introduced within the framework of pseudodifferential operators assuming non-spherical (orbital) geometry.
5. An important feature of our contribution are the so-called Meissl schemata which are graphical illustrations for the conversion of data both on different heights (terrestrial level, satellite orbit) and of different degrees of derivative of the gravitational potential. The comparison between data on the (spherical) Earth's surface and the orbital sphere was primarily carried out by Meissl (1971) and has been transformed by Rummel [60, 61, 63] and by Rummel and van Gelderen [64, 65] into a more general framework concerning relations between different gravity quantities in the framework of outer harmonics. One of our objectives is the extension of the Meissl schemata to the concept of multiscale decomposition of scalar functions, vector, and tensor fields. In principle, we follow the ideas of mathematical classification first presented in [19, 29, 32–34] for the scalar case and extended in the Ph.D.-thesis [58] to the vector and tensor approach.

2. Current state of gravity field determination

Positioning systems are ideally located as far as possible from the Earth, whereas gravity field sensors are ideally located as close as possible to the Earth. Following these basic principles, various positioning and gravity field determination techniques have been designed. Sensors may be sensitive to local or global features of the gravity field. Considering the spatial location of the data, we may distinguish between terrestrial (surface), airborne, and spaceborne methods. Regarding the data type we have various measurement principles of the gravity field (see, for example, [9–11, 51] and the references therein for more details) leading to different types of data.

2.1. Important geodetic observables

(a) *Gravity Measurements:* The force of gravity provides a directional structure to the space above the Earth's surface. It is tangential to the vertical plumb lines and perpendicular to all (level) equipotential surfaces. Any water surface at rest is part of a level surface. Level (equipotential) surfaces are ideal reference surfaces, for example, for heights. The geoid is defined as that level surface of the gravity field which best fits the mean sea level. Gravity vectors can be measured by absolute or relative gravimeters. The highest available accuracy relative gravity measurements are conducted at the Earth's surface. Measurements on ships and in aircrafts deliver reasonably good data only after the removal of inertial noise. Gravity data are converted into gravity anomalies by subtracting a corresponding reference potential derived from a simple gravity field model associated to an, e.g., ellipsoidal surface (see also Appendix A). Gravity anomalies are furthermore converted into mean gravity anomalies by a proper averaging process over well defined areas. It should be pointed out that the distribution of Earth's gravity data on a global scale is far from being homogeneous with large gaps, in particular over oceans but also over land. In addition, the quality of the data is very distinct. Thus, terrestrial gravity data coverage now and in the foreseeable future is far from being satisfactory for the global purpose of geoidal determination (at an accuracy of essentially less than one centimeter).

(b) *Vertical Deflections.* The direction of the gravity vector can be obtained by astronomical positioning. Measurements are only possible on the Earth's surface. Observations of the gravity vector are converted into so-called vertical deflections by subtracting a corresponding reference direction derived from a simple gravity field model associated to an ellipsoidal surface. Vertical deflections are tangential fields of the anomalous potential in a spherical Earth's model. Due to the high measurement effort required to acquire these types of data compared to a gravity measurement, the data density of vertical deflections is much less than that of gravity anomalies. Gravitational field determination based on the observation of vertical deflections and combined with gravity is feasible in smaller areas with good data coverage.

(c) *Satellite Radar Altimetry.* Satellite radar altimetry has demonstrated an impressive capability of mapping the surface of the oceans. The ocean surface is a good approximation of an equipotential surface and, as such, its offset from the geoid at mean sea level (mean in terms of time) is called sea surface topography. This offset, which can be as large as two meters, reflects many effects including the variables salinity, ocean temperature, ocean currents, variable atmospheric conditions such as wind and air pressure perturbations, tides, etc. Since the sea surface topography refers to the geoid, the precise and sufficiently detailed knowledge of the geoid is mandatory.

(d) *Global Gravitational Field Models.* On the basis of all satellite data, collected over the last decades in orbits at different altitudes and inclinations, only long wavelength components of the global gravity field can be recovered. There are two reasons for this fact: First, an orbit as such is rather insensitive to local features of the gravitational field, and this insensitivity increases with increasing orbit altitude. Second, the satellites which can and are being used are flying at altitudes which are too high for a better purpose such as local gravimetry. Therefore, satellite-only global gravity field models are reliable to a moderate maximum degree expressed in a potential representation in terms of spherical harmonics. Considering the shortcomings of satellite-only gravity field models and of the information content of surface data, several institutions have been working for many years on the combination of both data sets. This work in geodesy has resulted in various gravitational field models in terms of spherical harmonics. All gravity field data available worldwide have entered into the production of this model. Therefore, such models represent the latest state of the art in global gravitational field knowledge.

2.2. Satellite concepts and airborne data

The three satellite concepts which are of importance for gravity field determination are satellite-to-satellite tracking in the high-low mode (SST hi-lo), satellite-to-satellite tracking in the low-low mode (SST lo-lo), and satellite gravity gradiometry (SGG). Common to all three concepts is that the determination of the Earth's gravitational field is based on the measurement of the relative motion (in the Earth's gravity field) of test masses.

1. *Satellite-to-Satellite Tracking.* In the case of SST hi-lo the low flying test mass is a low earth orbiter (LEO) and the high flying test masses are the satellites of the GNSS-system (i.e., GPS, GLONASS, Galileo, and Beidou). As, for example, the GNSS-receiver mounted on the LEO always "contacts" four or even more of the GNSS satellites the relative motion of the LEO can be monitored three-dimensionally, i.e., in all three coordinate directions. The lower the orbit of the LEO the higher is its sensitivity with respect to the spatial variations of the gravitational forces but to skin forces as well (atmospheric drag, solar radiation, albedo, etc.). The latter have either to be compensated for by a drag-free mechanism or be measured by a three axis accelerometer.

Also the high orbiters, the GNSS satellites, are affected by non-gravitational forces. However the latter can be modeled quite well. They affect mainly the very long spatial scales, and to a large extent their effect averages out. In addition, the ephemerides of the GNSS satellites are determined very accurately by the large network of ground stations. In the case of SST lo-lo the relative motion between two LEOs, chasing each other, is measured with highest precision. The quantity of interest is the relative motion of the centre of mass of the two satellites. Again, the effect of non-gravitational forces on the two spacecraft either has to be compensated actively or be measured.

2. *Satellite Gravity Gradiometry.* The satellite gravity gradiometry technique is the measurement of the relative acceleration, not between free falling test masses like satellites, but of test masses at different locations inside one satellite. Each test mass is enclosed in a housing and kept levitated (floating, without ever touching the walls) by a capacitive or inductive feedback mechanism. The difference in feedback signals between two test masses is proportional to their relative acceleration and exerted purely by the differential gravitational field. Non-gravitational acceleration of the spacecraft affects all accelerometers inside the satellite in the same manner and so ideally drops out during differencing. The rotational motion of the satellite affects the measured differences. However, the rotational signal (angular velocities and accelerations) can be separated from the gravitational signal, if acceleration differences are taken in all possible (spatial) combinations (= full tensor gradiometer). In order to achieve a higher sensitity, an orbit as low as possible is of great importance.

In a unified view on spaceborne missions (see, e.g., [9–11, 51]), one can argue that the basic observable in all three cases is gravitational acceleration. In the case of SST hi-lo, with the motion of the high orbiting GNSS satellites assumed to be perfectly known, this corresponds to an in situ 3-D acceleration measurement in the LEO. For SST lo-lo it is the measurement of acceleration difference over the intersatellite distance and in the line-of-sight (LOS) of the LEOs. Finally, in the case of gradiometry, it is the measurement of acceleration differences in 3-D over the tiny baseline of the gradiometer. In short we are confronted with the following situation:

SST hi-lo:	3-D acceleration	=	gravitational gradient,
SST lo-lo:	acceleration difference	=	difference in gradient,
SGG:	differential	=	gradient of gradient ("tensor").

As explained in more detail by W. Freeden [19], in mathematical sense, it is a transition from the first derivative of the gravitational potential via a difference in the first derivative to the second derivative. The guiding parameter that determines sensitivity with respect to the spatial scales of the Earth's gravitational potential is the distance between the test masses, being almost infinite for SST hi-lo and almost zero for gradiometry.

3. *Airborne Gravimetry.* Airborne gravimetry is a highly sensitive detection method of the gravitational potential of the Earth by a gravity accelerometer mostly for regional and/or local purposes. Proposals to implement airborne gravimetry go back to the late fifties of the last century, and first flight experiments were already done in the early sixties. A major obstacle of such techniques at that time was the inaccuracy of navigational information (e.g., velocity and acceleration of the space vehicle) which is needed to obtain the desired precision. Although at an appropriate level of accuracy airborne gravimetry is vastly superior in economy and efficiency to pointwise terrestrial methods, there were serious doubts in the seventies and eighties of ever achieving useful results. In the early nineties, however, great advances in GNSS technology opened new ways to resolve the navigational problems. More explicitly, altitude, position, and velocity of the airborne gravity system become sufficiently computable from the inertial measurements updated by GNSS carrier phase and Doppler observations. Vehicle accelerations are derivable from GNSS data only, so that in a third step the airborne gravity disturbance is determinable from the difference between the force vector and the GNSS-derived acceleration vector. Nowadays, some industrial companies are perfecting their system concepts by paying careful attention to the operational conditions under which an airborne gravimeter works best, also for progress in gravimetric exploration.

All in all, over the last decades, geoscientists have realized the great complexity of the Earth and its environment. In particular, the knowledge of the gravity potential and its level (equipotential) surfaces have become an important issue. It was realized that dedicated highly accurate gravity field sensors, when operating in an isolated manner, have their shortcomings, and combining data from different sensors is therefore the way forward. At this stage of development, the global determination of the Earth's gravitational field is a mathematical challenge which should include the numerical progress obtainable by modern multiscale approximation.

2.3. Gravity field applications

The knowledge of the gravitational field of the Earth is of great importance for many applications from which we only mention some significant examples (cf. [19, 61]):

(i) *Geodesy and Civil Engineering.* Accurate heights are needed for civil constructions, mapping, etc. They are obtained by leveling, a very time consuming and expensive procedure. Nowadays, geometric heights can be obtained fast and efficiently from space positioning (GNSS). The geometric heights are convertible to leveled heights by subtracting the precise geoid, which is achieved by a high resolution gravitational potential. To be more specific, in those areas where good gravity information is available already, the future data information will eliminate all medium and long wavelength distortions in unsurveyed areas. For example, GNSS (GPS, GLONASS, Galileo, or Beidou) together with today's satellite missions provide high quality height information at global scale.

(ii) *Satellite Orbits.* For any positioning from space, the uncertainty in the orbit of the spacecraft is the limiting factor. The spaceborne techniques eliminate basically all gravitational uncertainties in satellite orbits.

(iii) *Solid Earth Physics.* The gravity anomaly field derivable from future satellite observations has its origin mainly in mass inhomogeneities of the continental and oceanic lithosphere. Together with height information and regional tomography, a much deeper understanding of tectonic processes is obtainable.

(iv) *Physical Oceanography.* Altimeter satellites in combination with a precise geoid deliver global dynamic ocean topography. From ocean topography, global surface circulation and its variations in time can be computed resulting in efficient ocean modeling. Circulation allows the determination of transport processes of, e.g., polluted material. Moreover, ocean modeling is an important indicator of climate change.

(v) *Earth System.* There is a growing awareness of global environmental problems (for example, the CO_2-question, the rapid decrease of rain forests, global sea level changes, etc.). What is the role of the airborne methods and satellite missions in this context? They do not tell us the reasons for physical processes, but it is essential to bring the phenomena into one system (e.g., to make sea level records comparable in different parts of the world). In other words, equipotential surfaces such as the geoid may be viewed as an almost static reference for many rapidly changing processes and at the same time as a "frozen picture" of tectonic processes that evolve in geological time spans.

(vi) *Exploration Geophysics and Prospecting.* Knowledge of local geologic structures can easily be gained by means of terrestrial and airborne data so gravity prospecting can be done over land or sea areas using different techniques and equipment. Terrestrial gravimetry was first applied to prospect for salt domes (e.g., in the Gulf of Mexico), and later for looking for anticlines in continental areas. In future, embedded in (regional) airborne and (global) spaceborne gravity information such as satellite-to-satellite tracking (SST) and/or satellite gravity gradiometry (SGG) (see, e.g., [19, 27, 32] and the references therein), new promising components in gravimetrically oriented modeling can be expected, for example, based on multiscale modeling providing reconstruction and decomposition of geological signatures.

2.4. Principles of multiscale approximation

Spaceborne observation combined with terrestrial and airborne activities provide huge datasets of the order of millions of data (see [9–11, 51, 63]). Standard mathematical theory and numerical methods are not at all adequate for the solution of data systems with such a structure, because these methods are not adapted to the specific properties of the data set. They quickly reach their capacity limit even on very powerful computers. An adequate reconstruction of the gravitational field from the huge and heterogeneous data material requires a careful multiscale analysis of the gravitational potential, fast solution techniques, and a proper stabilization of the inverse character of satellite problems by regularization. In order to achieve these objectives various strategies and structures must be introduced reflecting the different aspects of geopotential determination. While global long-wavelength modeling can be adequately done by use of spherical harmonic expansions, it becomes more and more obvious that harmonic splines and/or wavelets are most likely the candidates for medium and short-wavelength approximation. The concept of harmonic wavelets, however, demands its own nature which only on exploration areas of small size may be developed to some extend from the theory in Euclidean spaces. Fundamental results known from the Euclidean wavelet approach have to be recovered. Nevertheless, the stage is set for working out and improving essential ideas and results involving harmonic wavelets. Why are harmonic wavelets important in future gravitational potential determination? Following [19], the answer is summarized in the following sentence:

Harmonic wavelets are "building blocks" that enable fast decorrelation of gravitational data. Thus three features are incorporated in this way of thinking about georelevant harmonic wavelets, namely basis property, decorrelation, and efficient algorithms. These aspects should be discussed in more detail:

(i) *Basis property*
Wavelets are building blocks for the approximation of arbitrary functions (signals). In mathematical understanding this formulation expresses that the set of wavelets forms a "frame" (see, e.g., [6] for details in classical one-dimensional theory).

(ii) *Decorrelation*
Wavelets possess the ability to decorrelate the signal. This means, that the representation of the signal via wavelet coefficients occurs in a "more constituting" form as in the original form reflecting a certain amount of space and frequency (more accurately, momentum) information. The decorrelation enables the extraction of specific information contained in a signal through a particular number of coefficients. Signals usually show a correlation in the frequency (momentum) domain as well as in the space domain. Obviously, since data points in a local neighborhood are stronger correlated as those data points far-off from each other, signal characteristics often appear in certain frequency bands. In order to analyze and reconstruct such signals, we need "auxiliary functions" providing localized information in the space as well as in the frequency domain. In applications, different approaches have been realized in the field of signal analysis before the occurrence of wavelets: on the one hand, the Fourier theory allows a trendsetting bandlimited decomposition, on the other hand, the Haar theory offers short-wavelets spacelimited decomposition. The (Heisenberg) uncertainty principle (see, e.g., [21]) tells us that a simultaneous sharp localization in frequency as well as space domain is exclusive. Even more within a "zooming-in process", the amount of

frequency as well as space contribution can be specified in quantitative way. A so-called scaling function forms a compromise in which a certain balanced amount of frequency and space localization in the sense of the uncertainty principle is realized. In consequence, each scaling function depends on two variables, namely a "shifting" and a scaling parameter, which control the amount of the space localization to be available at the price of the frequency localization, and vice versa. Associated to each scaling function is a wavelet function, which here is simply understood to be the difference of two successive scaling functions. All in all, filtering (convolution) with a scaling function takes the part of a lowpass filter, while convolution with the corresponding wavelet function provides a bandpass filtering. A multiscale approximation of a signal is the successive execution of an efficient evaluation process by use of scaling and wavelet functions which show more and more space localization at the cost of frequency localization. The wavelet transform within a multiscale approximation lays the foundation for the decorrelation of a signal.

(iii) *Efficient algorithms*
Wavelet transformation provides efficient algorithms because of the space-localizing character. The successive decomposition of the signal by use of wavelets at different scales offers the advantage for efficient and economic numerical calculation (e.g., tree algorithm). The detail information stored in the wavelet coefficients leads to a reconstruction from a rough to a fine resolution and to a decomposition from fine to rough resolution in form of tree algorithms. In particular, the decomposition algorithm is an excellent tool for the post-processing of a signal into "constituting blocks" by decorrelation, e.g., the specification of signature bands corresponding to certain geological formations.

3. Geodetically relevant Sobolev spaces

We start our mathematical foundation of Meissl schemata by introducing some basic information related to the theory of geodetic observables within the framework of Sobolev spaces. We adopt the following general scheme of notation which is non-standard in geodesy, but extremely helpful in establishing Meissl schemata especially for the vectorial and tensorial framework. Capital letters (F, G, $\ldots$) are used for scalar functions, small letters (f, g, $\ldots$) represent vector fields and small boldface letters ($\mathbf{f}$, $\mathbf{g}$, $\ldots$) represent tensor fields of second rank. As usual, a scalar function having k continuous derivatives is said to be of class $C^{(k)}$ whereas L^2 denotes the Hilbert space of square integrable functions. A vector field having k continuous derivatives is said to be of class $c^{(k)}$ and l^2 denotes the Hilbert space of square-integrable vector fields. Finally, the space of all tensor fields having k continuous derivatives is denoted by $\mathbf{c}^{(k)}$ and $\mathbf{l}^2$ denotes the Hilbert space of all square-integrable tensor fields.

$\Sigma \subset \mathbb{R}^3$ is called a regular surface if Σ is the boundary of a regular region $\Sigma^{\text{int}} \subset \mathbb{R}^3$, i.e., $\Sigma = \partial\Sigma^{\text{int}}$, with the following properties (cf. [20]):

(i) Σ constitutes an orientable piecewise smooth Lipschitzian manifold of dimension 2.
(ii) The origin is contained in Σ^{int}.
(iii) Σ divides $\mathbb{R}^3$ into the "inner space" Σ^{int} and the "outer space" $\Sigma^{\text{ext}} = \mathbb{R}^3 \backslash \overline{\Sigma^{\text{int}}}$, $\overline{\Sigma^{\text{int}}} = \Sigma^{\text{int}} \cup \Sigma$.

Georelevant regular surfaces Σ are, for example, the sphere, the ellipsoid, the telluroid, the geoid, and the regular Earth's surface.

The geometric concept to be discussed in our approach is as follows (see Figure 3.1): Σ denotes the Earth's surface which we assume to be known and

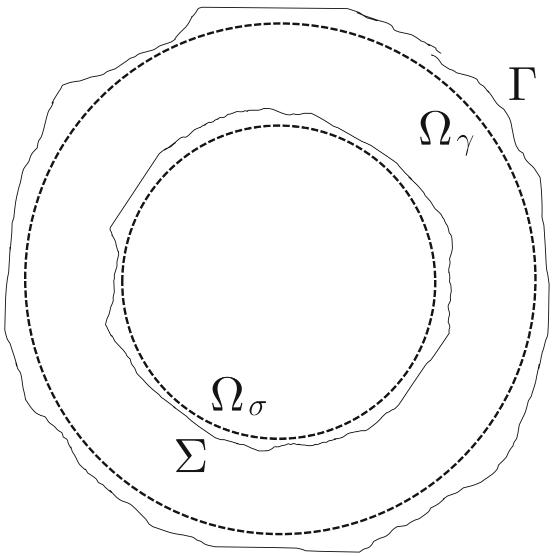


FIGURE 3.1. Geometric concept characterizing the surface of the Earth Σ and the orbit of a satellite Γ.

regular. Γ is the orbit of a satellite which is not necessarily a closed surface. σ is the radius of a so-called Runge (in the jargon of geodesy, Bjerhammar) sphere inside the Earth, that is $\sigma < \alpha = \inf_{x \in \Sigma} |x|$. The value γ is a lower bound of the lowest possible altitude of the satellite, i.e., $\gamma < \inf_{x \in \Gamma} |x|$. $\Omega_\sigma^{\text{ext}} = \{x \in \mathbb{R}^3 \ : \ |x| > \sigma\}$ denotes the outer space of the sphere Ω_σ with radius σ around the origin 0, whereas Σ^{ext} denotes the outer space of the (actual) Earth.

Let $V : \Omega_\sigma^{\text{ext}} \to \mathbb{R}$, $v : \Omega_\sigma^{\text{ext}} \to \mathbb{R}^3$, and $\mathbf{v} : \Omega_\sigma^{\text{ext}} \to \mathbb{R}^3 \otimes \mathbb{R}^3$, respectively, be a scalar, vector, and tensor field on the set $\Omega_\sigma^{\text{ext}}$. We say that V, v, $\mathbf{v}$, respectively, are *harmonic* on $\Omega_\sigma^{\text{ext}}$ if V, v, $\mathbf{v}$ are twice continuously differentiable on $\Omega_\sigma^{\text{ext}}$ and $\Delta V = 0$, $\Delta v = 0$, $\Delta \mathbf{v} = 0$ on $\Omega_\sigma^{\text{ext}}$.

Without proof we mention some well-known theorems concerning harmonic fields on $\Omega_\sigma^{\mathrm{ext}}$ (for the proofs see, for example, [20, 38, 47]):

(1) Every harmonic field in $\Omega_\sigma^{\mathrm{ext}}$ is analytic in $\Omega_\sigma^{\mathrm{ext}}$, i.e., every harmonic field is determined by its local properties.

(2) *Harnack's convergence theorem:* Let $V_\delta : \Omega_\sigma^{\mathrm{ext}} \to \mathbb{R}$, $v_\delta : \Omega_\sigma^{\mathrm{ext}} \to \mathbb{R}^3$, and $\mathbf{v}_\delta : \Omega_\sigma^{\mathrm{ext}} \to \mathbb{R}^3 \otimes \mathbb{R}^3$, respectively, be harmonic on $\Omega_\sigma^{\mathrm{ext}}$ for each value δ ($0 < \delta < \delta_0$), and regular at infinity. Moreover, let

$$V_\delta \to V, \quad \delta \to 0,\ \delta > 0, \tag{3.1}$$

$$v_\delta \to v, \quad \delta \to 0,\ \delta > 0, \tag{3.2}$$

$$\mathbf{v}_\delta \to \mathbf{v}, \quad \delta \to 0,\ \delta > 0, \tag{3.3}$$

uniformly on each subset K of $\Omega_\sigma^{\mathrm{ext}}$ with $\mathrm{dist}(\overline{K}, \partial\Omega_\sigma^{\mathrm{ext}}) > 0$. Then $V : \Omega_\sigma^{\mathrm{ext}} \to \mathbb{R}$, $v : \Omega_\sigma^{\mathrm{ext}} \to \mathbb{R}^3$, and $\mathbf{v} : \Omega_\sigma^{\mathrm{ext}} \to \mathbb{R}^3 \otimes \mathbb{R}^3$, respectively, is harmonic on $\Omega_\sigma^{\mathrm{ext}}$ and regular at infinity.

(3) Let $V : \overline{\Omega_\sigma^{\mathrm{ext}}} \to \mathbb{R}$ be twice continuously differentiable on $\Omega_\sigma^{\mathrm{ext}}$ and continuous on $\overline{\Omega_\sigma^{\mathrm{ext}}}$, i.e., $V \in C^{(0)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \cap C^{(2)}(\Omega_\sigma^{\mathrm{ext}})$, harmonic on $\Omega_\sigma^{\mathrm{ext}}$, and regular at infinity. Then the *maximum/minimum principle* tells us that

$$\sup_{x \in \overline{\Omega_\sigma^{\mathrm{ext}}}} |V(x)| \leq \sup_{x \in \Omega_\sigma} |V(x)| \,. \tag{3.4}$$

(4) There is a so-called fundamental solution (singularity function) $S : x \mapsto |x-y|^{-1}$, $x \neq y$, with respect to the Laplace operator Δ such that *the fundamental theorem of potential theory*

$$\int_{\partial\Omega_\sigma^{\mathrm{ext}}} \left(\frac{1}{|x-y|} \frac{\partial V}{\partial \nu}(y) - V(y) \frac{\partial}{\partial \nu_y} \frac{1}{|x-y|} \right) d\omega(y) = \begin{cases} -4\pi V(x), & x \in \Omega_\sigma^{\mathrm{ext}}, \\ -2\pi V(x), & x \in \partial\Omega_\sigma^{\mathrm{ext}}, \\ 0, & x \notin \overline{\Omega_\sigma^{\mathrm{ext}}}, \end{cases}$$

holds true.

3.1. Scalar outer harmonic and Sobolev theory

As already explained, we let $\Omega_\sigma \subset \mathbb{R}^3$ be the sphere around the origin with radius $\sigma > 0$, $\Omega_\sigma^{\mathrm{int}}$ is the inner space of Ω_σ, and $\Omega_\sigma^{\mathrm{ext}}$ is the outer space. We let $\Omega = \Omega_1$. By virtue of the isomorphism $\Omega \ni \xi \mapsto \sigma\xi \in \Omega_\sigma$ we assume functions $F : \Omega_\sigma \to \mathbb{R}$ to be defined on Ω. It is clear that the function spaces defined on Ω admit their natural generalizations as spaces of functions defined on Ω_σ. We have, for example, $C^{(\infty)}(\Omega_\sigma)$, $L^p(\Omega_\sigma)$, etc.

Let $\{Y_{n,m}\}_{n \in \mathbb{N}_0; m=1,\ldots,2n+1}$ be an L^2-orthonormal system of (surface) spherical harmonics. Obviously, such an $L^2(\Omega)$-orthonormal system of spherical harmonics forms an orthogonal system on Ω_σ (with respect to $(\cdot,\cdot)_{L^2(\Omega_\sigma)}$). More explicitly, we have

$$(Y_{n,k}, Y_{p,q})_{L^2(\Omega_\sigma)} = \int_{\Omega_\sigma} Y_{n,k}\left(\frac{x}{|x|}\right) Y_{p,q}\left(\frac{x}{|x|}\right) d\omega(x) = \sigma^2 \delta_{n,p}\delta_{k,q}, \tag{3.5}$$

where $\delta_{n,p}$ is the Kronecker symbol and $d\omega$ is the surface element. With the relationship $\xi \leftrightarrow \sigma\xi$, the *surface gradient* $\nabla^{*;\sigma}$ and the *Beltrami operator* $\Delta^{*;\sigma}$ on Ω_σ, respectively, have the representation $\nabla^{*;\sigma} = (1/\sigma)\nabla^{*;1} = (1/\sigma)\nabla^*$, $\Delta^{*;\sigma} = (1/\sigma^2)\Delta^{*;1} = (1/\sigma^2)\Delta^*$, where ∇^*, Δ^* are the surface gradient and the Beltrami operator of the unit sphere Ω.

We now introduce the system $\{Y^\sigma_{n,k}\}_{n=0,1,\ldots;\, k=1,\ldots,2n+1}$ by letting

$$Y^\sigma_{n,k}(x) = \frac{1}{\sigma} Y_{n,k}\left(\frac{x}{|x|}\right), \quad x \in \Omega_\sigma. \tag{3.6}$$

Due to (3.5) the system $\{Y^\sigma_{n,k}\}_{n=0,1,\ldots;k=1,\ldots,2n+1}$ is an orthonormal basis in $L^2(\Omega_\sigma)$:

$$L^2(\Omega_\sigma) = \overline{\operatorname{span}_{\substack{n=0,1,\ldots;\\ k=1,\ldots,2n+1}} (Y^\sigma_{n,k})}^{\|\cdot\|_{L^2(\Omega_\sigma)}}. \tag{3.7}$$

The system $\{H_{n,m}(\sigma;\cdot)\}_{n\in\mathbb{N}_0;m=1,\ldots,2n+1}$, of *scalar outer harmonics* defined by

$$H_{n,m}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n+1} Y_{n,m}\left(\frac{x}{|x|}\right), \quad x \in \overline{\Omega^{\text{ext}}_\sigma},$$

satisfies the following properties:

- $H_{n,m}(\sigma;\cdot)$ is of class $C^{(\infty)}(\Omega^{\text{ext}}_\sigma)$,
- $H_{n,m}(\sigma;\cdot)$ is harmonic in $\Omega^{\text{ext}}_\sigma$, i.e., $\Delta_x H_{n,m}(\sigma;x) = 0$ for $x \in \Omega^{\text{ext}}_\sigma$,
- $H_{n,m}$ is regular at infinity, i.e., $|H_{n,m}(\sigma;x)| = \mathcal{O}(|x|^{-1})$, $x| \to \infty$,
- $H_{n,m}(\sigma;\cdot)|_{\Omega_\sigma} = \frac{1}{\sigma} Y_{n,m}$,
- $\int_{\Omega_\sigma} H_{n,m}(\sigma;x)H_{k,l}(\sigma;x)d\omega(x) = \delta_{n,k}\delta_{m,l}$.

As it is well known (cf., e.g., [32, 57]), the *addition theorem of outer harmonics* reads as follows:

$$\sum_{m=1}^{2n+1} H_{n,m}(\sigma;x)H_{n,m}(\sigma;y) = \frac{2n+1}{4\pi\sigma^2}\left(\frac{\sigma^2}{|x|\;|y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right), \tag{3.8}$$

for all $(x,y) \in \overline{\Omega^{\text{ext}}_\sigma} \times \overline{\Omega^{\text{ext}}_\sigma}$ and $n \in \mathbb{N}_0$, where P_n denotes the Legendre polynomial of degree n. $Harm_n(\overline{\Omega^{\text{ext}}_\sigma})$ denotes the space of all outer harmonics of order n, $n \in \mathbb{N}_0$:

$$Harm_n(\overline{\Omega^{\text{ext}}_\sigma}) = \operatorname{span}_{m=1,\ldots,2n+1}(H_{n,m}(\sigma;\cdot)).$$

It is well known that $\dim(Harm_n(\overline{\Omega^{\text{ext}}_\sigma})) = 2n+1$. We let $Harm_{p,\ldots,q}(\overline{\Omega^{\text{ext}}_\sigma})$ be the space of all linear combinations of the functions $H_{n,m}(\sigma;\cdot)$ on $\overline{\Omega^{\text{ext}}_\sigma}$, $n = p,\ldots,q$, $m = 1,\ldots,2n+1$, i.e.,

$$Harm_{p,\ldots,q}(\overline{\Omega^{\text{ext}}_\sigma}) = \bigoplus_{n=p}^{q} Harm_n(\overline{\Omega^{\text{ext}}_\sigma}).$$

The space $Harm_{p,\ldots,q}(\overline{\Omega_\sigma^{\rm ext}})$ has the reproducing kernel $K_{Harm_{p,\ldots,q}(\overline{\Omega_\sigma^{\rm ext}})}(\cdot,\cdot)$ given by

$$\begin{aligned} K_{Harm_{p,\ldots,q}(\overline{\Omega_\sigma^{\rm ext}})}(x,y) &= \sum_{n=p}^{q}\sum_{m=1}^{2n+1} H_{n,m}(\sigma;x)H_{n,m}(\sigma;y) \\ &= \sum_{n=p}^{q} \frac{2n+1}{4\pi\sigma^2}\left(\frac{\sigma^2}{|x|\,|y|}\right)^2 P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right). \end{aligned} \tag{3.9}$$

$Pot(\Sigma^{\rm ext})$ denotes the space of all functions (potentials) $U:\Sigma^{\rm ext}\to\mathbb{R}$ with

- $U\in C^{(2)}(\Sigma^{\rm ext})$,
- U satisfies the Laplace equation in the outer space, i.e., $\Delta_x U(x)=0$, $x\in\Sigma^{\rm ext}$,
- U is regular at infinity, i.e., $|U(x)|=O(|x|^{-1})$, $|x|\to\infty$.

As usual, for $k=0,1,\ldots$, we let $Pot^{(k)}(\overline{\Sigma^{\rm ext}})$ be the space of functions $F:\overline{\Sigma^{\rm ext}}\to\mathbb{R}$ such that $F|_{\Sigma^{\rm ext}}\in Pot(\Sigma^{\rm ext})$ and $F\in C^{(k)}(\overline{\Sigma^{\rm ext}})$, in brief,

$$Pot^{(k)}(\overline{\Sigma^{\rm ext}}) = Pot(\Sigma^{\rm ext})\cap C^{(k)}(\overline{\Sigma^{\rm ext}}). \tag{3.10}$$

It is known from [13] and [17] that

$$L^2(\Sigma) = \overline{\operatorname{span}_{\substack{n=0,1,\ldots;\\ m=1,\ldots,2n+1}} (H_{n,m}(\sigma;\cdot))|_\Sigma}^{\|\cdot\|_{L^2(\Sigma)}}, \tag{3.11}$$

$$C^{(0)}(\Sigma) = \overline{\operatorname{span}_{\substack{n=0,1,\ldots;\\ m=1,\ldots,2n+1}} (H_{n,m}(\sigma;\cdot))|_\Sigma}^{\|\cdot\|_{C^{(0)}(\Sigma)}}. \tag{3.12}$$

Furthermore (cf. [13]),

$$Pot^{(0)}(\overline{\Sigma^{\rm ext}}) = \overline{\operatorname{span}_{\substack{n=0,1,\ldots;\\ m=1,\ldots,2n+1}} (H_{n,m}(\sigma;\cdot))|_{\overline{\Sigma^{\rm ext}}}}^{\|\cdot\|_{C^{(0)}(\overline{\Sigma^{\rm ext}})}}. \tag{3.13}$$

Next we introduce *Sobolev spaces* $\mathcal{H}(\overline{\Omega_\sigma^{\rm ext}})$ (cf. [14]). We start with a general definition based on the concept of summable sequences, give some examples for spaces with a reproducing kernel structure, and, finally, introduce the well-known $\mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}})$-spaces.

The introduction of the Sobolev spaces may be based on a linear space $\mathcal{A}$ consisting of all sequences $\{A_n\}$ of real numbers $A_n, n=0,1,\ldots$, i.e.,

$$\mathcal{A} = \{\{A_n\}\ :\ A_n\in\mathbb{R},\ n=0,1,\ldots\}.$$

For given sequences $\{A_n\},\{B_n\}\in\mathcal{A}$ we denote by $\mathcal{N}(B_n^{-1}A_n)$ the set of all non-negative integers n for which $B_nA_n^{-1}$ exists and is different from 0. Let $\mathcal{N}_0(B_n^{-1}A_n)$ denote the complement of $\mathcal{N}(B_n^{-1}A_n)$ in $\mathbb{N}_0$. Consequently, it follows that $\mathbb{N}_0 = \mathcal{N}(B_n^{-1}A_n)\cup\mathcal{N}_0(B_n^{-1}A_n)$ and $\mathcal{N}(B_n^{-1}A_n)\cap\mathcal{N}_0(B_n^{-1}A_n)=\emptyset$. In particular, if $\{B_n\}$ is chosen such that $B_n=1$ for all $n\in\mathbb{N}_0$, $\mathcal{N}(A_n)$ is the set of all integers $n\in\mathbb{N}_0$ for which $A_n\neq 0$, and $\mathcal{N}_0(A_n)$ is the set of all integers $n\in\mathbb{N}_0$ with $A_n=0$. Further on $\mathcal{N}(A_n)$ is always assumed to be non-void. Moreover, we write $\mathcal{N}$ instead of $\mathcal{N}(A_n)$ if no confusion is likely to arise.

Consider the set $\mathcal{E}(\overline{\Omega_\sigma^{\mathrm{ext}}})\left(=\mathcal{E}(\{A_n\};\overline{\Omega_\sigma^{\mathrm{ext}}})\right)$ of all functions $F \in Pot^{(\infty)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ of the form

$$F = \sum_{n\in\mathcal{N}} \sum_{m=1}^{2n+1} F^\wedge(n,m) H_{n,m}(\sigma;\cdot) \tag{3.14}$$

with

$$F^\wedge(n,m) = F^{\wedge_{L^2(\Omega_\sigma)}}(n,m) = \int_{\Omega_\sigma} F(y) H_{n,m}(\sigma;y)\, d\omega(y)$$

satisfying

$$\sum_{n\in\mathcal{N}} \sum_{m=1}^{2n+1} A_n^2\ (F^\wedge(n,m))^2 < \infty \tag{3.15}$$

(note that $\Sigma_{n\in\mathcal{N}}$ means that the sum is extended over all non-negative integers n with $n \in \mathcal{N}$). From the Cauchy–Schwarz inequality it follows that

$$\left|\sum_{n\in\mathcal{N}} \sum_{m=1}^{2n+1} A_n^2\ F^\wedge(n,m) G^\wedge(n,m)\right| \tag{3.16}$$

$$\leq \left(\sum_{n\in\mathcal{N}} \sum_{m=1}^{2n+1} A_n^2\ (F^\wedge(n,m))^2\right)^{1/2} \left(\sum_{n\in\mathcal{N}} \sum_{m=1}^{2n+1} A_n^2\ (G^\wedge(n,m))^2\right)^{1/2}$$

for all $F, G \in \mathcal{E}(\overline{\Omega_\sigma^{\mathrm{ext}}})$, hence, the left-hand side of (3.16) is finite whenever each member of the right-hand side is finite. This is the reason why we are able to impose on $\mathcal{E}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ an inner product $(\cdot,\cdot)_{\mathcal{H}(\{A_n\};\overline{\Omega_\sigma^{\mathrm{ext}}})}$ by letting

$$(F,G)_{\mathcal{H}(\{A_n\};\overline{\Omega_\sigma^{\mathrm{ext}}})} = \sum_{n\in\mathcal{N}} \sum_{m=1}^{2n+1} A_n^2\ F^\wedge(n,m) G^\wedge(n,m). \tag{3.17}$$

The associated norm is given by

$$\|F\|_{\mathcal{H}(\{A_n\};\overline{\Omega_\sigma^{\mathrm{ext}}})} = \left(\sum_{n\in\mathcal{N}} \sum_{m=1}^{2n+1} A_n^2\ (F^\wedge(n,m))^2\right)^{1/2}. \tag{3.18}$$

Summarizing our results we therefore obtain the following definition.

Definition 3.1. The *Sobolev space* $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ (more accurately: $\mathcal{H}(\{A_n\};\overline{\Omega_\sigma^{\mathrm{ext}}})$) is the completion of $\mathcal{E}(\overline{\Omega_\sigma^{\mathrm{ext}}})(=\mathcal{E}(\{A_n\};\overline{\Omega_\sigma^{\mathrm{ext}}}))$ under the norm $\|\cdot\|_{\mathcal{H}(\{A_n\};\overline{\Omega_\sigma^{\mathrm{ext}}})}$:

$$\mathcal{H}(\{A_n\};\overline{\Omega_\sigma^{\mathrm{ext}}}) = \overline{\mathcal{E}(\{A_n\};\overline{\Omega_\sigma^{\mathrm{ext}}})}^{\|\cdot\|_{\mathcal{H}(\{A_n\};\overline{\Omega_\sigma^{\mathrm{ext}}})}}.$$

$\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ equipped with the inner product corresponding to the norm (3.18) is a Hilbert space. The system $\{H_{n,m}^{*\{A_n\}}(\sigma;\cdot)\}$ given by

$$H_{n,m}^{*\{A_n\}}(\sigma;x) = A_n^{-1} H_{n,m}(\sigma;x), \quad x \in \overline{\Omega_\sigma^{\mathrm{ext}}}, \tag{3.19}$$

is a Hilbert basis. We simply write $H_{n,m}^*(\sigma;\cdot)$ instead of $H_{n,m}^{*\{A_n\}}(\sigma;\cdot)$ if no confusion is likely to arise.

Consider the Beltrami operator $\Delta^{*;\sigma}$ on the sphere Ω_σ. We know that

$$\Delta^{*;\sigma} Y_{n,m} = \frac{1}{\sigma^2}\Delta^* Y_{n,m} = -\frac{1}{\sigma^2} n(n+1) Y_{n,m}$$

for $n \in \mathbb{N}_0$; $m = 1, \ldots, 2n+1$ (note that $\Delta^{*;1} = \Delta^*$). Thus we formally have

$$\left(-\Delta^{*;\sigma} + \frac{1}{4\sigma^2}\right)^{s/2} Y_{n,k} = \left(\frac{n+\frac{1}{2}}{\sigma}\right)^s Y_{n,m}$$

and

$$\left(\left(-\Delta^{*;\sigma} + \frac{1}{4\sigma^2}\right)^{s/2} F\right)^\wedge (n,m) = \left(\frac{n+\frac{1}{2}}{\sigma}\right)^s F^\wedge(n,m)$$

for all $n \in \mathbb{N}_0$; $m = 1, \ldots, 2n+1$.

Definition 3.2. For any given value $s \in \mathbb{R}$, the *Sobolev space* $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ is the completion of $\mathcal{E}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ under the norm $\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}$:

$$\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \overline{\mathcal{E}(\overline{\Omega_\sigma^{\mathrm{ext}}})}^{\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}}.$$

$\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ equipped with the inner product $(\cdot,\cdot)_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}$ is a Hilbert space. The system $\{H^s_{n,m}(\sigma;\cdot)\}$ given by

$$H^s_{n,m}(\sigma;x) = \left(\frac{\sigma}{n+\frac{1}{2}}\right)^s H_{n,m}(\sigma;x), \quad x \in \overline{\Omega_\sigma^{\mathrm{ext}}}, \tag{3.20}$$

is a Hilbert basis.

Hence, the norm in $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ reads as follows:

$$\|F\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} = \left(\int_{\Omega_\sigma} \left(\left(-\Delta_x^{*;\sigma} + \frac{1}{4\sigma^2}\right)^{s/2} F(x)\right)^2 d\omega(x)\right)^{1/2}. \tag{3.21}$$

$\mathcal{H}_0(\overline{\Omega_\sigma^{\mathrm{ext}}})$ may be understood as the space of all harmonic functions in $\Omega_\sigma^{\mathrm{ext}}$, regular at infinity, corresponding to L^2-restrictions (note that the potentials in $\mathcal{H}_0(\overline{\Omega_\sigma^{\mathrm{ext}}})$ are uniquely determined by their L^2-(Dirichlet) boundary conditions on Ω_σ). According to our construction, $Pot^{(\infty)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ is a dense subspace of $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ for each s. If $t < s$, then $\|F\|_{\mathcal{H}_t(\overline{\Omega_\sigma^{\mathrm{ext}}})} \leq \|F\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}$ and $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \mathcal{H}_t(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

If we associate to U the outer harmonic expansion (3.14) it is of fundamental importance to know when the series (3.14) converges uniformly on the whole set $\overline{\Omega_\sigma^{\mathrm{ext}}}$. To this end we need the concept of summable sequences.

Definition 3.3. A sequence $\{A_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ is called summable if

$$\sum_{n=0}^{\infty} \frac{2n+1}{A_n^2} < \infty. \tag{3.22}$$

Lemma 3.4 (Sobolev Lemma). *Assume that the sequences* $\{A_n\}_{n\in\mathbb{N}_0}, \{B_n\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ *are such that* $\{B_n^{-1}A_n\}_{n\in\mathbb{N}_0}$ *is summable. Then each* $F \in \mathcal{H}\left(\{B_n^{-1}A_n\}; \overline{\Omega_\sigma^{\text{ext}}}\right)$ *corresponds to a potential of class* $Pot^{(0)}(\overline{\Omega_\sigma^{\text{ext}}})$.

The Sobolev Lemma which is proved in [19] states that in the case of summability of the sequence $\{B_n^{-1}A_n\}_{n\in\mathbb{N}_0}$, the Fourier series in terms of the basis functions $H_{n,m} \in \mathcal{H}\left(\{B_n^{-1}A_n\}; \overline{\Omega_\sigma^{\text{ext}}}\right)$ is continuous on the boundary Ω_σ. In particular, we have the following statement (cf. [19]).

Lemma 3.5. *If* $U \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, $s > k+1$, *then* U *corresponds to a potential of class* $Pot^{(k)}(\overline{\Omega_\sigma^{\text{ext}}})$.

3.2. Vectorial outer harmonic and Sobolev theory

We now extend the theory of scalar outer harmonics and scalar Sobolev spaces to the vectorial case. We use a system of vector spherical harmonics (cf. [21]) in order to generate the set of vector outer harmonics in such a way, that the Laplace equation is fulfilled componentwise.

Let $\{\tilde{y}_{n,m}^{(i)}\}_{i=1,2,3;n\in\mathbb{N}_{0_i};m=1,\ldots,2n+1}$ be a set of vector spherical harmonics satisfying the condition of being a set of eigenfunctions of the Beltrami operator, with

$$0_i = \begin{cases} 0, & i=1, \\ 1, & i=2,3. \end{cases} \tag{3.23}$$

(see, e.g., [21, 32, 58], for a detailed introduction and profound discussion of these vector spherical harmonics). In the nomenclature of [32], the vector outer harmonics $h_{n,m}^{(i)}(\sigma;\cdot)$ of degree n and kind i are defined by

$$h_{n,m}^{(1)}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n+2} \tilde{y}_{n,m}^{(1)}\left(\frac{x}{|x|}\right), \quad n=0,1,\ldots;m=1,\ldots,2n+1, \tag{3.24}$$

$$h_{n,m}^{(2)}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n} \tilde{y}_{n,m}^{(2)}\left(\frac{x}{|x|}\right), \quad n=1,2,\ldots;m=1,\ldots,2n+1, \tag{3.25}$$

$$h_{n,m}^{(3)}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n+1} \tilde{y}_{n,m}^{(3)}\left(\frac{x}{|x|}\right), \quad n=1,2,\ldots;m=1,\ldots,2n+1, \tag{3.26}$$

for $x \in \overline{\Omega_\sigma^{\text{ext}}}$. The following properties are satisfied:

- $h_{n,m}^{(i)}(\sigma;\cdot)$ is of class $c^{(\infty)}(\Omega_\sigma^{\text{ext}})$,
- $\Delta_x h_{n,m}^{(i)}(\sigma;x) = 0$ for $x\in\Omega_\sigma^{\text{ext}}$, i.e., every component function $h_{n,m}^{(i)} \cdot \varepsilon^k$ satisfies the Laplace equation,
- $h_{n,m}^{(i)}$ is regular at infinity, i.e., $|h_{n,m}^{(i)}(\sigma;x)| = \mathcal{O}(|x|^{-1})$, $|h_{n,m}^{(2)}(\sigma\cdot x)| = \mathcal{O}(|x|^{-2})$, $|x| \to \infty$
- $h_{n,m}^{(i)}(\sigma;\cdot)|_{\Omega_\sigma} = (1/\sigma)\tilde{y}_{n,m}^{(i)}$,
- $(h_{n,m}^{(i)}(\sigma;\cdot), h_{l,s}^{(j)}(\sigma;\cdot))_{l^2(\Omega_\sigma)} = \int_{\Omega_\sigma} h_{n,m}^{(i)}(\sigma;x) h_{l,s}^{(j)}(\sigma;x)\, d\omega(x) = \delta_{i,j}\delta_{n,l}\delta_{m,s}$.

We introduce

$$harm^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \overline{\mathrm{span}_{\substack{n=0_i,\ldots;\\ m=1,\ldots,2n+1}} h^{(i)}_{n,m}(\sigma;\cdot)}^{\|\cdot\|_{c^{(0)}(\overline{\Omega_\sigma^{\mathrm{ext}}})}}, \tag{3.27}$$

$$harm(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \overline{\mathrm{span}_{\substack{i=1,2,3;n=0_i,\ldots;\\ m=1,\ldots,2n+1}} h^{(i)}_{n,m}(\sigma;\cdot)}^{\|\cdot\|_{c^{(0)}(\overline{\Omega_\sigma^{\mathrm{ext}}})}}. \tag{3.28}$$

Some results concerning addition theorems for outer harmonics using Legendre tensors and Legendre vectors can be found in the Ph.D.-thesis [58] and are not discussed here.

Lemma 3.6. *Let* $\{H_{n,m}(\sigma;\cdot)\}_{n\in\mathbb{N}_0;m=1,\ldots,2n+1}$ *be a system of scalar outer harmonics. Then*

$$\overline{\mathrm{span}\{H_{n,m}(\sigma;\cdot)\varepsilon^i|_\Sigma\}_{i=1,2,3}}^{\|\cdot\|_{l^2(\Sigma)}} = l^2(\Sigma),$$

$$\overline{\mathrm{span}\{H_{n,m}(\sigma;\cdot)\varepsilon^i|_\Sigma\}_{i=1,2,3}}^{\|\cdot\|_{c^{(0)}(\Sigma)}} = c^{(0)}(\Sigma).$$

Theorem 3.7. *Let* $\{h^{(i)}_{n,m}(\sigma;\cdot)\}_{\substack{i=1,2,3;n=0_i,\ldots;\\ m=1,\ldots,2n+1}}$ *be a system of vector outer harmonics as defined in* (3.24)–(3.26)*. Then the following statements hold true:*

$$l^2(\Sigma) = \overline{\mathrm{span}_{\substack{i=1,2,3;n=0_i,\ldots;\\ m=1,\ldots,2n+1}} (h^{(i)}_{n,m}(\sigma;\cdot))|_\Sigma}^{\|\cdot\|_{l^2(\Sigma)}},$$

$$c^{(0)}(\Sigma) = \overline{\mathrm{span}_{\substack{i=1,2,3;n=0_i,\ldots;\\ m=1,\ldots,2n+1}} (h^{(i)}_{n,m}(\sigma;\cdot))|_\Sigma}^{\|\cdot\|_{c^{(0)}(\Sigma)}}.$$

In order to define the vectorial potential space $pot(\Sigma^{\mathrm{ext}})$ we need the divergence and curl operator, which are defined by

$$\mathrm{div}\, f(x) = \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i}(x), \quad f = \sum_{i=1}^{3} F_i \varepsilon^i, \tag{3.29}$$

and

$$(\mathrm{curl}\, f(x))_i = \sum_{j,k=1}^{3} \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j}(x), \tag{3.30}$$

where ε_{ijk} is the alternator defined by

$$\varepsilon_{ijk} = \begin{cases} +1, & (i,j,k) \text{ is an even permutation of } (1,2,3), \\ -1, & (i,j,k) \text{ is an odd permutation of } (1,2,3), \\ 0, & (i,j,k) \text{ is not a permutation of } (1,2,3). \end{cases} \tag{3.31}$$

By $pot(\Sigma^{\mathrm{ext}})$ we denote the space of all vector fields $f : \Sigma^{\mathrm{ext}} \to \mathbb{R}^3$ satisfying the following properties:

(i) $f \in c^{(1)}(\Sigma^{\mathrm{ext}})$,
(ii) f is a harmonic vector field: $\mathrm{div} f = 0$, $\mathrm{curl} f = 0$ in Σ^{ext},
(iii) f is regular at infinity: $|f(x)| = \mathcal{O}(|x|^{-2})$, $|x| \to \infty$.

Furthermore, we let

$$pot^{(k)}(\overline{\Sigma^{\mathrm{ext}}}) = pot(\Sigma^{\mathrm{ext}}) \cap c^{(k)}(\overline{\Sigma^{\mathrm{ext}}}), \tag{3.32}$$

which is meant in the same sense as we explained in the scalar case. It is well known (see, e.g., [38]), that every function $f \in c^{(k)}(\Sigma^{\mathrm{ext}})$ satisfying $\mathrm{curl} f = 0$ is the gradient of a function $V \in C^{(k+1)}(\Sigma^{\mathrm{ext}})$: $f = \nabla V$. As a consequence, we get that every $f \in pot(\Sigma^{\mathrm{ext}})$ can be represented as a gradient field $f = \nabla V$, where $V \in Pot(\Sigma^{\mathrm{ext}})$, and vice versa. Furthermore, it is obvious, that a function $f \in pot(\Sigma^{\mathrm{ext}})$ of the form $f = \sum_{i=1}^{3} F_i \varepsilon^i$ fulfills $F_i \in Pot(\Sigma^{\mathrm{ext}})$.

For arbitrary $\varepsilon > 0$, we have an integer $N = N(\varepsilon)$ and coefficients $a_{n,m}$, $n = 0, \ldots, N$; $m = 1, \ldots, 2n+1$, such that

$$\sup_{x \in \Sigma} \left| F(x) - \sum_{n=0}^{N} \sum_{m=1}^{2n+1} a_{n,m} H_{n,m}(\sigma; x) \right| < \varepsilon. \tag{3.33}$$

For the gradient of $H_{n,m}(\sigma; \cdot)$ we obtain

$$\nabla_x H_{n,m}(\sigma; x) = C\, h^{(1)}_{n,m}(\sigma; x), \tag{3.34}$$

with a constant factor C, which leads us to (cf. [25])

$$pot^{(0)}(\overline{\Sigma^{\mathrm{ext}}}) = \overline{\operatorname{span}_{\substack{n \in \mathbb{N}_0;\\ m=1,\ldots,2n+1}} (h^{(1)}_{n,m}(\sigma; \cdot))|_{\overline{\Sigma^{\mathrm{ext}}}}}^{\|\cdot\|_{c^{(0)}(\overline{\Sigma^{\mathrm{ext}}})}} \tag{3.35}$$

(Runge–Walsh approximation property).

In analogy to the scalar case, we define Sobolev spaces for vector fields. We do not restrict our considerations to $pot^{(\infty)}(\overline{\Omega^{\mathrm{ext}}_\sigma})$ as a reference space for the definition of vectorial Sobolev spaces, because in this case only the $h^{(1)}_{n,m}$-part would be taken into account.

Consider the space a defined by

$$a = \{\{a_n\} \,|\, a_n = \left(A^{(1)}_n, A^{(2)}_n, A^{(3)}_n\right)^T \in \mathbb{R}^3,\ A^{(i)}_n \neq 0,\ n \in \mathbb{N}_0\}. \tag{3.36}$$

Obviously, we have $\{A^{(i)}_n\}_{n \in \mathbb{N}_0} \in \mathcal{A}$ for $i \in \{1, 2, 3\}$.

For $\{a_n\}_{n \in \mathbb{N}_0} \in a$ we define

$$e^{(i)}(\overline{\Omega^{\mathrm{ext}}_\sigma}) = \left\{ f \in harm^{(i)}(\overline{\Omega^{\mathrm{ext}}_\sigma}) \ : \ \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} |A^{(i)}_n|^2 (f, h^{(i)}_{n,m})^2_{l^2(\Omega_\sigma)} < \infty \right\}, \tag{3.37}$$

$i \in \{1, 2, 3\}$. Equipped with the inner product

$$(f, g)_{h(\overline{\Omega^{\mathrm{ext}}_\sigma})} = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} |A^{(i)}_n|^2 (f, h^{(i)}_{n,m})_{l^2(\Omega_\sigma)} (g, h^{(i)}_{n,m})_{l^2(\Omega_\sigma)}, \tag{3.38}$$

$f, g \in e^{(i)}(\overline{\Omega^{\mathrm{ext}}_\sigma})$, the space $e^{(i)}(\overline{\Omega^{\mathrm{ext}}_\sigma})$ becomes a pre-Hilbert space. We define the Sobolev space $h^{(i)}(\overline{\Omega^{\mathrm{ext}}_\sigma}) = h^{(i)}(\{A^{(i)}_n\}; \overline{\Omega^{\mathrm{ext}}_\sigma})$ to be the completion of $e^{(i)}(\overline{\Omega^{\mathrm{ext}}_\sigma})$

under the norm $\|\cdot\|_{h(\overline{\Omega_\sigma^{\rm ext}})}$, which denotes the norm associated to $(\cdot,\cdot)_{h(\overline{\Omega_\sigma^{\rm ext}})}$:

$$h^{(i)}(\{A_n^{(i)}\};\overline{\Omega_\sigma^{\rm ext}}) = \overline{e^{(i)}(\overline{\Omega_\sigma^{\rm ext}})}^{\|\cdot\|_{h(\overline{\Omega_\sigma^{\rm ext}})}}. \tag{3.39}$$

We use the following notation

$$h(\overline{\Omega_\sigma^{\rm ext}}) = h(\{a_n\};\overline{\Omega_\sigma^{\rm ext}}) = \bigoplus_{i=1}^{3} h^{(i)}(\overline{\Omega_\sigma^{\rm ext}}) = \bigoplus_{i=1}^{3} h^{(i)}(\{A_n^{(i)}\};\overline{\Omega_\sigma^{\rm ext}}). \tag{3.40}$$

The space $h(\overline{\Omega_\sigma^{\rm ext}})$ equipped with the inner product $(\cdot,\cdot)_{h(\overline{\Omega_\sigma^{\rm ext}})}$ is a Hilbert space with Hilbert basis $\{h_{n,m}^{(i)*\{A_n^{(i)}\}}(\sigma;\cdot)\}_{i=1,2,3;\,n=0_i,\dots;\,m=1,\dots,2n+1}$ given by

$$h_{n,m}^{(i)*\{A_n^{(i)}\}}(\sigma;x) = (A_n^{(i)})^{-1} h_{n,m}^{(i)}(\sigma;x), \quad x\in\overline{\Omega_\sigma^{\rm ext}}. \tag{3.41}$$

We can, therefore, expand a function $f\in h(\overline{\Omega_\sigma^{\rm ext}})$ as a Fourier series in terms of the basis $h_{n,m}^{(i)*\{A_n^{(i)}\}}$:

$$f = \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{m=1}^{2n+1} f^{(i)\wedge_{h(\{a_n\};\overline{\Omega_\sigma^{\rm ext}})}}(n,m) h_{n,m}^{(i)*\{A_n^{(i)}\}}, \tag{3.42}$$

where

$$f^{(i)\wedge_{h(\{a_n\};\overline{\Omega_\sigma^{\rm ext}})}}(n,m) = f^{(i)\wedge}(n,m) = (f, h_{n,m}^{(i)*\{A_n^{(i)}\}})_{h(\overline{\Omega_\sigma^{\rm ext}})}. \tag{3.43}$$

In analogy to the scalar spaces $\mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}})$, we define the vectorial spaces $h_s(\overline{\Omega_\sigma^{\rm ext}})$ by

$$h_s^{(i)}(\overline{\Omega_\sigma^{\rm ext}}) = h^{(i)}\left(\left\{\left(\frac{n+\frac{1}{2}}{\sigma}\right)^s\right\};\overline{\Omega_\sigma^{\rm ext}}\right), \tag{3.44}$$

$$h_s(\overline{\Omega_\sigma^{\rm ext}}) = \bigoplus_{i=1}^{3} h_s^{(i)}(\overline{\Omega_\sigma^{\rm ext}}). \tag{3.45}$$

The space $h_s(\overline{\Omega_\sigma^{\rm ext}})$ equipped with the inner product $(\cdot,\cdot)_{h_s(\overline{\Omega_\sigma^{\rm ext}})}$ is a Hilbert space with Hilbert basis $\{h_{n,m}^{(i)s}(\sigma;\cdot)\}_{i=1,2,3;\,n=0_i,\dots;\,m=1,\dots,2n+1}$ given by

$$h_{n,m}^{(i)s}(\sigma;x) = \left(\frac{\sigma}{n+\frac{1}{2}}\right)^s h_{n,m}^{(i)}(\sigma;x), \quad x\in\overline{\Omega_\sigma^{\rm ext}}. \tag{3.46}$$

In the case of the space $h_0(\overline{\Omega_\sigma^{\rm ext}})$ we understand the norm $\|\cdot\|_{h_0(\overline{\Omega_\sigma^{\rm ext}})}$ to be the $\|\cdot\|_{l^2(\Omega_\sigma)}$- norm.

Next, the scalar Sobolev Lemma 3.4 will be extended to vector fields.

Definition 3.8. A sequence $\{a_n\}_{n\in\mathbb{N}_0}\in a$ is called summable if

$$\sum_{n=0_i}^{\infty}\frac{2n+1}{\left(A_n^{(i)}\right)^2}<\infty, \tag{3.47}$$

for $i=1,2,3$.

In the sequel, $\{b_n^{-1}\}_{n\in\mathbb{N}_0} \in a$ means the sequence given by

$$b_n^{-1} = \left(\left(B_n^{(1)}\right)^{-1}, \left(B_n^{(2)}\right)^{-1}, \left(B_n^{(3)}\right)^{-1} \right)^T, \tag{3.48}$$

and

$$b_n^{-1} a_n = \left(A_n^{(1)} \left(B_n^{(1)}\right)^{-1}, A_n^{(2)} \left(B_n^{(2)}\right)^{-1}, A_n^{(3)} \left(B_n^{(3)}\right)^{-1} \right)^T. \tag{3.49}$$

Lemma 3.9 (Vectorial Sobolev Lemma). *Assume, that $\{a_n\}_{n\in\mathbb{N}_0}, \{b_n\}_{n\in\mathbb{N}_0} \in a$ are sequences such that $\{b_n^{-1}a_n\}_{n\in\mathbb{N}_0} \in a$ is summable. Then each $f \in h(\{b_n^{-1}a_n\}; \overline{\Omega_\sigma^{\mathrm{ext}}})$ corresponds to a function of class harm$(\overline{\Omega_\sigma^{\mathrm{ext}}})$.*

3.3. Tensorial outer harmonic and Sobolev theory

The extension of vectorial to tensorial theory is straightforward (see [21, 32, 58]). With the help of a system $\{\tilde{\mathbf{y}}_{n,m}^{(i,k)}\}$ of tensor spherical harmonics we can derive a set of tensor outer harmonics $\{\mathbf{h}_{n,m}^{(i,k)}(\sigma;\cdot)\}$ satisfying the Laplace equation componentwise.

Let $\{\tilde{\mathbf{y}}_{n,m}^{(i,k)}\}_{i,k=1,2,3;n\in\mathbb{N}_0;m=1,\ldots,2n+1}$ with

$$0_{ik} = \begin{cases} 0, & (i,k) \in \{(1,1),(2,1),(3,1)\}, \\ 1, & (i,k) \in \{(1,2),(1,3),(2,3),(3,3)\}, \\ 2, & (i,k) \in \{(2,2),(3,2)\}, \end{cases} \tag{3.50}$$

be a set of tensorial spherical harmonics satisfying the condition of being eigenfunctions of the Beltrami operator (see, e.g., the Ph.D.-thesis [58] for a detailed introduction and profound discussion of these tensor spherical harmonics). The tensor outer harmonics $\mathbf{h}_{n,m}^{(i,k)}(\sigma;\cdot)$ of degree n and kind (i,k) are then defined by

$$\mathbf{h}_{n,m}^{(1,1)}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n+3} \tilde{\mathbf{y}}_{n,m}^{(1,1)}\left(\frac{x}{|x|}\right), \tag{3.51}$$

$$\mathbf{h}_{n,m}^{(1,2)}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n+1} \tilde{\mathbf{y}}_{n,m}^{(1,2)}\left(\frac{x}{|x|}\right), \tag{3.52}$$

$$\mathbf{h}_{n,m}^{(2,1)}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n+1} \tilde{\mathbf{y}}_{n,m}^{(2,1)}\left(\frac{x}{|x|}\right), \tag{3.53}$$

$$\mathbf{h}_{n,m}^{(2,2)}(R;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n-1} \tilde{\mathbf{y}}_{n,m}^{(2,2)}\left(\frac{x}{|x|}\right), \tag{3.54}$$

$$\mathbf{h}_{n,m}^{(3,3)}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n+1} \tilde{\mathbf{y}}_{n,m}^{(3,3)}\left(\frac{x}{|x|}\right), \tag{3.55}$$

$$\mathbf{h}_{n,m}^{(1,3)}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n+2} \tilde{\mathbf{y}}_{n,m}^{(1,3)}\left(\frac{x}{|x|}\right), \tag{3.56}$$

$$\mathbf{h}_{n,m}^{(2,3)}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n} \tilde{\mathbf{y}}_{un,m}^{(2,3)}\left(\frac{x}{|x|}\right), \tag{3.57}$$

$$\mathbf{h}_{n,m}^{(3,1)}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n+2} \tilde{\mathbf{y}}_{n,m}^{(3,1)}\left(\frac{x}{|x|}\right), \tag{3.58}$$

$$\mathbf{h}_{n,m}^{(3,2)}(\sigma;x) = \frac{1}{\sigma}\left(\frac{\sigma}{|x|}\right)^{n} \tilde{\mathbf{y}}_{n,m}^{(3,2)}\left(\frac{x}{|x|}\right), \tag{3.59}$$

where $x \in \overline{\Omega_\sigma^{\text{ext}}}$, $n = 0_{ik}, \ldots$; $m = 1, \ldots, 2n+1$. The following properties are satisfied:

- $\mathbf{h}_{n,m}^{(i,k)}(\sigma;\cdot)$ is of class $\mathbf{c}^{(\infty)}(\Omega_\sigma^{\text{ext}})$,
- $\Delta_x \mathbf{h}_{n,m}^{(i,k)}(\sigma;x) = 0$ for $x \in \Omega_\sigma^{\text{ext}}$, i.e., the component functions of $\mathbf{h}_{n,m}^{(i,k)}(\sigma;\cdot)$ fulfill the Laplace equation,
- $\mathbf{h}_{n,m}^{(i,k)}$ is regular at infinity, i.e., $|\mathbf{h}_{n,m}^{(i,k)}(\sigma;x)| = \mathcal{O}(|x|^{-3}), \quad |x| \to \infty$.
- $\mathbf{h}_{n,m}^{(i,k)}(\sigma;\cdot)|_{\Omega_\sigma} = (1/\sigma)\tilde{\mathbf{y}}_{n,m}^{(i,k)}$,
- $(\mathbf{h}_{n,m}^{(i,k)}(\sigma;\cdot), \mathbf{h}_{l,s}^{(p,q)}(\sigma;\cdot))_{\mathbf{l}^2(\Omega_\sigma)} = \int_{\Omega_\sigma} \mathbf{h}_{n,m}^{(i,k)}(\sigma;x)\mathbf{h}_{l,s}^{p,q}(\sigma;x)d\omega(x)$ $= \delta_{i,p}\delta_{k,q}\delta_{n,l}\delta_{m,s}$.

Moreover, we define

$$\mathbf{harm}^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}}) = \overline{\operatorname{span}_{\substack{n=0_{ik}\ldots;\\ m=1,\ldots,2n+1}} \mathbf{h}_{n,m}^{(i,k)}(\sigma;\cdot)}^{\|\cdot\|_{\mathbf{c}^{(0)}(\overline{\Omega_\sigma^{\text{ext}}})}}, \tag{3.60}$$

$$\mathbf{harm}(\overline{\Omega_\sigma^{\text{ext}}}) = \overline{\operatorname{span}_{\substack{i,k\in\{1,2,3\};n=0_{ik}\ldots;\\ m=1,\ldots,2n+1}} \mathbf{h}_{n,m}^{(i,k)}(\sigma;\cdot)}^{\|\cdot\|_{\mathbf{c}^{(0)}(\overline{\Omega_\sigma^{\text{ext}}})}}. \tag{3.61}$$

Some results concerning addition theorems for outer harmonics can be formulated both for the tensor product of two tensor outer harmonics and for the product of a scalar and a tensor outer harmonic. They can be found in the Ph.D.-thesis [58] and are not discussed in this contribution.

Lemma 3.10. *Let* $\{H_{n,m}(\sigma;\cdot)\}_{n\in\mathbb{N}_{0_{ik}};m=1,\ldots,2n+1}$ *be a system of scalar outer harmonics. Then*

$$\overline{\operatorname{span}\{H_{n,m}(\sigma;\cdot)\varepsilon^i \otimes \varepsilon^k|_\Sigma\}}^{\|\cdot\|_{\mathbf{l}^2(\Sigma)}} = \mathbf{l}^2(\Sigma), \tag{3.62}$$

$$\overline{\operatorname{span}\{H_{n,m}(\sigma;\cdot)\varepsilon^i \otimes \varepsilon^k|_{\Sigma)}\}}^{\|\cdot\|_{\mathbf{c}^{(0)}(\Sigma)}} = \mathbf{c}^{(0)}(\Sigma). \tag{3.63}$$

Theorem 3.11. *Let* $\{\mathbf{h}_{n,m}^{(i,k)}\}_{\substack{i,k=1,2,3;n=0_{ik},\ldots;\\ m=1,\ldots,2n+1}}$ *be a system of tensor outer harmonics. Then the following statements hold true:*

$$\mathbf{l}^2(\Sigma) = \overline{\operatorname{span}_{\substack{i,k=1,2,3;n=0_{ik},\ldots;\\ m=1,\ldots,2n+1}} (\mathbf{h}_{n,m}^{(i,k)}(\sigma;\cdot))|_\Sigma}^{\|\cdot\|_{\mathbf{l}^2(\Sigma)}}, \tag{3.64}$$

$$\mathbf{c}(\Sigma) = \overline{\operatorname{span}_{\substack{i,k=1,2,3;n=0_{ik},\ldots;\\ m=1,\ldots,2n+1}} (\mathbf{h}_{n,m}^{(i,k)}(\sigma;\cdot))|_\Sigma}^{\|\cdot\|_{\mathbf{c}(\Sigma)}}. \tag{3.65}$$

In order to define a tensorial counterpart $\mathbf{pot}(\overline{\Sigma^{\text{ext}}})$ of the space $pot(\overline{\Sigma^{\text{ext}}})$, we need the divergence and the curl operator of tensor fields. Having (3.29) in mind, we define $\operatorname{div}\mathbf{f}$ by

$$(\operatorname{div}\mathbf{f}(x))_i = \sum_{j=1}^{3} \frac{\partial F_{i,j}}{\partial x_j}(x), \quad \mathbf{f} = \sum_{i,j=1}^{3} F_{i,j}\varepsilon^i \otimes \varepsilon^j. \tag{3.66}$$

Furthermore, based on (3.30) we have the following definition of $\operatorname{curl}\mathbf{f}$:

$$(\operatorname{curl}\mathbf{f}(x))_{i,j} = \sum_{p,k=1}^{3} \varepsilon_{ipk} \frac{\partial F_{j,k}}{\partial x_p}(x). \tag{3.67}$$

The space $\mathbf{pot}(\Sigma^{\text{ext}})$ denotes the space of all tensor fields $\mathbf{f} : \Sigma^{\text{ext}} \to \mathbb{R}^3 \otimes \mathbb{R}^3$ satisfying the following properties:

(i) $\mathbf{f} \in \mathbf{c}^{(1)}(\Sigma^{\text{ext}})$,

(ii) $\mathbf{f}$ is a harmonic tensor field: $\operatorname{div}\mathbf{f} = 0$, $\operatorname{curl}\mathbf{f} = 0$ in Σ^{ext},

(iii) $\mathbf{f}$ is regular at infinity: $|\mathbf{f}(x)| = \mathcal{O}(|x|^{-3})$, $|x| \to \infty$.

Furthermore, we let

$$\mathbf{pot}^{(k)}(\overline{\Sigma^{\text{ext}}}) = \mathbf{pot}(\Sigma^{\text{ext}}) \cap \mathbf{c}^{(k)}(\overline{\Sigma^{\text{ext}}}), \tag{3.68}$$

which we understand in the same sense as in the scalar and vectorial case. As shown, e.g., in [38], every tensor function $\mathbf{f} \in \mathbf{c}^{(k)}(\Sigma^{\text{ext}})$ with $\operatorname{curl}\mathbf{f} = 0$ is the gradient of a vector field $v \in c^{(k+1)}(\Sigma^{\text{ext}})$:

$$\mathbf{f} = \nabla v, \tag{3.69}$$

where ∇v is the tensor of second rank defined by

$$(\nabla_x v)_{ij}(x) = \frac{\partial v_i}{\partial x_j}(x). \tag{3.70}$$

Therefore, every member $\mathbf{v} \in \mathbf{pot}(\Sigma^{\text{ext}})$ can be represented as a gradient field $\mathbf{v} = \nabla v$, where v is of class $pot(\Sigma^{\text{ext}})$, and vice versa. As a consequence of this, in connection with the fact that every $v \in pot(\Sigma^{\text{ext}})$ can be represented as a gradient field $v = \nabla V$ with $V \in Pot(\Sigma^{\text{ext}})$, we finally get that a tensor field $\mathbf{v} \in \mathbf{pot}(\Sigma^{\text{ext}})$ can be represented as the Hesse tensor of a scalar field $V \in Pot(\Sigma^{\text{ext}})$:

$$\mathbf{v} = \nabla \otimes \nabla V, \tag{3.71}$$

and vice versa.

It is obvious, that $\mathbf{f} \in \mathbf{pot}(\Sigma^{\text{ext}})$ of the form $\mathbf{f} = \sum_{i,k=1}^{3} F_{i,k}\varepsilon^i \otimes \varepsilon^k$ fulfills $F_{i,k} \in Pot(\Sigma^{\text{ext}})$. In addition, we are able to show that

$$\mathbf{pot}^{(0)}(\overline{\Sigma^{\text{ext}}}) = \overline{\operatorname{span}_{\substack{n\in\mathbb{N}_0;\\ m=1,\dots,2n+1}} (\mathbf{h}^{(1,1)}_{n,m}(\sigma;\cdot))|_{\overline{\Sigma^{\text{ext}}}}}^{\|\cdot\|_{\mathbf{c}^{(0)}(\overline{\Sigma^{\text{ext}}})}} \tag{3.72}$$

(Runge–Walsh approximation property).

Our purpose is now to define Sobolev spaces for tensor fields in analogy to the vectorial Sobolev spaces. We introduce the linear space $\mathbf{a}$ in the following way:

$$\mathbf{a} = \{\{\mathbf{a}_n\} \mid \mathbf{a}_n \in \mathbb{R}^3 \otimes \mathbb{R}^3,\ A_n^{(i,k)} \neq 0,\ n \in \mathbb{N}_0; m = 1,\dots,2n+1; i,k \in \{1,2,3\}\}, \tag{3.73}$$

where

$$\mathbf{a}_n = \begin{pmatrix} A_n^{(1,1)} & A_n^{(1,2)} & A_n^{(1,3)} \\ A_n^{(2,1)} & A_n^{(2,2)} & A_n^{(2,3)} \\ A_n^{(3,1)} & A_n^{(3,2)} & A_n^{(3,3)} \end{pmatrix}, \tag{3.74}$$

with $\{A_n^{(i,k)}\}_{n\in\mathbb{N}_0} \in \mathcal{A}$ for $i,k \in \{1,2,3\}$.

Let us now consider a sequence $\{\mathbf{a}_n\}_{n\in\mathbb{N}_0} \in \mathbf{a}$. Then we define

$$\mathbf{e}^{(i,k)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \left\{ \mathbf{f} \in \mathbf{harm}^{(i,k)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \; : \; \sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} |A_n^{(i,k)}|^2 (\mathbf{f}, \mathbf{h}_{n,m}^{(i,k)})^2_{\mathbf{l}^2(\Omega_\sigma)} < \infty \right\}, \tag{3.75}$$

$i,k \in \{1,2,3\}$. Equipped with the inner product

$$(\mathbf{f},\mathbf{g})_{\mathbf{h}(\overline{\Omega_\sigma^{\mathrm{ext}}})} = \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} |A_n^{(i,k)}|^2 (\mathbf{f}, \mathbf{h}_{n,m}^{(i,k)})_{\mathbf{l}^2(\Omega_\sigma)} (\mathbf{g}, \mathbf{h}_{n,m}^{(i,k)})_{\mathbf{l}^2(\Omega_\sigma)}, \tag{3.76}$$

$\mathbf{f},\mathbf{g} \in \mathbf{e}^{(i,k)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$, the space $\mathbf{e}^{(i,k)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ becomes a pre-Hilbert space. We define the Sobolev space $\mathbf{h}^{(i,k)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \mathbf{h}^{(i,k)}(\{A_n^{(i,k)}\};\overline{\Omega_\sigma^{\mathrm{ext}}})$ to be the completion of $\mathbf{e}^{(i,k)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ under the norm $\|\cdot\|_{\mathbf{h}(\overline{\Omega_\sigma^{\mathrm{ext}}})}$, which denotes the norm associated to $(\cdot,\cdot)_{\mathbf{h}(\overline{\Omega_\sigma^{\mathrm{ext}}})}$:

$$\mathbf{h}^{(i,k)}(\{A_n^{(i,k)}\};\overline{\Omega_\sigma^{\mathrm{ext}}}) = \overline{\mathbf{e}^{(i,k)}(\overline{\Omega_\sigma^{\mathrm{ext}}})}^{\|\cdot\|_{\mathbf{h}(\{\overline{\Omega_\sigma^{\mathrm{ext}}})}}. \tag{3.77}$$

We use the following notation

$$\mathbf{h}(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \bigoplus_{i,k=1}^{3} \mathbf{h}^{(i,k)}(\overline{\Omega_\sigma^{\mathrm{ext}}}). \tag{3.78}$$

The space $\mathbf{h}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ equipped with the inner product $(\cdot,\cdot)_{\mathbf{h}(\{\overline{\Omega_\sigma^{\mathrm{ext}}})}$ is a Hilbert space. The system $\{\mathbf{h}_{n,m}^{(i,k)*\{A_n^{(i,k)}\}}(\sigma;\cdot)\}_{\substack{i,k\in\{1,2,3\};n\in\mathbb{N}_{0_{ik}};\\ m=1,\ldots,2n+1}}$, given by

$$\mathbf{h}_{n,m}^{(i,k)*\{A_n^{(i,k)}\}}(\sigma;x) = (A_n^{(i,k)})^{-1} \mathbf{h}_{n,m}^{(i,k)}(\sigma;x), \quad x \in \overline{\Omega_\sigma^{\mathrm{ext}}}, \tag{3.79}$$

represents an $\mathbf{h}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-orthonormal Hilbert basis in $\mathbf{h}(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

As a consequence, we can expand a function $\mathbf{f} \in \mathbf{h}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ as a Fourier series in terms of the basis $\mathbf{h}_{n,m}^{(i,k)*\{A_n^{(i,k)}\}}$:

$$\mathbf{f} = \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} \mathbf{f}^{(i,k)\wedge_{\mathbf{h}(\{\mathbf{a}_n\};\overline{\Omega_\sigma^{\mathrm{ext}}})}}(n,m) \mathbf{h}_{n,m}^{(i,k)*\{A_n^{(i,k)}\}}, \tag{3.80}$$

where

$$\mathbf{f}^{(i,k)\wedge_{\mathbf{h}(\{\mathbf{a}_n\};\overline{\Omega_\sigma^{\mathrm{ext}}})}}(n,m) = \mathbf{f}^{(i,k)\wedge}(n,m) = (\mathbf{f}, \mathbf{h}_{n,m}^{(i,k)*\{A_n^{(i,k)}\}})_{\mathbf{h}(\overline{\Omega_\sigma^{\mathrm{ext}}})}. \tag{3.81}$$

Finally, in analogy to the vectorial spaces $h_s^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$, we define

$$\mathbf{h}_s^{(i,k)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \mathbf{h}^{(i,k)}\left(\left\{\left(\frac{n+\frac{1}{2}}{\sigma}\right)^s\right\};\overline{\Omega_\sigma^{\mathrm{ext}}}\right), \tag{3.82}$$

$$\mathbf{h}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \bigoplus_{i,k=1}^{3} \mathbf{h}_s^{(i,k)}(\overline{\Omega_\sigma^{\mathrm{ext}}}). \tag{3.83}$$

The space $\mathbf{h}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ equipped with the inner product $(\cdot,\cdot)_{\mathbf{h}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}$ is a Hilbert space. The system $\{\mathbf{h}_{n,m}^{(i,k)s}(\sigma;\cdot)\}_{i,k\in\{1,2,3\};\, n\in\mathbb{N}_{0_{ik}};\, m=1,\ldots,2n+1}$, given by

$$\mathbf{h}_{n,m}^{(i,k)s}(\sigma;x) = \left(\frac{\sigma}{n+\frac{1}{2}}\right)^s \mathbf{h}_{n,m}^{(i,k)}(\sigma;x), \quad x\in\overline{\Omega_\sigma^{\mathrm{ext}}}, \tag{3.84}$$

represents an $\mathbf{h}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-orthonormal Hilbert basis in $\mathbf{h}(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

Our next goal is to extend the Sobolev Lemma 3.4 to tensor fields.

Definition 3.12. A sequence $\{\mathbf{a}_n\}_{n\in\mathbb{N}_0}\in\mathbf{a}$ is called summable if

$$\sum_{n=0_{ik}}^{\infty} \frac{2n+1}{\left(A_n^{(i,k)}\right)^2} < \infty \tag{3.85}$$

for $i,k\in\{1,2,3\}$.

In the sequel, $\{\mathbf{b}_n^{-1}\}_{n\in\mathbb{N}_0}\in\mathbf{a}$ represents the sequence given by

$$\mathbf{b}_n^{-1} = \begin{pmatrix} \left(B_n^{(1,1)}\right)^{-1} & \left(B_n^{(1,2)}\right)^{-1} & \left(B_n^{(1,3)}\right)^{-1} \\ \left(B_n^{(2,1)}\right)^{-1} & \left(B_n^{(2,2)}\right)^{-1} & \left(B_n^{(2,3)}\right)^{-1} \\ \left(B_n^{(3,1)}\right)^{-1} & \left(B_n^{(3,2)}\right)^{-1} & \left(B_n^{(3,3)}\right)^{-1} \end{pmatrix}, \tag{3.86}$$

and $\{\mathbf{a}_n^{-1}\mathbf{b}_n\}_{n\in\mathbb{N}_0}\in\mathbf{a}$ is given by

$$\mathbf{b}_n^{-1}\mathbf{a}_n = \begin{pmatrix} A_n^{(1,1)}\left(B_n^{(1,1)}\right)^{-1} & A_n^{(1,2)}\left(B_n^{(1,2)}\right)^{-1} & A_n^{(1,3)}\left(B_n^{(1,3)}\right)^{-1} \\ A_n^{(2,1)}\left(B_n^{(2,1)}\right)^{-1} & A_n^{(2,2)}\left(B_n^{(2,2)}\right)^{-1} & A_n^{(2,3)}\left(B_n^{(2,3)}\right)^{-1} \\ A_n^{(3,1)}\left(B_n^{(3,1)}\right)^{-1} & A_n^{(3,2)}\left(B_n^{(3,2)}\right)^{-1} & A_n^{(3,3)}\left(B_n^{(3,3)}\right)^{-1} \end{pmatrix}. \tag{3.87}$$

Lemma 3.13 (Tensorial Sobolev Lemma). *Assume, that the sequences* $\{\mathbf{a}_n\}_{n\in\mathbb{N}_0}$, $\{\mathbf{b}_n\}_{n\in\mathbb{N}_0}\in\mathbf{a}$ *are such that* $\{\mathbf{b}_n^{-1}\mathbf{a}_n\}_{n\in\mathbb{N}_0}\in\mathbf{a}$ *is summable. Then each* $\mathbf{f}\in\mathbf{h}\left(\{\mathbf{b}_n^{-1}\mathbf{a}_n\};\overline{\Omega_\sigma^{\mathrm{ext}}}\right)$ *corresponds to a function of class* $\mathbf{harm}(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

4. Pseudodifferential operators and geodetic nomenclature

All gravitational information under discussion in physical geodesy leads to operator equations relating the (disturbing) potential to geodetically relevant observables. In physical geodesy, one can think of observables as operating on an "input signal" F (e.g., the (disturbing) potential) to produce an (scalar, vectorial or tensorial) output signal of the form

$$\Lambda F = G \tag{4.1}$$

(for example, geoidal undulation, gravity anomaly, radial or tangential derivatives), where Λ is a certain (scalar, vectorial or tensorial) operator. Note, that later on we will differentiate in our notation weather we deal with scalar, vectorial or tensorial observables, but in this introductory part of the text for reason of readability we do not distinguish the geodetic quantities. Fortunately, it is the case in geodetic applications involving the (disturbing) potential that large portions of interest can be well approximated by operators that represent linear, rotation-invariant pseudodifferential operators.

The standard pseudodifferential operators Λ occurring in physical geodesy (cf. [69]) have to reflect the Pizzetti concept (cf. [36, 59]):

1. The mass within the reference ellipsoid for establishing the disturbing potential F is equal to the mass of the Earth.
2. The center of the reference ellipsoid coincides with the center of the Earth.
3. The value of the potential on the geoidal surface and the value of the normal potential on the reference ellipsoidal surface are the same.
4. There are no masses outside the geoid (remove-restore-principle from masses outside the geoid).
5. The constructive approximation is simplified for reasons of computational economy from an ellipsoidal to a spherical framework by Runge–Walsh justification (see the contribution [4] in this volume).

The presentation of the classical quantities in gravitational potential determination can be formulated within the framework of pseudodifferential operators. To be more concrete, in our approach we deal with radial, tangential and mixed (first- and second-order) derivatives of the Earth gravitational potential. Two important properties have to be taken into account specifying the operators which we study in the sequel. On the one hand, the mathematical modeling should lead to a consistent setup. It turns out that this requirement is, in fact, assured by the operators. On the other hand, we demand the assigned operators to be isotropic for structural reasons. In consequence (see also [63]), the (scalar) tangential derivatives $\frac{\partial}{\partial \varphi}$ and $\frac{\partial}{\partial t}$ are of no interest for us because they do not lead to isotropic operators in a scalar framework. Instead of using scalar tangential operators we decide to go over to the vectorial (and tensorial) tangential derivative using the surface gradient ∇^*. Indeed, we want to point out that we have the choice between two viable variants namely either to develop a scalar anisotropic theory for component modeling, or to turn over to vectorial/tensorial isotropic theory. In this contribution, we prefer the second variant, expecting that the development of a vector/tensor theory provides us with a versatile tool for modeling geodetically relevant vector and tensor fields and solving the SST and SGG problem in a simply structured isotropic framework. The observables we discuss are presented in Tables 1, 2 and 3.

Quantity	Operator	Symbol	Order
gravity anomaly	Λ_A	$\frac{n-1}{\sigma}$	1
geoid undulations	Λ_U	σ^2	0
Stokes operator	Λ_{St}	$\frac{\sigma}{n-1}$	-1
first radial derivative	$\Lambda_{\frac{\partial}{\partial r}}$	$-\frac{n+1}{\sigma}$	1
second radial derivative	$\Lambda_{\frac{\partial^2}{\partial r^2}}$	$\frac{(n+1)(n+2)}{\sigma^2}$	2
upward continuation	Λ_{UPC}	$\left(\frac{\sigma}{\gamma}\right)^n$	$-\infty$
scalar SST	Λ_{SST}	$\left(\frac{\sigma}{\gamma}\right)^n \frac{n+1}{\gamma}$	$-\infty$
scalar SGG	Λ_{SGG}	$\left(\frac{\sigma}{\gamma}\right)^n \frac{(n+1)(n+2)}{\gamma^2}$	$-\infty$

TABLE 1. Scalar geodetic observables leading to isotropic pseudodifferential operators (note that the symbol is given with respect to $H_{n,m}$).

4.1. Scalar theory

We start with the scalar definition and give some examples.

Definition 4.1. Let $\mathcal{H}_s(\overline{\Omega_\tau^{\text{ext}}})$ and $\mathcal{H}_s(\overline{\Omega_\rho^{\text{ext}}})$ be Sobolev spaces, $\tau, \rho > 0$. Furthermore, let $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ be a sequence of real numbers. The operator $\Lambda : \mathcal{H}_s(\overline{\Omega_\tau^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\rho^{\text{ext}}})$ defined by

$$\Lambda F = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \Lambda^\wedge(n) F^\wedge(n,m) H^s_{n,m}(\rho;\cdot) \tag{4.2}$$

is called a *scalar pseudodifferential operator* of order t, if

$$\lim_{n\to\infty} \frac{|\Lambda^\wedge(n)|}{\left(n+\frac{1}{2}\right)^t} = \text{const} \neq 0 \tag{4.3}$$

for some $t \in \mathbb{R}$. The sequence $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ is called the *symbol* of Λ. Moreover, if the limit relation

$$\lim_{n\to\infty} \frac{|\Lambda^\wedge(n)|}{\left(n+\frac{1}{2}\right)^t} = 0 \tag{4.4}$$

holds for all $t \in \mathbb{R}$, then the operator is called a *pseudodifferential operator of order* $-\infty$.

Quantity	Operator	Symbol	Order
first tangential derivative	∇^*	$\frac{n}{\sigma}\sqrt{\frac{n+1}{2n+1}}$, $\frac{n+1}{\sigma}\sqrt{\frac{n}{2n+1}}$, 0	1
second mixed derivative	$\nabla^* \frac{\partial V}{\partial r}$,	$\frac{n(n+1)}{\sigma^2}\sqrt{\frac{n+1}{2n+1}}$, $\frac{(n+1)^2}{\sigma^2}\sqrt{\frac{n}{2n+1}}$, 0	2
vectorial SST	λ_{SST}	$\left(\frac{\sigma}{\gamma}\right)^n \frac{n}{\gamma}\sqrt{\frac{n+1}{2n+1}}$, $\left(\frac{\sigma}{\gamma}\right)^n \frac{n+1}{\gamma}\sqrt{\frac{n}{2n+1}}$, 0	$-\infty$,
vectorial SGG	λ_{SGG}	$\left(\frac{\sigma}{\gamma}\right)^n \frac{n(n+1)}{\gamma^2}\sqrt{\frac{n+1}{2n+1}}$ $\left(\frac{\sigma}{\gamma}\right)^n \frac{(n+1)^2}{\gamma}\sqrt{\frac{n}{2n+1}}$, 0	$-\infty$

TABLE 2. Vectorial geodetic observables leading to isotropic pseudodifferential operators (note that the symbol is given with respect to $h^{(i)}_{n,m}$, $i = 1, 2, 3$ from top to down for each operator).

Note that the convergence of the series in (4.2) is understood in $\mathcal{H}_s(\overline{\Omega^{\text{ext}}_\rho})$-topology. As an immediate consequence (cf. [69]), we have the important relation

$$\Lambda H^s_{n,m}(\tau;\cdot) = \Lambda^\wedge(n) H^s_{n,m}(\rho;\cdot). \tag{4.5}$$

In other words, we have the requirement that the outer harmonics are the eigenfunctions of the operator Λ, and the invertibility has to be controlled by the invertibility of the values $\Lambda^\wedge(n)$, $n \in \mathbb{N}_0$. The symbol has many appealing properties (cf. [69]): It is easily seen that

$$(\Lambda' + \Lambda'')^\wedge(n) = (\Lambda')^\wedge(n) + (\Lambda'')^\wedge(n), \tag{4.6}$$

$$(\Lambda'\Lambda'')^\wedge(n) = (\Lambda')^\wedge(n)(\Lambda'')^\wedge(n), \tag{4.7}$$

for all $n \in \mathbb{N}_0$.

As any "output function" (output signal) can be expanded into an orthogonal series of outer harmonics

$$G = \Lambda F = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \Lambda^\wedge(n) F^\wedge(n,m) H^s_{n,m}(\rho;\cdot) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} G^\wedge(n,m) H^s_{n,m}(\rho;\cdot) \tag{4.8}$$

Quantity	Operator	Symbol	Order
second tangential derivative	$\nabla^* \otimes \nabla^*$	$\frac{n(n+1)}{\sigma^2(2n+1)(2n+3)}\sqrt{(n+2)(n+1)(2n+1)(2n+3)}$, $\frac{-(n+1)(n-1)}{\sigma^2(2n-1)(2n+1)}\sqrt{3}n^2$, 0, $\frac{-n(n+2)}{\sigma^2(2n+3)(2n+1)}(n+1)\sqrt{(2n+1)(2n+3)}$, $\frac{n(n+1)(n+2)}{\sigma^2(2n-1)(2n+1)}\sqrt{n(n-1)(2n-1)(2n+1)}$, 0, for $(i,k) \in \{(2,3),(3,1),(3,2),(3,3)\}$	2
tensorial SGG	$\boldsymbol{\lambda}_{SGG}$	$\left(\frac{\sigma}{\gamma}\right)^n \frac{n(n+1)}{\gamma^2(2n+1)(2n+3)}\sqrt{(n+2)(n+1)(2n+1)(2n+3)}$, $-\left(\frac{\sigma}{\gamma}\right)^n \frac{-(n+1)(n-1)}{\gamma^2(2n-1)(2n+1)}\sqrt{3}n^2$, 0, $-\left(\frac{\sigma}{\gamma}\right)^n \frac{-n(n+2)}{\gamma^2(2n+3)(2n+1)}(n+1)\sqrt{(2n+1)(2n+3)}$, $\left(\frac{\sigma}{\gamma}\right)^n \frac{n(n+1)(n+2)}{\sigma^2(2n-1)(2n+1)}\sqrt{n(n-1)(2n-1)(2n+1)}$, 0, for $(i,k) \in \{(2,3),(3,1),(3,2),(3,3)\}$	$-\infty$

TABLE 3. Tensorial geodetic observables leading to isotropic pseudodifferential operators (note that the symbol is given with respect to $\mathbf{h}^{(i,k)}_{n,m}$, $i,k = 1,2,3$, from top to down $((1,1),(1,2),\ldots,(3,2),(3,3))$ for each operator).

in the sense of $\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\rho^{\mathrm{ext}}})}$, we are confronted with a spectral representation of the form

$$G^\wedge(n,m) = (\Lambda F)^\wedge(n,m) = \Lambda^\wedge(n)\ F^\wedge(n,m), \quad n \in \mathbb{N}_0,\ k = 1,\ldots,2n+1. \tag{4.9}$$

This means that the "amplitude spectrum" $\{G^\wedge(n,m)\}$ of the response of Λ is described in terms of the amplitude spectrum of functions (signals) F by a simple multiplication by the "transfer" $\Lambda^\wedge(n)$.

The following list contains (scalar) pseudodifferential operators which are of importance for geodetic applications.
Consider a potential F of the class $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, that is

$$F = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} F^\wedge(n,m) H^s_{n,m}(\sigma;\cdot), \tag{4.10}$$

where we use the geometric concept as explained in Section 3 and shown in Figure 3.1.

(i) *Gravity Anomalies*. The problem of determining the disturbing potential U with $\Lambda(U) = F$ from prescribed gravity anomalies F is the "fundamental problem of classical physical geodesy" (see, e.g., [37, 43, 53, 69]). The operator related to gravity anomalies $\Lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ has the symbol

$$\Lambda^\wedge(n) = \frac{n-1}{\sigma}. \tag{4.11}$$

(ii) *Geoid Undulations*. The operator related to geoid undulations $\Lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ has the symbol

$$\Lambda^\wedge(n) = \sigma^2. \tag{4.12}$$

(iii) *Stokes Operator*. This operator is defined by

$$\Lambda(F)(x) = \frac{\sigma}{4\pi}\int_{\Omega_\sigma} St(x,y)F(y), d\omega(y), \quad x \in \Omega_\sigma \tag{4.13}$$

where $St(\cdot,\cdot)$ is the Stokes kernel (cf. [32, 68, 69]).
The Stokes operator $\Lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ has the symbol

$$(\Lambda)^\wedge(n) = \begin{cases} 0\,, & \text{for} \quad n=1 \\ \frac{\sigma}{n-1}\,, & \text{for} \quad n = 0,2,3,4,\ldots\ . \end{cases} \tag{4.14}$$

(iv) *Upward Continuation Operator*. The upward continuation operator associates to $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ the solution ΛF of the Dirichlet problem $\Lambda F \in Pot^{(0)}(\overline{\Omega_\gamma^{\text{ext}}})$ corresponding to the boundary values $(\Lambda F)|_{\Omega_\gamma} = F|_{\Omega_\gamma}$. The upward continuation operator $\Lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$ has the symbol

$$\Lambda^\wedge(n) = \left(\frac{\sigma}{\gamma}\right)^n, \quad n \in \mathbb{N}_0. \tag{4.15}$$

The upward continuation operator indeed plays an important role in the mathematical treatment of spaceborne problems, since it relates potential values at height σ to potential values at height $\gamma (> \sigma)$.

(v) *Operator of the (Negative) First-order Radial Derivative on* Ω_σ. This operator associates to $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ the solution ΛF of the Dirichlet problem $\Lambda F \in Pot^{(0)}(\overline{\Omega_\sigma^{\text{ext}}})$ corresponding to the boundary values $(\Lambda F)|_{\Omega_\sigma} = -\frac{\partial}{\partial r}F|_{\Omega_\sigma}$. Λ is a pseudodifferential operator of order 1 with symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ given by

$$\Lambda^\wedge(n) = \frac{n+1}{\sigma}, \quad n \in \mathbb{N}_0. \tag{4.16}$$

In fact, Λ is the "harmonic continuation" of the radial derivative on Ω_σ into the outer space $\Omega_\sigma^{\text{ext}}$ and is important in case of the SST problem.

(vi) *Operator of the Second-order Radial Derivative on* Ω_σ. This operator associates to $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ the solution ΛF of the Dirichlet problem $\Lambda F \in Pot^{(0)}(\overline{\Omega_\sigma^{\text{ext}}})$ corresponding to the boundary values $(\Lambda F)|_{\Omega_\sigma} = \frac{\partial^2}{\partial r^2}F|_{\Omega_\sigma}$. Λ is a pseudodifferential operator of order 2 with symbol $\{\Lambda^\wedge(n)\}_{n\in\mathbb{N}_0}$ given by

$$\Lambda^\wedge(n) = \frac{(n+1)(n+2)}{\sigma^2}, \quad n \in \mathbb{N}_0. \tag{4.17}$$

Λ is the "harmonic continuation" of the second radial derivative on Ω_σ into the outer space $\Omega_\sigma^{\text{ext}}$ and is important in case of the SGG problem.

4.2. Vectorial theory

We now introduce vectorial pseudodifferential operators and give two examples.

Definition 4.2. Let $\mathcal{H}_s(\overline{\Omega_\tau^{\mathrm{ext}}})$ be a scalar Sobolev space and $h_s^{(i)}(\overline{\Omega_\rho^{\mathrm{ext}}})$ a vectorial Sobolev space, $\tau, \rho > 0$, $i \in \{1,2,3\}$. Furthermore, let $\{\lambda^{(i)\wedge}(n)\}_{n\in\mathbb{N}_{0_i}}$ be a sequence of real numbers for $i = 1,2,3$. The operator $\lambda^{(i)} : \mathcal{H}_s(\overline{\Omega_\tau^{\mathrm{ext}}}) \to h_s^{(i)}(\overline{\Omega_\rho^{\mathrm{ext}}})$ defined by

$$\lambda^{(i)}F = \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} \lambda^{(i)\wedge}(n) F^\wedge(n,m) h_{n,m}^{(i)s}(\rho;\cdot) \tag{4.18}$$

is called a *vectorial pseudodifferential operator* of kind i and order t, if

$$\lim_{n\to\infty} \frac{|\lambda^{(i)\wedge}(n)|}{(n+\frac{1}{2})^t} = \text{ const } \neq 0 \tag{4.19}$$

for some $t \in \mathbb{R}$. Moreover, if the limit relation

$$\lim_{n\to\infty} \frac{|\lambda^{(i)\wedge}(n)|}{(n+\frac{1}{2})^t} = 0 \tag{4.20}$$

holds for all $t \in \mathbb{R}$, then the operator $\lambda^{(i)}$ is called a *vectorial pseudodifferential operator of kind i and order $-\infty$*. The sequence $\{\lambda^{(i)\wedge}(n)\}$ is called the *symbol* of $\lambda^{(i)}$. Further on, the operator $\lambda : \mathcal{H}_s(\overline{\Omega_\tau^{\mathrm{ext}}}) \to h_s(\overline{\Omega_\rho^{\mathrm{ext}}})$ defined by

$$\lambda = \sum_{i=1}^{3} \lambda^{(i)}, \tag{4.21}$$

is called a *vectorial pseudodifferential operator of order* t, where $t = \max_{i=1}^{3}$ (*order of* $\lambda^{(i)}$). Moreover, if the limit relation

$$\lim_{n\to\infty} \frac{|\lambda^{(i)\wedge}(n)|}{(n+\frac{1}{2})^t} = 0 \tag{4.22}$$

holds for all $t \in \mathbb{R}$, and all $i \in \{1,2,3\}$, then the operator λ is called a *vectorial pseudodifferential operator of order $-\infty$*.

We now give two examples of vectorial pseudodifferential operators which are important for geodetic applications. We use the surface gradient on the sphere Ω_σ defined by

$$\nabla^{*;\sigma} = \frac{1}{\sigma}\nabla^*. \tag{4.23}$$

(iv) *The Operator of the First-order Tangential Derivatives on* Ω_σ. This operator associates to $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ the solution λF of the Dirichlet problem $\lambda F \in h_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ corresponding to the boundary value $(\lambda F)|_{\Omega_\sigma} = \nabla^{*,\sigma} F|_{\Omega_\sigma}$. λ is a pseudodifferential operator of order 1 with symbol $\{\lambda^{(i)\wedge}(n)\}_{n\in\mathbb{N}_{0_i}}$ given by

$$\lambda^{(i)\wedge}(n) = \begin{cases} \frac{n}{\sigma}\sqrt{\frac{n+1}{2n+1}}, & i=1, \\ \frac{n+1}{\sigma}\sqrt{\frac{n}{2n+1}}, & i=2, \\ 0, & i=3. \end{cases} \tag{4.24}$$

In fact, Λ is the "harmonic continuation" of the tangential derivative on Ω_σ into the outer space $\Omega_\sigma^{\text{ext}}$ and is important in case of the SST problem.

(v) *The Operator of the (Negative) Second-order Mixed Derivatives on* Ω_σ. This operator associates to $F\in\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ the solution λF of the Dirichlet problem $\lambda F\in h_s(\overline{\Omega_\sigma^{\text{ext}}})$ corresponding to the boundary values $(\lambda F)|_{\Omega_\sigma}=-\frac{\partial}{\partial r}\nabla_\xi^{*,\sigma}F|_{\Omega_\sigma}$. λ is a pseudodifferential operator of second order with symbol $\{\lambda^{(i)\wedge}(n)\}_{n\in\mathbb{N}_{0_i}}$ given by

$$\lambda^{(i)\wedge}(n) = \begin{cases} \frac{n(n+1)}{\sigma^2}\sqrt{\frac{n+1}{2n+1}}, & i=1, \\ \frac{(n+1)^2}{\sigma^2}\sqrt{\frac{n}{2n+1}}, & i=2, \\ 0, & i=3. \end{cases} \tag{4.25}$$

Λ is the "harmonic continuation" of the second-order mixed derivatives on Ω_σ into the outer space $\Omega_\sigma^{\text{ext}}$ and is important in case of the SGG problem.

4.3. Tensorial theory

The introduction of tensorial pseudodifferential operators is straightforward.

Definition 4.3. Let $\mathcal{H}_s(\overline{\Omega_\tau^{\text{ext}}})$ be a scalar Sobolev space and $\mathbf{h}_s^{(i,k)}(\overline{\Omega_\rho^{\text{ext}}})$ a tensorial Sobolev space, $\tau,\rho>0$, $i,k\in\{1,2,3\}$. Furthermore, for $i,k\in\{1,2,3\}$, let $\boldsymbol{\lambda}^{(i,k)\wedge}(n)_{n\in\mathbb{N}_{0_{ik}}}$ be a sequence of real numbers. The operator $\boldsymbol{\lambda}^{(i,k)\wedge}(n)$: $\mathcal{H}_s(\overline{\Omega_\tau^{\text{ext}}})\to\mathbf{h}_s^{(i,k)}(\overline{\Omega_\rho^{\text{ext}}})$ defined by

$$\boldsymbol{\lambda}^{(i,k)}F=\sum_{n=0_{ik}}^{\infty}\sum_{m=1}^{2n+1}\boldsymbol{\lambda}^{(i,k)\wedge}(n)\}F^\wedge(n,m)\mathbf{h}_{n,m}^{(i,k)s}(\rho;\cdot) \tag{4.26}$$

is called a *tensorial pseudodifferential operator* of kind (i,k) and order t, if the limit relation

$$\lim_{n\to\infty}\frac{|\boldsymbol{\lambda}^{(i,k)\wedge}(n)|}{(n+\frac{1}{2})^t} = \text{const} \neq 0 \tag{4.27}$$

is satisfied for some $t\in\mathbb{R}$. Moreover, if the limit relation

$$\lim_{n\to\infty}\frac{|\boldsymbol{\lambda}^{(i,k)\wedge}(n)|}{(n+\frac{1}{2})^t}=0 \tag{4.28}$$

holds for all $t\in\mathbb{R}$, then the operator $\boldsymbol{\lambda}$ is called a *pseudodifferential operator of kind* (i,k) *and order* $-\infty$. The sequence $\{\boldsymbol{\lambda}^{(i,k)\wedge}(n)\}$ is called the (*spherical*) *symbol* of $\boldsymbol{\lambda}^{(i,k)}$. Further on, the operator $\boldsymbol{\lambda} : \mathcal{H}_s(\overline{\Omega_\tau^{\text{ext}}}) \to \mathbf{h}_s(\overline{\Omega_\rho^{\text{ext}}})$ defined by

$$\boldsymbol{\lambda} = \sum_{i=1}^{3}\sum_{k=1}^{3} \boldsymbol{\lambda}^{(i,k)}, \tag{4.29}$$

is called a *tensorial pseudodifferential operator of order* t, where $t = \max_{i,k=1}^{3}$ (*order of* $\boldsymbol{\lambda}^{(i,k)}$). Moreover, if the limit relation

$$\lim_{n\to\infty} \frac{|\, \boldsymbol{\lambda}^{(i,k)\wedge}(n)|}{(n+\frac{1}{2})^t} = 0 \tag{4.30}$$

holds for all $t \in \mathbb{R}$, and all $i, k \in \{1, 2, 3\}$, then the operator $\boldsymbol{\lambda}$ is called a *pseudodifferential operator of order* $-\infty$.

Finally, we mention one important example.

(iv) *The Operator of the Second-order Tangential Derivatives on* Ω_σ. This operator associates to $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ the solution $\boldsymbol{\lambda}F$ of the Dirichlet problem $\boldsymbol{\lambda}F \in \mathbf{h}_s(\overline{\Omega_\sigma^{\text{ext}}})$ corresponding to the boundary values $(\boldsymbol{\lambda}F)|_{\Omega_\sigma} = \nabla^{*,\sigma} \otimes \nabla^{*,\sigma} F|_{\Omega_\sigma}$. It is a pseudodifferential operator of order 2 with the symbol $\{\boldsymbol{\lambda}^{(i,k)\wedge}(n)\}_{n\in\mathbb{N}_{0_{ik}}}$ given by

$$\boldsymbol{\lambda}^{(i,k)\wedge}(n) \tag{4.31}$$
$$= \begin{cases} \frac{n(n+1)}{\sigma^2(2n+1)(2n+3)}\sqrt{(n+2)(n+1)(2n+1)(2n+3)}, & (i,k) = (1,1), \\ \frac{-(n+1)(n-1)}{\sigma^2(2n-1)(2n+1)}\sqrt{3}n^2, & (i,k) = (1,2), \\ \frac{-n(n+2)}{\sigma^2(2n+3)(2n+1)}(n+1)\sqrt{(2n+1)(2n+3)}, & (i,k) = (2,1), \\ \frac{n(n+1)(n+2)}{\sigma^2(2n-1)(2n+1)}\sqrt{n(n-1)(2n-1)(2n+1)}, & (i,k) = (2,2), \\ 0, & \text{else.} \end{cases}$$

Λ is the “harmonic continuation” of the second-order tangential derivatives on Ω_σ into the outer space $\Omega_\sigma^{\text{ext}}$ and is important in case of the SGG problem.

5. Reproducing kernel structure and observational functionals

Of great importance for our considerations are Sobolev spaces equipped with a reproducing kernel structure. The importance of the reproducing kernel lies in the fact that it determines the norm of the dual space. Furthermore, no computational work must be done to evaluate inner products involving reproducing kernel expressions. Within this section, we focus on scalar theory and essentially follow [19]. The extension to vectorial and tensorial reproducing kernel Sobolev spaces is not hard to perform.

5.1. Reproducing Hilbert spaces

Theorem 5.1. *Let the sequence $\{A_n\}$ be summable in the sense of Definition* 3.3. *Then $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ (more explicitly, $\mathcal{H}(\{A_n\};\overline{\Omega_\sigma^{\mathrm{ext}}})$) is a Hilbert subspace of the space $Pot^{(0)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$. The space $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ has the reproducing kernel function*

$$K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot) : \overline{\Omega_\sigma^{\mathrm{ext}}} \times \overline{\Omega_\sigma^{\mathrm{ext}}} \to \mathbb{R}$$

given by

$$K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(x,y) = \sum_{n\in\mathcal{N}(A_n)} \sum_{m=1}^{2n+1} H_{n,m}^{*\{A_n\}}(\sigma;x) H_{n,m}^{*\{A_n\}}(\sigma;y),$$

$x, y \in \overline{\Omega_\sigma^{\mathrm{ext}}}$.

If $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ has a reproducing kernel, then the Fourier (orthogonal) expansion of a potential in terms of the Hilbert basis $\{H_{n,k}^*(\sigma;\cdot)\}$ in $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ converges uniformly on the domain $\overline{\Omega_\sigma^{\mathrm{ext}}}$ (cf. [3, 7]). To be more specific, the relation

$$\lim_{N\to\infty} \left\| F - \sum_{\substack{n\in\mathcal{N}\\ n\le N}} \sum_{m=1}^{2n+1} F^\wedge(n,m) H_{n,m}^{*\{A_n\}}(\sigma;\cdot) \right\|_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})} = 0$$

implies

$$\lim_{N\to\infty} \sup_{x\in\overline{\Omega_\sigma^{\mathrm{ext}}}} \left| F(x) - \sum_{\substack{n\in\mathcal{N}\\ n\le N}} \sum_{m=1}^{2n+1} F^\wedge(n,m) H_{n,m}^*(\sigma;x) \right| = 0.$$

The representer of a bounded linear functional $\mathcal{L}$ on $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ has a simple expression. More explicitly, $L(x) = \mathcal{L}K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,x), x \in \overline{\Omega_\sigma^{\mathrm{ext}}}$, is in $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$, and for all $F \in \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ we have $\mathcal{L}F = (F,L)_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}$ (note that x is held fixed and $\mathcal{L}$ is applied to $K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,x)$ as a function of the first variable). Obviously, $(L,L)_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})} = \mathcal{L}\mathcal{L}K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot) = (\mathcal{L},\mathcal{L})_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})^*}$. The *dual space* $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})^*$ of $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ (i.e., the space of all linear bounded functionals on $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$) is a Hilbert space with respect to $\|\cdot\|_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})^*} = (\cdot,\cdot)^{\frac{1}{2}}_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})^*}$; the spaces $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ and $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})^*$ are known as isomorphic and isometric (see, e.g., [7]).

Reproducing kernel representations may be used to act as basis system in reproducing Sobolev spaces.

Theorem 5.2. *Let $\{A_n\}$ be summable in the sense of Definition* 3.3. *Assume that X is a countable dense set of points on a regular surface $\Xi \subset \overline{\Omega_\sigma^{\mathrm{ext}}}$ (for example, Runge sphere Ω_σ, real Earth's surface Σ). Then*

$$\overline{\mathrm{span}_{x\in X} K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(x,\cdot)}^{\|\cdot\|_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}} = \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}}).$$

Theorem 5.2 allows an obvious generalization by means of bounded linear functionals on $\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})$.

Theorem 5.3. *Let $\{A_n\}$ be summable. Assume that X is a countable dense set of linear functionals in $\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})^*$. Then*

$$\overline{\operatorname{span}_{\mathcal{L}\in X}\mathcal{L}K_{\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})}(\cdot,\cdot)}^{\|\cdot\|_{\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})}} = \mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}}).$$

The set of all finite linear combinations of outer harmonics is dense in the space $Pot^{(0)}(\overline{\Omega_\sigma^{\text{ext}}})$ in the sense of $\|\cdot\|_{C^{(0)}(\overline{\Omega_\sigma^{\text{ext}}})}$. Hence, $\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})$ is a dense subset of $Pot^{(0)}(\overline{\Omega_\sigma^{\text{ext}}})$, too. This leads us to the following corollary.

Corollary 5.4. *Under the assumption of Theorem* 5.3

$$\overline{\operatorname{span}_{\mathcal{L}\in X}\mathcal{L}K_{\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})}(\cdot,\cdot)}^{\|\cdot\|_{C^{(0)}(\overline{\Omega_\sigma^{\text{ext}}})}} = Pot^{(0)}(\overline{\Omega_\sigma^{\text{ext}}}).$$

Next we come to the problem of specifying certain types of sequences $\{A_n\}$ such that $\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})(=\mathcal{H}(\{A_n\};\overline{\Omega_\sigma^{\text{ext}}}))$ is a reproducing kernel Hilbert space. We restrict ourselves to those kernel functions which are usable later on in multiscale approximation. Other types of kernel functions which are known from spline interpolation or smoothing procedures (see, for example, [14–16, 18, 20, 49, 55, 56, 72]) are not discussed here.

Our list of (reproducing) kernel functions is divided into two parts, namely *bandlimited kernel functions* such as Shannon's kernel, smoothed Shannon kernels, etc., and *non-bandlimited kernel functions* such as rational kernel functions, exponential kernel functions, (smoothed) Haar kernel functions, etc.

5.2. Bandlimited kernel functions

These kernel functions are characterized by the property that only a finite number of coefficients A_n does not vanish. Consequently, the reproducing kernel Hilbert space is of finite dimension.

At this stage two important cases of bandlimited kernels should be mentioned:

(a) *The Shannon Kernel* (see Figure 5.1). For a non-negative integer N we let

$$A_n = \begin{cases} 1, & n\in[0,N+1), \\ 0, & n\in[N+1,\infty), \end{cases}$$

i.e., $\mathcal{N}(A_n)=\{0,\dots,N\}$. Obviously, the reproducing kernel Hilbert space $\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})$ is equal to the space $Harm_{0,\dots,N}(\overline{\Omega_\sigma^{\text{ext}}})$ of outer harmonics of degree $\le N$. The reproducing kernel function $K_{\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})}(\cdot,\cdot):\overline{\Omega_\sigma^{\text{ext}}}\times\overline{\Omega_\sigma^{\text{ext}}}\to\mathbb{R}$, i.e., the Shannon kernel, reads as follows:

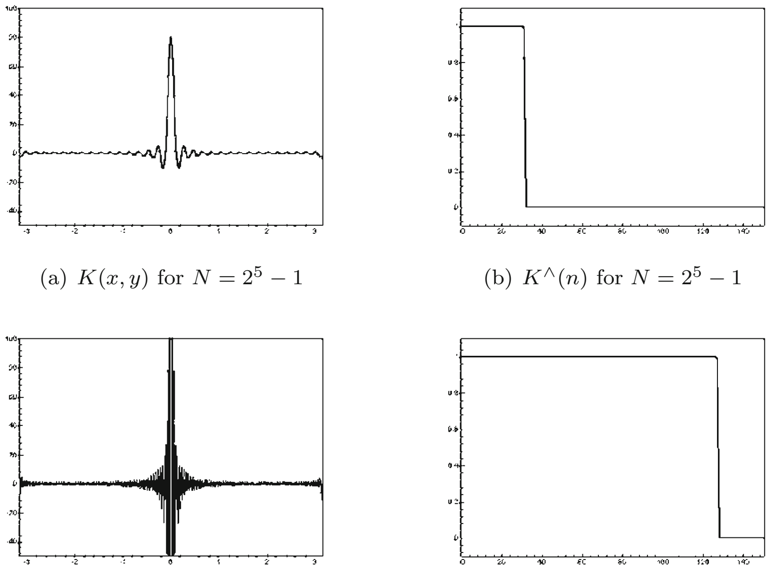
$$\begin{aligned} K_{\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})}(x,y) &= \sum_{0\le n\le N}\sum_{m=1}^{2n+1} H^*_{n,m}(\sigma;x)H^*_{n,m}(\sigma;y) \\ &= \sum_{0\le n\le N}\frac{2n+1}{4\pi\sigma^2}\left(\frac{\sigma^2}{|x|\,|y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right). \end{aligned} \tag{5.1}$$

Observing the well-known recursion relation for Legendre polynomials

$$(n+1)(P_{n+1}(t)-P_n(t))-n(P_n(t)-P_{n-1}(t)) = (2n+1)(t-1)P_n(t), \quad n \geq 1, \tag{5.2}$$

we obtain for $(x,y) \in \Omega_\sigma \times \Omega_\sigma$

$$\left(\frac{x}{|x|}\cdot\frac{y}{|y|}-1\right) K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(x,y) = \frac{N+1}{4\pi\sigma^2}\left(P_{N+1}\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) - P_N\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\right). \tag{5.3}$$



(a) $K(x,y)$ for $N = 2^5 - 1$

(b) $K^\wedge(n)$ for $N = 2^5 - 1$

(c) $K(x,y)$ for $N = 2^7 - 1$

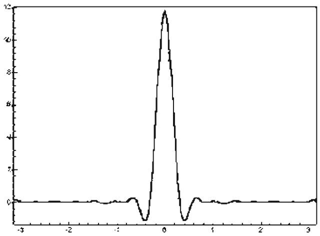
(d) $K^\wedge(n)$ for $N = 2^7 - 1$

FIGURE 5.1. Shannon kernel with $N = 2^5 - 1$ (above) and $N = 2^7 - 1$ (below): space domain, i.e., $K(x,y)$ for $(x,y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $K^\wedge(n) = A_n$ (right).

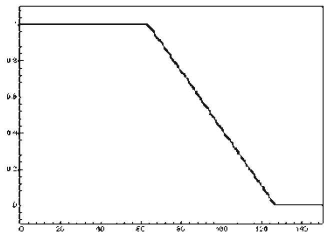
(b) *Smoothed Shannon Kernels* (see Figure 5.2). For (fixed) non-negative integers N, M with $N > M+1$ we let

$$A_n = \begin{cases} 1, & n \in [0, M+1), \\ \frac{N-m}{N-M}, & n \in [M+1, N+1), \\ 0, & n \in [N+1, \infty). \end{cases}$$

Of course, many other suitable choices can be found for practical purposes.



(a) $K(x,y)$



(b) $K^\wedge(n)$

FIGURE 5.2. Smoothed Shannon kernel with $M = 2^6$ and $N = 2^7 - 1$: space domain, i.e., $K(x,y)$ for $(x,y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $K^\wedge(n) = A_n$ (right).

5.3. Non-bandlimited kernel functions

All non-bandlimited kernels share the property that an infinite number of coefficients A_n is different from zero. The corresponding reproducing Hilbert kernel spaces are infinite-dimensional. We mention rational kernels, exponential kernels, and "locally supported" kernels, i.e., (smoothed) Haar kernels.

(a) *Rational Kernels* (see Figure 5.3). Let $\{A_n\}$ be a sequence of real numbers A_n satisfying the following conditions:
 (i) $n \mapsto A_n^2$, $n \in \mathbb{N}_0$, is a (real) rational function (in the integer variable n).
 (ii) There exist two positive constants C, C' with

$$C\left(\frac{n+(\frac{1}{2})}{\sigma}\right)^{2+\varepsilon} \leq A_n^2 \leq C'\left(\frac{n+(\frac{1}{2})}{\sigma}\right)^{\alpha} \tag{5.4}$$

 for some $\varepsilon > 0$, $\alpha \geq 2+\varepsilon$.

Then the norm reads

$$\|F\|^2_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})} = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} A_n^2 \; \left(F^\wedge(n,m)\right)^2 .$$

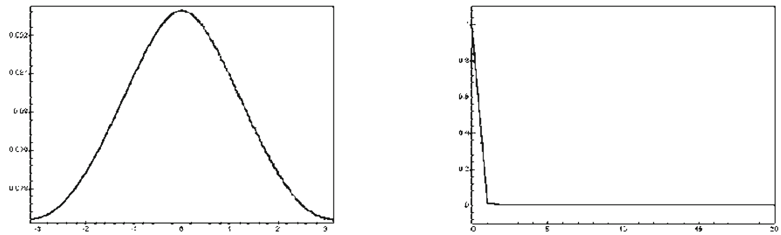
For the reproducing kernel in $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ we find the representation

$$K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(x,y) = \sum_{n=0}^{\infty} \frac{1}{A_n^2}\frac{2n+1}{4\pi\sigma^2}\left(\frac{\sigma^2}{|x|\,|y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right).$$

$\left\{A_n\left(\frac{n+\frac{1}{2}}{\sigma}\right)^{-\beta}\right\}$ is summable for all $\beta < \varepsilon/2$.

(b) *Exponential Kernels*. An alternative to come to candidates of reproducing kernel sum representations with an exponential rate of convergence is to use a sequence $\{A_n\}$ of the form

$$A_n = \left(\frac{\sigma}{\sigma'}\right)^n B_n, \quad n \in \mathcal{N}, \tag{5.5}$$



(a) $K(x,y)$ (b) $K^\wedge(n)$

FIGURE 5.3. Rational kernel with $A_n^2 = (1+n)^{-s}$, $s = 6.5$: space domain, i.e., $K(x,y)$ for $(x,y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $K^\wedge(n) = A_n$ (right).

with $\sigma' < \sigma$ and B_n satisfying

$$0 < B_n^2 \leq C' \left(\frac{n + (\frac{1}{2})}{\sigma} \right)^{\alpha} \tag{5.6}$$

for all $n \in \mathcal{N}$, some value α and a positive constant C'. The radius $\sigma'(< \sigma)$ should be taken close to the value σ (i.e., σ' is assumed to be the radius of a Runge sphere so that σ/σ' is close to 1). It is evident that an “inner radius” σ' gives additional flexibility in choosing the norm of the Hilbert space and also results in more general sequences $\{A_n\}$ being possible. On the other hand, the radius σ' appears as an artificial value in the infinite sum of the kernel to force an exponential rate of sum convergence. In conclusion, the sequence $\left\{ A_n \left(\frac{n+\frac{1}{2}}{\sigma} \right)^{-\beta} \right\}$ is summable for every β.

Kernel representations of type (5.5) for $(x,y) \in \overline{\Omega_\sigma^{\mathrm{ext}}} \times \overline{\Omega_\sigma^{\mathrm{ext}}}$

$$K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(x,y) = \sum_{n \in \mathcal{N}} \frac{1}{B_n^2} \frac{2n+1}{4\pi\sigma'^2} \left(\frac{\sigma'^2}{|x|\,|y|} \right)^{n+1} P_n \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right)$$

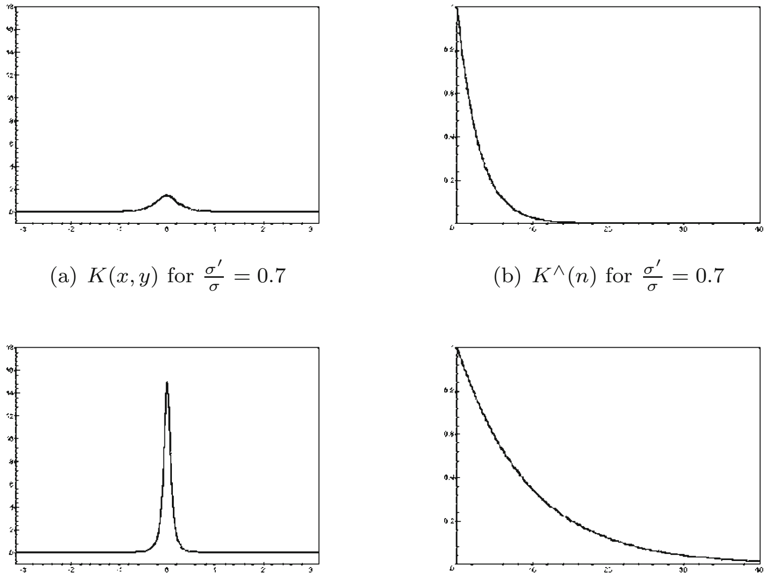
are well known from geophysical applications (see, for example, [14, 32, 55, 72]).

Far- and Near-Field Methods as well as Multipole Methods are explained in the Ph.D.-thesis [39] and can also be found in [24, 40, 41] and in the contribution [42] in this volume.

Of particular importance for purposes of minimum norm (spline) interpolation and smoothing (cf., e.g., [14–16, 18, 72]) are kernels, which are available in terms of elementary functions. We only mention here (cf. [52]):

(i) *Abel–Poisson kernel* (see Figure 5.4):

$$B_n^2 = 1, \quad n \in \mathbb{N}_0. \tag{5.7}$$



(a) $K(x,y)$ for $\frac{\sigma'}{\sigma} = 0.7$

(b) $K^\wedge(n)$ for $\frac{\sigma'}{\sigma} = 0.7$

(c) $K(x,y)$ for $\frac{\sigma'}{\sigma} = 0.9$

(d) $K^\wedge(n)$ for $\frac{\sigma'}{\sigma} = 0.9$

FIGURE 5.4. Abel–Poisson kernel with $\frac{\sigma'}{\sigma} = 0.7$ (above) and $\frac{\sigma'}{\sigma} = 0.9$ (below): space domain, i.e., $K(x,y)$ for $(x,y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $K^\wedge(n) = A_n$ (right).

The kernel reads as follows:

$$K_{\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})}(x,y) = \frac{|x|\,|y|}{4\pi\sigma'^2}\,\frac{|x|^2|y|^2 - \sigma'^4}{(L(x,y))^{3/2}}, \quad x,y \in \overline{\Omega_\sigma^{\text{ext}}},$$

where we have used the abbreviation

$$L(x,y) = |x|^2|y|^2 - 2\sigma'^2 x\cdot y + \sigma'^4.$$

(ii) *"Singularity kernel"* (see Figure 5.5)

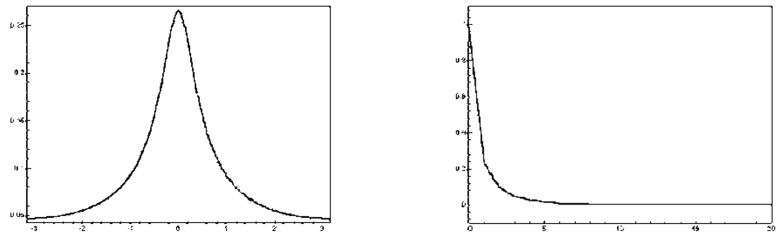
$$B_n^2 = (2n+1)/2, \quad n \in \mathbb{N}_0. \tag{5.8}$$

The kernel is given by

$$K_{\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})}(x,y) = \frac{1}{4\pi}\,\frac{1}{(L(x,y))^{\frac{1}{2}}}, \quad x,y \in \overline{\Omega_\sigma^{\text{ext}}}.$$

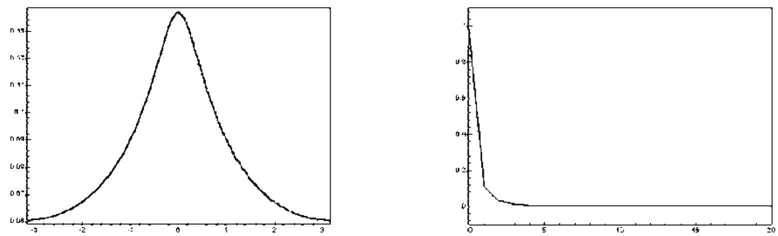
(iii) *"Logarithmic kernel"* (see Figure 5.6)

$$B_n^2 = (2n+1)(n+1), \quad n \in \mathbb{N}_0. \tag{5.9}$$



(a) $K(x,y)$ (b) $K^\wedge(n)$

Figure 5.5. Singularity kernel with $\frac{\sigma'}{\sigma} = 0.7$: space domain, i.e., $K(x,y)$ for $(x,y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $K^\wedge(n) = A_n$ (right).



(a) $K(x,y)$ (b) $K^\wedge(n)$

Figure 5.6. Logarithmic kernel with $\frac{\sigma'}{\sigma} = 0.7$: space domain, i.e., $K(x,y)$ for $(x,y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $K^\wedge(n) = A_n$ (right).

Now we have

$$K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(x,y) = \frac{1}{4\pi\sigma'^2} \ln\left(1 + \frac{2\sigma'^2}{M(x,y)}\right), \quad x,y \in \overline{\Omega_\sigma^{\mathrm{ext}}},$$

with

$$M(x,y) = (L(x,y))^{\frac{1}{2}} + |x|\,|y| - \sigma'^2.$$

(c) *"Locally Supported" Kernels* (*Smoothed Haar Kernels,* see Figure 5.7): Consider the piecewise polynomial function $B_h^{(k)} : [-1,+1] \to \mathbb{R}$, $k = 0,1,\dots$ and $h \in (0,1)$ given by

$$B_h^{(k)}(t) = \begin{cases} 0, & t \in [-1,h), \\ \frac{(t-h)^k}{(1-h)^k}, & t \in [h,1], \end{cases} \tag{5.10}$$

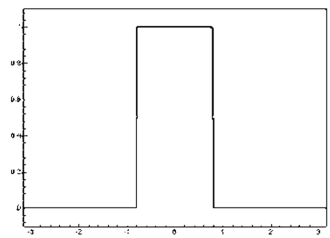
(cf. [5, 20, 21, 26, 35, 67]). Let $\xi \in \Omega = \Omega_1$ be fixed. Then the ξ-zonal function $B_h^{(k)}(\xi\ \cdot) : \Omega \to \mathbb{R}$ has a local support. More explicitly, the support of $B_h^{(k)}(\xi\ \cdot)$ is the cap with centre ξ characterized by

$$\mathrm{supp} B_h^{(k)}(\xi\ \cdot) = \{\eta \in \Omega\ :\ h \leq \xi \cdot \eta \leq 1\}.$$

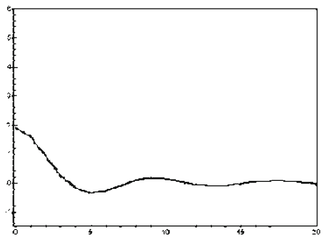
The ξ-zonal function $B_h^{(0)}(\xi\ \cdot) : \Omega \to \mathbb{R}$ given by

$$B_h^{(0)}(\xi \cdot \eta) = \begin{cases} 0 & \text{for} \quad \xi \cdot \eta \in [-1, h), \\ 1 & \text{for} \quad \xi \cdot \eta \in [h, 1]. \end{cases}$$

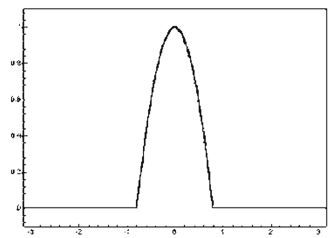
is called the *Haar kernel at position* $\xi \in \Omega$, while $B_h^{(k)}(\xi\ \cdot)$, $k > 0$, are called *"smoothed" Haar kernels at position* $\xi \in \Omega$.



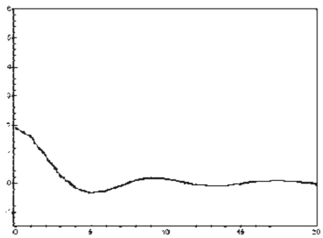
(a) Haar kernel $K(x, y)$



(b) Symbol $K^\wedge(n)$ of the Haar kernel



(c) Smoothed Haar kernel $K(x, y)$



(d) Symbol $K^\wedge(n)$ of the smoothed Haar kernel

FIGURE 5.7. Haar kernel (above) and smoothed Haar kernel (below) with $h = 0.7$: space domain, i.e., $K(x, y)$ for $(x, y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $K^\wedge(n) = A_n$ (right).

An easy calculation shows that the *iterated "Haar kernel"*

$$(B_h^{(k)})^{(2)}(\xi\ \cdot) = (B_h^{(k)} *_{L^2(\Omega)} B_h^{(k)})(\xi\ \cdot)$$

also has a cap with centre ξ as a local support:

$$\mathrm{supp}(B_h^{(k)})^{(2)}(\xi\ \cdot) = \{\eta \in \Omega\ :\ 2h^2 - 1 \leq \xi \cdot \eta \leq 1\}.$$

Expanding $B_h^{(k)}$ in terms of Legendre polynomials we obtain

$$B_h^{(k)} = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi} (B_h^{(k)})^\wedge(n) P_n, \tag{5.11}$$

where

$$(B_h^{(k)})^\wedge(n) = 2\pi \int_{-1}^{+1} \left(\frac{t-h}{1-h}\right)^k P_n(t)\, dt, \quad n = 0, 1, \ldots.$$

The recurrence formulae for Legendre polynomials give us

$$(k+1)(B_h^{(k)})^\wedge(0) = 2\pi(1-h), \tag{5.12}$$

$$(k+2)(B_h^{(k)})^\wedge(1) = (k+1+h)(B_h^{(k)})^\wedge(0), \tag{5.13}$$

$$\begin{aligned}(n+k+2)(B_h^{(k)})^\wedge(n+1) &= (2n+1)h(B_h^{(k)})^\wedge(n) \\ &\quad + (k+1-n)(B_h^{(k)})^\wedge(n-1)\end{aligned} \tag{5.14}$$

(for more details the reader is referred to [26]).

For $k = 0$ it is easy to see that $\left|(B_h^{(0)})^\wedge(n)\right| = \mathcal{O}(n^{-3/2})$, $n \to \infty$. Moreover, from the recurrence relations Eqs. (5.12)–(5.14) it follows that

$$\left|(B_h^{(k)})^\wedge(n)\right| = \mathcal{O}(n^{-(3/2)-k}), \quad n \to \infty.$$

Furthermore, [67] has shown the following statements:

(i) $(B_h^{(k)})^\wedge(n) \neq 0$ for $n = 0, 1, \ldots, k+2$.

(ii) For $n \geq k+2$, $(B_h^{(k)})^\wedge(n) = 0$ if and only if $C_{n-k-1}^{k+\frac{3}{2}}(h) = 0$ (where $C_m^{k+\frac{3}{2}}$ is the Gegenbauer polynomial of order m with respect to $k+\frac{3}{2}$).

This leads us to the following result: For $k \geq 0$, $h \in (0,1)$, the sequence

$$A_n = \begin{cases} ((B_h^{(k)})^\wedge(n))^{-1}, & n \in \mathcal{N}, \\ 0, & n \in \mathcal{N}_0 \end{cases} \tag{5.15}$$

is summable.

In case of locally supported kernels we have the following lemma:

Lemma 5.5. *$\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}}) \subset Pot^{(0)}(\overline{\Omega_\sigma^{\text{ext}}})$, as defined by* (5.15), *is a reproducing kernel Hilbert space with the reproducing kernel*

$$K_{\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})}(x,y) = \sum_{n\in\mathcal{N}} \left(\left(B_h^{(k)}\right)^{(2)}\right)^\wedge (n) \frac{2n+1}{4\pi\sigma^2} \left(\frac{\sigma^2}{|x|\,|y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right). \tag{5.16}$$

Moreover, for $x = \sigma\xi$, $y = \sigma\eta$, we have

$$\sigma^2 K_{\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})}(x,y)\Big|_{\substack{|x|=\sigma,\\ |y|=\sigma}} = \left(B_h^{(k)}\right)^{(2)}\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right) = \left(B_h^{(k)}\right)^{(2)}(\xi\cdot\eta),$$

where

$$\operatorname{supp}\left(B_h^{(k)}\right)^{(2)}\left(\cdot\,\frac{x}{|x|}\right)=\left\{y\in\Omega_\sigma\ :\ 2h^2-1\le\frac{x}{|x|}\cdot\frac{y}{|y|}\le 1\right\}.$$

In other words, reproducing kernel Hilbert spaces of potentials defined on and outside the sphere Ω_σ are found such that the "restriction" $(x,y)\mapsto K_{\mathcal{H}(\overline{\Omega_\sigma^{\rm ext}})}(x,y)$, $(x,y)\in\Omega_\sigma\times\Omega_\sigma$, is a locally supported (zonal) function on Ω_σ (note that $(B_h^{(k)})^{(2)}(\xi\cdot\eta)$ is a zonal function, i.e., depends only on the scalar product of the unit vectors ξ and η).

6. Ill-posedness of the satellite problems

The question of subsets $X\subset\overline{\Omega_\gamma^{\rm ext}}$ on which observations are required in order to uniquely determine the potential $F|_{\overline{\Sigma^{\rm ext}}}$, is answered in this section. In order to handle existence and stability of the solution we give a reformulation of the pseudodifferential operators as convolution operators.

6.1. Scalar SST and SGG problem

Throughout the remaining part of this contribution, the sequence $\{A_n\}\in\mathcal{A}$ generating the reference space $\mathcal{H}(\overline{\Omega_\sigma^{\rm ext}})$ for gravitational field determination is assumed to satisfy the so-called '*consistency conditions*':

Definition 6.1. A sequence $\{A_n\}\in A$ is said to satisfy the *consistency conditions* (CC1) and (CC2) relative to $[\sigma,\sigma^{\rm inf})$, if the following conditions are satisfied:

(CC1) A_n is different from 0 for all $n\in\mathbb{N}_0$, i.e.,

$$A_n\neq 0,\quad n=0,1,\ldots,\tag{6.1}$$

and

(CC2) there exists a value τ with $\sigma\le\tau<\sigma^{\rm inf}$ such that

$$\sum_{n=0}^{\infty}(2n+1)\left(\frac{\sigma}{\tau}\right)^n\frac{1}{A_n^2}<\infty.\tag{6.2}$$

The "downward continuation problem" of determining the potential $F\in\mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}})$ from "satellite data" $G\in\mathcal{H}_s(\overline{\Omega_\gamma^{\rm ext}})$ reads as follows.

(i) (*Scalar*) *SST Problem* (*Corresponding to the First-order Radial Derivative*). Let the values $G(x)$, $x\in X$, for some subset $X\subset\overline{\Omega_\gamma^{\rm ext}}$ be known from a function G of the class $\mathcal{H}_s(\overline{\Omega_\gamma^{\rm ext}})$. We search for a potential $F|_{\overline{\Sigma^{\rm ext}}}$ with F being from $\mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}})$ that fulfills the (scalar) SST operator equation with the SST operator $\Lambda:\mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}})\to\mathcal{H}_s(\overline{\Omega_\gamma^{\rm ext}})$ given by

$$\Lambda F(x)=G(x),\quad x\in X,\tag{6.3}$$

where

$$(\Lambda F)(x)=\left(-\frac{x}{|x|}\cdot\nabla_x\right)F(x)|_{|x|=\gamma}=G(x),\quad x\in X.\tag{6.4}$$

Equation (6.4) means that the SST operator is the composition of the radial derivative and the upward continuation operator. Having in mind that the symbol of a pseudodifferential operator $\Lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$ satisfies $\Lambda H_{n,m}^s(\sigma;\cdot) = \Lambda^\wedge(n) H_{n,m}^s(\gamma;\cdot)$, we have

$$\Lambda^\wedge(n) = \frac{n+1}{\gamma}\left(\frac{\sigma}{\gamma}\right)^n, \quad n = 0,1,\ldots, \tag{6.5}$$

and the SST operator is given by

$$\Lambda F(x) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \Lambda^\wedge(n) F^\wedge(n,m) H_{n,m}^s(\gamma;x). \tag{6.6}$$

(ii) (*Scalar*) *SGG problem* (*Corresponding to the Second-order Radial Derivative*). Let the values $G(x)$, $x \in X$, for some subset $X \subset \overline{\Omega_\gamma^{\text{ext}}}$ be known from a function G of the class $\mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$. We search for a potential $F|_{\overline{\Sigma^{\text{ext}}}}$ with F being from $\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ such that

$$\Lambda F(x) = G(x), \quad x \in X, \tag{6.7}$$

where the SGG operator $\Lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$ with the symbol

$$\Lambda^\wedge(n) = \frac{(n+1)(n+2)}{\gamma^2}\left(\frac{\sigma}{\gamma}\right)^n, \quad n = 0,1,\ldots, \tag{6.8}$$

is given by

$$\begin{aligned}\Lambda F(x) &= \left(-\frac{x}{|x|}\cdot\nabla_x\right)\left(-\frac{x}{|x|}\cdot\nabla_x\right) F(x)|_{|x|=\gamma} \\ &= \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \Lambda^\wedge(n) F^\wedge(n,m) H_{n,m}^s(\gamma;x).\end{aligned} \tag{6.9}$$

In the case of combined SST/SGG data we have the following formulation in terms of pseudodifferential operators.

(iii) *Combined* (*scalar*) *SST/SGG problem.* Let the values $G_1(x)$, $x \in X_1 \subset \overline{\Omega_\gamma^{\text{ext}}}$ and $G_2(x)$, $x \in X_2 \subset \overline{\Omega_\gamma^{\text{ext}}}$ be known from a function of class $\mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$. Let the symbols of the two corresponding pseudodifferential operators Λ_1 and Λ_2 be given by

$$\Lambda_1^\wedge(n) = \left(\frac{\sigma}{\gamma}\right)^n \frac{n+1}{\gamma}, \qquad n = 0,1,\ldots \quad \text{for SST}, \tag{6.10}$$

$$\Lambda_2^\wedge(n) = \left(\frac{\sigma}{\gamma}\right)^n \frac{(n+1)(n+2)}{\gamma^2}, \quad n = 0,1,\ldots \quad \text{for SGG}. \tag{6.11}$$

Find a potential $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})|_{\overline{\Sigma^{\text{ext}}}}$ such that

$$(\Lambda_1 F)(x) = G_1(x), \quad x \in X_1, \tag{6.12}$$

$$(\Lambda_2 F)(x) = G_2(x), \quad x \in X_2. \tag{6.13}$$

In order to give an answer to the question of subsets $X \subset \overline{\Omega_\gamma^{\rm ext}}$ on which data are necessary to assure uniqueness of the solution F, we define $\mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}})$-*fundamental systems.*

Definition 6.2. A system $X = \{x_n\}_{n=0,1,\ldots}$ of points $x_n \in \overline{\Omega_\gamma^{\rm ext}}$ is called an $\mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}})$-fundamental system in $\overline{\Omega_\gamma^{\rm ext}}$, if the conditions $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}})$ and $F(x_n) = 0$ for $n = 0, 1, \ldots$ imply $F = 0$.

For fundamental systems we get the following uniqueness theorems which are proved in the Ph.D.-thesis [58].

Theorem 6.3. *Let $X = \{x_n\}_{n=0,1,\ldots}$ be an $\mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}})$-fundamental system in $\overline{\Omega_\gamma^{\rm ext}}$. Then the potential $F|_{\overline{\Sigma^{\rm ext}}}$ solving the (scalar) SST or SGG problem is uniquely defined.*

Theorem 6.4. *Let $X_1 \subset \overline{\Omega_\sigma^{\rm ext}}$, $X_2 \subset \overline{\Omega_\sigma^{\rm ext}}$ such that $X = X_1 \cup X_2 = \{x_n\}_{n=0,1,\ldots}$ is an $\mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}})$-fundamental system in $\overline{\Omega_\gamma^{\rm ext}}$. Then the potential $F|_{\overline{\Sigma^{\rm ext}}}$ solving the combined (scalar) SST/SGG problem is uniquely defined.*

In order to present the results concerning the ill-posedness of the satellite problems, we essentially follow [19]. We reformulate the SST and SGG problem as a convolution equation using kernel functions.

Definition 6.5. Let $\alpha,\ \beta \in \mathbb{R}$, $\alpha \geq \sigma$ and $\beta \geq \sigma$. Then any kernel $K^{\alpha,\beta}(\cdot,\cdot) : \overline{\Omega_\alpha^{\rm ext}} \times \overline{\Omega_\beta^{\rm ext}} \to \mathbb{R}$ of the form

$$K^{\alpha,\beta}(x,y) = \sum_{n=0}^{\infty} K^\wedge(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\alpha;x) H_{n,m}^s(\beta;y) \tag{6.14}$$

$(x,y) \in \overline{\Omega_\alpha^{\rm ext}} \times \overline{\Omega_\beta^{\rm ext}}$, is called an $\mathcal{H}_{\alpha,\beta}$-kernel.

The sequence $\{(K^{\alpha,\beta})^\wedge(n)\}_{n\in\mathbb{N}_0}$ with $(K^{\alpha,\beta})^\wedge(n) = \left(\frac{\alpha\beta}{\sigma^2}\right)^n K^\wedge(n)$, $n = 0, 1, \ldots$, is called the (α,β)-*symbol of the* $\mathcal{H}_{\alpha,\beta}$-*kernel* $K^{\alpha,\beta}(\cdot,\cdot)$. The (σ,σ)-symbol of the $\mathcal{H}_{\alpha,\beta}$-kernel $K^{\alpha,\beta}(\cdot,\cdot)$ is simply called the *symbol of the* $\mathcal{H}_{\alpha,\beta}$-*kernel.*

Definition 6.6. An $\mathcal{H}_{\alpha,\beta}$-kernel $K^{\alpha,\beta}(\cdot,\cdot)$ with symbol $\{K^\wedge(n)\}_{n=0,1,\ldots}$ is called admissible, if the following conditions are satisfied:

(i) $\sum_{n=0}^{\infty} (K^\wedge(n))^2 < \infty$,

(ii) $\sum_{n=0}^{\infty} (2n+1)\,(K^\wedge(n))^2 \left(\frac{\sigma}{n+\frac{1}{2}}\right)^{2s} < \infty$.

The first property in Definition 6.6 ensures that $K^\wedge(n) \to 0$ as $n \to \infty$, whereas the second condition implies the following lemma.

Lemma 6.7. *Let $\alpha, \beta \in \mathbb{R}$, $\alpha \geq \sigma$, $\beta \geq \sigma$.*

(i) *If $K^{\alpha,\beta}(\cdot,\cdot)$ is an admissible $\mathcal{H}_{\alpha,\beta}$-kernel with the symbol $\{K^\wedge(n)\}_{n=0,1,\ldots}$, then $K^{\alpha,\beta}(x,\cdot)$ is an element of $\mathcal{H}_s(\overline{\Omega_\beta^{\rm ext}})$ for every (fixed) $x \in \overline{\Omega_\alpha^{\rm ext}}$.*

(ii) *If $K^{\alpha,\beta}(\cdot,\cdot)$ is an admissible $\mathcal{H}_{\alpha,\beta}$-kernel with the symbol $\{K^\wedge(n)\}_{n=0,1,\ldots}$, then $K^{\alpha,\beta}(\cdot,y)$ is an element of $\mathcal{H}_s(\overline{\Omega_\alpha^{\mathrm{ext}}})$ for every (fixed) $x \in \overline{\Omega_\beta^{\mathrm{ext}}}$.*

Suppose now that F, G are elements of class $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$. Then we understand the $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$-*convolution* of F and G simply to be the inner product in $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, i.e.:

$$F * G = (F, G)_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}. \tag{6.15}$$

(More precisely, we had to write $F * G = F *_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} G$.) By definition, we let

$$F^\wedge(n,k) = F * H_{n,k}^s(\sigma;\cdot) \tag{6.16}$$

for $n \in \mathcal{N}(A_n)$; $k = 1, \ldots, 2n+1$. It follows from (6.15) via the Parseval identity that

$$F * G = \sum_{n \in \mathcal{N}} \sum_{k=1}^{2n+1} F^\wedge(n,k) G^\wedge(n,k),$$

for $F, G \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

We now define the convolution of an admissible $\mathcal{H}_{\alpha,\beta}$-kernel against a function $F \in \mathcal{H}_s(\overline{\Omega_\beta^{\mathrm{ext}}})$ as follows:

$$\begin{aligned}(K^{\alpha,\beta} * F)(x) &= K^{\alpha,\beta}(x,\cdot) * F \\ &= \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} K^\wedge(n) F^\wedge(n,m) H_{n,m}^s(\alpha;x), \quad x \in \overline{\Omega_\alpha^{\mathrm{ext}}}.\end{aligned} \tag{6.17}$$

It directly follows that $(K^{\alpha,\beta} * F)^\wedge(n,m) = K^\wedge(n) F^\wedge(n,m)$ and $K^{\alpha,\beta} * F \in \mathcal{H}_s(\overline{\Omega_\alpha^{\mathrm{ext}}})$. In analogous way we define the convolution of an $\mathcal{H}_{\alpha,\beta}$-kernel $K^{\alpha,\beta}(\cdot,\cdot)$ against a function $F \in \mathcal{H}_s(\overline{\Omega_\alpha^{\mathrm{ext}}})$ by

$$\begin{aligned}(K^{\alpha,\beta} * F)(y) &= K^{\alpha,\beta}(\cdot,y) * F \\ &= \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} K^\wedge(n) F^\wedge(n,m) H_{n,m}^s(\beta;y), \quad y \in \overline{\Omega_\beta^{\mathrm{ext}}},\end{aligned} \tag{6.18}$$

and $K^{\alpha,\beta} * F$ is an element of $\mathcal{H}_s(\overline{\Omega_\beta^{\mathrm{ext}}})$.

If L, K are admissible $\mathcal{H}_{\sigma,\sigma}$-kernels, then the $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$-*convolution* $L * K$ is defined by

$$(L * K)(x,y) = (L(x,\cdot), K(\cdot,y))_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}, \quad (x,y) \in \overline{\Omega_\sigma^{\mathrm{ext}}} \times \overline{\Omega_\sigma^{\mathrm{ext}}}.$$

Obviously, $(L * K)(\cdot,\cdot)$ is an admissible $\mathcal{H}_{\sigma,\sigma}$-kernel, and it is not difficult to see that

$$(L * K)^\wedge(n) = L^\wedge(n) K^\wedge(n), \quad n \in \mathcal{N}\left((K^\wedge(n) L^\wedge(n))^{-1} A_n\right).$$

We usually write $K^{(2)}(\cdot,\cdot) = (K * K)(\cdot,\cdot)$ to indicate the convolution of an $\mathcal{H}_{\sigma,\sigma}$-kernel with itself. $K^{(2)}(\cdot,\cdot) = (K * K)(\cdot,\cdot)$ is said to be the *iterated kernel* of $K(\cdot,\cdot)$. More generally, $K^{(p)}(\cdot,\cdot) = (K^{(p-1)} * K)(\cdot,\cdot)$ for $p = 2, 3, \ldots$, and

$K^{(1)}(\cdot,\cdot) = K(\cdot,\cdot)$ for $p = 1$. Obviously, we have

$$(K^{(2)})^\wedge(n) = (K^\wedge(n))^2.$$

In order to give an answer to the question of ill-posedness of the (scalar) SST or SGG problem, the continuity of the inverse additionally has to be investigated. The answer to this question requires the reformulation of the problem as convolution equation. Starting from a pseudodifferential operator $\Lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$ given by

$$\Lambda F = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \Lambda^\wedge(n) F^\wedge(n,m) H^s_{n,m}(\gamma;\cdot), \tag{6.19}$$

we can interpret the symbol of the pseudodifferential operator as the symbol of an $\mathcal{H}_{\sigma,\gamma}$-kernel $(K^\Lambda)^{\sigma,\gamma}$ presuming that the symbol satisfies the admissibility conditions. The pseudodifferential operator is then given by the convolution identity

$$\Lambda F(x) = (K^\Lambda)^{\sigma,\gamma}(\cdot,x) * F, \quad x \in \overline{\Omega_\gamma^{\text{ext}}}, \tag{6.20}$$

for $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, where $(K^\Lambda)^\wedge(n) = \Lambda^\wedge(n)$, $n = 0, 1, \ldots$. Obviously, we have

$$\begin{aligned}(K^\Lambda)^{\sigma,\gamma}(\cdot,x) * H^s_{n,m}(\sigma;\cdot) &= (K^\Lambda)^{\gamma,\sigma}(x,\cdot) * H^s_{n,m}(\sigma;\cdot)\\ &= \Lambda^\wedge(n) H^s_{n,m}(\gamma;x),\end{aligned} \tag{6.21}$$

for all $n \in \mathbb{N}$; $m = 1, \ldots, 2n+1$, or, equivalently,

$$\Lambda H^s_{n,m}(\sigma;\cdot) = \Lambda^\wedge(n) H^s_{n,m}(\gamma;\cdot). \tag{6.22}$$

Having a look at the (scalar) SST and SGG operator, we get the following result.

Theorem 6.8. *The $\mathcal{H}_{\sigma,\gamma}$-kernel $(K^\Lambda)^{\sigma,\gamma}$ defined by the symbol*

$$\Lambda^\wedge(n) = \begin{cases} \left(\frac{\sigma}{\gamma}\right)^n \frac{n+1}{\gamma}, & n = 0, 1, \ldots \quad \textit{for SST},\\ \left(\frac{\sigma}{\gamma}\right)^n \frac{(n+1)(n+2)}{\gamma^2}, & n = 0, 1, \ldots \quad \textit{for SGG}, \end{cases} \tag{6.23}$$

is admissible, if $\left\{\left(\frac{n+\frac{1}{2}}{\sigma}\right)^s\right\}$ is summable in the sense of Eq. (3.3).

Theorem 6.9. *Let $\Lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$ be a pseudodifferential operator with $(K^\Lambda)^{\sigma,\gamma}$ satisfying the admissibility conditions. Then the pseudodifferential operator Λ is bounded and $\|\Lambda\| = \max_{n\in\mathbb{N}_0} |\Lambda^\wedge(n)|$. Further on, Λ is an injective operator.*

From functional analysis (see, e.g., [70, 77]), we know that the SST and SGG operators are compact as being so-called Hilbert–Schmidt operators. Summing up the preceding considerations we finally get the following result.

Theorem 6.10. *Let*

$$\Lambda F = G, \quad F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}), \quad G \in \mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}}), \tag{6.24}$$

be the (scalar) SST or SGG problem. Then Λ is a compact operator with infinite-dimensional range. Furthermore, Λ^{-1} is not bounded on $\mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$. The (scalar) SST or SGG problem is solvable if and only if

$$\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left(\frac{G^\wedge(n,m)}{\Lambda^\wedge(n)}\right)^2 < \infty. \tag{6.25}$$

Remembering Hadamard's definition of a well-posed problem (existence, uniqueness and continuity of the inverse), we consequently see that the (scalar) SST or SGG problem is ill posed, as it violates the first and third condition.

6.2. Vectorial SST and SGG problem

Following [58], we additionally formulate uniqueness results for the (vectorial) SST and SGG problems. Let $\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ be a (scalar) Sobolev space with $\left(\frac{n+\frac{1}{2}}{\sigma}\right)^s$ satisfying the consistency condition (CC2) relative to $[\sigma,\tau)$ (see Eq. (6.2)). Further on, let $h_s^{(i)}(\overline{\Omega_\gamma^{\text{ext}}})$, $i=1,2$, be (vectorial) Sobolev spaces. Then the "downward continuation problem" of determining the potential $F\in\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ from "satellite data" $g\in h_s^{(1)}(\overline{\Omega_\gamma^{\text{ext}}})\oplus h_s^{(2)}(\overline{\Omega_\gamma^{\text{ext}}})$ reads as follows.

(i) (*Vectorial*) *SST problem* (*Corresponding to the First-order Tangential Derivative*). Let the values $g(x)$, $x\in X$, for some subset $X\subset\overline{\Omega_\gamma^{\text{ext}}}$ be known from a function g of the class $h_s^{(1)}(\overline{\Omega_\gamma^{\text{ext}}})\oplus h_s^{(2)}(\overline{\Omega_\gamma^{\text{ext}}})$. We search for a potential $F|_{\overline{\Sigma^{\text{ext}}}}$ with F being of the class $\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ such that

$$\lambda F(x) = g(x), \quad x\in X, \tag{6.26}$$

where the SST Operator $\lambda:\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})\to h_s^{(1)}(\overline{\Omega_\gamma^{\text{ext}}})\oplus h_s^{(2)}(\overline{\Omega_\gamma^{\text{ext}}})$ is given by

$$(\lambda F)(x) = \nabla_\xi^{*,\sigma}F(x)|_{|x|=\gamma}, \tag{6.27}$$

with $x=|x|\xi$. Observing the symbol

$$\lambda^{(i)\wedge}(n) = \begin{cases} \left(\frac{\sigma}{\gamma}\right)^n \frac{n}{\gamma}\sqrt{\frac{n+1}{2n+1}}, & i=1;\quad n=0,1,\ldots,\\ \left(\frac{\sigma}{\gamma}\right)^n \frac{n+1}{\gamma}\sqrt{\frac{n}{2n+1}}, & i=2;\quad n=1,2,\ldots, \end{cases} \tag{6.28}$$

the (vectorial) SST operator can be written as

$$\lambda F(x) = \sum_{i=1}^{2}\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\lambda^{(i)\wedge}(n)F^\wedge(n,m)h_{n,m}^{(i)s}(\gamma;x). \tag{6.29}$$

In the case of SGG-data the mixed derivatives can be handled within vectorial framework.

(ii) (*Vectorial*) *SGG problem* (*Corresponding to the Second-order Mixed Derivatives*). Let the values $g(x)$, $x\in X$, for some subset $X\subset\overline{\Omega_\gamma^{\text{ext}}}$ be known from a function g of the class $h_s^{(1)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \oplus h_s^{(2)}(\overline{\Omega_\gamma^{\mathrm{ext}}})$. We search for a potential $F|_{\overline{\Sigma^{\mathrm{ext}}}}$ with F being of the class $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ such that

$$\lambda F(x) = g(x), \quad x \in X, \tag{6.30}$$

where the SGG operator $\lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to h_s^{(1)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \oplus h_s^{(2)}(\overline{\Omega_\gamma^{\mathrm{ext}}})$ with symbol

$$\lambda^{(i)\wedge}(n) = \begin{cases} \left(\frac{\sigma}{\gamma}\right)^n \frac{n(n+1)}{\gamma^2}\sqrt{\frac{n+1}{2n+1}}, & i=1; \quad n=0,1,\ldots, \\ \left(\frac{\sigma}{\gamma}\right)^n \frac{(n+1)^2}{\gamma^2}\sqrt{\frac{n}{2n+1}}, & i=2; \quad n=1,2,\ldots, \end{cases} \tag{6.31}$$

is given by

$$\sum_{i=1}^{2}\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} \lambda^{(i)\wedge}(n) F^\wedge(n,m) h_{n,m}^{(i)s}(\gamma;x). \tag{6.32}$$

In order to give an answer to the question of subsets $X \subset \overline{\Omega_\gamma^{\mathrm{ext}}}$ on which data are necessary to get uniqueness of the solution F, we define $h_s^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-*fundamental systems.*

Definition 6.11. A system $X = \{x_n\}_{n=0,1,\ldots}$ of points $x_n \in \overline{\Omega_\sigma^{\mathrm{ext}}}$ is called an $h_s^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-fundamental system in $\overline{\Omega_\sigma^{\mathrm{ext}}}$, if the conditions $g \in h_s^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ and $g(x_n) = 0$ for $n \in \mathbb{N}_0$ imply $g = 0$, $i \in \{1,2,3\}$. Further on, X is called an $h_s^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \oplus h_s^{(j)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-fundamental system, if $g \in h_s^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \oplus h_s^{(j)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ and $g(x_n) = 0$ for $n \in \mathbb{N}_0$ imply $g = 0$ for $i,j \in \{1,2,3\}$ with $i \neq j$.

We now obtain the following uniqueness theorem.

Theorem 6.12. *Let $X = \{x_n\}_{n=0,1,\ldots}$ be an $h_s^{(1)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \oplus h_s^{(2)}(\overline{\Omega_\gamma^{\mathrm{ext}}})$-fundamental system in $\overline{\Omega_\gamma^{\mathrm{ext}}}$. Then the potential $F|_{\overline{\Sigma^{\mathrm{ext}}}}$ solving the (vectorial) SST or SGG problem is uniquely defined up to an additive constant C.*

Definition 6.13. Let $\alpha, \beta \in \mathbb{R}$, $\alpha \geq \sigma$ and $\beta \geq \sigma$. Then any kernel $k^{(i),\alpha,\beta}(\cdot,\cdot) : \overline{\Omega_\alpha^{\mathrm{ext}}} \times \overline{\Omega_\beta^{\mathrm{ext}}} \to \mathbb{R}^3$ of the form

$$k^{(i),\alpha,\beta}(x,y) = \sum_{n=0_i}^{\infty} k^{(i)\wedge}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\alpha;x) h_{n,m}^{(i)s}(\beta;y), \tag{6.33}$$

$(x,y) \in \overline{\Omega_\alpha^{\mathrm{ext}}} \times \overline{\Omega_\beta^{\mathrm{ext}}}$, is called an $h_{\alpha,\beta}^{(i)}$-kernel. Furthermore,

$$k^{\alpha,\beta}(x,y) = \sum_{i=1}^{3} k^{(i),\alpha,\beta}(x,y), \tag{6.34}$$

$(x,y) \in \overline{\Omega_\alpha^{\mathrm{ext}}} \times \overline{\Omega_\beta^{\mathrm{ext}}}$, is called an $h_{\alpha,\beta}$-kernel.

The sequence $\{\left(k^{(i),\alpha,\beta}\right)^\wedge(n)\}_{n\in\mathbb{N}_{0_i}}$ with

$$\left(k^{(i),\alpha,\beta}\right)^\wedge(n) = \left(\frac{\alpha\beta}{\sigma^2}\right)^n k^{(i)\wedge}(n), \quad n = 0_i, \ldots, \tag{6.35}$$

is called the (α,β)-*symbol of the* $h^{(i)}_{\alpha,\beta}$-*kernel* $k^{(i),\alpha,\beta}(\cdot,\cdot)$. The (σ,σ)-symbol of the $h^{(i)}_{\alpha,\beta}$-kernel $k^{(i),\alpha,\beta}(\cdot,\cdot)$ is simply called the *symbol of the* $h^{(i)}_{\alpha,\beta}$-*kernel.*

Definition 6.14. An $h^{(i)}_{\alpha,\beta}$-kernel $k^{(i),\alpha,\beta}(\cdot,\cdot)$ with symbol $\{k^{(i)\wedge}(n)\}_{n=0_i,\ldots}$ is called admissible, if the following conditions are satisfied:

(i) $\sum_{n=0_i}^{\infty}(k^{(i)\wedge}(n))^2<\infty$,

(ii) $\sum_{n=0_i}^{\infty}(2n+1)\left(k^{(i)\wedge}(n)\right)^2\left(\frac{\sigma}{n+\frac{1}{2}}\right)^{2s}<\infty$,

(iii) (a) $\sum_{n=0}^{\infty}(2n+1)(2n+3)\left(k^{(1)\wedge}(n)\right)^2\left(\frac{\sigma}{n+\frac{1}{2}}\right)^{2s}<\infty$,

(b) $\sum_{n=1}^{\infty}(2n+1)(2n-1)\left(k^{(2)\wedge}(n)\right)^2\left(\frac{\sigma}{n+\frac{1}{2}}\right)^{2s}<\infty$,

(c) $\sum_{n=1}^{\infty}(2n+1)(2n+1)\left(k^{(3)\wedge}(n)\right)^2\left(\frac{\sigma}{n+\frac{1}{2}}\right)^{2s}<\infty$.

Furthermore, the $h_{\alpha,\beta}$-kernel is called admissible, if the $h^{(i)}_{\alpha,\beta}$-kernels, $i\in\{1,2,3\}$, are admissible.

The second and the third condition imply the following lemma.

Lemma 6.15. *Let $\alpha,\beta\in\mathbb{R}$, $\alpha\geq\sigma$, $\beta\geq\sigma$.*

(i) *If $k^{(i),\alpha,\beta}(\cdot,\cdot)$ is an admissible $h^{(i)}_{\alpha,\beta}$-kernel with the symbol $\{k^{(i)\wedge}(n)\}_{n=0_i,\ldots}$, then $k^{(i),\alpha,\beta}(x,\cdot)$ is an element of $h^{(i)}_s(\overline{\Omega^{\mathrm{ext}}_\beta})$ for every (fixed) $x\in\overline{\Omega^{\mathrm{ext}}_\alpha}$.*

(ii) *If $k^{(i),\alpha,\beta}(\cdot,\cdot)$ is an admissible $h^{(i)}_{\alpha,\beta}$-kernel with the symbol $\{k^{(i)\wedge}(n)\}_{n=0_i,\ldots}$, then the component functions $k^{(i),\alpha,\beta}(\cdot,y)^T\varepsilon^l$ are elements of $\mathcal{H}_s(\overline{\Omega^{\mathrm{ext}}_\alpha})$ for every (fixed) $x\in\overline{\Omega^{\mathrm{ext}}_\beta}$, $l\in\{1,2,3\}$.*

Our next step is the definition of the convolution of an admissible $h^{(i)}_{\alpha,\beta}$-kernel against a function $f\in h_s(\overline{\Omega^{\mathrm{ext}}_\beta})$ as follows:

$$\begin{aligned}(k^{(i),\alpha,\beta}*f)(x)&=k^{(i),\alpha,\beta}(x,\cdot)*f\\&=\sum_{n=0_i}^{\infty}\sum_{m=1}^{2n+1}k^{(i)\wedge}(n)f^{(i)\wedge}(n,m)H^s_{n,m}(\alpha;x),\quad x\in\overline{\Omega^{\mathrm{ext}}_\alpha}.\end{aligned}\tag{6.36}$$

It directly follows that $(k^{(i),\alpha,\beta}*f)^\wedge(n,m)=k^{(i)\wedge}(n)f^{(i)\wedge}(n,m)$, $n=0_i$, $i\in\{1,2,3\}$, and $k^{(i),\alpha,\beta}*f\in\mathcal{H}_s(\overline{\Omega^{\mathrm{ext}}_\alpha})$. In an analogous way we define the convolution of an $h_{\alpha,\beta}$-kernel $k^{\alpha,\beta}(\cdot,\cdot)$ against a function $F\in\mathcal{H}_s(\overline{\Omega^{\mathrm{ext}}_\alpha})$ by

$$\begin{aligned}(k^{\alpha,\beta}\star F)(y)&=k^{\alpha,\beta}(\cdot,y)\star F\\&=\sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{m=1}^{2n+1}k^{(i)\wedge}(n)F^\wedge(n,m)h^{(i)s}_{n,m}(\beta;y),\quad y\in\overline{\Omega^{\mathrm{ext}}_\beta},\end{aligned}\tag{6.37}$$

and $k^{\alpha,\beta}\star F$ is an element of $h_s(\overline{\Omega^{\mathrm{ext}}_\beta})$.

Our next purpose is to present the formulation of the vectorial SST respectively SGG operators with the help of convolutions. This enables us to give an answer to the question of continuity of the inverse. We start from a pseudodifferential operator $\lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to h_s(\overline{\Omega_\gamma^{\mathrm{ext}}})$ given by

$$\lambda F = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} \lambda^{(i)\wedge}(n) F^\wedge(n,m) h_{n,m}^{(i)s}(\gamma;\cdot), \tag{6.38}$$

and interpret the symbol of the pseudodifferential operator as the symbol of an $h_{\sigma,\gamma}$-kernel $(k^\lambda)^{\sigma,\gamma}$ presuming that the symbol satisfies the admissibility conditions. The pseudodifferential operator is then given by the convolution identity

$$\lambda F(x) = (k^\lambda)^{\sigma,\gamma}(\cdot, x) \star F, \quad x \in \overline{\Omega_\gamma^{\mathrm{ext}}}, \tag{6.39}$$

for $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, where $(k^\lambda)^{(i)\wedge}(n) = \lambda^{(i)\wedge}(n)$, $i = 1,2,3$; $n = 0_i, \ldots$. Obviously, we have

$$(k^{\lambda^{(i)}})^{\sigma,\gamma}(\cdot, x) \star H_{n,m}^s(\sigma;\cdot) = \lambda^{(i)\wedge}(n) h_{n,m}^{(i)s}(\gamma; x), \tag{6.40}$$

for all $i = 1,2,3$; $n = 0_i, \ldots$; $m = 1, \ldots, 2n+1$, or, equivalently

$$\lambda^{(i)} H_{n,m}^s(\sigma;\cdot) = \lambda^{(i)\wedge}(n) h_{n,m}^{(i)s}(\gamma;\cdot). \tag{6.41}$$

Having a look at the (vectorial) SST and SGG operator, we get the following result.

Theorem 6.16. *The $h_{\sigma,\gamma}$-kernel $(k^\lambda)^{\sigma,\gamma}$ defined by the symbol*

$$(k^\lambda)^{(1)\wedge}(n) = \lambda^{(1)\wedge}(n) = \begin{cases} \left(\frac{\sigma}{\gamma}\right)^n \frac{n}{\gamma}\sqrt{\frac{n+1}{2n+1}}, & n = 0,1,\ldots \quad \textit{for SST}, \\ \left(\frac{\sigma}{\gamma}\right)^n \frac{n(n+1)}{\gamma^2}\sqrt{\frac{n+1}{2n+1}}, & n = 0,1,\ldots \quad \textit{for SGG}, \end{cases} \tag{6.42}$$

and

$$(k^\lambda)^{(2)\wedge}(n) = \lambda^{(2)\wedge}(n) = \begin{cases} \left(\frac{\sigma}{\gamma}\right)^n \frac{n+1}{\gamma}\sqrt{\frac{n}{2n+1}}, & n = 1,2,\ldots \quad \textit{for SST}, \\ \left(\frac{\sigma}{\gamma}\right)^n \frac{(n+1)^2}{\gamma^2}\sqrt{\frac{n}{2n+1}}, & n = 1,2,\ldots \quad \textit{for SGG}, \end{cases} \tag{6.43}$$

is admissible, if $\{\left(\frac{n+\frac{1}{2}}{\sigma}\right)^s\}$ is summable and satisfies, in addition, condition (iii) *in Definition* 6.14*.*

Theorem 6.17. *Let $\lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to h_s(\overline{\Omega_\gamma^{\mathrm{ext}}})$ be a pseudodifferential operator with $(k^\lambda)^{\sigma,\gamma}$ satisfying the admissibility conditions, and $\lambda^{(i)\wedge}(n) \neq 0$, $i \in \{1,2,3\}$, $n = 0_i, \ldots$. Then the pseudodifferential operator λ is bounded and*

$$\|\lambda\| = \max_{n \in \mathbb{N}_0} \left| \sum_{i=1}^{3} \lambda^{(i)\wedge}(n) \right|, \tag{6.44}$$

where we let $\lambda^{(2)\wedge}(0) = \lambda^{(3)\wedge}(0) = 0$. Further on, λ is an injective operator.

Finally, we get the following result.

Theorem 6.18. *Let*

$$\lambda F = g, \quad F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}),\ g \in h_s^{(1)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \oplus h_s^{(2)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \tag{6.45}$$

be the (vectorial) SST or SGG problem. Then λ is a compact operator with infinite-dimensional range. Furthermore, λ^{-1} is not bounded on $h_s^{(1)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \oplus h_s^{(2)}(\overline{\Omega_\gamma^{\mathrm{ext}}})$. The SST/SGG problem is solvable if and only if

$$\sum_{i=1}^{2} \sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} \left(\frac{g^{(i)\wedge}(n,m)}{\lambda^{(i)\wedge}(n)} \right)^2 < \infty. \tag{6.46}$$

We consequently get that the (vectorial) SST/SGG problem is ill posed because existence and continuity of the inverse are violated.

6.3. Tensorial SGG problem

The formulation of the definitions and theorems for the tensorial case is straightforward. Let $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ be a (scalar) Sobolev space satisfying the consistency condition (CC2) relative to $[\sigma, \tau)$ (see Eq. (6.2)). Further on, let $\mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\mathrm{ext}}})$, $(i,k) \in \{(1,1),(1,2),(2,1),(2,2)\}$, be (tensorial) Sobolev spaces. Then the "downward continuation problem" of determining the potential $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ from "satellite data" $\mathbf{g} \in \mathbf{h}_s^{SGG}(\overline{\Omega_\gamma^{\mathrm{ext}}})$, where we use the abbreviation

$$\mathbf{h}_s^{SGG}(\overline{\Omega_\gamma^{\mathrm{ext}}}) = \mathbf{h}^{(1,1)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \oplus \mathbf{h}_s^{(1,2)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \oplus \mathbf{h}_s^{(2,1)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \oplus \mathbf{h}_s^{(2,2)}(\overline{\Omega_\gamma^{\mathrm{ext}}}), \tag{6.47}$$

reads as follows.

(i) (*Tensorial*) *SGG problem* (*Corresponding to the Second-order Tangential Derivative*). Let the values $\mathbf{g}(x)$, $x \in X$, for some subset $X \subset \overline{\Omega_\gamma^{\mathrm{ext}}}$ be known from a function $\mathbf{g}$ of the class $\mathbf{h}_s^{SGG}(\overline{\Omega_\gamma^{\mathrm{ext}}})$. We search for a potential $F|_{\overline{\Sigma^{\mathrm{ext}}}}$ with F being from $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ such that

$$\boldsymbol{\lambda} F(x) = \mathbf{g}(x), \quad x \in X, \tag{6.48}$$

where the SGG operator $\boldsymbol{\lambda} : \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to \mathbf{h}_s^{SGG}(\overline{\Omega_\gamma^{\mathrm{ext}}})$ is given by

$$(\boldsymbol{\lambda} F)(x) = \nabla^{*,\sigma} \otimes \nabla^{*,\sigma} F(x)|_{|x|=\gamma}, \tag{6.49}$$

with $x = |x|\xi$. With the help of the symbol

$$\boldsymbol{\lambda}^{(i,k)\wedge}(n) = \begin{cases} \left(\frac{\sigma}{\gamma}\right)^n \frac{n(n+1)}{\gamma^2(2n+1)(2n+3)} \sqrt{\nu_n^{(1,1)}}, & (i,k) = (1,1), \quad n = 0,1,\ldots, \\ -\left(\frac{\sigma}{\gamma}\right)^n \frac{(n+1)(n-1)}{\gamma^2((2n-1)(2n+1)} \sqrt{\nu_n^{(1,2)}}, & (i,k) = (1,2), \quad n = 1,2,\ldots, \\ -\left(\frac{\sigma}{\gamma}\right)^n \frac{n(n+2)}{\gamma^2(2n+3)(2n+1)} \sqrt{\nu_n^{(2,1)}}, & (i,k) = (2,1), \quad n = 0,1,\ldots, \\ \left(\frac{\sigma}{\gamma}\right)^n \frac{n(n+1)(n+2)}{\gamma^2(2n-1)(2n+1)} \sqrt{\nu_n^{(2,2)}}, & (i,k) = (2,2), \quad n = 2,3,\ldots, \end{cases} \tag{6.50}$$

with

$$\nu_n^{(1,1)} = (n+1)(n+2)(2n+1)(2n+3), \tag{6.51}$$

$$\nu_n^{(1,2)} = 3n^4, \tag{6.52}$$

$$\nu_n^{(2,1)} = (n+1)^2(2n+1)(2n+3), \tag{6.53}$$

$$\nu_n^{(2,2)} = n(n-1)(2n-1)(2n+1), \tag{6.54}$$

the SGG operator can be written as

$$\boldsymbol{\lambda} F(x) = \sum_{(i,k)\in\mathcal{I}^{SGG}} \sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} \boldsymbol{\lambda}^{(i,k)\wedge}(n) F^{\wedge}(n,m) \mathbf{h}_{n,m}^{(i,k)s}(\gamma;x), \tag{6.55}$$

where $\mathcal{I}^{SGG} = \{(1,1),(1,2),(2,1),(2,2)\}$ is the index set for the tensorial SGG problem.

In order to give an answer to the question of subsets $X \subset \overline{\Omega_\gamma^{\text{ext}}}$ on which data are necessary to get uniqueness of the solution F, we define $\mathbf{h}_s^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})$-*fundamental systems.*

Definition 6.19. A system $X = \{x_n\}_{n=0,1,\ldots}$ of points $x_n \in \overline{\Omega_\sigma^{\text{ext}}}$ is called an $\mathbf{h}_s^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})$-fundamental system in $\overline{\Omega_\sigma^{\text{ext}}}$, if the conditions $\mathbf{g} \in \mathbf{h}_s^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})$ and $\mathbf{g}(x_n) = 0$ for $n \in \mathbb{N}_0$ imply $\mathbf{g} = 0$, $i,k \in \{1,2,3\}$. In analogy the fundamental systems are defined for spaces which are direct sums of the spaces $\mathbf{h}_s^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})$, $i,k \in \{1,2,3\}$.

As in the scalar and vectorial case we have the following theorem.

Theorem 6.20. *Let* $X = \{x_n\}_{n=0,1,\ldots}$ *be an* $\mathbf{h}_s^{SGG}(\overline{\Omega_\gamma^{\text{ext}}})$*-fundamental system in* $\overline{\Omega_\gamma^{\text{ext}}}$*. Then the potential* $F|_{\overline{\Sigma^{\text{ext}}}}$ *solving the* (*tensorial*) *SGG problem is uniquely defined up to a term of the form*

$$V(x) = \sum_{n=0}^{1} \sum_{m=1}^{2n+1} c_{nm} \left(\frac{\sigma}{|x|}\right)^{n+1} \frac{1}{\sigma} Y_{n,m}\left(\frac{x}{|x|}\right), \quad x \in \overline{\Omega_\sigma^{\text{ext}}}, \tag{6.56}$$

for constants $c_{01}, c_{11}, c_{12}, c_{13} \in \mathbb{R}$.

We finally shortly present the results using the reformulation as convolution equation.

Definition 6.21. Let $\alpha, \beta \in \mathbb{R}$, $\alpha \geq \sigma$ and $\beta \geq \sigma$. Then any kernel $\mathbf{k}^{(i,k),\alpha,\beta}(\cdot,\cdot) : \overline{\Omega_\alpha^{\text{ext}}} \times \overline{\Omega_\beta^{\text{ext}}} \to \mathbb{R}^3 \otimes \mathbb{R}^3$ of the form

$$\mathbf{k}^{(i,k),\alpha,\beta}(x,y) = \sum_{n=0_{ik}}^{\infty} \mathbf{k}^{(i,k)\wedge}(n) \sum_{m=1}^{2n+1} H_{n,m}^{s}(\alpha;x) \mathbf{h}_{n,m}^{(i,k)s}(\beta;y), \tag{6.57}$$

$(x,y) \in \overline{\Omega_\alpha^{\mathrm{ext}}} \times \overline{\Omega_\beta^{\mathrm{ext}}}$, is called an $\mathbf{h}_{\alpha,\beta}^{(i,k)}$-kernel. Furthermore,

$$\mathbf{k}^{\alpha,\beta}(x,y) = \sum_{i,k=1}^{3} \mathbf{k}^{(i,k),\alpha,\beta}(x,y), \tag{6.58}$$

$(x,y) \in \overline{\Omega_\alpha^{\mathrm{ext}}} \times \overline{\Omega_\beta^{\mathrm{ext}}}$, is called an $\mathbf{h}_{\alpha,\beta}$-kernel.

The sequence $\{\left(\mathbf{k}^{(i,k),\alpha,\beta}\right)^\wedge(n)\}_{n\in\mathbb{N}_{0_{ik}}}$ with

$$\left(\mathbf{k}^{(i,k),\alpha,\beta}\right)^\wedge(n) = \left(\frac{\alpha\beta}{\sigma^2}\right)^n \mathbf{k}^{(i,k)\wedge}(n), \quad n = 0_{ik}, \ldots, \tag{6.59}$$

is called the (α,β)-*symbol of the* $\mathbf{h}_{\alpha,\beta}^{(i,k)}$-*kernel* $\mathbf{k}^{(i,k),\alpha,\beta}(\cdot,\cdot)$. The (σ,σ)-symbol of the $\mathbf{h}_{\alpha,\beta}^{(i,k)}$-kernel $\mathbf{k}^{(i,k),\alpha,\beta}(\cdot,\cdot)$ is simply called the *symbol of the* $\mathbf{h}_{\alpha,\beta}^{(i,k)}$-*kernel.*

Definition 6.22. An $\mathbf{h}_{\alpha,\beta}^{(i,k)}$-kernel $\mathbf{k}^{(i,k),\alpha,\beta}(\cdot,\cdot)$ with the symbol $\{\mathbf{k}^{(i,k)\wedge}(n)\}_{n=0_{ik},\ldots}$ is called admissible, if the following conditions are satisfied:

(i) $\sum_{n=0_{ik}}^\infty (\mathbf{k}^{(i,k)\wedge}(n))^2 < \infty$,

(ii) $\sum_{n=0_{ik}}^\infty (2n+1)(\mathbf{k}^{(i,k)\wedge}(n))^2 \left(\frac{\sigma}{n+\frac12}\right)^{2s} < \infty$,

(iii) (a) $\sum_{n=0}^\infty (2n+1)(2n+5)(\mathbf{k}^{(1,1)\wedge}(n))^2 \left(\frac{\sigma}{n+\frac12}\right)^{2s} < \infty$,

(b) $\sum_{n=0_{ik}}^\infty (2n+1)(2n+3)(\mathbf{k}^{(i,k)\wedge}(n))^2 \left(\frac{\sigma}{n+\frac12}\right)^{2s} < \infty$, $(i,k) \in \{(1,3),(3,1)\}$,

(c) $\sum_{n=0_{ik}}^\infty (2n+1)(2n+1)(\mathbf{k}^{(i,k)\wedge}(n))^2 \left(\frac{\sigma}{n+\frac12}\right)^{2s} < \infty$, $(i,k) \in \{(1,2),(2,1),(3,3)\}$,

(d) $\sum_{n=0_{ik}}^\infty (2n+1)(2n-1)(\mathbf{k}^{(i,k)\wedge}(n))^2 \left(\frac{\sigma}{n+\frac12}\right)^{2s} < \infty$, $(i,k) \in \{(2,3),(3,2)\}$,

(e) $\sum_{n=2}^\infty (2n+1)(2n-3)(\mathbf{k}^{(2,2)\wedge}(n))^2 \left(\frac{\sigma}{n+\frac12}\right)^{2s} < \infty$.

Furthermore, the $\mathbf{h}_{\alpha,\beta}$-kernel is called admissible, if all $\mathbf{h}_{\alpha,\beta}^{(i,k)}$-kernels, $i,k \in \{1,2,3\}$, are admissible.

The second and the third condition imply the following lemma.

Lemma 6.23. *Let* $\alpha,\beta \in \mathbb{R}$, $\alpha \geq \sigma$, $\beta \geq \sigma$.

1. *If the kernel* $\mathbf{k}^{(i,k),\alpha,\beta}(\cdot,\cdot)$ *is an admissible* $\mathbf{h}_{\alpha,\beta}^{(i,k)}$-*kernel with the symbol given by* $\{\mathbf{k}^{(i,k)\wedge}(n)\}_{n=0_{ik},\ldots}$, *then* $\mathbf{k}^{(i,k),\alpha,\beta}(x,\cdot)$ *is an element of* $\mathbf{h}_s^{(i,k)}(\overline{\Omega_\beta^{\mathrm{ext}}})$ *for every (fixed)* $x \in \overline{\Omega_\alpha^{\mathrm{ext}}}$.
2. *If the kernel* $\mathbf{k}^{(i,k),\alpha,\beta}(\cdot,\cdot)$ *is an admissible* $\mathbf{h}_{\alpha,\beta}^{(i,k)}$-*kernel with the symbol* $\{\mathbf{k}^{(i,k)\wedge}(n)\}_{n=0_{ik},\ldots}$, *then the component functions* $\mathbf{k}^{(i,k),\alpha,\beta}(\cdot,y)\cdot\varepsilon^j \otimes \varepsilon^l$ *are elements of* $\mathcal{H}_s(\overline{\Omega_\alpha^{\mathrm{ext}}})$ *for every (fixed)* $x \in \overline{\Omega_\beta^{\mathrm{ext}}}$, $j,l \in \{1,2,3\}$.

We now define the convolution of an admissible $\mathbf{h}_{\alpha,\beta}^{(i,k)}$-kernel against a function $\mathbf{f} \in \mathbf{h}_s(\overline{\Omega_\beta^{\text{ext}}})$ as follows:

$$
\begin{aligned}
(\mathbf{k}^{(i,k),\alpha,\beta} * \mathbf{f})(x) &= \mathbf{k}^{(i,k),\alpha,\beta}(x,\cdot) * \mathbf{f} \qquad (6.60)\\
&= \sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} \mathbf{k}^{(i,k)\wedge}(n)\mathbf{f}^{(i,k)\wedge}(n,m) H_{n,m}^s(\alpha;x), \quad x \in \overline{\Omega_\alpha^{\text{ext}}}.
\end{aligned}
$$

It follows directly that $(\mathbf{k}^{(i,k),\alpha,\beta} * \mathbf{f})^\wedge(n,m) = \mathbf{k}^{(i,k)\wedge}(n)\mathbf{f}^{(i,k)\wedge}(n,m)$, $n = 0_{ik}$, $i,k \in \{1,2,3\}$, and $\mathbf{k}^{(i,k),\alpha,\beta} * \mathbf{f} \in \mathcal{H}_s(\overline{\Omega_\alpha^{\text{ext}}})$. In an analogous way we define the convolution of an $\mathbf{h}_{\alpha,\beta}$-kernel $\mathbf{k}^{\alpha,\beta}(\cdot,\cdot)$ against a function $F \in \mathcal{H}_s(\overline{\Omega_\alpha^{\text{ext}}})$ by

$$
\begin{aligned}
(\mathbf{k}^{\alpha,\beta} \star F)(y) &= \mathbf{k}^{\alpha,\beta}(\cdot,y) \star F \qquad (6.61)\\
&= \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} \mathbf{k}^{(i,k)\wedge}(n) F^\wedge(n,m) \mathbf{h}_{n,m}^{(i,k)s}(\beta;y), \quad y \in \overline{\Omega_\beta^{\text{ext}}},
\end{aligned}
$$

and $\mathbf{k}^{\alpha,\beta} \star F$ is an element of $\mathbf{h}_s(\overline{\Omega_\beta^{\text{ext}}})$. Our next purpose is to present the formulation of the tensorial SGG operator with the help of convolutions. This enables us to give an answer to the question of continuity of the inverse. We start from a pseudodifferential operator $\boldsymbol{\lambda} : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathbf{h}_s(\overline{\Omega_\gamma^{\text{ext}}})$ given by

$$
\boldsymbol{\lambda} F = \sum_{i,k=1}^{3} \sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} \boldsymbol{\lambda}^{(i,k)\wedge}(n) F^\wedge(n,m) \mathbf{h}_{n,m}^{(i,k)s}(\gamma;\cdot), \qquad (6.62)
$$

and interpret the symbol of the pseudodifferential operator as the symbol of an $\mathbf{h}_{\sigma,\gamma}$-kernel $(\mathbf{k}^{\boldsymbol{\lambda}})^{\sigma,\gamma}$ presuming that the symbol satisfies the admissibility conditions. The pseudodifferential operator is then given by the convolution identity

$$
\boldsymbol{\lambda} F(x) = (\mathbf{k}^{\boldsymbol{\lambda}})^{\sigma,\gamma}(\cdot,x) \star F, \quad x \in \overline{\Omega_\gamma^{\text{ext}}}, \qquad (6.63)
$$

for $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, where $(\mathbf{k}^{\boldsymbol{\lambda}})^{(i,k)\wedge}(n) = \boldsymbol{\lambda}^{(i,k)\wedge}(n)$, $i,k = 1,2,3; n = 0_{ik},\ldots$. Obviously, we have

$$
(\mathbf{k}^{\boldsymbol{\lambda}})^{\sigma,\gamma}(\cdot,x) \star H_{n,m}^s(\sigma;\cdot) = \boldsymbol{\lambda}^{(i,k)\wedge}(n)\mathbf{h}_{n,m}^{(i,k)s}(\gamma;x), \qquad (6.64)
$$

for all $i,k = 1,2,3$; $n = 0_{ik},\ldots$; $m = 1,\ldots,2n+1$, or, equivalently,

$$
\boldsymbol{\lambda}^{(i,k)} H_{n,m}^s(\sigma;\cdot) = \boldsymbol{\lambda}^{(i,k)\wedge}(n)\mathbf{h}_{n,m}^{(i,k)s}(\gamma;\cdot). \qquad (6.65)
$$

Having a look at the (tensorial) SGG operator, we get the following result.

Theorem 6.24. *The* $\mathbf{h}^{SGG}_{\sigma,\gamma}$*-kernel* $(\mathbf{k}\,\boldsymbol{\lambda})^{\sigma,\gamma}$ *defined by the symbol*

$$\boldsymbol{\lambda}^{(i,k)\wedge}(n) = \begin{cases} \left(\frac{\sigma}{\gamma}\right)^n \frac{n(n+1)}{\gamma^2(2n+1)(2n+3)}\sqrt{\nu_n^{(1,1)}}, & (i,k)=(1,1),\\ -\left(\frac{\sigma}{\gamma}\right)^n \frac{(n+1)(n-1)}{\gamma^2((2n-1)(2n+1)}\sqrt{\nu_n^{(1,2)}}, & (i,k)=(1,2),\\ -\left(\frac{\sigma}{\gamma}\right)^n \frac{n(n+2)}{\gamma^2(2n+3)(2n+1)}\sqrt{\nu_n^{(2,1)}}, & (i,k)=(2,1),\\ \left(\frac{\sigma}{\gamma}\right)^n \frac{n(n+1)(n+2)}{\gamma^2(2n-1)(2n+1)}\sqrt{\nu_n^{(2,2)}}, & (i,k)=(2,2), \end{cases} \tag{6.66}$$

is admissible, if $\left(\frac{n+\frac{1}{2}}{\sigma}\right)^s$ *is summable and satisfies, in addition, condition* (iii) *in Definition* 6.22.

We finally get the following results.

Theorem 6.25. *Let* $\boldsymbol{\lambda} : \mathcal{H}(\overline{\Omega^{\text{ext}}_\sigma}) \to \mathbf{h}(\overline{\Omega^{\text{ext}}_\gamma})$ *be a pseudodifferential operator with* $(\mathbf{k}\,\boldsymbol{\lambda})^{\sigma,\gamma}$ *satisfying the admissibility conditions, and* $\boldsymbol{\lambda}^{(i,k)\wedge}(n) \neq 0$, $i \in \{1,2,3\}$, $n = 0_{ik}, \ldots$ *. Then the pseudodifferential operator* $\boldsymbol{\lambda}$ *is bounded and*

$$\|\boldsymbol{\lambda}\| = \max_{n\in\mathbb{N}_0} \left| \sum_{i,k=1}^{3} \boldsymbol{\lambda}^{(i,k)\wedge}(n) \right|, \tag{6.67}$$

where the sum has to be understood in the same sense as in the vectorial case. Further on, $\boldsymbol{\lambda}$ *is an injective operator.*

Theorem 6.26. *Let*

$$\boldsymbol{\lambda}F = \mathbf{g}, \quad F \in \mathcal{H}_s(\overline{\Omega^{\text{ext}}_\sigma}), \quad \mathbf{g} \in \mathbf{h}^{SGG}_s(\overline{\Omega^{\text{ext}}_\gamma}), \tag{6.68}$$

be the (*tensorial*) *SGG problem. Then* $\boldsymbol{\lambda}$ *is a compact operator with infinite-dimensional range. Furthermore,* $\boldsymbol{\lambda}^{-1}$ *is not bounded on* $\mathbf{h}^{SGG}_s(\overline{\Omega^{\text{ext}}_\gamma})$*. The SGG problem is solvable if and only if*

$$\sum_{(i,k)\in\mathcal{I}^{SGG}} \sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} \left(\frac{\mathbf{g}^{(i,k)\wedge}(n,m)}{\boldsymbol{\lambda}^{(i,k)\wedge}(n)} \right)^2 < \infty. \tag{6.69}$$

We consequently have that the (tensorial) SGG problem is ill posed because existence and continuity of the inverse are violated.

7. Geodetically oriented wavelet approximation

In this section we present a multiscale approach based on wavelet approximation. Note that all modern multiscale approaches have a conception of wavelets as constituting multiscale building blocks in common, which provide a fast and efficient way to decorrelate a given signal data set. As already mentioned in Section 2.4, this characterization contains three basic attributes (basis property, decorrelation and efficient algorithms), which are common features of all classical wavelets and form the key for a variety of applications, particularly for signal reconstruction and decomposition, thresholding, data compression, denoising, etc.

7.1. Scalar wavelet theory

We start with the presentation of the scalar theory, where we follow the approach given in [19]. First, we define an $\mathcal{H}_{\sigma,\sigma}$-multiresolution analysis. We use the abbreviation $\Phi^{(2)}(\cdot,\cdot) = (\Phi * \Phi)(\cdot,\cdot)$, where Φ is an $\mathcal{H}_{\sigma,\sigma}$-kernel.

Definition 7.1. Let $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be a family of admissible $\mathcal{H}_{\sigma,\sigma}$-kernels as defined in Definition 6.6. Then the family $\{\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})\}_{j\in\mathbb{N}_0}$ of scale spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ defined by

$$\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \{\Phi_j^{(2)} * F \ : \ F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})\}, \tag{7.1}$$

is called an $\mathcal{H}_{\sigma,\sigma}$*-multiresolution analysis*, if the following properties are satisfied:

(i) $\mathcal{V}_0(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \cdots \subset \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \mathcal{V}_{j+1}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \cdots \subset \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$,

(ii) $\overline{\bigcup_{j\in\mathbb{N}_0} \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})}^{\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}} = \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

Wavelet analysis is based on the idea of splitting the function into a lowpass part and several bandpass parts. The so-called scaling function corresponds to the lowpass filter, whereas the bandbass filters are the shifted and dilated versions of the wavelet, which are defined as differences between successive scaling functions with the help of a so-called *refinement equation*.

Definition 7.2. A family $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ of sequences $\{\varphi_j(n)\}_{n\in\mathbb{N}_0}$ is called a *generator of a scaling function*, if it satisfies the following requirements:

(i) $(\varphi_j(0))^2 = 1$, for all $j \in \mathbb{N}_0$,

(ii) $(\varphi_j(n))^2 \le (\varphi_{j'}(n))^2$, for all $j, j' \in \mathbb{N}_0$ with $j \le j'$ and all $n \in \mathbb{N}$,

(iii) $\lim_{j\to\infty} (\varphi_j(n))^2 = 1$, for all $n \in \mathbb{N}$.

Based on the definition of a generator of a scaling function, we now introduce $\mathcal{H}_{\sigma,\sigma}$-scaling functions.

Definition 7.3. A family $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ of $\mathcal{H}_{\sigma,\sigma}$-kernels $\Phi_j(\cdot,\cdot)$ defined by $\Phi_j^\wedge(n) = \varphi_j(n)$, $n, j \in \mathbb{N}_0$, i.e.,

$$\Phi_j(x,y) = \sum_{n=0}^{\infty} \varphi_j(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\sigma;x) H_{n,m}^s(\sigma;y), \quad x, y \in \overline{\Omega_\sigma^{\mathrm{ext}}}, \tag{7.2}$$

is called an $\mathcal{H}_{\sigma,\sigma}$*-scaling function*, if it satisfies the following properties:

(i) $\Phi_j(\cdot,\cdot)$ is an admissible $\mathcal{H}_{\sigma,\sigma}$-kernel for every $j \in \mathbb{N}_0$ (in the sense of Definition 6.6),

(ii) $\{\Phi_j^\wedge(n)_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ constitutes a generator of a scaling function (in the sense of Definition 7.2).

The following theorem shows the approximation property of an $\mathcal{H}_{\sigma,\sigma}$-scaling function.

Theorem 7.4. *Let $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be an $\mathcal{H}_{\sigma,\sigma}$-scaling function. Then*

$$\lim_{j\to\infty} \|F - \Phi_j^{(2)} * F\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})} = 0 \tag{7.3}$$

holds for all $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$.

We now introduce the dilation and the shifting operator in order to define an $\mathcal{H}_{\sigma,\sigma}$-approximate identity. Let $J, J_1, J_2 \in \mathbb{N}_0$ and $x \in \overline{\Omega_\sigma^{\text{ext}}}$. Then we define the *dilation operator* D_{J_1} and the *shifting operator* S_x by

$$D_{J_1} : \Phi_{J_2}(\cdot,\cdot) \mapsto D_{J_1}\Phi_{J_2}(\cdot,\cdot) = \Phi_{J_1+J_2}(\cdot,\cdot), \tag{7.4}$$

$$S_x : \Phi_J(\cdot,\cdot) \mapsto S_x\Phi_J(\cdot,\cdot) = \Phi_J(x,\cdot). \tag{7.5}$$

The shifting operator S_y acting on the second variable is defined in an analogous way. Note that by definition $\Phi_J(\cdot,\cdot) = D_J\Phi_0(\cdot,\cdot)$ for any $J \in \mathbb{N}_0$.

Definition 7.5. Let $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be an $\mathcal{H}_{\sigma,\sigma}$-scaling function. Then $\{P_j\}_{j\in\mathbb{N}_0}$ with $P_j : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ defined by

$$\begin{aligned} P_j(F)(x) &= \left(S_x D_j \Phi_0^{(2)}(\cdot,\cdot), F\right)_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})} \\ &= \left(\Phi_j^{(2)}(x,\cdot), F\right)_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})} \\ &= (\Phi_j^{(2)} * F)(x), \end{aligned} \tag{7.6}$$

for $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, $x \in \overline{\Omega_\sigma^{\text{ext}}}$, is called an $\mathcal{H}_{\sigma,\sigma}$*-approximate identity.*

The kernel Φ_0 is called mother kernel of the $\mathcal{H}_{\sigma,\sigma}$-scaling function. Theorem 7.4 leads to

$$\lim_{j\to\infty} \|F - P_j(F)\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})} = 0. \tag{7.7}$$

The following theorem clarifies the connection between the concept of multiresolution analysis and the scaling functions.

Theorem 7.6. *Let $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be an $\mathcal{H}_{\sigma,\sigma}$-scaling function. Then $\{\mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})\}_{j\in\mathbb{N}_0}$ forms an $\mathcal{H}_{\sigma,\sigma}$-multiresolution analysis.*

We now turn to the definition of the primal and dual wavelet.

Definition 7.7. Let $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be an $\mathcal{H}_{\sigma,\sigma}$-scaling function. Then the families of $\mathcal{H}_{\sigma,\sigma}$-kernels $\{\Psi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$, $\{\tilde{\Psi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ given by

$$\Psi_j^\wedge(n) = \psi_j(n), \quad n, j \in \mathbb{N}_0, \tag{7.8}$$

$$\tilde{\Psi}_j^\wedge(n) = \tilde{\psi}_j(n), \quad n, j \in \mathbb{N}_0, \tag{7.9}$$

are called (*primal*) $\mathcal{H}_{\sigma,\sigma}$*-wavelet* and *dual* $\mathcal{H}_{\sigma,\sigma}$*-wavelet*, respectively, if all $\mathcal{H}_{\sigma,\sigma}$-kernels $\Psi_j(\cdot,\cdot)$, $\tilde{\Psi}_j(\cdot,\cdot)$, $j \in \mathbb{N}_0$, are admissible and the symbols $\{\psi_j(n)\}$, $\{\tilde{\psi}_j(n)\}$, in addition, satisfy the (scalar) refinement equation

$$\tilde{\psi}_j(n)\psi_j(n) = (\varphi_{j+1}(n))^2 - (\varphi_j(n))^2 \tag{7.10}$$

for all $j, n \in \mathbb{N}_0$.

The following equation is a direct consequence of the refinement equation:

$$(\varphi_{J+1}(n))^2 = (\varphi_0(n))^2 + \sum_{j=0}^{J} \tilde{\psi}_j(n)\psi_j(n), \quad J \in \mathbb{N}_0, \tag{7.11}$$

for all $n \in \mathbb{N}_0$. This property finally leads to the reconstruction formula which states how the original function $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ can be derived from a lowpass part and the corresponding bandpass parts (see Theorem 7.9).

We now turn to the definition of the wavelet transform. To this end we define $\mathbb{N}_{-1} = \mathbb{N}_0 \cup \{-1\}$ and let $\psi_{-1}(n) = \tilde{\psi}_{-1}(n) = \varphi_0(n)$, for $n \in \mathbb{N}_0$, $\Psi_{-1}(\cdot,\cdot) = \tilde{\Psi}_{-1}(\cdot,\cdot) = \Phi_0(\cdot,\cdot)$. This abbreviation simplifies our notation. Then we define the space

$$\mathcal{H}_s(\mathbb{N}_{-1} \times \overline{\Omega_\sigma^{\mathrm{ext}}}) = \{H : \mathbb{N}_{-1} \times \overline{\Omega_\sigma^{\mathrm{ext}}} \to \mathbb{R} \ : \ \sum_{j=-1}^{\infty} (H(j;\cdot), H(j;\cdot))_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} < \infty\} \tag{7.12}$$

with inner product

$$(H_1, H_2)_{\mathcal{H}_s(\mathbb{N}_{-1} \times \overline{\Omega_\sigma^{\mathrm{ext}}})} = \sum_{j=-1}^{\infty} (H_1(j;\cdot), H_2(j;\cdot))_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} \tag{7.13}$$

and corresponding norm

$$\|H\|_{\mathcal{H}_s(\mathbb{N}_{-1} \times \overline{\Omega_\sigma^{\mathrm{ext}}})} = \left(\sum_{j=-1}^{\infty} \|H(j;\cdot)\|^2_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} \right)^{1/2}. \tag{7.14}$$

With the help of the dilation operator D_j and the shifting operator S_y we introduce the following abbreviation:

$$\Psi_{j;y}(\cdot) = \Psi_j(\cdot, y) = S_y \Psi_j(\cdot,\cdot) = S_y D_j \Psi_0(\cdot,\cdot), \tag{7.15}$$

$$\tilde{\Psi}_{j;y}(\cdot) = \tilde{\Psi}_j(\cdot, y) = S_y \tilde{\Psi}_j(\cdot,\cdot) = S_y D_j \tilde{\Psi}_0(\cdot,\cdot). \tag{7.16}$$

Definition 7.8. Let $\{\Psi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_{-1}}$ be a (primal) $\mathcal{H}_{\sigma,\sigma}$-wavelet. Then

$$WT : \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to \mathcal{H}_s(\mathbb{N}_{-1} \times \overline{\Omega_\sigma^{\mathrm{ext}}}),$$

defined by

$$(WT)(F)(j;y) = (\Psi_{j;y}, F)_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})} = (\Psi_j * F)(y), \tag{7.17}$$

is called $\mathcal{H}_{\sigma,\sigma}$-wavelet transform of F at position $y \in \overline{\Omega_\sigma^{\mathrm{ext}}}$ and scale $j \in \mathbb{N}_{-1}$.

Having the definition of the scale spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ in mind, we now define the detail spaces $\mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ at scale j by

$$\mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \left\{ \tilde{\Psi}_j * \Psi_j * F \ : \ F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \right\}, \quad j \in \mathbb{N}_0. \tag{7.18}$$

Theorem 7.9 (Scalar Reconstruction Formula for the Outer Space). *Let the families $\{\Psi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ and $\{\tilde{\Psi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$, respectively, be a (primal) $\mathcal{H}_{\sigma,\sigma}$-wavelet and its dual corresponding to an $\mathcal{H}_{\sigma,\sigma}$-scaling function $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$. Then*

$$F = \sum_{j=-1}^{\infty} \tilde{\Psi}_j * \Psi_j * F \tag{7.19}$$

holds for all $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ (in $\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})}$-sense).

We now solve the (scalar) SST or SGG problem using bandlimited harmonic wavelets. First, we define $\mathcal{H}_{\alpha,\alpha}$-scaling functions with the help of a generator of a scaling function $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$. Since the generator does not depend on σ, we can directly extend the theory to the case of $\mathcal{H}_{\alpha,\alpha}$-scaling functions $\Phi_j^{\alpha,\alpha}$ with $\alpha \geq \sigma$:

$$\Phi_j^{\alpha,\alpha}(x,y) = \sum_{n=0}^{\infty} \varphi_j(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\alpha;x) H_{n,m}^s(\alpha;y), \tag{7.20}$$

where

$$(\Phi_j^{\alpha,\alpha})^\wedge(n) = \varphi_j(n). \tag{7.21}$$

As a consequence, Theorem 7.4 is valid substituting σ by α. Furthermore, the definition of the scale spaces can be directly transferred in the following way:

$$\mathcal{V}_j(\overline{\Omega_\alpha^{\text{ext}}}) = \{(\Phi_j^{(2)})^{\alpha,\alpha} * F \;:\; F \in \mathcal{H}_s(\overline{\Omega_\alpha^{\text{ext}}})\}, \tag{7.22}$$

where

$$(\Phi_j^{(2)})^{\alpha,\alpha} = \Phi_j^{\alpha,\alpha} * \Phi_j^{\alpha,\alpha}. \tag{7.23}$$

The system $\{\mathcal{V}_j(\overline{\Omega_\alpha^{\text{ext}}})\}$ of scale spaces forms a multiresolution analysis due to Theorem 7.6. We now investigate the solution of the restriction of an operator $\Lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$ to a scale space $\mathcal{V}_j$:

$$\Lambda : \mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{V}_j(\overline{\Omega_\gamma^{\text{ext}}}). \tag{7.24}$$

Note that $\Lambda(\mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})) \subset \mathcal{V}_j(\overline{\Omega_\gamma^{\text{ext}}})$ is automatically fulfilled, because every $F \in \mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})$ of the form

$$F = \Phi_j^{(2)} * Q, \quad Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \tag{7.25}$$

with Fourier coefficients $F^\wedge(n,m) = (\varphi_j^\wedge(n))^2 Q^\wedge(n,m)$ leads to

$$\begin{aligned}
\Lambda F(x) &= \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \Lambda^\wedge(n) F^\wedge(n,m) H_{n,m}^s(\gamma;x) \\
&= \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \Lambda^\wedge(n) (\varphi_j^\wedge(n))^2 Q^\wedge(n,m) H_{n,m}^s(\gamma;x) \\
&= (\Phi_j^{(2)})^{\gamma,\gamma} * (\Lambda Q) = (\Phi_j^{(2)})^{\gamma,\gamma} * G,
\end{aligned} \tag{7.26}$$

where we let $G = \Lambda Q \in \mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$. Thus, we get the following theorem.

Theorem 7.10. *The restriction of the operator* $\Lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$ *to a scale space* $\mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})$, $j \in \mathbb{N}_0$, *i.e.*,

$$\Lambda|_{\mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})} : \mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{V}_j(\overline{\Omega_\gamma^{\text{ext}}}) \tag{7.27}$$

is injective. Moreover, we have the following results:

(i) *If the families* $\{\{\psi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ *and* $\{\{\tilde{\psi}_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ *are bandlimited (for example,* $\psi_j(n) = \tilde{\psi}_j(n) = 0$ *for all* $n \geq 2^j$*), then the restricted operator is even bijective. To be more specific, for* $G \in \mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$ *the unique solution* $F_j \in \mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})$, $j \in \mathbb{N}_0$, *of the equation*

$$\Lambda F_j = (\Phi_j^{(2)})^{\gamma,\gamma} * G \tag{7.28}$$

is given by

$$F_j = (\Phi_j^{(2)})^{\sigma,\sigma} * Q, \tag{7.29}$$

where $Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ *is given by*

$$Q^\wedge(n,m) = \begin{cases} \frac{G^\wedge(n,m)}{\Lambda^\wedge(n)}, & n \in [0, 2^j), \\ 0, & n \in [2^j, \infty), \end{cases} \tag{7.30}$$

$n = 0, 1, \ldots; m = 1, \ldots, 2n+1.$

(ii) *If the families* $\{\{\psi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ *and* $\{\{\tilde{\psi}_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ *are not bandlimited, the equation*

$$\Lambda F_j = (\Phi_j^{(2)})^{\gamma,\gamma} * G \tag{7.31}$$

has a solution $F_j \in \mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})$ *provided that* $G \in \mathcal{H}_s^\Lambda(\overline{\Omega_\sigma^{\text{ext}}})$, *where* $\mathcal{H}_s^\Lambda(\overline{\Omega_\sigma^{\text{ext}}})$ *is a suitable Sobolev space (see the Ph.D.-thesis* [58] *for a detailed introduction). In this case, the unique solution of the equation is given by*

$$F_j = (\Phi_j^{(2)})^{\sigma,\sigma} * Q, \tag{7.32}$$

where $Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ *is obtainable by*

$$Q^\wedge(n,m) = \frac{G^\wedge(n,m)}{\Lambda^\wedge(n)}, \tag{7.33}$$

$n = 0, 1, \ldots;\ m = 1, \ldots, 2n+1.$

We now define the primal wavelets $\{\Psi_j^{\alpha,\alpha}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ and the dual wavelets $\{\tilde{\Psi}_j^{\alpha,\alpha}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ for $\alpha \geq \sigma$ in the way as we did in the case of the scaling functions and get

$$\Psi_j^{\alpha,\alpha}(x,y) = \sum_{n=0}^{\infty} \psi_j(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\alpha;x) H_{n,m}^s(\alpha;y), \tag{7.34}$$

$$\tilde{\Psi}_j^{\alpha,\alpha}(x,y) = \sum_{n=0}^{\infty} \tilde{\psi}_j(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\alpha;x) H_{n,m}^s(\alpha;y), \tag{7.35}$$

where

$$(\Psi_j^{\alpha,\alpha})^\wedge(n) = \psi_j(n), \quad (\tilde{\Psi}_j^{\alpha,\alpha})^\wedge(n) = \tilde{\psi}_j(n). \tag{7.36}$$

The detail spaces are defined in canonical manner:

$$\mathcal{W}_j(\overline{\Omega_\alpha^{\mathrm{ext}}}) = \{(\Psi_j * \tilde{\Psi}_j)^{\alpha,\alpha} * F \ : \ F \in \mathcal{H}_s(\overline{\Omega_\alpha^{\mathrm{ext}}})\}, \tag{7.37}$$

where

$$(\Psi_j * \tilde{\Psi}_j)^{\alpha,\alpha} = \Psi_j^{\alpha,\alpha} * \tilde{\Psi}_j^{\alpha,\alpha}. \tag{7.38}$$

The reconstruction formula given in Theorem 7.9 is valid substituting $\tilde{\Psi}_j * \Psi * F$ by $(\tilde{\Psi}_j * \Psi)^{\alpha,\alpha} * F$. Theorem 7.10 can now be transferred to the restriction on detail spaces and we get the following theorem.

Theorem 7.11. *The restriction of the operator* $\Lambda : \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\gamma^{\mathrm{ext}}})$ *to a detail space* $\mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, *i.e.,*

$$\Lambda|_{\mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})} : \mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to \mathcal{W}_j(\overline{\Omega_\gamma^{\mathrm{ext}}}) \tag{7.39}$$

is injective. Moreover, we have the following results:

(i) *If the family* $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ *is bandlimited (for example,* $\varphi_j(n) = 0$ *for all* $n \geq 2^j$*), then the restricted operator is even bijective. To be more specific, for* $G \in \mathcal{H}_s(\overline{\Omega_\gamma^{\mathrm{ext}}})$ *the unique solution* $H_j \in \mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, *of the equation*

$$\Lambda H_j = (\tilde{\Psi}_j * \Psi_j)^{\gamma,\gamma} * G \tag{7.40}$$

is given by

$$H_j = (\tilde{\Psi}_j * \Psi_j)^{\sigma,\sigma} * Q, \tag{7.41}$$

where $Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ *is obtainable by*

$$Q^\wedge(n,m) = \begin{cases} \frac{G^\wedge(n,m)}{\Lambda^\wedge(n)}, & n \in [0, 2^{j+1}), \\ 0, & n \in [2^{j+1}, \infty), \end{cases} \tag{7.42}$$

$n \in \mathbb{N}_0$; $m = 1, \ldots, 2n+1$.

(ii) *If the family* $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ *is non-bandlimited, the equation*

$$\Lambda H_j = (\tilde{\Psi}_j * \Psi_j)^{\gamma,\gamma} * G \tag{7.43}$$

has a solution $H_j \in \mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ *provided that* $G \in \mathcal{H}_s^\Lambda(\overline{\Omega_\sigma^{\mathrm{ext}}})$, *where* $\mathcal{H}_s^\Lambda(\overline{\Omega_\sigma^{\mathrm{ext}}})$ *is a suitable Sobolev space (cf. the Ph.D.-thesis [58] for a detailed definition). In this case, the unique solution of the equation is given by*

$$H_j = (\tilde{\Psi}_j * \Psi_j)^{\sigma,\sigma} * Q, \tag{7.44}$$

where $Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ *is obtainable by*

$$Q^\wedge(n,m) = \frac{G^\wedge(n,m)}{\Lambda^\wedge(n)}, \tag{7.45}$$

$n \in \mathbb{N}_0$; $m = 1, \ldots, 2n+1$.

Up to now, we have summarized some results about the filtered solution, i.e., the solution when we restrict the operator to scale or detail spaces. In the case of the unfiltered solution, we have the following theorem.

Theorem 7.12. *Let $G \in \mathcal{H}_s(\overline{\Omega_\gamma^{\mathrm{ext}}})$ satisfy the condition $G \in \mathrm{im}(\Lambda)$. Then the unique solution $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ of the equation $\Lambda F = G$ is given by*

$$F^\wedge(n,m) = \frac{G^\wedge(n,m)}{\Lambda^\wedge(n)}, \tag{7.46}$$

$n \in \mathbb{N}_0$; $m = 1, \ldots, 2n+1$.

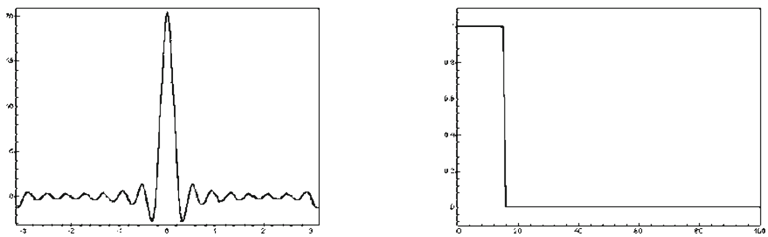
Examples for scaling functions

To make the preceding considerations more concrete, we would like to show that all reproducing kernel functions introduced in Section 5 may be used as $\mathcal{H}_{\sigma,\sigma}$-scaling functions. We essentially follow [19] and distinguish in accordance with Definition 7.2 two cases, viz. (1) *bandlimited $\mathcal{H}_{\sigma,\sigma}$-scaling functions* and (2) *non-bandlimited $\mathcal{H}_{\sigma,\sigma}$-scaling functions.*

(1) *Bandlimited $\mathcal{H}_{\sigma,\sigma}$-scaling Functions.* Suppose that $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ is a Sobolev space (satisfying the consistency conditions (CC1) and (CC2) relative to $[\sigma, \sigma^{\mathrm{inf}})$). Consider sequences $\{\varphi_j(n)\}_{n\in\mathbb{N}_0}$ with "local support" (for example, $\varphi_j(n) = 0$ for all $n \geq 2^j$, $j \in \mathbb{N}_0$). Thus all members $\Phi_j(\cdot,\cdot)$ of an associated $\mathcal{H}_{\sigma,\sigma}$-scaling function $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ with $(\Phi_j)^\wedge(n) = \varphi_j(n)$, $n \in \mathbb{N}_0$, are bandlimited. This allows to deal with finite-dimensional scale spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$. Consequently, all spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ have finite-dimensional basis systems.

(1a) *Shannon $\mathcal{H}_{\sigma,\sigma}$-scaling function* (see Figure 7.1). Consider the family

$$\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$$



(a) $\Phi_4(x,y)$ (b) $\varphi_4(n)$

FIGURE 7.1. Shannon $\mathcal{H}_{\sigma,\sigma}$-scaling function for $j = 4$ and $A_n = 1$: space domain, i.e., $\Phi_j(x,y)$ for $(x,y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $\varphi_j(n)$ (right).

given by

$$\varphi_j(n) = \begin{cases} 1, & n \in [0, 2^j), \\ 0, & n \in [2^j, \infty). \end{cases}$$

The family $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ forms a generator of a scaling function in the sense of Definition 7.2. The $\mathcal{H}_{\sigma,\sigma}$-scaling function $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ reads as follows:

$$\Phi_j(x,y) = \sum_{n\le 2^j-1} \frac{1}{A_n^2} \frac{2n+1}{4\pi\sigma^2} \left(\frac{\sigma^2}{|x|\,|y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right),$$

$(x,y) \in \overline{\Omega_\sigma^{\mathrm{ext}}} \times \overline{\Omega_\sigma^{\mathrm{ext}}}$. A remarkable property is that $\Phi_j(\cdot,\cdot)$ coincides with its iterations:

$$\Phi_j^{(k)}(\cdot,\cdot) = (\Phi_j *_{\mathcal{H}}\ \Phi_j^{(k-1)})(\cdot,\cdot), \quad k = 2,3,\ldots\ .$$

The scale spaces

$$\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) = P_j(\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})) = \bigoplus_{n\le 2^j-1} Harm_n(\overline{\Omega_\sigma^{\mathrm{ext}}}), \quad j \in \mathbb{N}_0,$$

satisfy the properties:

(i) $\mathcal{V}_0(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \cdots \subset \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \mathcal{V}_{j+1}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \cdots \subset \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$,

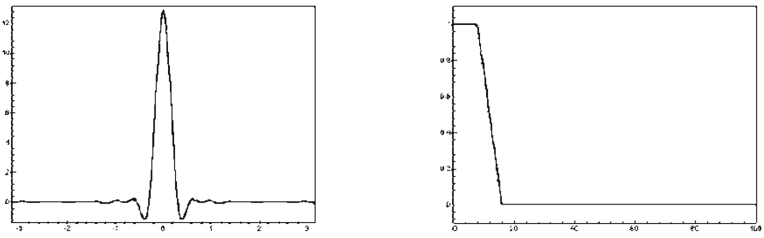
(ii) $\overline{\bigcup_{j\in\mathbb{N}_0} \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})}^{\|\cdot\|_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}} = \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$,

(iii) $\bigcap_{j\in\mathbb{N}_0} V_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) = Harm_0(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

The multiresolution analysis is orthogonal. As a matter of fact, the Shannon "detail spaces" $\mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \mathcal{V}_{j+1}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \ominus \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ of different scales j do not have any common frequencies. Consequently, the orthogonality of the outer harmonics immediately implies the orthogonality of the Shannon detail spaces. The scale spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, form an $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-multiresolution analysis. Apart from this, it can be even verified that the decomposition of the scale space $\mathcal{V}_{j+1}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ into the scale space $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ and the detail space $\mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ is orthogonal. This orthogonality of the decomposition easily follows from the already known fact that

$$\begin{aligned} \mathcal{V}_{j+1}(\overline{\Omega_\sigma^{\mathrm{ext}}}) &= \bigoplus_{0\le n\le 2^{j+1}-1} Harm_n(\overline{\Omega_\sigma^{\mathrm{ext}}}) \\ &= \bigoplus_{0\le n\le 2^j-1} Harm_n(\overline{\Omega_\sigma^{\mathrm{ext}}}) \oplus \bigoplus_{2^j\le n\le 2^{j+1}-1} Harm_n(\overline{\Omega_\sigma^{\mathrm{ext}}}) \\ &= \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) \oplus \mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}). \end{aligned} \tag{7.47}$$

On the one hand, the orthogonal structure of the Shannon multiresolution analysis seems to be very profitable. On the other hand, it is not surprising that the Shannon $\mathcal{H}_{\sigma,\sigma}$-scaling function shows strong oscillations. This is the price to be paid for the sharp separation "in momentum space". For numerical purposes it is often advisable to discuss "smoothed versions" of the Shannon



(a) $\Phi_4(x,y)$ (b) $\varphi_4(n)$

FIGURE 7.2. Smoothed Shannon $\mathcal{H}_{\sigma,\sigma}$-scaling function for $j=4$ and $A_n=1$, $h=\frac{1}{2}$: space domain, i.e., $\Phi_j(x,y)$ for $(x,y)\in\Omega_\sigma\times\Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $\varphi_j(n)$ (right).

kernels. But this automatically implies the loss of the orthogonality in the multiresolution analysis.

(1b) *Smoothed Shannon $\mathcal{H}_{\sigma,\sigma}$-scaling Function* (see Figure 7.2). For fixed $h\in[0,1)$ we now consider the family $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ given by

$$\varphi_j(n)=\begin{cases}1, & n\in[0,2^jh),\\ \frac{1-2^{-j}n}{1-h}, & n\in[2^jh,2^j),\\ 0, & n\in[2^j,\infty).\end{cases}$$

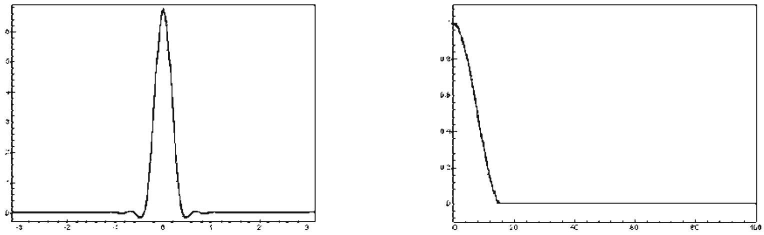
The family $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ defines a generator of an $\mathcal{H}_{\sigma,\sigma}$-scaling function. Obviously, $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ with $(\Phi_j)^\wedge(n)=\varphi_j(n)$ for $n,j\in\mathbb{N}_0$ is an $\mathcal{H}_{\sigma,\sigma}$-scaling function. Clearly, for each $n\in\mathbb{N}_0$, $\{\varphi_j(n)\}_{j\in\mathbb{N}_0}$ is monotonously increasing. The kernels $\Phi_j(\cdot,\cdot)\colon\overline{\Omega_\sigma^{\mathrm{ext}}}\times\overline{\Omega_\sigma^{\mathrm{ext}}}\to\mathbb{R}$ read as follows:

$$\Phi_j(x,y)=\sum_{n\le 2^j-1}\frac{2n+1}{4\pi\sigma^2}\frac{\varphi_j(n)}{A_n^2}\left(\frac{\sigma^2}{|x|\,|y|}\right)^{n+1}P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right).$$

The value $h\in[0,1)$ represents a "control parameter" of the smoothing effect of the $\mathcal{H}_{\sigma,\sigma}$-scaling function $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$. The scale spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j\in\mathbb{N}_0$, form an $\mathcal{H}_{\sigma,\sigma}$-multiresolution analysis. This multiresolution analysis, however, is *not* orthogonal, since $\mathcal{V}_{j+1}(\overline{\Omega_\sigma^{\mathrm{ext}}})=\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})+\mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j\in\mathbb{N}_0$, cannot be understood as orthogonal sum decomposition.

(1c) *<u>C</u>ubic <u>P</u>olynomial (CP) $\mathcal{H}_{\sigma,\sigma}$-scaling Function* (see Figure 7.3). In order to gain a higher intensity of the smoothing effect than in the case of the $\mathcal{H}_{\sigma,\sigma}$-scaling function (1b), we introduce a function $\varphi_0:[0,\infty)\to\mathbb{R}$ in such a way that $\varphi_0|_{[0,1]}$ coincides with the uniquely determined cubic polynomial $p:[0,1]\to[0,1]$ with the properties:

$$p(0)=1,\ p(1)=0,\quad p'(0)=0,\ p'(1)=0.$$



(a) $\Phi_4(x,y)$ (b) $\varphi_4(n)$

FIGURE 7.3. CP $\mathcal{H}_{\sigma,\sigma}$-scaling function for $j = 4$ and $A_n = 1$: space domain, i.e., $\Phi_j(x,y)$ for $(x,y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $\varphi_j(n)$ (right).

It is not difficult to see that these properties are fulfilled by

$$p(t) = (1-t)^2(1+2t), \quad t \in [0,1].$$

This leads us to a function $\varphi_0 : [0,\infty) \to \mathbb{R}$ given by

$$\varphi_0(t) = \begin{cases} (1-t)^2(1+2t), & t \in [0,1), \\ 0, & t \in [1,\infty). \end{cases}$$

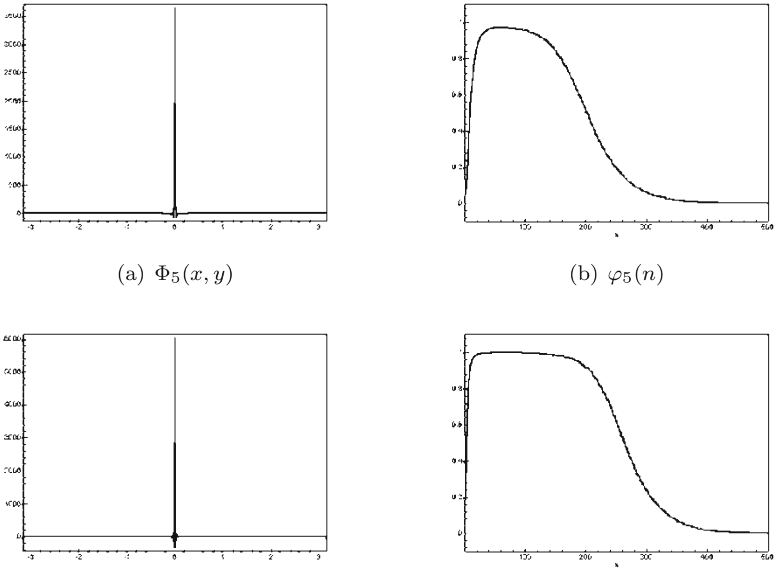
It is obvious that φ_0 is a monotonously decreasing function. In [31] a construction principle of deriving scaling functions from a "mother function" $\varphi_0 : [0,\infty) \to \mathbb{R}$ by letting $\varphi_j(t) = \varphi_0(2^{-j}t), t \in [0,\infty)$, is described and we thus define the family $\{\{\varphi_j\}_{j\in\mathbb{N}_0}\}_{n\in\mathbb{N}_0}$ with $\varphi_j(t) = \varphi_0(2^{-j}t)$, $t \in [0,\infty)$, by

$$\varphi_j(t) = \varphi_0(2^{-j}t) = \begin{cases} (1-2^{-j}t)^2(1+2^{-j+1}t), & t \in [0,2^j), \\ 0, & t \in [2^j,\infty). \end{cases}$$

$\{\varphi_j(n)\}_{j\in\mathbb{N}_0}$ is a monotonously increasing sequence for each $n \in \mathbb{N}_0$, hence, $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ defines an $\mathcal{H}_{\sigma,\sigma}$-scaling function. The finite-dimensional scale spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})$, $j \in \mathbb{N}_0$, represent a non-orthogonal $\mathcal{H}_{\sigma,\sigma}$-multiresolution analysis.

Finally, it should be remarked that one can think of other ways to "smooth" the Shannon generator but these are not discussed.

(2) *Non-bandlimited $\mathcal{H}_{\sigma,\sigma}$-scaling functions.* Next we take a look at non-bandlimited generators of scaling functions. In other words, all $\mathcal{H}_{\sigma,\sigma}$-scaling functions $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ discussed in the following share the property that their "generators" $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ have a "global support". Since there are only a few conditions for a family $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ to generate an $\mathcal{H}_{\sigma,\sigma}$-scaling function, there are various possibilities for its concrete realization. In our approach we concentrate on three types: Tikhonov, rational, and exponential $\mathcal{H}_{\sigma,\sigma}$-scaling functions.



(a) $\Phi_5(x,y)$ (b) $\varphi_5(n)$

(c) $\Phi_7(x,y)$ (d) $\varphi_7(n)$

FIGURE 7.4. Tikhonov $\mathcal{H}_{\sigma,\sigma}$-scaling function for $j=5$ (above) and $j=7$ (below) and $A_n = 1$: space domain, i.e., $\Phi_j(x,y)$ for $(x,y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $\varphi_j(n)$ (right).

(2a) *Tikhonov $\mathcal{H}_{\sigma,\sigma}$-scaling Function* (see Figure 7.4). Consider the family

$$\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$$

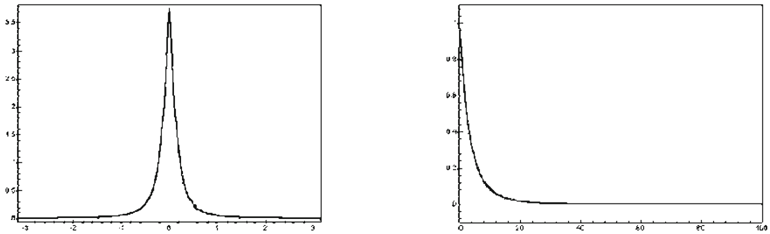
given by

$$\varphi_j(n) = \begin{cases} 1, & n = 0, \\ \left(\frac{\tau_n^2}{\tau_n^2 + (2^{-j})^2}\right)^{1/2}, & n = 1, 2, \ldots, \end{cases} \tag{7.48}$$

where the sequence $\{\tau_n\}_{n\in\mathbb{N}_0}$ with $\tau_n \neq 0$ for all $n \in \mathbb{N}_0$ is given in such a way that

$$\text{(i)} \qquad \sum_{n=0}^{\infty} \tau_n^2 < \infty\,, \qquad \text{(ii)} \qquad \sum_{n=0}^{\infty} (2n+1)\left(\frac{\tau_n}{A_n}\right)^2 < \infty.$$

It is not hard to see that the family $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ constitutes an $\mathcal{H}_{\sigma,\sigma}$-scaling function. The Tikhonov $\mathcal{H}_{\sigma,\sigma}$-scaling function plays an important role in the theory of regularization wavelets.



(a) $\Phi_4(x, y)$ (b) $\varphi_4(n)$

FIGURE 7.5. Rational $\mathcal{H}_{\sigma,\sigma}$-scaling function for $j = 4$ and $A_n = 1$, $\tau = 5$: space domain, i.e., $\Phi_j(x, y)$ for $(x, y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $\varphi_j(n)$ (right).

(2b) *Rational $\mathcal{H}_{\sigma,\sigma}$-scaling Functions* (see Figure 7.5). Consider $\varphi_j : [0, \infty) \to \mathbb{R}$ given by

$$\varphi_j(t) = (1 + 2^{-j} t)^{-\tau}, \quad t \in [0, \infty), \quad \tau > 1. \tag{7.49}$$

Clearly, for all values $\tau > 1$, the family $\{\{\varphi_j(n)\}_{n \in \mathbb{N}_0}\}_{j \in \mathbb{N}_0}$ forms a generator of a scaling function. All functions φ_j, $j \in \mathbb{N}_0$, define admissible $\mathcal{H}_{\sigma,\sigma}$-kernels $\Phi_j(\cdot, \cdot)$, $j \in \mathbb{N}_0$, if, in addition, $\tau > 1$ is chosen in such a way that

$$\sum_{n=0}^{\infty} (2n + 1) \frac{(1 + 2^{-j} n)^{-2\tau}}{A_n^2} < \infty \tag{7.50}$$

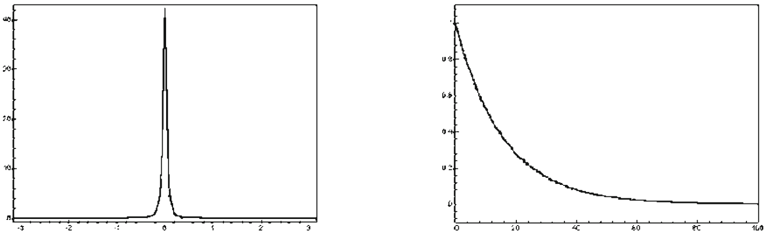
for $j \in \mathbb{N}_0$. For example, in the case of $\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, i.e., $A_n = \left(\frac{n+\frac{1}{2}}{\sigma}\right)^s$ for $n = 0, 1, \ldots$, we find $s + \tau > 1$ to satisfy the estimate (7.50). More generally, $(1 + n)^{-2\tau} A_n^{-2} = \mathcal{O}(n^{-2-\varepsilon})$ for $n \to \infty$ with $\varepsilon > 0$ together with $\tau > 1$ is a sufficient condition to define an admissible $\mathcal{H}_{\sigma,\sigma}$-kernel $\Phi_j(\cdot, \cdot)$, $j \in \mathbb{N}_0$. The $\mathcal{H}_{\sigma,\sigma}$-scaling function $\{\Phi_j(\cdot, \cdot)\}_{j \in \mathbb{N}_0}$ consists of the kernels

$$\Phi_j(x, y) = \sum_{n=0}^{\infty} \frac{(1 + 2^{-j} n)^{-\tau}}{A_n^2} \frac{2n + 1}{4\pi\sigma^2} \left(\frac{\sigma^2}{|x|\,|y|}\right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right),$$

$(x, y) \in \overline{\Omega_\sigma^{\text{ext}}} \times \overline{\Omega_\sigma^{\text{ext}}}$. The functions φ_j, $j \in \mathbb{N}_0$, are monotonously decreasing on the interval $[0, \infty)$ for all values $\tau > 1$ and all $j \in \mathbb{N}_0$. Therefore, the scale spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})$ form an $\mathcal{H}_{\sigma,\sigma}$-multiresolution analysis provided that both $\tau > 1$ and the summability condition (7.50) is valid.

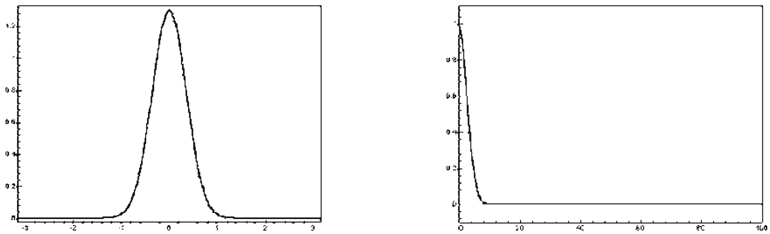
(2c) *Exponential $\mathcal{H}_{\sigma,\sigma}$-scaling Functions* (see Figures 7.6 and 7.7). Choose $\varphi_j : [0, \infty) \to \mathbb{R}$, $j \in \mathbb{N}_0$, to be defined by

$$\varphi_j(t) = e^{-2^{-j} H(t)}, \qquad t \in [0, \infty), \tag{7.51}$$



(a) $\Phi_4(x, y)$ (b) $\varphi_4(n)$

FIGURE 7.6. Abel–Poisson $\mathcal{H}_{\sigma,\sigma}$-scaling function for $j = 4$ and $A_n = 1$, $\tau = 1$: space domain, i.e., $\Phi_j(x, y)$ for $(x, y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $\varphi_j(n)$ (right).



(a) $\Phi_4(x, y)$ (b) $\varphi_4(n)$

FIGURE 7.7. Gauss–Weierstraß $\mathcal{H}_{\sigma,\sigma}$-scaling function for $j = 4$ and $A_n = 1$, $\tau = 1$: space domain, i.e., $\Phi_j(x, y)$ for $(x, y) \in \Omega_\sigma \times \Omega_\sigma$ in sectional representation (left) and frequency domain, i.e., $\varphi_j(n)$ (right).

where $H : [0, \infty) \to [0, \infty)$ satisfies the properties:

- $H \in C^{(\infty)}[0, \infty)$,
- $H(0) = 0$,
- $H(t) > 0 \quad \text{for} \quad t > 0$,
- $H(t) < H(t') \quad \text{whenever} \quad 0 < t < t'$.

The sequence $\{\varphi_j(n)\}_{j \in \mathbb{N}_0}$ is monotonously increasing for each $n \in \mathbb{N}_0$. The functions φ_j, $j \in \mathbb{N}_0$, define an $\mathcal{H}_{\sigma,\sigma}$-scaling function $\{\Phi_j(\cdot, \cdot)\}_{j \in \mathbb{N}_0}$ by letting $(\Phi_j)^\wedge(n) = \varphi_j(n)$, $n \in \mathbb{N}_0$, provided that $\Phi_j(\cdot, \cdot)$, $j \in \mathbb{N}_0$, are admissible $\mathcal{H}_{\sigma,\sigma}$-kernel functions. It is not hard to see that

$$
\begin{aligned}
&(\Phi_j * \Phi_j)(x, y) \\
&\quad = \sum_{n=0}^{\infty} \frac{(e^{-2^{-j} H(n)})^2}{A_n^2} \frac{2n+1}{4\pi\sigma^2} \left(\frac{\sigma^2}{|x|\,|y|} \right)^{n+1} P_n\left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right)
\end{aligned}
$$

$$= \sum_{n=0}^{\infty} \frac{e^{-2^{-(j-1)}H(n)}}{A_n^2} \frac{2n+1}{4\pi\sigma^2} \left(\frac{\sigma^2}{|x|\,|y|}\right)^{n+1} P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)$$
$$= \Phi_{j-1}(x,y) \tag{7.52}$$

holds for all $j \in \mathbb{N}$ and all $(x,y) \in \overline{\Omega_\sigma^{\mathrm{ext}}} \times \overline{\Omega_\sigma^{\mathrm{ext}}}$. The scale spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ constitute an $\mathcal{H}_{\sigma,\sigma}$-multiresolution analysis. Altogether we find the following result for exponential $\mathcal{H}_{\sigma,\sigma}$-scaling functions: The family $\{P_j\}_{j\in\mathbb{N}_0}$ of operators $P_j : \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ (defined by $P_j(F) = \Phi_j^{(2)} * F$, $F \in \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$) forms an $\mathcal{H}_{\sigma,\sigma}$-contracting approximate identity (called the *exponential* $\mathcal{H}_{\sigma,\sigma}$-*contracting approximate identity*), i.e., the following properties are satisfied:

(i) P_j is a bounded linear operator for every $j \in \mathbb{N}_0$ and $P_\infty = I$ (identity),
(ii) $P_{j-1} = P_j P_j$ for all $j \in \mathbb{N}_0$,
(iii) $\lim_{j\to\infty} \|F - P_j(F)\|_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})} = 0$ for all $F \in \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$,
(iv) $\|P_j(F)\|_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})} \le \|F\|_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}$ for all $j \in \mathbb{N}_0$, $F \in \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

As examples we mention the *Abel–Poisson* $\mathcal{H}_{\sigma,\sigma}$*-contracting approximate identity* given by $H(t) = \alpha t$, $\alpha > 0$, and the *Gauss–Weierstraß* $\mathcal{H}_{\sigma,\sigma}$*-contracting approximate identity* given by $H(t) = \alpha t(t+1)$, $\alpha > 0$.

Remark 7.13. Non-bandlimited scaling functions become bandlimited ones by suitable truncation in momentum space. To be more specific, if $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ is a non-bandlimited $\mathcal{H}_{\sigma,\sigma}$-scaling function, then $\{\Gamma_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ given by $(\Gamma_j)^\wedge(n) = (\Phi_j)^\wedge(n)$ for $n \in [0, 2^j)$ and $(\Gamma_j)^\wedge(n) = 0$ for $n \in [2^j, \infty)$ represents a bandlimited $\mathcal{H}_{\sigma,\sigma}$-scaling function.

We now explain the connection between the solution in the scale spaces and the unfiltered solution.

Theorem 7.14. *Suppose that G is of class $\mathcal{H}_s^\Lambda(\overline{\Omega_\gamma^{\mathrm{ext}}})$. Let $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ be the unique solution of $\Lambda F = G$. Then*

$$F_j = (\Phi_j^{(2)})^{\sigma,\sigma} * F \tag{7.53}$$

is the unique solution in $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ of the equation

$$\Lambda F_j = (\Phi_j^{(2)})^{\gamma,\gamma} * G \tag{7.54}$$

for every $j \in \mathbb{N}_0$. Furthermore, the limit relation

$$\lim_{J\to\infty} (\Phi_J^{(2)})^{\sigma,\sigma} * F = F \tag{7.55}$$

holds (in $\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}$-sense).

In the case of bandlimited scaling functions, the preceding theorem shows that the (scalar) SST or SGG problem is well posed: A unique solution always exists and due to the finite dimension of the scale spaces the solution is also stable. According to the multiscale approach the solution in the scale space is given by adding the solution of the corresponding detail spaces to the solution of the scale space of a lower scale. Because of the limit relation given in Theorem 7.14 the filtered solutions converge to the unfiltered solution in the Sobolev space $\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$. If we now turn to non-bandlimited scaling functions, the stability of the solution cannot be ensured, because the (scalar) SST or SGG problem is an exponentially ill-posed problem with unbounded inverse operator Λ^{-1}. In order to obtain a well-posed problem, we have to replace the inverse operator by an appropriate bounded operator, that is we have to use a regularization of Λ^{-1}.

Definition 7.15. A family of linear operators $S_j : \mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, $j \in \mathbb{N}_0$, is called a *regularization* of Λ^{-1}, if it satisfies the following properties:

(i) S_j is bounded on $\mathcal{H}_s(\overline{\Omega_\gamma^{\text{ext}}})$ for all $j \in \mathbb{N}_0$,

(ii) for any member $G \in im(\Lambda)$, the limit relation $\lim_{J\to\infty} S_J G = \Lambda^{-1} G$ holds (in $\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})}$-sense).

The function $F_J = S_J G$ is called the *J-level regularization of* $\Lambda^{-1}G$. In our approach we want to represent the J-level regularization with the help of harmonic wavelets which guarantees that we can calculate the $J+1$-level regularization by adding the corresponding detail information to the J-level regularization. In order to formulate the multiscale regularization concept, we start with the definition of a generator of a regularization scaling function by modifying Definition 7.2.

Definition 7.16. A family $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ of sequences $\{\varphi_j(n)\}_{n\in\mathbb{N}_0}$ is called a *generator of a regularization scaling function with respect to* Λ^{-1}, if it satisfies the following requirements:

(i) $(\varphi_j(0))^2 = \frac{1}{\Lambda^\wedge(0)}$, for all $j \in \mathbb{N}_0$,

(ii) $(\varphi_j(n))^2 \le (\varphi_{j'}(n))^2$, for all $j, j' \in \mathbb{N}_0$ with $j \le j'$ and all $n \in \mathbb{N}$,

(iii) $\lim_{j\to\infty} (\varphi_j(n))^2 = \frac{1}{\Lambda^\wedge(n)}$, for all $n \in \mathbb{N}$.

Now we are able to define the decomposition and reconstruction regularization scaling functions in such a way that the corresponding convolutions lead to the J-level approximation of $\Lambda^{-1}G$, $G \in \text{im}(\Lambda)$.

Definition 7.17. Let $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ be a generator of a regularization scaling function with respect to Λ^{-1}. Then a family $\{{}^d\Phi_j^{\sigma,\gamma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ of admissible $\mathcal{H}_{\sigma,\gamma}$-kernels given by

$$ {}^d\Phi_j^{\sigma,\gamma}(x,z) = \sum_{n=0}^{\infty} \varphi_j(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\sigma;x) H_{n,m}^s(\gamma;z), \tag{7.56} $$

$(x,z) \in \overline{\Omega_\sigma^{\text{ext}}} \times \overline{\Omega_\gamma^{\text{ext}}}$, is called a *decomposition regularization $\mathcal{H}_{\sigma,\gamma}$-scaling function with respect to* Λ^{-1}, whereas a family $\{{}^r\Phi_j^{\sigma,\sigma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ of admissible $\mathcal{H}_{\sigma,\sigma}$-kernels given by

$$ {}^r\Phi_j^{\sigma,\sigma}(x,y) = \sum_{n=0}^{\infty} \varphi_j(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\sigma;x) H_{n,m}^s(\sigma;y), \tag{7.57} $$

$(x,y) \in \overline{\Omega_\sigma^{\mathrm{ext}}} \times \overline{\Omega_\sigma^{\mathrm{ext}}}$ is called a *reconstruction regularization $\mathcal{H}_{\sigma,\sigma}$-scaling function with respect to* Λ^{-1}.

Obviously, the regularization scaling functions fulfill

$$ {}^d\Phi_j^{\sigma,\gamma}(x,\cdot) \in \mathcal{H}_s(\overline{\Omega_\gamma^{\mathrm{ext}}}), \quad x \in \overline{\Omega_\sigma^{\mathrm{ext}}},\ j \in \mathbb{N}_0, \tag{7.58} $$

$$ {}^r\Phi_j^{\sigma,\sigma}(x,\cdot) \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}), \quad x \in \overline{\Omega_\sigma^{\mathrm{ext}}},\ j \in \mathbb{N}_0. \tag{7.59} $$

As already stated, we obtain the following theorem:

Theorem 7.18. *Let $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ be a generator of a regularization scaling function with respect to Λ^{-1}. If we define the admissible $\mathcal{H}_{\sigma,\gamma}$-kernel $({}^r\Phi_j * {}^d\Phi_j)^{\sigma,\gamma}(\cdot,\cdot)$ by*

$$ ({}^r\Phi_j * {}^d\Phi_j)^{\sigma,\gamma}(x,z) = {}^r\Phi_j^{\sigma,\sigma}(x,\cdot) * {}^d\Phi_j^{\sigma,\gamma}(\cdot,z), \tag{7.60} $$

$(x,z) \in \overline{\Omega_\sigma^{\mathrm{ext}}} \times \overline{\Omega_\gamma^{\mathrm{ext}}}$, then

$$ F_J = ({}^r\Phi_J * {}^d\Phi_J)^{\sigma,\gamma} * G, \quad G \in \mathcal{H}_s(\overline{\Omega_\gamma^{\mathrm{ext}}}), \tag{7.61} $$

represents the J-level regularization of $\Lambda^{-1}G$.

If, in addition, $G \in \mathrm{im}(\Lambda) = \mathcal{H}_s^\Lambda(\overline{\Omega_\gamma^{\mathrm{ext}}})$, then

$$ \lim_{J\to\infty} \|F_J - \Lambda^{-1}G\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} = 0. \tag{7.62} $$

If we define the convolution operators $S_J : \mathcal{H}_s(\overline{\Omega_\gamma^{\mathrm{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $J \in \mathbb{N}_0$, by

$$ S_J(G) = ({}^r\Phi_J * {}^d\Phi_J)^{\sigma,\gamma} * G, \tag{7.63} $$

and introduce the scale spaces$S_J(\mathrm{im}(\Lambda))$ as follows

$$ S_J(\mathrm{im}(\Lambda)) = \{({}^r\Phi_J * {}^d\Phi_J)^{\sigma,\gamma} * G \ :\ G \in \mathrm{im}(\Lambda)\}, \tag{7.64} $$

the following theorem holds.

Theorem 7.19. *The scale spaces satisfy the following properties:*

(i) $S_0(\mathrm{im}(\Lambda)) \subset \cdots \subset S_J(\mathrm{im}(\Lambda)) \subset S_{J'}(\mathrm{im}(\Lambda)) \subset \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $J \leq J'$, *i.e., for any right-hand side $G \in \mathrm{im}(\Lambda)$ of the* (*scalar*) *SST or SGG problem, all J-level regularizations with fixed parameter J are sampled in a scale space $S_J(\mathrm{im}(\Lambda))$ with the above property,*

(ii) $\overline{\bigcup_{J=0}^\infty S_J(\mathrm{im}(\Lambda))}^{\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}} = \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

A set of subspaces of $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ satisfying the conditions of Theorem 7.19 is called *regularization $\mathcal{H}_{\sigma,\gamma}$-multiresolution analysis* (*RMRA*) *of the* (*scalar*) *SST or SGG problem.*

We now turn to the definition of regularization wavelets following the procedure described in the case of regularization scaling functions. Obviously, we have to define decomposition and reconstruction regularization wavelets.

Definition 7.20. Let $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ be a generator of a regularization scaling function with respect to Λ^{-1}. Then the generating symbols $\{\psi_j(n)\}_{n\in\mathbb{N}_0}$ and $\{\tilde{\psi}_j(n)\}_{n\in\mathbb{N}_0}$ of the corresponding regularization wavelets are defined by the refinement equation (7.10). The admissible $\mathcal{H}_{\sigma,\gamma}$-kernel $\{{}^d\Psi_j^{\sigma,\gamma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ given by

$$ {}^d\Psi_j^{\sigma,\gamma}(x,z) = \sum_{n=0}^{\infty} \psi_j(n) \sum_{m=1}^{2n+1} H^s_{n,m}(\sigma;x) H^s_{n,m}(\gamma;z), \tag{7.65} $$

$(x,z) \in \overline{\Omega^{\text{ext}}_\sigma} \times \overline{\Omega^{\text{ext}}_\gamma}$ is called the *decomposition regularization $\mathcal{H}_{\sigma,\gamma}$-wavelet*, while the admissible $\mathcal{H}_{\sigma,\sigma}$-kernel $\{{}^r\tilde{\Psi}_j^{\sigma,\sigma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ given by

$$ {}^r\tilde{\Psi}_j^{\sigma,\sigma}(x,y) = \sum_{n=0}^{\infty} \tilde{\psi}_j(n) \sum_{m=1}^{2n+1} H^s_{n,m}(\sigma;x) H^s_{n,m}(\sigma;y), \tag{7.66} $$

$(x,y) \in \overline{\Omega^{\text{ext}}_\sigma} \times \overline{\Omega^{\text{ext}}_\sigma}$ is called the *reconstruction regularization $\mathcal{H}_{\sigma,\sigma}$-wavelet.*

We now define the convolution operators $T_j : \mathcal{H}_s(\overline{\Omega^{\text{ext}}_\gamma}) \to \mathcal{H}_s(\overline{\Omega^{\text{ext}}_\sigma})$, $j \in \mathbb{N}_0$, by

$$ T_j(G) = ({}^r\tilde{\Psi}_j * {}^d\Psi_j)^{\sigma,\gamma} * G, \quad G \in \mathcal{H}_s(\overline{\Omega^{\text{ext}}_\gamma}). \tag{7.67} $$

Obviously, due to the refinement equation, the operator S_{J+1} can be represented in the form

$$ S_{J+1} = S_0 + \sum_{j=0}^{J} T_j. \tag{7.68} $$

Thus, we now introduce the *detail spaces* $T_J(\text{im}(\Lambda))$ by

$$ T_J(\text{im}(\Lambda)) = \left\{ ({}^r\tilde{\Psi}_J * {}^d\Psi_J)^{\sigma,\gamma} * G \ : \ G \in \text{im}(\Lambda) \right\}. \tag{7.69} $$

The space $T_J(\text{im}(\Lambda))$ contains the detail information which has to be added in order to turn from the J-level regularization to the $J+1$-level regularization:

$$ S_{J+1}(\text{im}(\Lambda)) = S_J(\text{im}(\Lambda)) + T_J(\text{im}(\Lambda)). \tag{7.70} $$

In general, the sum is neither direct nor orthogonal.

Theorem 7.21. *Let $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ be a generator of a regularization scaling function with respect to Λ^{-1}. Suppose that $\{\{\psi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$, $\{\{\tilde{\psi}_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ are the generating symbols of the corresponding regularization wavelets. Furthermore, let G be of class $\mathcal{H}_s(\overline{\Omega^{\text{ext}}_\gamma})$. Define the* regularization $\mathcal{H}_{\sigma,\gamma}$-wavelet transform at scale $j \in \mathbb{N}_0$ and position $x \in \overline{\Omega^{\text{ext}}_\sigma}$ *by*

$$ (\text{RWT})(G)(j;x) = {}^d\Psi_J^{\sigma,\gamma}(x,\cdot) * G, \quad G \in \mathcal{H}_s(\overline{\Omega^{\text{ext}}_\gamma}). \tag{7.71} $$

Then

$$ F_J = \left({}^r\Phi_0 * {}^d\Phi_0\right)^{\sigma,\gamma} * G + \sum_{j=0}^{J-1} {}^r\tilde{\Psi}_J^{\sigma,\sigma} * (RWT)(G)(j;\cdot) \tag{7.72} $$

is the J-level regularization of the (scalar) SST or SGG problem satisfying

$$\lim_{J\to\infty} \|F_J - \Lambda^{-1}G\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} = 0 \tag{7.73}$$

provided that $G \in \mathrm{im}(\Lambda) = \mathcal{H}_s^{\Lambda}(\overline{\Omega_\gamma^{\mathrm{ext}}})$.

Some examples of regularization wavelets and numerical calculations can be found in [19], where, in addition, all the above-mentioned theorems are proved.

7.2. Vectorial wavelet theory

We now give the extension of the scalar wavelet theory to the vectorial case. First we define vectorial scaling functions and wavelets. The reconstruction formula is the main result stating how the function can be split into a lowpass part and an infinite sum of bandpass parts. Then we solve the (vectorial) SST or SGG problem defining regularization wavelets. We use the notation $\hat{\Phi}_j^{(i)} \star \hat{\Phi}_j^{(i)} * f$ instead of $\hat{\Phi}_j^{(i)} \star (\hat{\Phi}_j^{(i)} * f)$, and $\hat{\Phi}_j \star \hat{\Phi}_j * f = \sum_{i=1}^{3} \hat{\Phi}_j^{(i)} \star \hat{\Phi}_j^{(i)} * f^{(i)}$.

Definition 7.22. Let $\{\hat{\Phi}_j^{(i)}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be a family of admissible $h_{\sigma,\sigma}^{(i)}$-kernels, $i \in \{1,2,3\}$. Then the family $\{\mathcal{V}_j^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})\}_{j\in\mathbb{N}_0}$ of scale spaces $\mathcal{V}_j^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ defined by

$$\mathcal{V}_j^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \{\hat{\Phi}_j^{(i)} \star \hat{\Phi}_j^{(i)} * f \ : \ f \in h_s(\overline{\Omega_\sigma^{\mathrm{ext}}})\}, \tag{7.74}$$

is called an $h_{\sigma,\sigma}^{(i)}$-*multiresolution analysis*, if the following properties are satisfied:

(i) $\mathcal{V}_0^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \cdots \subset \mathcal{V}_j^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \mathcal{V}_{j+1}^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \cdots \subset h_s^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$,

(ii) $\overline{\bigcup_{j\in\mathbb{N}_0} \mathcal{V}_j^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})}^{\|\cdot\|_{h_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}} = h_s^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

Definition 7.23. Let $\{\hat{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be a family of admissible $h_{\sigma,\sigma}$-kernels. The set of scale spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ defined by

$$\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \{\hat{\Phi}_j \star \hat{\Phi}_j * f \ : \ f \in h_s(\overline{\Omega_\sigma^{\mathrm{ext}}})\} \tag{7.75}$$

is called an $h_{\sigma,\sigma}$-multiresolution analysis, if $\{\mathcal{V}_j^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})\}_{j\in\mathbb{N}_0}$ is an $h_s^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-multiresolution analysis for every $i \in \{1,2,3\}$.

Our next purpose is to define scaling functions.

Definition 7.24. A family $\{\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$ of sequences $\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}$ is called a *generator of a scaling function of kind* i, $i \in \{1,2,3\}$, if it satisfies the following requirements:

(i) $(\varphi_j^{(i)}(0_i))^2 = 1$ for all $j \in \mathbb{N}_0$,

(ii) $(\varphi_j^{(i)}(n))^2 \le \left(\varphi_{j'}^{(i)}(n)\right)^2$ for all $j, j' \in \mathbb{N}_0$ with $j \le j'$ and all $n \in \mathbb{N}_{0_i+1}$,

(iii) $\lim_{j\to\infty} (\varphi_j^{(i)}(n))^2 = 1$ for all $n \in \mathbb{N}_{0_i+1}$.

Furthermore, the family $\{\{\{\varphi^{(i)}(n)\}_{i\in\{1,2,3\}}\}_{n\in\mathbb{N}_{0_i+1}}\}_{j\in\mathbb{N}_0}$ is called a *generator of a scaling function*, if $\{\{\varphi^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$ are generators of a scaling function of kind i, $i \in \{1,2,3\}$.

Based on the definition of a generator of a scaling function, we now introduce $h_{\sigma,\sigma}$-scaling functions.

Definition 7.25. A family $\{\hat{\Phi}_j^{(i)}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ of $h^{(i)}$-kernels $\hat{\Phi}_j^{(i)}(\cdot,\cdot)$ defined by

$$\hat{\Phi}_j^{(i)\wedge}(n) = \varphi_j^{(i)}(n), \quad j \in \mathbb{N}_0,\ n \in \mathbb{N}_{0_i},\ i \in \{1,2,3\},$$

i.e.,

$$\hat{\Phi}_j^{(i)}(x,y) = \sum_{n=0_i}^{\infty} \varphi_j^{(i)}(n) \sum_{m=1}^{2n+1} H_{n,m}^{s}(\sigma;x) h_{n,m}^{(i)s}(\sigma;y), \quad x,y \in \overline{\Omega_\sigma^{\mathrm{ext}}}, \tag{7.76}$$

is called an $h_{\sigma,\sigma}^{(i)}$*-scaling function*, if it satisfies the following properties:

(i) $\hat{\Phi}_j^{(i)}(\cdot,\cdot)$ is an admissible $h_{\sigma,\sigma}^{(i)}$-kernel for every $j \in \mathbb{N}_0$,

(ii) $\{\{\hat{\Phi}_j^{(i)\wedge}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$ constitutes a generator of a scaling function of kind i.

Furthermore, the family $\{\hat{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ of $h_{\sigma,\sigma}$-kernels $\hat{\Phi}_j(\cdot,\cdot)$ is called an $h_{\sigma,\sigma}$-scaling function, if $\{\hat{\Phi}_j^{(i)}\}_{j\in\mathbb{N}_0}$ are $h_{\sigma,\sigma}^{(i)}$-scaling functions for $i \in \{1,2,3\}$.

The following approximation property can be derived.

Theorem 7.26. *Let* $\{\hat{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ *be an* $h_{\sigma,\sigma}$*-scaling function. Then*

$$\lim_{j\to\infty} \|f - \hat{\Phi}_j \star \hat{\Phi}_j * f\|_{h_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} = 0 \tag{7.77}$$

holds for all $f \in h_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

Definition 7.27. Let $\{\hat{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be an $h_{\sigma,\sigma}$-scaling function. Then $\{P_j\}_{j\in\mathbb{N}_0}$ with $P_j : h_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to h_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ defined by

$$P_j(f)(x) = \hat{\Phi}_j \star \hat{\Phi}_j * f, \quad f \in h_s(\overline{\Omega_\sigma^{\mathrm{ext}}}),\ x \in \overline{\Omega_\sigma^{\mathrm{ext}}}, \tag{7.78}$$

is called an $h_{\sigma,\sigma}$*-approximate identity.*

The kernel $\hat{\Phi}_0$ is called the mother kernel of the $h_{\sigma,\sigma}$-scaling function.

Theorem 7.28. *Let* $\{\hat{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ *be an* $h_{\sigma,\sigma}$*-scaling function. Then* $\{\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})\}_{j\in\mathbb{N}_0}$ *defined in* (7.75) *forms an* $h_{\sigma,\sigma}$*-multiresolution analysis.*

We are now at the point to define the (primal/dual) wavelet with the help of the bilinear refinement equation.

Definition 7.29. Let $\{\hat{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be an $h_{\sigma,\sigma}$-scaling function. Then the families of $h_{\sigma,\sigma}$-kernels $\{\hat{\Psi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$, $\{\tilde{\hat{\Psi}}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ given by

$$(\hat{\Psi}_j)^{(i)\wedge}(n) = \psi_j^{(i)}(n), \quad j\in\mathbb{N}_0,\ n\in\mathbb{N}_{0_i},\ i\in\{1,2,3\}, \tag{7.79}$$

$$(\tilde{\hat{\Psi}}_j)^{(i)\wedge}(n) = \tilde{\psi}_j^{(i)}(n), \quad j\in\mathbb{N}_0,\ n\in\mathbb{N}_{0_i},\ i\in\{1,2,3\}, \tag{7.80}$$

are called (*primal*) $h_{\sigma,\sigma}$*-wavelet* and *dual* $h_{\sigma,\sigma}$*-wavelet*, respectively, if all $h_{\sigma,\sigma}$-kernels $\hat{\Psi}_j(\cdot,\cdot)$, $\tilde{\hat{\Psi}}_j(\cdot,\cdot)$, $j\in\mathbb{N}_0$, are admissible and the symbols $\{\psi_j^{(i)}(n)\}$, $\{\tilde{\psi}_j^{(i)}(n)\}$, in addition, satisfy the (vectorial) refinement equation

$$\tilde{\psi}_j^{(i)}(n)\psi_j^{(i)}(n) = (\varphi_{j+1}^{(i)}(n))^2 - (\varphi_j^{(i)}(n))^2 \tag{7.81}$$

for all $j\in\mathbb{N}_0$, $n\in\mathbb{N}_{0_i}$, $i\in\{1,2,3\}$.

The following equation can directly be seen:

$$(\varphi_{J+1}^{(i)}(n))^2 = (\varphi_0^{(i)}(n))^2 + \sum_{j=0}^{J}\tilde{\psi}_j^{(i)}(n)\psi_j^{(i)}(n), \quad J\in\mathbb{N}_0, \tag{7.82}$$

for all $n\in\mathbb{N}_{0_i}$. We now define the wavelet transform. To this end we let $\psi_{-1}^{(i)}(n) = \tilde{\psi}_{-1}^{(i)}(n) = \varphi_0^{(i)}(n)$ and $\hat{\Psi}_{-1}(\cdot,\cdot) = \tilde{\hat{\Psi}}_{-1}(\cdot,\cdot) = \hat{\Phi}_0(\cdot,\cdot)$ for $n\in\mathbb{N}_{0_i}$, $i\in\{1,2,3\}$. We remember that we have already defined the space $\mathcal{H}_s(\mathbb{N}_{-1}\times\overline{\Omega_\sigma^{\mathrm{ext}}})$ (see Eqs. (7.12)–(7.14))

Definition 7.30. Let $\{\hat{\Psi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_{-1}}$ be a (primal) $h_{\sigma,\sigma}$-wavelet. Then $(WT)^{(i)}$: $h_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to \mathcal{H}_s(\mathbb{N}_{-1}\times\overline{\Omega_\sigma^{\mathrm{ext}}})$ defined by

$$(WT)^{(i)}(f)(j;y) = (\hat{\Psi}_j^{(i)} * f)(y) \tag{7.83}$$

is called $h_{\sigma,\sigma}$-wavelet transform of kind i of f at position $y\in\overline{\Omega_\sigma^{\mathrm{ext}}}$ and scale $j\in\mathbb{N}_{-1}$.

As usual, we define the detail space $\mathcal{W}_j^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ at scale j by

$$\mathcal{W}_j^{(i)}(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \left\{\tilde{\hat{\Psi}}_j^{(i)} \star \hat{\Psi}_j^{(i)} * f \ :\ f\in h_s(\overline{\Omega_\sigma^{\mathrm{ext}}})\right\}, \tag{7.84}$$

and

$$\mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \left\{\tilde{\hat{\Psi}}_j \star \hat{\Psi}_j * f \ :\ f\in h_s(\overline{\Omega_\sigma^{\mathrm{ext}}})\right\}. \tag{7.85}$$

Theorem 7.31 (Vectorial Reconstruction Formula for the Outer Space). *Let the families* $\{\hat{\Psi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ *and* $\{\tilde{\hat{\Psi}}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$*, respectively, be a (primal)* $h_{\sigma,\sigma}$*-wavelet and its dual corresponding to an* $h_{\sigma,\sigma}$*-scaling function* $\{\hat{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$*. Then*

$$f = \sum_{j=-1}^{\infty}\tilde{\hat{\Psi}}_j \star \hat{\Psi}_j * f \tag{7.86}$$

holds for all $f\in h_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ *(in* $\|\cdot\|_{h_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}$*-sense).*

Our next purpose is to solve the (vectorial) SST or SGG problem with the help of bandlimited harmonic wavelets. First, we transfer the theory of $h^{(i)}_{\sigma,\sigma}$-scaling functions to the case of $h^{(i)}_{\alpha,\alpha}$-scaling functions $\hat{\Phi}_j^{(i),\alpha,\alpha}$ with $\alpha \geq \sigma$:

$$\hat{\Phi}_j^{(i),\alpha,\alpha}(x,y) = \sum_{n=0_i}^{\infty} \varphi_j^{(i)}(n) \sum_{m=1}^{2n+1} H^s_{n,m}(\alpha;x) h^{(i)s}_{n,m}(\alpha;y), \tag{7.87}$$

where

$$(\hat{\Phi}_j^{(i),\alpha,\alpha})^\wedge(n) = \varphi_j^{(i)}(n). \tag{7.88}$$

Obviously, Theorem 7.26 can be directly transferred substituting σ by α. The scale spaces are defined in the following way:

$$\mathcal{V}_j^{(i)}(\overline{\Omega_\alpha^{\text{ext}}}) = \{\hat{\Phi}_j^{(i),\alpha,\alpha} \star \hat{\Phi}_j^{(i),\alpha,\alpha} * f \ : \ f \in h_s(\overline{\Omega_\alpha^{\text{ext}}})\}. \tag{7.89}$$

The system $\{\mathcal{V}_j^{(i)}(\overline{\Omega_\alpha^{\text{ext}}})\}$ of scale spaces forms a multiresolution analysis.

Theorem 7.32. *The restriction of the operator $\lambda^{(i)} : \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to h_s^{(i)}(\overline{\Omega_\gamma^{\text{ext}}})$ to a scale space $\mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})$, $j \in \mathbb{N}_0$, i.e.,*

$$\lambda^{(i)}|_{\mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})} : \mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{V}_j^{(i)}(\overline{\Omega_\gamma^{\text{ext}}}), \tag{7.90}$$

is injective for $i = 1$, whereas in the case of $i \in \{2,3\}$ the Fourier coefficient of degree 0 cannot be recovered and the Fourier coefficients of degree $n \geq 1$ are uniquely defined. Moreover, we have the following results:

(i) *If the families $\{\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$, $i \in \{1,2,3\}$, and $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ are bandlimited (for example, $\varphi_j^{(i)}(n) = \varphi_j(n) = 0$ for all $n \geq 2^j$), then the restricted operator is even bijective (in the sense described above). To be more specific, for $g^{(i)} \in h_s^{(i)}(\overline{\Omega_\gamma^{\text{ext}}})$ the (in the case of $i = 2,3$ up to Fourier coefficients of degree 0) unique solution $F_j \in \mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})$, $j \in \mathbb{N}_0$, of the equation*

$$\lambda^{(i)} F_j = \hat{\Phi}_j^{(i),\gamma,\gamma} \star \hat{\Phi}_j^{(i),\gamma,\gamma} * g^{(i)} \tag{7.91}$$

is given by

$$F_j = \Phi_j^{\sigma,\sigma} * \Phi_j^{\sigma,\sigma} * Q, \tag{7.92}$$

where $Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ is obtainable by

$$Q^\wedge(n,m) = \begin{cases} \frac{g^{(i)\wedge}(n,m)}{\lambda^{(i)\wedge}(n)}, & n \in [0_i, 2^j), \\ 0, & n \in [2^j, \infty). \end{cases} \tag{7.93}$$

(ii) *If the families $\{\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$, $i \in \{1,2,3\}$, and $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ are non-bandlimited, the equation*

$$\lambda^{(i)} F_j = \hat{\Phi}_j^{(i),\gamma,\gamma} \star \hat{\Phi}_j^{(i),\gamma,\gamma} * g^{(i)} \tag{7.94}$$

has a solution $F_j \in \mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})$ provided that $g^{(i)} \in h_s^{(i)\Lambda}(\overline{\Omega_\gamma^{\text{ext}}})$, where $h_s^{(i)\Lambda}(\overline{\Omega_\gamma^{\text{ext}}})$ is a suitable Sobolev space (see the Ph.D.-thesis [58] for more details). In this *case, the (in the case of $i = 2, 3$ up to Fourier coefficients of degree 0) unique solution is given by*

$$F_j = \Phi_j^{\sigma,\sigma} * \Phi_j^{\sigma,\sigma} * Q, \tag{7.95}$$

where $Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ is obtainable in spectral language by

$$Q^\wedge(n,m) = \frac{g^{(i)\wedge}(n,m)}{\lambda^{(i)\wedge}(n)}, \tag{7.96}$$

$n = 0_i, \ldots; m = 1, \ldots, 2n+1.$

The following corollary shows that in the case of general operators $\lambda = \sum_{i=1}^3 \lambda^{(i)}$ we have to claim an additional assumption onto the function g.

Corollary 7.33. *The restriction of the operator $\lambda = \sum_{i=1}^3 \lambda^{(i)}$ to a scale space $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, i.e.,*

$$\lambda|_{\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})} : \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to \bigoplus_{i=1}^{3} \mathcal{V}_j^{(i)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \tag{7.97}$$

has, in general, no solution. Under the assumption $\varphi_j^{(i)}(n) = \varphi_j(n)$, $i \in \{1,2,3\}$, we have to claim, in addition, that

$$\frac{g^{(i)\wedge}(n,m)}{\lambda^{(i)\wedge}(n)} = \frac{g^{(l)\wedge}(n,m)}{\lambda^{(l)\wedge}(n)}, \tag{7.98}$$

with $i, l \in \{1,2,3\}$; $n = \max\limits_{i,l \in \{1,2,3\}}(0_i, 0_l), \ldots$; $m = 1, \ldots, 2n+1$.

Then the results in Theorem 7.32 can directly be transferred.

Note that according to Theorem 7.32 the restriction of a pseudodifferential operator of kind i to a scale space $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ is injective. Therefore, in the case of a pseudodifferential operator $\lambda = \sum_{i=1}^3 \lambda^{(i)}$ each pseudodifferential operator $\lambda^{(i)}$ leads to a unique solution. The additional assumption (7.98) is thus necessary, in order to guarantee that the pseudodifferential operators of kind i do not lead to different solutions.

With the help of the refinement equation (7.81) we now define the primal wavelets $\{\hat{\Psi}_j^{(i),\alpha,\alpha}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ and the dual wavelets $\{\tilde{\hat{\Psi}}_j^{(i),\alpha,\alpha}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ for $\alpha \geq \sigma$, $i \in \{1,2,3\}$:

$$\hat{\Psi}_j^{(i),\alpha,\alpha}(x,y) = \sum_{n=0_i}^{\infty} \psi_j^{(i)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\alpha;x) h_{n,m}^{(i)s}(\alpha;y), \tag{7.99}$$

$$\tilde{\hat{\Psi}}_j^{(i),\alpha,\alpha}(x,y) = \sum_{n=0_i}^{\infty} \tilde{\psi}_j^{(i)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\alpha;x) h_{n,m}^{(i)s}(\alpha;y), \tag{7.100}$$

where

$$(\hat{\Psi}_j^{(i),\alpha,\alpha})^\wedge(n) = \psi_j^{(i)}(n), \quad (\tilde{\hat{\Psi}}_j^{(i),\alpha,\alpha})^\wedge(n) = \tilde{\psi}_j^{(i)}(n). \tag{7.101}$$

The detail spaces are defined in canonical manner:

$$\mathcal{W}_j^{(i)}(\overline{\Omega_\alpha^{\mathrm{ext}}}) = \{\hat{\Psi}_j^{(i),\alpha,\alpha} \star \tilde{\hat{\Psi}}_j^{(i),\alpha,\alpha} * f \ : \ f \in h_s(\overline{\Omega_\alpha^{\mathrm{ext}}})\}. \tag{7.102}$$

Theorem 7.31 can be directly transferred by substituting the convolutions with respect to the sphere Ω_σ by the corresponding convolutions with respect to the sphere Ω_α. We now transfer Theorem 7.32 to the case of the detail spaces and get the following theorem, where we use the terms injectivity, bijectivity, and uniqueness in the same sense as before (i.e., up to Fourier coefficients of degree 0 in the case of $i = 2, 3$).

Theorem 7.34. *The restriction of the operator $\lambda^{(i)} : \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to h_s^{(i)}(\overline{\Omega_\gamma^{\mathrm{ext}}})$ to a detail space $\mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, i.e.,*

$$\lambda^{(i)}|_{\mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})} : \mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to \mathcal{W}_j^{(i)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \tag{7.103}$$

with $\psi_j(n) = \psi_j^{(i)}(n)$ is injective. Moreover, we have the following results:

(i) *If the families $\{\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$, $i \in \{1,2,3\}$, and $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ are bandlimited (for example, $\varphi_j^{(i)}(n) = \varphi_j(n) = 0$ for all $n \geq 2^j$), then the restricted operator is even bijective. To be more specific, for $g^{(i)} \in h_s^{(i)}(\overline{\Omega_\gamma^{\mathrm{ext}}})$ the unique solution $H_j \in \mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, of the equation*

$$\lambda^{(i)} H_j = \tilde{\hat{\Psi}}_j^{(i),\gamma,\gamma} \star \hat{\Psi}_j^{(i),\gamma,\gamma} * g^{(i)} \tag{7.104}$$

is given by

$$H_j = \tilde{\Psi}_j^{\sigma,\sigma} * \Psi_j^{\sigma,\sigma} * Q, \tag{7.105}$$

where $Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ is obtainable by

$$Q^\wedge(n,m) = \begin{cases} \frac{g^{(i)\wedge}(n,m)}{\lambda^{(i)\wedge}(n)}, & n \in [0_i, 2^{j+1}), \\ 0, & n \in [2^{j+1}, \infty). \end{cases} \tag{7.106}$$

(ii) *If the families $\{\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$, $i \in \{1,2,3\}$, and $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ are non-bandlimited, the equation*

$$\lambda^{(i)} H_j = \tilde{\hat{\Psi}}_j^{(i),\gamma,\gamma} \star \hat{\Psi}_j^{(i),\gamma,\gamma} * g^{(i)} \tag{7.107}$$

has a solution $H_j \in \mathcal{W}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ provided that the condition

$$\sum_{n=0_i}^{\infty} \sum_{m=1}^{2n+1} \frac{g^{(i)\wedge}(n,m)}{\lambda^{(i)\wedge}(n)} < \infty \tag{7.108}$$

is satisfied for $g^{(i)} \in h_s^{(i)}(\overline{\Omega_\gamma^{\mathrm{ext}}})$. In this case, the unique solution of the equation is given by

$$H_j = \tilde{\Psi}_j^{\sigma,\sigma} * \Psi_j^{\sigma,\sigma} * Q, \tag{7.109}$$

where $Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ *is obtainable by*

$$Q^\wedge(n,m) = \frac{g^{(i)\wedge}(n,m)}{\lambda^{(i)\wedge}(n)}, \tag{7.110}$$

$n = 0_i, \ldots; m = 1, \ldots, 2n+1.$

Corollary 7.35. *The restriction of the operator* $\lambda = \sum_{i=1}^{3} \lambda^{(i)}$ *to a detail space* $\mathcal{W}_j(\overline{\Omega_\sigma^{\text{ext}}})$, $j \in \mathbb{N}_0$, *i.e.,*

$$\lambda|_{\mathcal{W}_j(\overline{\Omega_\sigma^{\text{ext}}})} : \mathcal{W}_j(\overline{\Omega_\sigma^{\text{ext}}}) \to \bigoplus_{i=1}^{3} \mathcal{W}_j^{(i)}(\overline{\Omega_\gamma^{\text{ext}}}) \tag{7.111}$$

has, in general, no solution. Under the assumption $\psi_j^{(i)}(n) = \psi_j(n)$ *and* $\tilde{\psi}_j^{(i)}(n) = \tilde{\psi}_j(n)$, $i \in \{1,2,3\}$, *we have to claim, in addition, that*

$$\frac{g^{(i)\wedge}(n,m)}{\lambda^{(i)\wedge}(n)} = \frac{g^{(l)\wedge}(n,m)}{\lambda^{(l)\wedge}(n)}, \tag{7.112}$$

with $i,l \in \{1,2,3\}$; $n = \max_{i,l \in \{1,2,3\}} (0_i, 0_l), \ldots; m = 1, \ldots, 2n+1.$

Then the results in Theorem 7.34 *can be directly transferred.*

Up to now, we have summarized some results about the filtered solution, i.e., the solution when we restrict the operator to scale or detail spaces. In this case, we have injectivity (in the case of $i = 2,3$ up to Fourier coefficients of degree 0) for the operators $\lambda^{(i)}$, whereas in the case of general operators $\lambda = \sum_{i=1}^{3} \lambda^{(i)}$ we have to claim that (7.98) is valid. In the case of the unfiltered solution, we obtain the following theorem.

Theorem 7.36. *Let* $g^{(i)} \in h_s^{(i)}(\overline{\Omega_\gamma^{\text{ext}}})$ *satisfy the condition* $g^{(i)} \in \operatorname{im}(\lambda^{(i)})$, $i \in \{1,2,3\}$. *Then the unique solution* $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ (*in the case of* $i = 2,3$ *up to Fourier coefficients of degree* 0) *of the equation* $\lambda^{(i)} F = g^{(i)}$ *is given by*

$$F^\wedge(n,m) = \frac{g^{(i)\wedge}(n,m)}{\lambda^{(i)\wedge}(n)}, \tag{7.113}$$

$n = 0_i, \ldots; m = 1, \ldots, 2n+1$. *In the case of the operator* $\lambda = \sum_{i=1}^{3} \lambda^{(i)}$ *we have to claim, in addition, that* (7.112) *holds in order to guarantee the solvability.*

Last, we explain the connection between the solution in the scale spaces and the unfiltered solution.

Theorem 7.37. *Suppose that* $g^{(i)}$ *is of the class* $h_s^{(i)\Lambda}(\overline{\Omega_\gamma^{\text{ext}}})$. *Let* $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ *be the unique* (*in the case of* $i = 2,3$ *up to Fourier coefficients of degree* 0) *solution of* $\lambda^{(i)} F = g^{(i)}$. *Then*

$$F_j = (\Phi_j^{(2)})^{\sigma,\sigma} * F \tag{7.114}$$

is the unique solution in $\mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})$ *of the equation*

$$\lambda^{(i)} F_j = \hat{\Phi}_j^{(i),\gamma,\gamma} \star \hat{\Phi}_j^{(i),\gamma,\gamma} * g^{(i)} \tag{7.115}$$

for every $j \in \mathbb{N}_0$. *Furthermore, the limit relation*

$$\lim_{J\to\infty} (\Phi_J^{(2)})^{\sigma,\sigma} * F = F \tag{7.116}$$

holds (in $\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})}$*-sense).*

The preceding theorem shows that in the case of bandlimited scaling functions the (vectorial) SST or SGG problem is well posed, because a unique solution always exists and due to the finite dimension of the scale spaces the solution is also stable. We now investigate the case of non-bandlimited scaling functions and it turns out that the stability cannot be ensured. The reason is that the (vectorial) SST or SGG problem is an exponentially ill-posed problem with unbounded inverse operator λ^{-1}. Therefore, we have to turn to regularization methods and replace the inverse operator by an appropriate bounded operator.

Definition 7.38. A family of linear operators $S_j^{(i)} : h_s^{(i)}(\overline{\Omega_\gamma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, $j \in \mathbb{N}_0$, is called a *regularization* of $(\lambda^{(i)})^{-1}$, $i \in \{1,2,3\}$, if it satisfies the following properties:

(i) $S_j^{(i)}$ is bounded on $h_s^{(i)}(\overline{\Omega_\gamma^{\text{ext}}})$ for all $j \in \mathbb{N}_0$,

(ii) for any member $g^{(i)} \in im(\lambda^{(i)})$, the limit relation

$$\lim_{J\to\infty} S_J^{(i)} g^{(i)} = (\lambda^{(i)})^{-1} g^{(i)} \tag{7.117}$$

holds (in $\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})}$-sense).

The operator $S : h_s(\overline{\Omega_\gamma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ given by $S|_{h_s^{(i)}(\overline{\Omega_\gamma^{\text{ext}}})} = S_j^{(i)}$ is called a regularization of λ^{-1}.

The function $F_J = S_J g$ is called the *J-level regularization of* $\lambda^{-1} g$, whereas $F_J^{(i)} = S_J^{(i)} g^{(i)}$ is called the *J-level regularization of* $(\lambda^{(i)})^{-1} g$. Within our multiscale approach, we now represent the $(J+1)$-level regularization using the J-level regularization by adding the corresponding detail information. To this end, we first introduce a multiscale regularization concept starting with the definition of a generator of a regularization scaling function.

Definition 7.39. A family $\{\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$ of sequences $\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}$, $i \in \{1,2,3\}$, is called a *generator of a regularization scaling function with respect to* $(\lambda^{(i)})^{-1}$, if it satisfies the following requirements:

(i) $(\varphi_j^{(i)}(0_i))^2 = \frac{1}{\lambda^{(i)\wedge}(0_i)}$ for all $j \in \mathbb{N}_0$,

(ii) $(\varphi_j^{(i)}(n))^2 \le (\varphi_{j'}^{(i)}(n))^2$ for all $j, j' \in \mathbb{N}_0$ with $j \le j'$ and all $n \in \mathbb{N}_{0_i+1}$,

(iii) $\lim_{j\to\infty} (\varphi_j^{(i)}(n))^2 = \frac{1}{\lambda^{(i)\wedge}(n)}$ for all $n \in \mathbb{N}_{0_i+1}$.

Furthermore, $\{\{\{\varphi_j^{(i)}(n)\}_{i\in\{1,2,3\}}\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$ is called a *generator of a regularization scaling function with respect to* λ^{-1}, if $(\lambda^{(i)})^{-1}$ is a generator of a regularization scaling function with respect to $(\lambda^{(i)})^{-1}$ for every $i = 1, 2, 3$.

We now define decomposition and reconstruction regularization scaling functions.

Definition 7.40. Let $\{\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$ be a generator of a regularization scaling function with respect to $(\lambda^{(i)})^{-1}$. Then a family $\{{}^d\hat{\Phi}_j^{(i),\sigma,\gamma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ of admissible $h_{\sigma,\gamma}^{(i)}$-kernels given by

$$ {}^d\hat{\Phi}_j^{(i),\sigma,\gamma}(x,z) = \sum_{n=0_i}^{\infty} \varphi_j^{(i)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\sigma;x) h_{n,m}^{(i)s}(\gamma;z), \tag{7.118} $$

$(x,z) \in \overline{\Omega_\sigma^{\text{ext}}} \times \overline{\Omega_\gamma^{\text{ext}}}$, is called a *decomposition regularization $h_{\sigma,\gamma}^{(i)}$-scaling function with respect to* $(\lambda^{(i)})^{-1}$, whereas a family $\{{}^r\hat{\Phi}_j^{(i),\sigma,\sigma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ of admissible $h_{\sigma,\sigma}^{(i)}$-kernels given by

$$ {}^r\hat{\Phi}_j^{(i),\sigma,\sigma}(x,y) = \sum_{n=0_i}^{\infty} \varphi_j^{(i)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\sigma;x) h_{n,m}^{(i)s}(\sigma;y), \tag{7.119} $$

$(x,y) \in \overline{\Omega_\sigma^{\text{ext}}} \times \overline{\Omega_\sigma^{\text{ext}}}$, is called a *reconstruction regularization $h^{(i),\sigma,\sigma}$-scaling function with respect to* $(\lambda^{(i)})^{-1}$.

We obtain the following theorem:

Theorem 7.41. *Let $\{\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$ be a generator of a regularization scaling function with respect to $(\lambda^{(i)})^{-1}$, $i \in \{1,2,3\}$. If we formally define*

$$ ({}^r\hat{\Phi}_j^{(i)} \star {}^d\hat{\Phi}_j^{(i)})^{\sigma,\gamma}(\cdot,\cdot) $$

by

$$ ({}^r\hat{\Phi}_j^{(i)} \star {}^d\hat{\Phi}_j^{(i)})^{\sigma,\gamma}(x,z) = {}^r\hat{\Phi}_j^{(i),\sigma,\sigma}(x,\cdot) \star {}^d\hat{\Phi}_j^{(i),\sigma,\gamma}(\cdot,z), \tag{7.120} $$

$(x,z) \in \overline{\Omega_\sigma^{\text{ext}}} \times \overline{\Omega_\gamma^{\text{ext}}}$, *then*

$$ F_J^{(i)} = ({}^r\hat{\Phi}_J^{(i)} \star {}^d\hat{\Phi}_J^{(i)})^{\sigma,\gamma} * g^{(i)}, \quad g^{(i)} \in h_s^{(i)}(\overline{\Omega_\gamma^{\text{ext}}}), \tag{7.121} $$

represents the J-level regularization of $(\lambda^{(i)})^{-1}g^{(i)}$. If, in addition, $g^{(i)} \in \operatorname{im}(\lambda^{(i)})$, then

$$ \lim_{J\to\infty} \|F_J^{(i)} - (\lambda^{(i)})^{-1}g^{(i)}\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})} = 0. \tag{7.122} $$

Furthermore,

$$ F_J = \sum_{i=1}^{3} ({}^r\hat{\Phi}_J^{(i)} \star {}^d\hat{\Phi}_J^{(i)})^{\sigma,\gamma} * g^{(i)}, \quad g = \sum_{i=1}^{3} g^{(i)} \in h_s(\overline{\Omega_\gamma^{\text{ext}}}), \tag{7.123} $$

represents the J-level regularization of $\lambda^{-1}g$. If, in addition, $g \in \operatorname{im}(\lambda)$, then

$$ \lim_{J\to\infty} \|F_J - \lambda^{-1}g\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})} = 0. \tag{7.124} $$

We now define the convolution operators $S_J^{(i)} : h_s^{(i)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $J \in \mathbb{N}_0$, by

$$S_J^{(i)}(g^{(i)}) = ({}^r\hat{\Phi}_J^{(i)} \star {}^d\hat{\Phi}_J^{(i)})^{\sigma,\gamma} * g^{(i)}, \tag{7.125}$$

whereas the convolution operator $S_J : h_s(\overline{\Omega_\gamma^{\mathrm{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $J \in \mathbb{N}_0$, is given by

$$S_J(g) = \sum_{i=1}^{3} S_J^{(i)}(g^{(i)}). \tag{7.126}$$

Furthermore, we introduce the corresponding scale spaces $S_J^{(i)}(\mathrm{im}(\lambda^{(i)}))$, $i \in \{1,2,3\}$, and $S_J(\mathrm{im}(\lambda))$ as follows

$$S_J^{(i)}(\mathrm{im}(\lambda^{(i)})) = \left\{ ({}^r\hat{\Phi}_J^{(i)} \star {}^d\hat{\Phi}_J^{(i)})^{\sigma,\gamma} * g^{(i)} \ : \ g^{(i)} \in \mathrm{im}(\lambda^{(i)}) \right\}, \tag{7.127}$$

$$S_J(\mathrm{im}(\lambda)) = \left\{ \sum_{i=1}^{3} ({}^r\hat{\Phi}_J^{(i)} \star {}^d\hat{\Phi}_J^{(i)})^{\sigma,\gamma} * g^{(i)} \ : \ g = \sum_{i=1}^{3} g^{(i)} \in \mathrm{im}(\lambda) \right\}. \tag{7.128}$$

Theorem 7.42. *The scale spaces satisfy the following properties:*

(i) $S_0^{(i)}(\mathrm{im}(\lambda^{(i)})) \subset \cdots \subset S_J^{(i)}(\mathrm{im}(\lambda^{(i)})) \subset S_{J'}^{(i)}(\mathrm{im}(\lambda^{(i)})) \subset \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $J \le J'$, *i.e., for any right-hand side* $g^{(i)} \in \mathrm{im}(\lambda^{(i)})$ *of the (vectorial) SST or SGG problem, all* J*-level regularizations with fixed parameter* J *are sampled in a scale space* $S_J^{(i)}(\mathrm{im}(\lambda^{(i)}))$ *with the above property,*

(ii) $\overline{\bigcup_{J=0}^{\infty} S_J^{(i)}(\mathrm{im}(\lambda^{(i)}))}^{\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}} = \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

Obviously, Theorem 7.42 is also valid substituting $S_J^{(i)}$ by S_J which leads to the following corollary.

Corollary 7.43. *The scale spaces satisfy the following properties:*

(i) $S_0(\mathrm{im}(\lambda)) \subset \cdots \subset S_J(\mathrm{im}(\lambda)) \subset S_{J'}(\mathrm{im}(\lambda)) \subset \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $J \le J'$, *i.e., for any right-hand side* $g \in \mathrm{im}(\lambda)$ *of the (vectorial) SST or SGG problem, all* J*-level regularizations with fixed parameter* J *are sampled in a scale space* $S_J(\mathrm{im}(\lambda))$ *with the above property,*

(ii) $\overline{\bigcup_{J=0}^{\infty} S_J(\mathrm{im}(\lambda))}^{\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}} = \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

A set of subspaces of $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ satisfying the conditions of Corollary 7.43 is called *regularization* $h_{\sigma,\gamma}$*-multiresolution analysis (RMRA) of the (vectorial) SST or SGG problem.*

Definition 7.44. Let $\{\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$ be a generator of a regularization scaling function with respect to $(\lambda^{(i)})^{-1}$. Then the generating symbols $\{\psi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}$, $\{\tilde{\psi}_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}$ of the corresponding regularization wavelets are defined by the refinement equation (7.81). The admissible $h_{\sigma,\gamma}^{(i)}$-kernels $\{{}^d\hat{\Psi}_j^{(i),\sigma,\gamma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ given by

$${}^d\hat{\Psi}_j^{(i),\sigma,\gamma}(x,z) = \sum_{n=0_i}^{\infty} \psi_j^{(i)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\sigma;x) h_{n,m}^{(i)s}(\gamma;z), \tag{7.129}$$

$(x,z) \in \overline{\Omega_\sigma^{\mathrm{ext}}} \times \overline{\Omega_\gamma^{\mathrm{ext}}}$, are called the *decomposition regularization $h^{(i)}_{\sigma,\gamma}$-wavelets*, while the admissible $h^{(i)}_{\sigma,\sigma}$-kernels $\{{}^r\tilde{\hat{\Psi}}_j^{(i),\sigma,\sigma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ given by

$$ {}^r\tilde{\hat{\Psi}}_j^{(i),\sigma,\sigma}(x,y) = \sum_{n=0_i}^{\infty} \tilde{\psi}_j^{(i)}(n) \sum_{m=1}^{2n+1} H^s_{n,m}(\sigma;x) h^{(i)s}_{n,m}(\sigma;y), \tag{7.130}$$

$(x,y) \in \overline{\Omega_\sigma^{\mathrm{ext}}} \times \overline{\Omega_\sigma^{\mathrm{ext}}}$ are called the *reconstruction regularization $h^{(i)}_{\sigma,\sigma}$-wavelets.*

We now define the convolution operators $T_j^{(i)} : h_s^{(i)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, $i = 1,2,3$, by

$$ T_j^{(i)}(g^{(i)}) = ({}^r\tilde{\hat{\Psi}}_j^{(i)} \star {}^d\hat{\Psi}_j^{(i)})^{\sigma,\gamma} * g^{(i)}, \quad g^{(i)} \in h_s^{(i)}(\overline{\Omega_\gamma^{\mathrm{ext}}}), \tag{7.131}$$

and the convolution operator $T_j : h_s(\overline{\Omega_\gamma^{\mathrm{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, by

$$ T_J(g) = \sum_{i=1}^{3} T_J^{(i)}(g^{(i)}). \tag{7.132}$$

Obviously, due to the refinement equation, the operators $S^{(i)}_{J+1}$ and S_{J+1} can be represented in the form

$$ S^{(i)}_{J+1} = S_0^{(i)} + \sum_{j=0}^{J} T_j^{(i)}, \tag{7.133}$$

$$ S_{J+1} = S_0 + \sum_{j=0}^{J} T_j. \tag{7.134}$$

Thus, we now introduce the *detail spaces* $T_J^{(i)}(\mathrm{im}(\lambda^{(i)}))$ and $T_J(\mathrm{im}(\lambda))$ by

$$ T_J^{(i)}(\mathrm{im}(\lambda^{(i)})) = \left\{ ({}^r\tilde{\hat{\Psi}}_J^{(i)} \star {}^d\hat{\Psi}_J^{(i)})^{\sigma,\gamma} * g^{(i)} \ : \ g^{(i)} \in \mathrm{im}(\lambda^{(i)}) \right\}, \tag{7.135}$$

$$ T_J(\mathrm{im}(\lambda)) = \left\{ \sum_{i=1}^{3} ({}^r\tilde{\hat{\Psi}}_J^{(i)} \star {}^d\hat{\Psi}_J^{(i)})^{\sigma,\gamma} * g^{(i)} \ : \ g = \sum_{i=1}^{3} g^{(i)} \in \mathrm{im}(\lambda) \right\}. \tag{7.136}$$

In terms of the multiscale concept, the space $T_J(\mathrm{im}(\lambda))$ contains the detail information which has to be added in order to turn from the J-level regularization to the $(J+1)$-level regularization:

$$ S_{J+1}(\mathrm{im}(\lambda)) = S_J(\mathrm{im}(\lambda)) + T_J(\mathrm{im}(\lambda)). \tag{7.137}$$

In general, the sum is neither direct nor orthogonal.

Theorem 7.45. *Let $\{\{\varphi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$ be a generator of a regularization scaling function with respect to $(\lambda^{(i)})^{-1}$, $i \in \{1,2,3\}$. Suppose that $\{\{\psi_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$,*

$\{\{\tilde{\psi}_j^{(i)}(n)\}_{n\in\mathbb{N}_{0_i}}\}_{j\in\mathbb{N}_0}$ *are the generating symbols of the corresponding regularization wavelets. Furthermore, let* $g^{(i)}$ *be of class* $h_s^{(i)}(\overline{\Omega_\gamma^{\text{ext}}})$*. Define the* regularization $h_{\sigma,\gamma}^{(i)}$-wavelet transform at scale $j\in\mathbb{N}_0$ and position $x\in\overline{\Omega_\sigma^{\text{ext}}}$ *by*

$$(RWT)(g^{(i)})(j;x) = {}^d\hat{\Psi}_j^{(i),\sigma,\gamma}(x,\cdot) * g^{(i)}, \quad g^{(i)}\in h_s^{(i)}(\overline{\Omega_\gamma^{\text{ext}}}). \tag{7.138}$$

Then

$$F_J = ({}^r\hat{\Phi}_0^{(i)} \star {}^d\hat{\Phi}_0^{(i)})^{\sigma,\gamma} * h^{(i)} + \sum_{j=0}^{J-1} {}^r\tilde{\hat{\Psi}}_j^{(i),\sigma,\sigma} \star (RWT)(g^{(i)})(j;\cdot)$$

is the J*-level regularization of the (vectorial) SST or SGG problem satisfying*

$$\lim_{J\to\infty} \|F_J - (\lambda^{(i)})^{-1} g^{(i)}\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})} = 0 \tag{7.139}$$

provided that $g^{(i)}\in\operatorname{im}(\lambda^{(i)})$.

7.3. Tensorial wavelet theory

The extension from vector to tensor theory is performed in this section. First, we define tensorial scaling functions and wavelets and give the reconstruction formula. The solution of the tensorial SGG problem is presented using regularization wavelets.

Definition 7.46. Let $\{\mathbf{\Phi}_j^{(i,k)}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$, $i,k\in\{1,2,3\}$, be a family of admissible $\mathbf{h}_{\sigma,\sigma}^{(i,k)}$-kernels. Then the family $\{\mathcal{V}_j^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})\}_{j\in\mathbb{N}_0}$ of scale spaces $\mathcal{V}_j^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})$ defined by

$$\mathcal{V}_j^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}}) = \{\mathbf{\Phi}_j^{(i,k)} \star \mathbf{\Phi}_j^{(i,k)} * \mathbf{f} \ : \ \mathbf{f}\in\mathbf{h}_s(\overline{\Omega_\sigma^{\text{ext}}})\}, \tag{7.140}$$

is called an $\mathbf{h}_{\sigma,\sigma}^{(i,k)}$*-multiresolution analysis*, if the following properties are satisfied:

(i) $\mathcal{V}_0^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}}) \subset \cdots \subset \mathcal{V}_j^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}}) \subset \mathcal{V}_{j+1}^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}}) \subset \cdots \subset \mathbf{h}_s^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})$,

(ii) $\overline{\bigcup_{j\in\mathbb{N}_0} \mathcal{V}_j^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})}^{\|\cdot\|_{\mathbf{h}_s(\overline{\Omega_\sigma^{\text{ext}}})}} = \mathbf{h}_s^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})$.

Definition 7.47. Let $\{\mathbf{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be a family of admissible $\mathbf{h}_{\sigma,\sigma}$-kernels. The set of scale spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}})$ defined by

$$\mathcal{V}_j(\overline{\Omega_\sigma^{\text{ext}}}) = \{\mathbf{\Phi}_j \star \mathbf{\Phi}_j * \mathbf{f} \ : \ \mathbf{f}\in\mathbf{h}_s(\overline{\Omega_\sigma^{\text{ext}}})\} \tag{7.141}$$

is called an $\mathbf{h}_{\sigma,\sigma}$-multiresolution analysis, if $\{\mathcal{V}_j^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})\}_{j\in\mathbb{N}_0}$ is an $\mathbf{h}_s^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})$-multiresolution analysis for every $i,k\in\{1,2,3\}$.

We now define the scaling functions.

Definition 7.48. A family $\{\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ of sequences $\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}$ is called a *generator of a scaling function of kind* (i,k), $i,k\in\{1,2,3\}$, if it satisfies the following requirements:

(i) $(\varphi_j^{(i,k)}(0_{ik}))^2 = 1$, for all $j \in \mathbb{N}_0$,

(ii) $(\varphi_j^{(i,k)}(n))^2 \leq \left(\varphi_{j'}^{(i,k)}(n)\right)^2$, for all $j, j' \in \mathbb{N}_0$ with $j \leq j'$ and all $n \in \mathbb{N}_{0_{ik}+1}$,

(iii) $\lim_{j\to\infty} (\varphi_j^{(i,k)}(n))^2 = 1$, for all $n \in \mathbb{N}_{0_{ik}+1}$.

Furthermore, the family $\{\{\{\varphi^{(i,k)}(n)\}_{i,k\in\{1,2,3\}}\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ is called a generator of a scaling function, if $\{\{\varphi^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ are generators of a scaling function of kind (i,k), $i,k \in \{1,2,3\}$.

Based on the definition of a generator of a scaling function, we now introduce $\mathbf{h}_{\sigma,\sigma}$-scaling functions.

Definition 7.49. A family $\{\boldsymbol{\Phi}_j^{(i)}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ of $\mathbf{h}^{(i,k)}$-kernels $\boldsymbol{\Phi}_j^{(i,k)}(\cdot,\cdot)$ defined by $\boldsymbol{\Phi}_j^{(i,k)\wedge}(n) = \varphi_j^{(i,k)}(n)$, $j \in \mathbb{N}_0$, $n \in \mathbb{N}_{0_{ik}}$, i.e.,

$$\boldsymbol{\Phi}_j^{(i,k)}(x,y) = \sum_{n=0_{ik}}^{\infty} \varphi_j^{(i,k)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\sigma;y)\mathbf{h}_{n,m}^{(i,k)s}(\sigma;x), \quad x,y \in \overline{\Omega_\sigma^{\mathrm{ext}}}, \tag{7.142}$$

is called an $\mathbf{h}_{\sigma,\sigma}^{(i,k)}$-*scaling function*, if it satisfies the following properties:

(i) $\boldsymbol{\Phi}_j^{(i,k)}(\cdot,\cdot)$ is an admissible $\mathbf{h}_{\sigma,\sigma}^{(i,k)}$-kernel for every $j \in \mathbb{N}_0$,

(ii) $\{\{\boldsymbol{\Phi}_j^{(i,k)\wedge}(n)_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ constitutes a generator of a scaling function of kind (i,k).

Furthermore, the family $\{\boldsymbol{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ of $\mathbf{h}_{\sigma,\sigma}$-kernels $\boldsymbol{\Phi}_j(\cdot,\cdot)$ is called an $\mathbf{h}_{\sigma,\sigma}$-scaling function, if $\{\boldsymbol{\Phi}_j^{(i,k)}\}_{j\in\mathbb{N}_0}$ are $\mathbf{h}_{\sigma,\sigma}^{(i,k)}$-scaling functions for $i,k \in \{1,2,3\}$.

As in the scalar and vectorial theory, the following approximation theorem is valid.

Theorem 7.50. *Let* $\{\boldsymbol{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ *be an* $\mathbf{h}_{\sigma,\sigma}$*-scaling function. Then*

$$\lim_{j\to\infty} \|\mathbf{f} - \boldsymbol{\Phi}_j \star \boldsymbol{\Phi}_j * \mathbf{f}\|_{\mathbf{h}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} = 0 \tag{7.143}$$

holds for all $\mathbf{f} \in \mathbf{h}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

Definition 7.51. Let $\{\boldsymbol{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be an $\mathbf{h}_{\sigma,\sigma}$-scaling function. Then $\{P_j\}_{j\in\mathbb{N}_0}$ with $P_j : \mathbf{h}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to \mathbf{h}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ defined by

$$P_j(\mathbf{f})(x) = \boldsymbol{\Phi}_j \star \boldsymbol{\Phi}_j * \mathbf{f}, \quad \mathbf{f} \in \mathbf{h}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}), \quad x \in \overline{\Omega_\sigma^{\mathrm{ext}}}, \tag{7.144}$$

is called an $\mathbf{h}_{\sigma,\sigma}$-*approximate identity.*

The kernel $\boldsymbol{\Phi}_0$ is called the mother kernel of the $\mathbf{h}_{\sigma,\sigma}$-scaling function. We obtain the following theorem.

Theorem 7.52. *Let* $\{\boldsymbol{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ *be an* $\mathbf{h}_{\sigma,\sigma}$*-scaling function. Then* $\{\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})\}_{j\in\mathbb{N}_0}$ *given in* (7.141) *forms an* $\mathbf{h}_{\sigma,\sigma}$*-multiresolution analysis.*

The next purpose is to define the primal and dual wavelet with the help of the tensorial refinement equation.

Definition 7.53. Let $\{\mathbf{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be an $\mathbf{h}_{\sigma,\sigma}$-scaling function. Then the families of $\mathbf{h}_{\sigma,\sigma}$-kernels $\{\mathbf{\Psi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$, $\{\tilde{\mathbf{\Psi}}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ given by

$$(\mathbf{\Psi}_j)^{(i,k)\wedge}(n) = \psi_j^{(i,k)}(n), \quad j\in\mathbb{N}_0,\ n\in\mathbb{N}_{0_{ik}},\ i,k\in\{1,2,3\}, \tag{7.145}$$

$$(\tilde{\mathbf{\Psi}}_j)^{(i,k)\wedge}(n) = \tilde{\psi}_j^{(i,k)}(n), \quad j\in\mathbb{N}_0,\ n\in\mathbb{N}_{0_{ik}},\ i,k\in\{1,2,3\}, \tag{7.146}$$

are called (*primal*) $\mathbf{h}_{\sigma,\sigma}$-*wavelet* and *dual* $\mathbf{h}_{\sigma,\sigma}$-*wavelet*, respectively, if all $\mathbf{h}_{\sigma,\sigma}$-kernels $\mathbf{\Psi}_j(\cdot,\cdot)$, $\tilde{\mathbf{\Psi}}_j(\cdot,\cdot)$, $j\in\mathbb{N}_0$, are admissible and the symbols $\{\psi_j^{(i,k)}(n)\}$, $\{\tilde{\psi}_j^{(i,k)}(n)\}$, in addition, satisfy the (tensorial) refinement equation

$$\tilde{\psi}_j^{(i,k)}(n)\psi_j^{(i,k)}(n) = (\varphi_{j+1}^{(i,k)}(n))^2 - (\varphi_j(n)^{(i,k)})^2 \tag{7.147}$$

for all $j\in\mathbb{N}_0$, $n\in\mathbb{N}_{0_{ik}}$, $i,k\in\{1,2,3\}$.

As a direct consequence we get the following equation:

$$(\varphi_{J+1}^{(i,k)}(n))^2 = (\varphi_0^{(i,k)}(n))^2 + \sum_{j=0}^{J} \tilde{\psi}_j^{(i,k)}(n)\psi_j^{(i,k)}(n), \quad J\in\mathbb{N}_0, \tag{7.148}$$

for all $n\in\mathbb{N}_{0_{ik}}$. We now define the wavelet transform. To this end we let $\psi_{-1}^{(i,k)}(n) = \tilde{\psi}_{-1}^{(i,k)}(n) = \varphi_0^{(i,k)}(n)$, for $n\in\mathbb{N}_{0_{ik}}$, $i,k\in\{1,2,3\}$, $\mathbf{\Psi}_{-1}(\cdot,\cdot) = \tilde{\mathbf{\Psi}}_{-1}(\cdot,\cdot) = \mathbf{\Phi}_0(\cdot,\cdot)$. We remember the space $\mathcal{H}(\mathbb{N}_{-1}\times\overline{\Omega_\sigma^{\text{ext}}})$ (see Eqs. (7.12)–(7.14)).

Definition 7.54. Let $\{\mathbf{\Psi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_{-1}}$ be a (primal) $\mathbf{h}_{\sigma,\sigma}$-wavelet. Then $(WT)^{(i,k)}$: $\mathbf{h}_s(\overline{\Omega_\sigma^{\text{ext}}}) \to \mathcal{H}_s(\mathbb{N}_{-1}\times\overline{\Omega_\sigma^{\text{ext}}})$ defined by

$$(WT)^{(i,k)}(\mathbf{f})(j;y) = (\mathbf{\Psi}_j^{(i,k)} * \mathbf{f})(y) \tag{7.149}$$

is called $\mathbf{h}_{\sigma,\sigma}$-wavelet transform if kind (i,k) of $\mathbf{f}$ at position $y\in\overline{\Omega_\sigma^{\text{ext}}}$ and scale $j\in\mathbb{N}_{-1}$.

As usual, we define the detail space $\mathcal{W}_j^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}})$ at scale j by

$$\mathcal{W}_j^{(i,k)}(\overline{\Omega_\sigma^{\text{ext}}}) = \{\tilde{\mathbf{\Psi}}_j^{(i,k)} \star \mathbf{\Psi}_j^{(i,k)} * \mathbf{f} \ :\ \mathbf{f}\in\mathbf{h}({}_s\overline{\Omega_\sigma^{\text{ext}}})\}, \tag{7.150}$$

and

$$\mathcal{W}_j(\overline{\Omega_\sigma^{\text{ext}}}) = \{\tilde{\mathbf{\Psi}}_j \star \mathbf{\Psi}_j * \mathbf{f} \ :\ \mathbf{f}\in\mathbf{h}_s(\overline{\Omega_\sigma^{\text{ext}}})\}. \tag{7.151}$$

Theorem 7.55 (Tensorial Reconstruction Formula for the Outer Space). *Let the families* $\{\mathbf{\Psi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ *and* $\{\tilde{\mathbf{\Psi}}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$*, respectively, be a (primal)* $\mathbf{h}_{\sigma,\sigma}$*-wavelet and its dual corresponding to an* $\mathbf{h}_{\sigma,\sigma}$*-scaling function* $\{\mathbf{\Phi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$*. Then*

$$\mathbf{f} = \sum_{j=-1}^{\infty} \tilde{\mathbf{\Psi}}_j \star \mathbf{\Psi}_j * \mathbf{f} \tag{7.152}$$

holds for all $\mathbf{f}\in\mathbf{h}_s(\overline{\Omega_\sigma^{\text{ext}}})$ *(in* $\|\cdot\|_{\mathbf{h}_s(\overline{\Omega_\sigma^{\text{ext}}})}$*-sense).*

We now solve the (tensorial) SGG problem using regularization wavelets. First, we transfer the theory of $\mathbf{h}^{(i,k)}_{\sigma,\sigma}$-scaling functions to the general case of $\mathbf{h}^{(i,k)}_{\alpha,\alpha}$-scaling functions $\mathbf{\Phi}_j^{(i,k),\alpha,\alpha}$ with $\alpha \geq \sigma$:

$$\mathbf{\Phi}_j^{(i,k),\alpha,\alpha}(x,y) = \sum_{n=0_{ik}}^{\infty} \varphi_j^{(i,k)}(n) \sum_{m=1}^{2n+1} H^s_{n,m}(\alpha;x)\mathbf{h}^{(i,k)s}_{n,m}(\alpha;y), \tag{7.153}$$

where

$$(\mathbf{\Phi}_j^{(i,k),\alpha,\alpha})^\wedge(n) = \varphi_j^{(i,k)}(n). \tag{7.154}$$

Theorem 7.50 can be directly transferred substituting σ by α. The scale spaces are defined in the following way:

$$\mathcal{V}_j^{(i,k)}(\overline{\Omega^{\text{ext}}_\alpha}) = \{\mathbf{\Phi}_j^{(i,k),\alpha,\alpha} \star \mathbf{\Phi}_j^{(i,k),\alpha,\alpha} * \mathbf{f} \ : \ \mathbf{f} \in \mathbf{h}_s(\overline{\Omega^{\text{ext}}_\alpha})\}. \tag{7.155}$$

The system $\{\mathcal{V}_j^{(i,k)}(\overline{\Omega^{\text{ext}}_\alpha})\}$ of scale spaces forms a multiresolution analysis.

Theorem 7.56. *The restriction of the operator* $\boldsymbol{\lambda}^{(i,k)} : \mathcal{H}_s(\overline{\Omega^{\text{ext}}_\sigma}) \to \mathbf{h}_s^{(i,k)}(\overline{\Omega^{\text{ext}}_\gamma})$ *to a scale space* $\mathcal{V}_j(\overline{\Omega^{\text{ext}}_\sigma})$, $j \in \mathbb{N}_0$, *i.e.,*

$$\boldsymbol{\lambda}^{(i,k)}|_{\mathcal{V}_j(\overline{\Omega^{\text{ext}}_\sigma})} : \mathcal{V}_j(\overline{\Omega^{\text{ext}}_\sigma}) \to \mathcal{V}_j^{(i,k)}(\overline{\Omega^{\text{ext}}_\gamma}), \tag{7.156}$$

is injective for $(i,k) \in \{(1,1), (2,1), (3,1)\}$, *whereas in the case of* $(i,k) \in \{(1,2), (1,3), (2,3), (3,3)\}$ *the Fourier coefficient of degree* 0 *cannot be recovered and the Fourier coefficients of degree* $n \geq 1$ *are uniquely defined. In the case of* $(i,k) \in \{(2,2),(3,2)\}$ *the Fourier coefficient of degree* 0 *and* 1 *cannot be recovered and the Fourier coefficients of degree* $n \geq 2$ *are uniquely defined (in the following text, injectivity, bijectivity and uniqueness is always used in this sense).*

Moreover, we have the following results:

(i) *If the families* $\{\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ *and* $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$, $i,k\in\{1,2,3\}$, *are bandlimited (for example,* $\varphi_j^{(i,k)}(n) = \varphi_j(n) = 0$ *for all* $n \geq 2^j$*), then the restricted operator is even bijective (in the sense described above). To be more specific, for* $\mathbf{g}^{(i,k)} \in \mathbf{h}_s^{(i,k)}(\overline{\Omega^{\text{ext}}_\gamma})$ *the unique solution* $F_j \in \mathcal{V}_j(\overline{\Omega^{\text{ext}}_\sigma})$, $j \in \mathbb{N}_0$, *of the equation*

$$\boldsymbol{\lambda}^{(i,k)} F_j = \mathbf{\Phi}_j^{(i,k),\gamma,\gamma} \star \mathbf{\Phi}_j^{(i,k),\gamma,\gamma} * \mathbf{g}^{(i,k)} \tag{7.157}$$

is given by

$$F_j = \Phi_j^{\sigma,\sigma} * \Phi_j^{\sigma,\sigma} * Q, \tag{7.158}$$

where $Q \in \mathcal{H}_s(\overline{\Omega^{\text{ext}}_\sigma})$ *is obtainable by*

$$Q^\wedge(n,m) = \begin{cases} \frac{\mathbf{g}^{(i,k)\wedge}(n,m)}{\boldsymbol{\lambda}^{(i,k)\wedge}(n)}, & n \in [0_{ik}, 2^j), \\ 0, & n \in [2^j, \infty). \end{cases} \tag{7.159}$$

(ii) *If the families* $\{\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$, $i,k\in\{1,2,3\}$, *and* $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ *are non-bandlimited, the equation*

$$\boldsymbol{\lambda}^{(i,k)}F_j = \boldsymbol{\Phi}_j^{(i,k),\gamma,\gamma} \star \boldsymbol{\Phi}_j^{(i,k),\gamma,\gamma} * \mathbf{g}^{(i,k)} \tag{7.160}$$

has a solution $F_j \in \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ *provided that* $\mathbf{g}^{(i,k)} \in \mathbf{h}_s^{(i,k)\Lambda}(\overline{\Omega_\gamma^{\mathrm{ext}}})$, *where* $\mathbf{h}_s^{(i,k)\Lambda}(\overline{\Omega_\gamma^{\mathrm{ext}}})$ *is an appropriate Sobolev space (see the Ph.D.-thesis* [58] *for more details). In this case, the unique solution of the equation is given by*

$$F_j = \Phi_j^{\sigma,\sigma} * \Phi_j^{\sigma,\sigma} * Q, \tag{7.161}$$

where $Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ *is obtainable in spectral language by*

$$Q^\wedge(n,m) = \frac{\mathbf{g}^{(i,k)\wedge}(n,m)}{\boldsymbol{\lambda}^{(i,k)\wedge}(n)}, \tag{7.162}$$

$n = 0_{ik}, \ldots; \ m = 1, \ldots, 2n+1$.

The following corollary shows that in the case of general operators $\boldsymbol{\lambda} = \sum_{i,k=1}^3 \boldsymbol{\lambda}^{(i,k)}$ we have to claim an additional assumption onto the function $\mathbf{g}$.

Corollary 7.57. *The restriction of the operator* $\boldsymbol{\lambda} = \sum_{i,k=1}^3 \boldsymbol{\lambda}^{(i,k)}$ *to a scale space* $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, *i.e.,*

$$\boldsymbol{\lambda}|_{\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})} : \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) \to \bigoplus_{i,k=1}^{3} \mathcal{V}_j^{(i,k)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \tag{7.163}$$

has, in general, no solution. Under the assumption $\varphi_j^{(i,k)}(n) = \varphi_j(n)$, $i,k \in \{1,2,3\}$, *we have to claim, in addition, that*

$$\frac{\mathbf{g}^{(i,k)\wedge}(n,m)}{\boldsymbol{\lambda}^{(i,k)\wedge}(n)} = \frac{\mathbf{g}^{(l,r)\wedge}(n,m)}{\boldsymbol{\lambda}^{(l,r)\wedge}(n)}, \tag{7.164}$$

with $i,k,l,r \in \{1,2,3\}$; $n = \max\limits_{i,k,l,r\in\{1,2,3\}}(0_{ik}, 0_{lr}), \ldots; \ m = 1, \ldots, 2n+1$. *Then the results in Theorem* 7.56 *can be directly transferred.*

With the help of the refinement equation (7.147) we now define the primal wavelets $\{\boldsymbol{\Psi}_j^{(i,k),\alpha,\alpha}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ and the dual wavelets $\{\tilde{\boldsymbol{\Psi}}_j^{(i,k),\alpha,\alpha}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ for $\alpha \geq \sigma$, $i,k \in \{1,2,3\}$:

$$\boldsymbol{\Psi}_j^{(i,k),\alpha,\alpha}(x,y) = \sum_{n=0_{ik}}^{\infty} \psi_j^{(i,k)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\alpha;x)\mathbf{h}_{n,m}^{(i,k)s}(\alpha;y), \tag{7.165}$$

$$\tilde{\boldsymbol{\Psi}}_j^{(i,k),\alpha,\alpha}(x,y) = \sum_{n=0_{ik}}^{\infty} \tilde{\psi}_j^{(i,k)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\alpha;x)\mathbf{h}_{n,m}^{(i,k)s}(\alpha;y), \tag{7.166}$$

where

$$(\boldsymbol{\Psi}_j^{(i,k),\alpha,\alpha})^\wedge(n) = \psi_j^{(i,k)}(n), \quad (\tilde{\boldsymbol{\Psi}}_j^{(i,k),\alpha,\alpha})^\wedge(n) = \tilde{\psi}_j^{(i,k)}(n). \tag{7.167}$$

The detail spaces are defined in canonical manner:

$$\mathcal{W}_j^{(i,k)}(\overline{\Omega_\alpha^{\rm ext}}) = \{\boldsymbol{\Psi}_j^{(i,k),\alpha,\alpha} \star \tilde{\boldsymbol{\Psi}}_j^{(i,k),\alpha,\alpha} * \mathbf{f} \ : \ \mathbf{f} \in \mathbf{h}_s(\overline{\Omega_\alpha^{\rm ext}})\}. \tag{7.168}$$

Theorem 7.55 can be directly transferred by substituting the convolutions with respect to the sphere Ω_σ by the corresponding convolutions with respect to the sphere Ω_α. We now transfer Theorem 7.56 to the detail spaces and get the following theorem, where we use the terms injectivity, bijectivity, and uniqueness in the same sense as before.

Theorem 7.58. *The restriction of the operator* $\boldsymbol{\lambda}^{(i,k)} : \mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}}) \to \mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\rm ext}})$ *to a detail space* $\mathcal{W}_j(\overline{\Omega_\sigma^{\rm ext}})$, $j \in \mathbb{N}_0$, *i.e.,*

$$\boldsymbol{\lambda}^{(i,k)}|_{\mathcal{W}_j(\overline{\Omega_\sigma^{\rm ext}})} \mathcal{W}_j(\overline{\Omega_\sigma^{\rm ext}}) \to \mathcal{W}_j^{(i,k)}(\overline{\Omega_\gamma^{\rm ext}}) \tag{7.169}$$

with $\psi_j(n) = \psi_j^{(i,k)}(n)$ *is injective. Moreover, we have the following results:*

(i) *If the families* $\{\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ *and* $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$, $i,k\in\{1,2,3\}$, *are bandlimited (for example,* $\varphi_j^{(i,k)}(n) = \varphi_j(n) = 0$ *for all* $n \geq 2^j$*), then the restricted operator is even bijective. To be more specific, for* $\mathbf{g}^{(i,k)} \in \mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\rm ext}})$ *the unique solution* $H_j \in \mathcal{W}_j(\overline{\Omega_\sigma^{\rm ext}})$, $j \in \mathbb{N}_0$, *of the equation*

$$\boldsymbol{\lambda}^{(i,k)} H_j = \tilde{\boldsymbol{\Psi}}_j^{(i,k),\gamma,\gamma} * \boldsymbol{\Psi}_j^{(i,k),\gamma,\gamma} * \mathbf{g}^{(i,k)} \tag{7.170}$$

is given by

$$H_j = \tilde{\Psi}_j^{\sigma,\sigma} * \Psi_j^{\sigma,\sigma} * Q, \tag{7.171}$$

where $Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\rm ext}})$ *is obtainable by*

$$Q^\wedge(n,m) = \begin{cases} \frac{\mathbf{g}^{(i,k)\wedge}(n,m)}{\boldsymbol{\lambda}^{(i,k)\wedge}(n)}, & n \in [0_{ik}, 2^{j+1}), \\ 0, & n \in [2^{j+1}, \infty). \end{cases} \tag{7.172}$$

(ii) *If the families* $\{\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$, $i,k\in\{1,2,3\}$, *and* $\{\{\varphi_j(n)\}_{n\in\mathbb{N}_0}\}_{j\in\mathbb{N}_0}$ *are non-bandlimited, the equation*

$$\boldsymbol{\lambda}^{(i,,k)} H_j = \tilde{\boldsymbol{\Psi}}_j^{(i,k),\gamma,\gamma} \star \boldsymbol{\Psi}_j^{(i,k),\gamma,\gamma} * \mathbf{g}^{(i,k)} \tag{7.173}$$

has a solution $H_j \in \mathcal{W}_j(\overline{\Omega_\sigma^{\rm ext}})$ *provided that the condition*

$$\sum_{n=0_{ik}}^{\infty} \sum_{m=1}^{2n+1} \frac{\mathbf{g}^{(i,k)\wedge}(n,m)}{\boldsymbol{\lambda}^{(i,k)\wedge}(n)} < \infty \tag{7.174}$$

is satisfied for $\mathbf{g}^{(i,k)} \in \mathbf{h_s}^{(i,k)}(\overline{\Omega_\gamma^{\rm ext}})$*. In this case, the unique solution of the equation is given by*

$$H_j = \tilde{\Psi}_j^{\sigma,\sigma} * \Psi_j^{\sigma,\sigma} * Q, \tag{7.175}$$

where $Q \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ *is obtainable by*

$$Q^\wedge(n,m) = \frac{\mathbf{g}^{(i,k)\wedge}(n,m)}{\boldsymbol{\lambda}^{(i,k)\wedge}(n)}, \tag{7.176}$$

$n = 0_{ik}, \ldots$; $m = 1, \ldots, 2n+1$.

Furthermore, we have the following corollary.

Corollary 7.59. *The restriction of the operator* $\boldsymbol{\lambda} = \sum_{i,k=1}^{3} \boldsymbol{\lambda}^{(i,k)}$ *to a detail space* $\mathcal{W}_j(\overline{\Omega_\sigma^{\text{ext}}})$, $j \in \mathbb{N}_0$, *i.e.,*

$$\boldsymbol{\lambda}|_{\mathcal{W}_j(\overline{\Omega_\sigma^{\text{ext}}})} : \mathcal{W}_j(\overline{\Omega_\sigma^{\text{ext}}}) \to \bigoplus_{i,k=1}^{3} \mathcal{W}_j^{(i,k)}(\overline{\Omega_\gamma^{\text{ext}}}) \tag{7.177}$$

has, in general, no solution. Under the assumption

$$\psi_j^{(i,k)}(n) = \psi_j(n) \quad \textit{and} \quad \tilde{\psi}_j^{(i,k)}(n) = \tilde{\psi}_j(n), \quad i,k \in \{1,2,3\},$$

we have to claim, in addition, that

$$\frac{\mathbf{g}^{(i,k)\wedge}(n,m)}{\boldsymbol{\lambda}^{(i,k)\wedge}(n)} = \frac{\mathbf{g}^{(l,r)\wedge}(n,m)}{\boldsymbol{\lambda}^{(l,r)\wedge}(n)}, \tag{7.178}$$

with $i,k,l,r \in \{1,2,3\}$; $n = \max\limits_{i,k,l,r}(0_{ik}, 0_{lr}), \ldots$; $m = 1, \ldots, 2n+1$. *Then the results in Theorem* 7.58 *can be directly transferred.*

Up to now, we have summarized some results about the filtered solution, i.e., the solution when we restrict the operator to the scale or detail spaces. In this case, the injectivity for the operators $\boldsymbol{\lambda}^{(i,k)}$ could be proved, whereas in the case of general operators $\boldsymbol{\lambda} = \sum_{i,k=1}^{3} \boldsymbol{\lambda}^{(i,k)}$ we have to claim that (7.164) is valid. In the case of the unfiltered solution, we obtain the following theorem.

Theorem 7.60. *Let* $\mathbf{g}^{(i,k)} \in \mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\text{ext}}})$ *satisfy the condition* $\mathbf{g} \in \operatorname{im}(\boldsymbol{\lambda}^{(i,k)})$, $i,k \in \{1,2,3\}$. *Then the unique solution* $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ *of the equation* $\boldsymbol{\lambda}^{(i,k)} F = \mathbf{g}^{(i,k)}$ *is given by*

$$F^\wedge(n,m) = \frac{\mathbf{g}^{(i,k)\wedge}(n,m)}{\boldsymbol{\lambda}^{(i,k)\wedge}(n)}, \tag{7.179}$$

$n = 0_{ik}, \ldots$; $m = 1, \ldots, 2n+1$. *In the case of the operator* $\boldsymbol{\lambda} = \sum_{i,k=1}^{3} \boldsymbol{\lambda}^{(i,k)}$ *we have to claim, in addition, that* (7.178) *holds in order to guarantee the solvability.*

Last, we explain the connection between the solution in the scale spaces and the unfiltered solution.

Theorem 7.61. *Suppose that* $\mathbf{g}^{(i,k)}$ *is of the class* $\mathbf{h}_s^{(i,k)\Lambda}(\overline{\Omega_\gamma^{\text{ext}}})$. *Let* $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$ *be the unique solution of* $\boldsymbol{\lambda}^{(i,k)} F = \mathbf{g}^{(i,k)}$. *Then*

$$F_j = (\Phi_j^{(2)})^{\sigma,\sigma} * F \tag{7.180}$$

is the unique solution in $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ *of the equation*

$$\boldsymbol{\lambda}^{(i,k)} F_j = \boldsymbol{\Phi}_j^{(i,k),\gamma,\gamma} \star \boldsymbol{\Phi}_j^{(i,k),\gamma,\gamma} * \mathbf{g}^{(i,k)} \tag{7.181}$$

for every $j \in \mathbb{N}_0$*. Furthermore, the limit relation*

$$\lim_{J\to\infty} (\Phi_J^{(2)})^{\sigma,\sigma} * F = F \tag{7.182}$$

holds (in $\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}$*-sense).*

The preceding theorem shows that in the case of bandlimited scaling functions the (tensorial) SGG-problem is well posed, because a unique solution always exists and due to the finite dimension of the scale spaces the solution is also stable. We now investigate the case of non-bandlimited scaling functions, where the stability cannot be ensured and we have to use regularization methods.

Definition 7.62. A family of linear operators $S_j^{(i,k)} : \mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, is called a *regularization* of $(\boldsymbol{\lambda}^{(i,k)})^{-1}$, $i,k \in \{1,2,3\}$, if it satisfies the following properties:

(i) $S_j^{(i,k)}$ is bounded on $\mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\mathrm{ext}}})$ for all $j \in \mathbb{N}_0$,

(ii) for any member $\mathbf{g}^{(i,k)} \in im(\boldsymbol{\lambda}^{(i,k)})$, the limit relation

$$\lim_{J\to\infty} S_J^{(i,k)} \mathbf{g}^{(i,k)} = (\boldsymbol{\lambda}^{(i,k)})^{-1} \mathbf{g}^{(i,k)} \tag{7.183}$$

holds (in $\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}$-sense).

The operator $S : \mathbf{h}_s(\overline{\Omega_\gamma^{\mathrm{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ given by $S|_{\mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\mathrm{ext}}})} = S_j^{(i,k)}$ is called a regularization of $\boldsymbol{\lambda}^{-1}$.

The function $F_J = S_J \mathbf{g}$ is called the *J-level regularization of* $\boldsymbol{\lambda}^{-1}\mathbf{g}$, whereas $F_J^{(i,k)} = S_J^{(i,k)} \mathbf{g}^{(i,k)}$ is called the *J-level regularization of* $(\boldsymbol{\lambda}^{(i,k)})^{-1}\mathbf{g}$. Within our multiscale approach, we now represent the $(J+1)$-level regularization using the J-level regularization by adding the corresponding detail information. To this end we first introduce a multiscale regularization concept starting with the definition of a generator of a regularization scaling function.

Definition 7.63. A family $\{\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ of sequences $\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}$, $i,k \in \{1,2,3\}$, is called a *generator of a regularization scaling function with respect to* $(\boldsymbol{\lambda}^{(i,k)})^{-1}$, if it satisfies the following requirements:

(i) $(\varphi_j^{(i,k)}(0_{ik}))^2 = \frac{1}{\boldsymbol{\lambda}^{(i,k)\wedge}(0_{ik})}$, for all $j \in \mathbb{N}_0$,

(ii) $(\varphi_j^{(i,k)}(n))^2 \leq (\varphi_{j'}^{(i,k)}(n))^2$, for all $j,j' \in \mathbb{N}_0$ with $j \leq j'$ and all $n \in \mathbb{N}_{0_{ik}+1}$,

(iii) $\lim_{j\to\infty} (\varphi_j^{(i,k)}(n))^2 = \frac{1}{(\boldsymbol{\lambda}^{(i,k)})^\wedge(n)}$, for all $n \in \mathbb{N}_{0_{ik}+1}$.

Furthermore, $\{\{\{\varphi_j^{(i,k)}(n)\}_{i,k\in\{1,2,3\}}\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ is called a *generator of a regularization scaling function with respect to* $\boldsymbol{\lambda}^{-1}$, if $(\boldsymbol{\lambda}^{(i,k)})^{-1}$ is a generator of a regularization scaling function with respect to $(\boldsymbol{\lambda}^{(i,k)})^{-1}$ for every $i,k = 1,2,3$.

We now define decomposition and reconstruction regularization scaling functions.

Definition 7.64. Let $\{\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ be a generator of a regularization scaling function with respect to $(\boldsymbol{\lambda}^{(i,k)})^{-1}$, $i,k\in\{1,2,3\}$.

Then a family $\{{}^d\boldsymbol{\Phi}_j^{(i,k),\sigma,\gamma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ of admissible $\mathbf{h}_{\sigma,\gamma}^{(i,k)}$-kernels given by

$$ {}^d\boldsymbol{\Phi}_j^{(i,k),\sigma,\gamma}(x,z) = \sum_{n=0_{ik}}^{\infty} \varphi_j^{(i,k)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\sigma;x)\mathbf{h}_{n,m}^{(i,k)s}(\gamma;z), \tag{7.184} $$

$(x,z)\in\overline{\Omega_\sigma^{\mathrm{ext}}}\times\overline{\Omega_\gamma^{\mathrm{ext}}}$, is called a *decomposition regularization* $\mathbf{h}_{\sigma,\gamma}^{(i,k)}$*-scaling function with respect to* $(\boldsymbol{\lambda}^{(i,k)})^{-1}$, whereas a family $\{{}^r\boldsymbol{\Phi}_j^{(i,k),\sigma,\sigma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ of admissible $\mathbf{h}_{\sigma,\sigma}^{(i,k)}$-kernels given by

$$ {}^r\boldsymbol{\Phi}_j^{(i,k),\sigma,\sigma}(x,y) = \sum_{n=0_{ik}}^{\infty} \varphi_j^{(i,k)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\sigma;x)\mathbf{h}_{n,m}^{(i,k)s}(\sigma;y), \tag{7.185} $$

$(x,y)\in\overline{\Omega_\sigma^{\mathrm{ext}}}\times\overline{\Omega_\sigma^{\mathrm{ext}}}$ is called a *reconstruction regularization* $\mathbf{h}_{\sigma,\sigma}^{(i,k)}$*-scaling function with respect to* $(\boldsymbol{\lambda}^{(i,k)})^{-1}$.

We obtain the following theorem:

Theorem 7.65. *Let* $\{\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ *be a generator of a regularization scaling function with respect to* $(\boldsymbol{\lambda}^{(i,k)})^{-1}$, $i,k\in\{1,2,3\}$*. If we formally define*

$$ ({}^r\boldsymbol{\Phi}_j^{(i,k)} \star {}^d\boldsymbol{\Phi}_j^{(i,k)})^{\sigma,\gamma}(\cdot,\cdot) $$

by

$$ ({}^r\boldsymbol{\Phi}_j^{(i,k)} \star {}^d\boldsymbol{\Phi}_j^{(i,k)})^{\sigma,\gamma}(x,z) = {}^r\boldsymbol{\Phi}_j^{(i,k),\sigma,\sigma}(x,\cdot) \star {}^d\boldsymbol{\Phi}_j^{(i,k),\sigma,\gamma}(\cdot,z), \tag{7.186} $$

$(x,z)\in\overline{\Omega_\sigma^{\mathrm{ext}}}\times\overline{\Omega_\gamma^{\mathrm{ext}}}$*, then*

$$ F_J^{(i,k)} = ({}^r\boldsymbol{\Phi}_J^{(i,k)} \star {}^d\boldsymbol{\Phi}_J^{(i,k)})^{\sigma,\gamma} * \mathbf{g}^{(i,k)}, \quad \mathbf{g}^{(i,k)} \in \mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\mathrm{ext}}}), \tag{7.187} $$

represents the J*-level regularization of* $(\boldsymbol{\lambda}^{(i,k)})^{-1}\mathbf{g}^{(i,k)}$*. If, in addition,* $\mathbf{g}^{(i,k)}\in\mathrm{im}(\boldsymbol{\lambda}^{(i,k)})$*, then*

$$ \lim_{J\to\infty} \|F_J^{(i,k)} - (\boldsymbol{\lambda}^{(i,k)})^{-1}\mathbf{g}^{(i,k)}\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} = 0. \tag{7.188} $$

Furthermore,

$$ F_J = \sum_{i,k=1}^{3} ({}^r\boldsymbol{\Phi}_J^{(i,k)} \star {}^d\boldsymbol{\Phi}_J^{(i,k)})^{\sigma,\gamma} * \mathbf{g}^{(i,k)}, \quad \mathbf{g} = \sum_{i,k=1}^{3} \mathbf{g}^{(i,k)} \in \mathbf{h}_s(\overline{\Omega_\gamma^{\mathrm{ext}}}), \tag{7.189} $$

represents the J*-level regularization of* $\boldsymbol{\lambda}^{-1}g$*. If, in addition,* $\mathbf{g}\in\mathrm{im}(\boldsymbol{\lambda})$*, then*

$$ \lim_{J\to\infty} \|F_J - \boldsymbol{\lambda}^{-1}\mathbf{g}\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} = 0. \tag{7.190} $$

We define the convolution operators $S_J^{(i,k)} : \mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\mathrm{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $J \in \mathbb{N}_0$, by

$$S_J^{(i,k)}(\mathbf{g}^{(i,k)}) = ({}^r\mathbf{\Phi}_J^{(i,k)} \star {}^d\mathbf{\Phi}_J^{(i,k)})^{\sigma,\gamma} * \mathbf{g}^{(i,k)}, \tag{7.191}$$

whereas the convolution operator $S_J : \mathbf{h}_s(\overline{\Omega_\gamma^{\mathrm{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $J \in \mathbb{N}_0$, is given by

$$S_J(\mathbf{g}) = \sum_{i,k=1}^{3} S_J^{(i,k)}(\mathbf{g}^{(i,k)}). \tag{7.192}$$

Furthermore, we introduce the corresponding scale spaces $S_J^{(i,k)}(\mathrm{im}(\boldsymbol{\lambda}^{(i,k)}))$, $i,k \in \{1,2,3\}$, and $S_J(\mathrm{im}(\boldsymbol{\lambda}))$ as follows

$$S_J^{(i,k)}(\mathrm{im}(\boldsymbol{\lambda}^{(i,k)})) = \left\{ ({}^r\mathbf{\Phi}_J^{(i,k)} \star {}^d\mathbf{\Phi}_J^{(i,k)})^{\sigma,\gamma} * \mathbf{g}^{(i,k)} \ : \ \mathbf{g}^{(i,k)} \in \mathrm{im}(\boldsymbol{\lambda}^{(i,k)}) \right\}, \tag{7.193}$$

$$S_J(\mathrm{im}(\boldsymbol{\lambda})) = \left\{ \sum_{i,k=1}^{3} ({}^r\mathbf{\Phi}_J^{(i,k)} \star {}^d\mathbf{\Phi}_J^{(i,k)})^{\sigma,\gamma} * \mathbf{g}^{(i,k)} : \mathbf{g} = \sum_{i,k=1}^{3} \mathbf{g}^{(i,k)} \in \mathrm{im}(\boldsymbol{\lambda}) \right\}. \tag{7.194}$$

Theorem 7.66. *The scale spaces satisfy the following properties:*

(i) $S_0^{(i,k)}(\mathrm{im}(\boldsymbol{\lambda}^{(i,k)})) \subset \cdots \subset S_J^{(i,k)}(\mathrm{im}(\boldsymbol{\lambda}^{(i,k)})) \subset S_{J'}^{(i,k)}(\mathrm{im}(\boldsymbol{\lambda}^{(i,k)})) \subset \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $J \leq J'$, *i.e., for any right-hand side* $\mathbf{g}^{(i,k)} \in \mathrm{im}(\boldsymbol{\lambda}^{(i,k)})$ *of the* (*tensorial*) *SGG problem, all J-level regularizations with fixed parameter J are sampled in a scale space* $S_J^{(i,k)}(\mathrm{im}(\boldsymbol{\lambda}^{(i,k)}))$ *with the above property,*

(ii) $\overline{\bigcup_{J=0}^{\infty} S_J^{(i,k)}(\mathrm{im}(\boldsymbol{\lambda}^{(i,k)}))}^{\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}} = \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

Obviously, Theorem 7.66 is also valid substituting $S_J^{(i,k)}$ by S_J which leads to the following corollary.

Corollary 7.67. *The scale spaces satisfy the following properties:*

(i) $S_0(\mathrm{im}(\boldsymbol{\lambda})) \subset \cdots \subset S_J(\mathrm{im}(\boldsymbol{\lambda})) \subset S_{J'}(\mathrm{im}(\boldsymbol{\lambda})) \subset \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $J \leq J'$, *i.e., for any right-hand side* $\mathbf{g} \in \mathrm{im}(\boldsymbol{\lambda})$ *of the* (*tensorial*) *SGG problem, all J-level regularizations with fixed parameter J are sampled in a scale space* $S_J(\mathrm{im}(\boldsymbol{\lambda}))$ *with the above property,*

(ii) $\overline{\bigcup_{J=0}^{\infty} S_J(\mathrm{im}(\boldsymbol{\lambda}))}^{\|\cdot\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}} = \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

A set of subspaces of $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ satisfying the conditions of Corollary 7.67 is called *regularization* $h_{\sigma,\gamma}$*-multiresolution analysis* (*RMRA*) *of the* (*tensorial*) *SGG problem.*

Definition 7.68. Let $\{\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ be a generator of a regularization scaling function with respect to $\left(\boldsymbol{\lambda}^{(i,k)}\right)^{-1}$. Then the generating symbols

$$\{\tilde{\psi}_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}, \ \{\psi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}$$

of the corresponding regularization wavelets are defined by the refinement equation

(7.147). The admissible $\mathbf{h}_{\sigma,\gamma}^{(i,k)}$-kernels $\{{}^d\boldsymbol{\Psi}_j^{(i,k),\sigma,\gamma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ given by

$$ {}^d\boldsymbol{\Psi}_j^{(i,k),\sigma,\gamma}(x,z) = \sum_{n=0_{ik}}^{\infty} \psi_j^{(i,k)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\sigma;x)\mathbf{h}_{n,m}^{(i,k)s}(\gamma;z), \tag{7.195} $$

$(x,z) \in \overline{\Omega_\sigma^{\text{ext}}} \times \overline{\Omega_\gamma^{\text{ext}}}$ are called the *decomposition regularization* $\mathbf{h}_{\sigma,\gamma}^{(i,k)}$*-wavelets*, while the admissible $\mathbf{h}_{\sigma,\sigma}^{(i,k)}$-kernels $\{{}^r\tilde{\boldsymbol{\Psi}}_j^{(i,k),\sigma,\sigma}(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ given by

$$ {}^r\tilde{\boldsymbol{\Psi}}_j^{(i,k),\sigma,\sigma}(x,y) = \sum_{n=0_{ik}}^{\infty} \tilde{\psi}_j^{(i,k)}(n) \sum_{m=1}^{2n+1} H_{n,m}^s(\sigma;x)\mathbf{h}_{n,m}^{(i,k)s}(\sigma;y), \tag{7.196} $$

$(x,y) \in \overline{\Omega_\sigma^{\text{ext}}} \times \overline{\Omega_\sigma^{\text{ext}}}$ are called the *reconstruction regularization* $\mathbf{h}_{\sigma,\sigma}^{(i,k)}$*-wavelets.*

We now define the convolution operators $T_j^{(i,k)} : \mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, $j \in \mathbb{N}_0$, $i,k = 1,2,3$, by

$$ T_j^{(i,k)}(\mathbf{g}^{(i,k)}) = ({}^r\tilde{\boldsymbol{\Psi}}_j^{(i,k)} \star {}^d\boldsymbol{\Psi}_j^{(i,k)})^{\sigma,\gamma} * \mathbf{g}^{(i,k)}, \quad \mathbf{g}^{(i,k)} \in \mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\text{ext}}}), \tag{7.197} $$

and the convolution operator $T_j : \mathbf{h}_s(\overline{\Omega_\gamma^{\text{ext}}}) \to \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, $j \in \mathbb{N}_0$, by

$$ T_J(\mathbf{g}) = \sum_{i,k=1}^{3} T_J^{(i,k)}(\mathbf{g}^{(i,k)}). \tag{7.198} $$

Obviously, due to the refinement equation the operators $S_{J+1}^{(i,k)}$ and S_{J+1} can be represented in the form

$$ S_{J+1}^{(i,k)} = S_0^{(i,k)} + \sum_{j=0}^{J} T_j^{(i,k)}, \tag{7.199} $$

$$ S_{J+1} = S_0 + \sum_{j=0}^{J} T_j. \tag{7.200} $$

Thus, we now introduce the *detail spaces* $T_J^{(i,k)}(\mathrm{im}(\boldsymbol{\lambda}^{(i,k)}))$ and $T_J(\mathrm{im}(\boldsymbol{\lambda}))$ by

$$ T_J^{(i,k)}(\mathrm{im}(\boldsymbol{\lambda}^{(i,k)})) = \left\{ ({}^r\tilde{\boldsymbol{\Psi}}_J^{(i,k)} \star {}^d\boldsymbol{\Psi}_J^{(i,k)})^{\sigma,\gamma} * \mathbf{g}^{(i,k)} : \mathbf{g}^{(i,k)} \in \mathrm{im}(\boldsymbol{\lambda}^{(i,k)}) \right\}, \tag{7.201} $$

$$ T_J(\mathrm{im}(\boldsymbol{\lambda})) = \left\{ \sum_{i,k=1}^{3} ({}^r\tilde{\boldsymbol{\Psi}}_J^{(i,k)} \star {}^d\boldsymbol{\Psi}_J^{(i,k)})^{\sigma,\gamma} * \mathbf{g}^{(i,k)} : \mathbf{g} = \sum_{i=1}^{3} \mathbf{g}^{(i,k)} \in \mathrm{im}(\boldsymbol{\lambda}) \right\}. \tag{7.202} $$

In terms of the multiscale concept, the space $T_J(\mathrm{im}(\boldsymbol{\lambda}))$ contains the detail information which has to be added in order to turn from the J-level regularization to the $(J+1)$-level regularization:

$$ S_{J+1}(\mathrm{im}(\boldsymbol{\lambda})) = S_J(\mathrm{im}(\boldsymbol{\lambda})) + T_J(\mathrm{im}(\boldsymbol{\lambda})). \tag{7.203} $$

In general, the sum is neither direct nor orthogonal.

Theorem 7.69. *Let* $\{\{\varphi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$ *be a generator of a regularization scaling function with respect to* $(\boldsymbol{\lambda}^{(i,k)})^{-1}$, $i,k\in\{1,2,3\}$. *Suppose that*

$$\{\{\psi_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0},\ \{\{\tilde{\psi}_j^{(i,k)}(n)\}_{n\in\mathbb{N}_{0_{ik}}}\}_{j\in\mathbb{N}_0}$$

are the generating symbols of the corresponding regularization wavelets. Furthermore, let $\mathbf{g}^{(i,k)}$ *be of the class* $\mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\mathrm{ext}}})$. *Define the* regularization $\mathbf{h}_{\sigma,\gamma}^{(i,k)}$-wavelet transform at scale $j\in\mathbb{N}_0$ and position $x\in\overline{\Omega_\sigma^{\mathrm{ext}}}$ *by*

$$(RWT)(\mathbf{g}^{(i,k)})(j;x) = {}^{d}\boldsymbol{\Psi}_j^{(i,k),\sigma,\gamma}(x,\cdot) * \mathbf{g}^{(i,k)}, \quad \mathbf{g}^{(i,k)}\in\mathbf{h}_s^{(i,k)}(\overline{\Omega_\gamma^{\mathrm{ext}}}). \tag{7.204}$$

Then

$$F_J = \left({}^{r}\boldsymbol{\Phi}_0^{(i,k)} \star {}^{d}\boldsymbol{\Phi}_0^{(i,k)}\right)^{\sigma,\gamma} * \mathbf{h}^{(i,k)} + \sum_{j=0}^{J-1} {}^{r}\tilde{\boldsymbol{\Psi}}_j^{(i,k),\sigma,\sigma} \star (RWT)(\mathbf{g}^{(i,k)})(j;\cdot) \tag{7.205}$$

is the J-level regularization of the (*tensorial*) *SGG problem satisfying*

$$\lim_{J\to\infty} \|F_J - (\boldsymbol{\lambda}^{(i,k)})^{-1}\mathbf{g}^{(i,k)}\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})} = 0 \tag{7.206}$$

provided that $\mathbf{g}^{(i,k)}\in\mathrm{im}(\boldsymbol{\lambda}^{(i,k)})$.

7.4. Combined outer harmonic and wavelet concept

In geodetic practice, there exists a variety of realizations of spherical harmonic models of the Earth's external gravitational potential. In [19] it is explained how to combine an outer harmonic model of fixed order m with a harmonic wavelet model. The justification for such a combined model is the fact that on the one hand the appropriate candidate for the approximation of the low frequency parts of the gravitational potential (i.e., global modeling) is a spherical harmonic (i.e., a multipole) model of moderate order m and on the other hand for the representation of the high frequency parts (i.e., local modeling) new wavelet techniques have to come into play (see also the investigations in spherical continuous wavelet theory [33, 34]).

Starting point of this model is the "refinement equation" (compare Eq. (7.10))

$$\tilde{\psi}_j(n)\psi_j(n) = (\varphi_{j+1}(n))^2 - (\varphi_j(n))^2.$$

It is clear that $\tilde{\psi}_j(n)\psi_j(n) = 0$ if and only if $(\varphi_{j+1}(n))^2 = (\varphi_j(n))^2$. Due to condition (i) in Definition 7.2, the wavelet (or its dual) satisfy the mean value condition $\psi_j(0) = 0$, i.e., it has to oscillate. For purposes of combined approximation we need, however, $(\varphi_{j+1}(n))^2 = (\varphi_j(n))^2$ for all $n\in[0,\ldots,m]$. Under these assumptions it may be guaranteed that the wavelets constructed in this way have more vanishing moments and we call them *wavelets of order m*. In [19] the reconstruction formula for such wavelets is studied in more detail. The transition of the combined outer harmonic and wavelet concept to the vectorial and tensorial case is also easy to perform.

8. Bandlimited Runge–Walsh multiscale approximation

In the previous sections we developed several methods of wavelet approximation. We briefly reformulate the main results: Let $\{\Psi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ be an $\mathcal{H}_{\sigma,\sigma}$-wavelet corresponding to an $\mathcal{H}_{\sigma,\sigma}$-scaling function $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$. Then any potential $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ can be expressed by a multiscale approximation given by

$$\Phi_0^{(2)} * F + \sum_{j=0}^{J-1} \tilde{\Psi}_j * \Psi_j * F, \quad F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}). \tag{8.1}$$

For a numerical realization, the discretization of the $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$-convolutions (i.e., the $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$-inner products) occurring in the J-level wavelet approximation is necessary. For that purpose we observe that any $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$-convolution is expressible as a bounded linear functional on $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$. Thus fully discretized wavelet approximation amounts to the problem of approximating a bounded linear functional (i.e., an $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$-inner product) by a linear combination of known bounded linear functionals. In this context it should again be mentioned that following our nomenclature an $\mathcal{H}_0(\overline{\Omega_\sigma^{\mathrm{ext}}})$-inner product can be identified with an ordinary integral over the sphere Ω_σ. Therefore, fully discretized $\mathcal{H}_0(\overline{\Omega_\sigma^{\mathrm{ext}}})$-wavelet approximation can be organized appropriately by numerical integration (cubature) over the sphere Ω_σ. Looking at the inner products in our general $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$-framework we are confronted with convolutions involving a pseudodifferential operator Λ with symbol $\Lambda^\wedge(n) = A_n$ for $n \in \mathbb{N}_0$. Their discretization requires the knowledge of linear (observational) functionals for the potential $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ under consideration. Usually, in gravitational field determination, these (observational) functionals are heterogeneous in nature. In addition, the approximate formulae have to be formulated in dependence on the scale parameter, since increasing space localization demands increasing data material.

All these requirements, however, do not lead to a unique procedure for discretizing $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$-convolutions. Many variants of approximate formulae are reasonable and conceivable. In fact, the choice of a suitable method is essentially dependent on the purpose for which scaling functions and wavelets are used. Unfortunately, it turns out that each of the discretization methods has its own drawback. Nevertheless, a lot of approximation schemata for $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$-convolutions can be found so that at least some of the requests can be fulfilled. As most important discretization rules we mention:

1. *Fast Fourier techniques and multipole techniques* (cf. [19, 39, 74]) are economical in time, but they are based on evaluation functionals on equiangular latitude-longitude grids. Thus the sample points are merely equidistributed on the (ϑ, φ)-parameter interval $[0,\pi] \times [0, 2\pi]$ in Euclidean space $\mathbb{R}^2$, but not on a sphere.
2. *Polynomial (i.e., outer harmonic) exact approximation of bandlimited functions* is a well-established tool for application to bandlimited potentials of moderate degree (cf. [12, 28, 29, 54]). The problem is that the preliminary

work includes the solution of a linear system of equations (which is full-sized and tends to be ill conditioned for an increasing number of nodal points). However, it can be shown that (outer harmonics) exact approximation of bandlimited potentials can be used very efficiently (without *a priori* solving any linear system) on equiangular grids (cf. [23, 39]).

3. Another method for the approximate evaluation of $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$-convolutions, which includes the exact approximation of bandlimited functions as a special case, is *harmonic spline exact best approximation* (cf. [12, 14, 19]). It can be applied appropriately for modeling the medium to short wavelength parts of a signal.
4. The *low discrepancy method* (cf. [21, 48]) represents an adequate tool if a great number of data is available, so that the solution of linear equations should be avoided. Sufficient accuracy can be guaranteed only if a high number of equidistributed data points are available. Thus it is of advantage for integrands of high complexity (e.g., short wavelength parts of a signal).

In what follows, it will be shown that both discretization techniques, i.e., outer harmonic and spline exact integration, lead to pyramid schemata adapted to the space localization properties of the potential we are interested in. To be more specific, the bandlimited variant of fast wavelet computation (based on the Shannon kernel and its modifications) can be based on outer harmonic exact formulae for the evaluation of $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$-inner products. It is proposed for the application to moderate phenomena of space localization (i.e., low-to-medium wavelength approximation) so that one can work with smaller data sets (cf. [31, 32]). In fast computation by bandlimited wavelets the number of wavelet coefficients is reduced, since they contain information of a more extended area. In addition, a certain spectral band is expressible exactly in terms of wavelets because of their bandlimited character. The non-bandlimited variant of fast wavelet evaluation (using non-bandlimited kernels such as Tikhonov, rational, exponential, and "locally supported" kernels (cf. [29, 31]) is meant for the application to seriously space localizing potentials (i.e., short wavelength approximation). In consequence, huge data sets can be handled since only a small subset of the data is needed for the purpose of numerical evaluation. On the other hand, a large number of wavelet coefficients is needed, since they only give local information related to a small area. Again, we are confronted with the drawback that large linear systems must be solved in an a priori step to obtain the weights in (spline exact) best approximation formulae. In the non-bandlimited case, however, panel clustering or sparse matrix techniques (cf. [23]) are efficiently applicable because of the strong space localization properties of the non-bandlimited kernel functions.

Next, the use of outer harmonic exact approximation will be discussed in more detail following [19]. A constructive version of the Runge–Walsh theorem will be developed in terms of bandlimited wavelets. The advantage is that when using bandlimited wavelets, we do not need the wavelet transform at all positions. It suffices to know a finite set of linear functionals for each scale J to evaluate the wavelet transform exactly. In conclusion, each J-level wavelet approximation $\Phi_J^{(2)} * F$ can be expressed exactly as a finite sum.

Our concept using bandlimited wavelets is presented under the assumption that the families $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$, $\{\Psi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$, and $\{\tilde{\Psi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ consist of bandlimited kernels such that

$$\varphi_j(n) \neq 0, \quad n = 0,\ldots,2^j-1 \tag{8.2}$$

and

$$\varphi_j(n) = 0, \quad n = 2^j, 2^j+1, \ldots. \tag{8.3}$$

In the following we use the notation

$$\mathcal{H}_{p,\ldots,q}(\overline{\Omega_\sigma^{\mathrm{ext}}}) = Harm_{p,\ldots,q}(\overline{\Omega_\sigma^{\mathrm{ext}}}). \tag{8.4}$$

Consequently, we have

$$\Phi_j(x,\cdot) \in \mathcal{H}_{0,\ldots,2^j-1}(\overline{\Omega_\sigma^{\mathrm{ext}}}), \tag{8.5}$$

and

$$\Psi_j(x,\cdot), \tilde{\Psi}_j(x,\cdot) \in \mathcal{H}_{0,\ldots,2^{j+1}-1}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \tag{8.6}$$

for all $x \in \overline{\Omega_\sigma^{\mathrm{ext}}}$. Thus the scale spaces and the detail spaces, respectively, fulfill the relations

$$\mathcal{V}_j = \mathcal{H}_{0,\ldots,2^j-1}(\overline{\Omega_\sigma^{\mathrm{ext}}}), \quad \mathcal{W}_j \subset \mathcal{H}_{0,\ldots,2^{j+1}-1}(\overline{\Omega_\sigma^{\mathrm{ext}}}). \tag{8.7}$$

Suppose now that there is known a set $\{v_1,\ldots,v_M\}$ of M values v_i, $i = 1,\ldots,M$, from a potential V (for example, the gravitational potential or the anomalous potential of the Earth) of class $Pot^{(0)}(\overline{\Sigma^{\mathrm{ext}}})$ corresponding to linear (observational) functionals $\mathcal{L}_1,\ldots,\mathcal{L}_M$. Then an extended version of Helly's theorem (cf. [76]) tells us that, corresponding to the potential $V \in Pot^{(0)}(\overline{\Sigma^{\mathrm{ext}}})$, there exists a member F (i.e., a Runge–Walsh approximation of the (anomalous) potential) of class $\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ such that $F|_{\overline{\Sigma^{\mathrm{ext}}}}$ is in an $(\varepsilon/2)$-neighbourhood to V (understood in uniform topology on $\overline{\Sigma^{\mathrm{ext}}}$) and $\mathcal{L}_iF = v_i$, $i = 1,\ldots,M$ (note that we may write more accurately $F_{0,\ldots,\infty}$ instead of F to indicate that all $Harm_n$-spaces generally contribute to the "nature" of F when the Earth's gravitational potential is required). Moreover, there exists an element $F_{0,\ldots,m}$ (i.e., a bandlimited approximation to the Runge–Walsh approximation) of class $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ such that the restriction $F_{0,\ldots,m}|_{\overline{\Sigma^{\mathrm{ext}}}}$ may be considered to be in $(\varepsilon/2)$-accuracy to $F|_{\overline{\Sigma^{\mathrm{ext}}}}$ uniformly on $\overline{\Sigma^{\mathrm{ext}}}$ and, in addition, $\mathcal{L}_iF_{0,\ldots,m} = \mathcal{L}_iF = v_i$, $i = 1,\ldots,M$. In other words, corresponding to a potential $V \in Pot^{(0)}(\overline{\Sigma^{\mathrm{ext}}})$ there exists on $\overline{\Sigma^{\mathrm{ext}}}$ a bandlimited potential in $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$, (namely, $F_{0,\ldots,m} \in \mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$) consistent with the original data in ε-accuracy (i.e., $v_i = \mathcal{L}_iF = \mathcal{L}_iF_{0,\ldots,m}$, $i = 1,\ldots,M$). This is the reason why we are interested in wavelet approximations of potentials $F_{0,\ldots,m}$ of class $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ uniformly on $\overline{\Sigma^{\mathrm{ext}}}$ from a finite set of functional values (note that, for the Earth's anomalous potential, the approximation consistent with the original data may be found in the class $\mathcal{H}_{2,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ which is a subspace of $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$).

Our strategy is to represent $F_{0,\dots,m} \in \mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ by a J-level approximation $\Phi_J^{(2)} * F_{0,\dots,m}$ with J chosen in such a way that $2^{J+1}-1 \geq m$ (note that $F_{0,\dots,m}$ coincides with $\Phi_{J+1}^{(2)} * F_{0,\dots,m}$ uniformly on $\overline{\Sigma^{\mathrm{ext}}}$ in the case of Shannon wavelets). We want to express the J-level wavelet approximation $\Phi_{J+1}^{(2)} * F_{0,\dots,m}$ of the potential $F_{0,\dots,m}$(with $2^{J+1}-1 \geq m$) *exactly* only by use of the M values $v_1,\dots,v_M$ corresponding to the linear functionals $\mathcal{L}_1,\dots,\mathcal{L}_M$.

First, our purpose is to apply outer harmonic based approximation formulae. To this end, we introduce fundamental systems of bounded linear functionals and derive some approximation formulae. Consider the matrix

$$\mathbf{m} = \begin{pmatrix} \mathcal{L}_1 H_{0,1}(\sigma;\cdot) & \dots & \mathcal{L}_N H_{0,1}(\sigma;\cdot) \\ \vdots & & \vdots \\ \mathcal{L}_1 H_{m,2m+1}(\sigma;\cdot) & \dots & \mathcal{L}_N H_{m,2m+1}(\sigma;\cdot) \end{pmatrix} \tag{8.8}$$

associated to a system of $N \geq \sum_{n=0}^{m}(2n+1) = (m+1)^2$ (linearly independent) bounded linear functionals $\mathcal{L}_1,\dots,\mathcal{L}_N$ on $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$. According to well-known arguments of approximation theory, the matrix (8.8) is not of maximal rank for all systems $\{\mathcal{L}_1,\dots,\mathcal{L}_N\}$, $N \geq (m+1)^2$. However, it is clear from a well-known construction principle (see, for example, [19]) that there exist systems $\{\mathcal{L}_1,\dots,\mathcal{L}_N\}$ possessing a non-degenerate matrix (8.8).

Definition 8.1. A system $\{\mathcal{L}_1,\dots,\mathcal{L}_N\}$ of $N \geq (m+1)^2$ bounded linear functionals on $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ is called an $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-*fundamental system*, if the conditions $F \in \mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ and $\mathcal{L}_i F = 0$, $i = 1,\dots,N$, imply $F = 0$.

From Definition 8.1 it is clear that the matrix (8.8) is of maximal rank $(m+1)^2$ if and only if $\{\mathcal{L}_1,\dots,\mathcal{L}_N\}$ is an $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-fundamental system. Moreover, it should be noted that the addition theorem of outer harmonics gives us

$$\mathbf{m}^T\mathbf{m} = \begin{pmatrix} \mathcal{L}_1\mathcal{L}_1 K_{\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot) & \dots & \mathcal{L}_1\mathcal{L}_N K_{\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot) \\ \vdots & & \vdots \\ \mathcal{L}_N\mathcal{L}_1 K_{\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot) & \dots & \mathcal{L}_N\mathcal{L}_N K_{\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot) \end{pmatrix}.$$

The Gram matrix $\mathbf{m}^T\mathbf{m}$ is regular if and only if the system $\{\mathcal{L}_1,\dots,\mathcal{L}_N\}$ is an $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-fundamental system. Moreover, it is clear that the property of $\{\mathcal{L}_1,\dots,\mathcal{L}_N\}$ of being an $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-fundamental system, is independent of the choice of the $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-orthonormal basis.

For later use we introduce the following definition.

Definition 8.2. Let Ξ be a regular surface with $\Xi \subset \overline{\Omega_\sigma^{\mathrm{ext}}}$.

Let $\{\mathcal{L}_1,\dots,\mathcal{L}_N\}$ be an $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-fundamental system of Dirichlet functionals $\mathcal{L}_1,\dots,\mathcal{L}_N$ on $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ (i.e., $\mathcal{L}_i F = F(y_i)$ for $y_i \in \Xi$, $i = 1,\dots,N$ and all $F \in \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$). Then the associated system $\{y_1,\dots,y_N\}$ is called an $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-*Dirichlet-fundamental system on* Ξ.

Let $\{\mathcal{L}_1, \dots, \mathcal{L}_N\}$ be an $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-fundamental system of Neumann functionals $\mathcal{L}_i$, $i = 1, \dots, N$ (i.e., $\mathcal{L}_i F = (\lambda \cdot (\nabla F))(y_i)$ for $y_i \in \Xi$ and all $F \in \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$) with $\lambda : \Xi \to \mathbb{R}^3$ being a unit vector field satisfying $\inf_{x \in \Xi} \nu(x) \cdot \lambda(x) > 0$ (where ν denotes the outer normal). Then the system $\{y_1, \dots, y_N\}$ is called an $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-*Neumann-fundamental system on* Ξ (*relative to* λ).

Let $\{\mathcal{L}_1, \dots, \mathcal{L}_N\}$ be an $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-fundamental system in the sense of Definition 8.1. Suppose that F is a potential of class $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$. Furthermore, let P be an element of $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ with the representation

$$P = \sum_{n=0}^{m} \sum_{l=1}^{2n+1} P^\wedge(n,l) H_{n,l}(\sigma; \cdot).$$

Then, for all solutions $a \in \mathbb{R}^N$, $a = (a_1, \dots, a_N)^T$, of the linear system

$$\sum_{k=1}^{N} a_k \mathcal{L}_k H_{n,l}(\sigma; \cdot) = P^\wedge(n,l), \tag{8.9}$$

$n = 0, \dots, m$; $l = 1, \dots, 2n+1$, we find

$$P = \sum_{k=1}^{N} a_k \sum_{n=0}^{m} \sum_{l=1}^{2n+1} \left(\mathcal{L}_k H_{n,l}(\sigma; \cdot)\right) H_{n,l}(\sigma; \cdot). \tag{8.10}$$

Observing this fact we get the following theorem.

Theorem 8.3. *Let* $\{\mathcal{L}_1, \dots, \mathcal{L}_N\}$ *be an* $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$*-fundamental system of bounded linear functionals on* $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$*. Then the identity*

$$F * P = \sum_{k=1}^{N} a_k \mathcal{L}_k F - \sum_{k=1}^{N} a_k \mathcal{L}_k K_{\mathcal{H}_{m+1,\dots,\infty}(\overline{\Omega_\sigma^{\mathrm{ext}}})} * F$$

holds for all $F \in \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ *and all solutions* $a \in \mathbb{R}^N$, $a = (a_1, \dots, a_N)^T$*, satisfying the linear system* (8.9)*.*

By virtue of the Cauchy–Schwarz inequality it follows from Theorem 8.3 that the estimate

$$\begin{aligned} &\left| F * P - \sum_{k=1}^{N} a_k \mathcal{L}_k F \right| \\ &\quad \le \left(\sum_{k=1}^{N} \sum_{s=1}^{N} a_k a_s \mathcal{L}_k \mathcal{L}_s K_{\mathcal{H}_{m+1,\dots,\infty}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot) \right)^{1/2} \|F\|_{\mathcal{H}_{m+1,\dots,\infty}(\overline{\Omega_\sigma^{\mathrm{ext}}})} \end{aligned} \tag{8.11}$$

holds for all $F \in \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ and all solutions $a \in \mathbb{R}^N$, $a = (a_1, \dots, a_N)^T$, satisfying (8.9). In particular, we have for $F \in \mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$

$$F * P = \sum_{k=1}^{N} a_k \mathcal{L}_k F, \tag{8.12}$$

since $\|F\|_{\mathcal{H}_{m+1,\ldots,\infty}(\overline{\Omega_\sigma^{\mathrm{ext}}})} = 0$. But this shows us that

$$K_{\mathcal{H}_{0,\ldots,m}} * P = \sum_{k=1}^{N} a_k \mathcal{L}_k K_{\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot) \tag{8.13}$$

holds for all $a \in \mathbb{R}^n$, $a = (a_1, \ldots, a_N)^T$, satisfying the linear equations (8.9).

Next we adopt a famous criterion due to [73] from Theorem 8.3.

Lemma 8.4. *The following statements are equivalent:*

(i) $\lim\limits_{N\to\infty} \sum\limits_{k=1}^{N} a_k \mathcal{L}_k H_{n,l}(\sigma;\cdot) = 0, \quad n = m+1, m+2, \ldots; \, l = 1, \ldots, 2n+1,$

(ii) $F * P = \lim\limits_{N\to\infty} \sum\limits_{k=1}^{N} a_k \mathcal{L}_k F, \quad F \in \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}}).$

As shown in [19], the definition of *fundamental systems* and *approximation formulae* leads us to exact approximation rules on $\mathcal{H}_{0,\ldots,2m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-spaces. To this end we have to summarize shortly some results concerning interpolation by outer harmonics (see [19]).

We start mentioning the *Shannon sampling* theorem for the finite-dimensional space $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

Lemma 8.5. *Let F be in $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$. Assume that $\{\mathcal{L}_1, \ldots, \mathcal{L}_N\}$ forms an $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-fundamental system. Then F can be reconstructed from its samples at the bounded linear functionals $\mathcal{L}_1, \ldots, \mathcal{L}_N$ by the following interpolation formula*

$$F(x) = \sum_{k=1}^{N} (\mathcal{L}_k F) P_k^N(x), \quad x \in \overline{\Omega_\sigma^{\mathrm{ext}}},$$

where the "Lagrangians" $P_k^N \in \mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $k = 1, \ldots, N$, are given by

$$P_k^N = \sum_{l=1}^{N} w_{l,k}^N \mathcal{L}_l K_{\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot)$$

and the coefficients $w_{l,k}^N$ have to satisfy the linear equations

$$\sum_{l=1}^{N} w_{l,k}^N \mathcal{L}_i \mathcal{L}_l K_{\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot) = \delta_{i,k},$$

$i, k = 1, \ldots, N.$

Next we come to some aspects on numerical integration on the sphere. Theorem 8.3 allows as special cases the following variants.

Lemma 8.6 (Koksma–Hlawka formula of approximation order 0). *Let F be of class $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ with $\{A_n\}$ being summable in the sense of Definition* 3.3. *Assume that*

$\{y_1^N, \ldots, y_N^N\}$ is a subset of points on Ω_σ. Then the integral formula

$$\begin{aligned} &\frac{1}{4\pi\sigma^2}\int_{\Omega_\sigma} F(y)\, d\omega(y) \\ &\quad = \sum_{k=1}^{N} w_k^N F(y_k^N) - \sum_{k=1}^{N} w_k^N \left(K_{\mathcal{H}_{1,\ldots,\infty}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot\,, y_k^N), F\right)_{\mathcal{H}_{1,\ldots,\infty}(\overline{\Omega_\sigma^{\mathrm{ext}}})} \end{aligned} \tag{8.14}$$

holds for all $w^N = (w_1^N, \ldots, w_N^N)^T$ with $\sum_{k=1}^N w_k^N = 1$ (e.g., $w_k^N = 1/N$).

Lemma 8.7 (Koksma–Hlawka formula of approximation order m). *Let F be a member of class $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ with $\{A_n\}$ being summable in the sense of Definition* 3.3*. Assume that $\{y_1^N, \ldots, y_N^N\} \subset \overline{\Omega_\sigma^{\mathrm{ext}}}$ is an $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-Dirichlet-fundamental system, i.e., a pointset on the sphere Ω_σ such that*

$$\begin{pmatrix} K_{\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(y_1^N, y_1^N) & \ldots & K_{\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(y_1^N, y_N^N) \\ \vdots & & \vdots \\ K_{\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(y_N^N, y_1^N) & \ldots & K_{\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(y_N^N, y_N^N) \end{pmatrix}$$

is regular. Then the integral formula

$$\begin{aligned} &\frac{1}{4\pi\sigma^2}\int_{\Omega_\sigma} F(y)\, d\omega(y) \\ &\quad = \sum_{k=1}^{N} w_k^N F(y_k^N) - \sum_{k=1}^{N} w_k^N \left(K_{\mathcal{H}_{m+1,\ldots,\infty}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot\,, y_k^N), F\right)_{\mathcal{H}_{m+1,\ldots,\infty}(\overline{\Omega_\sigma^{\mathrm{ext}}})} \end{aligned} \tag{8.15}$$

holds for all $w^N = (w_1^N, \ldots, w_N^N)^T$, satisfying

$$\sum_{l=1}^{N} w_l^N = 1, \tag{8.16}$$

$$\sum_{l=1}^{N} w_l^N H_{n,k}(\sigma; y_l^N) = 0, \quad n = 1, \ldots, m,\ k = 1, \ldots, 2n+1. \tag{8.17}$$

Finally we are interested in an extension of the Koksma–Hlawka formula for spherical integrals (see Lemma 8.7) to $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-inner products. To this end we understand the summable sequence $\{A_n\}$ generating the reference space $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ to be the symbol of a pseudodifferential operator A with $AH_{n,k}(\sigma;\cdot) = A^\wedge(n)H_{n,k}(\sigma;\cdot) = A_n H_{n,k}(\sigma;\cdot)$ for all $n \in \mathbb{N}_0$; $k = 1, \ldots, 2n+1$. Then the framework of the space $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ tells us that

$$\begin{aligned} F * P &= \sum_{n=0}^{m}\sum_{k=1}^{2n+1} F^\wedge(n,k)P^\wedge(n,k) \\ &= \int_{\Omega_\sigma} (AF)(y)(AP)(y)\, d\omega(y) \end{aligned} \tag{8.18}$$

holds for all $F \in \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ and $P \in \mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$. Moreover, we see that

$$\int_{\Omega_\sigma} (AF)(y)(AP)(y)\, d\omega(y) = \int_{\Omega_\sigma} F(y)(A^2P)(y)\, d\omega(y).$$

Clearly, A^2P is a member of class $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ (as defined in the foregoing). Assuming F to be of class $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $F(A^2P)|_{\overline{\Omega_\sigma^{\mathrm{ext}}}}$ is the product of two elements of class $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$, hence, $F(A^2P)|_{\overline{\Omega_\sigma^{\mathrm{ext}}}}$ is a member of class $\mathcal{H}_{0,\ldots,2m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$. In connection with Lemma 8.7 this leads us to the following result.

Lemma 8.8. *Let F and P be elements of class $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$.*

Assume that $\{y_1^N,\ldots,y_N^N\} \subset \Omega_\sigma$ is an $\mathcal{H}_{0,\ldots,2m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-Dirichlet-fundamental system on Ω_σ (with $N \geq (2m+1)^2$). Then the identity

$$F * P = \sum_{k=1}^{N} w_k^N F(y_k^N)(A^2P)(y_k^N)$$

holds for all $w^N = (w_1^N,\ldots,w_N^N)^T$ satisfying

$$\sum_{l=1}^{N} w_l^N = 1, \tag{8.19}$$

$$\sum_{l=1}^{N} w_l^N H_{n,k}(\sigma; y_l^N) = 0, \quad n = 1,\ldots,2m;\ k = 1,\ldots,2n+1. \tag{8.20}$$

In particular, we have

$$K_{\mathcal{H}_{0,\ldots,m}} * F = \sum_{k=1}^{N} w_k^N F(y_k^N) K_{Harm_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot, y_k^N).$$

Lemma 8.8 is an essential tool for the development of "tree algorithms" (pyramid schemata) in bandlimited harmonic wavelet theory.

Lemma 8.9. *Let the system $\{y_1^M,\ldots,y_M^M\} \subset \Omega_\sigma$, $M = (2m+1)^2$, define an $\mathcal{H}_{0,\ldots,2m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-Dirichlet-fundamental system. Furthermore, suppose that $P_{0,\ldots,m}$, $Q_{0,\ldots,m}$, respectively, are elements of class $\mathcal{H}_{0,\ldots,m}(\Omega_\sigma)$. Then the identity*

$$P_{0,\ldots,m} * Q_{0,\ldots,m} = \sum_{n=1}^{M} b_n^M P_{0,\ldots,m}(y_n^M)(A^2Q)_{0,\ldots,m}(y_n^M) \tag{8.21}$$

holds for all weights $b_1^M,\ldots,b_M^M$ satisfying

$$\begin{aligned}\sum_{r=1}^{M} b_r^M K_{\mathcal{H}_{0,\ldots,2m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(y_i^M, y_r^M)&\\ = \int_{\Omega_\sigma} K_{\mathcal{H}_{0,\ldots,2m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(y_i^M, x)\, d\omega(x), &\quad i = 1,\ldots,M.\end{aligned} \tag{8.22}$$

Furthermore, we have the following results.

Lemma 8.10. *Let* $\{\mathcal{L}_1^M, \ldots, \mathcal{L}_M^M\}$, $M = (m+1)^2$, *be an* $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$*-fundamental system, and suppose that* $P_{0,\ldots,m}$ *and* $Q_{0,\ldots,m}$ *are members of* $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$*. Then the identity*

$$P_{0,\ldots,m} * Q_{0,\ldots,m} = \sum_{n=0}^{m} \sum_{k=1}^{2n+1} \sum_{r=1}^{M} d_r^{n,k} \left(Q_{0,\ldots,m} * H_{n,k}(\sigma;\cdot)\right) \mathcal{L}_r^M P_{0,\ldots,m} \tag{8.23}$$

holds for all weights $d_1^{n,k}, \ldots, d_M^{n,k}$; $n = 0, \ldots, m$; $k = 1, \ldots, 2n+1$, *satisfying the linear equations*

$$\sum_{r=1}^{M} d_r^{n,k} \mathcal{L}_r^M H_{l,i}(\sigma;\cdot) = \delta_{n,l}\delta_{k,i},$$

$l = 0, \ldots, m$; $i = 1, \ldots, 2l+1$.

In order to reduce the number of weights in our approximation rules we formulate the following lemma.

Lemma 8.11. *Under the assumptions of Lemma* 8.10, *the formula*

$$Q_{0,\ldots,m} * P_{0,\ldots,m} = \sum_{r=1}^{M} d_r^M \mathcal{L}_r^M P_{0,\ldots,m} \tag{8.24}$$

holds for all weights $d_1^M, \ldots, d_M^M$ *satisfying the linear equations*

$$\begin{aligned}
&\sum_{r=1}^{M} d_r^M \mathcal{L}_i^M \mathcal{L}_r^M K_{\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot) \\
&\quad = \sum_{n=0}^{m} \sum_{k=1}^{2n+1} \left(\mathcal{L}_i^M H_{n,k}(\sigma;\cdot)\right) Q_{0,\ldots,m} * H_{n,k}(\sigma;\cdot) = \mathcal{L}_i^M Q_{0,\ldots,m},
\end{aligned} \tag{8.25}$$

$i = 1, \ldots, M$.

It should be mentioned that on the one hand the number of integration weights is reduced, but on the other hand the integration weights depend on $Q_{0,\ldots,m}$. Other variants of discretization rules have been presented by W. Freeden and W. Schneider [30] which allow different aspects of approximation. In this work, however, we restrict ourselves to the above results (more explicitly, Lemma 8.9, Lemma 8.10, Lemma 8.11) based on linear systems of $\mathcal{O}(M)$-dimension.

In what follows the Runge concept is of basic interest. Once again, it tells us that to any potential $V \in Pot^{(0)}(\overline{\Sigma^{\mathrm{ext}}})$ (for example, the Earth's gravitational potential) there exists a function F (namely, a Runge–Walsh approximation) harmonic in $\Omega_\sigma^{\mathrm{ext}}$ and being regular at infinity in the sense that the absolute error becomes arbitrarily small on the whole space $\overline{\Sigma^{\mathrm{ext}}}$. In this formulation as we already mentioned, the Runge–Walsh theorem is a pure existence theorem. It guarantees only the existence of an approximating potential and does not provide a method to find it. The theorem merely describes the theoretical background of approximating a potential by another one defined on a larger harmonicity domain. The results developed now, however, enable us to derive a constructive version of the Runge–Walsh theorem by means of a J-level wavelet approximation when the potential F we are looking for is assumed to be a member of class $\mathcal{H}(\overline{\Omega_\sigma^{\rm ext}})|_{\overline{\Sigma^{\rm ext}}}$ (note that $\mathcal{H}(\overline{\Omega_\sigma^{\rm ext}})|_{\overline{\Sigma^{\rm ext}}}$is a uniformly dense subset of $Pot^{(0)}(\overline{\Sigma^{\rm ext}})$). Essential tools of our considerations are the approximation formulae formulated above.

Theorem 8.12. *Let $\{\mathcal{L}_1^M,\ldots,\mathcal{L}_M^M\}$, $M=(m+1)^2$, be an $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\rm ext}})$-fundamental system. Furthermore, suppose that $\{y_1^{M_j},\ldots,y_{M_j}^{M_j}\}\subset\Omega_\sigma$, $M_j=(2m_j+1)^2$, define $\mathcal{H}_{0,\ldots,2m_j}(\overline{\Omega_\sigma^{\rm ext}})$-Dirichlet-fundamental systems for $j=0,\ldots,J$. Moreover, assume that from a potential $F_{0,\ldots,m}\in Harm_{0,\ldots,m}(\overline{\Omega_\sigma^{\rm ext}})$ there are known the data $\mathcal{L}_i^M F_{0,\ldots,m}=v_i$, $i=1,\ldots,M$. Then, under our assumption of bandlimited wavelets, the fully discrete J-level wavelet approximation of $F_{0,\ldots,m}$ reads as follows:*

(α)
$$\begin{aligned}\Phi_J^{(2)} * F_{0,\ldots,m}\\ &= \sum_{n=1}^{M_0} b_n^0 \sum_{k=0}^{m}\sum_{l=1}^{2k+1}\sum_{s=1}^{M} d_s^{k,l} A_k^2 \varphi_0(k) v_s H_{k,l}(\sigma;y_n^{M_0})\Phi_0(y_n^{M_0},\cdot)\\ &\quad+\sum_{j=0}^{J-1}\sum_{n=1}^{M_j} b_n^j \sum_{k=0}^{m}\sum_{l=1}^{2k+1}\sum_{s=1}^{M} d_s^{k,l} A_k^2 \psi_j(k) v_s H_{k,l}(\sigma;y_n^{M_j})\tilde{\Psi}_j(y_n^{M_j},\cdot), \qquad (8.26)\end{aligned}$$

where the weights $d_1^{k,l},\ldots,d_M^{k,l}$; $k=0,\ldots,m$; $l=1,\ldots,2k+1$, satisfy the linear equations

$$\sum_{s=1}^{M} d_s^{k,l}\mathcal{L}_s^M H_{n,i}(\sigma;\cdot)=\delta_{n,k}\delta_{i,l}, \qquad (8.27)$$

$n=0,\ldots,m$; $i=1,\ldots,2n+1$, and $b_1^j,\ldots,b_{M_j}^j$; $j=0,\ldots,J$, satisfy the linear equations

$$\sum_{n=1}^{M_j} b_n^j K_{\mathcal{H}_{0,\ldots,2m_j}(\overline{\Omega_\sigma^{\rm ext}})}(y_i^{M_j},y_n^{M_j}) = \int_{\Omega_\sigma} K_{\mathcal{H}_{0,\ldots,2m_j}(\overline{\Omega_\sigma^{\rm ext}})}(y_i^{M_j},x)\,d\omega(x), \qquad (8.28)$$

$i=1,\ldots,M_j$.

(β)
$$\Phi_J^{(2)} * F_{0,\ldots,m} = \sum_{n=1}^{M_0} b_n^0 \sum_{s=1}^{M}\tilde{d}_s^{0,n} v_s \Phi_0(y_n^{M_0},\cdot) + \sum_{j=0}^{J-1}\sum_{n=1}^{M_j} b_n^j\sum_{s=1}^{M} d_s^{j,n} v_s\tilde{\Psi}_j(y_n^{M_j},\cdot), \qquad (8.29)$$

where the weights $\tilde{d}_1^{0,n},\ldots,\tilde{d}_M^{0,n}$; $n=1,\ldots,M_0$, satisfy the linear equations

$$\sum_{s=1}^{M}\tilde{d}_s^{0,n}\mathcal{L}_i^M\mathcal{L}_s^M K_{Harm_{0,\ldots,m}(\overline{\Omega_\sigma^{\rm ext}})}(\cdot,\cdot) = \mathcal{L}_i^M(A^2\Phi_0)(y_n^{M_0},\cdot), \qquad (8.30)$$

$i = 1, \ldots, M$, and the weights $d_1^{j,n}, \ldots, d_M^{j,n}$; $j = 0, \ldots, J$; $n = 1, \ldots, M_j$, satisfy

$$\sum_{s=1}^{M} d_s^{j,n} \mathcal{L}_i^M \mathcal{L}_s^M K_{Harm_{0,\ldots,m(\overline{\Omega_\sigma^{\text{ext}}})}}(\cdot,\cdot) = \mathcal{L}_i^M (A^2 \Psi_j)(y_n^{M_j}, \cdot), \tag{8.31}$$

$i = 1, \ldots M$, and the coefficients $b_1^j, \ldots, b_{M_j}^j$; $j = 0, \ldots, J$ satisfy the linear system (8.28).

It should be remarked that a great number of linear systems must be solved in an a priori step. But if we look carefully we realize that we are always confronted with the same coefficient matrix. Having inverted the coefficient matrix once, all weights for numerical integration can be obtained by a matrix-vector multiplication and stored elsewhere (in an *a priori* step for computation). In addition, it should be mentioned that the solution of the linear systems determining the weights of the reconstruction step (8.28) can be avoided completely if we place the knots for numerical integration of the wavelet coefficients for each detail step $j = 0, \ldots, J-1$ on a special longitude-latitude grid on the sphere Ω_σ. The corresponding set of integration weights for reconstruction purposes are explicitly available without solving any linear system (for more details concerning numerical integration the reader is referred, e.g., to a paper due to Driscoll Healy [8]).

Until now the linear (observational) functionals have not been specified in more detail in our bandlimited wavelet approach presented above. In fact, the different types of linear functionals enable us to develop three important variants of wavelet approximation in the reality of gravitational potential determination:

(1) *Terrestrial-only Multiscale Approximation.* The linear functionals are understood to represent gravity observations (function values and/or derivatives) related to locations on the Earth's surface. If the data material is homogeneous, i.e., the linear functionals are all of the same type, terrestrial-only approximation reduces to the wavelet solution of a boundary-value problem of potential theory from discretely given data.
(2) *Spaceborne-only Multiscale Approximation.* In this case the linear functionals are understood to represent data measured by spacecraft in locations of $\overline{\Omega_\gamma^{\text{ext}}}$. As result we get a spaceborne-only approximation.

In practice, however, we are confronted with the situation that terrestrial, airborne as well as spaceborne data are available in gravitational potential determination (cf. [1, 2, 19, 22, 32, 45, 46, 50, 60, 62, 63, 66, 71, 75]). As a matter of fact, there are some areas on the continents (for example, some parts of Australia, Europe, and North-America), where the gravity field has been surveyed in much detail. Thus it is reasonable that such areas may be used for the verification or the calibration of the results obtained from spaceborne data.

(3) *Combined Multiscale Approximation.* Linear functionals representing terrestrial, airborne, and spaceborne observations are taken into account, i.e., numerical computation is required for a heterogeneous data set.

8.1. Runge–Walsh wavelet approximation of classical boundary value problems corresponding to regular surfaces

The wavelet representations (Theorem 8.12) of a bandlimited potential from a given finite set of linear functionals admit a variety of applications. The list includes the following examples of classical boundary value problems:

(i) *Dirichlet Problem.* First we are interested in the wavelet approximation $\Phi_J^{(2)} * F_{0,\ldots,m}$ of the solution of the exterior Dirichlet problem

$$F_{0,\ldots,m}|_{\overline{\Sigma^{\text{ext}}}} \in Harm_{0,\ldots,m}(\overline{\Sigma^{\text{ext}}}), \quad F_{0,\ldots,m}|_{\Sigma} = G_{0,\ldots,m}.$$

under the knowledge of the $M = (m+1)^2$ boundary data

$$v_i = \mathcal{L}_i^M F_{0,\ldots,m} = F_{0,\ldots,m}(x_i^M) = G_{0,\ldots,m}(x_i^M), \quad i = 1, \ldots, M.$$

Theorem 8.13. *Under the assumptions of Theorem* 8.12 *the fully discrete J-level wavelet approximation of the solution of the exterior Dirichlet problem $F_{0,\ldots,m}|_{\overline{\Sigma^{\text{ext}}}} \in Harm_{0,\ldots,m}(\overline{\Sigma^{\text{ext}}})$, $(F_{0,\ldots,m})|_{\Sigma} = G_{0,\ldots,m}$ reads as follows:*

(α)
$$\begin{aligned}\Phi_J^{(2)} * F_{0,\ldots,m} &\\ = \sum_{n=1}^{M_0} b_n^0 \sum_{k=0}^{m} \sum_{l=1}^{2k+1} \sum_{s=1}^{M} d_s^{k,l} A_k^2 \varphi_0(k) G_{0,\ldots,m}(x_s^M) H_{k,l}(\sigma; y_n^{M_0}) \Phi_0(y_n^{M_0}, \cdot) &\\ + \sum_{j=0}^{J-1} \sum_{n=1}^{M_j} b_n^j \sum_{k=0}^{m} \sum_{l=1}^{2k+1} \sum_{s=1}^{M} d_s^{k,l} A_k^2 \psi_j(k) G_{0,\ldots,m}(x_s^M) H_{k,l}(\sigma; y_n^{M_j}) \tilde{\Psi}_j(y_n^{M_j}; \cdot)&\end{aligned} \tag{8.32}$$

(β)
$$\begin{aligned}\Phi_J^{(2)} * F_{0,\ldots,m} = &\sum_{m=1}^{M_0} b_n^0 \sum_{s=1}^{M} \tilde{d}_s^{0,n} G_{0,\ldots,m}(x_s^M) \Phi_0(y_n^{M_0}, \cdot) \\ &+ \sum_{j=0}^{J-1} \sum_{n=1}^{M_j} b_n^j \sum_{s=1}^{M} d_s^{j,n} G_{0,\ldots,m}(x_s^M) \tilde{\Psi}_j(y_n^{M_j}, \cdot).\end{aligned} \tag{8.33}$$

The formulae (α), (β) of Theorem 8.13 are especially valid on the regular (Earth's) surface Σ, i.e., we automatically obtain by $\Phi_J^{(2)} * F_{0,\ldots,m}|_{\Sigma}$ a J-level wavelet approximation of the "boundary function" $F_{0,\ldots,m}|_{\Sigma} = G_{0,\ldots,m}$ (by applying Shannon wavelets we even know that $\Phi_J^{(2)} * F_{0,\ldots,m} = F_{0,\ldots,m}$). In other words, a wavelet representation of a (bandlimited) function on regular surfaces has been found from a discrete data set of function values.

By treating non-bandlimited potentials $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, $s > 1$, the developed integration formulae are only valid in approximate sense. To be more concrete, if $\Phi_J^{(2)} * F$ denotes the J-level wavelet approximation we actually calculate an approximation $\Phi_J^{(2)} * F_{0,\ldots,m}$ by performing the numerical integration methods in (α), (β) of Theorem 8.13. Since this approximation also is harmonic in Σ^{ext} the biggest absolute error between $\Phi_J^{(2)} * F$ and its numerical approximation $\Phi_J^{(2)} * F_{0,\ldots,m}$ is attained at the boundary Σ. Thus, the numerical error can be estimated by the use of the following theorem (cf. [29, 30]).

Theorem 8.14. *Let* F *satisfy* $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}}), F|_\Sigma = G$, $s > 1$. *Furthermore, assume that* $X_M^\Sigma = \{x_1^M, \ldots, x_M^M\} \subset \Sigma$, $M = (m+1)^2$, *is an* $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$*-Dirichlet-fundamental system on* Σ. *Then, for any* $Q \in \mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$, *we have*

$$\left| \int_{\Omega_\sigma} F(x)Q(x)\ d\omega(x) - \sum_{r=1}^{M} d_r G(x_r^M) \right| \leq \frac{C}{m^{s-1}} \left(\sum_{r=1}^{M} |d_r^M| \right) \|F\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})}, \tag{8.34}$$

where C *is a constant depending only on* s *and* $d_1^M, \ldots, d_M^M$ *are the weights of the integration rule.*

(ii) *Neumann Problem.* Now we are interested in the wavelet approximation $\Phi_J^{(2)} * F_{0,\ldots,m}$ of the solution of the oblique Neumann problem

$$F_{0,\ldots,m}|_{\overline{\Sigma^{\mathrm{ext}}}} \in Harm_{0,\ldots,m}(\overline{\Sigma^{\mathrm{ext}}}), \qquad \frac{\partial F_{0,\ldots,m}}{\partial \lambda} = G_{0,\ldots,m},$$

under the knowledge of the $M = (m+1)^2$ boundary data

$$v_i = \mathcal{L}_i^M F_{0,\ldots,m} = \frac{\partial F_{0,\ldots,m}}{\partial \lambda}(x_i^M) = G_{0,\ldots,m}(x_i^M), \quad i = 1, \ldots, M,$$

where $\lambda : \Sigma \to \mathbb{R}^3$ is a $C^{[1,\rho)}$-unit vector field (such that $0 < \rho < 1$ for $\lambda \neq \nu$ and $\rho = 0$ for $\lambda = \nu$) forming an angle with the outer normal ν satisfying

$$\inf_{x \in \Sigma} \nu(x) \cdot \lambda(x) > 0 \tag{8.35}$$

at any point of Σ.

Note that the boundedness of the linear functionals of the oblique derivative on Σ follows from well-known arguments (cf. [16, 18, 20]).

For the decomposition step we need in contrast to the Dirichlet problem an integration method in terms of oblique derivatives on Σ. From our results we obtain a fully discrete wavelet approximation for the solution of the exterior Neumann problem.

Theorem 8.15. *Let* $X_M^\Sigma = \{x_1^M, \ldots, x_M^M\} \subset \Sigma$, $M = (m+1)^2$, *be an* $\mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$*-Neumann-fundamental system on* Σ. *Furthermore, let* $X_{M_j} = \{y_1^{M_j}, \ldots, y_{M_j}^{M_j}\}$, $M_j = (2m_j+1)^2$, *be* $\mathcal{H}_{0,\ldots,2m_j}(\overline{\Omega_\sigma^{\mathrm{ext}}})$*-Neumann-fundamental systems on* Ω_σ *for* $j = 0, \ldots, J$. *Moreover, assume that from a function* $F_{0,\ldots,m} \in \mathcal{H}_{0,\ldots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ *there are known the oblique derivatives* $G_{0,\ldots,m} = (\partial F_{0,\ldots,m}/\partial\lambda)$ *at all points of* X_M^Σ. *Then, under our assumption of bandlimited wavelets, the fully discrete* J*-level wavelet approximation of the solution of the exterior Neumann problem* $F_{0,\ldots,m} \in$

$\mathcal{H}_{0,\dots,m}(\overline{\Sigma^{\mathrm{ext}}})$, $(\partial F_{0,\dots,m})/\partial\lambda = G_{0,\dots,m}$ *reads as follows:*

(α) $\Phi_J^{(2)} * F_{0,\dots,m}$

$$= \sum_{n=1}^{M_0} b_n^0 \sum_{k=0}^{m}\sum_{l=1}^{2k+1}\sum_{s=1}^{M} d_s^{k,l} A_k^2 \varphi_0(k) G_{0,\dots,m}(x_s^M) H_{k,l}(\sigma; y_n^{M_0})\Phi_0(y_n^{M_0},\cdot)$$
$$+ \sum_{j=0}^{J-1}\sum_{n=1}^{M_j} b_n^j \sum_{k=0}^{m}\sum_{l=1}^{2k+1}\sum_{s=1}^{M} d_s^{k,l} A_k^2 \psi_j(k) G_{0,\dots,m}(x_s^M) H_{k,l}(\sigma; y_n^{M_j})\tilde{\Psi}_j(y_n^{M_j},\cdot), \tag{8.36}$$

where the weights $d_1^{k,l},\dots,d_M^{k,l}$*;* $k=0,\dots,m$*;* $l=1,\dots,2k+1$ *have to satisfy the linear equations*

$$\sum_{s=1}^{M} d_s^{k,l} \frac{\partial H_{n,i}(\sigma; x_s^M)}{\partial\lambda} = \delta_{n,k}\delta_{i,l}, \quad n=0,\dots,\ m; i=1,\dots,2n+1,$$

and $b_1^j,\dots,b_M^j$*,* $j=0,\dots,J$ *must satisfy the linear equations* (8.28).

(β)

$$\Phi_J^{(2)} * F_{0,\dots,m}$$
$$= \sum_{n=1}^{M_0} b_n^0 \sum_{s=1}^{M} \tilde{d}_s^{0,n} G_{0,\dots,m}(x_s^M)\Phi_0(y_n^{M_0},\cdot)$$
$$+ \sum_{j=0}^{J-1}\sum_{n=1}^{M_j} b_n^j \sum_{s=1}^{M} d_s^{j,n} G_{0,\dots,m}(x_s^M)\tilde{\Psi}_j(y_n^{M_j},\cdot), \tag{8.37}$$

where the weights $\tilde{d}_1^{0,n},\dots,\tilde{d}_M^{0,n}$*;* $n=1,\dots,M_0$*, have to satisfy the linear equations*

$$\sum_{s=1}^{M} \tilde{d}_s^{0,n} \frac{\partial}{\partial\lambda_{y_i^M}}\frac{\partial}{\partial\lambda_{y_s^M}} K_{\mathcal{H}_{0,\dots,m}}(\overline{\Omega_\sigma^{\mathrm{ext}}})(\cdot,\cdot) = \frac{\partial}{\partial\lambda_{y_i^M}}(A^2\Phi_0)(y_n^{M_0},\cdot), \tag{8.38}$$

$i=1,\dots,M$*, and the weights* $d_1^{j,n},\dots,d_M^{j,n}$*;* $j=0,\dots,J$*;* $n=1,\dots,M_j$*, must satisfy*

$$\sum_{s=1}^{M} d_s^{j,n} \frac{\partial}{\partial\lambda_{y_i^M}}\frac{\partial}{\partial\lambda_{y_s^M}} K_{\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot) = \frac{\partial}{\partial\lambda_{y_i^M}}(A^2\Psi_j)(y_n^{M_j},\cdot), \tag{8.39}$$

$i=1,\dots,M$*, and* $b_1^j,\dots,b_{M_j}^j$*;* $j=0,\dots,J$*, satisfy the linear equations* (8.28).

The formulae (α), (β) of Theorem 8.15 are especially valid on Σ. Thus, we obtain by $\partial(\Phi_J^{(2)} * F_{0,\dots,m})/\partial\lambda$ a J-level wavelet approximation of $G_{0,\dots,m} = \partial F_{0,\dots,m}/\partial\lambda$.

In order to examine the error in the integration formulae when we turn over to non-bandlimited potentials we finally mention the following theorem.

Theorem 8.16. *Let F satisfy $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, $\frac{\partial F}{\partial \lambda} = G$, $s > 2$. Furthermore, let $X_M^\Sigma = \{x_1^M, \dots, x_M^M\} \subset \Sigma$, $M = (m+1)^2$, be an $\mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\text{ext}}})$-Neumann-fundamental system on Σ. Then, for any $Q \in \mathcal{H}_{0,\dots,m}(\overline{\Omega_\sigma^{\text{ext}}})$, we have*

$$\left| \int_{\Omega_\sigma} F(x)Q(x)\ d\omega(x) - \sum_{r=1}^{M} d_r^M G(x_r^M) \right| \leq \frac{C}{m^{s-2}} \left(\sum_{r=1}^{M} |d_r^M| \right) \|F\|_{\mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})}, \tag{8.40}$$

where C is a constant depending only on s and $d_1^M, \dots, d_M^M$ are the weights of the integration rule.

Hence, by treating non-bandlimited potentials $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\text{ext}}})$, $s > 2$, we obtain in similarity to the Dirichlet case a J-level wavelet approximation by performing the numerical rules as indicated by (α), (β) of Theorem 8.15, and the numerical errors can be estimated using Theorem 8.16.

Remark 8.17. The existence of all types of fundamental systems to be needed in our preceding approximation rules is guaranteed by a well-known induction procedure (as described, for example in [21, 24, 57]. Furthermore, more detailed remainder estimates for the integration formulae can be found in [28]).

8.2. Pyramid schemata based on outer harmonic exact approximation

Our purpose now is to use two variants of exact (outer harmonic) approximation to derive tree algorithms, i.e., pyramid schemata for fast evaluation of bandlimited potentials. Without loss of generality, we assume that $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$, $\{\Psi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$, and $\{\tilde{\Psi}_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$ are families of bandlimited kernels satisfying the conditions (8.2) and (8.3). Variant 1 is based on the ideas of Lemma 8.8 using evaluation (i.e. Dirichlet functionals) on a sphere, while Variant 2 is based on the Shannon sampling Theorem 8.5 in terms of linear functionals. Both variants are particularly suitable for application to medium wavelength parts of a signal (potential). As shown in [19], Variant 2 can be extended to non-bandlimited potentials. This variant is therefore also suitable for the transition from medium to short wavelength parts of a signal (potential).

Variant 1. The key ideas of our first discretization method using outer harmonic exact approximation formulae are based on the following observations:

(1) For some suitably large J, the scale space $\mathcal{V}_{J+1}(\overline{\Omega_\sigma^{\text{ext}}}) = \mathcal{H}_{0,\dots,2^{J+1}-1}(\overline{\Omega_\sigma^{\text{ext}}})$ is "sufficiently close" to $\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})$. Consequently, for each potential $F \in \mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})$, there exists a bandlimited potential of class $\mathcal{V}_{J+1}(\overline{\Omega_\sigma^{\text{ext}}})$ such that the error between F and $\Phi_{J+1}^{(2)} * F$ (understood in $\|\cdot\|_{\mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})}$-topology) is negligible. This is the reason why the input data $v_l^{N_J}$, $l = 1, \dots, N_J$, are assumed to be given from a potential of class $\mathcal{V}_{J+1}(\overline{\Omega_\sigma^{\text{ext}}})$ (for the remainder of this subsection).

(2) For $j = 0,\ldots,J$, the generating coefficients $b_l^{N_j}$ and nodal points $y_l^{N_j} \in \Omega_\sigma$ of the exact outer harmonic formulae of order $2^{j+2}-2(=2\cdot(2^{j+1}-1))$ (cf. Lemma 8.8) are determined such that

$$K_{\mathcal{H}_{0,\ldots,2^j-1}(\overline{\Omega_\sigma^{\text{ext}}})} * P = \sum_{l=1}^{N_j} b_l^{N_j} K_{Harm_{0,\ldots,2^j-1}(\overline{\Omega_\sigma^{\text{ext}}})}(\cdot, y_l^{N_j}) P(y_l^{N_j})$$

holds for all $P \in \mathcal{H}_{0,\ldots,2^j-1}(\overline{\Omega_\sigma^{\text{ext}}})$ with $N_j \geq ((2^{j+2}-2)+1)^2 = (2^{j+2}-1)^2$. The coefficients $b_l^{N_j}$ may be calculated from the linear equations

$$\begin{aligned}\sum_{l=1}^{N_j} b_l^{N_j} K_{\mathcal{H}_{0,\ldots,2^{j+2}-2}(\overline{\Omega_\sigma^{\text{ext}}})}(y_i^{N_j}, y_l^{N_j})& \\ = \frac{1}{4\pi\sigma^2}\int_{\Omega_\sigma} K_{\mathcal{H}_{0,\ldots,2^{j+2}-2}(\overline{\Omega_\sigma^{\text{ext}}})}(x, y_i^{N_j})\, d\omega(x),& \qquad (8.41)\end{aligned}$$

$i = 1,\ldots,N_j$, in an a priori step and stored elsewhere.

Our goal is to show that all convolutions occurring in the J-level wavelet approximation of a bandlimited potential (of order $2^{J+1}-1$) can be evaluated *exactly* by means of outer harmonic approximation formulae. As a matter of fact, what we realize is the following *pyramid scheme*: Starting from a sufficiently large J, there exist vectors $a^{N_j} \in \mathbb{R}^{N_j}$, $j = 0,\ldots,J$ (being, of course, dependent on the potential $F \in \mathcal{H}(\overline{\Omega_\sigma^{\text{ext}}})$ under consideration) such that the following statements hold true:

(i) For $j = 0,\ldots,J$, all wavelet coefficients can be calculated via the formulae

$$(WT)(F)(j;\cdot) = \sum\nolimits_{i=1}^{N_j} a_i^{N_j} \Psi_j(\cdot, y_i^{N_j}).$$

(ii) The vectors $a^j \in \mathbb{R}^{N_j}$ are obtainable from $a^{j+1} \in \mathbb{R}^{N_{j+1}}$ by recursion:

$$a_i^{N_j} = b_i^{N_j} \sum\nolimits_{l=1}^{N_{j+1}} a_l^{N_{j+1}} K_{Harm_{0,\ldots,2^{j+1}-1}(\overline{\Omega_\sigma^{\text{ext}}})}(y_i^{N_j}, y_l^{N_{j+1}}),$$

$i = 1,\ldots,N_j$.

(iii) The vectors satisfy, in addition, the identities

$$\Phi_{j+1}^{(2)} * F = \sum_{i=1}^{N_j} a_i^{N_j} \Phi_{j+1}^{(2)}(\cdot, y_i^{N_j})$$

and

$$(\tilde{\Psi}_j * \Psi_j) * F = \sum_{i=1}^{N_j} a_i^{N_j} (\tilde{\Psi}_j * \Psi_j)(\cdot, y_i^{N_j}).$$

Our considerations are divided into two parts, viz. the initial step concerning the scale level J and the pyramid step establishing the recursion relation.

The Initial Step. Observing the exact (outer harmonic) formulae we obtain from Lemma 8.8 for all potentials $F \in \mathcal{V}_{J+1}(\overline{\Omega_\sigma^{\text{ext}}}) = \mathcal{H}_{0,\ldots,2^{J+1}-1}(\overline{\Omega_\sigma^{\text{ext}}})$

$$K_{\mathcal{H}_{0,\ldots,2^{J+1}-1}(\overline{\Omega_\sigma^{\text{ext}}})} * F = \sum_{l=1}^{N_J} b_l^{N_J} F(y_l^{N_J}) K_{Harm_{0,\ldots,2^{J+1}-1}(\overline{\Omega_\sigma^{\text{ext}}})}(\cdot, y_l^{N_J}).$$

It follows that $a^{N_J} \in \mathbb{R}^{N_J}$, $a^{N_J} = (a_1^{N_J}, \ldots, a_{N_J}^{N_J})^T$, given by

$$a_l^{N_J} = b_l^{N_J} F(y_l^{N_J}) = b_l^{N_J} v_l^{N_J}, \quad l = 1, \ldots, N_J, \tag{8.42}$$

satisfies the equation

$$K_{\mathcal{H}_{0,\ldots,2^{J+1}-1}(\overline{\Omega_\sigma^{\text{ext}}})} * F = \sum_{i=1}^{N_J} a_i^{N_J} K_{Harm_{0,\ldots,2^{J+1}-1}(\overline{\Omega_\sigma^{\text{ext}}})}(\cdot, y_i^{N_J}).$$

Note that the coefficients $a_i^{N_J}$ are dependent on F. Again Lemma 8.8 now implies the following result.

Lemma 8.18. *Let F be of class $\mathcal{V}_{J+1}(\overline{\Omega_\sigma^{\text{ext}}}) = \mathcal{H}_{0,\ldots,2^{J+1}-1}(\overline{\Omega_\sigma^{\text{ext}}})$. Suppose that $K(\cdot,\cdot)$ is (an $\mathcal{H}_{\sigma,\sigma}$-kernel) such that $K^\wedge(n) = 0$ for all $n > 2^{J+1} - 1$. Then the coefficients* (8.42) *satisfy the equation*

$$K * F = \sum_{i=1}^{N_J} a_i^{N_J} A^2 K(\cdot, y_i^{N_J}).$$

It should be noted that

$$A^2 K(x,y) = \sum_{n=0}^{2^{J+1}-1} A_n^2 K^\wedge(n) \sum_{k=1}^{2n+1} H_{n,k}^*(\sigma;x) H_{n,k}(\sigma;y) \tag{8.43}$$

for all $(x,y) \in \overline{\Omega_\sigma^{\text{ext}}} \times \overline{\Omega_\sigma^{\text{ext}}}$. Furthermore, the vector a^{N_J} is independent of the choice of the $\mathcal{H}_{\sigma,\sigma}$-kernel $K(\cdot,\cdot)$.

As special cases we obtain from Lemma 8.18 the following identities:

$$\Phi_{J+1} * F = \sum_{i=1}^{N_J} a_i^{N_J} A^2 \Phi_{J+1}(\cdot, y_i^{N_J}), \tag{8.44}$$

$$(\Phi_{J+1} * \Phi_{J+1}) * F = \sum_{i=1}^{N_J} a_i^{N_J} A^2 (\Phi_{J+1} * \Phi_{J+1})(\cdot, y_i^{N_J}), \tag{8.45}$$

and

$$\Psi_J * F = \sum_{i=1}^{N_J} a_i^{N_J} A^2 \Psi_J(\cdot, y_i^{N_J}), \tag{8.46}$$

$$(\tilde{\Psi}_J * \Psi_J) * F = \sum_{i=1}^{N_J} a_i^{N_J} A^2 (\tilde{\Psi}_J * \Psi_J)(\cdot, y_i^{N_J}). \tag{8.47}$$

The Pyramid Step. An essential tool for the pyramid step is the following lemma.

Lemma 8.19. *Let F be of class $\mathcal{V}_{J+1}(\overline{\Omega_\sigma^{\mathrm{ext}}})$. Suppose that $K(\cdot,\cdot)$ is an $\mathcal{H}_{\sigma,\sigma}$-kernel with $K^\wedge(n) = 0$ for all $n > 2^J - 1$. Then the vector $a^{N_{J-1}} \in \mathbb{R}^{N_{J-1}}$, $a^{N_{J-1}} = (a_1^{N_{J-1}}, \ldots, a_{N_{J-1}}^{N_{J-1}})^T$, given by*

$$a_i^{N_{J-1}} = b_i^{N_{J-1}} (K_{\mathcal{H}_{0,\ldots,2^J-1}(\overline{\Omega_\sigma^{\mathrm{ext}}})} * F)(y_i^{N_{J-1}}), \quad i = 1, \ldots, N_{J-1},$$

satisfies the equation

$$K * F = \sum_{i=1}^{N_{J-1}} a_i^{N_{J-1}} A^2 K(\cdot, y_i^{N_{J-1}}).$$

Suppose that $K(\cdot,\cdot)$ satisfies the assumption of Lemma 8.19. Looking at our foregoing results we notice that there are two ways of discretizing an $\mathcal{H}$-convolution $K * F$. On the one hand we obtain from Lemma 8.18

$$K * F = \sum_{i=1}^{N_J} a_i^{N_J} A^2 K(\cdot, y_i^{N_J}) \tag{8.48}$$

with coefficients $a_1^{N_J}, \ldots, a_{N_J}^{N_J}$ given by

$$a_i^{N_J} = b_i^{N_J} F(y_i^{N_J}) = b_i^{N_J} v_i^{N_J}, \quad i = 1, \ldots, N_J. \tag{8.49}$$

It is remarkable that the coefficients are independent of the choice of the kernel $K(\cdot,\cdot)$. As particularly important case we mention

$$K_{\mathcal{H}_{0,\ldots,2^J-1}(\overline{\Omega_\sigma^{\mathrm{ext}}})} * F = \sum_{i=1}^{N_J} a_i^{N_J} K_{Harm_{0,\ldots,2^J-1}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(y_i^{N_J}, \cdot). \tag{8.50}$$

On the other hand, we are able to deduce from Lemma 8.19 that

$$K * F = \sum_{i=1}^{N_{J-1}} a_i^{N_{J-1}} A^2 K(\cdot, y_i^{N_{J-1}}) \tag{8.51}$$

with coefficients $a_1^{N_{J-1}}, \ldots, a_{N_{J-1}}^{N_{J-1}}$ given by

$$a_i^{N_{J-1}} = b_i^{N_{J-1}} (K_{\mathcal{H}_{0,\ldots,2^J-1}(\overline{\Omega_\sigma^{\mathrm{ext}}})} * F)(y_i^{N_{J-1}}), \tag{8.52}$$

$i = 1, \ldots, N_{J-1}$. Inserting (8.50) into (8.52) we find

$$a_i^{N_{J-1}} = b_i^{N_{J-1}} \sum_{l=1}^{N_J} a_l^{N_J} K_{Harm_{0,\ldots,2^J-1}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(y_i^{N_{J-1}}, y_l^{N_J}) \tag{8.53}$$

for $i = 1, \ldots, N_{J-1}$. In other words, the coefficients $a_i^{N_{J-1}}$ can be calculated recursively. Moreover, the coefficients are independent of the special choice of the kernel

$K(\cdot,\cdot)$. This finally leads us to the following discretization of the $\mathcal{H}$-convolutions

$$\Phi_J * F = \sum_{i=1}^{N_{J-1}} a_i^{N_{J-1}} A^2 \Phi_J(\cdot, y_i^{N_{J-1}}), \tag{8.54}$$

$$(\Phi_J * \Phi_J) \ *_{\mathcal{H}} \ F = \sum_{i=1}^{N_{J-1}} a_i^{N_{J-1}} A^2 (\Phi_J * \Phi_J)(\cdot, y_i^{N_{J-1}}), \tag{8.55}$$

and

$$\Psi_{J-1} * F = \sum_{i=1}^{N_{J-1}} a_i^{N_{J-1}} A^2 \Psi_{J-1}(\cdot, y_i^{N_{J-1}}), \tag{8.56}$$

$$(\tilde{\Psi}_{J-1} * \Psi_{J-1}) * F = \sum_{i=1}^{N_{J-1}} a_i^{N_{J-1}} A^2 (\tilde{\Psi}_{J-1} * \Psi_{J-1})(\cdot, y_i^{N_{J-1}}). \tag{8.57}$$

In conclusion, we end up with the following pyramid scheme for the decomposition of a potential F:

$$\begin{array}{ccccccccc} F & \longrightarrow & a^{N_J} & \longrightarrow & a^{N_{J-1}} & \longrightarrow & \cdots & \longrightarrow & a^{N_0} \\ & & \downarrow & & \downarrow & & & & \downarrow \\ & & (WT)(F)(J;\cdot) & & (WT)(F)(J-1;\cdot) & & & & (WT)(F)(0;\cdot). \end{array}$$

The reconstruction of the wavelet coefficients can be performed as described before via the formula

$$\begin{aligned} R_j(F) &= \tilde{\Psi}_j * (WT)(F)(j;\cdot) \\ &= \sum_{i=1}^{N_j} b_i^{N_j} (WT)(F)(j; y_i^{N_j}) A^2 \tilde{\Psi}_j(\cdot, y_i^{N_j}). \end{aligned} \tag{8.58}$$

This leads us to the following scheme:

$$\begin{array}{cccccccc} (WT)(F)(0;y_i^{N_0}) & & & (WT)(F)(1;y_i^{N_1}) & & & & \\ \downarrow & & & \downarrow & & & & \\ R_0(F) & & & R_1(F) & & & & \\ & \searrow & & & \searrow & & & \\ P_0(F) & \rightarrow & + & P_1(F) & \rightarrow & + & \cdots & . \end{array}$$

According to our approach the wavelet transform $(WT)(F)(j;\cdot)$ is given by the coefficients $a_1^{N_j}, \ldots, a_{N_j}^{N_j}$. This also enables us to reconstruct the potential only by use of the coefficients $a_i^{N_j}$, rather than calculating the wavelet coefficients of F:

$$R_j(F) = \sum_{i=1}^{N_j} a_i^{N_j} A^2 (\tilde{\Psi}_j * \Psi_j)(\cdot, y_i^{N_j}).$$

Thus the decomposition and reconstruction, respectively, can be simplified as follows:

$$F \to a^{N_J} \to a^{N_{J-1}} \to \cdots \to a^{N_0}$$

and

$$\begin{array}{ccccccccccc}
a^{N_0} & & & & a^{N_1} & & & & a^{N_2} & & \\
\downarrow & & & & \downarrow & & & & \downarrow & & \\
R_0(F) & & & & R_1(F) & & & & R_2(F) & & \\
& \searrow & & & & \searrow & & & & \searrow & \\
P_0(F) & \rightarrow & + & \rightarrow & P_1(F) & \rightarrow & + & \rightarrow & P_2(F) & \rightarrow & + \rightarrow \cdots .
\end{array}$$

That means the reconstruction of the potential is not performed with $\tilde{\Psi}_j$. Instead we have used the $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$-convolution $\tilde{\Psi}_j * \Psi_j$. Of particular significance is that the vectors a^{N_j} do not depend on the special choice of the bandlimited scaling function. As a matter of fact, we are able to reconstruct the potential with respect to different types of wavelets just by use of the vectors a^{N_j}.

Remark 8.20. The critical point of our pyramid scheme is the determination of the coefficients $b_l^{N_j}$, $j = 0, \ldots, J$, from the linear system (8.41) which provides outer harmonic exactness up to the order $2^{j+2} - 2$. It should be mentioned that the solution of this linear system can be avoided completely if we place the knots for each detail step $j = 0, \ldots, J$ on a spherical longitude-latitude grid on the sphere Ω_σ. The corresponding set of weights is explicitly available without solving any linear system from results due to [8].

Variant 2. In what follows we use outer harmonic exact approximation (Lemma 8.5) to develop a bandlimited variant of the pyramid scheme based on the Shannon sampling theorem. Our approach consists of the following steps:

(i) According to our bandlimited wavelet approach the (reference) Sobolev space $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ is subdivided by a nested sequence of 2^{2j}-dimensional scale spaces $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ as follows: $\cdots \subset \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \mathcal{V}_{j+1}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \cdots \subset \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

(ii) $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, can be identified with the set

$$\mathcal{H}_{0,\ldots,2^j-1}(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \mathcal{H}(\{A_n/(\varphi_j(n))^2\}; \overline{\Omega_\sigma^{\mathrm{ext}}}),$$

and $\Phi_j^{(4)}(\cdot,\cdot)$ is the uniquely determined reproducing kernel in $(\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}), (\cdot,\cdot)_{\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})})$ with $(\cdot,\cdot)_{\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})}$ given by

$$(\cdot,\cdot)_{\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})} = (\cdot,\cdot)_{\mathcal{H}(\{A_n/(\varphi_j(n))^2\};\overline{\Omega_\sigma^{\mathrm{ext}}})}.$$

(iii) For each $j \in \mathbb{N}_0$, consider sequences $\{\mathcal{L}_1^{N_j}, \ldots, \mathcal{L}_{N_j}^{N_j}\}$ of $N_j \geq 2^{2j}$ (linearly independent) bounded linear functionals on $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ such that

$$\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \operatorname{span}\left(\mathcal{L}_1^{N_j}\Phi_j^{(4)}(\cdot,\cdot), \ldots, \mathcal{L}_{N_j}^{N_j}\Phi_j^{(4)}(\cdot,\cdot)\right).$$

Then it also follows that

$$\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \operatorname{span}\left(\mathcal{L}_1^{N_j}\Phi_j^{(2)}(\cdot,\cdot), \ldots, \mathcal{L}_{N_j}^{N_j}\Phi_j^{(2)}(\cdot,\cdot)\right).$$

(iv) $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, $j \in \mathbb{N}_0$, can be identified with the set $\mathcal{H}(\{A_n/\varphi_j(n)\};\overline{\Omega_\sigma^{\mathrm{ext}}})$, and $\Phi_j^{(2)}(\cdot,\cdot)$ is the reproducing kernel in $\left(\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}), (\cdot,\cdot)_{\mathcal{V}_j^{(1/2)}(\overline{\Omega_\sigma^{\mathrm{ext}}})}\right)$ with $(\cdot,\cdot)_{\mathcal{V}_j^{(1/2)}(\overline{\Omega_\sigma^{\mathrm{ext}}})}$ defined by

$$(\cdot,\cdot)_{\mathcal{V}_j^{(1/2)}(\overline{\Omega_\sigma^{\mathrm{ext}}})} = (\cdot,\cdot)_{\mathcal{H}(\{A_n/\varphi_j(n)\};\overline{\Omega_\sigma^{\mathrm{ext}}})}.$$

The key idea of our fast evaluation method using the Shannon sampling theorem in terms of linear functionals is based on the following observations:

(1) For some suitably large J, the scale space $\mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}})$ is "sufficiently close" to $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$. Consequently, for each $F \in \mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$, there exists a function of class $\mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}})$ such that the error between F and $\Phi_J^{(2)} * F$ (understood in $\|\cdot\|_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}$-topology) is negligible. This is the reason why the input data $v_k^{N_J} = \mathcal{L}_k^{N_J} F$, $k = 1,\ldots,N_J$, are assumed to be of a potential F of class $\mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}})$ for the remainder of this subsection.

(2) For $j = 0,\ldots,J$, consider sequences $\{\mathcal{L}_1^{N_j},\ldots,\mathcal{L}_{N_j}^{N_j}\}$ of $N_j \geq 2^{2j}$ (linearly independent) bounded linear functionals on $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ such that

$$\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \mathcal{H}_{0,\ldots,2^j-1}(\overline{\Omega_\sigma^{\mathrm{ext}}}) = \mathrm{span}\left(\mathcal{L}_1^{N_j}\Phi_j^{(2)}(\cdot,\cdot),\ldots,\mathcal{L}_{N_j}^{N_j}\Phi_j^{(2)}(\cdot,\cdot)\right).$$

In an *a priori* step the coefficients $w_{l,k}^{N_j}$ have to be determined from the systems of linear equations (see Lemma 8.5)

$$\sum_{l=1}^{N_j} w_{l,k}^{N_j}\mathcal{L}_i^{N_j}\mathcal{L}_l^{N_j}\Phi_j^{(2)}(\cdot,\cdot) = \delta_{i,k}, \quad i,k = 1,\ldots,N_j,$$

$j = 0,\ldots,J$, and can be stored elsewhere. Looking carefully at the linear systems, it can be recognized that the coefficients $w_{l,k}^{N_j}$ do not depend on the particular function F under consideration, but only on the chosen linear functionals and pointsets.

Next our considerations are divided into two parts, viz. the initial step concerning the scale level J and the pyramid step establishing the recursion relation.

The Initial Step. The exact approximation

$$J_{N_J}S = \sum_{i=1}^{N_J} a_i^{N_J}\mathcal{L}_i^{N_J}S, \quad S \in \mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}}),$$

to the bounded linear functionals $\mathcal{L}$ on $\mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}})$ defined by

$$\mathcal{L}S = (S,F)_{\mathcal{V}_J^{(1/2)}(\overline{\Omega_\sigma^{\mathrm{ext}}})} = S \ *_{\mathcal{V}_J^{(1/2)}} F, \quad S \in \mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}}),\ F \in \mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}}),$$

is given by

$$a_i^{N_J} = \sum_{k=1}^{N_J} w_{i,k}^{N_J}\mathcal{L}\mathcal{L}_k^{N_J}\Phi_J^{(2)}(\cdot,\cdot), \quad i = 1,\ldots,N_J.$$

Note that in order to clarify the convolution we use a lower index at the symbol "$*$" in the following text if necessary. In accordance with our assumption $F \in \mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}})$ and the reproducing property of $\Phi_J^{(2)}(\cdot,\cdot)$ in $\mathcal{V}_J^{(1/2)}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ we see that $\Phi_J^{(2)} \; *_{\mathcal{V}_J^{(1/2)}} F = F$. Thus we find

$$a_i^{N_J} = \sum_{k=1}^{N_J} w_{i,k}^{N_J} (\mathcal{L}_k^{N_J} \Phi_J^{(2)}(\cdot,\cdot) \; *_{\mathcal{V}_J^{(1/2)}} F) = \sum_{k=1}^{N_J} w_{i,k}^{N_J} \mathcal{L}_k^{N_J} F = \sum_{k=1}^{N_J} w_{i,k}^{N_J} v_k^{N_J}$$

for $i = 1, \ldots, N_J$. This leads us to the following conclusion.

Lemma 8.21. *If F is a member of class $\mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}})$, then the identity*

$$S \; *_{\mathcal{V}_J^{(1/2)}} \; F = \sum_{i=1}^{N_J} a_i^{N_J} \mathcal{L}_i^{N_J} S$$

holds for all $S \in \mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}})$.

Lemma 8.21 immediately enables us to formulate the following lemma.

Lemma 8.22. *Let F be a member of class $\mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}})$, then the identity*

$$K * F = \sum_{i=1}^{N_J} a_i^{N_J} \mathcal{L}_i^{N_J} K(\cdot,\cdot)$$

holds for all $\mathcal{H}_{\sigma,\sigma}$-kernels $K(\cdot,\cdot)$ with $K^\wedge(n) = 0$ for $n = 2^J, ^J+1, \ldots$.

The next theorem clarifies the remarkable consequences for our wavelet concept.

Theorem 8.23. *Under the assumptions of Lemma 8.22 we have*

$$\Phi_J * F = \sum_{i=1}^{N_J} a_i^{N_J} \mathcal{L}_i^{N_J} \Phi_J(\cdot,\cdot), \tag{8.59}$$

$$(\Phi_J * \Phi_J) * F = \sum_{i=1}^{N_J} a_i^{N_J} \mathcal{L}_i^{N_J} (\Phi_J * \Phi_J)(\cdot,\cdot), \tag{8.60}$$

and

$$\Psi_{J-1} * F = \sum_{i=1}^{N_J} a_i^{N_J} \mathcal{L}_i^{N_J} \Psi_{J-1}(\cdot,\cdot), \tag{8.61}$$

$$(\tilde{\Psi}_{J-1} * \Psi_{J-1}) * F = \sum_{i=1}^{N_J} a_i^{N_J} \mathcal{L}_i^{N_J} (\tilde{\Psi}_{J-1} * \Psi_{J-1})(\cdot,\cdot). \tag{8.62}$$

In conclusion, the vector $a^{N_J} = (a_1^{N_J}, \ldots, a_{N_J}^{N_J})^T \in \mathbb{R}^{N_J}$ does not depend on the special choice of the $\Phi_J^{(2)}(\cdot,\cdot)$-kernel in $\mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}})$. Wavelet transform, lowpass, and bandpass filter can be computed by use of the same set of coefficients.

The Pyramid Step. This step provides an algorithm such that $a^{N_J} \in \mathbb{R}^{N_J}$ serves as starting vector for $a^{N_j} \in \mathbb{R}^{N_j}$, $j = 0, \ldots, J-1$, which fulfill the following properties:

(i) The vectors a^{N_j} satisfy

$$\Phi_j^{(2)} * F = \sum_{i=1}^{N_j} a_i^{N_j} \mathcal{L}_i^{N_j} \Phi_j^{(2)}(\cdot, \cdot),$$

$j = 0, \ldots, J$.

(ii) The wavelet transforms are given by

$$\Psi_{j-1} * F = \sum_{i=1}^{N_j} a_i^{N_j} \mathcal{L}_i^{N_j} \Psi_{j-1}(\cdot, \cdot),$$

$j = 1, \ldots, J$.

(iii) The vector a^{N_j} is obtainable from $a^{N_{j+1}}$, $j = 0, \ldots, J-1$, by recursion.

In the remainder of this section the properties (i), (ii) and (iii) are described in more detail. The exact approximations J_{N_j}, $j = 0, \ldots, J-1$,

$$J_{N_j} S = \sum_{i=1}^{N_j} a_i^{N_j} \mathcal{L}_i^{N_j} S, \quad S \in \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$$

to the bounded linear functional $\mathcal{L}$ on $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$ defined by

$$\mathcal{L}S = S \ *_{\mathcal{V}_j^{(1/2)}} \ (\Phi_j^{(2)} \ *_{\mathcal{H}} \ F), \quad S \in \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}), \ F \in \mathcal{V}_J(\overline{\Omega_\sigma^{\mathrm{ext}}}),$$

(note that $\Phi_j^{(2)} \ *_{\mathcal{H}} \ F \in \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$) are given by the coefficients

$$a_l^{N_j} = \sum_{i=1}^{N_j} w_{l,i}^{N_j} \mathcal{L}_i^{N_j} \Phi_j^{(2)}(\cdot, \cdot), \quad l = 1, \ldots, N_j.$$

Consequently it is easily seen that for $l = 1, \ldots, N_j$

$$a_l^{N_j} = \sum_{i=1}^{N_j} w_{l,i}^{N_j} \mathcal{L}_i^{N_j} (\Phi_j^{(2)}(\cdot, \cdot) * F).$$

Thus we obtain the following lemma.

Lemma 8.24. *If F is a member of class $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, then the identity*

$$S \ *_{\mathcal{V}_j^{(1/2)}} \ (\Phi_j^{(2)} \ *_{\mathcal{H}} \ F) = \sum_{i=1}^{N_j} a_i^{N_j} \mathcal{L}_i^{N_j} S$$

holds for all $S \in \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$. In particular,

$$\Phi_j^{(2)} \ *_{\mathcal{H}} \ F = \sum_{i=1}^{N_j} a_i^{N_j} \mathcal{L}_i^{N_j} \Phi_j^{(2)}(\cdot, \cdot).$$

By the same arguments as given in the last subsection we obtain the following lemma.

Lemma 8.25. *Let F be a function of class $\mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}})$, then the identity*

$$K * F = \sum_{i=1}^{N_j} a_i^{N_j} \mathcal{L}_i^{N_j} K(\cdot,\cdot)$$

holds for all $\mathcal{H}_{\sigma,\sigma}$-kernels $K(\cdot,\cdot)$ with $K^\wedge(n) = 0$, $n = 2^j, 2^j+1, \ldots$.

Finally we get the following results.

Theorem 8.26. *Under the assumptions of Lemma 8.25 we have*

$$\Phi_j * F = \sum_{i=1}^{N_j} a_i^{N_j} \mathcal{L}_i^{N_j} \Phi_j(\cdot,\cdot),$$

$$(\Phi_j * \Phi_j) * F = \sum_{i=1}^{N_j} a_i^{N_j} \mathcal{L}_i^{N_j} (\Phi_j * \Phi_j)(\cdot,\cdot),$$

and

$$\Psi_{j-1} * F = \sum_{i=1}^{N_j} a_i^{N_j} \mathcal{L}_i^{N_j} \Psi_{j-1}(\cdot,\cdot),$$

$$(\tilde{\Psi}_{j-1} * \Psi_{j-1}) * F = \sum_{i=1}^{N_j} a_i^{N_j} \mathcal{L}_i^{N_j} (\tilde{\Psi}_{j-1} * \Psi_{j-1})(\cdot,\cdot).$$

From Theorem 8.26 we are able to deduce that

$$\Phi_{J-1}^{(2)} * F = \sum_{i=1}^{N_{J-1}} a_i^{N_{J-1}} \mathcal{L}_i^{N_{J-1}} \Phi_{J-1}^{(2)}(\cdot,\cdot), \tag{8.63}$$

where

$$a_l^{N_{J-1}} = \sum_{i=1}^{N_{J-1}} w_{l,i}^{N_{J-1}} \mathcal{L}_i^{N_{J-1}} (\Phi_{J-1}^{(2)}(\cdot,\cdot) * F). \tag{8.64}$$

On the other hand, by virtue of Lemma 8.22, we have

$$\Phi_{J-1}^{(2)} \ * \ F = \sum_{i=1}^{N_J} a_i^{N_J} \mathcal{L}_i^{N_J} \Phi_{J-1}^{(2)}(\cdot,\cdot). \tag{8.65}$$

Combining (8.64) and (8.65) we obtain

$$a_l^{N_{J-1}} = \sum_{i=1}^{N_{J-1}} \sum_{k=1}^{N_J} w_{l,i}^{N_{J-1}} a_k^{N_J} \mathcal{L}_i^{N_{J-1}} \mathcal{L}_k^{N_J} \Phi_{J-1}^{(2)}(\cdot,\cdot) \tag{8.66}$$

for $l = 1,\ldots,N_{J-1}$. Assuming the sets $\{\mathcal{L}_1^{N_j},\ldots,\mathcal{L}_{N_j}^{N_j}\}$ to be hierarchical, i.e., $\mathcal{L}_i^{N_j} = \mathcal{L}_i^{N_{j+1}}$, $i = 1,\ldots,N_j$; $j = 0,\ldots,J-1$, and observing the symmetry of the matrix $(w_{l,i}^{N_{J-1}})$ we gain a reduction of computational costs as follows:

$$\begin{aligned} a_l^{N_{J-1}} &= \sum_{i=1}^{N_{J-1}} \sum_{k=1}^{N_J} w_{i,l}^{N_{J-1}} a_k^{N_J} \mathcal{L}_i^{N_{J-1}} \mathcal{L}_k^{N_J} \Phi_{J-1}^{(2)}(\cdot,\cdot) \\ &= \sum_{i=1}^{N_{J-1}} \sum_{k=1}^{N_{J-1}} w_{i,l}^{N_{J-1}} a_k^{N_J} \mathcal{L}_i^{N_{J-1}} \mathcal{L}_k^{N_{J-1}} \Phi_{J-1}^{(2)}(\cdot,\cdot) \\ &\quad + \sum_{i=1}^{N_{J-1}} \sum_{k=N_{J-1}+1}^{N_J} w_{i,l}^{N_{J-1}} a_k^{N_J} \mathcal{L}_i^{N_{J-1}} \mathcal{L}_k^{N_J} \Phi_{J-1}^{(2)}(\cdot,\cdot) \\ &= a_l^{N_J} + \sum_{i=1}^{N_{J-1}} \sum_{k=N_{J-1}+1}^{N_J} w_{i,l}^{N_{J-1}} a_k^{N_J} \mathcal{L}_i^{N_{J-1}} \mathcal{L}_k^{N_J} \Phi_{J-1}^{(2)}(\cdot,\cdot). \end{aligned}$$

The recursion relation (8.66) leads us to the following *decomposition scheme*:

$$\begin{array}{ccccccc} F & \rightarrow & a^{N_J} & \rightarrow & a^{N_{J-1}} & \rightarrow \cdots & a^{N_0} \\ & & \downarrow & & \downarrow & & \downarrow \\ & & (WT)(F)(J;\cdot) & & (WT)(F)(J-1;\cdot) & & (WT)(F)(0;\cdot). \end{array}$$

The bandpass filter $R_j(F)$ can be deduced from the formula

$$R_j(F) = \tilde{\Psi}_j * (WT)(F)(j;\cdot) = \sum_{i=1}^{N_j} a_i^{N_j} \mathcal{L}_i^{N_j} (\tilde{\Psi}_j * \Psi_j)(\cdot,\cdot). \tag{8.67}$$

This allows the following *reconstruction scheme* of F:

$$\begin{array}{ccccccccccccc} a^{N_0} & & & & a^{N_1} & & & & a^{N_2} & & & & \\ \downarrow & & & & \downarrow & & & & \downarrow & & & & \\ R_0(F) & & & & R_1(F) & & & & R_2(F) & & & & \\ & \searrow & & & & \searrow & & & & \searrow & & & \\ P_0(F) & \longrightarrow & + & \longrightarrow & P_1(F) & \longrightarrow & + & \longrightarrow & P_2(F) & \longrightarrow & + & \longrightarrow & \cdots . \end{array}$$

We have seen that the vectors a^{N_j} do *not* depend on the special choice of the scaling function $\{\Phi_j(\cdot,\cdot)\}_{j\in\mathbb{N}_0}$. In other words, we are able to reconstruct a function with respect to different wavelets just by the knowledge of the vectors a^{N_j}.

Let us finally make some comments concerning the pyramid schemata:

(1) In signal processing a variant of the pyramid scheme is known as subband coding. This technique was originally studied before wavelet theory. The decomposition step consists of applying a lowpass and a bandpass filter followed by downsampling; the reconstruction consists of upsampling followed by filtering.

(2) Any bandlimited potential can be reconstructed exactly via the pyramid scheme by use of bandlimited wavelets (see also [67]). In this case spline exact approximation coincides with polynomial (i.e., outer harmonic) exact approximation. The scale and detail spaces are finite-dimensional so that the

detail information of a potential is only determined by a finite number of wavelet coefficients for each scale.

(3) In case of evaluation functionals and (radial) derivatives at certain points on a sphere $\Omega_r, r \geq \sigma$, the numerical effort can be drastically reduced by three integration procedures on the sphere. The first method is to use gridded pointsystems and then to apply FFT-techniques (cf. the Ph.D.-thesis [74]). The second technique is to use a suitable Gauss-quadrature rule in north-south direction. The third method is to apply the idea of fast summation and panel clustering (cf. [23, 39]). For more details concerning numerical integration on the sphere the reader is referred to [21, 44].

(4) The pyramid scheme provides a powerful tool in interpreting and constructing lowpass and bandpass filters. The wavelets localize in space and frequency. This makes wavelets particularly useful for data compression. Compression techniques aim at reducing storage requirements and speeding up read or write operations to or from disks. For the compression scheme we are ready to accept an error as long as the quality after compression is acceptable.

(5) Another application is, that for the evaluation of a potential or its derivatives at a point, only wavelet coefficients close to the point have to be taken into account. This enables us to observe local features of the geopotential in a global model.

Example. In the foregoing we have seen that bandlimited harmonic wavelets provide "building blocks" that enable fast decorrelation of geopotential data. Next we are interested in discussing the concept of multiresolution analysis from practical point of view. To be more specific, the multiresolution analysis "looks at" the Earth's gravitational potential through a microscope, whose resolution gets finer and finer. Thus it associates to the gravitational potential a sequence of smoothed versions, labelled by the scale parameter. This aspect is illustrated by the figures below for the (bandlimited) EGM96 model. The computation has been performed on the basis of the CP-wavelets following Variant 1.

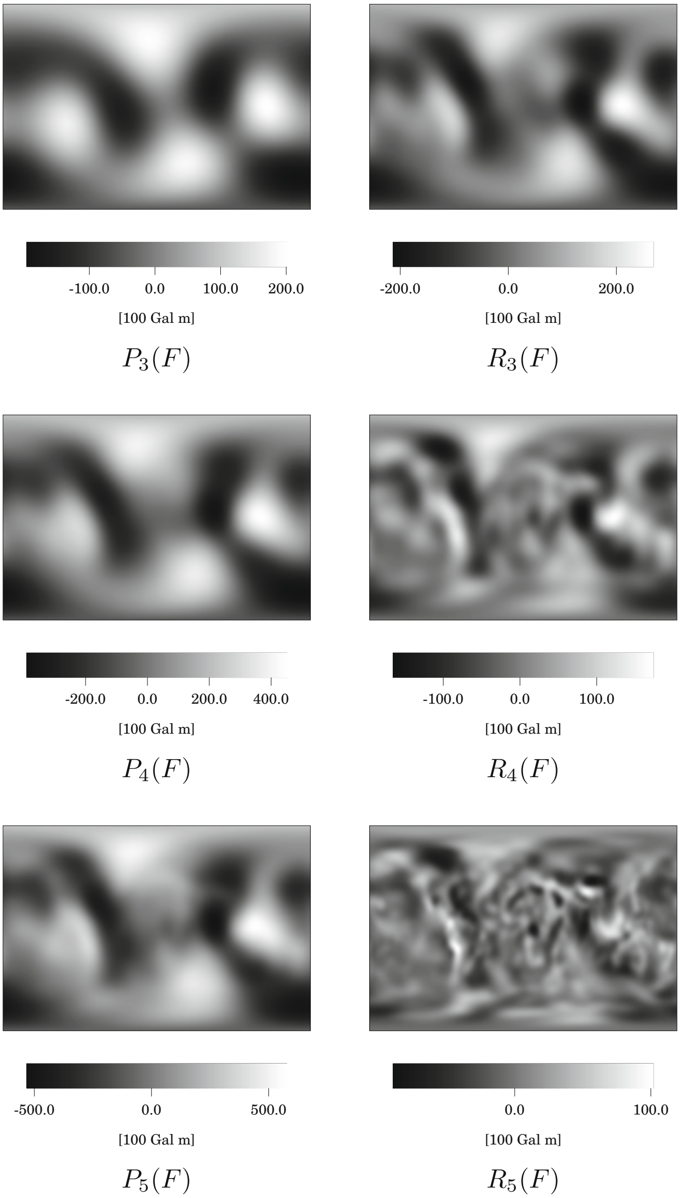


FIGURE 8.1. EGM96 CP-wavelet representation at height 0 km.

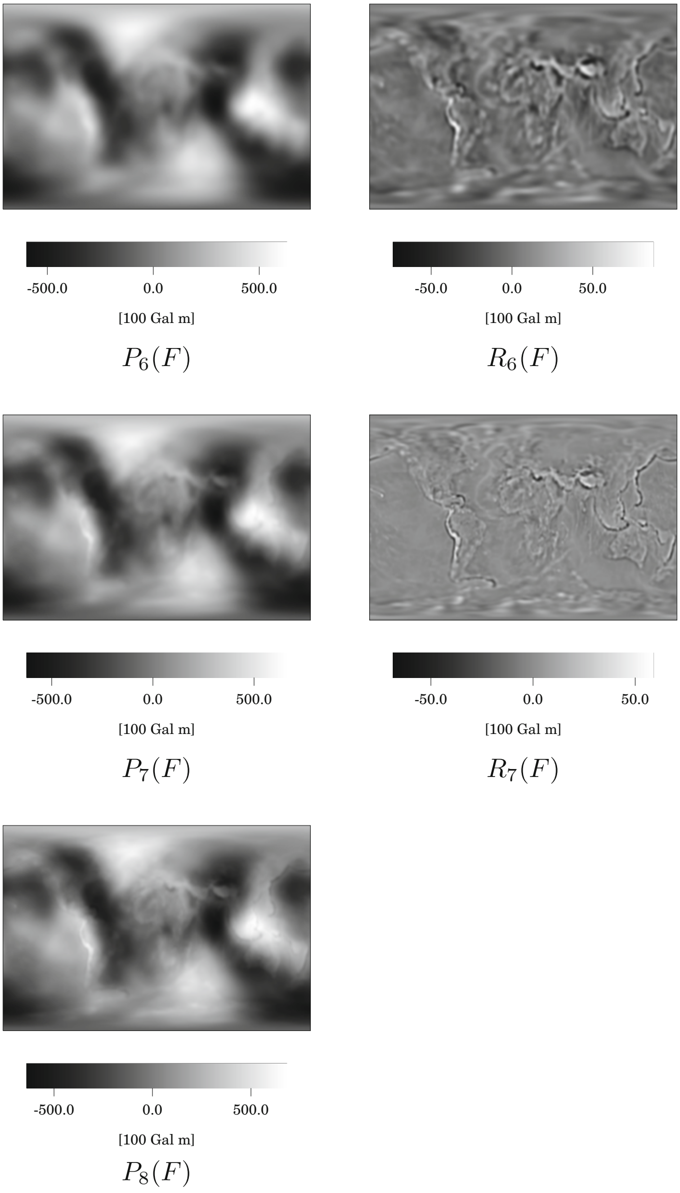


FIGURE 8.2. EGM96 CP-wavelet representation at height 0 km (cont.).

9. Illustrations of Meissl schemata

In this section we derive Meissl schemata for the SST and SGG operators (cf. [20, 32, 58]). In our contribution we focus on the gravitational potential, but obviously, the results are also valid for the disturbing potential.

9.1. Meissl schemata based on outer harmonic framework

We start from the scalar Fourier expansion of the gravitational potential V in terms of outer harmonics

$$V(x) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} V^{\wedge}(n,m) H^s_{n,m}(\sigma;\cdot). \tag{9.1}$$

If the observables are given both at minimum satellites altitude γ and at minimum Earth's radius σ (see Figure 3.1), the symbols of the pseudodifferential operators for the SST and SGG problem can be arranged in a Meissl scheme. The symbols at the arrows indicate how the Fourier coefficients of degree n change at the transition form one quantity to another. In order to avoid confusion the corresponding basis functions are also given. In the case of radial derivatives we remember that the basis system $H_{n,m}$ fulfills

$$H_{n,m}(\sigma;\cdot)|_{\Omega_\sigma} = (1/R)Y_{n,m}. \tag{9.2}$$

Therefore, we get the Meissl scheme for radial derivatives given in Figure 9.1.

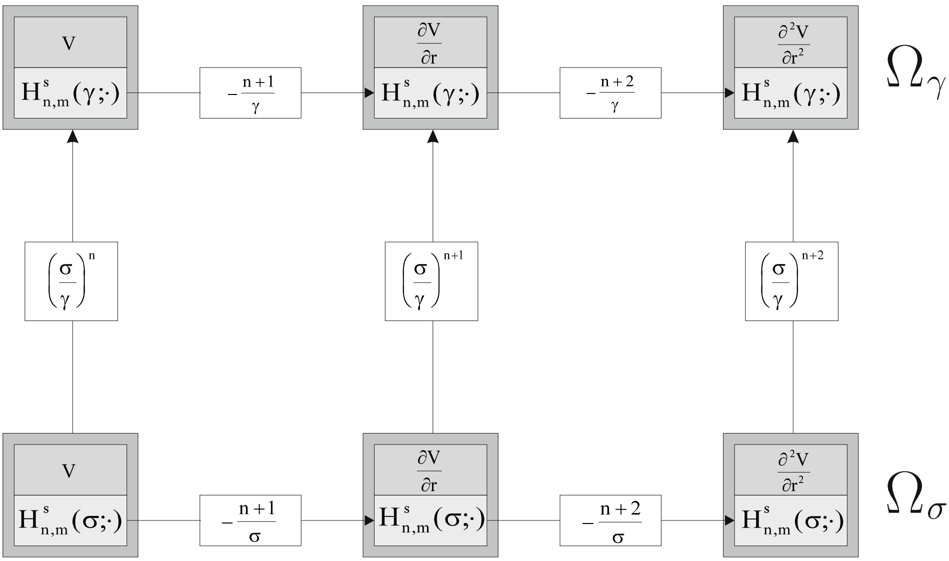


FIGURE 9.1. Meissl scheme for radial derivatives.

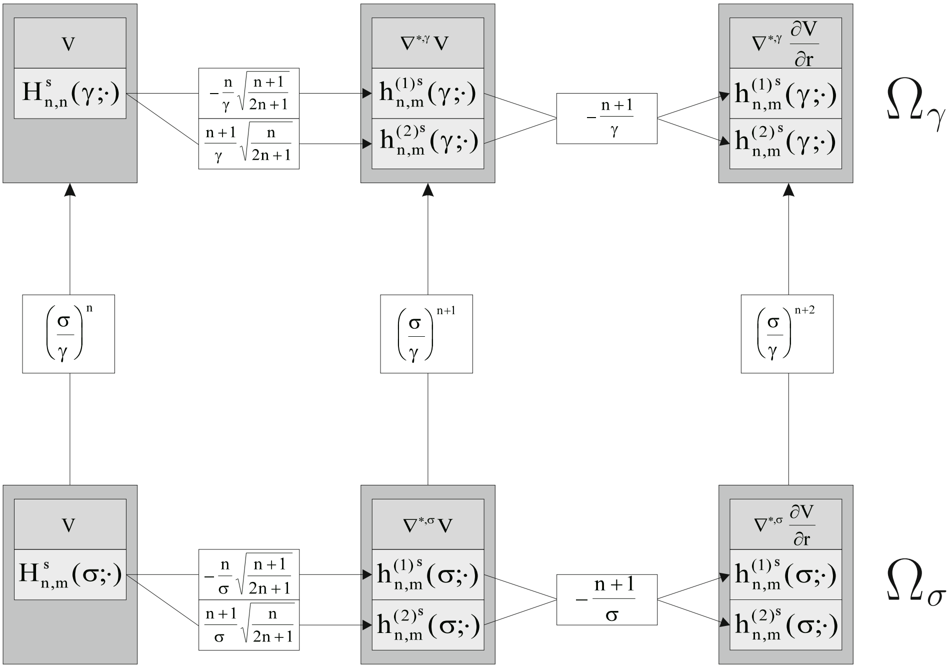


FIGURE 9.2. Meissl scheme for first-order tangential derivatives and second-order mixed derivatives.

If vectorial observables are investigated, we need that

$$o^{(2)}Y_{n,m} = -n\sqrt{\frac{n+1}{2n+1}}\tilde{y}^{(1)}_{n,m} + (n+1)\sqrt{\frac{n}{2n+1}}\tilde{y}^{(2)}_{n,m}, \tag{9.3}$$

which yields the Meissl schemata in Figures 9.2 and 9.3.

Finally, in the case of second-order tangential derivatives $(\nabla^* \otimes \nabla^*)$ we calculate

$$\nabla^* \otimes \nabla^* \tilde{y}^{(1)}_{n,m} = \rho_n^{(1,1)} \frac{n+1}{2n+3}\tilde{\mathbf{y}}^{(1,1)}_{n,m} + \rho_n^{(2,1)}\frac{n+2}{2n+3}\tilde{\mathbf{y}}^{(2,1)}_{n,m} + \rho_n^{(2,2)}\frac{2(n+1)}{(2n+1)(2n-1)}\tilde{\mathbf{y}}^{(2,2)}_{n,m} \tag{9.4}$$

and

$$\begin{aligned}\nabla^* \otimes \nabla^* \tilde{y}^{(2)}_{n,m} = {} & \tau_n^{(1,1)}(-1)\frac{2n(n+1)}{(2n+1)(2n+3)}\tilde{\mathbf{y}}^{(1,1)}_{n,m} + \tau_n^{(1,2)}\frac{n-1}{(2n-1)(2n+1)}\tilde{\mathbf{y}}^{(1,2)}_{n,m} \\ & + \tau_n^{(2,1)}\frac{2n(n+2)}{(2n+3)(2n+1)}\tilde{\mathbf{y}}^{(2,1)}_{n,m} + \tau_n^{(2,2)}(-1)\frac{n}{(2n-1)(2n+1)}\tilde{\mathbf{y}}^{(2,2)}_{n,m},\end{aligned} \tag{9.5}$$

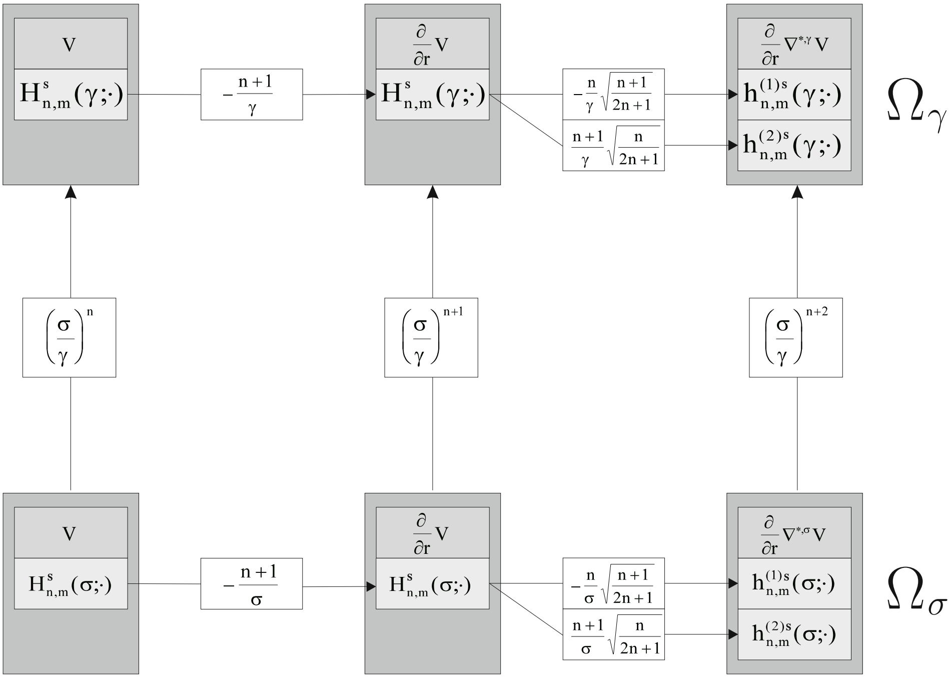


FIGURE 9.3. Meissl scheme for first-order radial derivatives and second-order mixed derivatives.

where the constants $\rho_n^{(i,k)}$ and $\tau_n^{(i,k)}$ are given by

$$\rho_n^{(i,k)} = \sqrt{\frac{\nu_n^{(i,k)}}{(2n+1)(n+1)}}, \tag{9.6}$$

$$\tau_n^{(i,k)} = \sqrt{\frac{\nu_n^{(i,k)}}{(2n+1)n}}. \tag{9.7}$$

In conclusion, we get the Meissl scheme for first- and second-order tangential derivatives (see Figure 9.4).

9.2. Meissl schemata based on kernel function framework

In order to derive Meissl schemata based on kernel functions we want to recapitulate the convolutions which are used in this section (see Table 4).

Our point of departure is the description of a function $F \in \mathcal{H}_s(\overline{\Omega_\sigma^{\mathrm{ext}}})$ in terms of outer harmonics

$$F(x) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} F^\wedge(n,m) H^s_{n,m}(\sigma;x), \tag{9.8}$$

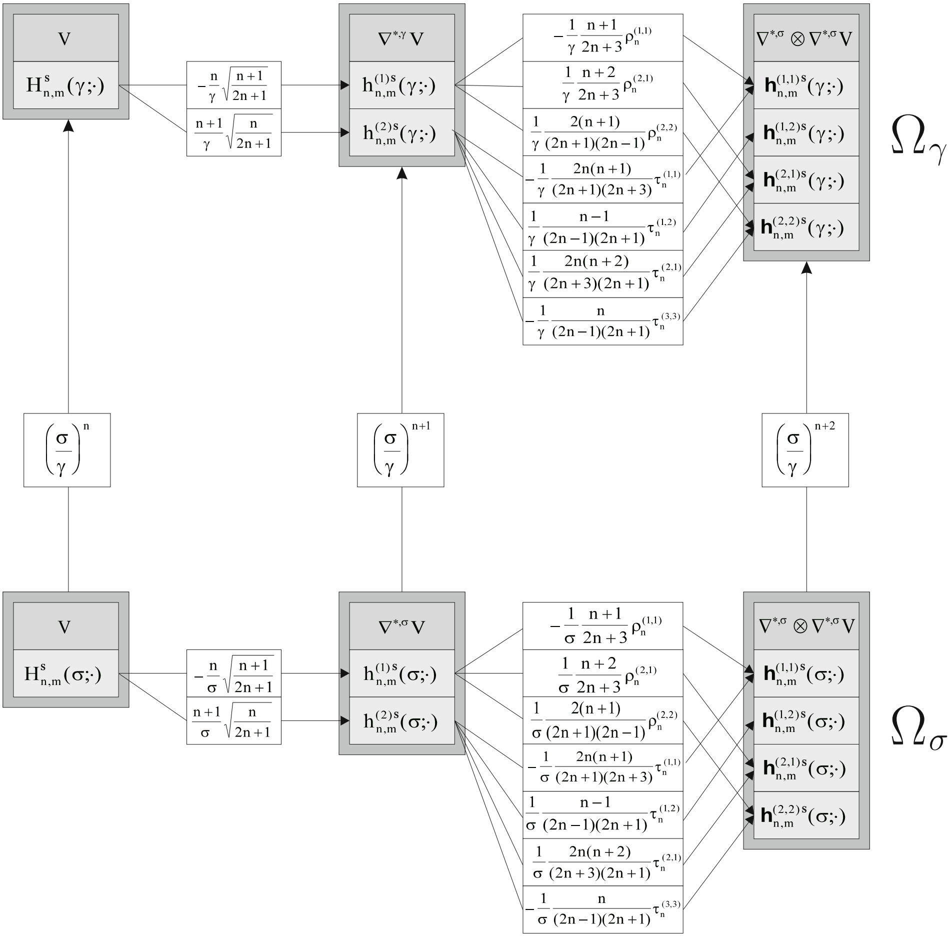


FIGURE 9.4. Meissl scheme for first- and second-order tangential derivatives.

$x \in \overline{\Omega_\sigma^{\mathrm{ext}}}$, and we first derive the kernel functions corresponding to the SST and SGG operators.

Scalar SST and SGG Operators

The SST and SGG operators are given by the convolution equation

$$\Lambda F(x) = (K^\Lambda)^{\sigma,\gamma}(\cdot, x) * F, \quad x \in \overline{\Omega_\gamma^{\mathrm{ext}}}, \tag{9.9}$$

where the symbol of the kernel $(K^\Lambda)^{\sigma,\gamma}$ is given by

$$(K^\Lambda)^\wedge(n) = \Lambda^\wedge(n) = \begin{cases} \left(\frac{\sigma}{\gamma}\right)^n \frac{n+1}{\gamma}, & n = 0, 1, \ldots \quad \text{for SST,} \\ \left(\frac{\sigma}{\gamma}\right)^n \frac{(n+1)(n+2)}{\gamma^2}, & n = 0, 1, \ldots \quad \text{for SGG.} \end{cases} \tag{9.10}$$

$K * F = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} K^\wedge(n)F^\wedge(n,m)H^s_{n,m}(\gamma;\cdot)$	$F, K(\cdot,y) \in \mathcal{H}_s(\overline{\Omega^{\mathrm{ext}}_\sigma})$
$k^{(i)} * f = \sum_{n=0_i}^{\infty}\sum_{m=1}^{2n+1} k^{(i)\wedge}(n)f^{(i)\wedge}(n,m)H^s_{n,m}(\gamma;\cdot)$	$f, k^{(i)}(\cdot,y) \in h^{(i)}_s(\overline{\Omega^{\mathrm{ext}}_\sigma})$
$k \star F = \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{m=1}^{2n+1} k^{(i)\wedge}(n)F^\wedge(n,m)h^{(i)s}_{n,m}(\gamma;\cdot)$	$F \in \mathcal{H}_s(\overline{\Omega^{\mathrm{ext}}_\sigma})$, $k(\cdot,y) \in h_s(\overline{\Omega^{\mathrm{ext}}_\sigma})$
$\mathbf{k}^{(i,k)} * \mathbf{f} = \sum_{n=\tilde{0}_{ik}}^{\infty}\sum_{m=1}^{2n+1} \mathbf{k}^{(i,k)\wedge}(n)\mathbf{f}^{(i,k)\wedge}(n,m)H^s_{n,m}(\gamma;\cdot)$	$\mathbf{f},\ \mathbf{k}^{(i,k)}(\cdot,y) \in \mathbf{h}^{(i,k)}_s(\overline{\Omega^{\mathrm{ext}}_\sigma})$
$\mathbf{k} \star F = \sum_{i,k=1}^{3}\sum_{n=\tilde{0}_{ik}}^{\infty}\sum_{m=1}^{2n+1} \mathbf{k}^{(i,k)\wedge}(n)F^\wedge(n,m)\mathbf{h}^{(i,k)s}_{n,m}(\gamma;\cdot)$	$F \in \mathcal{H}_s(\overline{\Omega^{\mathrm{ext}}_\sigma})$, $\mathbf{k}(\cdot,y) \in \mathbf{h}_s(\overline{\Omega^{\mathrm{ext}}_\sigma})$

TABLE 4. List of the convolutions.

Vectorial SST and SGG Operators

In the vectorial case we have

$$\lambda F(x) = (k^{\lambda})^{\sigma,\gamma}(\cdot,x) \star F, \quad x \in \overline{\Omega^{\mathrm{ext}}_\gamma}, \tag{9.11}$$

with the symbol $(k^\lambda)^{(i)\wedge}(n)$ given by

$$(k^\lambda)^{(1)\wedge}(n) = \lambda^{(1)\wedge}(n) = \begin{cases} -\left(\frac{\sigma}{\gamma}\right)^{n}\frac{n}{\gamma}\sqrt{\frac{n+1}{2n+1}}, & n = 1,2,\ldots \quad \text{for SST,} \\ -\left(\frac{\sigma}{\gamma}\right)^{n+1}\frac{n(n+1)}{\gamma^2}\sqrt{\frac{n+1}{2n+1}}, & n = 1,2,\ldots \quad \text{for SGG,} \end{cases} \tag{9.12}$$

and

$$(k^\lambda)^{(2)\wedge}(n) = \lambda^{(2)\wedge}(n) = \begin{cases} \left(\frac{\sigma}{\gamma}\right)^{n}\frac{n+1}{\gamma}\sqrt{\frac{n}{2n+1}}, & n = 1,2,\ldots \quad \text{for SST,} \\ \left(\frac{\sigma}{\gamma}\right)^{n+1}\frac{(n+1)^2}{\gamma^2}\sqrt{\frac{n}{2n+1}}, & n = 1,2,\ldots \quad \text{for SGG,} \end{cases} \tag{9.13}$$

and $(k^\lambda)^{(3)\wedge}(n) = 0$.

Tensorial SGG Operator

This operator is given by

$$\boldsymbol{\lambda} F(x) = (\mathbf{k}^{\boldsymbol{\lambda}})^{\sigma,\gamma}(\cdot,x) \star F, \quad x \in \overline{\Omega^{\mathrm{ext}}_\gamma}, \tag{9.14}$$

where we have the symbol

$$(\mathbf{k}\,\boldsymbol{\lambda})^{(i,k)\wedge}(n) = \boldsymbol{\lambda}^{(i,k)\wedge}(n) = \begin{cases} \left(\frac{\sigma}{\gamma}\right)^n \frac{1}{\gamma^2}\frac{n(n+1)}{(2n+1)(2n+3)}\sqrt{\nu_n^{(1,1)}}, & (i,k)=(1,1),\\ \left(\frac{\sigma}{\gamma}\right)^n \frac{1}{\gamma^2}\frac{-(n+1)(n-1)}{((2n-1)(2n+1)}\sqrt{\nu_n^{(1,2)}}, & (i,k)=(1,2),\\ \left(\frac{\sigma}{\gamma}\right)^n \frac{1}{\gamma^2}\frac{-n(n+2)}{(2n+3)(2n+1)}\sqrt{\nu_n^{(2,1)}}, & (i,k)=(2,1),\\ \left(\frac{\sigma}{\gamma}\right)^n \frac{1}{\gamma^2}\frac{n(n+1)(n+2)}{(2n-1)(2n+1)}\sqrt{\nu_n^{(2,2)}}, & (i,k)=(2,2),\\ 0, & \text{else}. \end{cases} \tag{9.15}$$

Upward Continuation Operators

The kernels of the (scalar) upward continuation operators K_U, $K_{U'}$, and $K_{U''}$ are given by

$$K_U(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left(\frac{\sigma}{\gamma}\right)^n H^s_{n,m}(\gamma;x)H^s_{n,m}(\sigma;y), \tag{9.16}$$

$$K_{U'}(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left(\frac{\sigma}{\gamma}\right)^{n+1} H^s_{n,m}(\gamma;x)H^s_{n,m}(\sigma;y), \tag{9.17}$$

$$K_{U''}(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left(\frac{\sigma}{\gamma}\right)^{n+2} H^s_{n,m}(\gamma;x)H^s_{n,m}(\sigma;y). \tag{9.18}$$

The upward continuation operators for vector and tensor fields can be introduced in the same way by use of the vectorial and tensorial basis functions $h^{s(i)}_{n,m}$ and $\mathbf{h}^{s(i,k)}_{n,m}$, $i,k \in \{1,2,3\}$.

The Meissl schemata for the scalar/vectorial/tensorial wavelets can now be derived as follows:

Scalar Meissl Scheme. From the reconstruction formula in the scalar case (7.9) we get

$$F(x) = \sum_{j=-1}^{\infty} \tilde{\Psi}_j * (WT)(F)(j;x) = \sum_{j=-1}^{\infty} (\tilde{\Psi}_j * \Psi_j * F)(x), \tag{9.19}$$

$x \in \overline{\Omega_\sigma^{\text{ext}}}$, whereas

$$\frac{\partial F}{\partial r}(x) = \sum_{j=-1}^{\infty}\left(\tilde{\Psi}_j * \Psi_j * K^\sigma_{\frac{\partial}{\partial r}} * F\right)(x), \tag{9.20}$$

where the kernel of the first radial derivative $K^\sigma_{\frac{\partial}{\partial r}}$ on the sphere Ω_σ is given by

$$K^\sigma_{\frac{\partial}{\partial r}}(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left(-\frac{n+1}{\sigma}\right) H^s_{n,m}(\sigma;x)H^s_{n,m}(\sigma;y). \tag{9.21}$$

The same calculation for the second radial derivative $\frac{\partial^2}{\partial r^2}$ leads to

$$\frac{\partial^2 F}{\partial r^2}(x) = \sum_{j=-1}^{\infty} \left(\tilde{\Psi}_j * \Psi_j * K^R_{\frac{\partial^2}{\partial r^2}} * F \right)(x), \tag{9.22}$$

where $K^\sigma_{\frac{\partial^2}{\partial r^2}}$ is given by

$$\begin{aligned} K^\sigma_{\frac{\partial^2}{\partial r^2}}(x,y) &= \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \frac{(n+1)(n+2)}{\sigma^2} H^s_{n,m}(\sigma;x) H^s_{n,m}(\sigma;y) \\ &= \left(K^\sigma_{\frac{\partial}{\partial r}} * \tilde{K}^\sigma_{\frac{\partial}{\partial r}} \right)(x,y), \end{aligned} \tag{9.23}$$

and the kernel $\tilde{K}^\sigma_{\frac{\partial}{\partial r}}$ is given by

$$\tilde{K}^\sigma_{\frac{\partial}{\partial r}}(x,y) = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \left(-\frac{n+2}{\sigma} \right) H^s_{n,m}(\sigma;x) H^s_{n,m}(\sigma;y). \tag{9.24}$$

Therefore, we get the Meissl scheme shown in Figure 9.5.

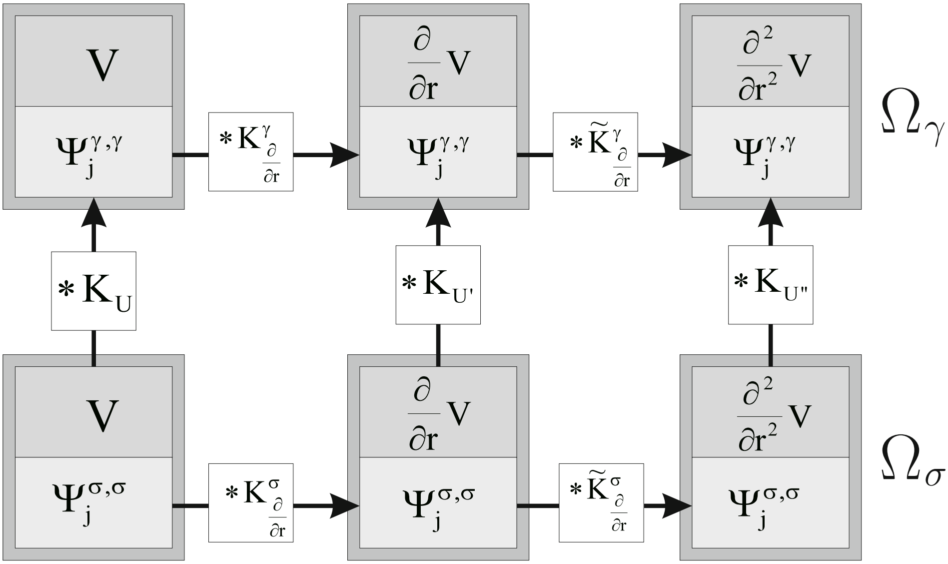


FIGURE 9.5. Meissl scheme for kernel functions (scalar case).

Scalar/Vectorial Meissl Scheme. The extension the the case of vectorial operators is straightforward:

$$o^{(2),\sigma} F(x) = \sum_{j=-1}^{\infty} \sum_{i=1}^{2} \left(\tilde{\Psi}_j^{(i)} \star \Psi_j^{(i)} * \left(k^{\sigma,(i)}_{o^{(2),\sigma}} \star F \right) \right)(x), \tag{9.25}$$

where the kernel functions $k^{\sigma,(i)}_{o^{(2)},\sigma}$ are given by

$$k^{\sigma,(1)}_{o^{(2)},\sigma}(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left(-\frac{n}{\sigma}\right)\sqrt{\frac{n+1}{2n+1}}h^{s(1)}_{n,m}(\sigma;x)H^s_{n,m}(\sigma;y), \tag{9.26}$$

$$k^{\sigma,(2)}_{o^{(2)},\sigma}(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\frac{n+1}{\sigma}\sqrt{\frac{n}{2n+1}}h^{s(2)}_{n,m}(\sigma;x)H^s_{n,m}(\sigma;y). \tag{9.27}$$

In the SGG case we calculate

$$o^{(2),\sigma}\frac{\partial F}{\partial r}(x) = \sum_{j=-1}^{\infty}\sum_{i=1}^{2}\left(\tilde{\Psi}^{(i)}_j \star \Psi^{(i)}_j * \left(k^{\sigma,(i)}_{o^{(2)}\frac{\partial}{\partial r}} \star F\right)\right)(x), \tag{9.28}$$

where the kernels $k^{\sigma,(i)}_{o^{(2)},\sigma\frac{\partial}{\partial r}}$ are given by

$$\begin{aligned} k^{\sigma,(1)}_{o^{(2)},\sigma\frac{\partial}{\partial r}}(x,y) &= \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left(-\frac{n+1}{\sigma}\right)\frac{n}{\sigma}\sqrt{\frac{n+1}{2n+1}}h^{s(1)}_{n,m}(\sigma;x)H^s_{n,m}(\sigma;y) \\ &= \left(k^{\sigma,(1)}_{o^{(2)},\sigma} \star K^{\sigma}_{\frac{\partial}{\partial r}}\right)(x,y), \end{aligned} \tag{9.29}$$

$$\begin{aligned} k^{\sigma,(2)}_{o^{(2)},\sigma\frac{\partial}{\partial r}}(x,y) &= \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\frac{n+1}{\sigma}\frac{n+1}{\sigma}\sqrt{\frac{n}{2n+1}}h^{s(2),\sigma}_{n,m}(\sigma;x)H^s_{n,m}(\sigma;y) \\ &= \left(k^{\sigma,(2)}_{o^{(2)}} \star K^{\sigma}_{\frac{\partial}{\partial r}}\right)(x,y). \end{aligned} \tag{9.30}$$

Summing up, we finally get the Meissl schemata given in Figures 9.6 and 9.7 for the vector approach.

Scalar/Vectorial/Tensorial Meissl Scheme. We get

$$\begin{aligned} &\nabla^{*,\sigma}\otimes\nabla^{*,\sigma}F(x) \\ &\quad = \sum_{j=-1}^{\infty}\sum_{\substack{(i,k)\in\\ \{(1,1),(1,2),(2,1),(2,2)\}}}\left(\tilde{\boldsymbol{\Psi}}^{(i,k)}_j \star \boldsymbol{\Psi}^{(i,k)}_j * \left(\mathbf{k}^{\sigma,(i,k)}_{\nabla^{*,\sigma}\otimes\nabla^{*,\sigma}} \star F\right)\right)(x), \end{aligned} \tag{9.31}$$

where the kernel functions $\mathbf{k}^{\sigma,(i,k)}_{\nabla^{*,\sigma}\otimes\nabla^{*,\sigma}}$ are given by

$$\mathbf{k}^{\sigma,(1,1)}_{\nabla^{s,\sigma}\otimes\nabla^{*,\sigma}}(x,y) = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\sqrt{\nu^{(1,1)}_n}\frac{n(n+1)}{\sigma^2(2n+1)(2n+3)}\mathbf{h}^{s(1,1)}_{n,m}(\sigma;x)H^s_{n,m}(\sigma;y), \tag{9.32}$$

$$\mathbf{k}^{\sigma,(1,2)}_{\nabla^{*,\sigma}\otimes\nabla^{*,R}}(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\left(-\sqrt{\nu^{(1,2)}_n}\right)\frac{(n-1)(n+1)}{\sigma^2(2n-1)(2n+1)}\mathbf{h}^{s(1,2)}_{n,m}(\sigma;x)H^s_{n,m}(\sigma;y), \tag{9.33}$$

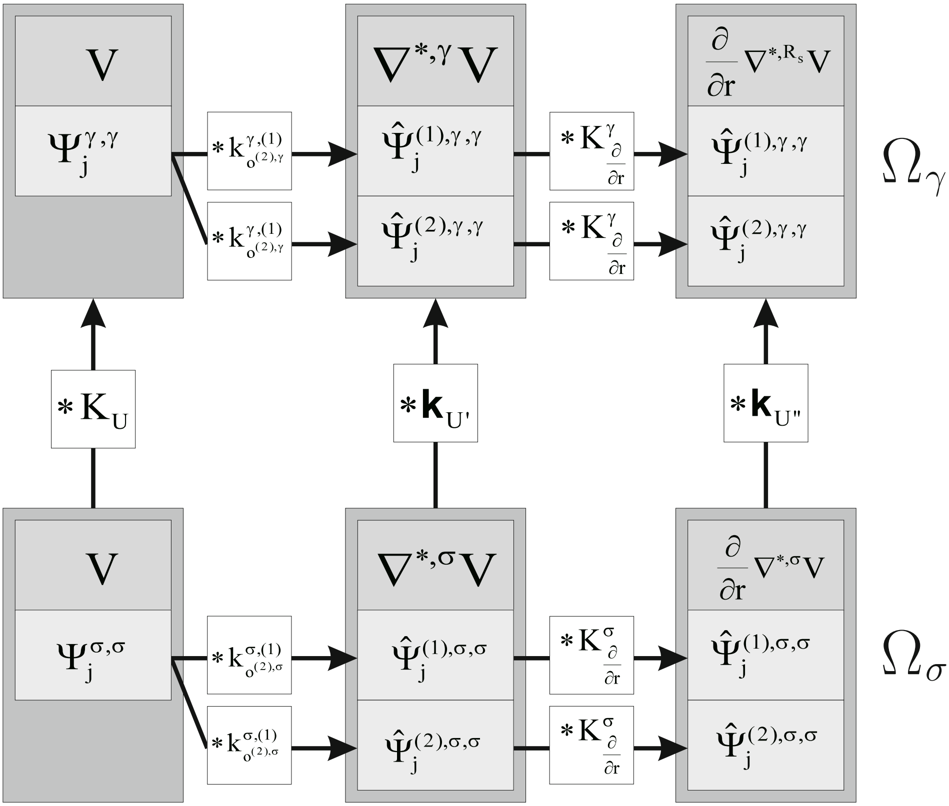


FIGURE 9.6. Meissl scheme for kernel functions (scalar/vectorial case).

$$\mathbf{k}^{\sigma,(2,1)}_{\nabla^{*,\sigma}\otimes\nabla^{*,\sigma}}(x,y)=\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\left(-\sqrt{\nu_n^{(2,1)}}\right)\frac{n(n+2)}{\sigma^2(2n+1)(2n+3)}\mathbf{h}^{s(2,1)}_{n,m}(\sigma;x)H^s_{n,m}(\sigma;y), \tag{9.34}$$

$$\mathbf{k}^{\sigma,(2,2)}_{\nabla^{*,\sigma}\otimes\nabla^{*,\sigma}}(x,y)=\sum_{n=2}^{\infty}\sum_{m=1}^{2n+1}\sqrt{\nu_n^{(2,2)}}\frac{n(n+1)(n+2)}{\sigma^2(2n-1)(2n+1)}\mathbf{h}^{s(2,2)}_{n,m}(\sigma;x)H^s_{n,m}(\sigma;y). \tag{9.35}$$

Note that the kernels $\mathbf{k}^{\sigma,(i,k)}_{\nabla^{*,\sigma}\otimes\nabla^{*,\sigma}}$, $(i,k)\in\{(1,1),(1,2),(2,1),(2,2)\}$ can be split into $\mathbf{k}^{\sigma,(i,k)}_{\nabla^{*,\sigma}\otimes\nabla^{*,\sigma}}=\sum_{l=1}^{2}\mathbf{k}^{\sigma,(i,k),(l)}_{\nabla^{*,\sigma}}\star k^{\sigma,(l)}_{o^{(2)},\sigma}$, where the kernels $\mathbf{k}^{\sigma,(i,k)(l)}_{\nabla^{*,\sigma}}$ are given by

$$\mathbf{k}^{\sigma,(1,1),(1)}_{\nabla^{*,\sigma}}=\sum_{n=0}^{\infty}\sum_{m=1}^{2n+1}\frac{n+1}{\sigma(2n+3)}\rho_n^{(1,1)}\mathbf{h}^{s(1,1)}_{n,m}(\sigma;x)h^{s(1)}_{n,m}(\sigma;y), \tag{9.36}$$

$$\mathbf{k}^{\sigma,(1,1),(2)}_{\nabla^{*,\sigma}}=\sum_{n=1}^{\infty}\sum_{m=1}^{2n+1}\left(-\frac{2n(n+1)}{\sigma(2n+1)(2n+3)}\right)\tau_n^{(1,1)}\mathbf{h}^{s(1,1)}_{n,m}(\sigma;x)h^{s(2)}_{n,m}(\sigma;y), \tag{9.37}$$

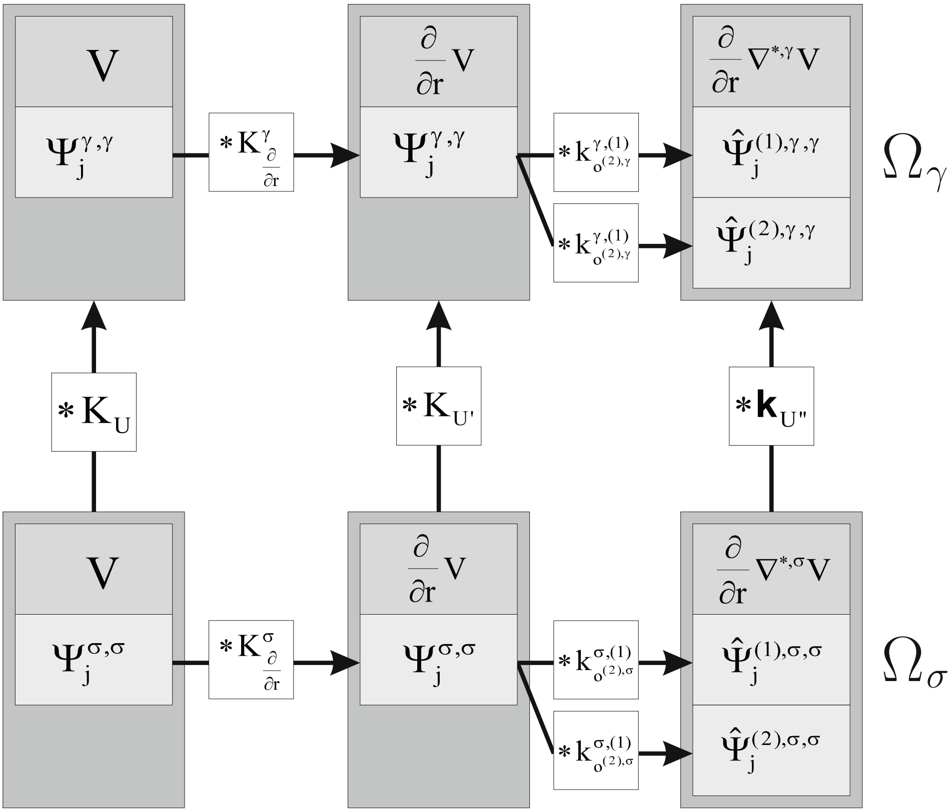


FIGURE 9.7. Meissl scheme for kernel functions (scalar/vectorial case).

$$\mathbf{k}^{\sigma,(1,2),(1)}_{\nabla^{*,\sigma}} = 0, \tag{9.38}$$

$$\mathbf{k}^{\sigma,(1,2),(2)}_{\nabla^{*,\sigma}} = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} \frac{n-1}{\sigma(2n-1)(2n+1)}\tau_n^{(1,2)}\mathbf{h}^{s(1,2)}_{n,m}(\sigma;x)h^{s(2)}_{n,m}(\sigma;y), \tag{9.39}$$

$$\mathbf{k}^{\sigma,(2,1),(1)}_{\nabla^{*,\sigma}} = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \frac{n+2}{\sigma(2n+3)}\rho_n^{(2,1)}\mathbf{h}^{s(2,1)}_{n,m}(\sigma;x)h^{s(1)}_{n,m}(\sigma;y), \tag{9.40}$$

$$\mathbf{k}^{\sigma,(2,1),(2)}_{\nabla^{*,\sigma}} = \sum_{n=1}^{\infty}\sum_{m=1}^{2n+1} \frac{2n(n+2)}{\sigma(2n+3)(2n+1)}\tau_n^{(2,1)}\mathbf{h}^{s(2,1)}_{n,m}(\sigma;x)h^{s(2)}_{n,m}(\sigma;y), \tag{9.41}$$

$$\mathbf{k}^{\sigma,(2,2),(1)}_{\nabla^{*,\sigma}} = \sum_{n=0}^{\infty}\sum_{m=1}^{2n+1} \frac{2(n+2)}{\sigma(2n-1)(2n+1)}\rho_n^{(2,2)}\mathbf{h}^{s(2,2)}_{n,m}(\sigma;x)h^{s(1)}_{n,m}(\sigma;y), \tag{9.42}$$

$$\mathbf{k}^{\sigma,(2,2),(2)}_{\nabla^{*,\sigma}} = \sum_{n=2}^{\infty}\sum_{m=1}^{2n+1} \left(-\frac{n}{\sigma(2n-1)(2n+1)}\right)\tau_n^{(2,2)}\mathbf{h}^{s(2,2)}_{n,m}(\sigma;x)h^{s(2)}_{n,m}(\sigma;y). \tag{9.43}$$

The convolution of the kernel $\mathbf{k}^{\sigma,(i,k),(l)} = \sum_{n=0}^{\infty} \sum_{m=1}^{2n+1} \mathbf{k}^{\sigma,(i,k),(l)\wedge}(n)\mathbf{h}^{s(i,k)}_{n,m} h^{s(l)}_{n,m}$ and the vector field $f^{(l)} \in h(\overline{\Omega^{\text{ext}}_\sigma})$ is given by

$$\mathbf{k}^{\sigma,(i,k),(l)} \star f^{(l)} = \sum_{n=\tilde{0}_{ik}}^{\infty} \sum_{m=1}^{2n+1} \mathbf{k}^{\sigma}, (i,k), (l) \wedge (n) f^{(l)\wedge_h}(n,m)\mathbf{h}^{s(i,k)}_{n,m}(\sigma;\cdot). \tag{9.44}$$

Thus, we get the Meissl scheme given in Figure 9.8.

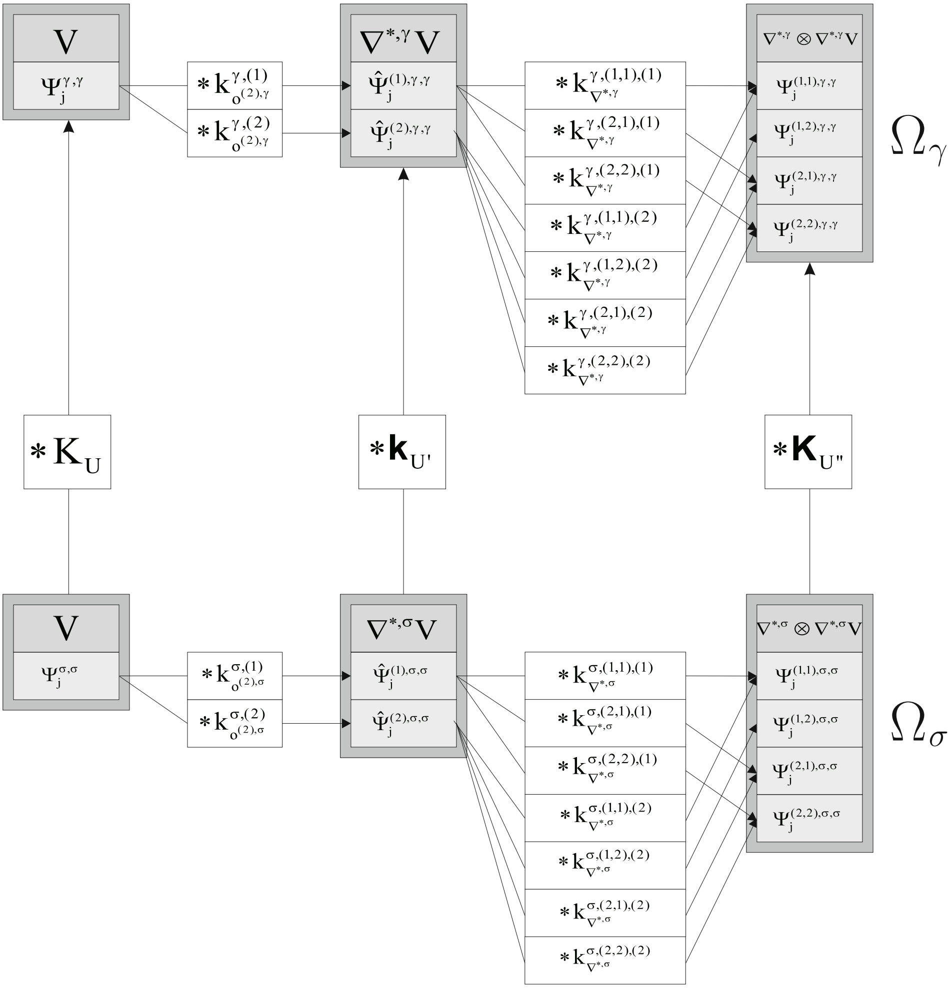


FIGURE 9.8. Meissl scheme for kernel functions (scalar/vectorial/tensorial case). (Note that the tensor-2 wavelets could not be written in bold letter for technical reasons.)

10. Conclusions

As already pointed out, accurate knowledge of the gravitational potential of the Earth is required in order to solve, for example, problems in geodesy, navigation, oceanography, solid Earth physics, and exploration geophysics. In physical geodesy it is the essential pre-stage of geoid computation. Earlier it was envisaged that the gravitational potential could be determinable as a solution of a boundary value problem. The classical problem was the Stokes problem, the boundary values were the gravity anomalies, for which the hitherto unrealistic assumption of global (terrestrial) coverage was required. But today we are confronted with the situation where also other quantities give information about the Earth's gravity potential, for example, gravity disturbance vector or second-order gradients of the disturbance potential from air- and spacecraft. In recent years the geometric shape of the Earth, continents and ocean surface, became measurable with unprecedented precision, due to the enormous progress of space methods like GNSS, VLBI, SLR, and satellite altimetry. The mathematical connection between the gravitational data within a georelevant geometry is the integrated concept. Usually, this concept is formulated in the framework of a reproducing kernel Hilbert space $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ consisting of potentials harmonic down to an internal (Runge) sphere Ω_σ. Mathematically, the gravitational (anomalous) potential of the Earth is assumed to be an element of such a space $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$. In the Hilbert space $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ any element may be represented by its expansion with respect to a complete system of kernel expressions $\mathcal{L}_i K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot)$ related to (linear) observables $\mathcal{L}_i$ on $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$. Because of the reproducing kernel structure imposed on $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$, orthonormalization of a finite system $\{\mathcal{L}_i K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot)\}_{i=1,\ldots,N}$ is equivalent to the spline problem of finding the minimum norm interpolant in the associated $\|\cdot\|_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}$-metric. When using minimum norm interpolation (or smoothing), however, the normal equation matrix $(\mathcal{L}_i\mathcal{L}_k K_{\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})}(\cdot,\cdot))_{i,k=1,\ldots,N}$ is in general a full matrix, reflecting the certain status of decorrelation guaranteed by the reproducing kernel (covariance function) under consideration. This problem causes numerical difficulties which may to a certain extend be overcome by several techniques (for example, fast summation, panel clustering, etc.). But the numerical obstacles are the main reasons why approximation methods of the Earth's gravitational field determination based on spline procedures could not keep pace with the increasing flow of observational information. In other words, the serious drawback of spline approximation is that there is no efficient transition from global to local modeling by only using one kernel (covariance) function with (fixed) space/momentum localization property.

The power of harmonic wavelets lies in the fact that kernel functions with variable space/momentum localization come into use according to a suitable dilation process. By using a sequence of more and more kernels reflecting the various levels of space/momentum localization the reference Sobolev space $\mathcal{H}(\overline{\Omega_\sigma^{\mathrm{ext}}})$ is decomposed into a nested sequence of approximating subspaces

$$\cdots\ \mathcal{V}_j(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \mathcal{V}_{j+1}(\overline{\Omega_\sigma^{\mathrm{ext}}}) \subset \cdots$$

reflecting the different stages of decorrelation. In doing so, harmonic wavelets may be used as mathematical means for breaking up a complicated function (such as the Earth's gravitational potential) into many simple pieces at different scales and positions. This allows multiresolution analysis and compression of data. The particular efficiency of wavelets is caused by the property that only a few wavelet coefficients in the wavelet table are needed in areas where the gravitational potential is "smooth", whereas stronger resolution of a complicated pattern is settled by a zooming-in capability. Wavelets offer canonical tools for combined terrestrial, airborne, and spaceborne data management under realistic assumptions imposed on the geometry of the Earth's surface and the "orbital configuration". Fast computation becomes available in form of tree algorithms. This enables gravitational potential determination with millions of data. Thus harmonic (regularization) wavelets are particularly important for inverse multiscale modeling of spaceborne data. In a subsequent step geoid computation can be based on a highly accurate gravitational potential derived from a homogeneous set of spaceborne data combined with terrestrial and/or airborne data.

For inverse multiscale modeling of spaceborne data two different ways of wavelet regularization are available, namely bandlimited truncated singular value decomposition and non-bandlimited regularization using, e.g., Tikhonov, rational, exponential, and "locally supported" kernels. In accordance with the uncertainty principle the different constituting elements of regularization may be explained as follows: *Non-bandlimited regularization wavelets* tend to be extremely space localizing. Thus huge data sets of irregular distribution can be handled since only data in a small neighborhood, whose size is determined by the particular choice of the wavelet type, is needed for the purpose of evaluating the wavelet coefficients. On the other hand, a large number of wavelet coefficients depending on the choice of the wavelet for the regularization is needed, since the wavelet coefficients only give local information of a small neighborhood. It appears that non-bandlimited regularization is an appropriate tool of local gravity surveys for oil and mineral exploration. However, little practical work has been done yet in this application area for non-synthetic data sets, although the use of linear functionals allows a very promising combination of terrestrial and/or airborne data within a unified setup in terms of wavelets. Moreover, fast summation techniques and panel clustering is adequately applicable in pyramid schemata.

Bandlimited regularization wavelets show more moderate phenomena of space localization so that one can work with smaller data sets in numerical evaluation. In consequence, the number of wavelet coefficients can be reduced, since they contain information of a more extended area. Moreover, a certain spectral band can be expressed exactly in terms of wavelets because of their bandlimited character even when the airborne data are combined with terrestrial information. Pyramid schemata can be based on exact (outer harmonic) approximation. In conclusion, dependent on the space/momentum character of the bandlimited wavelets inverse multiscale gravity modeling of spaceborne data can be handled successfully by multiresolution analysis.

Finally, it should be pointed out that our approach is given within a spherical context. Geodesists sometimes believe that ellipsoidal reference surfaces in combination with ellipsoidal harmonics might be the better choice. No doubt, an ellipsoidally reflected multiscale formulation is mathematically interesting and geodetically relevant. However, its numerical realization is by far more complicated than the spherical oriented variant chosen for our study here. As a matter of fact, Meissl schemata are involved with gravitational quantities not including the centrifugal influence. In this case, however, Runge–Walsh methods corresponding to Runge–Walsh (Bjerhammar) spheres form an adequate alternative which, in the opinion of the authors, is superior when numerical purposes come into play because of the much more efficient and economical structure inherent in spherical framework. Even better, Runge–Walsh procedures are not only applicable for ellipsoidal reference surfaces, but also for geometrically complicated reference surfaces such as telluroid, or (co)geoid.

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11. Appendix A: List of basic gravity field quantities

The list of this appendix essentially follows [ESA1]. It provides an introductory collection of quantities used in classical geodesy that could not be explained throughout the paper:

Definition	**Observation method**
Gravity potential, W: Sum of the gravitational and the centrifugal potential.	Differences between values in two points observed by levelling.
Equipotential surface: Surface where the gravity potential is equal to a constant.	Points on one surface determined regionally by tide-gauges, which define the regional mean sea level.
Height datum: The equipotential surface best agreeing with local mean sea level.	Mean sea-level calculated from tide-gauges for a specific time period.
Geoid: The equipotential surface which agrees with global mean sea level.	The equipotential surface which agrees with a global set of tide-gauges and leveling bench-marks.

Gravity: Magnitude of gradient of the gravity potential at Earth's surface and of the gravitational potential in the outer space.	Observed by absolute (e.g., free fall experiment) or relative (as a difference) spring gravimeter.
Gravity gradient: Derivatives of the gravity vector, i.e., second-order derivatives of W.	Certain linear combinations measured by torsion-balance at Earth's surface, by difference between accelerometers in space (gradiometry).
Mean Earth Ellipsoid: Ellipse rotated around the ε^3-axis, with center at the Earth's gravity center.	Surface which gives best fit to mean sea-level, and which has centrum in the gravity centre.
Height above ellipsoid: Height above mean Earth ellipsoid measured along the normal to ellipsoid.	Observed indirectly by GPS from cartesian coordinates.
Geoid height: Height of a point on the geoid above the reference ellipsoid.	Observed by GPS at tide-gauge or at leveling point.
Orthometric height: Height from geoid measured along a plumb-line (often height above mean sea-level).	Observed by leveling and converted to metric units by dividing with gravity.
GNSS: A satellite navigation system with global coverage.	GNSS: GPS, GLONASS, Galileo or Beidou.
Gravity anomaly: A model gravity potential with a reference ellipsoid as an equipotential surface is used to calculate normal gravity (needed is latitude and orthometric height).	It is a value derived by subtracting measured and normal gravity. The normal gravity is calculated in a point with the ellipsoidal height put equal to the orthometric height.

12. Appendix B: List of basic units in gravitational field theory

Units and orders in gravity field theory are the following: The gravity is expressed in m/s^2 or in milligal (1 mgal$= 10^{-5}$ m/s^2); the mean Earth gravity is about 981 000 mgal, and varies from 978 100 mgal to 983 200 mgal from equator to pole due to the Earth's flattening and rotation. Deviations due to density inhomogenities, mountain ridges, etc. range from tens to hundreds of milligals. On the other hand, the excursions of the geoid, measured from the mean Earth ellipsoid, amount to about -105 and $+90$ meters. Gravity gradients are expressed in Eőtvős ($1E = 10^{-9}$ s^2). The largest component is the vertical gravity gradient, being on Earth's surface of about $3000E$ (gravity changes by $3 \cdot 10^{-6}$ m/s^2 per meter of elevation). The horizontal components are approximately half this size, mixed gradients are below $100E$ for the normal field. Gravity gradient anomalies can be much larger and reach about $1000E$ in mountainous areas (for more details see, for example, [R4]).

	SI units	**traditional**
gravity		
	$10^{-2}ms^{-2}$	1 Gal
	$10^{-5}ms^{-2}$	1 mGal
	$10^{-8}ms^{-2}$	1μ Gal
gravity potential		
	$10m^2s^{-2}$	1 kGal $\cdot m$
gravity gradients		
	$10^{-9}s^{-2}$	$1E$

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