On the Line-Symmetry of Self-motions of Linear Pentapods

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Abstract We show that all self-motions of pentapods with linear platform of Type 1 and Type 2 can be generated by line-symmetric motions. Thus this paper closes a gap between the more than 100 year old works of Duporcq and Borel and the extensive study of line-symmetric motions done by Krames in the 1930s. As a consequence we also get a new solution set for the Borel Bricard problem. Moreover we discuss the reality of self-motions and give a sufficient condition for the design of linear pentapods of Type 1 and Type 2, which have a self-motion free workspace.

1 Introduction

The geometry of a linear pentapod is given by the five base anchor points M_i in the fixed system Σ_0 and by the five collinear platform anchor points m_i in the moving system Σ (for $i = 1, \ldots, 5$). Each pair $(\mathsf{M}_i, \mathsf{m}_i)$ of corresponding anchor points is connected by a SPS-leg, where only the prismatic joint is active.

If the geometry of the linear pentapod is given as well as the lengths R_i of the five pairwise distinct legs, it has generically mobility 1. This degree of freedom corresponds to the rotational motion about the carrier line p of the five platform anchor points. As this rotation is irrelevant for applications with axial symmetry (e.g. 5-axis milling, laser or water-jet engraving/cutting, spot-welding, spray-based painting, etc.), these manipulators are of great practical interest. Nevertheless configurations should be avoided where the linear pentapod gains an additional uncontrollable mobility, which is referred as self-motion.

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1.1 Review on Self-motions of Linear Pentapods

The self-motions of linear pentapods represent interesting solutions to a problem posed 1904 by the French Academy of Science for the *Prix Vaillant*, which is also known as Borel-Bricard problem (cf. [2, 3]). This still unsolved kinematic challenge reads as follows: *"Determine and study all displacements of a rigid body in which distinct points of the body move on spherical paths."*

For the special case of five collinear points the Borel-Bricard problem was studied by Darboux [5, p. 222], Mannheim [6, p. 180ff] and Duporcq [7] (see also Bricard [3, Chap. III]). A contemporary and accurate reexamination of these old results, which also takes the coincidence of platform anchor points into account, was done in [1] yielding a full classification of linear pentapods with self-motions.

Beside the architecturally singular linear pentapods [1, Corollary 1] and some trivial cases with pure rotational self-motions [1, Designs α, β, γ] or pure translational ones [1, Theorem 1] there only remain the following three designs:

Under a self-motion each point of the line p has a spherical (or planar) trajectory. The locus of the corresponding sphere centers is a cubic space curve P, where the mapping from p to P is named σ. P intersects the ideal plane in one real point W and two conjugate complex ideal points, where the latter ones are the cyclic points I and J of a plane orthogonal to the direction of W. P is therefore a so-called straight cubic circle. The following subcases can be distinguished:

- P is irreducible:
  - σ maps the ideal point U of p to W (Type 5 according to [1]).
  - σ maps U to a finite point of P (Type 1 according to [1]).
- P splits up into a circle and a line, which is orthogonal to the carrier plane of the circle and intersects the circle in a point Q. Moreover σ maps U to a point on the circle different from Q (Type 2 according to [1]).

1.2 Basics on Line-Symmetric Motions

Krames (e.g. [4, 10]) studied special one-parametric motions (*Symmetrische Schrotung* in German), which are obtained by reflecting the moving system Σ in the generators of a ruled surface of the fixed system Σ_0, which is the so called *basic surface*. These so-called *line-symmetric motions* were also studied by Bottema and Roth [8, Sect. 7 of Chap. 9], who gave an intuitive algebraic characterization in terms of Study parameters $(e_0 : e_1 : e_2 : e_3 : f_0 : f_1 : f_2 : f_3)$, which are shortly repeated next.

All real points of the Study parameter space P^7 (7-dimensional projective space), which are located on the so-called Study quadric $\Psi : \sum_{i=0}^{3} e_i f_i = 0$, correspond to an Euclidean displacement with exception of the 3-dimensional subspace $e_0 = e_1 = e_2 = e_3 = 0$, as its points cannot fulfill the condition $N \neq 0$ with $N := e_0^2 + e_1^2 +$

$e_2^2 + e_3^2$. The translation vector $\mathbf{s} := (s_1, s_2, s_3)^T$ and the rotation matrix $\mathbf{R}$ of the corresponding Euclidean displacement $\mathbf{m}_i \mapsto \mathbf{R}\mathbf{m}_i + \mathbf{s}$ are given for $N = 1$ by:

$$s_1 = -2(e_0 f_1 - e_1 f_0 + e_2 f_3 - e_3 f_2), \quad s_2 = -2(e_0 f_2 - e_2 f_0 + e_3 f_1 - e_1 f_3),$$
$$s_3 = -2(e_0 f_3 - e_3 f_0 + e_1 f_2 - e_2 f_1),$$
$$\mathbf{R} = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix} = \begin{pmatrix} e_0^2 + e_1^2 - e_2^2 - e_3^2 & 2(e_1 e_2 - e_0 e_3) & 2(e_1 e_3 + e_0 e_2) \\ 2(e_1 e_2 + e_0 e_3) & e_0^2 - e_1^2 + e_2^2 - e_3^2 & 2(e_2 e_3 - e_0 e_1) \\ 2(e_1 e_3 - e_0 e_2) & 2(e_2 e_3 + e_0 e_1) & e_0^2 - e_1^2 - e_2^2 + e_3^2 \end{pmatrix}.$$

There always exists a moving frame (in dependence of a given fixed frame) in a way that $e_0 = f_0 = 0$ holds for a line-symmetric motion. Then $(e_1 : e_2 : e_3 : f_1 : f_2 : f_3)$ are the Plücker coordinates (according to the convention used in [8]) of the generators of the basic surface with respect to the fixed frame.

1.3 Line-Symmetric Self-motions of Linear Pentapods

It is well known (cf. [7, Sect. 15], [3, Sect. 12]) that the self-motions of Type 5 are obtained by restricting the Borel-Bricard motions[1] (also known as BB-I motions) to a line. Note that Krames gave a detailed discussion of this special case in [4, Sect. 5], where he also pointed out the line-symmetry of BB-I motions.

Beside these BB-I motions, there also exist line-symmetric motions (so-called BB-II motions), where every point of a hyperboloid carrying two reguli of lines has a spherical path. It is known (cf. [9, p. 24] and [10, p. 188]) that the corresponding sphere centers of lines, belonging to one regulus,[2] constitute irreducible straight cubic circles, which imply examples of Type 1 self-motions. It should be noted that there also exist degenerated cases where the hyperboloid splits up into the union two orthogonal planes, which contain examples of Type 2 self-motions.

A simple count of free parameters shows that not all self-motions of Type 1 (5-parametric set[3] of motions where all points of a line have spherical paths) can be generated by BB-II motions (which produce only a 4-parametric set[4]). The same argumentation holds for Type 2 self-motions and the mentioned degenerated case.

As a consequence the question arise whether all self-motions of linear pentapods of Type 1 and Type 2 can be generated by line-symmetric motions. If this is the case

[1] These are the only non-trivial motions where every point of the moving space has a spherical trajectory (cf. [3, Chap. VI]).

[2] The corresponding sphere centers of lines belonging to the other regulus are again located on lines (cf. [9, p. 24]), which imply linear pentapods with an architecturally singular design.

[3] With respect to the notation introduced in Sect. 2 these five parameters are C, a_r, a_c, a_4 and p_5 or R_1 (cf. Eq. (7)) by canceling the factor of similarity by setting $A = 1$.

[4] These are the parameters a, c, g, k used in [9, Sect. 2.3].

we can apply a construction proposed by Krames [4, p. 416], which is discussed in Sect. 4, yielding new solutions to the Borel-Bricard problem.

Finally it should be noted that a detailed review on line-symmetric motions with spherical trajectories is given in [11, Sect. 1].

2 On the Line-Symmetry of Type 1 and Type 2 Self-motions

For our calculations we do not select arbitrary pairs $(\mathsf{m}_i, \mathsf{M}_i)$ of p and P, which are in correspondence with respect to σ ($\Leftrightarrow \sigma(\mathsf{m}_i) = \mathsf{M}_i$), but choose the following special ones:

M_4 equals W, M_2 coincides with I and M_3 with J. The corresponding platform anchor points are denoted by m_4, m_2 and m_3, respectively. As M_i are ideal points the corresponding points m_i are not running on spheres but in planes orthogonal to the direction of M_i. Therefore these three point pairs imply three so-called Darboux conditions Ω_i for $i = 2, 3, 4$. Moreover we denote U as m_5 and its corresponding finite point under σ by M_5. This point pair describes a so-called Mannheim condition Π_5 (which is the inverse of a Darboux condition). The pentapod is completed by a sphere condition Λ_1 of any pair of corresponding finite points m_1 and M_1.

In [1] we have chosen the fixed frame $\mathscr{F}_0$ in a way that M_1 equals its origin and M_4 coincides with the ideal point of the z-axis. Moreover we located the moving frame $\mathscr{F}$ in a way that p coincides with the x-axis, where m_1 equals its origin.

For the study at hand it is advantageous to select a different set of fixed and moving frames $\mathscr{F}_0'$ and $\mathscr{F}'$, respectively:

- As M_2 and M_3 coincides with the cyclic points, we can assume without loss of generality (w.l.o.g.) that M_5 is located in the xz-plane (as a rotation about the z-axis does not change the coordinates of $\mathsf{M}_1, \dots, \mathsf{M}_4$). Moreover we want to apply a translation in a way that M_5 is in the origin of the new fixed frame $\mathscr{F}_0'$. Summed up the coordinates with respect to $\mathscr{F}_0'$ read as:

$$\mathbf{M}_5 = (0, 0, 0), \quad \mathbf{M}_1 = (A, 0, C) \quad \text{with} \quad A \neq 0 \tag{1}$$

  as $A = 0$ implies a contradiction to the properties of P for Type 1 and Type 2 pentapods given in Sect. 1.1. Moreover, M_2, M_3 and M_4 are the ideal points in direction $(1, i, 0)^T$, $(1, -i, 0)^T$ and $(0, 0, 1)^T$, respectively.
- With respect to $\mathscr{F}_0'$ the location of p is undefined, but the coordinates $\mathbf{m}_i$ of m_i can be parametrized as follows for $i = 1, \dots, 4$:

$$\mathbf{m}_i = \mathbf{n} + (a_i - a_r)\mathbf{d} \quad \text{with} \quad a_1 = 0,\ a_2 = a_r + ia_c,\ a_3 = a_r - ia_c \tag{2}$$

  where $a_r, a_c \in \mathbb{R}$ and $a_c \neq 0$ holds. m_5 is the ideal point in direction of the unit-vector $\mathbf{d} = (d_1, d_2, d_3)^T$, which obtains the rational homogeneous parametrization of the unit-sphere, i.e.

$$d_1 = \frac{2h_0h_1}{h_0^2+h_1^2+h_2^2}, \quad d_2 = \frac{2h_0h_2}{h_0^2+h_1^2+h_2^2}, \quad d_3 = \frac{h_1^2+h_2^2-h_0^2}{h_0^2+h_1^2+h_2^2}. \tag{3}$$

Now we are looking for the point $\mathbf{n} = (n_1, n_2, n_3)^T$ and the direction $(h_0 : h_1 : h_2)$ in a way that for the self-motion of the pentapod $e_0 = f_0 = 0$ holds. We can discuss Type 1 and Type 2 at the same time, just having in mind that $a_4 \neq 0 \neq C$ has to hold for Type 1 and $a_4 = 0 = C$ for Type 2 (according to [1]).

By setting $\mathbf{r}_i := (r_{i1}, r_{i2}, r_{i3})^T$ for $i = 1, 2, 3$ the Darboux and Mannheim constraints with respect to $\mathscr{F}_0'$ and $\mathscr{F}'$ can be written as:

$$\Omega_2 : (s_1 + \mathbf{r}_1\mathbf{m}_2) - i(s_2 + \mathbf{r}_2\mathbf{m}_2) - p_2N = 0, \qquad \Omega_4 : (s_3 + \mathbf{r}_3\mathbf{m}_4) - p_4N = 0, \tag{4}$$

$$\Omega_3 : (s_1 + \mathbf{r}_1\mathbf{m}_3) + i(s_2 + \mathbf{r}_2\mathbf{m}_3) - p_3N = 0, \qquad \Pi_5 : (\mathbf{Rd})(\mathbf{s} + \mathbf{Rp}_5)N^{-1} = 0, \tag{5}$$

with $\mathbf{p}_5 = \mathbf{n} + (p_5 - a_r)\mathbf{d}$, which is the coordinate vector of the intersection point of the Mannheim plane and p with respect to $\mathscr{F}'$. Moreover $(p_j, 0, 0)^T$ for $j = 2, 3$ (resp. $(0, 0, p_4)^T$) are the coordinates of the intersection point of the Darboux plane and the x-axis (resp. z-axis) of $\mathscr{F}_0'$.

Remark 1 As from the Mannheim constraint Π_5 of Eq. (5) the factor N cancels out, all four constraints $\Omega_2, \Omega_3, \Omega_4, \Pi_5$ are homogeneous quadratic in the Study parameters and especially linear in $f_0, \ldots, f_3$. ◇

According to [1, Theorems 13 and 14] the leg-parameters $p_2, \ldots, p_5, R_1$ have to fulfill the following necessary and sufficient conditions for the self-mobility (over $\mathbb{C}$) of a linear pentapod of Type 1 and Type 2, respectively:

$$p_2 = \frac{Aa_3v}{(a_3-a_4)^2}, \qquad p_3 = \frac{Aa_2v}{(a_2-a_4)^2}, \qquad p_4 = -\frac{Ca_4v}{(a_2-a_4)(a_3-a_4)}, \tag{6}$$

$$(a_2 - a_4)^2(a_3 - a_4)^2 \left[2wp_5 - vR_1^2 - (2w - va_4)a_4\right] + vw^2(A^2 + C^2) = 0, \tag{7}$$

with $v := a_2 + a_3 - 2a_4$ and $w := a_2a_3 - a_4^2$. Therefore if we set p_2, p_3, p_4 as given in Eq. (6) then only one condition in p_5 and R_1 remains in Eq. (7). Therefore these pentapods have a 1-dimensional set of self-motions.

Theorem 1 *Each self-motion of a linear pentapod of Type 1 and Type 2 can be generated by a 1-dimensional set of line-symmetric motions. For the special case $p_5 = a_4 = a_r$ this set is even 2-dimensional.*

Proof W.l.o.g. we can set $e_0 = 0$ as any two directions $\mathbf{d}$ of p can be transformed into each other by a half-turn about their enclosed bisecting line. Note that this line is not uniquely determined if and only if the two directions are antipodal.

W.l.o.g. we can solve $\Psi, \Omega_2, \Omega_3, \Omega_4$ for f_0, f_1, f_2, f_3 and plug the obtained expressions into Π_5, which yields in the numerator a homogeneous quartic polynomial $G[1563]$ in e_1, e_2, e_3, where the number in the brackets gives the number of terms. Moreover the numerator of the obtained expression for f_0 is denoted by $F[600]$, which is a homogeneous cubic polynomial in e_1, e_2, e_3.

General Case ($v \neq 0$)**:** The condition $G = 0$ already expresses the self-motion as G equals Λ_1 if we solve Eq. (7) for R_1. Moreover $F = 0$ has to hold if the self-motion of the line p can be generated by a line-symmetric motion. As for any solution ($e_1 : e_2 : e_3$) of $F = 0$ also $G = 0$ has to hold, G has to split into F and a homogeneous linear factor L in e_1, e_2, e_3.

Now $L = 0$ cannot correspond to a self-motion of the linear pentapod, but has to arise from the ambiguity in representing a direction of p mentioned at the beginning of the proof. This can be argued indirectly as follows:

Assumed $L = 0$ implies a self-motion, then it has to be a Schönflies motion (with a certain direction v of the rotation axis) due to $e_0 = 0$. As under such a motion the angle enclosed by v and p remains constant[5] the ideal point U of p has to be mapped by σ to the ideal point V of v. This implies that V has to coincide with W, which can only be the case for pentapods of Type 5; a contradiction.

Therefore there has to exist a pose of p during the self-motion, where it is oppositely oriented with respect to the fixed frame and moving frame, respectively. As a consequence we can set $L = d_1e_1 + d_2e_2 + d_3e_3$ which yields the ansatz $\Delta : \lambda LF - G = 0$. The resulting set of four equations arising from the coefficients of $e_1^3e_2, e_1^3e_3, e_1e_3^3$ and $e_2e_3^3$ of Δ has the unique solution:

$$n_1 = a_c d_2, \quad n_2 = -a_c d_1, \quad n_3 = (a_r - a_4)d_3, \quad \lambda = 2(h_0^2 + h_1^2 + h_2^2). \tag{8}$$

Now Δ splits up into $(e_1^2 + e_2^2 + e_3^2)^2(h_0^2 + h_1^2 + h_2^2)H[177]$, where H is homogeneous of degree 4 in h_0, h_1, h_2. For more details on $H = 0$ please see Remark 3, which is given right after this proof.

Remark 2 Note that all self-motions of the general case can be parametrized as the resultant of G and the normalizing condition $N - 1$ with respect to e_i yields a polynomial, which is only quadratic in e_j for pairwise distinct $i, j \in \{1, 2\}$. ◇

Special Case ($v = 0$)**:** If $v = 0$ holds, we cannot solve Eq. (7) for R_1. The conditions $v = 0$ and Eq. (7) imply $p_5 = a_4 = a_r$. Now G is fulfilled identically and the self-motion is given by $\Lambda_1 = 0$, which is of degree 4 in e_1, e_2, e_3. Moreover for this special case $F = 0$ already holds for $\mathbf{n}$ given in Eq. (8). Therefore any direction $(h_0 : h_1 : h_2)$ for p can be chosen in order to fix the line-symmetric motion. □

Remark 3 $H = 0$ represents a planar quartic curve, which can be verified to be entirely circular. Moreover $H = 0$ can be solved linearly for p_5. The corresponding graph is illustrated in Fig. 1.

If we reparametrize the $h_0h_1h_2$-plane in terms of homogenized polar coordinates by:

[5] This angle condition can be seen as the limit of the sphere condition (cf. [12, Sect. 4.1]).

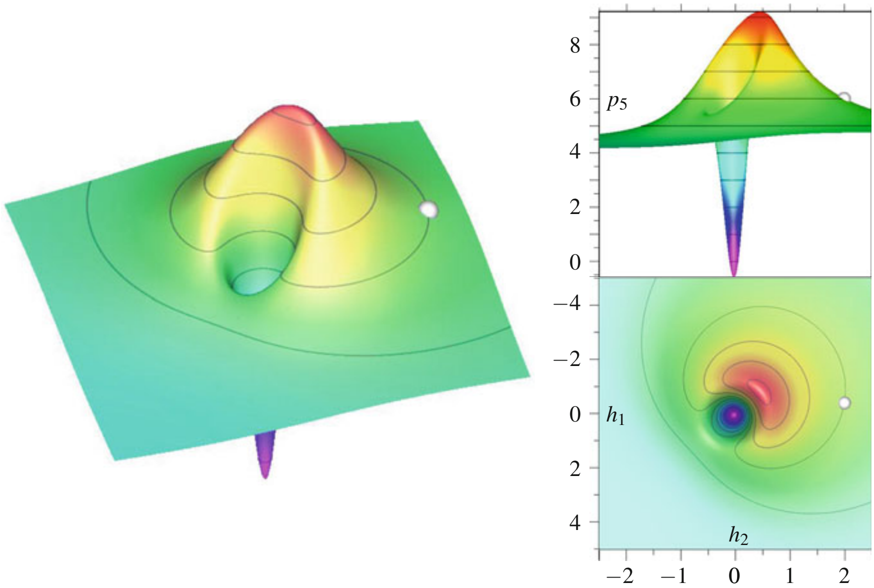

Fig. 1 For a type 1 pentapod with self-motion given by the parameters $a_4 = 2$, $A = -1$, $C = -5$, $a_r = 7$ and $a_c = 4$, the graph of p_5 in dependency of h_1 and h_2 with $h_0 = 1$ is displayed in the axonometric view on the *left* and in the front resp. *top* view on the *right* side. The highlighted point at height 6 corresponds to the values $h_1 = -\frac{489262}{226525} + \frac{488}{226525}\sqrt{675091}$ and $h_2 = \frac{535336}{226525} + \frac{446}{226525}\sqrt{675091}$

$$h_0 = (\tau_1^2 + \tau_0^2)\rho_0, \qquad h_1 = (\tau_1^2 - \tau_0^2)\rho_1, \qquad h_2 = 2\tau_0\tau_1\rho_1, \tag{9}$$

where $(\tau_0, \tau_1) \neq (0, 0) \neq (\rho_0, \rho_1)$ and $\tau_0, \tau_1, \rho_0, \rho_1 \in \mathbb{R}$ hold, then H factors into $(\tau_0^2 + \tau_1^2)^3(H_2\tau_1^2 + H_1\tau_0\tau_1 + H_0\tau_0^2)$ with

$$\begin{aligned} H_1 &= 8\rho_0\rho_1 A(a_4 - a_r)(\rho_1^2 + \rho_0^2)(a_r^2 - a_4^2 + a_c^2)a_c, \\ H_0 - H_2 &= 8\rho_0\rho_1 A(a_4 - a_r)(\rho_1^2 + \rho_0^2)[a_r(a_r - a_4)^2 + a_c^2(a_r - 2a_4)], \\ H_0 + H_2 &= 2\left[(a_r - a_4)^2 + a_c^2\right][2a_4(\rho_1^4 - \rho_0^4)(a_4 - a_r)C \\ &\quad + \left((a_r - a_4)^2 + a_c^2\right)\left((\rho_0^4 + \rho_1^4)(a_4 - p_5) + 2\rho_0^2\rho_1^2(2a_r - a_4 - p_5)\right)]. \end{aligned} \tag{10}$$

Therefore this equation can be solved quadratically for the homogeneous parameter $\tau_0 : \tau_1$. Note that the value p_5 is fixed during a self-motion. ◇

3 On the Reality of Type 1 and Type 2 Self-motions

A similar computation to [1, Example 1] shows that for any real point $\mathsf{p}_t \in \mathsf{p}$ with $t \in \mathbb{R}$ and coordinate vector $\mathbf{p}_t = \mathbf{n} + (t - a_r)\mathbf{d}$ with respect to $\mathscr{F}'$ the corresponding real point $\mathsf{P}_t \in \mathsf{P}$ has the following coordinate vector $\mathbf{P}_t$ with respect to $\mathscr{F}_0'$:

$$\mathbf{P}_t = \left(\frac{A(a_r^2+a_c^2-ta_r)}{(t-a_r)^2+a_c^2}, -\frac{Aa_ct}{(t-a_r)^2+a_c^2}, \frac{Ca_4}{a_4-t} \right)^T. \tag{11}$$

As $L = 0$ corresponds with one configuration of the self-motion we can compute the locus $\mathscr{E}_t$ of p_t with respect to $\mathscr{F}_0'$ under the 1-parametric set of self-motions by the variation of $(h_0 : h_1 : h_2)$ within $L = 0$. Moreover due to the mentioned ambiguity we can select an arbitrary solution $(e_0 : e_1 : e_2)$ for $L = 0$ fulfilling the normalization condition $N = 1$; e.g.:

$$e_1 = \frac{h_2}{\sqrt{h_1^2+h_2^2}}, \quad e_2 = -\frac{h_1}{\sqrt{h_1^2+h_2^2}} \quad \text{and} \quad e_3 = 0. \tag{12}$$

Now the computation of $\mathbf{R}\mathbf{p}_t + \mathbf{s}$ yields a rational quadratic parametrization of $\mathscr{E}_t$ in dependency of $(h_0 : h_1 : h_2)$.

Note that this approach also includes the special case $(v = 0)$ as there always exists a value for R_1^2 (in dependency of $(h_0 : h_1 : h_2)$) in a way that $\Lambda_1 = 0$ holds.

For $t \neq a_4$ all $\mathscr{E}_t$ are ellipsoids of rotation (see Fig. 2a), which have the same center point C and axis of rotation c. In detail, C is the point of the straight cubic circle (11) for the value $t = c$ with $c := \frac{a_4^2-a_c^2-a_r^2}{2(a_4-a_r)}$ (for $a_4 = a_r$ we get $c = \infty$ thus $\mathsf{p}_\infty = \mathsf{U} = \mathsf{m}_5$ holds, which implies $\mathsf{C} = \mathsf{M}_5$) and c is parallel to the z-axis of $\mathscr{F}_0'$. Moreover the vertices on c have distance $|a_4 - t|$ from C and the squared radius of the equator circle equals $(a_r - t)^2 + a_c^2$. Note that for $a_4 \neq a_r$ the only sphere within the described set of ellipsoids is $\mathscr{E}_c$. For $a_4 = a_r$ no such sphere exists.

$\mathscr{E}_{a_4}$ is a circular disc in the Darboux plane $z = p_4$ (w.r.t. $\mathscr{F}_0'$) centered in C.

Remark 4 The existence of these ellipsoids was already known to Duporcq [7, Sect. 9], who used them to show that the spherical trajectories are algebraic curves of degree 4 (intersection curve of $\mathscr{E}_t$ and the sphere Φ_t centered in P_t illustrated in Fig. 2b). ◇

Based on this geometric property, recovered by line-symmetric motions, we can formulate the condition for the self-motion to be real as follows:

- $w \neq 0$: We can reduce the problem to a planar one by intersecting the plane spanned by $\mathsf{P}_0 = \mathsf{M}_1$ and c with $\mathscr{E}_0$ and the sphere with radius R_1 centered in P_0. Now there exists an interval $I_0 =]I_-, I_+[$ such that for $R_1 \in I_0$ the two resulting conics have at least two distinct real intersection points. It is well known (e.g. [14]) that the computation of the limits I_- and I_+ of the reality interval I_0 leads across an algebraic problem of degree 4 (explicitly solvable). Thus for a real self-motion we have to choose $R_1 \in I_0$ and solve Eq. (7) for p_5.

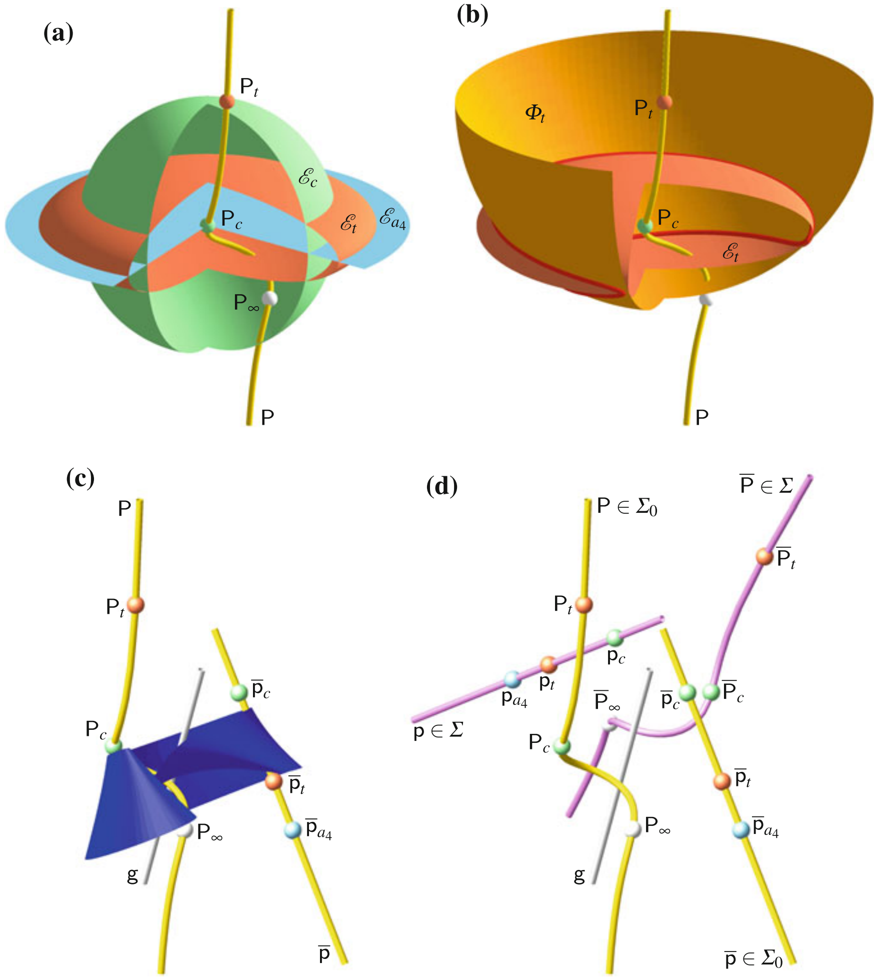

Fig. 2 Type 1 pentapod with self-motion given by $a_4 = 2$, $A = -1$, $C = -5$, $a_r = 7$ and $a_c = 4$. **a** The loci $\mathscr{E}_{a_4}$, $\mathscr{E}_c$ and $\mathscr{E}_t$ with $t = \frac{69}{20}$ are sliced (along the not drawn axis of rotation c) in order to visualize their positioning with respect to the cubic P on which the points $\mathsf{P}_\infty = \sigma(\mathsf{U})$, $\mathsf{P}_c = \mathsf{C}$ and P_t are highlighted. Note that $\mathsf{P}_{a_4} = \mathsf{W}$ is the real ideal point of P. **b** By setting $p_5 = 6$ a one-parametric self-motion μ is fixed. The trajectory of p_t under μ is illustrated as the intersection curve of $\mathscr{E}_t$ and the sphere Φ_t centered in P_t. **c** A strip of the basic surface of μ is illustrated for the value highlighted in Fig. 1. In addition P and $\overline{\mathsf{p}}$ are visualized, where the latter denotes the pose of p such that its half-turns about the generators of the basic surface yield the self-motion μ. **d** Krames's construction is illustrated with respect to the generator g of the basic surface: As P_{a_4} (resp. $\overline{\mathsf{p}}_\infty$) is the real ideal point of P (resp. $\overline{\mathsf{p}}$), the trajectory of p_{a_4} (resp. $\overline{\mathsf{P}}_\infty$) under μ is planar. The (Mannheim) plane $\in \Sigma$, which contains the point P_∞ (resp. $\overline{\mathsf{p}}_{a_4}$) and is orthogonal to the direction of the real ideal point p_∞ (resp. $\overline{\mathsf{P}}_{a_4}$) of p (resp. $\overline{\mathsf{P}}$) in the displayed pose, slides through the point P_∞ (resp. $\overline{\mathsf{p}}_{a_4}$) during the complete motion μ

- $w = 0$: Now P_0 coincides with C and the interval collapses to the single value $R_1 = |a_4|$, which can be seen from Eq. (7). Moreover p_5 can be chosen arbitrarily.

These considerations also show that any pentapod of Type 1 and 2 has real self-motions if the leg-parameters are chosen properly. Note that this is e.g. not the case for some designs of Type 5 pentapods described in [1, Sect. 6], where it was also proven that pentapods with self-motions have a quartically solvable direct kinematics. It is possible to use this advantage (closed form solution) of pentapods with self-motions without any risk,[6] by designing linear pentapods of Type 1 and Type 2, which are guaranteed free of self-motions within their workspace.

A sufficient condition for that is that (at least) for one of the five legs $\mathsf{p}_t\mathsf{P}_t$ of the pentapod the corresponding reality interval I_t is disjoint with the interval of the maximal and minimal leg length implied by the mechanical realization. This condition for a self-motion free workspace gets especially simple if $\mathsf{p}_c\mathsf{P}_c$ is this leg.

Remark 5 Due to limitation of pages, we refer for detailed examples to the paper's corresponding arXiv version [13], which also show that for the general case ($v \neq 0$) the basic surface is of degree 5 (see Fig. 2c) and that a general point has a trajectory of degree 6 under the corresponding line-symmetric motion.[7] Note that the latter also holds for a general point of the cubic $\overline{\mathsf{P}}$ explained in the next section. ◇

4 Conclusion and Open Problem

Krames [4, p. 416] outlined the following construction (see Fig. 2d): Assume that p is in an arbitrary pose of the self-motion μ with respect to P, where g denotes the generator of the basic surface, which corresponds to this pose. Moreover $\overline{\mathsf{p}}$ and $\overline{\mathsf{P}}$ are obtained by the reflexion of p and P, respectively, with respect to g, where $\overline{\mathsf{p}}$ belongs to the fixed system Σ_0 and $\overline{\mathsf{P}}$ to the moving system Σ. Then under the self-motion μ also the points of $\overline{\mathsf{P}}$ are located on spheres with centers on the line $\overline{\mathsf{p}}$.

We can apply this construction for each line-symmetric motion of Theorem 1, which yields new solutions for the Borel Bricard problem, with the exception of one special case where $\mathsf{W} \in \overline{\mathsf{p}}$ holds (i.e. $h_1 = h_2 = 0$ or $h_0 = 0$), which was already given by Borel in [2, Case Fa4]. Moreover for this case Borel noted that beside p and $\overline{\mathsf{P}}$ only two imaginary planar cubic curves ($\in$ isotropic planes through p) run on spheres. The example of [13] shows that this also holds true for the general case.

Thus the problem remains to determine all line-symmetric motions of Theorem 1 where additional real points (beside those of p and $\overline{\mathsf{P}}$) run on spheres. Until now the only known examples with this property are the BB-II motions (cf. Sect. 1.3).

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[6] A self-motion is dangerous as it is uncontrollable and thus a hazard to man and machine.

[7] Note that all basic surfaces and trajectories can be parametrized due to Remark 2.

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