Chapter Eight: Finite Projective Geometries

Section 1. Lions and Ponies

Many students have difficulties with proofs. This is understandable since the concept of proofs lies in the heart of mathematics, and proofs are not the most intuitive things to do. On top of that, proofs cannot exist in a vacuum, and often, the subject matter adds to the difficulty.

What we will do now is to consider a situation which does not apparently involve any mathematical concepts. We will then do some proofs without being encumbered by having to deal with contents. Thus we can focus on the proof process itself.

A certain community of lions and ponies is defined by the following postulates.
(1) There are at least two lions.
(2) Each lion has bitten at least three ponies.
(3) For any pair of lions, there is exactly one pony that both have bitten.
(4) For any pair of ponies, there is at least one lion that has bitten both.

So far, there is nothing that would scare students, apart from being bitten by lions. Nevertheless, from this humble beginning, we can derive many interesting results.

Theorem 1.
For any lion, there is at least one pony that it has not bitten.

Theorem 1 is essentially a negative statement. To prove it, we let L be a given lion. We have to find a pony which it has not bitten. Where do we begin? We know nothing about this community except for the four postulates which define it. All of them are positive statements.

In the absence of any good ideas, we may use an indirect approach. Assume the opposite of what we have to prove, and see what may be wrong with it. So we suppose that L has bitten every pony. By Postulate 1, there is another lion M. By Postulate 3, there is exactly one pony which both L and M have bitten. Since L has bitten every pony, this means that M can have bitten only one pony.

However, Postulate 2 states clearly that M has bitten at least three ponies. What is going on? What we have is a contradiction. Where does it arise? Did we make any mistakes in our logical reasoning? Going over the steps carefully, they are all correct. Hence the problem arises from our initial assumption, that L has bitten every pony. It follows that L has not bitten every pony, and Theorem 1 has been proved.

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We can now turn this argument around and give a direct proof of Theorem 1. For a given lion L, we wish to prove that there is a pony which it has not bitten. By Postulate 1, there is another lion M. By Postulate 2, M has bitten at least three ponies P, Q and R. By Postulate 3, there is exactly one pony, say P, which both L and M have bitten. Then Q is a pony which L has not bitten.

Theorem 2.
For any pony, there is at least one lion that has not bitten it.

Theorem 2 is obtained from Theorem 1 by interchanging the roles of the lions and ponies. It is called the *dual* of Theorem 1. Of course, Theorem 1 is also the dual of Theorem 2.

Again, we start with an indirect approach. Assume that there is a pony P that every lion has bitten. By Postulate 1, there are two lions L and M. By Postulate 2, L has bitten a pony Q other than P and M has bitten a pony R other than P. By Postulate 3, L has not bitten R and M has not bitten Q. Hence Q and R are not the same pony. By Postulate 4, there is a lion N that has bitten both Q and R. Hence N and L are two different lions but they have both bitten P and Q. This contradicts Postulate 3.

Very minor changes yield a direct proof. By Postulate 1, there are two lions L and M. If one of them has not bitten P, there is nothing else to prove. So assume that they both have. By Postulate 2, L has bitten a pony Q other than P and M has bitten a pony R other than P. By Postulate 3, L has not bitten R and M has not bitten Q. Hence Q and R are not the same pony. By Postulate 4, there is a lion N that has bitten both Q and R. Hence N and L are two different lions. Since they have both bitten Q, N is a lion which has not bitten P by Postulate 3.

Theorem 3.
For any pair of lions, there is at least one pony that neither has bitten.

Let L and M be any pair of lions. By Postulate 3, there is a pony P that both L and M have bitten. By Theorem 2, there is a lion N that has not bitten P. By Postulate 2, N has bitten at least three ponies. By Postulate 3, L has bitten at most one of these three and so has M. Hence there is a pony that neither L nor M has bitten.

Theorem 4.
For any pair of ponies, there is at least one lion that has not bitten either.

Let P and Q be any pair of ponies. By Postulate 4, there is a lion L that has bitten both of them. By Postulate 2, this lion has bitten a third pony R. By Theorem 1, there is a pony S that L has not bitten. By Postulate 4, there is a lion M that has bitten both R and S. By Postulate 3, M has not bitten either P or Q.

Just as Theorems 1 and 2 are the duals of each other, Theorems 3 and 4 are the duals of each other.

We have done some reasonably sophisticated proofs without delving into any formal subject matter. Clearly, there cannot be a community in the real world where lions bite ponies according to a set of postulates. Nevertheless, one should raise the question whether such an abstract structure can in fact exist. If not, all our proofs so far are for nothing.

Before we pursue this angle, let us recall that duality is a central concept in our community of lions and ponies. Duality allows us to obtain two results for the price of one, which is a very good thing. It was this search for duality that led to a very important development in the history of mathematics.

In Euclidean geometry, the first postulate is that every two points determine a line. The dual of this result is that every two lines determine a point. It is almost true, except in the case where the lines are parallel to each other. If we wish this dual to hold true, then parallel lines must also meet at some point.

In everyday life, the two rails of a straight railway track must be parallel, as otherwise any train running on them must derail. However, in a painting or photograph of a railway track, the two rails may perhaps not come to a point, but they are most certainly not parallel. They come closer and closer to each other as they recede into the background, which leads to the saying that *parallel lines meet at infinity.*

So we add a point at infinity to each line in the Euclidean plane. We call them *dieal* points, to distinguish them from the ordinary points. Parallel lines have the same ideal point while non-parallel lines have different ideal points. In a sense, an ideal point represents the common slope of a set of parallel lines. In this extended plane, every two lines determine a point.

We have plugged one hole, but we may have cracked another leak. Is it still true that every two points determine a line?

If both points are ordinary points, then they determine a line as usual. If one point is ordinary and the other is ideal, they still determine a line. This is the familiar point-slope formula in analytic geometry. However, if both points are ideal, they do not determine a line.

To plug this new hole, we now add an *ideal* line which passes through all the ideal points and only the ideal points. So two ideal points will determine the ideal line. Moreover, the ideal line and an ordinary line determines the ideal point on that ordinary line, so we have indeed achieved duality.

This new plane is called the *projective* plane, and a new branch of mathematics called *projective* geometry is born. The quest for duality, though apparently an exercise in abstraction, does have important consequences.

The church paintings (only the churches could afford paintings in those days) up until about the twelfth century did not have depth perception. This was because the concept of perspective views had not yet existed. Paintings since then have benefited from the development of projective geometry.

We now see that if we replace the ponies by points and the lions by lines, our community becomes a projective geometry. However, it is clear that we intend only to have a finite number of lions and a finite number of ponies. So we turn our attention to the subject of *finite* geometry.

Figure 8.1 shows the simplest finite model of the Euclidean plane. There are four points A, B, C and D, with coordinates $(x, y) = (0, 0)$, (1,0), (0,1) and (1,1) respectively There are six lines AB ($y = 0$), CD ($y = 1$), AC ($x = 0$), BD ($x = 1$), AD ($x + y = 0$) and BC ($x + y = 1$). Note that addition is in modulo 2, so that the point (1,1) lies on $x + y = 0$.

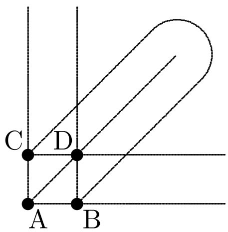

Figure 8.1

Clearly, AB and CD are parallel to each other, as are AC and BD. Moreover, AD and BC are also parallel to each other, as they do not intersect at any of the points in this geometry. We have deliberately drawn BC in such a way to emphasize this point. We call this finite plane the *affine* plane of *order* 2.

We now add an ideal point E to AB and CD, an ideal point F to AC and BD, an ideal point G to AD and BC, and an ideal line passing through E, F and G. This extended plane, shown on the left of Figure 8.2, is called the *projective* plane of order 2. It is redrawn in a more stylish form on the right of Figure 8.2, where the line BC is still drawn as a curve. This is a famous example called the *Fano* plane.

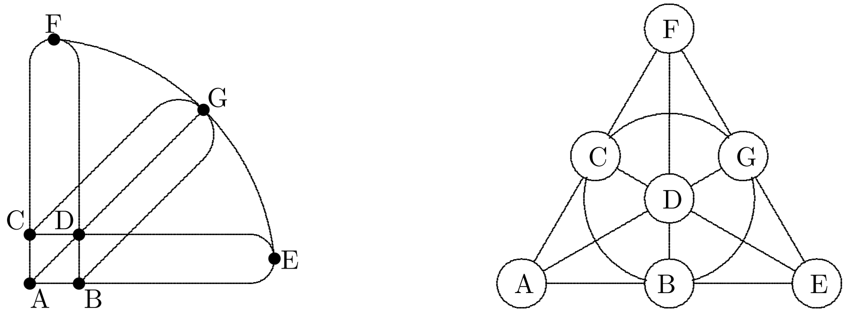

Figure 8.2

Now let the seven points represent the ponies and let the seven lines represent the lions. Each lion bites the three ponies represented by points on the line representing the lion. We can verify that the four postulates of the community of lions and ponies indeed hold. They now become the postulates of the projective geometry, and we restate them as follows.

(1) There are at least two lines.
(2) Each line passes through at least three points.
(3) For any pair of lines, there is exactly one point that both pass through.
(4) For any pair of points, there is at least one line that passes through both.

Figure 8.3 shows the projective plane of order 3, with the affine plane of order 3 embedded in it.

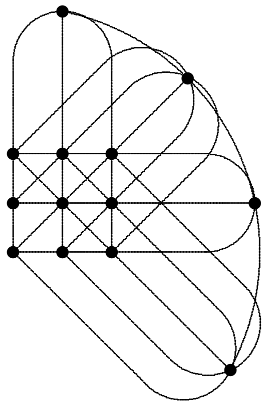

Figure 8.3

We must exercise caution in trying to construct an affine plane of order 4. It would appear natural to take the points with coordinates $(x, y) = (0, 0)$, (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3), (2,0), (2,1), (2,2), (2,3), (3,0), (3,1), (3,2) and (3,3). The lines may be divided into the following five groups of parallel lines.

$y = 0$	$x = 0$	$y = x$	$y = 2x$	$y = 3x$
$y = 1$	$x = 1$	$y = x + 1$	$y = 2x + 1$	$y = 3x + 1$
$y = 2$	$x = 2$	$y = x + 2$	$y = 2x + 2$	$y = 3x + 2$
$y = 3$	$x = 3$	$y = x + 3$	$y = 2x + 3$	$y = 3x + 3$

Consider the line $y = 2x$ which passes through the four points (0,0), (1,2), (2,0) and (3,2). It will intersect the line $y = 0$ in two points, namely, (0,0) and (2,0), and this violates the first postulate of Euclidean geometry. The reason for this is that 4 is not a prime number, so that we can have $2 \times 2 \equiv 0 \pmod 4$. Nevertheless, there is an affine plane of order 4, but it is based on a concept called *Galois* theory which we will not discuss here.

In general, if n is a prime number, then we can construct an affine plane of order n in the usual way. It has n^2 points and $n^2 + n$ lines which may be divided into $n + 1$ classes of n parallel lines. Each line passes through n points and each point lies on $n + 1$ lines. By adding an ideal point to each line and joining them by an ideal line, we have a projective plane of order n. It has $n^2 + n + 1$ points and $n^2 + n + 1$ lines. Each line passes through $n + 1$ points and each point lies on $n + 1$ lines. Observe the duality between the points and the lines here.

To conclude this section, let us return to the community of lions and ponies and work on some counting problems. Let L be a lion and P be a pony such that L has not bitten P. We claim that the number of ponies that L has bitten is equal to the number of lions that have bitten P.

Let M be another lion which has bitten P and Q be another pony L has bitten. By Postulate 3 and its dual (proof left as an exercise), there is exactly one pony that both L and M have bitten and exactly one lion that has bitten both P and Q. Hence the ponies that L has bitten and the lions that have bitten P can be paired off, and their numbers are equal. This justifies our claim.

We now prove that each lion has bitten the same number of ponies. Let L and M be any two lions. We have proved earlier that there is a pony P that neither has bitten. Hence the number of ponies that L has bitten is equal to the number of ponies that M has bitten. This follows from the claim above since both are equal to the number of lions that have bitten P. In an analogous manner, we can prove that each pony has been bitten by the same number of lions.

Suppose one of the lions, say L, has bitten exactly $n + 1$ ponies. Then every lion has bitten exactly $n + 1$ ponies. Now there is a pony P that L has not bitten. It follows that exactly $n + 1$ lions has bitten P, so that each pony has been bitten by exactly $n + 1$ lions.

Consider the $n+1$ ponies that L has bitten. Each has been bitten by n other lions. This yields a count of $n(n+1)$ lions besides L. By Postulate 3, each lion other than L has bitten exactly one of these ponies, and is therefore counted exactly once. Hence there are exactly n^2+n+1 lions. In an analogous manner, we can prove that the total number of ponies is also n^2+n+1.

The community of lions and ponies was introduced by me in [3], with the ponies replaced by lambs. In thinking that the lambs were the most likely candidates to have been bitten, I totally missed the *point*.

Section 2. Starwars

Space Station Intelligentia received a call for help on the hyperradio from Spaceship Academia. Captain Philip said, “We are on the way home after a successful promotion of higher learning in distant star systems. We are surrounded by a Kleingon Fleet. Because we are on a peaceful mission, we are unarmed. Please send a relief force.”

“Unfortunately, due to budget cuts, there are no other Spaceships on base at the moment,” said Commander Gilbert. “Can you hold out?”

“Affirmative,” said Captain Philip, “but we cannot disengage. It would help if you can get a Space Cannon to us.”

“No problem. We will send one over by a Space Pod.”

“Hang on a minute! Oh, no! There is a Space Tetropus in the Kleingon Fleet. It can grab one Space Pod at a time.”

“I will send two Space Pods, each carrying a Space Cannon,” said Commander Gilbert.

“Do not do that! Repeat! Do not do that!” Captain Philip said urgently. “If a Space Cannon falls into the hands of the Kleingons, we are history. It is too powerful even for us.”

“I will send two, but only one has a Space Cannon. We will get the empty one to nudge up to the Space Tetropus.”

“I don’t think we can assume that the Space Tetropus is stupid.”

“I wish I have one hundred Space Pods that I can send at the same time. With only one of them carrying a Space Cannon, our chance of success is 99%.”

“That is easy for you to say, safely on the Space Station. Up here in the Spaceship, we do not like the 1% chance of failure.”

“I will get back to you as soon as possible.”

Commander Gilbert consulted Lieutenant Kenneth, the scientific advisor. He said, “We can break up a Space Cannon into two parts and send them separately. This way, the Kleingons can only get half of it, which is of absolutely no use to them.”

“Unfortunately, Spaceship Academia will not get too much out of the other half either. However, your idea is an excellent one. If we break up two Space Cannons into two parts in identical fashion and send them by four Space Pods, the Kleingons will still be out of luck, while Spaceship Academia will have enough parts to reassemble a complete one.”

The two officers were very pleased with their plan. However, when they tried to put it in operation, they found that there were only three Space Pods available on base.

Lieutenant Kenneth thought for a while and said, "Our main difficulty is not knowing which Space Pod the Space Tetropus will take. So to minimize our loss, we should distribute the Space Cannons as evenly as possible among the Space Pods."

"We must use two Space Cannons," said Commander Gilbert, "as we are bound to lose some parts. However, if we do not put the same part in the same Space Pod, we cannot lose them both. So two Space Cannons is exactly what we need to use."

"With two Space Cannons and three Space Pods, each Space Pod should carry two-thirds of a Space Cannon. So this means breaking up a Space Cannon into three parts, in identical fashion. Let us call them A, B and C. The first Space Pod will carry A and B, the second B and C, and the third C and A. So Spaceship Academia can still get a complete Space Cannon, while the Kleingons can only get two-thirds of one."

"It would be best if we do not break up the Space Cannons into too many parts. Couldn't we still have done it with only two?"

"No. Since we have four copies and three Space Pods, one of them must carry two. These must be different as there is no point in any Space Pod carrying two identical parts. If the Space Tetropus grabs this one, the Kleingons can put the two parts together to get a complete Space Cannon."

"I guess you are right," said Commander Gilbert. "It is lucky that we have three Space Pods. Had there been only two, we could not have done anything."

"Yes, each of Spaceship Academia and the Kleingons will get one. Either both have a chance of getting a complete Space Cannon, or neither has, which is definitely not good for us."

"Let us stop theorizing and put our plan to work. We cannot count on Spaceship Academia holding out forever against the Kleingons."

This was done, and soon words came over the hyperradio that all was well. Before long, Spaceship Academia was docking at Space Station Intelligentia. Commander Gilbert and Lieutenant Kenneth welcomed Captain Philip's safe return.

"That was a close call," reported Captain Philip. "The Kleingons were about to replace the Space Tetropus with a Space Octopus, which can grab two Space Pods at a time."

“This is serious,” said Commander Gilbert. “Let us go to work at once and figure out a way around it, rather than wait until we have to face the situation.”

“To begin with,” said Lieutenant Kenneth, “we have to break up three Space Cannons. This way, we cannot lose every copy of any part. On the other hand, we do not need to break up more than three, as that will only make things easier for the Kleingons.”

“Also, each Space Cannon must be broken up into at least three parts,” Captain Philip said. “If there are only two, the Space Octopus can just nab one Space Pod carrying each part, and the Kleingons will have a complete Space Cannon. If we break it up into exactly three parts, we will need nine Space Pods so that each one will carry one part. Nothing less will do.”

“We seldom have that many Space Pods on base,” Commander Gilbert pointed out. “What is the smallest number of Space Pods that can carry out a successful convoy?”

“It has to be five or more. If we send only four, each side will get two, and that is bad news. It is the same argument which explains why two Space Pods are not enough for getting around a Space Tetropus.”

“Are five Space Pods enough though?” Commander Gilbert pressed the point.

Nobody had an answer for a few days. Then Commander Gilbert found a diagram which his son Atticus drew.

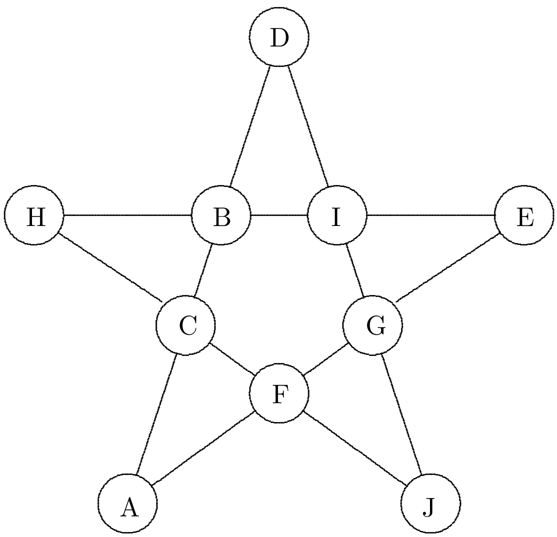

Figure 8.4

It was a regular five-pointed star, the emblem of Space Station Intelligentia. Atticus had labeled the ten points of intersections A, B, C, D, E, F, G, H, I and J in some random order, as shown in Figure 8.4.

"That gives me an idea," said Lieutenant Kenneth. "Let us break up each of three Space Cannons into ten parts labeled A to J. Let each of the five lines in Figure 8.4 represent a Space Pod, carrying all the parts whose labels do not appear on that line. Call DA line 1, AE line 2, EH line 3, HJ line 3 and JD line 5. Here is a list of the parts carried by the five Space Pods."

Space Pod 1: E, F, G, H, I and J.
Space Pod 2: B, C, D, H, I and J.
Space Pod 3: A, C, D, F, G and J.
Space Pod 4: A, B, D, E, G and I.
Space Pod 5: A, B, C, E, F and H.

"Why would this work?" questioned Captain Philip.

"I see," said Commander Gilbert. "Whichever two Space Pods the Space Octopus captures, the two lines representing them will intersect. So it will be missing the part represented by that intersection. This is brilliant."

"Yes, indeed," agreed Captain Philip. "However, let us see if we can still work something out even if we have not been blessed with this divine revelation. Consider all possible scenarios. The Space Octopus may nab Space Pods 1 and 2, 1 and 3, 1 and 4, 1 and 5, 2 and 3, 2 and 4, 2 and 5, 3 and 4, 3 and 5, or 4 and 5. So for any of these ten pairs, there must be at least one part neither of which is carrying."

"Going back to what I said earlier," chimed in Lieutenant Kenneth, "we must have three copies of each part. Let the parts be labeled from A on. If Space Pods 1 and 2 are missing part A, then Space Pods 3, 4 and 5 must have it."

"This means that we cannot have two different pairs missing the same part," said Captain Gilbert, "even if the pairs overlap. In other words, we must break up each Space Cannon into ten parts, so that each of the ten pairs will be missing a different part."

"So 1 and 3 must be missing some part other than A, say B," said Captain Philip. "Then 2, 4 and 5 must have B. I think this will work. Let us draw a chart to show what each Space Pod should be carrying."

"Look!" exclaimed Lieutenant Kenneth. "You have come up with exactly the same solution obtained earlier!"

"What?" said Captain Philip. "Yes, I do believe you are right. It is nice to be able to get this in two ways."

Captured	1	1	1	1	2	2	2	3	3	4	S
Space Pods	2	3	4	5	3	4	5	4	5	5	P
Parts					E	F	G	H	I	J	1
carried		B	C	D				H	I	J	2
by each	A		C	D		F	G			J	3
of the	A	B		D	E		G		I		4
Space Pods	A	B	C		E	F		H			5

"I think you should write a book called *MENSA for Dummies* and put this in," laughed Commander Gilbert. "It is very pedantic and not brilliant at all, but it does work all the same."

The officers did not have much time to enjoy their discovery. Words just came in that the Kleingons had stepped up the arms race. Instead of replacing the Space Tetropus by a Space Octopus, they had replaced it by a Space Dodecopus. It had twelve arms and could grab three Space Pods at a time.

"Well," said Lieutenant Kenneth, "we must use four Space Cannons. We can break up each of them into only four parts if we have sixteen Space Pods."

"In view of the new development, we should have our budget increased. However, I seriously doubt that we could afford sixteen Space Pods. The minimum number of Space Pods we must have is seven," said Commander Gilbert.

"This would be a nightmare," complained Captain Philip. "Here is a list of thirty-five trios of seven Space Pods. This means we have to break up each Space Cannon into thirty-five parts. I am not sure if we can put them back together again."

(1,2,3)	(1,2,4)	(1,2,5)	(1,2,6)	(1,2,7)	(1,3,4)	(1,3,5)
(1,3,6)	(1,3,7)	(1,4,5)	(1,4,6)	(1,4,7)	(1,5,6)	(1,5,7)
(1,6,7)	(2,3,4)	(2,3,5)	(2,3,6)	(2,3,7)	(2,4,5)	(2,4,6)
(2,4,7)	(2,5,6)	(2,5,7)	(2,6,7)	(3,4,5)	(3,4,6)	(3,4,7)
(3,5,6)	(3,5,7)	(3,6,7)	(4,5,6)	(4,5,7)	(4,6,7)	(5,6,7)

The new budget allowed for eight Space Pods, one more than the absolute minimum.

"We have to break up each Space Cannon into at least fourteen parts," lamented Lieutenant Kenneth. "Suppose we have only thirteen parts. Then there are fifty-two pieces of equipments. On the average, each of the eight Space Pods must carry more than six pieces. Hence some Space Pod must carry at least seven parts."

"Let us assume that the Space Dodecopus will grab this one," said Commander Gilbert, picking up the train of thought. "Then the six parts it is missing are carried by the other seven Space Pods. Now there are twenty-four pieces of equipment. On the average, each of these seven Space Pods must carry more than three pieces, so that some Space Pod must carry at least four parts."

"Let this be the second Space Pod captured by the Space Dodecopus," said Captain Philip. "Now the two parts it is still missing are carried by the six Space Pods. Since there are eight pieces of equipment, some Space Pod must carry two pieces. If the Space Dodecopus captures this one too, it will have all the parts to put a Space Cannon back together.".

"Well, fourteen parts is much better than thirty-five parts," remarked Lieutenant Kenneth, "but can we do it with only fourteen parts?"

It was quite a while before any progress was made. The officers remembered the Finite Projective Geometry course they took during basic training. In particular, they recalled the Fano plane. They took two copies of it and labeled the points A, B, C, D, E, F and G in one, and T, U, V, W, X, Y and Z in the other, as shown in Figure 8.5.

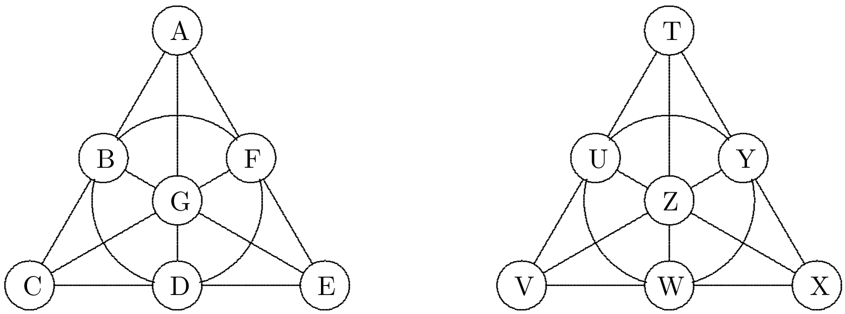

Figure 8.5

Captain Philip said, "Let line 1 be ABC, line 2 be CDE, line 3 be EFA, line 4 be ADG, line 5 by CFG, line 6 be BEG and line 7 be BDF. Following our earlier example, each line represents a Space Pod carrying all the parts whose labels do not appear on that line in the first copy. In addition, it carries all the parts whose labels do appear on that line in the second copy. The last Space Pod carries every part in the second copy. Here is a list of the parts carried by the eight Space Pods."

Space Pod 1: D, E, F, G, T, U and V.
Space Pod 2: A, B, F, G, V, W and X.
Space Pod 3: A, B, C, D, T, X and Y.
Space Pod 4: B, C, E, F, T, W and Z.
Space Pod 5: A, B, D, E, V, Y and Z.
Space Pod 6: A, C, D, F, U, X and Z.
Space Pod 7: A, C, E, G, U, W and Y.
Space Pod 8: T, U, V, W, X, Y and Z.

"Suppose the Space Dodecopus captures Space Pod 8. Then it captures only two of the other seven," said Lieutenant Kenneth. "The two lines representing them will intersect. So the Space Dodecopus will be missing the part represented by that intersection in the first copy."

"On the other hand," observed Commander Gilbert, "suppose that the Space Dodecopus does not capture Space Pod 8. In order for it to have all of T, U, V, W, X, Y and Z, they have to capture three Space Pods represented by lines all passing through the same point. However, the Space Dodecopus will be missing the part represented by that intersection in the first copy."

"So it works," said Captain Philip. "Let us hope that this stupid war ends soon."

Remark:
The problem in this section appears in [5] with a different story line. The results appeared in [2], a paper by Circle members Gilbert Lee, Kenneth Ng and Philip Stein. The extension into the Space Dodecopus are a small part of [1], a paper by Sven Chou and Jason Liao, two members of Chiu Chang Mathematical Circle. By the way, Kleingon was not a misspelling of Klingon of *Star Trek* fame. The word referred to an underling of a certain Albertan politician at the time, who was at odds with education.

Section 3. Convenient Buildings

A building is said to be *convenient* if for any two floors, there is at least one elevator which stops on both of them. Suppose the building has m elevators each of which stops on n floors. There are no restrictions on the choice of these floors. They do not have to be consecutive, and need not include the ground floor. What is the maximum number $f(m,n)$ of floors in this convenient building?

To establish the answer to this or any extremal problem, we need to do two things. First, we must show by an explicit construction that the answer can be attained. The finite projective planes would be useful for this purpose. Second, we must prove by a general argument that the answer cannot be improved.

We first prove three useful preliminary results.

Observation 1. $f(m+1,n) \geq f(m,n)$.

Proof:
Having an extra elevator never hurts, though it may not help.

Observation 2. $f(m,n+1) \geq f(m,n)+1$.

Proof:
The extra stop for each elevator can all be on a new floor.

Observation 3. $f(m,kn) \geq kf(m,n)$.

Proof:
Pile k copies of a convenient building with $f(m,n)$ floors on top of one another to form a building with $kf(m,n)$ floors and connect the corresponding elevators in each copy so that each stops on kn floors. The same elevator which links the i-th and j-th floors in each copy will link the i-th floor of any copy to the j-th floor of any other copy. Thus the new building is convenient, and we have $f(m,kn) \geq kf(m,n)$.

We now study the function $f(m,n)$ by keeping m constant.

For $m=1$, we have $f(1,n)=n$. The building can certainly have n floors. If it has more, the elevator will not stop on some floor. No elevator will stop on both this floor and any other floor.

For $m=2$, we still have $f(2,n)=n$. By Observation 1, $f(2,n) \geq f(1,n)=n$. If the building has more floors, each elevator will not stop on some floor. If they skip different floors, no elevator will stop on both. If they skip the same floor, no elevator will stop on both this floor and any other floor.

The first interesting case is $m = 3$. Let there be three floors 1, 2 and 3. Let the first elevator stop on floors 1 and 2, the second elevator on floors 1 and 3, and the third elevator on 2 and 3. Thus we have the lower bound $f(3, 2) \geq 3$. This is a *perfect* building because there is no duplication of services.

We next prove that $f(3, 2k) = 3k$. By $f(3, 2) \geq 3$ and Observation 3, $f(3, 2k) \geq 3k$. The total number of stops is $6k$. If each floor is served by at least 2 elevators, then the number of floors is at most $3k$. If some floor is served by at most 1 elevator, it can be linked to at most $2k - 1$ other floors. Counting this floor, the building can have at most $2k$ floors. It follows that $f(3, 2k) = 3k$.

We now prove that $f(3, 2k + 1) = 3k + 1$. By Observation 2,

$$f(3, 2k + 1) \geq f(3, 2k) + 1 = 3k + 1.$$

The total number of stops is $6k + 3$. If each floor is served by at least 2 elevators, then the number of floors is at most $3k + 1$. If some floor is served by at most 1 elevator, it can be linked to at most $2k$ other floors. Counting this floor, the building can have at most $2k + 1$ floors. It follows that $f(3, 2k + 1) = 3k + 1$.

The cases $m = 4$ and $m = 5$ are slightly more difficult because of the absence of perfect buildings. The next perfect building occurs at $m = 6$. Here the floors are 1, 2, 3 and 4, and each of the six elevators stop on a different pair of the four floors, namely, (1,2), (1,3), (1,4), (2,3), (2,4) and (3,4). This yields the lower bound $f(6, 2) \geq 4$.

We next prove that $f(6, 2k) = 4k$. By $f(6, 2) \geq 4$ and Observation 3, $f(6, 2k) \geq 4k$. The total number of stops is $12k$. If each floor is served by at least 3 elevators, then the number of floors is at most $4k$. If some floor is served by at most 2 elevators, it can be linked to at most $4k - 2$ other floors. Counting this floor, the building can have at most $4k - 1$ floors. It follows that $f(6, 2k) = 4k$.

We now prove that $f(6, 2k+1) \leq 4k+2$. Observe that the total number of stops is $12k + 6$. If each floor is served by at least 3 elevators, then the number of floors is at most $4k + 2$. If some floor is served by at most 2 elevators, it can be linked to at most $4k$ other floors. Counting this floor, the building can have at most $4k + 1$ floors.

Finally, we give a general construction to show that $f(6, 2k+1) \geq 4k+2$. Let the floors be $a_1, a_2, \ldots, a_k$, $b_1, b_2, \ldots, b_k$, $c_1, c_2, \ldots, c_k$, $d_1, d_2, \ldots, d_k$, e and f. Let the first elevator stop at all the a's and b's, the second at all the a's and c's, the third at all the a's and d's, the fourth at all the b's and c's, the fifth at all the b's and d's, and the sixth at all the c's and d's. Then these $4k$ floors are all linked.

If we add e as the last stop of the first and sixth elevator and f as the last stop of the second and fifth elevator, they are also linked to the other $4k$ floors. However, e and f are not linked. So we replace d_k by f in the sixth elevator. This destroys the links between d_k on the one hand and e and the c's on the other. The remedy is to add e as the last stop of the third elevator and d_k as the last stop of the fourth elevator. It follows that $f(6, 2k+1) = 4k+2$.

The Fano plane is the next example of a perfect building. It leads to the lower bound $f(7, 3) \geq 7$. By this and Observation 3, we have $f(7, 3k) \geq 7k$. Now the total number of stops is $21k$. If each floor is served by at least 3 elevators, then the number of floors is at most $7k$. If some floor is served by at most 2 elevators, it can be linked to at most $6k - 2$ other floors. Counting this floor, the building can have at most $6k - 1$ floors. It follows that $f(7, 3k) = 7k$.

Our final result is that $f(7, 3k+2) = 7k+4$. To prove that $f(7, 3k+2) \leq 7k+4$, observe that the total number of stops is $21k+14$. If each floor is served by at least 3 elevators, then the number of floors is at most $7k+4$. If some floor is served by at most 2 elevators, it can be linked to at most $6k+2$ other floors. Counting this floor, the building can have at most $6k+3$ floors.

We now give a general construction to show that $f(7, 3k+2) \geq 7k+4$. Let the floors be $a_1, a_2, \ldots, a_{k+1}, b_1, b_2, \ldots, b_{k+1}, c_1, c_2, \ldots, c_{k+1}, d_1, d_2, \ldots, d_{k+1}, e_1, e_2, \ldots, e_k, f_1, f_2, \ldots, f_k$ and $g_1, g_2, \ldots, g_k$. Let the first elevator stop at all the a's, b's and e's, the second at all the a's, c's and f's, the third at all the a's, d's and g's, the fourth at all the b's, c's and g's, the fifth at all the b's, d's and f's, the sixth at all the c's, d's and e's, and the seventh at all the e's, f's and g's. Then all the floors are linked, with two wasted stops in the seventh elevator. It follows that $f(7, 3k+2) = 7k+4$.

We are unable to determine $f(7, 3k+1)$.

The results in this section are contained in [4], the work of Jerry Lo, a member of Chiu Chang Mathematical Circle, and Circle David Rhee.

Exercises

1. Prove the duals of the four postulates of the community of lions and ponies:
   (a) There are at least two ponies.
   (b) Each pony has been bitten by at least three lions.
   (c) For any pair of ponies, there is exactly one lion that has bitten both.
   (d) For any pair of lions, there is at least one pony that both have bitten.
2. Show that to get around the Space Dodecapus with nine Space Pods, it is enough to break up each of four Space Cannons into twelve parts.
3. Prove that $7k+1 \leq f(7,3k+1) \leq 7k+2$ where $f(m,n)$ is the function associated with convenient buildings.

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