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On the Non-uniqueness of Gravitational and Magnetic Field Data Inversion (Survey Article)

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Abstract. The gravitational and the magnetic field of the Earth represent some of the most important observables of the geosystem. The inversion of these fields reveals hidden structures and dynamics at the surface or in the interior of the Earth (or other celestial bodies). However, the inversions of both fields suffer from a severe non-uniqueness of the solutions. In this paper, we present a generalized approach which includes the inversion of gravitational and magnetic field data. Amongst others, uniqueness constraints are proposed and compared. This includes the surface density ansatz (also known as the thin layer assumption). We characterize the null space of the considered class of inverse problems via an appropriate orthonormal basis system. Further, we expand the reconstructable part of the solution by means of orthonormal bases and reproducing kernels. One result is that information on the radial dependence of the solution is lost in the observables. As an illustration of the non-uniqueness, we show examples of anomalies which cannot be disclosed from the inversion of gravitational data. This paper is intended to be a theoretical reference work on the inversion of gravitational but also magnetic field data of the Earth.

1. Introduction

Numerous tasks in mathematical geodesy involve the regularization of ill-posed inverse problems. The reason is obvious: neither the interior of the Earth nor the Earth's surface in its entirety are accessible for exploration. However, the demand for more accurate and more localized models has dramatically increased for the last decades. As a consequence, numerous large data sets of various observables have been generated, also by means of satellite missions. These data sets often provide us with the possibility to derive models for non-observable, but urgently needed geodetic fields. Examples are the quantification of mass transports due to climate change or other phenomena (GRACE (Gravity Recovery And Climate Experiment) data are well appropriate for this purpose, see, e.g., [11, 26, 29, 52, 53]) and the modeling of those layers of the Earth which contribute to the magnetic field (this can be done with SWARM data, see, e.g., [37, 43, 50]).

This survey article presents a generalized approach which comprises, in particular, the inversion of gravitational or magnetic field data. In the former case, the unknown is the mass density distribution of the Earth's body or its surface. In the latter case, the unknown is considered to be the electric current distribution inside. In this sense, this paper is an extension of the survey article [33] on inverse gravimetry. One benefit of the generalized approach is that it makes it easier to transfer theoretical knowledge and numerical methods from one problem to the other within the considered class of problems. For example, it was shown in [23] and [33] that such a transfer yields novel achievements. Furthermore, our generalized approach also enables us to set the surface mass density approach (also known as the thin layer assumption) into the same concept with the inversion for volumetric density distribution – two approaches which have often been used parallelly and independently (see, e.g., [33, 52]). Since this paper addresses primarily a geodetic audience, we focus on the relevant facts and their interpretation. For the detailed mathematical theory including the proofs, we recommend to use the paper [34] as a supplement.

Note that the considered inversion of magnetic field data is motivated by the inversion of MEG (magnetoencephalography) data, as it occurs in medical imaging (see also [23] and the references therein). Thus, it does not represent a typical inverse problem in geomagnetics, where, for instance, material parameters like the magnetization or the susceptibility are the unknowns and not the current (see, e.g., [46]). However, the inversion of the magnetic field for currents in the interior might be interesting for investigating the outer core. Nevertheless, there is still an obvious limitation of our generalized approach with respect to the practical applicability in geomagnetics. On the other hand, reversing the point of view, the generalized approach shows a perspective how methods from medical imaging (which exist in a vast variety) could be transferred to geodetic and geophysical inverse problems.

The content and the outline of the paper are as follows: in Section 2, we summarize some basic fundamentals, like the definition of the function spaces and the orthogonal polynomials which we need.

In Section 3, we formulate the generalized class of inverse problems which represents the central theme of this paper. Then, we discuss two particular cases: the inversion of the gravitational field (this is known as the inverse gravimetric problem) and the inversion of the Bio–Savart operator of a magnetic field for getting the current distribution inside (we call this the inverse magnetic problem). With this in mind, every theoretical result that we present here for the generalized problem is valid for these two particular applications, and the derived formulae can be directly used for the precise problem by inserting the associated parameters. In Subsection 3.2, we derive a spectral relation between the given field and the unknown field. This relation directly shows the problem of the non-uniqueness which is linked to the insufficiently identifiable radial parametrization of the solution.

In Section 4, we introduce a class of orthonormal basis systems on a 3-dimensional ball. One particular instance of this class yields the well-known system of harmonic and anharmonic functions which have been used for the inverse gravimetric problem. We include some plots of the basis functions and show that the basis is appropriate for separating the solution into its projections on the null space of the solution (i.e., the indeterminable part of the solution) and on the orthogonal complement (the components of the solution which are uniquely constrained by the given data). We also show graphical illustrations of phantoms which occur, that is, examples of anomalies inside the Earth which cannot be distinguished if only gravitational data are available.

In Section 5, we discuss several modeling assumptions which can be used to obtain a unique solution: a minimum norm constraint, a harmonicity constraint, a layer density constraint and the surface density (i.e., thin layer) constraint, which is common for the identification of water mass transports.

2. Preliminaries

In this work, the set of positive integers is denoted by $\mathbb{N}$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Moreover, $\mathbb{R}$ represents the set of real numbers. The Euclidean standard $\mathbb{R}^3$-scalar product (dot product) is denoted by $\cdot$ and the cross product by $\times$. The norm associated to the Euclidean dot product is represented by $|x| := \sqrt{x \cdot x}$, $x \in \mathbb{R}^3$. Furthermore, the sphere with radius R is denoted by $\Omega_R := \left\{x \in \mathbb{R}^3 \;\middle|\; |x| = R\right\}$ and the corresponding (closed) ball is denoted by $\mathcal{B} := \left\{x \in \mathbb{R}^3 \;\middle|\; |x| \leq R\right\}$. For $R = 1$, we often use the abbreviation $\Omega := \Omega_1$. By $\mathcal{S} := \Omega_\beta$, with $\beta > R$, we denote a particular sphere in the exterior of $\mathcal{B}$. This could, for example, represent a satellite altitude or the location of airborne data.

A function $F\colon \mathcal{G} \to \mathbb{R}$ possessing k continuous derivatives on the open set $\mathcal{G} \subset \mathbb{R}^n$ is of class $\mathrm{C}^{(k)}(\mathcal{G})$, for $0 \leq k \leq \infty$. Furthermore, for a measurable set $\mathcal{G} \subset \mathbb{R}^n$, $\mathrm{L}^2(\mathcal{G})$ stands for the space of all square-integrable functions (more precisely, some equivalence classes of such functions). $\mathrm{L}^2(\mathcal{G})$ is a Hilbert space with the inner product

$$\langle F, G\rangle_{\mathrm{L}^2(\mathcal{G})} := \int_{\mathcal{G}} F(x)G(x)\,\mathrm{d}\mu(x), \qquad F,\ G \in \mathrm{L}^2(\mathcal{G}),$$

and the norm

$$\|F\|_{\mathrm{L}^2(\mathcal{G})} = \left(\int_{\mathcal{G}} F(x)^2\,\mathrm{d}\mu(x)\right)^{1/2}, \qquad F \in \mathrm{L}^2(\mathcal{G}),$$

where μ is an appropriate measure, like a surface measure ω if $\mathcal{G}$ is a surface. For a mathematically accurate definition of the space, see, for example, [42].

With $P_m^{(\alpha,\beta)}$, we denote the Jacobi polynomials, where $\alpha, \beta > -1$. They are uniquely determined by the conditions that

1. each $P_m^{(\alpha,\beta)}$ is a polynomial of degree m,
2. for all m, $n \in \mathbb{N}_0$ with $m \neq n$,

$$\left\langle P_m^{(\alpha,\beta)}, P_n^{(\alpha,\beta)} \right\rangle_{\alpha,\beta} := \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x) \, \mathrm{d}x = 0, \tag{1}$$

3. and for each $m \in \mathbb{N}_0$, we set $P_m^{(\alpha,\beta)}(1) = \binom{m+\alpha}{m}$.

For $\alpha = \beta = 0$, the Jacobi polynomials coincide with the Legendre polynomials. For further properties and the $\mathrm{L}^2[0,R]$-norm of Legendre, or (more generally) Jacobi polynomials, see [24, 36, 49].

3. Generalization of gravitational and magnetic field inversion

3.1. A class of inverse problems and examples

Within this paper, we consider a class of inverse problems which are given by a Fredholm integral operator of the first kind $T\colon \mathrm{L}^2(\mathcal{B}) \to \mathrm{L}^2(\mathcal{S})$

$$T\colon D \mapsto \int_{\mathcal{B}} D(x) k(x, \cdot) \, \mathrm{d}x = V \tag{2}$$

with an integral kernel $k\colon \mathcal{B} \times \mathcal{S} \to \mathbb{R}$ of the form

$$k(x,y) := \sum_{i=0}^{\infty} c_i \frac{|x|^{l_i}}{|y|^{i+1}} P_i\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right), \tag{3}$$

which is defined for all $(x,y) \in \mathrm{dom}(k)$, where the domain of the kernel k is given by

$$\mathrm{dom}(k) := \{(x,y) \in \mathcal{B} \times \mathcal{S} \mid x \neq 0 \text{ if there exists } i \in \mathbb{N}_0 \text{ with } l_i < 0\}.$$

In this setting, the right-hand side V in Equation (2) is given and the function D is unknown. It is the aim to reconstruct D in $\mathcal{B}$ from knowledge of V on $\mathcal{S}$. In order to have a well-defined integral kernel, which means that the series representation in (3) converges, k has to fulfil certain assumptions:

Assumption 3.1.

1. *The sequence* $(c_i)_{i\in\mathbb{N}_0}$ *is a real and bounded sequence (i.e., there exists* $c \in \mathbb{R}^+$ *such that* $\sup_{i\in\mathbb{N}_0} |c_i| \leq c$*).*
2. *The sequence of real exponents* $(l_i)_{i\in\mathbb{N}_0}$ *satisfies* $\inf_{i\in\mathbb{N}_0} l_i \geq -1$.
3. *The sequence* $(l_i)_{i\in\mathbb{N}_0}$ *fulfils the condition* $\sup_{i\in\mathbb{N}_0} R^{l_i - i} < \infty$.

Note, that the third condition implies

$$R^{i-l_i} = \frac{1}{R^{l_i-i}} \geq \frac{1}{\sup_{i\in\mathbb{N}} R^{l_i-i}} > 0.$$

This kind of integral equation arises in many areas, for example, in geosciences and medical imaging. Two examples for this inverse problem are given below.

For both, Example 3.2 (inverse gravimetric problem) and Example 3.3 (inverse magnetic problem), the conditions of Assumption 3.1 are fulfilled. In the first particular case, that is, $l_i = i$ and $c_i = \gamma$ for all $i \in \mathbb{N}_0$, the integral kernel is well known. In this case, we directly obtain $k(x, y) = \frac{\gamma}{|x-y|}$ for $|x| < |y|$, due to the identity

$$\sum_{i=0}^{\infty} \frac{|x|^i}{|y|^{i+1}} P_i \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right) = \frac{1}{|x-y|} \qquad \text{for } |x| < |y|. \tag{4}$$

Example 3.2 (The Inverse Gravimetric Problem). For the inverse gravimetric problem, the kernel and the integral operator are given by

$$T^{\mathrm{G}} \colon D \mapsto \int_{\mathcal{B}} D(x) k^{\mathrm{G}}(x, \cdot) \, \mathrm{d}x,$$
$$k^{\mathrm{G}}(x, y) := \frac{\gamma}{|x-y|} = \gamma \sum_{i=0}^{\infty} \frac{|x|^i}{|y|^{i+1}} P_i \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right),$$

where $x \in \mathcal{B}$, $y \in \mathcal{S}$, P_i denotes the Legendre polynomial of degree i and γ is the gravitational constant. $T^{\mathrm{G}} D$ is known as the gravitational potential or the Newton potential. The associated inverse problem $T^{\mathrm{G}} D = V$ represents the reconstruction of a (volumetric) mass density function from the gravitational potential, which is important, for example, for the detection of particular anomalies or mass transports. For the latter, time series of potential models have been provided, for instance, by the GRACE mission, see [10]. Note that the determination of a surface density can be regarded as a particular modeling in this context.

This problem first occurs in the works of Stokes [47] and has been widely discussed since then (see also the survey article [33]).

Example 3.3 (The Inverse Magnetic Problem). To compute the magnetic field B caused by electric sources inside a body, the quasi-static approximation of Maxwell's equation is often used, see [39].

$$E = -\nabla U \quad \text{on } \mathcal{B}, \qquad \nabla \cdot B = 0 \quad \text{on } \mathcal{B},$$
$$\nabla \times B = 0 \quad \text{on } \mathcal{S}, \qquad \nabla \times B = \mu_0 J^{\mathrm{T}} \quad \text{on } \mathcal{B},$$

where E is the electric field, U is the electric potential, $J^{\mathrm{T}} = J^{\mathrm{P}} + \sigma E$ is the total current with the primary current J^{P} and the Ohmic current σE, σ is the conductivity, and μ_0 is the permeability. It is common to use the Biot–Savart operator instead of Maxwell's equations to describe the relation between the current and the magnetic field

$$B(x) = \frac{\mu_0}{4\pi} \int_{\mathcal{B}} J^{\mathrm{T}}(y) \times \frac{x-y}{|x-y|} \, \mathrm{d}y. \tag{5}$$

In this case, we want to recover a particular component of the electric current inside $\mathcal{B}$ (which could be the Earth (in particular the outer core)). Note that this geophysical problem is closely related to a problem in medical imaging, where neuronal currents are determined from magnetoencephalography (MEG) data, see, for example, [19]. In some applications, only the reconstruction of the primary current instead of the total current or the induced current is of interest. After splitting the current in this sense and assuming a ball-shaped conductor consisting of spherical shells Ω_j with constant conductivities σ_j, one obtains the Geselowitz' formula (see [25])

$$\begin{aligned} B(x) &= \frac{\mu_0}{4\pi}\int_{\mathcal{B}} J^{\mathrm{P}}(y)\times\frac{x-y}{|x-y|^3}\,\mathrm{d}y \\ &\quad - \frac{\mu_0}{4\pi}\sum_j(\sigma_{j-1}-\sigma_j)\int_{\Omega_{j-1}} V(y)n(y)\times\frac{x-y}{|x-y|^3}\,\mathrm{d}\omega(y), \end{aligned}$$

where n is the normal vector on the surface Ω_j. With the identity in (4) and after further calculations, see [23], one gets a relation for the magnetic potential $(B=\nabla V)$

$$V(y)=\frac{1}{4\pi}\int_{\mathcal{B}}\nabla_x\cdot(J^{\mathrm{P}}(x)\times x)\sum_{i=0}^{\infty}\frac{|x|^i}{|y|^{i+1}(i+1)}P_i\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\mathrm{d}x.$$

More precisely, the vectorial current J^{P} inside $\mathcal{B}$ can be decomposed via two scalar-valued (up to an additional constant unique) functions F and G and a scalar-valued unique function J^r (see, e.g., [23]) as follows:

$$J^{\mathrm{P}}(r\xi)=\frac{1}{r}\nabla_\xi^* G(r\xi)-\frac{1}{r}\mathrm{L}_\xi^*F(r\xi)+J^r(r\xi)\xi.$$

Here, $\mathcal{B}\setminus\{0\}\ni x=r\xi$ with $\xi\in\Omega$ and $r=|x|$, ∇_ξ^* is the surface gradient, and $\mathrm{L}_\xi^*:=\xi\times\nabla_\xi^*$ is the surface curl operator on the unit sphere. Due to [45] and the above decomposition, the relation between the current and the magnetic potential V in a spherical model can be described by

$$V(y)=\frac{1}{4\pi}\int_{\mathcal{B}}\Delta^*_{\frac{x}{|x|}}F(x)\sum_{i=0}^{\infty}\frac{|x|^{i-1}}{|y|^{i+1}(i+1)}P_i\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\mathrm{d}x,$$

where $\Delta^*_{\frac{x}{|x|}}$ denotes the Beltrami operator.

Hence, only the function F and, therefore, only one tangential component of the current can be reconstructed. We use now the abbreviation $D(x):=\Delta^*_{\frac{x}{|x|}}F(x)$ such that for the inverse magnetic problem (as we call the problem here), the kernel and the integral operator are given by

$$T^{\mathrm{M}}\colon D\mapsto\int_{\mathcal{B}}D(x)k^{\mathrm{M}}(x,\cdot)\,\mathrm{d}x, \tag{6}$$

$$k^{\mathrm{M}}(x,y):=\frac{1}{4\pi}\sum_{i=0}^{\infty}\frac{|x|^{i-1}}{|y|^{i+1}(i+1)}P_i\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right), \tag{7}$$

where $x\in\mathcal{B}\setminus\{0\}$, $y\in\mathcal{S}$.

This particular modeling of the inversion of magnetic data has been used for data from MEG, as we indicated above. For two reasons, we consider a discussion to be useful: The magnetic field of a ball-shaped domain with a current in the interior is also relevant in geodesy, and there is a close link to the inverse gravimetric problem as our generalized approach suggests.

We can find further properties of the integral kernel in (3). An estimate shows that the kernel function $k(\cdot, y)$, for each fixed $y \in \mathcal{S}$, is a function in $\mathrm{L}^2(\mathcal{B})$. Indeed (with $x = r\xi$, $r \in [0, R]$, $\xi \in \Omega$) we get, using Assumption 3.1 and the fact that $|P_i(t)| \leq 1$ for all $i \in \mathbb{N}_0$ and all $t \in [-1, 1]$, the estimate

$$\begin{aligned}\int_{\mathcal{B}} (k(x,y))^2 \,\mathrm{d}x &= \int_{\mathcal{B}} \left(\sum_{i=0}^{\infty} c_i \frac{|x|^{l_i}}{|y|^{i+1}} P_i\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right)\right)^2 \mathrm{d}x \\ &\leq c^2 \int_{\mathcal{B}} \left(\sum_{i=0}^{\infty} \frac{|x|^{l_i}}{|y|^{i+1}} \right)^2 \mathrm{d}x = 4\pi c^2 \int_0^R r^2 \left(\sum_{i=0}^{\infty} \frac{r^{l_i}}{|y|^{i+1}} \right)^2 \mathrm{d}r \\ &= 4\pi c^2 \int_0^R \left(\sum_{i=0}^{\infty} \frac{r^{l_i+1}}{|y|^{i+1}} \right)^2 \mathrm{d}r \leq 4\pi R c^2 \left(\sup_{n \in \mathbb{N}_0} R^{l_n - n} \right)^2 \left(\sum_{i=0}^{\infty} \frac{R^{i+1}}{|y|^{i+1}} \right)^2 < \infty.\end{aligned}$$

The last series is convergent and, hence, finite, since it is a geometric series. With similar calculations one can prove that the interchanging between the series and the integration over $\mathcal{B}$ was allowed.

Besides the well-definition of the integral kernel, we need the existence of the integral in (2) to obtain a well-defined problem. We will later see that this is achieved if some technical conditions are fulfilled. On the other hand, for the well-posedness of the problem (in the sense of Hadamard), three questions are important.

- Does, for every right-hand side V in (2), a solution D exist?
- Is there not more than one solution D for a given V?
- Is the problem stable, that is, does D depend continuously on the data V?

The question about the non-uniqueness of the solution for the above mentioned problems has been discussed comprehensively in literature. One of the first works is the paper due to Stokes [47] for the inverse gravimetric problem. Further publications are, for example, [4, 6, 8, 48]. For a survey article on this topic, see [33]. For the inverse magnetic problem (with a focus on medical imaging), see [13–15, 19–22, 45].

In the following sections, we want to derive a possibility to characterize the null space, or in other words we want to describe the part of the solution which is non-reconstructable. We also want to formulate additional conditions to guarantee the uniqueness of the solution. For this, we need more knowledge of the forward problem.

3.2. Derivation of a spectral relation

In this subsection, it is our aim to derive an equation which connects the spherical harmonics coefficients of the given function V and the unknown function D. With this spectral relation, we are able to give answers to the questions concerning the ill-posedness of the problem. For this purpose, we analyze the forward problem. The following considerations are motivated by a similar result for the particular case of the inverse gravimetric problem, see [33]. We assume that we can choose basis functions for D which are separable into a radial and an angular part such that D is expandable in an $\mathrm{L}^2(\mathcal{B})$-convergent spherical harmonics series

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} D_{n,j}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right). \tag{8}$$

Here, $Y_{n,j}$ denotes the spherical harmonics of degree n and order j, which are an orthonormal basis for $\mathrm{L}^2(\Omega)$. Furthermore, $D_{n,j}(r)$, $r \in [0, R]$, represents the spherical harmonics coefficients for the case that D is restricted to the sphere around the origin with radius r.

By virtue of the weak convergence in Hilbert spaces, we know that

$$\begin{aligned}\int_{\mathcal{B}} D(x)F(x)\,\mathrm{d}x &= \int_{\mathcal{B}} \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} D_{n,j}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right) F(x)\,\mathrm{d}x \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_{\mathcal{B}} D_{n,j}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right) F(x)\,\mathrm{d}x\end{aligned}$$

for all functions $F \in \mathrm{L}^2(\mathcal{B})$. In particular, this holds true for the integral kernel $k(\cdot, y) \in \mathrm{L}^2(\mathcal{B})$ for all $y \in \mathcal{S}$. Inserting the expansion (8) in (2) and using the abbreviation $y = |y|\,\eta$, $x = r\xi$ with η, $\xi \in \Omega$, we get

$$\begin{aligned}V(y) &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{i=0}^{\infty} \int_0^R r^2 D_{n,j}(r) \frac{c_i r^{l_i}}{|y|^{i+1}}\,\mathrm{d}r \int_{\Omega} P_i\left(\xi \cdot \eta\right) Y_{n,j}\left(\xi\right)\,\mathrm{d}\omega(\xi) \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \sum_{i=0}^{\infty} \frac{c_i}{|y|^{i+1}} \int_0^R r^{l_i+2} D_{n,j}(r)\,\mathrm{d}r\,\frac{4\pi}{2n+1}\,\delta_{i,n} Y_{n,j}\left(\eta\right) \\ &= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \left(\int_0^R r^{l_n+2} D_{n,j}(r)\,\mathrm{d}r\right) \frac{4\pi c_n}{(2n+1)\,|y|^{n+1}} Y_{n,j}\left(\eta\right).\end{aligned} \tag{9}$$

In the first step the reproducing property of the reproducing kernel for the spherical harmonics of degree n, given by

$$\Omega^2 \ni (\xi, \eta) \mapsto \frac{2n+1}{4\pi} P_n(\xi \cdot \eta),$$

is used. More precisely,

$$\frac{2n+1}{4\pi}\int_\Omega P_i(\xi\cdot\eta)Y_{n,j}(\xi)\,\mathrm{d}\omega(\xi) = Y_{n,j}(\eta)\delta_{i,n} \tag{10}$$

for all $\eta \in \Omega$. We also remark that the existence of the integral in (2) only depends on the existence of the integral of the radial part and the convergence of the series in (9). Regarding the latter, we obtain a pointwise convergence of (9) for $y \in \mathcal{S}$, since the following estimate of the summands in (9) (note that $\max_{\xi\in\Omega}|Y_{n,j}(\xi)| \le \sqrt{(2n+1)/(4\pi)}$ for all $n \in \mathbb{N}_0$) holds true:

$$\begin{aligned}
&\left|\int_0^R r^{l_n+2}D_{n,j}(r)\,\mathrm{d}r\ \frac{4\pi c_n}{2n+1}\ |y|^{-n-1}Y_{n,j}\left(\frac{y}{|y|}\right)\right| \\
&\quad\le \left(\frac{R^{2l_n+3}}{2l_n+3}\int_0^R r^2\left(D_{n,j}(r)\right)^2\,\mathrm{d}r\right)^{1/2}\frac{4\pi c}{2n+1}\ |y|^{-n-1}\sqrt{\frac{2n+1}{4\pi}} \\
&\quad\le c\left(\frac{R^{2l_n+3}}{R^{2n+2}(2l_n+3)}\int_0^R r^2\left(D_{n,j}(r)\right)^2\,\mathrm{d}r\frac{4\pi}{2n+1}\right)^{1/2}\left(\frac{R}{|y|}\right)^{n+1}.
\end{aligned}$$

The right-hand side is bounded for all $n \in \mathbb{N}_0$, due to the conditions on $(l_n)_{n\in\mathbb{N}}$ (see Assumption 3.1, items 2 and 3) and the convergence of the Parseval identity of $D \in \mathrm{L}^2(\mathcal{B})$. Hence, the series (9) is dominated by a geometric series for all $y \in \mathcal{S}$ (i.e., $|y| > R$).

We are also able to extend the function V onto Ω_R. In addition, for $V|_{\Omega_R}$, we obtain the $\mathrm{L}^2(\Omega_R)$-convergence of the series representation in Equation (9). This convergence is a direct consequence of the Cauchy–Schwarz inequality and the Parseval identity (note that $\{\frac{1}{R}Y_{n,j}(\frac{\cdot}{R})\}_{n\in\mathbb{N}_0,j=1,\dots,2n+1}$ is an orthonormal basis of $\mathrm{L}^2(\Omega_R)$), since

$$\begin{aligned}
\|V|_{\Omega_R}\|^2_{\mathrm{L}^2(\Omega_R)} &= \sum_{n=0}^\infty\sum_{j=1}^{2n+1}\left(\int_0^R r^{l_n+2}D_{n,j}(r)\,\mathrm{d}r\right)^2\left(\frac{4\pi c_n}{(2n+1)R^n}\right)^2 \\
&\le \sum_{n=0}^\infty\sum_{j=1}^{2n+1}\left(\int_0^R \frac{r^{2l_n+2}}{R^{2n}}\,\mathrm{d}r\right)\left(\int_0^R r^2(D_{n,j}(r))^2\,\mathrm{d}r\right)\left(\frac{4\pi c}{2n+1}\right)^2 \\
&\le 16\pi^2c^2\sum_{n=0}^\infty\sum_{j=1}^{2n+1}\frac{R^{2l_n+3-2n}}{2l_n+3}\left(\int_0^R r^2(D_{n,j}(r))^2\,\mathrm{d}r\right) \\
&\le 16\pi^2c^2R^3\sup_{n\in\mathbb{N}_0}\frac{R^{2l_n-2n}}{2l_n+3}\,\|D\|^2_{\mathrm{L}^2(\mathcal{B})} < \infty.
\end{aligned}$$

Hence, Equation (9) is valid pointwise on $\mathcal{S}$ and in the sense of $\mathrm{L}^2(\Omega_R)$ on Ω_R.

In order to find a direct relation between the Fourier coefficients of the given function V and the unknown function D, we consider the Fourier coefficients of V restricted to the sphere Ω_R. This relation can be seen directly from (9).

Theorem 3.4. *Consider the orthonormal basis system on Ω_R given by the set of functions $\{\frac{1}{R}Y_{n,j}(\frac{\cdot}{R})\}_{n\in\mathbb{N}_0,j=1,\ldots,2n+1}$. Then, the Fourier coefficients of V defined by $V_{n,j} := \langle V|_{\Omega_R}, \frac{1}{R}Y_{n,j}\left(\frac{\cdot}{R}\right)\rangle_{\mathrm{L}^2(\Omega_R)}$ satisfy the identity*

$$V_{n,j} = \left(\int_0^R r^{l_n+2} D_{n,j}(r)\,\mathrm{d}r\right) \frac{4\pi c_n}{(2n+1)R^n}.$$

for all $n \in \mathbb{N}_0$, $j = 1,\ldots,2n+1$. This yields the equation

$$\frac{(2n+1)R^n}{4\pi c_n} V_{n,j} = \int_0^R r^{l_n+2} D_{n,j}(r)\,\mathrm{d}r, \qquad \text{if } c_n \neq 0, \tag{11}$$

otherwise $V_{n,j} = 0$ with $j = 1,\ldots,2n+1$, respectively.

The relation from Theorem 3.4 allows an infinite number of choices for $D_{n,j}$ and, hence, the solution D cannot be uniquely determined by the function $V|_{\Omega_R}$. For the inverse gravimetric problem, the last relation is well known, see, for example, [35, 38, 41], and for the inverse magnetic problem for $R = 1$, see for instance [21]. Analogously, we obtain with (remember that $\mathcal{S} = \Omega_\beta$, $\beta > R$) for all $n \in \mathbb{N}_0$, $j = 1,\ldots,2n+1$

$$\begin{aligned} V_{n,j}^{\mathcal{S}} &:= \left\langle V|_{\mathcal{S}}, \frac{1}{\beta}Y_{n,j}\left(\frac{\cdot}{\beta}\right)\right\rangle_{\mathrm{L}^2(\mathcal{S})} \\ &= \left(\int_0^R r^{l_n+2} D_{n,j}(r)\,\mathrm{d}r\right) \frac{4\pi c_n}{(2n+1)\beta^n} = V_{n,j}\left(\frac{R}{\beta}\right)^n \end{aligned} \tag{12}$$

the spherical harmonics coefficients of V with respect to an orthonormal basis system on $\mathcal{S}$. Hence, we have a direct relation between the singular values of the Fredholm integral operator T and the spherical harmonic coefficients $V_{n,j}$. The additional factor $(\frac{R}{\beta})^n$ symbolizes the upward continuation from $\mathcal{S}$ to Ω_R. The upward continuation does not effect the null space of the operator T at all. Due to this property and the aim to keep the formulae simple, we analyze Equation (11) further and keep in mind that we can consequently deduce properties of T via Equation (12).

Note that (11) shows, in particular, the degree of freedom with respect to the radial part of D, since $V_{n,j}$ is some weighted radial mean of $D_{n,j}(r)$. On the other hand, one can expect a one-to-one relation for the angular dependence of V and D.

4. Investigation of the homogeneous problem

In order to obtain a unique solution, an appropriate modeling is required, that is, the solution space has to be restricted by certain constraints. Before this can be done (in Section 5), we have to study the null space $\ker T$, that is, the space of all D with $TD = 0$. Note that, due to the linearity, all solutions of $TD = V$ are given by $\tilde{D} + D_0$, with arbitrary $D_0 \in \ker T$, for a particular solution $\tilde{D}$ of $TD = V$.

4.1. Some orthonormal basis functions on the ball

It is our aim to characterize the null space, that is, the so-called kernel of the Fredholm integral operator of the first kind, in order to describe the non-reconstructable parts of the solution. For the separation of $\mathrm{L}^2(\mathcal{B})$ into the null space and the orthogonal complement we need an appropriate basis for $\mathrm{L}^2(\mathcal{B})$.

For the ball, there are several known basis systems available. For the construction of these systems see, for example, [1, 7, 17, 30, 32, 51]. We analogously use the idea to combine an orthonormal basis system on the unit sphere with one on the interval $[0, R]$, to construct a basis system on the ball. The $\mathrm{L}^2(\mathcal{B})$-orthonormal system used here is a generalization of the system which was introduced in [17] and [7].

For $x \in \mathcal{B} \setminus \{0\}$, it is given by

$$G_{m,n,j}(x) \coloneqq \gamma_{m,n} P_m^{(0,l_n+1/2)} \left(2\frac{|x|^2}{R^2} - 1 \right) \frac{|x|^{l_n}}{R^{l_n}} Y_{n,j}\left(\frac{x}{|x|} \right), \tag{13}$$

with $m, n \in \mathbb{N}_0$, $j = 1, \ldots, 2n+1$, where $\{P_m^{(\alpha,\beta)}\}_{m\in\mathbb{N}_0}$ are the Jacobi polynomials and $\gamma_{m,n}$ are normalization constants with

$$\gamma_{m,n} \coloneqq \sqrt{\frac{4m + 2l_n + 3}{R^3}}\,. \tag{14}$$

Since $\alpha = 0$ in Equation (13) and $P_m^{(0,l_n+1/2)}(1) = 1$ for all $m, n \in \mathbb{N}_0$, we get $G_{m,n,j}|_{\Omega_R} = \gamma_{m,n} Y_{n,j}(\frac{\cdot}{R})$.

The functions in (13) were called $G^{\mathrm{I}}_{m,n,j}$ in [31] and [32] in the case of $l_n = n$ (remember that this setting corresponds to the inverse gravimetric problem).

A continuous expansion of our functions $G_{m,n,j}$ on the domain $\mathcal{B}$ is possible, if all exponents l_n, $n \in \mathbb{N}$, are positive. Otherwise we obtain a singularity at the origin of the functions $G_{m,n,j}$ for negative values of l_n and a discontinuity at the very same place in the case $l_n = 0$ for $n > 0$. For the theory stated in this paper, this is not a problem, since $G_{m,n,j}$ remains square-integrable for $-1 \leq l_n$ (as we required).

As we claimed above, the functions $G_{m,n,j}$ for $m, n \in \mathbb{N}_0$, $j = 1, \ldots, 2n+1$ given in (13) build an orthonormal basis for $\mathrm{L}^2(\mathcal{B})$. This property can easily be verified by calculating the inner products and using a formula for a weighted L^2-norm of Jacobi polynomials (see, e.g., [36]). With the $\mathrm{L}^2(\Omega)$-orthogonality of the spherical harmonics and the substitution $r = R\sqrt{(1+z)/2}$, we obtain

$$\begin{aligned}
&\langle G_{m,n,j}, G_{\mu,\nu,\iota} \rangle_{\mathrm{L}^2(\mathcal{B})} \\
&\quad = \gamma_{m,n}\gamma_{\mu,\nu}\delta_{\nu,n}\delta_{\iota,j} \int_0^R \frac{r^{2l_n+2}}{R^{2l_n}} P_m^{(0,l_n+1/2)}\left(2\frac{r^2}{R^2} - 1\right) P_\mu^{(0,l_n+1/2)}\left(2\frac{r^2}{R^2} - 1\right) \mathrm{d}r \\
&\quad = \gamma_{m,n}\gamma_{\mu,n}\delta_{\nu,n}\delta_{\iota,j} \frac{R^3}{2^{l_n+5/2}} \int_{-1}^1 (1+z)^{l_n+1/2} P_m^{(0,l_n+1/2)}(z) P_\mu^{(0,l_n+1/2)}(z)\,\mathrm{d}z
\end{aligned}$$

$$= \gamma_{m,n}\gamma_{\mu,n}\delta_{\nu,n}\delta_{\iota,j}\frac{R^3}{2^{l_n+5/2}}\frac{2^{l_n+3/2}}{2m+l_n+3/2}\,\delta_{\mu,m}$$

$$= \gamma_{m,n}^2\delta_{\nu,n}\delta_{\iota,j}\frac{R^3}{4m+2l_n+3}\,\delta_{\mu,m} = \delta_{\mu,m}\delta_{\nu,n}\delta_{\iota,j}.$$

Thus, the set $\{G_{m,n,j}\}_{m,n\in\mathbb{N}_0,j=1,\ldots,2n+1}$ is $\mathrm{L}^2(\mathcal{B})$-orthonormal. Moreover, the spherical harmonics are complete in $\mathrm{L}^2(\Omega)$ and the Jacobi polynomials are complete with respect to the inner product in (1) such that the system

$$\{G_{m,n,j}\}_{m,n\in\mathbb{N}_0,j=1,\ldots,2n+1}$$

is complete in $\mathrm{L}^2(\mathcal{B})$ and constitutes an orthonormal basis.

Some of the functions $G^{\mathrm{I}}_{m,n,j}$ (i.e., in the case of $l_n = n$) are shown in Figures 1 and 2. For $m = 0$, the functions $G^{\mathrm{I}}_{0,n,j}$ are inner harmonics, hence they are harmonic, and attain their maximum and minimum on the boundary. A selection of the functions corresponding to the inverse magnetic problem, where $l_n = n-1$, is shown in Figures 3 and 4. The singularity (for $n = 0$, i.e., $l_0 = -1$) at the origin is visible in Figures 3 (A) and (C) and Figure 4 (B).

4.2. Splitting the basis into the null space and its complement

With the orthonormal basis introduced in Subsection 4.1, we are now able to expand the functions $D_{n,j}$ in (8) for all $n \in \mathbb{N}_0$ and $j = 1,\ldots,2n+1$ and we obtain

$$D_{n,j}(r) = \frac{r^{l_n}}{R^{l_n}}\sum_{m=0}^{\infty} d_{m,n,j}\gamma_{m,n}P_m^{(0,l_n+1/2)}\left(2\frac{r^2}{R^2}-1\right), \tag{15}$$

where $d_{m,n,j} \coloneqq \langle D, G_{m,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})}$ and $\gamma_{m,n}$ is given in (14).

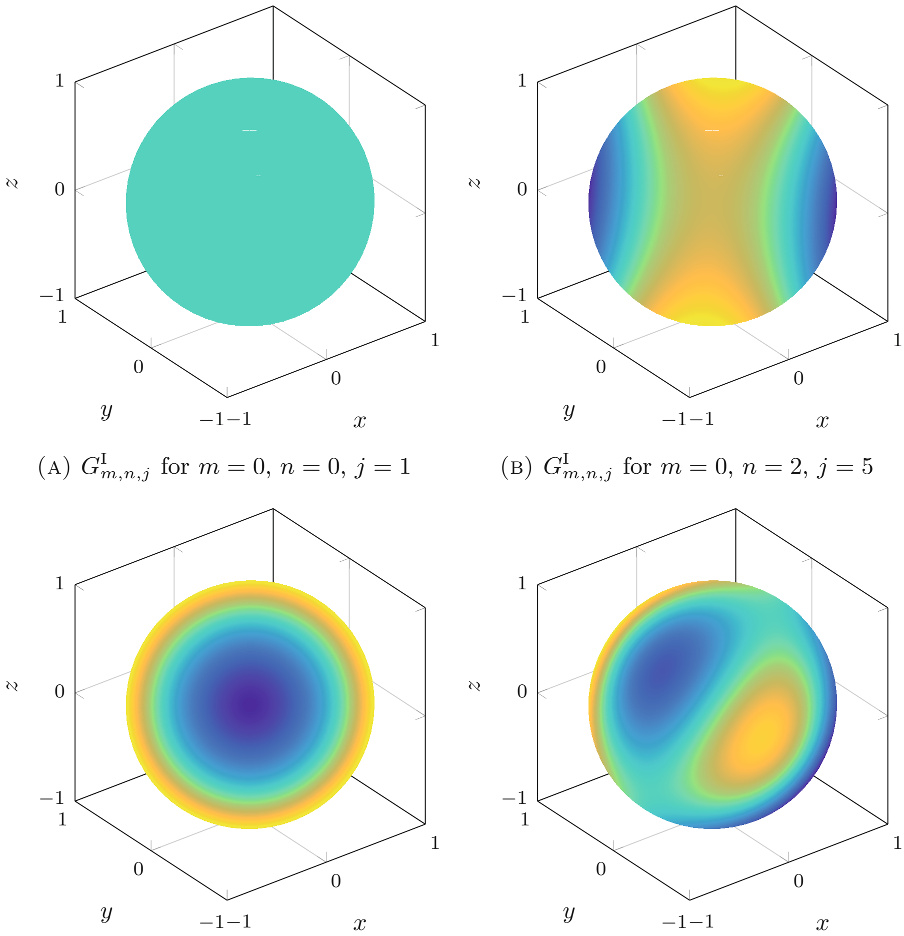
For further investigations of the forward problem, we use the representation of (the known function) V in (9), where we have already calculated the integral over the angular part. For the remaining integral over the radial part, we use the precise representation of $D_{n,j}$ in (15) and the orthogonality of the Jacobi polynomials.With the substitution $r = R\sqrt{(1+z)/2}$, $\mathrm{d}r = \frac{R}{4}\left(\frac{2}{1+z}\right)^{1/2}\mathrm{d}z$, we get

$$\int_0^R r^{l_n+2}D_{n,j}(r)\,\mathrm{d}r = \int_0^R \frac{r^{2l_n+2}}{R^{l_n}}\sum_{m=0}^{\infty} d_{m,n,j}\gamma_{m,n}P_m^{(0,l_n+1/2)}\left(2\frac{r^2}{R^2}-1\right)\mathrm{d}r$$

$$= \frac{R^{3+l_n}}{2^{l_n+5/2}}\sum_{m=0}^{\infty} d_{m,n,j}\gamma_{m,n}\int_{-1}^{1}(1+z)^{l_n+1/2}\,P_m^{(0,l_n+1/2)}(z)\,\mathrm{d}z = \frac{R^{3+l_n}}{2l_n+3}d_{0,n,j}\gamma_{0,n}.$$

Inserting the latter result in (9), we eventually obtain (remember the definition of $\gamma_{m,n}$ in (14))

$$V(y) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1}\left(\int_0^R r^{l_n+2}D_{n,j}(r)\,\mathrm{d}r\right)\frac{4\pi c_n}{(2n+1)\left|y\right|^{n+1}}Y_{n,j}\left(\frac{y}{|y|}\right)$$



(A) $G^{\mathrm{I}}_{m,n,j}$ for $m = 0$, $n = 0$, $j = 1$

(B) $G^{\mathrm{I}}_{m,n,j}$ for $m = 0$, $n = 2$, $j = 5$

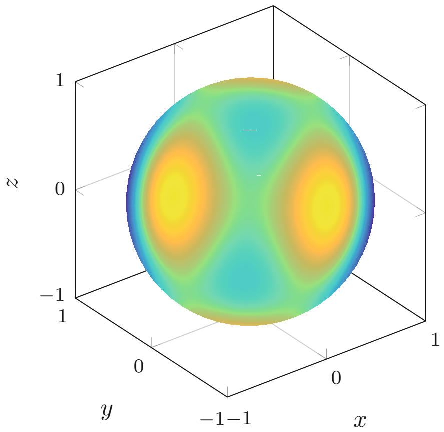
(C) $G^{\mathrm{I}}_{m,n,j}$ for $m = 1$, $n = 0$, $j = 1$

(D) $G^{\mathrm{I}}_{m,n,j}$ for $m = 1$, $n = 1$, $j = 3$

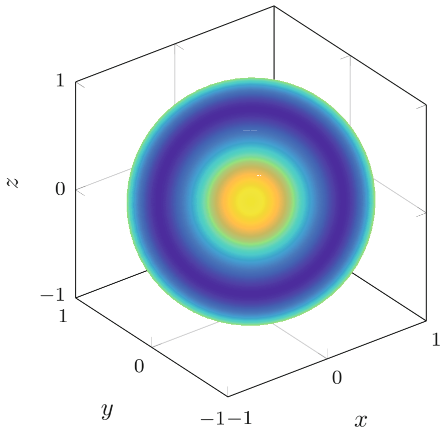
FIGURE 1. The functions $G_{m,n,j}$ in the case $l_n = n$ (also called $G^{\mathrm{I}}_{m,n,j}$) for different parameters m, n, j are plotted at the plane through the origin with normal vector $(1, 1, -1)^{\mathrm{T}}$. For the particular parameters, see the respective caption. The maximum is always yellow and the minimum is blue (see also [32, 34]).

$$= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{R^{3+l_n} 4\pi c_n}{(2l_n + 3)(2n + 1)|y|^{n+1}} d_{0,n,j} \gamma_{0,n} Y_{n,j}\left(\frac{y}{|y|}\right)$$

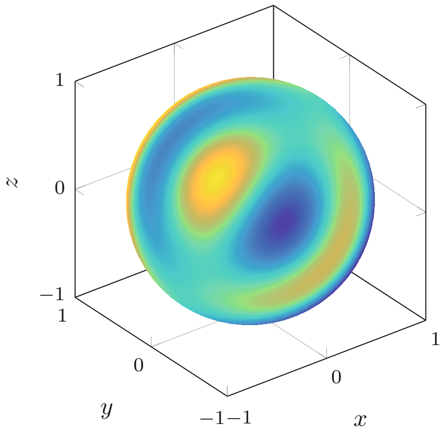
$$= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{4\pi c_n R^{l_n}}{(2n + 1)\,|y|^{n+1}} d_{0,n,j} \gamma_{0,n}^{-1} Y_{n,j}\left(\frac{y}{|y|}\right). \quad (16)$$



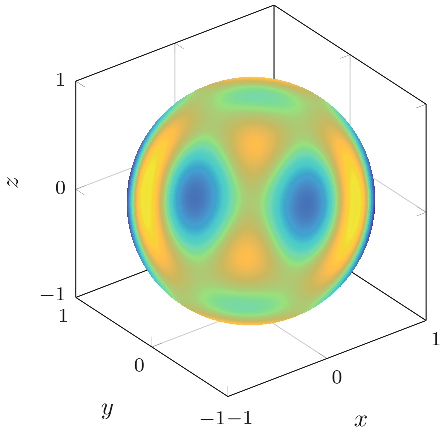
(A) $G^{\mathrm{I}}_{m,n,j}$ for $m = 1$, $n = 2$, $j = 5$



(B) $G^{\mathrm{I}}_{m,n,j}$ for $m = 2$, $n = 0$, $j = 1$



(C) $G^{\mathrm{I}}_{m,n,j}$ for $m = 2$, $n = 1$, $j = 3$

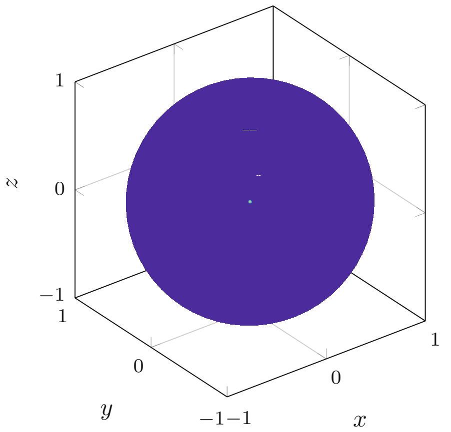


(D) $G^{\mathrm{I}}_{m,n,j}$ for $m = 2$, $n = 2$, $j = 5$

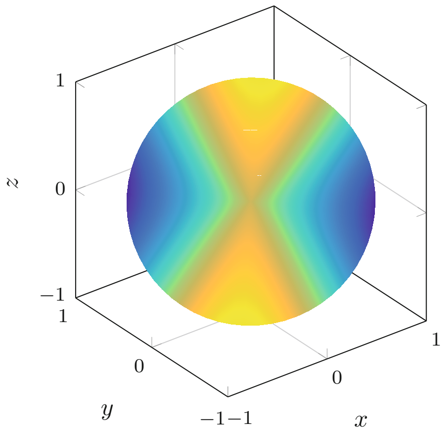
FIGURE 2. The functions $G_{m,n,j}$ in the case $l_n = n$ (also called $G^{\mathrm{I}}_{m,n,j}$) for different parameters m, n, j are plotted at the plane through the origin with normal vector $(1, 1, -1)^{\mathrm{T}}$. For the particular parameters, see the respective caption. The maximum is always yellow and the minimum is blue (see also [32, 34]).

Hence, $G_{m,n,j}$ is in the null space of the operator T with the kernel from (3), if and only if $m > 0$ or $c_n = 0$. Examples of functions in the null space are given in Figures 2 and 4 (for different inverse problems). The function plotted in Figure 5 is not in the null space.

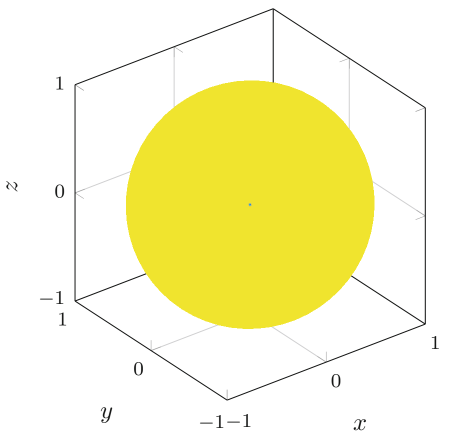
Since $\mathrm{L}^2(\mathcal{B})$ is the direct sum of the null space $\ker T$ and its orthogonal complement, the obtained result allows a precise characterization of the null space



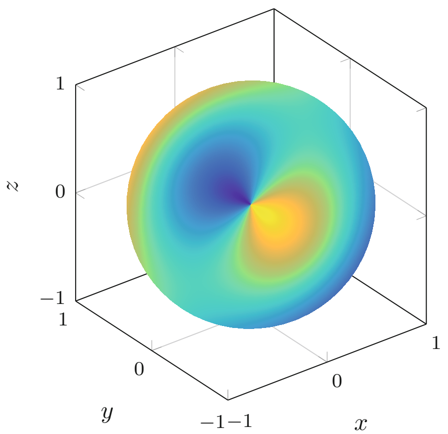
(A) $G_{m,n,j}$ for $m = 0$, $n = 0$, $j = 1$



(B) $G_{m,n,j}$ for $m = 0$, $n = 2$, $j = 5$



(C) $G_{m,n,j}$ for $m = 1$, $n = 0$, $j = 1$



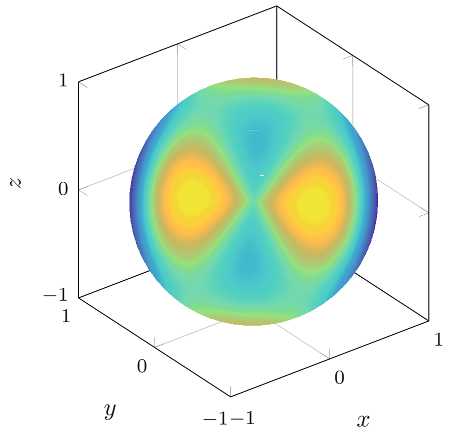
(D) $G_{m,n,j}$ for $m = 1$, $n = 1$, $j = 3$

FIGURE 3. The functions $G_{m,n,j}$ in the case $l_n = n - 1$ for different parameters m, n, j are plotted at the plane through the origin with normal vector $(1, 1, -1)^{\mathrm{T}}$. For the particular parameters, see the respective caption. The maximum is always yellow and the minimum is blue (see also [34]).

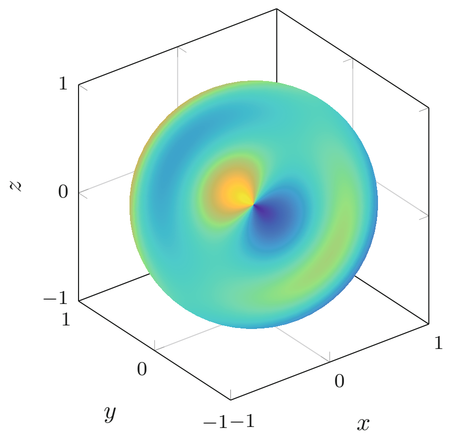
of the corresponding Fredholm integral operator as

$$\ker T = \overline{\operatorname{span}\left\{G_{m,n,j} \;\middle|\; m \geq 1, n \in \mathbb{N}_0, j = 1, \ldots, 2n+1 \text{ or } c_n = 0\right\}}^{\|\cdot\|_{\mathrm{L}^2(\mathcal{B})}}. \quad (17)$$

For the inverse gravimetric problem ($l_n = n$), we can deduce the well-known fact that the null space can be described as the set of all anharmonic functions, which are the elements of the orthogonal complement of the set of all harmonic



(A) $G_{m,n,j}$ for $m = 1$, $n = 2$, $j = 5$ (B) $G_{m,n,j}$ for $m = 2$, $n = 0$, $j = 1$



(C) $G_{m,n,j}$ for $m = 2$, $n = 1$, $j = 3$ (D) $G_{m,n,j}$ for $m = 2$, $n = 2$, $j = 5$

FIGURE 4. The functions $G_{m,n,j}$ in the case $l_n = n - 1$ for different parameters m, n, j are plotted at the plane through the origin with normal vector $(1, 1, -1)^{\mathrm{T}}$. For the particular parameters, see the respective caption. The maximum is always yellow and the minimum is blue (see also [34]).

functions. That is,

$$\ker T^{\mathrm{G}} = \overline{\operatorname{span}\left\{G^{\mathrm{I}}_{m,n,j} \;\middle|\; m \geq 1\right\}}^{\|\cdot\|_{\mathrm{L}^2(\mathcal{B})}} = \left\{F \in \mathrm{C}^{(2)}(\mathcal{B}) \;\middle|\; \Delta F = 0\right\}^{\perp_{\mathrm{L}^2(\mathcal{B})}},$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ represents the Laplace operator. In this case, the functions $G^{\mathrm{I}}_{0,n,j}$, $n \in \mathbb{N}_0$, $j = 1, \ldots, 2n+1$ are the inner harmonics and, therefore,

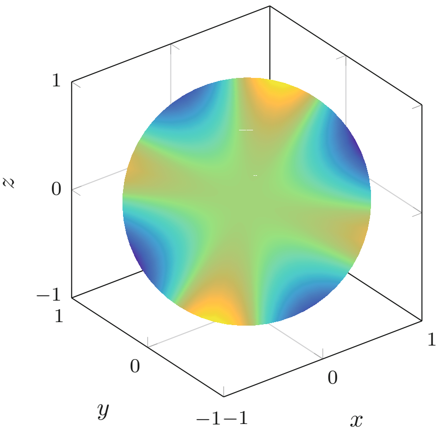


FIGURE 5. The function $G_{0,4,8}$, which is not in the null space of the Fredholm integral operator T for $l_n = n - 1$.

form a basis for the set of all harmonic functions on the ball:

$$G^{\mathrm{I}}_{0,n,j}(x) = \sqrt{\frac{2n+3}{R}} \frac{|x|^n}{R^{n+1}} Y_{n,j}\left(\frac{x}{|x|}\right), \qquad x \in \mathcal{B}.$$

For some particular cases of the considered Fredholm integral operators, we are also able to find a characterization of the null space via an elliptic partial differential equation.

For this purpose, we consider the particular integral kernel

$$k(x,y) := \sum_{i=0}^{\infty} c_i \frac{|x|^{i+\kappa}}{|y|^{i+1}} P_i\left(\frac{x}{|x|} \cdot \frac{y}{|y|}\right), \qquad (x,y) \in \mathrm{dom}(k),$$

for a fixed $\kappa \in [-1, \infty)$ and $c_i \neq 0$ for all $i \in \mathbb{N}_0$. Note that in the case of the inverse gravimetric problem $\kappa = 0$ and in the case of the inverse magnetic problem $\kappa = -1$. We have already proven that the orthogonal complement of the null space of the corresponding operator T is given by the set

$$(\ker T)^{\perp_{\mathrm{L}^2(\mathcal{B})}} = \overline{\mathrm{span}\left\{ G_{m,n,j} \,\middle|\, m = 0, n \in \mathbb{N}_0, j = 1, \ldots, 2n+1 \text{ and } c_n \neq 0 \right\}}^{\|\cdot\|_{\mathrm{L}^2(\mathcal{B})}}.$$

Now, we define an elliptic partial differential operator $\tilde{\Delta}$ by

$$\tilde{\Delta} F(r\xi) := \Delta\left(r^{-\kappa} F(r\xi)\right) = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*_\xi\right)\left(r^{-\kappa} F(r\xi)\right).$$

Using the product rule for the derivative, we get

$$\begin{aligned}\tilde{\Delta}F(r\xi) &= \left(-\kappa(-\kappa-1)r^{-\kappa-2} - 2\kappa r^{-\kappa-1}\frac{\partial}{\partial r} + r^{-\kappa}\frac{\partial^2}{\partial r^2} - 2\kappa r^{-\kappa-2}\right.\\ &\qquad \left. +2r^{-\kappa-1}\frac{\partial}{\partial r} + r^{-\kappa-2}\Delta^*_\xi\right) F(r\xi)\\ &= \left(r^{-\kappa}\frac{\partial^2}{\partial r^2} + 2(1-\kappa)r^{-\kappa-1}\frac{\partial}{\partial r} + \kappa(\kappa-1)r^{-\kappa-2} + r^{-\kappa-2}\Delta^*_\xi\right) F(r\xi).\end{aligned}$$

In the particular case of the inverse gravimetric problem (i.e., $\kappa = 0$) this reduces to

$$\tilde{\Delta}F(r\xi) = \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta^*_\xi\right) F(r\xi) = \Delta F(r\xi),$$

and the differential operator corresponding to the inverse magnetic problem (i.e., $\kappa = -1$) is given by

$$\tilde{\Delta}F(r\xi) = \left(r\frac{\partial^2}{\partial r^2} + 4\frac{\partial}{\partial r} + \frac{2}{r} + \frac{1}{r}\Delta^*_\xi\right) F(r\xi).$$

In order to get a new characterization of the null space, we apply the differential operator to the basis functions $G_{0,n,j}$ for $n \in \mathbb{N}_0$, $j = 1, \ldots, 2n+1$ and obtain

$$\begin{aligned}\tilde{\Delta}G_{0,n,j}(r\xi) &= \tilde{\Delta}\left(\gamma_{0,n}\left(\frac{r}{R}\right)^{n+\kappa} Y_{n,j}(\xi)\right) = \gamma_{0,n}\Delta\left(r^{-\kappa}\left(\frac{r}{R}\right)^{n+\kappa} Y_{n,j}(\xi)\right)\\ &= \frac{\gamma_{0,n}}{R^{n+\kappa}}\Delta\left(r^n Y_{n,j}(\xi)\right) = 0,\end{aligned}$$

since the mapping $r\xi \mapsto r^n Y_{n,j}(\xi)$ is a harmonic function for all $n \in \mathbb{N}_0$, $j = 1, \ldots, 2n+1$. In analogy, $\tilde{\Delta}G_{m,n,j} \not\equiv 0$ for $m \geq 1$, $n \in \mathbb{N}_0$, $j = 1, \ldots, 2n+1$ follows by similar considerations. This means that $\tilde{\Delta}F$ is equal to zero if and only if $r\xi \mapsto r^{-\kappa}F(r\xi)$ is a harmonic function, that is, is contained in

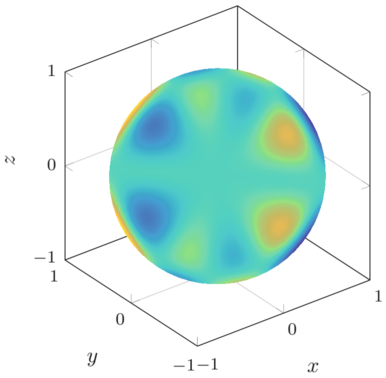
$$\operatorname{span}\left\{G^{\mathrm{I}}_{0,n,j}\right\}_{n\in\mathbb{N}_0, j=1\ldots,2n+1}.$$

Since this is equivalent to expanding $F(r\xi)$ in terms of $r^\kappa G^{\mathrm{I}}_{0,n,j}(r\xi)$ and $l_n = n+\kappa$ here, our definition in (13) leads us to the following result.

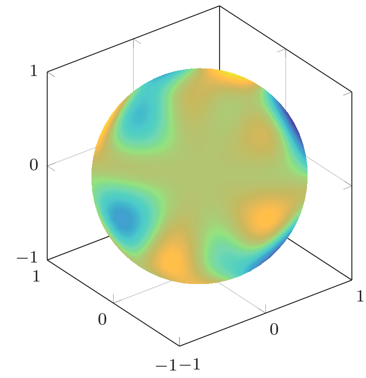
Theorem 4.1. *If we assume that there exists a fixed parameter $\kappa \geq -1$ such that $l_n = n+\kappa$ for all $n \in \mathbb{N}_0$ and that $c_n \neq 0$ for all $n \in \mathbb{N}_0$, then*

$$\begin{aligned}\ker T &= \overline{\operatorname{span}\left\{G_{m,n,j} \mid m > 0, n \in \mathbb{N}_0, j = 1, \ldots, 2n+1\right\}}^{\|\cdot\|_{\mathrm{L}^2(\mathcal{B})}}\\ &= \left\{F\colon \mathcal{B} \to \mathbb{R} \;\middle|\; (r\xi \mapsto r^{-\kappa}F(r\xi)) \in \mathrm{C}^{(2)}(\mathcal{B}) \text{ and } \tilde{\Delta}F = 0\right\}^{\perp_{\mathrm{L}^2(\mathcal{B})}}. \qquad (18)\end{aligned}$$

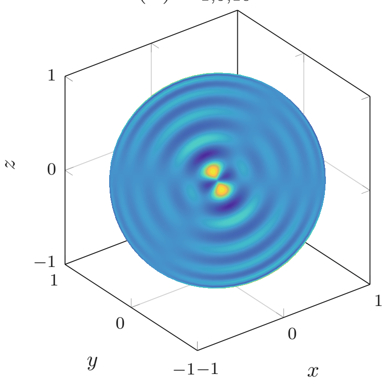
After having given two mathematical characterizations of the null space in Equation (18) for a particular case (i.e., $l_n = n+\kappa$) and one characterization in Equation (17) for the general case, we want to demonstrate what kind of functions D generate the same forward solution V.



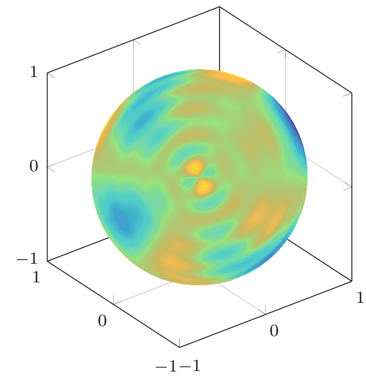
(A) $G_{1,5,10}$



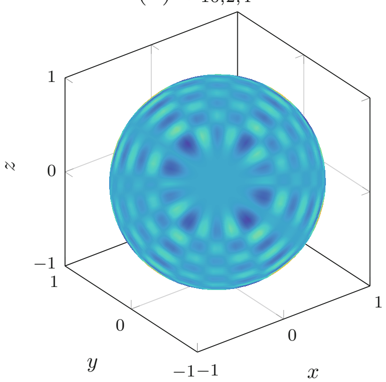
(B) Sum of the function in Fig. 5 and (A)



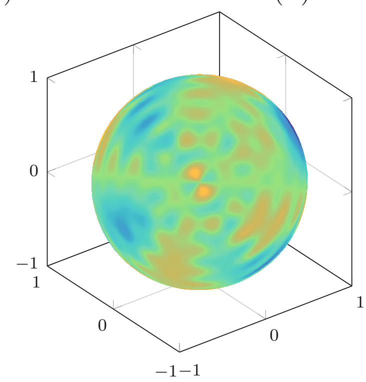
(C) $G_{10,2,4}$



(D) Sum of the function in (B) and (C)

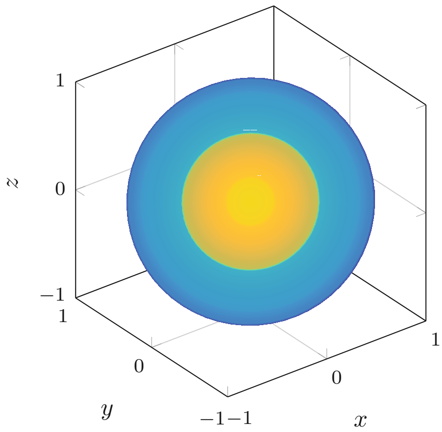


(E) $G_{7,10,4}$

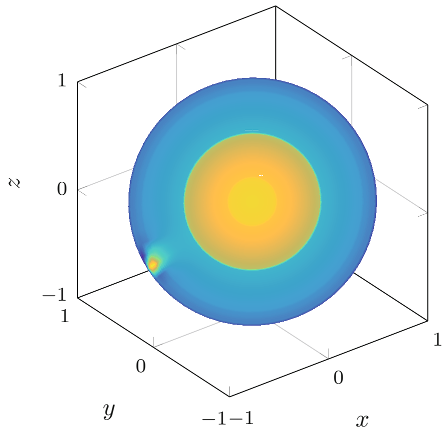


(F) Sum of the function in (D) and (E)

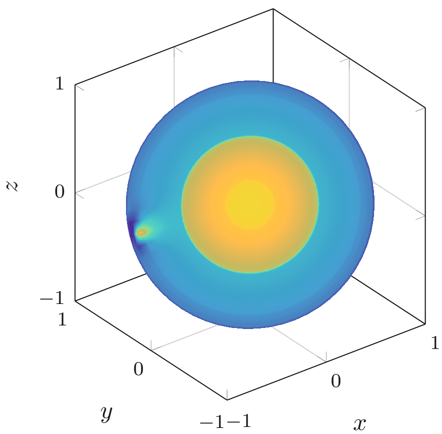
FIGURE 6. Several functions from the null space of T, that is, they generate the solution $V = 0$ (left column), and the sum of these functions with $G_{0,4,8} \notin \ker T$ (right column) which generate the same right-hand side $V = TG_{0,4,8}$, that is, the same data for the inverse problem.



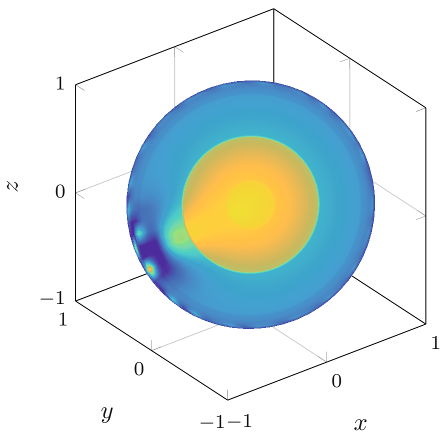
(A) The PREM model



(B) Sum of PREM and $K(z_1, \cdot)$



(C) Sum of PREM and $K(z_2, \cdot)$



(D) Sum of PREM and $K(z_1, \cdot)+K(z_2, \cdot)+K(z_3, \cdot)$

FIGURE 7. The density of the PREM model added to several functions from the null space of T^{G}. They all generate the same gravitational potential. Here, $z_i \in \mathcal{B}$, $i = 1, 2, 3$ are fixed.

For this purpose, we consider the function $G_{0,4,8}$ plotted in Figure 5, which is not in the null space of the operator, that means this function generates the result $TG_{0,4,8} = V \neq 0$. Then, we add several functions from the null space (see Figures 6 (A), (C), and (E)) to $G_{0,4,8}$. The results are shown in Figures 6 (B), (D), and (F). Keep in mind that all functions in the left column of Figure 6 generate the zero potential and all functions in the right column of Figure 6 generate the same forward solution $V = TG_{0,4,8}$. Similarly, we proceed in Figure 7, where linear combinations of functions $K(z_i, \cdot)$, $z_i \in \mathcal{B} \setminus \{0\}$, with

$$K(z_i, x) := \sum_{n=0}^{100} \sum_{j=1}^{2n+1} (0.95)^{1+n} G_{1,n,j}(x) G_{1,n,j}(z_i), \qquad x \in \mathcal{B} \setminus \{0\},$$

are added to the density D of the PREM model, see [18]. Again, $K(z_i, \cdot)$ can be extended onto $\mathcal{B}$, if $l_n \geq 0$ for all $n \in \mathbb{N}_0$. Note that $K(z_i, \cdot) \in \ker T$ for all $z_i \in \mathcal{B} \setminus \{0\}$ such that, again, there is no difference between the potentials generated by PREM (see Figure 7 (A)) and the potentials generated by the perturbed mass densities in Figures 7 (B), (C), and (D).

Hence, the solution of the inverse problem from Equation (2) is not unique, since we can always add functions from the null space to it without changing the function V. In particular, Figure 7 shows that certain kinds of mass anomalies (in the interior of the Earth) remain completely concealed if gravitational data are used solely.

Now we can sum up our results and give an answer to the three questions about the well-posedness of the problem posed in Section 3.

Theorem 4.2. *Let the operator* $T \colon \mathrm{L}^2(\mathcal{B}) \to \mathrm{L}^2(\mathcal{S})$ *be given by*

$$T \colon D \mapsto \int_{\mathcal{B}} D(x) k(x, \cdot) \, \mathrm{d}x. \tag{19}$$

with an integral kernel $k \colon \mathcal{B} \times \mathcal{S} \to \mathbb{R}$ *of the form*

$$k(x, y) := \sum_{i=0}^{\infty} c_i \frac{|x|^{l_i}}{|y|^{i+1}} P_i \left(\frac{x}{|x|} \cdot \frac{y}{|y|} \right), \qquad x \in \mathcal{B} \setminus \{0\}, y \in \mathcal{S},$$

satisfying Assumption 3.1*. Moreover, let the following three conditions be fulfilled (by the function* V*):*

- *The restriction* $V|_{\Omega_R}$ *of* V *is an* $\mathrm{L}^2(\Omega_R)$*-function.*
- *The spherical harmonics coefficients* $V_{n,j}$ *of* V *fulfil a summability condition*

$$\sum_{\substack{n=0 \\ c_n \neq 0}}^{\infty} n^2 (2l_n + 3) R^{2n - 2l_n} c_n^{-2} \sum_{j=1}^{2n+1} V_{n,j}^2 < \infty.$$

- *The function* V *is harmonic in the exterior of* $\mathcal{B}$*, that is,* $\Delta V(y) = 0$ *for all* $y \in \mathbb{R}^3 \setminus \mathcal{B}$*, and regular at infinity, that is,* $|V(y)| = \mathcal{O}(|y|^{-1})$ *and* $|\nabla V(y)| = \mathcal{O}(|y|^{-2})$ *for* $|y| \to \infty$*.*

Then both inverse problems, which are, the recovery of $D \in \mathrm{L}^2(\mathcal{B})$ *from either given values of* $V|_{\Omega_R}$ *or the upward continued potential* $V|_{\mathcal{S}}$ *are ill posed, since their solutions are not unique. However, in both cases, the solution exists under these conditions but is not stable.*

The second condition in Theorem 4.2 is also known as the Picard condition.In several cases, for example, the inverse gravimetric problem (i.e., $l_n = n$, $c_n = \gamma$ for all $n \in \mathbb{N}_0$), the Picard condition implies $V|_{\Omega_R} \in \mathrm{L}^2(\Omega_R)$. For the inverse gravimetric problem the Picard condition is satisfied, since the (empirical) Kaula rule of thumb holds:

$$\sum_{j=1}^{2n+1} \langle V|_{\Omega_R}, Y_{n,j} \rangle^2_{\mathrm{L}^2(\Omega_R)} = \mathcal{O}(\vartheta^{n+1} n^{-3}), \qquad n \to \infty,$$

for a constant $\vartheta \in]0,1[$, see, for example, [28] or [44]. Note that the Picard condition is necessary for the existence of the solution. Since this condition is not necessarily satisfied by every $V|_{\Omega_R} \in \mathrm{L}^2(\Omega_R)$, also this criterion by Hadamard may be violated.

We want to discuss the instability of the solution in detail using the following example.

Example 4.3. Let a family of functions be defined by

$$V_n(y) := \frac{1}{n} \frac{\beta^n}{|y|^{n+1}} Y_{n,1}\left(\frac{y}{|y|}\right), \qquad y \in \mathcal{S}, \text{ for all } n \in \mathbb{N}_0.$$

Since $\{\frac{1}{\beta} Y_{n,1}(\frac{\cdot}{\beta})\}_{n \in \mathbb{N}_0}$ is an $\mathrm{L}^2(\mathcal{S})$-orthonormal system, we get

$$\|V_n\|_{\mathrm{L}^2(\mathcal{S})} = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

Hence, the norms build a null sequence. Using Equation (16), we see that

$$D_n(x) := \frac{\sqrt{2n+3}\,(2n+1)}{4\pi R^{3/2} n} \left(\frac{\beta}{R}\right)^n G_{0,n,1}(x)$$

yields $TD_n = V_n$ in the case of $l_n = n$, $c_n = 1$. In addition, we obtain that the sequence of norms diverges, since $\beta > R$ and

$$\begin{aligned} \|D_n\|_{\mathrm{L}^2(\mathcal{B})} &= \frac{\sqrt{2n+3}(2n+1)}{4\pi R^{3/2} n} \left(\frac{\beta}{R}\right)^n \|G_{0,n,1}\|_{\mathrm{L}^2(\mathcal{B})} \\ &= \frac{\sqrt{2n+3}(2n+1)}{4\pi R^{3/2} n} \left(\frac{\beta}{R}\right)^n \to \infty \quad \text{as } n \to \infty. \end{aligned}$$

Thus, small changes in the potential V yield large changes in the solution D and, hence, the problem is not stable. Note that this instability is already given for the case of terrestrial data, which means that it is not (only) caused by the instability of the downward continuation.

4.3. Expansion of the solution in reproducing kernel based functions

In certain cases, it can be of interest to expand the unknown function D in terms of appropriate reproducing kernels instead of orthonormal basis functions. Reproducing kernels are localized in contrast to the global orthonormal basis functions from the previous subsection (see also the paper by Freeden, Michel and Simons in this handbook). In addition, the problems due to the discontinuity at the origin can be avoided by using this approach. For a more general introduction into reproducing kernels and reproducing kernel Hilbert spaces, see, for a general setting, [2, 3, 5, 16], for reproducing kernel Hilbert spaces on the ball.

Let $\mathcal{H} \coloneqq \mathcal{H}((A_{m,n}), \mathcal{B}) \subset \mathrm{L}^2(\mathcal{B})$, with the real sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$, be defined as

$$\mathcal{H}((A_{m,n}), \mathcal{B}) \coloneqq \overline{\left\{ F \in \mathrm{L}^2(\mathcal{B}) \;\middle|\; \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} A_{m,n}^2 \langle F, G_{m,n,j} \rangle_{\mathrm{L}^2(\mathcal{B})}^2 < \infty \right\}}^{\|\cdot\|_{\mathcal{H}}},$$

with

$$\|F\|_{\mathcal{H}}^2 \coloneqq \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} A_{m,n}^2 \langle F, G_{m,n,j} \rangle_{\mathrm{L}^2(\mathcal{B})}^2, \qquad F \in \mathcal{H}.$$

The inner product in $\mathcal{H}$ is then given by

$$\langle F, G \rangle_{\mathcal{H}} = \sum_{m,n=0}^{\infty} \sum_{j=1}^{2n+1} A_{m,n}^2 \langle F, G_{m,n,j} \rangle_{\mathrm{L}^2(\mathcal{B})} \langle G, G_{m,n,j} \rangle_{\mathrm{L}^2(\mathcal{B})} \tag{20}$$

for all F, $G \in \mathcal{H}$.

If the sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ fulfils a certain summability condition, see, for more details, [34], then $\mathcal{H}$ is a reproducing kernel Hilbert space. Due to the property of the sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$, the evaluation functional in $\mathcal{H}$ is continuous. The reproducing kernel of $\mathcal{H}$ is given by $K\colon (\mathcal{B} \setminus \{0\}) \times (\mathcal{B} \setminus \{0\}) \to \mathbb{R}$ with

$$K(z, x) \coloneqq \sum_{\substack{m,n=0;\\ A_{m,n}\neq 0}}^{\infty} \sum_{j=1}^{2n+1} A_{m,n}^{-2} G_{m,n,j}(x) G_{m,n,j}(z), \qquad z,\ x \in \mathcal{B} \setminus \{0\}. \tag{21}$$

Again, in certain cases of $G_{m,n,j}$, the definition of K on $\mathcal{B} \times \mathcal{B}$ is valid.

The kernel K has the reproducing property, that is,

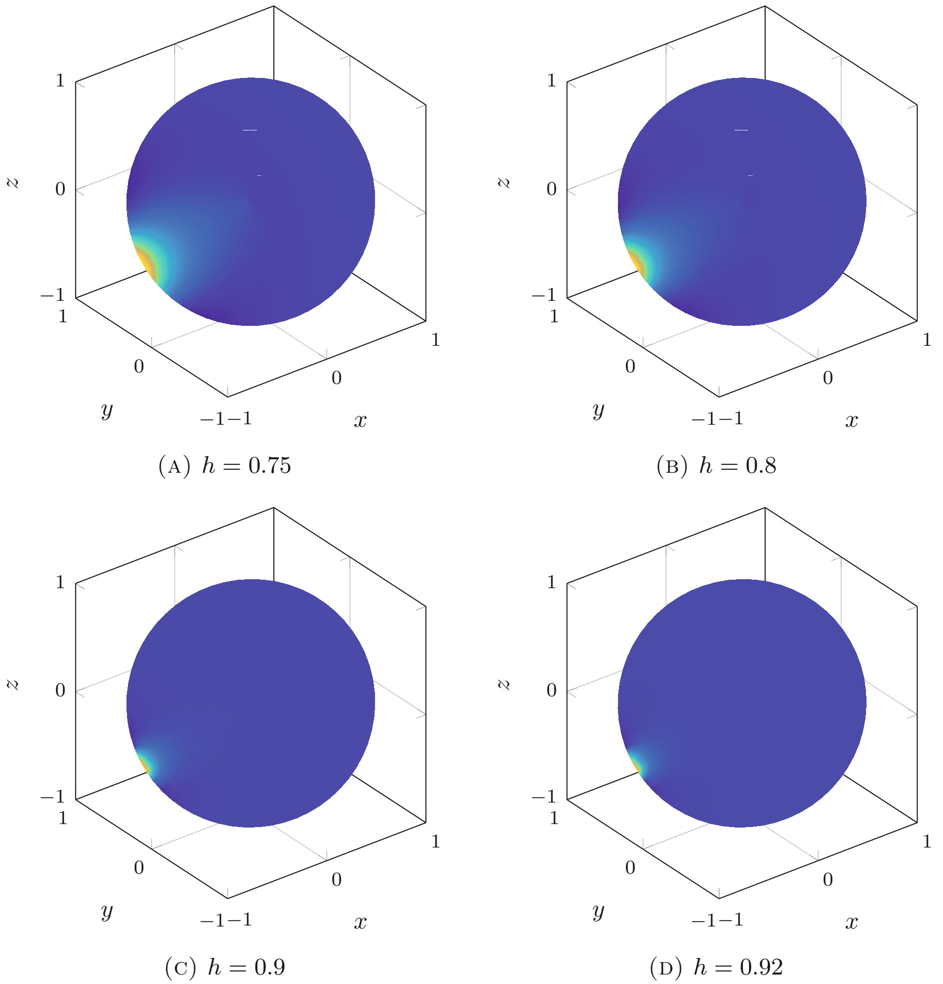
$$\langle F, K(z, \cdot) \rangle_{\mathcal{H}} = F(z) \qquad \text{for all } F \in \mathcal{H} \text{ and all } z \in \mathcal{B} \setminus \{0\}.$$

In our setting, the first input argument z denotes the (fixed) centre of the kernel, that is, the position in the ball where the kernel is located. Some examples of reproducing kernels with the same centre and different sequences $(A_{m,n})_{m,n\in\mathbb{N}_0}$ are plotted in Figure 8. As one can see, the discontinuity at the origin is, at least visibly, smoothed away.

Let the set $\{y_1, \ldots, y_\ell\} \subset \mathcal{S}$, $\ell \in \mathbb{N}$, contain our measuring positions. We define linear functionals by $\mathcal{F}^\nu F \coloneqq (TF)(y_\nu)$ for $\nu = 1, \ldots, \ell$. In other words, the functionals $\mathcal{F}^\nu$ are the evaluations of our operator T applied to an (unknown) function F at the measuring positions y_ν, $\nu = 1, \ldots, \ell$. The data collected at the sensor positions are given by $v_\nu = V(y_\nu)$. The functionals $\mathcal{F}^\nu$ are linear, since they are the composition of the linear operator T and the linear evaluation functional.

If the function F is an element of the Sobolev space $\mathcal{H}((A_{m,n}), \mathcal{B})$ with a sequence $(A_{m,n})_{m,n\in\mathbb{N}_0}$ fulfilling the summability condition

$$\sum_{n=0}^{\infty} A_{0,n}^{-2} \frac{1}{(2n+1)(2l_n+3)} < \infty,$$



(A) $h = 0.75$ (B) $h = 0.8$

(C) $h = 0.9$ (D) $h = 0.92$

FIGURE 8. Reproducing kernel $K(z_1, \cdot)$ for several $(A_{m,n})_{m,n\in\mathbb{N}_0}$ with $A_{m,n}^{-2} = (Cn+1)h^{2(m+n)}\delta_{m,0}$ at a fixed centre $z_1 \in \mathcal{B}\backslash\{0\}$, a sufficiently large constant C, and the functions $G_{m,n,j}$ in the case $l_n = n-1$.

then the functionals $\mathcal{F}^\nu$ are also continuous (with $y_\nu = r_\nu\xi_\nu$) for $\nu = 1, \dots, \ell$, since

$$\begin{aligned}
|\mathcal{F}^\nu F|^2 &= |(TF)(y_\nu)|^2 = \left|\left(T\left(\sum_{m=0}^\infty\sum_{n=0}^\infty\sum_{j=1}^{2n+1}\langle F, G_{m,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})}G_{m,n,j}\right)\right)(y_\nu)\right|^2 \\
&= (4\pi)^2\left|\sum_{n=0}^\infty\sum_{j=1}^{2n+1}\langle F, G_{0,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})}\frac{c_nR^{l_n}}{(2n+1)|r_\nu|^{n+1}}\gamma_{0,n}^{-1}Y_{n,j}(\xi_\nu)\frac{A_{0,n}}{A_{0,n}}\right|^2 \\
&\leq (4\pi)^2\left(\sum_{n=0}^\infty\sum_{j=1}^{2n+1}\langle F, G_{0,n,j}\rangle_{\mathrm{L}^2(\mathcal{B})}^2A_{0,n}^2\right)\sum_{n=0}^\infty\sum_{j=1}^{2n+1}\left(\frac{c_nR^{l_n}}{A_{0,n}(2n+1)|r_\nu|^{n+1}}\gamma_{0,n}^{-1}Y_{n,j}(\xi_\nu)\right)^2
\end{aligned}$$

$$\leq (4\pi c)^2 \|F\|^2_{\mathcal{H}} \left(\sup_{n\in\mathbb{N}_0} R^{l_n-n}\right)^2 \sum_{n=0}^{\infty} \frac{R}{A^2_{0,n}(2n+1)^2(2l_n+3)} \frac{2n+1}{4\pi}$$
$$\leq 4\pi c^2 R \|F\|^2_{\mathcal{H}} \left(\sup_{n\in\mathbb{N}_0} R^{l_n-n}\right)^2 \sum_{n=0}^{\infty} \frac{1}{A^2_{0,n}(2n+1)(2l_n+3)} < \infty,$$

due to (16), the Cauchy–Schwarz inequality, the definition of the inner product in $\mathcal{H}$ in (20), and Assumption 3.1.

We can apply these functionals to the kernel with respect to z and obtain the following result by using Equation (16) and the addition theorem for spherical harmonics. The interchanging of limits (in the series) and the integral, which is needed in this calculation, is allowed due to the previous estimates. Hence,

$$\begin{aligned}
\mathcal{F}^\nu_z K(z,x) &= \left[\int_{\mathcal{B}} K(z,x) \sum_{i=0}^{\infty} c_i \frac{|z|^{l_i}}{|y|^{i+1}} P_i\left(\frac{z}{|z|}\cdot\frac{y}{|y|}\right) \mathrm{d}z\right]_{y=y_\nu} \\
&= \sum_{\substack{m,n=0;\\ A_{m,n}\neq 0}}^{\infty} \sum_{j=1}^{2n+1} A^{-2}_{m,n} G_{m,n,j}(x) \mathcal{F}^\nu_z G_{m,n,j}(z) \\
&= \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} A^{-2}_{0,n} \gamma_{0,n} \frac{|x|^{l_n}}{R^{l_n}} Y_{n,j}\left(\frac{x}{|x|}\right) \gamma^{-1}_{0,n} \frac{4\pi c_n R^{l_n}}{(2n+1)|y_\nu|^{n+1}} Y_{n,j}\left(\frac{y_\nu}{|y_\nu|}\right) \\
&= \sum_{n=0}^{\infty} A^{-2}_{0,n} c_n \frac{|x|^{l_n}}{|y_\nu|^{n+1}} P_n\left(\frac{x}{|x|}\cdot\frac{y_\nu}{|y_\nu|}\right).
\end{aligned}$$

It is known that we can construct an expansion for the solution, see [3, 23] by

$$D(x) = \sum_{\nu=1}^{\ell} a_\nu \mathcal{F}^\nu_z K(z,x). \tag{22}$$

Our aim is to determine the corresponding coefficients a_ν, $\nu = 1,\ldots,\ell$. Applying the functional on both sides, we obtain for $\iota = 1,\ldots,\ell$

$$\begin{aligned}
\mathcal{F}^\iota_x D(x) = v_\iota &= \sum_{\nu=1}^{\ell} a_\nu \mathcal{F}^\iota_x \mathcal{F}^\nu_z K(z,x) \\
&= \sum_{\nu=1}^{\ell} a_\nu \sum_{n=0}^{\infty} A^{-2}_{0,n} \gamma^{-2}_{0,n} \frac{4\pi c_n^2}{2n+1} \frac{R^{2l_n}}{|y_\iota|^{n+1}|y_\nu|^{n+1}} P_n\left(\frac{y_\iota}{|y_\iota|}\cdot\frac{y_\nu}{|y_\nu|}\right).
\end{aligned}$$

This linear system is uniquely solvable, which means that the expansion in (22) is unique, if the linear and continuous functionals $\mathcal{F}^\nu$, $\nu = 1,\ldots,\ell$ are linearly independent, see [2]. Among all solutions $D \in \mathcal{H}$ with $\mathcal{F}^\nu D = v_\nu$ for $\nu = 1,\ldots,\ell$, the solution in (22) uniquely minimizes the norm $\|\cdot\|_{\mathcal{H}}$ induced by the inner product in (20). These are basically the ideas of a spline interpolation method (for further details, see [2] and [9]).

5. Constraints for the uniqueness of the solution

In the previous section, we have shown that we cannot expect a unique solution of the Fredholm integral equation of the first kind stated in (2). Hence, in practice, additional conditions are necessary to impose uniqueness. Some possible uniqueness constraints are now discussed. The most approaches are generalizations of the results in [33]. More precisely, we present the minimum norm condition, a generalization of the harmonicity constraint, and the layer density constraint. In addition, we discuss the surface density approach.

5.1. Minimum norm constraint

As we have seen, we are not able to obtain a uniquely determined solution without additional assumptions or information. A widespread approach to force uniqueness is the minimum norm condition (see, e.g., [40]). The following result is a generalization of the theorem concerning the minimum norm solution of the inverse gravimetric problem, see [33] and the references therein. Throughout this subsection, we assume that the conditions in Theorem 4.2 are fulfilled and, hence, a solution of the inverse problem exists.

Recall Equation (11), which is repeated below for convenience:

$$\frac{(2n+1)R^n}{4\pi c_n} V_{n,j} = \int_0^R r^{l_n+2} D_{n,j}(r)\,\mathrm{d}r, \qquad \text{if } c_n \neq 0,$$

$V_{n,j} = 0$ for all $j = 1, \ldots, 2n+1$ otherwise. $D_{n,j}$ is originated by the (in $\mathrm{L}^2(\mathcal{B})$ convergent) series

$$D(x) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} D_{n,j}(|x|) Y_{n,j}\left(\frac{x}{|x|}\right).$$

The minimum norm conditionis fulfilled, if among all $D \in \mathrm{L}^2(\mathcal{B})$ with $V = \int_{\mathcal{B}} D(x) k(x,\cdot)\,\mathrm{d}x$, we choose the one with the minimum (squared) norm

$$\|D\|^2_{\mathrm{L}^2(\mathcal{B})} = \int_{\mathcal{B}} (D(x))^2\,\mathrm{d}x = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \int_0^R r^2 (D_{n,j}(r))^2\,\mathrm{d}r.$$

If we minimize this expression, we obtain the following minimization problem for each $n \in \mathbb{N}_0$ and $j = 1, \ldots, 2n+1$:

$$\begin{aligned} &\text{minimize} && \int_0^R r^2 (D_{n,j}(r))^2\,\mathrm{d}r, \\ &\text{subject to} && \int_0^R r^{l_n+2} D_{n,j}(r)\,\mathrm{d}r = \frac{2n+1}{4\pi c_n} R^n V_{n,j}, \qquad \text{if } c_n \neq 0. \end{aligned}$$

Note that the side condition drops out in the case $c_n = 0$ such that the unconstrained minimizer $D_{n,j} \equiv 0$ occurs. With the substitution $F_{n,j}(r) := r D_{n,j}(r)$, the problem above is equivalent to

$$\begin{aligned}&\text{minimize} && \int_0^R (F_{n,j}(r))^2 \,\mathrm{d}r,\\ &\text{subject to} && \int_0^R r^{l_n+1} F_{n,j}(r)\,\mathrm{d}r = \frac{2n+1}{4\pi c_n} R^n V_{n,j}, \qquad \text{if } c_n \neq 0.\end{aligned}$$

We now apply an orthogonal decomposition in $\mathrm{L}^2[0,R]$ to $F_{n,j}$ in the sense that $F_{n,j}(r) = \alpha_{n,j} r^{l_n+1} + H_{n,j}(r)$, where $\int_0^R r^{l_n+1} H_{n,j}(r)\,\mathrm{d}r = 0$. With this ansatz, our minimization problem reads

$$\begin{aligned}&\text{minimize} && \alpha_{n,j}^2 \int_0^R r^{2l_n+2}\,\mathrm{d}r + \|H_{n,j}\|^2_{\mathrm{L}^2[0,R]},\\ &\text{subject to} && \alpha_{n,j} \int_0^R r^{2l_n+2}\,\mathrm{d}r = \frac{2n+1}{4\pi c_n} R^n V_{n,j}, \qquad \text{if } c_n \neq 0.\end{aligned}$$

Since the side condition is independent of $H_{n,j}$, we see that $H_{n,j} \equiv 0$ yields the unique minimum, for which we have

$$\alpha_{n,j} = (2l_n+3)\frac{2n+1}{4\pi c_n}\frac{R^n}{R^{2l_n+3}} V_{n,j}, \qquad \text{if } c_n \neq 0$$

and $\alpha_{n,j} = 0$ for all $j = 1, \ldots, 2n+1$, if $c_n = 0$. We summarize our results in the following theorem.

Theorem 5.1. *Let the conditions on V from Theorem* 4.2 *be fulfilled. Then, among all $D \in \mathrm{L}^2(\mathcal{B})$ with $V = \int_{\mathcal{B}} D(x)k(x,\cdot)\,\mathrm{d}x$, the $\mathrm{L}^2(\mathcal{B})$-convergent series,*

$$\begin{aligned}D(x) &= \sum_{\substack{n=0\\ c_n\neq 0}}^{\infty} \sum_{j=1}^{2n+1} (2l_n+3)\frac{2n+1}{4\pi c_n} R^{n-l_n-3} V_{n,j} \frac{|x|^{l_n}}{R^{l_n}} Y_{n,j}\left(\frac{x}{|x|}\right)\\ &= \sum_{\substack{n=0\\ c_n\neq 0}}^{\infty} \sum_{j=1}^{2n+1} \sqrt{\frac{2l_n+3}{R^3}}\,\frac{2n+1}{4\pi c_n} R^{n-l_n} V_{n,j} G_{0,n,j}(x),\end{aligned} \tag{23}$$

is the unique minimizer of the functional

$$\mathcal{F}(D) := \int_{\mathcal{B}} (D(x))^2\,\mathrm{d}x.$$

In the particular case of $l_n = n$ and $c_n = \gamma$ for all $n \in \mathbb{N}_0$, it can be proven that the harmonic solution is equivalent to the minimum norm solution, see [33]. This particular solution of the inverse gravimetric problem is then given by

$$D(x) = \frac{1}{\gamma}\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \sqrt{\frac{2n+3}{R^3}}\,\frac{2n+1}{4\pi} V_{n,j} G^{\mathrm{I}}_{0,n,j}(x), \qquad x \in \mathcal{B}.$$

The convergence of the series in (23) can be proven using the orthonormality of the $G_{m,n,j}$ functions, since the Parseval identity yields

$$\|D\|^2_{\mathrm{L}^2(\mathcal{B})} = \sum_{\substack{n=0\\c_n\neq 0}}^{\infty} \sum_{j=1}^{2n+1} \frac{2l_n+3}{R^3} \left(\frac{2n+1}{4\pi c_n}\right)^2 R^{2n-2l_n} V^2_{n,j}.$$

Comparing this with Theorem 4.2, we achieve that the series in (23) converges if and only if V fulfils the Picard condition, that is,

$$\sum_{\substack{n=0\\c_n\neq 0}}^{\infty} \frac{n^2(2l_n+3)}{c_n^2 R^{2(l_n-n)}} \sum_{j=1}^{2n+1} V^2_{n,j} < \infty. \tag{24}$$

5.2. A generalization of the harmonicity constraint

In [33], the quasi-harmonic solution, which had already been discussed in the literature, was seized on. In this case, functions of the kind $x \mapsto |x|^{n+p} Y_{n,j}(\frac{x}{|x|})$, $x \in \mathcal{B}$, for a fixed $p \in \mathbb{R}_0^+$ are used as basis functions. We consider here the generalized case of a basis $\{B_{n,j}\}_{n\in\mathbb{N}_0, j=1,\ldots,2n+1}$ given by

$$B_{n,j}(x) := \frac{|x|^{k_n}}{R^{k_n+1}} Y_{n,j}\left(\frac{x}{|x|}\right), \qquad n \in \mathbb{N}_0, j = 1,\ldots,2n+1$$

with a preliminarily chosen sequence $(k_n)_{n\in\mathbb{N}_0} \subset \mathbb{R}$ and the additional condition that $2k_n + 3 > 0$ for all $n \in \mathbb{N}_0$. This condition guarantees that these functions have a finite $\mathrm{L}^2(\mathcal{B})$-norm. The orthogonality is a direct consequence of the $\mathrm{L}^2(\Omega)$-orthogonality of the spherical harmonics $Y_{n,j}$, since

$$\begin{aligned}
\langle B_{n,j}, B_{\nu,\iota}\rangle_{\mathrm{L}^2(\mathcal{B})} &= \int_{\mathcal{B}} \frac{|x|^{k_n}}{R^{k_n+1}} Y_{n,j}\left(\frac{x}{|x|}\right) \frac{|x|^{k_\nu}}{R^{k_\nu+1}} Y_{\nu,\iota}\left(\frac{x}{|x|}\right) \mathrm{d}x\\
&= \int_0^R \frac{r^{2k_n+2}}{R^{2k_n+2}}\,\mathrm{d}r\ \delta_{n,\nu}\delta_{j,\iota}\\
&= \frac{R^{2k_n+3}}{(2k_n+3)R^{2k_n+2}}\ \delta_{n,\nu}\delta_{j,\iota}\\
&= \frac{R}{2k_n+3}\ \delta_{n,\nu}\delta_{j,\iota}.
\end{aligned}$$

In the case $k_n = n$, the subspace spanned by this basis is the set of all harmonic functions and in the case $k_n = n + p$ we get the quasi-harmonic setting.

In contrast to the previous subsection, we have to assume slightly different properties of V. However, note that Assumption 3.1 is still valid.

Assumption 5.2. *We suppose that*

- *the restriction* $V|_{\Omega_R}$ *of* V *is an* $\mathrm{L}^2(\Omega_R)$*-function,*

- *the summability condition*

$$\sum_{\substack{n=0\\c_n\neq 0}}^{\infty} R^{2n-2l_n}\frac{n^2(l_n+k_n+3)^2}{c_n^2(2k_n+3)}\sum_{j=1}^{2n+1} V_{n,j}^2<\infty$$

  is fulfilled,
- *V is harmonic in the outer space, that is, $\Delta V(y)=0$ for all $y\in\mathbb{R}^3\setminus\mathcal{B}$,*
- *V is regular at infinity.*

With the orthogonal basis $\{B_{n,j}\}_{n\in\mathbb{N}_0,j=1,\dots,2n+1}$, the density D can be represented by the expansion

$$D(x)=\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} d_{n,j}\sqrt{\frac{2k_n+3}{R}}B_{n,j}(x),\qquad x\in\mathcal{B}\setminus\{0\},\tag{25}$$

in the sense of $\mathrm{L}^2(\mathcal{B})$. In accordance with the notations above, we have

$$D_{n,j}(r)=d_{n,j}\sqrt{\frac{2k_n+3}{R}}\frac{r^{k_n}}{R^{k_n+1}},\qquad r\in[0,R].$$

Thus, the relation between the Fourier coefficients of V and $D_{n,j}$ in (11) becomes for all $j=1,\dots,2n+1$

$$\begin{aligned}\frac{(2n+1)R^n}{4\pi c_n}V_{n,j}&=\int_0^R d_{n,j}\sqrt{\frac{2k_n+3}{R}}\frac{r^{l_n+k_n+2}}{R^{k_n+1}}\,\mathrm{d}r\\&=d_{n,j}\sqrt{\frac{2k_n+3}{R}}\frac{R^{l_n+k_n+3}}{(l_n+k_n+3)R^{k_n+1}}\\&=d_{n,j}\sqrt{\frac{2k_n+3}{R}}\frac{R^{l_n+2}}{(l_n+k_n+3)},\qquad \text{if } c_n\neq 0,\end{aligned}\tag{26}$$

and $V_{n,j}=0$, if $c_n=0$. Solving (26) for $d_{n,j}$ and inserting the result in (25), we obtain

$$\begin{aligned}D(x)&=\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} d_{n,j}\sqrt{\frac{2k_n+3}{R}}\frac{|x|^{k_n}}{R^{k_n+1}}Y_{n,j}\left(\frac{x}{|x|}\right)\\&=\sum_{\substack{n=0\\c_n\neq 0}}^{\infty}\frac{2n+1}{4\pi c_n}(l_n+k_n+3)R^{n-l_n-2}\frac{|x|^{k_n}}{R^{k_n+1}}\sum_{j=1}^{2n+1}V_{n,j}Y_{n,j}\left(\frac{x}{|x|}\right)+\tilde{D}\\&=\sum_{\substack{n=0\\c_n\neq 0}}^{\infty}\frac{2n+1}{4\pi c_n}(l_n+k_n+3)R^{n-l_n-2}\sum_{j=1}^{2n+1}V_{n,j}B_{n,j}(x)+\tilde{D},\end{aligned}$$

where $\tilde{D}\in\overline{\operatorname{span}\{B_{n,j}\mid n\in\mathbb{N}_0 \text{ with } c_n=0,\ j=1,\dots,2n+1\}}^{\|\cdot\|_{\mathrm{L}^2(\mathcal{B})}}$ is arbitrary. The convergence of the series is guaranteed by the summability conditions on V. Summarizing these results, we get the next theorem.

Theorem 5.3. *Let $c_n \neq 0$ for all $n \in \mathbb{N}_0$, and let Assumptions* 3.1 *and* 5.2 *be fulfilled. Then the unique solution $D \in U$, where the $\mathrm{L}^2(\mathcal{B})$-subspace U has the basis $\{B_{n,j}\}_{n\in\mathbb{N}_0, j=1,\ldots,2n+1}$, of the inverse problem*

$$\int_{\mathcal{B}} D(x)k(x,y)\,\mathrm{d}x = V(y) \qquad \text{in } \overline{\mathbb{R}^3 \setminus \mathcal{B}},$$

with $(x,y) \in \mathrm{dom}(k)$ is given by

$$D(x) = \sum_{n=0}^{\infty} \frac{2n+1}{4\pi c_n}(l_n + k_n + 3)R^{n-l_n}\frac{|x|^{k_n}}{R^{k_n+3}}\sum_{j=1}^{2n+1} V_{n,j}Y_{n,j}\left(\frac{x}{|x|}\right),$$

in the sense of $\mathrm{L}^2(\mathcal{B})$.

In [33], the biharmonic solution was also considered. In this case, the needed radial basis is given by the sum of two radial parts. An approach for a generalization of this ansatz is given by the sum of $K \in \mathbb{N}$ different radial parts, that is, $\{(\sum_{i=1}^{K} |\cdot|^{k_{i,n}})Y_{n,j}(\frac{\cdot}{|\cdot|})\}_{n\in\mathbb{N}_0, j=1,\ldots,2n+1}$. However, without any additional information, a unique solution cannot be obtained in this case (see also the result for the biharmonic solution in [33]).

5.3. Layer density constraint

As we have seen above, the non-uniqueness is primarily a matter of the radial parametrization of the solution D. For this reason and in view of the fact that, for example, lithospheric heterogeneities are particularly interesting with respect to their lateral structure, we consider here the (thin) spherical shell

$$\Omega_{[\tau,\tau+\varepsilon]} \coloneqq \left\{x \in \mathbb{R}^3 : 0 < \tau \leq |x| \leq \tau + \varepsilon \leq R\right\},$$

for $\tau > 0$ and $\varepsilon > 0$. We are interested in finding a solution D which consists of purely laterally inhomogeneous anomalies in $\Omega_{[\tau,\tau+\varepsilon]}$. This kind of uniqueness constraint was, for example, used in [23] for the inverse magnetic problem.

For the layer density constraint, we assume that the density $D \in \mathrm{L}^2(\mathcal{B})$ has (again) the form

$$D(x) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} D_{n,j}(|x|)Y_{n,j}\left(\frac{x}{|x|}\right), \qquad x \in \mathcal{B}, \tag{27}$$

where now

$$D_{n,j}(r) \coloneqq \kappa d_{n,j}\chi_{[\tau,\tau+\varepsilon]}(r), \qquad r \in [0,R], \tag{28}$$

for all $n \in \mathbb{N}_0$, $j = 1,\ldots,2n+1$, and χ is the characteristic function (i.e., $\chi_A(x) = 0$ if $x \notin A$ and $\chi_A(x) = 1$ if $x \in A$). The normalization constant κ is chosen as

$$\kappa \coloneqq \sqrt{\frac{3}{(\tau+\varepsilon)^3 - \tau^3}}.$$

Assumption 5.4. *For the function V, we now assume that*

- *the restriction $V|_{\Omega_R}$ of V is an $\mathrm{L}^2(\Omega_R)$-function,*

- *the summability condition*

$$\sum_{\substack{n=0\\c_n\neq 0}}^{\infty} \frac{n^2 l_n^2 R^{2n}}{((\tau+\varepsilon)^{l_n+3}-\tau^{l_n})^2 c_n^2} \sum_{j=1}^{2n+1} V_{n,j}^2 < \infty$$

is fulfilled,
- V *is harmonic in the outer space, that is,* $\Delta V(y) = 0$ *for all* $y \in \mathbb{R}^3 \setminus \mathcal{B}$,
- V *is regular at infinity.*

Using (11) and the desired representation of D, we have

$$\begin{aligned}\frac{(2n+1)R^n}{4\pi c_n} V_{n,j} &= \int_0^R r^{l_n+2} D_{n,j}(r)\,\mathrm{d}r\\ &= \kappa \int_0^R r^{l_n+2} d_{n,j}\chi_{[\tau,\tau+\varepsilon]}(r)\,\mathrm{d}r\\ &= \kappa d_{n,j}\,\frac{(\tau+\varepsilon)^{l_n+3}-\tau^{l_n+3}}{l_n+3}.\end{aligned}$$

This yields, for all $j = 1, \ldots, 2n+1$,

$$\kappa d_{n,j} = \frac{(2n+1)R^n}{4\pi c_n} V_{n,j}\,\frac{l_n+3}{(\tau+\varepsilon)^{l_n+3}-\tau^{l_n+3}}, \qquad \text{if } c_n \neq 0,$$

and $V_{n,j} = 0$, if $c_n = 0$. We insert this in Equations (27) and (28) and directly obtain, for all $x \in \mathcal{B}$,

$$\begin{aligned}D(x) &= \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} d_{n,j}\kappa\chi_{[\tau,\tau+\varepsilon]}(|x|)Y_{n,j}\left(\frac{x}{|x|}\right)\\ &= \sum_{\substack{n=0\\c_n\neq 0}}^{\infty}\sum_{j=1}^{2n+1} \frac{(2n+1)(l_n+3)}{4\pi c_n}\,\frac{R^n}{(\tau+\varepsilon)^{l_n+3}-\tau^{l_n+3}} V_{n,j}\chi_{[\tau,\tau+\varepsilon]}(|x|)Y_{n,j}\left(\frac{x}{|x|}\right) + \tilde{D},\end{aligned}$$

where

$$\tilde{D} \in \overline{\operatorname{span}\left\{ D_{n,j}(|\cdot|)Y_{n,j}\left(\frac{\cdot}{|\cdot|}\right) \,\middle|\, n\in\mathbb{N}_0 \text{ with } c_n = 0,\ j=1,\ldots,2n+1\right\}}^{\|\cdot\|_{\mathrm{L}^2(\mathcal{B})}}$$

can be chosen arbitrarily.

Theorem 5.5. *Let* $c_n \neq 0$ *for all* $n \in \mathbb{N}_0$ *and let Assumptions* 3.1 *and* 5.4 *be fulfilled. Then the unique solution under the layer density constraint is given by*

$$D(x) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \frac{R^n}{4\pi c_n}\,\frac{(2n+1)(l_n+3)}{(\tau+\varepsilon)^{l_n+3}-\tau^{l_n+3}} V_{n,j}\,\chi_{[\tau,\tau+\varepsilon]}(|x|)Y_{n,j}\left(\frac{x}{|x|}\right) \qquad (29)$$

in the sense of $\mathrm{L}^2(\mathcal{B})$.

Moreover, under the conditions in Assumption 5.4, the corresponding potential V possesses the following outer harmonics expansion

$$V(y) = \kappa \sum_{n=0}^{\infty} \frac{4\pi c_n}{(2n+1)(l_n+3)} \left((\tau+\varepsilon)^{l_n+3} - \tau^{l_n+3}\right) |y|^{-n-1} \sum_{j=1}^{2n+1} d_{n,j} Y_{n,j}\left(\frac{y}{|y|}\right).$$

This series fulfils the condition of Assumption 5.4, that is, $V|_{\Omega_R} \in \mathrm{L}^2(\Omega_R)$:

$$\begin{aligned}\|V|_{\Omega_R}\|^2_{\mathrm{L}^2(\Omega_R)} &= \kappa^2 \sum_{n=0}^{\infty} \left(\frac{4\pi c_n}{(2n+1)(l_n+3)} \left((\tau+\varepsilon)^{l_n+3} - \tau^{l_n+3}\right)\right)^2 R^{-2n} \sum_{j=1}^{2n+1} d_{n,j}^2 \\ &\leq 16\pi^2 c^2 \kappa^2 \sum_{n=0}^{\infty} \frac{\left(R^{l_n+3} + R^{l_n+3}\right)^2}{(2n+1)^2(l_n+3)^3 R^{2n}} \sum_{j=1}^{2n+1} d_{n,j}^2 \\ &\leq 64\pi^2 c^2 \kappa^2 \sup_{n\in\mathbb{N}_0} \left(\frac{R^{2l_n+6-2n}}{(2n+1)^2(l_n+3)^2}\right) \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} d_{n,j}^2 < \infty.\end{aligned}$$

For this estimate, we used the boundedness of the sequence $(c_n)_{n\in\mathbb{N}_0}$ (given by Assumption 3.1, item 1), the boundedness of the supremum in the latter estimate (given by Assumption 3.1, items 2 and 3), and the square-integrability of D.

5.4. Surface density

In inverse gravimetry, in particular, it is reasonable to consider a surface density instead of a density on the entire ball $\mathcal{B}$. In a time-variable gravity field (with relatively short time scales) most of the changes occur on the (Earth's) surface or at least on layers very close to it. So, if one is interested in anomalies as deviations from a reference model, which could be an annual mean, for instance, these anomalies can be typically found on the surface of the underlying body.

So far, in our general setup, we have

$$V(y) = (TD)(y) = \int_{\mathcal{B}} D(x)k(x,y)\,\mathrm{d}x. \tag{30}$$

Since the operator T is linear and continuous, we can also read the equation above in distributional sense. For the mathematical theory of distributions and, in this context, the definition of test functions, the reader is referred to [27]. In other words, we can look at Equation (30) as an application of a regular distribution D applied to the kernel k, that is

$$V(y) = (TD)(y) = \langle D, k(\cdot,y)\rangle. \tag{31}$$

Actually, we have a regular distribution $\mathfrak{D}$ with

$$\mathfrak{D}\varphi := \langle D, \varphi\rangle$$

for all test functions[1] φ, which is uniquely determined by the function D and vice versa (at least almost everywhere). Thus, the distribution can be, in fact,

[1] Actually, the function $k(\cdot,y)$ is not a test function, but the domain of $\mathfrak{D}$ can be extended such that $\mathfrak{D}k(\cdot,y)$ makes sense and equals $(TD)(y)$.

represented by the function D itself and the distinction is commonly omitted. Now, one can think of replacing the regular distribution and also allow singular distributions. For our purposes, a very useful singular distribution is $F\delta_{\Omega_R}$, which is a variation of the well known delta distribution and is given by

$$\langle F\delta_{\Omega_R}, \varphi\rangle := \int_{\Omega_R} F(x)\varphi(x)\,\mathrm{d}\omega(x),$$

for an arbitrary, over Ω_R square-integrable, function F and for every test function φ. In that case, we have (cf. Equation (31))

$$\tilde{V}(y) := \langle D\delta_{\Omega_R}, k(\cdot, y)\rangle = \int_{\Omega_R} D(x)k(x,y)\,\mathrm{d}\omega(x).$$

Conclusively, with our previous considerations, we get

$$\begin{aligned}\tilde{V}(y) &= \sum_{n=0}^{\infty} c_n \frac{R^{l_n}}{|y|^{n+1}} \int_{\Omega_R} D(x) P_n\left(\frac{x}{|x|}\cdot\frac{y}{|y|}\right)\mathrm{d}\omega(x)\\ &= \sum_{n=0}^{\infty} c_n \frac{R^{l_n}}{|y|^{n+1}} \int_{\Omega} D(R\xi) P_n\left(\xi\cdot\frac{y}{|y|}\right) R^2\,\mathrm{d}\omega(\xi).\end{aligned}$$

With the addition theorem for spherical harmonics and the ansatz (8), it follows that

$$\begin{aligned}\tilde{V}(y) &= \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \frac{4\pi c_n}{2n+1}\frac{R^{l_n+2}}{|y|^{n+1}} Y_{n,j}\left(\frac{y}{|y|}\right)\int_{\Omega} D(R\xi) Y_{n,j}(\xi)\,\mathrm{d}\omega(\xi)\\ &= \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \frac{4\pi c_n}{2n+1} R^{l_n-n+2} D_{n,j}(R)\left(\frac{R}{|y|}\right)^n \frac{1}{|y|} Y_{n,j}\left(\frac{y}{|y|}\right).\end{aligned}$$

Consequently, we find the Fourier coefficients

$$\tilde{V}_{n,j} = \frac{4\pi c_n}{2n+1} R^{l_n-n+2} D_{n,j}(R)$$

which in other words means that, for $c_n \neq 0$,

$$\frac{(2n+1)R^n}{4\pi c_n}\tilde{V}_{n,j} = R^{l_n+2} D_{n,j}(R). \tag{32}$$

As we see, this problem is again uniquely solvable (if $c_n \neq 0$ for all $n \in \mathbb{N}_0$) and in the particular case of the inverse gravimetric problem, the coefficients read

$$\frac{2n+1}{4\pi\gamma}\tilde{V}_{n,j} = R^2 D_{n,j}(R). \tag{33}$$

Theorem 5.6. *Let $D_{n,j}$ be given according to* (32) *and $c_n \neq 0$ for all $n \in \mathbb{N}_0$. Further, let $\tilde{V}$ be a harmonic function in the exterior of Ω_R which is regular at infinity with $\tilde{V}|_{\Omega_R} \in \mathrm{L}^2(\Omega_R)$ and*

$$\sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \frac{n^2 R^{2n-2l_n}}{c_n^2}\tilde{V}_{n,j}^2 < \infty.$$

Then a distributional solution of the Fredholm integral equation of the first kind in (2) *is given by*

$$D\delta_{\Omega_R} = \left(\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{(2n+1)R^{n-l_n}}{4\pi R^2 c_n} \tilde{V}_{n,j} Y_{n,j} \left(\frac{\cdot}{R} \right) \right) \delta_{\Omega_R}.$$

In the inverse gravimetric problem, as the typical application of the surface density approach, we have the following setting. Let $\bar{\rho}\colon \mathcal{B} \to \mathbb{R}$ be a density given by an arbitrary reference model of the Earth, for example, the Preliminary Reference Earth Model (PREM), see [18]. The corresponding gravitational potential is given by

$$\bar{V} = \gamma \int_{\mathcal{B}} \frac{\bar{\rho}(x)}{|x - \cdot|} \, \mathrm{d}x$$

and describes a part of the potential that does not change in the associated time span. The entire measured potential is given by $V = \bar{V} + \tilde{V}$, where $\tilde{V}$ are the relevant occurring changes in the gravitational potential. That is, we are here looking for a surface density $\sigma\colon \Omega_R \to \mathbb{R}$ with

$$\tilde{V} = V - \bar{V} = \gamma R^2 \int_{\Omega_R} \frac{\sigma(x)}{|x - \cdot|} \, \mathrm{d}\omega(x),$$

which causes these changes of the potential. By virtue of Equation (33), we know that the Fourier coefficients of the surface density are given by

$$\sigma_{n,j} = \frac{(2n+1)}{4\pi\gamma R^2} \tilde{V}_{n,j} \tag{34}$$

for all $n \in \mathbb{N}_0$ and all $j = 1, \ldots, 2n+1$. Chao [12] also proved that this problem is uniquely solvable. The obtained formula (34) coincides with the formulae which are commonly used in geodesy for a surface density ansatz or thin layer assumption, respectively, as originally proposed in [52].

6. Conclusions

We observed similarities between the inverse gravimetric and the inverse magnetic problem by considering both as particular cases of a kind of a master inverse problem. With this approach, a larger class of data inversion problems can be analyzed and solved all at once. A particular focus of the paper was the complete analysis of the non-uniqueness of the solution of all inverse problems of the investigated type. This analysis was based on something like a fundamental equation for the Fourier coefficients of the given data and the solution. The construction of a particular and appropriate orthonormal system on the ball enabled us to further understand the relation of the solution and the data. With this basis system and an adequate expansion in the data space, we characterized the null space of the Fredholm integral operator of the first kind in detail and calculated the singular system. Such a knowledge is an essential prerequisite for a series of regularization methods for inverse problems.

Furthermore, using the derived singular value decomposition, we also proved that this kind of inverse problem is unstable, that is, the inverse operator is unbounded. It also turned out that all considered problems have in common that most of the radial information gets lost. The ill-posedness of the considered problems is severely aggravated by the fact that the null space of the operator is infinite-dimensional, and, hence, the solution of the inverse problem is not unique. For this reason, we discussed four different additional conditions in order to obtain a unique solution: the minimum norm condition, a generalization of the harmonicity constraint, the layer density condition, and the surface density approach. In the particular case of the inverse gravimetric problem, our results coincide with the corresponding well-known results and in the case of the inverse magnetic problem, we found new results.

References

[1] Abramo, L.R., Reimberg, P.H., Xavier, H.S. (2010) CMB in a box: causal structure and the Fourier–Bessel expansion. Phys. Rev. D 82:043510

[2] Amirbekyan, A. (2007) The Application of Reproducing Kernel Based Spline Approximation to Seismic Surface and Body Wave Tomography: Theoretical Aspects and Numerical Results. Ph.D.-thesis, University of Kaiserslautern, Geomathematics Group, https://kluedo.ub.uni-kl.de/frontdoor/index/index/docId/1872

[3] Amirbekyan, A., Michel, V. (2008) Splines on the three-dimensional ball and their application to seismic body wave tomography. Inverse Probl. 24:015022

[4] Anger, G. (1990) Inverse Problems in Differential Equations. Akademie-Verlag, Berlin

[5] Aronszajn, N. (1950) Theory of reproducing kernels. Trans. Am. Math. Soc. 68:337–404

[6] Ballani, L., Stromeyer, D. (1992) On the structure of uniqueness in linear inverse source problems, Vieweg, Braunschweig, pp. 85–98. Theory and Practice of Geophysical Data Inversion

[7] Ballani, L., Engels, J., Grafarend, E.W. (1993) Global base functions for the mass density in the interior of a massive body (Earth). Manuscr. Geodaet. 18:99–114

[8] Barzaghi, R., Sansò, F. (1986) Remarks on the inverse gravimetric problem. Boll. geod. e sci. affini XLV:203–216

[9] Berkel, P. (2009) Multiscale Methods for the Combined Inversion of Normal Mode and Gravity Variations. Ph.D.-thesis, University of Kaiserslautern, Geomathematics Group

[10] Center for Space Research (2002) GRACE. http://www.csr.utexas.edu/grace/overview.html, [Online; accessed 30-August-2015]

[11] Chao, B.F., Dehant, V., Gross, R.S., Ray, R.D., Salstein, D., Watkins, M., Wilson, C. (2000) Space geodesy monitors mass transports in global geophysical fluids. EOS 81:247–250

[12] Chao, B.F. (2005) On inversion for mass distribution from global (time-variable) gravity field, J. Geodyn., 39:223–230

[13] Dassios, G., Fokas, A.S. (2009) Electro-magneto-encephalography for a three-shell model: dipoles and beyond for the spherical geometry. Inverse Probl. 25:035001

[14] Dassios, G., Fokas, A.S. (2013) The definite non-uniqueness results for deterministic EEG and MEG data. Inverse Probl. 29:065012

[15] Dassios, G., Fokas, A.S., Kariotou, F. (2005) On the non-uniqueness of the inverse MEG problem. Inverse Probl. 21:L1–L5

[16] Davis, P.J. (1975) Interpolation and Approximation. Dover Publications, New York

[17] Dufour, H.M. (1977) Fonctions orthogonales dans la sphère. Résolution théorique du problème du potential terrestre. B. Geod. 51:227–237

[18] Dziewonski, A.M., Anderson, D.L. (1981) Preliminary reference Earth model. Physics of the Earth and Planetary Interiors 25:297–356

[19] Fokas, A.S. (2009) Electro-magneto-encephalography for a three-shell model: distributed current in arbitrary, spherical and ellipsoidal geometries. J. R. Soc. Interface 6:479–488

[20] Fokas, A.S., Kurylev, Y. (2012) Electro-magneto-encephalography for the three-shell model: minimal L^2-norm in spherical geometry. Inverse Probl. 28:035010

[21] Fokas, A.S., Gel-fand, I.M., Kurylev, Y. (1996) Inversion method for magnetoencephalography. Inverse Probl. 2:L9–L11

[22] Fokas, A.S., Kurylev, Y., Marinakis, V. (2004) The unique determination of neuronal currents in the brain via magnetoencephalography. Inverse Probl. 20:1067–1082

[23] Fokas, A.S., Hauk, O., Michel, V. (2012) Electro-magneto-encephalography for the three-shell model: numerical implementation via splines for distributed current in spherical geometry. Inverse Probl. 28:035009

[24] Freeden, W., Gutting, M. (2013) Special Functions of Mathematical (Geo-)Physics. Birkhäuser, Basel

[25] Geselowitz, D.B. (1970) On the magnetic field generated outside an inhomogeneous volume conductor by internal current sources. IEEE Trans. Magn. 6:346–347

[26] Han, S., Shum, C., Bevis, M., Ji, C., Kuo, C. (2006) Crustal dilatation observed by GRACE after the 2004 Sumatra-Andaman earthquake. Science 313:658–662

[27] Hörmander, L. (1983) The Analysis of Linear Partial Differential Operators I. Springer, Berlin

[28] Kaula, W.M. (1966) Theory of Satellite Geodesy. Blaisdell, Waltham

[29] Kusche, J., Schrama, E. (2005) Surface mass redistribution inversion from global GPS deformation and Gravity Recovery and Climate Experiment (GRACE) gravity data. J. Geophys. Res. 110:B09409

[30] Leistedt, B., McEwen, J.D. (2012) Exact wavelets on the ball. IEEE Trans. Signal Process 60:6257–6269

[31] Michel, V. (2005) Wavelets on the 3-dimensional ball. Proc. Appl. Math. Mech. 5:775–776

[32] Michel, V. (2013) Lectures on Constructive Approximation. Fourier, Spline, and Wavelet Methods on the Real Line, the Sphere, and the Ball. Birkhäuser, Boston

[33] Michel, V., Fokas, A.S. (2008) A unified approach to various techniques for the non-uniqueness of the inverse gravimetric problem and wavelet-based methods. Inverse Probl. 24:045019

[34] Michel, V., Orzlowski, S. (2016) On the null space of a class of Fredholm integral equations of the first kind. J Inverse Ill-Posed Probl 24:687–710

[35] Moritz, H. (1990) The Figure of the Earth. Theoretical Geodesy of the Earth's Interior. Wichmann Verlag, Karlsruhe
[36] Nikiforov, A.F., Uvarov, V.B. (1988) Special Functions of Mathematical Physics. A Unified Introduction with Applications. Birkhäuser, Basel
[37] Olsen, N., et al (2013) The Swarm Satellite Constellation Application and Research Facility (SCARF) and Swarm data products. Earth Planets Space 65:1189–1200
[38] Pizzetti, P. (1910) Intorno alle possibili distribuzioni della massa nell'interno della terra. Annali di Mat., Milano XVII:225–258
[39] Plonsey, R. (1969) Biomagnetic Phenomena. McGraw-Hill, New York
[40] Rieder, A. (2003) Keine Probleme mit Inversen Problemen. Vieweg, Wiesbaden
[41] Rubincam, D.P. (1979) Gravitational potential energy of the Earth: a spherical harmonics approach. J. Geophys. Res.-Sol. Ea 84:6219–6225
[42] Rudin, W. (1991) Functional Analysis, 2nd edn. McGraw-Hill, Inc.
[43] Sabaka, T., Olsen, N. (2006) Enhancing comprehensive inversions using the Swarm constellation. Earth Planets Space 58:371–395
[44] Sansò, F., Rummel, R. (1997) Geodetic boundary value problems in view of the one centimeter geoid, Lect. Notes Earth Sci., vol 65. Springer, Berlin, Heidelberg
[45] Sarvas, J. (1987) Basic mathematical and electromagnetic concepts of the biomagnetic inverse problem. Phy. Med. Biol. 32:11–22
[46] Schnetzler, C. (1985) An estimation of continental crust magnetization and susceptibility from Magsat data for the conterminous United States. J. Geophys. Res. 90:2617–2620
[47] Stokes, G.G. (1867) On the internal distribution of matter which shall produce a given potential at the surface of a gravitating mass. Proc. Royal Soc. 15:482–486
[48] Stromeyer, D., Ballani L. (1984) Uniqueness of the inverse gravimetric problem for point mass models. Manuscr. Geodaet. 9:125–136
[49] Szegö, G. (1975) Orthogonal Polynomials. American Mathematical Society, Providence, Rhode Island
[50] Thébault, E., Purucker, M., Whaler, K.A., Langlais, B., Sabako, T.J. (2010) The magnetic field of the Earth's lithosphere. Space Sci. Rev. 155:95–127
[51] Tscherning, C.C. (1996) Isotropic reproducing kernels for the inner of a sphere or spherical shell and their use as density covariance functions. Math. Geol. 28:161–168
[52] Wahr, J., Molenaar, M., Bryan, F. (1998) Time variability of the Earth's gravity field: hydrological and oceanic effects and their possible detection using GRACE. J. Geophys. Res. 103B(12):30205–30229
[53] Wouters, B., Chambers, D., Schrama, E.J.O., (2008) GRACE observes small-scale mass loss in Greenland. Geophys. Res. Lett. 35:L20501

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