Numerical Modeling of Micropolar Thin Elastic Plates

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Abstract. System of equations describing micropolar elastic thin plates is written in symmetric hyperbolic form. For the solution of dynamic problems in the framework of micropolar thin plates the numerical algorithm is proposed. The algorithm based on the two-cyclic splitting method in combination with monotone finite-difference 1D schemes is proposed. The results of computations of problems on distributed impulse action loads are shown.

Keywords: Cosserat continuum · Micropolar plates · Elasticity · Finite -difference scheme · Dynamic problems

1 Introduction

Thin-walled structures such as rods, plates and shells are widely used in civil engineering, aero-space industry, medical and biological fields as basic structural elements. Structure is one of the most important indicator of the quality of materials directly influencing on theirs strength characteristics. Depending on type of the material and the scope of research in practical problems it is necessary to take into account the sctructure on nano-, micro- or mesolevel. To describe complex inner structure of a material the construction of new models of micropolar media is required. In micropolar or Cosserat continuum in addition to translational motion characterized with the velocity vector independent small rotations of particles are considered [1,2]. And together with the antisymmetric stress tensor antisymmetric couple stress tensor is introduced.

In papers [3–6] numerical solution of three-dimensional dynamic problems of Cosserat continuum was presented. In particular, it is shown that in Cosserat continuum there is a resonant frequency depending only on the inertial properties of particles and the elasticity parameters of the material. The present paper gives the results of numerical modeling of micropolar thin elastic plates.

There are some approaches of constructing two-dimensional mathematical models of micropoar plates. Within the framework of the direct approach the plate is modeled as a deformable surface with material points; see for example [7–9] and references therein. Another approach is based on the reduction

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of three-dimensional micropolar continuum equations. Various averaging procedures together with the approximation of the displacements and rotations in the thickness direction are applied; see for example [9–13] and references therein. In present paper assumptions on linear approximation of translation and rotation together with the through-the thickness integration procedure are made.

2 Mathematical Model

2.1 Equations of Three-Dimensional Cosserat Continuum

In Cosserat continuum translational motion denoted by u and independent rotations of particles denoted by φ are considered. The stress state of the material is characterized by the antisymmetric stress tensor σ and antisymmetric couple stress tensor m. The complete system of equations in three-dimensional Cosserat model consists of the motion equations, the kinematic equations and the generalised law of linear elasticity theory [2]

$$\begin{aligned}
&\rho\ddot{u} = \nabla\cdot\sigma + \rho g, \quad j\ddot{\varphi} = \nabla\cdot m - 2\sigma_x + jq,\\
&\Lambda = \nabla u + \varphi, \quad M = \nabla\varphi,\\
&\sigma = \lambda II\cdot\cdot\Lambda^S + 2\mu\Lambda^S + 2\alpha\Lambda^A,\\
&m = \beta II\cdot\cdot M^S + 2\gamma M^s + 2\varepsilon M^A.
\end{aligned} \tag{1}$$

Here g and q are the mass force and couple vectors, ρ is the material density, j is the inertial parameter equal to the product of the inertia moment of a particle about the axis through its center of gravity and the numbers of particles in unit volume. The formula $r = \sqrt{5j/(2\rho)}$ is valid to estimate the linear parameter of material microstructure. Constants λ and μ are the Lame parameters, and $\alpha, \beta, \gamma, \varepsilon$ are the phenomenological elasticity coefficients for an isotropic material. Λ and M are the strain and wryness tensors, the superscripts S and A correspond to the symmetric and antisymmetric tensor components respectively. The antisymmetric component is identified with its corresponding vector. A dot above a symbol denotes the derivative with respect to time t.

Boundary conditions have the following form in terms of translations and rotations

$$u = u^0, \quad \varphi = \varphi^0$$

or stresses

$$n\cdot\sigma = p^0, \quad n\cdot m = q^0,$$

where u^0, φ^0 are given functions, p^0 and q^0 are the surfaces forces and surface couples acting on a part of a boundary of micropolar body.

2.2 Equations of Two-Dimensional Micropolar Plates

The transition to the two-dimensional equations is based on the linear in x_3 approximation of the displacement and rotation with independent integration of motion equations in (1) through the plate thickness. Let the isotropic plate-like body occupy the volume $V = \{(x_1, x_2, x_3) \in R^3 : (x_1, x_2) \in S \subset R^2, x_3 \in [-h, h]\}$. Here h is a half of the plate thickness, $h = \text{const}$. Let us assume that the plate thickness $2h$ is small compared to characteristic sizes of a plate. For translations and rotations of the plate-like body the following aproximation is made

$$
\begin{aligned}
&u_i(t, x_1, x_2, x_3) = x_3 \psi_i(t, x_1, x_2), \quad i = 1, 2 \\
&u_3(t, x_1, x_2, x_3) = w(t, x_1, x_2), \\
&\varphi_i(t, x_1, x_2, x_3) = \omega_i(t, x_1, x_2), \\
&\varphi_3(t, x_1, x_2, x_3) = \omega_3(t, x_1, x_2) + x_3 \theta(t, x_1, x_2).
\end{aligned} \tag{2}
$$

Normal to the midplane displacement u_3 and rotations φ_1, φ_2 are independent on coordinate x_3. Hence, there are 7 kinematically independent scalar fields: ψ_1, ψ_2, w, φ_1, φ_2, φ_3, θ. In static case when ω_3 is equal to zero kinematical assumptions (2) coincide with hypotheses in [12].

On the front planes of the plate $x_3 = \pm h$ boundary conditions are assumed to be homogeneous, stress and couple stress tensors are zero.

Stress and couple stress tensor in constitutive Eq. (1) after the integration through the thickness are written in terms of stress and couple stress resultants

$$
N_{ij} =< \sigma_{ij} >, \quad T_{ij} =< x_3\, \sigma_{ij} >, \quad L_{ij} =< m_{ij} >, \quad K_{ij} =< x_3\, m_{ij} >,
$$

where $< (\dots) >= \int\limits_{-h}^{h} (\dots)\, dx_3$. Here and below indices $i, j = 1, 2,\ i \neq j$.

In terms of velocities and angular velocities

$$
\Psi_i = \dot{\psi}_i, \quad W = \dot{w}, \quad \Omega_i = \dot{\omega}_i, \quad \Omega_3 = \dot{\omega}_3, \quad \Theta = \dot{\theta},
$$

integration of the motion equations in (1) leads to the following motion equations:

$$
\begin{aligned}
&\frac{2h^3}{3} \rho \dot{\Psi}_i = T_{1i,1} + T_{2i,2} - N_{3i}, \quad 2h\rho \dot{W} = N_{13,1} + N_{23,2}, \\
&2hj\dot{\Omega}_i = L_{1i,1} + L_{2i,2} + (-1)^j (N_{j3} - N_{3j}), \\
&2hj\dot{\Omega}_3 = L_{13,1} + L_{23,2} + N_{12} - N_{21}, \\
&\frac{2h^3}{3} j\dot{\Theta} = K_{13,1} + K_{23,2} + T_{12} - T_{21} - L_{33}.
\end{aligned} \tag{3}
$$

The subscripts after a comma denote the partial derivatives with respect to the corresponding coordinate.

Elasticity relations after the integration are as follows

$$
\begin{aligned}
&\dot{T}_{ii} = \frac{2h^3}{3}\left(\lambda\left(\Psi_{1,1}+\Psi_{2,2}\right)+2\mu\Psi_{i,i}\right), \quad \dot{T}_{33} = \frac{2h^3}{3}\lambda\left(\Psi_{1,1}+\Psi_{2,2}\right),\\
&\dot{T}_{ij} = \frac{2h^3}{3}\left((\mu+\alpha)\Psi_{j,i}+(\mu-\alpha)\Psi_{i,j}+(-1)^i\cdot 2\alpha\Theta\right),\\
&\dot{N}_{i3} = 2h\left((\mu+\alpha)W_{,i}+(\mu-\alpha)\Psi_i+(-1)^j\cdot 2\alpha\Omega_j\right),\\
&\dot{N}_{3i} = 2h\left((\mu-\alpha)W_{,i}+(\mu+\alpha)\Psi_i+(-1)^i\cdot 2\alpha\Omega_j\right),\\
&\dot{L}_{ii} = 2h\left(\beta(\Omega_{1,1}+\Omega_{2,2}+\Theta)+2\gamma\Omega_{i,i}\right),\\
&\dot{L}_{33} = 2h\left(\beta(\Omega_{1,1}+\Omega_{2,2}+\Theta)+2\gamma\Theta\right),\\
&\dot{L}_{ij} = 2h\left((\gamma+\varepsilon)\Omega_{j,i}+(\gamma-\varepsilon)\Omega_{i,j}\right),\\
&\dot{K}_{i3} = \frac{2h^3}{3}(\gamma+\varepsilon)\Theta_{,i}, \quad \dot{K}_{3i} = \frac{2h^3}{3}(\gamma-\varepsilon)\Theta_{,i},\\
&\dot{L}_{i3} = 2h(\gamma+\varepsilon)\Omega_{3,i}, \quad \dot{L}_{3i} = 2h(\gamma-\varepsilon)\Omega_{3,i},\\
&\dot{N}_{ij} = (-1)^i 4h\alpha\Omega_3.
\end{aligned}
\tag{4}
$$

From the system (3)–(4) two independent systems are derived. One of them contains equations for Ω_3, N_{ij}, L_{i3}, L_{3i}:

$$
\begin{aligned}
&2hj\dot{\Omega}_3 = L_{13,1}+L_{23,2}+N_{12}-N_{21},\\
&\dot{L}_{i3} = 2h(\gamma+\varepsilon)\Omega_{3,i}, \quad \dot{L}_{3i} = 2h(\gamma-\varepsilon)\Omega_{3,i},\\
&\dot{N}_{ij} = (-1)^i 4h\alpha\Omega_3.
\end{aligned}
\tag{5}
$$

System (5) may be written as two-dimensional Klein-Gordon equation for angular velocity Ω_3:

$$
\ddot{\Omega}_3 = \frac{\gamma+\varepsilon}{j}\left(\Omega_{3,11}+\Omega_{3,22}\right)-4\frac{\alpha}{j}\Omega_3.
$$

The second independent system from (3)–(4) includes 24 equations for 24 unknowns and can be written in matrix form

$$
A\frac{\partial U}{\partial t} = B^1\frac{\partial U}{\partial x_1}+B^2\frac{\partial U}{\partial x_2}+QU+G,
\tag{6}
$$

where U is the vector-function

$$
U = (\Psi_i, W, \Omega_i, \Theta, T_{ii}, T_{33}, T_{ij}, N_{i3}, N_{3i}, L_{ii}, L_{33}, L_{ij}, K_{i3}, K_{3i}).
$$

The matrix coefficients A, B_1, B_2 containing elasticity parameters of a material are symmetric, and Q is antisymmetric. G is the given vector of mass forces and couples. The matrix A is positive definite if its diagonal blocks are positive definite. According to the Sylvester criterion this condition restricts the admissible values of the material parameters:

$$
3\lambda+2\mu>0, \quad \mu,\alpha>0, \quad 3\beta+2\gamma>0, \quad \gamma,\varepsilon>0.
\tag{7}
$$

If inequalities (7) are fulfilled system (6) is hyperbolic in the sense of Friedrichs. The potential energy of elastic deformation is a positive-definite quadratic form and conservation law is fulfilled:

$$\frac{\partial(UAU)}{\partial t} = \frac{\partial(UB_1U)}{\partial x_1} + \frac{\partial(UB_2U)}{\partial x_2}.$$

The characteristic properties of this system are described by the equation

$$\det(cA + n_1B_1 + n_2B_2) = 0, \quad n_1^2 + n_2^2 = 1.$$

Its positive roots are the velocities of longitudinal waves c_p, transverse waves c_s, torsional waves c_m, and rotational waves c_ω. These roots are

$$c_p = \sqrt{\frac{\lambda + 2\mu}{\rho}}, \quad c_s = \sqrt{\frac{\mu + \alpha}{\rho}}, \quad c_m = \sqrt{\frac{\beta + 2\gamma}{j}}, \quad c_\omega = \sqrt{\frac{\gamma + \varepsilon}{j}}.$$

For hyperbolic system (6) the boundary-value problem with initial conditions $U(0,x) = U^0(x)$ and dissipative boundary conditions is well-posed. In particular among the dissipative conditions there are the conditions in terms of velocities

$$\Psi_i = \Psi_i^0, \quad W = w^0, \quad \Omega_i = \Omega_i^0, \quad \Omega_3 = \Omega_3^0, \quad \Theta = \Theta^0,$$

or stresses

$$\begin{aligned} &n_1T_{1i} + n_2T_{2i} = p_i, \quad n_1N_{13} + n_2N_{23} = p_3, \\ &n_1L_{1i} + n_2L_{2i} = q_i, \quad n_1L_{13} + n_2L_{23} = q_3, \quad n_1K_{13} + n_2K_{23} = k, \end{aligned}$$

where Ψ_i^0, W^0, Ω_i^0, Ω_3^0 and Θ^0 are given functions, p_i, p_3, q_i, q_3, k are given surface forces and couples acting on part of the boundary of micropolar body.

3 Numerical Modeling

3.1 Numerical Algorithm

The algorithm of numerical solution of linear system (6) is based on two-cyclic splitting method with respect to spatial variables and time. On time interval $(t, t + \triangle t)$ the method consists of five stages: the solution of a one-dimensional problem in the x_1 direction on time interval $(t, t + \triangle t/2)$; similar stage in the x_2 direction; the stage of solution of a system of linear ordinary differential equations with matrix Q with full time-step; and two stages of repeated recalculations of a problem in the x_2 and x_1 directions respectively on time interval $(t + \triangle t/2, t + \triangle t)$. As applied to system (6) the splitting method has the following form:

$$\begin{aligned} &A\dot{U}^1 = B^1U^1_{,1} + G^1, \quad U^1(t,x) = U(t,x), \\ &A\dot{U}^2 = B^2U^2_{,2} + G^2, \quad U^2(t,x) = U^1(t + \triangle t/2, x), \\ &A\dot{U}^3 = QU^3, \quad U^3(t,x) = U^2(t + \triangle t/2, x), \\ &A\dot{U}^4 = B^2U^4_{,2} + G^2, \quad U^4(t + \triangle t/2, x) = U^3(t + \triangle t, x), \\ &A\dot{U}^5 = B^1U^5_{,1} + G^1, \quad U^5(t + \triangle t/2, x) = U^4(t + \triangle t, x). \end{aligned}$$

Vector-function U^1 is taken from the prevous time step. At $t = 0$ it is taken from the initial conditions. The unknown value $U(t + \triangle t, x)$ is $U^5(t + \triangle t, x)$, $G^1 + G^2 = G$.

At the third stage Crank-Nickolson finite-difference scheme with full time step is used. Each of four remaining one-dimensional problems are solved with the help of explicit monotone ENO-scheme of "predictor-corrector" type. This scheme is a generalization of Godunov collapse of the gap scheme.

The two-cyclic splitting method ensures the stability of a numerical solution provided Courant-Friedrichs-Levy stability condition for one-dimensional systems is fulfilled. It has second order of accuracy if second-order schemes at its stages are used.

The verification of the algorithm is performed by comparig the results of numerical computations with the exact solution describing wave propagation in micropolar plate.

3.2 Numerical Results

The results of numerical computations of the elastic waves propagation in a rectangular thin plate are presented in Figs. 1 and 2. Top and bottom sides of the plate are nonreflecting boundaries. The right side of the plate is fixed. On the left side distributed periodic load of Λ-impulses of T_{11} is given. The area of action load is one third of a side in the central part. As a result of impulse load a sequence of loading and unloading waves propagates over the material. Waves are generated at the points of the boundary of the area of action load on the left side of the plate. In the first case (Fig. 1) a single wave induced by the loading impulse is observed, in the second case (Fig. 2) three waves caused by three impulses propagate.

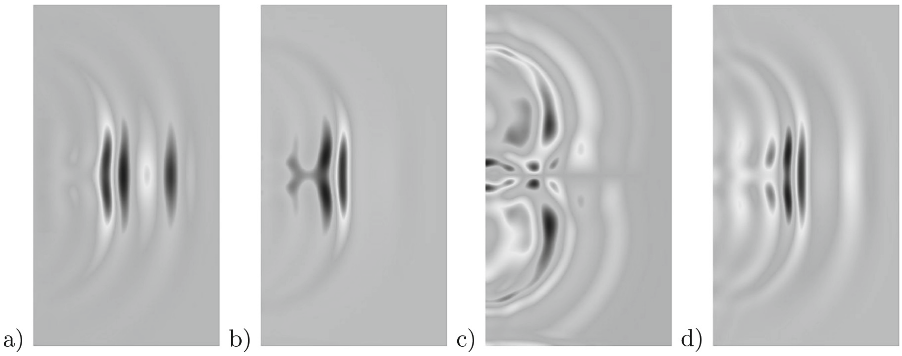

Fig. 1. The action of Λ-shaped impulse of T_{11}: level curves of the couple stress resultant T_{11} (a), velocity W (b), angular velocities Ω_1 (c) and Ω_2 (d)

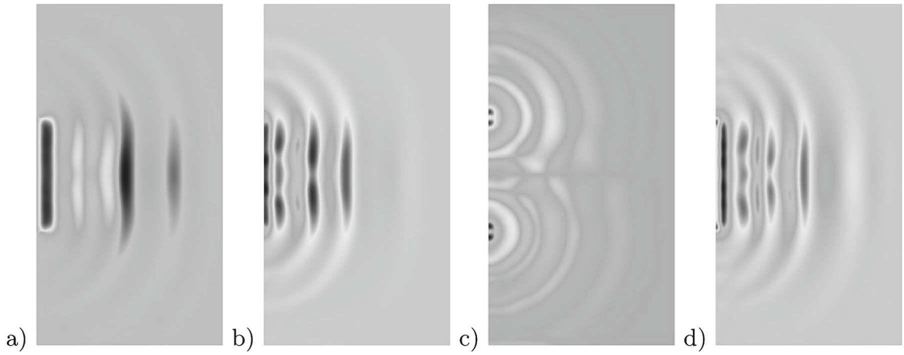

Fig. 2. The action of Λ-shaped impulses of T_{11}: level curves of the couple stress resultant T_{11} (a), velocity W (b), angular velocities Ω_1 (c) and Ω_2 (d)

The level curves of the couple stress resultant T_{11}, velocity W, angular velocities Ω_1 and Ω_2 are shown in Figs. 1 and 2 for the time moment $t = 18$ μs. The calculations are performed on a rectangular plate of sides 0.05×0.1 m for synthetic polyurethane. Material parameters are taken from [14]. Velocities of elastic waves are $c_p = 2687$, $c_s = 1394$, $c_\omega = 893$ m/s. The plate thickness $2h = 5$ mm. Characteristic scale of the microstructure of a material is $r = 0.15$ mm. The uniform difference grid used in computations consists of 1000×1000 cells with a mesh size of 0.1 mm less than r. On coarser grids calculations with satisfactory accur acy may not be performed.

4 Conclusions

In this paper basic equations of micropolar elasticity theory of thin plates are considered. These relations are constructed with assumptions on linear approximation of translation and rotation together with the through-the thickness integration procedure. The potential energy of elastic deformation is a positive-definite quadratic form and conservation law is fulfilled. System of the equations is written in symmetric hyperbolic form that is convenient for numerical computations. Numerical algorithm based on two-cyclic splitting method in combination with monotone finite-difference scheme can be used for the solution of dynamic problems of shock, impulse and concentrated action loads. The results of numerical computations of dynamic problem on disrtibuted periodic impulse load show the oscillatory nature of the solution.

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