Chapter 2
Linear Optimization

In this chapter we consider linear programming problems (LPPs). We show the practical significance of LPPs through a number of energy-related examples, providing a precise formulation for such problems. We then analyze the geometric and algebraic features of generic LPPs, using some of the energy-related examples as specific cases. We then describe a well known solution algorithm, which is based on the algebraic features of LPPs, show how to perform a sensitivity analysis, and provide and discuss the dual form of an LPP. We finally conclude with a number of practical observations and end-of-chapter exercises.

2.1 Motivating Examples

This introductory section provides a number of energy-related motivating examples for the use of linear optimization. It illustrates that optimization is an everyday endeavor.

2.1.1 Electricity-Production Problem

An electricity producer operates two production facilities that have capacities of 12 and 16 units per hour, respectively. This producer sells the electricity produced at $1 per unit per hour. The two production facilities share a cooling system that restricts their operation, from above and below. More specifically, the sum of the hourly output from facility 2 and twice the hourly output of facility 1 must be at least 8 units. Moreover, the sum of the hourly output from facility 2 and two-thirds of the hourly output from facility 1 must be no more than 18 units. The producer wishes to

determine hourly production from the two facilities to maximize total revenues from energy sales.

To formulate this and any optimization problem, there are three basic problem elements that must be identified. The first is the **decision variables**, which represent the decisions being made in the problem. In essence, the decision variables represent the elements of the system being modeled that are under the decision maker's control, in the sense that their values can be changed. This should be contrasted with problem data or parameters, which are fixed and cannot be changed by the decision maker. In the context of our electricity-production problem, the decisions being made are how many units to produce from each production facility in each hour. We denote these decisions by the two variables, x_1 and x_2, being cognizant of what units these production decisions are being measured in.

The second element of an optimization problem is the **objective function**. The objective function is the metric upon which the decision variables are chosen. Depending on the problem context, the objective is either being minimized or maximized. An optimization problem will often be referred to as either a minimization or maximization problem, depending on what 'direction' the objective function is being optimized in. An important property of an LPP is that the objective function must be *linear* in the decision variables. In the electric-production problem, we are told that the objective is to maximize total revenues. Because the two production facilities sell their outputs, which are represented by x_1 and x_2, at a unit price of \$1 per unit per hour, the objective function can be written as:

$$\max_{x_1,x_2} \ 1x_1 + 1x_2.$$

We typically write the decision variables underneath the 'min' or 'max' operator in the objective function, to make it easy for anyone to know what the problem variables are.

The final problem element is any **constraint**. The constraints indicate what, if any, restrictions there are on the decision variables. Most constraints are given in a problem's description. For instance, we are told that the two facilities have production limits of 12 and 16 units per hour. We can express these restrictions as the two constraints:

$$x_1 \leq 12,$$

and:

$$x_2 \leq 16.$$

We are also told that there are upper and lower limits imposed by the shared cooling system. These restrictions can be expressed mathematically as:

$$\frac{2}{3}x_1 + x_2 \leq 18,$$

and:

$$2x_1 + x_2 \geq 8.$$

In addition to these four explicit constraints, we know that it is physically impossible for either facility to produce a negative amount of electricity. Although this is not given as an explicit restriction in the problem description, we must include the following two non-negativity constraints:

$$x_1, x_2 \geq 0,$$

to complete the problem formulation.

Taking all of these together, the problem formulation can be written compactly as:

$$\max_{x_1, x_2} z = x_1 + x_2 \tag{2.1}$$

$$\text{s.t.} \ \ \frac{2}{3}x_1 + x_2 \leq 18 \tag{2.2}$$

$$2x_1 + x_2 \geq 8 \tag{2.3}$$

$$x_1 \leq 12 \tag{2.4}$$

$$x_2 \leq 16 \tag{2.5}$$

$$x_1, x_2 \geq 0. \tag{2.6}$$

The z in (2.1) represents the objective function value. The abbreviation 's.t.' stands for 'subject to,' and denotes that the following lines have problem constraints. The total set of constraints (2.2)–(2.6) define the problem's **feasible region** or feasible set. The feasible region is the set of values that the decision variables can take and satisfy the problem constraints. Importantly, it should be clear that an optimal solution of the problem must satisfy the constraints, meaning it must belong to the feasible region.

All of the constraints in an LPP must be one of three types: (i) less-than-or-equal-to inequalities, (ii) greater-than-or-equal-to inequalities, or (iii) equalities. Moreover, all of the constraints of an LPP must be *linear* in the decision variables. No other constraint types can be used in an LPP. Constraints (2.2), (2.4), and (2.5) are examples of less-than-or-equal-to inequalities while constraints (2.3) and (2.6) are examples of greater-than-or-equal-to constraints. This problem does not include any equality constraints, however, the Natural Gas-Transportation Problem, which is introduced in Section 2.1.2, does.

It is also important to stress that LPPs cannot include any *strict* inequality constraints. That is to say, each inequality constraint must either have a '$\leq$' or '$\geq$' in it and cannot have a '$<$' or '$>$.' The reason for this is that a strict inequality can give us an LPP that does not have a well defined optimal solution. To see a very simple example of this, consider the following optimization problem with a single variable, which we denote as x:

$$\min_{x} \; z = x$$

$$\text{s.t.} \;\; x > 0.$$

Examining this problem should reveal that it is impossible to find an optimal value for x. This is because for any strictly positive value of x, regardless of how close to 0 it is, $x/2$ will also be feasible and give a slightly smaller objective-function value. This simple example illustrates why we do not allow strict inequality constraints in LPPs.

If a problem does require a strict inequality, we can usually approximate it using a weak inequality based on physical realities of the system being modeled. For instance, suppose that in the Electricity-Production Problem we are told that facility 1 must produce a strictly positive amount of electricity. This constraint would take the form:

$$x > 0, \tag{2.7}$$

which cannot be included in an LPP. Suppose that the control system on the facility cannot realistically allow a production level less than 0.0001 units. Then, we can instead substitute strict inequality (2.7) with:

$$x \geq 0.0001,$$

which is a weak inequality that can be included in an LPP.

Figure 2.1 shows a geometrical representation of the Electricity-Production Problem. The two axes represent different values for the two decision variables. The boundary defined by each constraint is given by a blue line and the two small arrows at the end of each line indicate which side of the line satisfies the constraint. For instance, the blue line that goes through the points $(x_1, x_2) = (0, 8)$ and $(x_1, x_2) = (4, 0)$ defines the boundary of constraint (2.3) and the arrows indicate that points above and to the right of this line satisfy constraint (2.3).

For a problem that has more than two dimensions (meaning that it has more than two decision variables) the boundary of each constraint defines a **hyperplane**. A hyperplane is simply a higher-dimensional analogue to a line. The set of points that satisfies an inequality constraint in more than two dimensions is called a **halfspace**.

The feasible region of the Electricity-Production Problem is the interior of a **polygon**. For problems that have more than two dimensions the feasible region becomes a **polytope**. A polytope is simply a higher-dimensional analogue to a polygon. An important property of LPPs is that because the problem constraints are all linear in the decision variables, the feasible region is always a polytope. It is important to stress, however, that the polytope is not necessarily **bounded**, as the one for the Electricity-Production Problem is. That is to say, for some problems there may be a direction in which the decision variables can keep increasing or decreasing without any limit. The implications of having a feasible region that is not bounded are discussed later in Section 2.3.1. The feasible region of the Electricity-Production Problem has six

Fig. 2.1 Geometrical representation of the Electricity-Production Problem
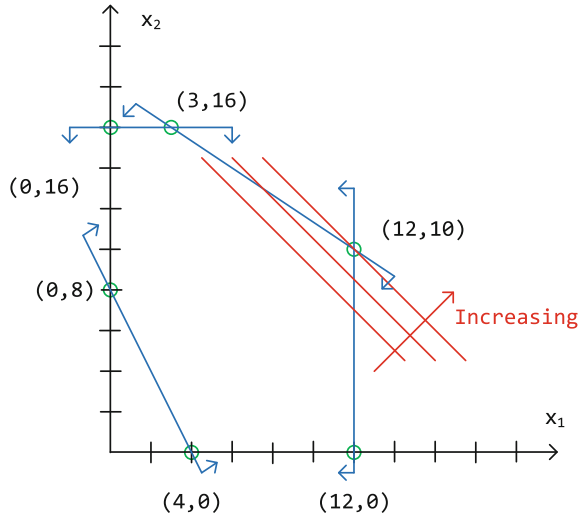
corners, which are (4, 0), (12, 0), (12, 10), (3, 16), (0, 16), and (0, 8). The corners of a polytope are often called **vertices** or **extreme points**.

The objective function is represented by its **contour plot**. The contour plot represents sets of values for the decision variables that give the same objective function value. The contour plot for the Electricity-Production Problem is represented by the parallel red lines in Figure 2.1. The red arrow that runs perpendicular to the red lines indicates the direction in which the objective function is increasing. An important property of LPPs is that because the objective function is linear in the decision variables, the contour plot is always parallel lines (for problems that have more than two dimensions, the contour plot is parallel hyperplanes) and the objective is always increasing/decreasing in the same direction that runs perpendicular to the contour plot.

The solution of this problem is easily obtained by inspecting Figure 2.1: the point in the feasible region that corresponds with the contour line with the highest value is vertex (12, 10). Thus, vertex (12, 10) is the optimal solution with an objective-function value of 22. We typically identify an optimal solution using stars, meaning that we write $(x_1^*, x_2^*) = (12, 10)$ and $z^* = 22$.

It is finally worth noting that the Electricity-Production Problem is a simplified example of what is known as a production-scheduling problem.

2.1.2 *Natural Gas-Transportation Problem*

A natural gas producer owns two gas fields and serves two markets. Table 2.1 summarizes the capacity of each field and Table 2.2 gives the demand of each market,

which must be satisfied exactly. Finally, Table 2.3 summarizes the per-unit cost of transporting gas from each field to each market. The company would like to determine how to transport natural gas from the two fields to the two markets to minimize its total transportation cost.

There are four decision variables in this problem, which are:

- $x_{1,1}$: units of natural gas transported from field 1 to market 1;
- $x_{1,2}$: units of natural gas transported from field 1 to market 2;
- $x_{2,1}$: units of natural gas transported from field 2 to market 1; and
- $x_{2,2}$: units of natural gas transported from field 2 to market 2.

Table 2.1 Capacity of each gas field in the Natural Gas-Transportation Problem

Field	Capacity [units]
1	7
2	12

Table 2.2 Demand of each market in the Natural Gas-Transportation Problem

Market	Demand [units]
1	10
2	8

Table 2.3 Transportation cost between each gas field and market [$/unit] in the Natural Gas-Transportation Problem

	Market 1	Market 2
Field 1	5	4
Field 2	3	6

For many problems, it can be cumbersome to list all of the problem variables explicitly. To shorten the variable definition, we can introduce **index sets**, over which the variables are defined. In this problem, the variables are indexed by two sets. The first is the field from which the gas is being transported and the second index set is the market to which it is being shipped. If we let i denote the index for the field and j the index for the market, we can more compactly define our decision variables as $x_{i,j}$, which represents the units of natural gas transported from field i to market j. Of course when defining the decision variables this way we know that $i = 1, 2$ (because there are two fields) and $j = 1, 2$ (because there are two markets). However, typically the two (or more) index sets over which a variable is defined do not necessarily have the same number of elements, as we have in this example.

The objective of this problem is to minimize total transportation cost, which is given by:

$$\min_x \ 5x_{1,1} + 4x_{1,2} + 3x_{2,1} + 6x_{2,2},$$

where we have listed the decision variables compactly as x, and have that $x = (x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2})$.

This problem has three types of constraints. The first are capacity limits on how much can be produced by each field:

$$x_{1,1} + x_{1,2} \leq 7,$$

and:

$$x_{2,1} + x_{2,2} \leq 12.$$

The second are constraints that ensure that the demand in each market is exactly satisfied:

$$x_{1,1} + x_{2,1} = 10,$$

and:

$$x_{1,2} + x_{2,2} = 8.$$

Note that these demand conditions are equality constraints. We finally need non-negativity constraints:

$$x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2} \geq 0.$$

The non-negativity constraints can be written more compactly either as:

$$x_{i,j} \geq 0, \forall\, i = 1, 2;\ j = 1, 2;$$

or as:

$$x \geq 0.$$

Taking all of these elements together, the entire LPP can be written as:

$$\min_{x} z = 5x_{1,1} + 4x_{1,2} + 3x_{2,1} + 6x_{2,2}$$
$$\text{s.t. } x_{1,1} + x_{1,2} \leq 7$$
$$x_{2,1} + x_{2,2} \leq 12$$
$$x_{1,1} + x_{2,1} = 10 \qquad\qquad (2.8)$$
$$x_{1,2} + x_{2,2} = 8 \qquad\qquad (2.9)$$
$$x_{i,j} \geq 0, \forall\, i = 1, 2;\ j = 1, 2.$$

The Natural Gas-Transportation Problem is a simplified instance of a transportation problem.

2.1.3   Gasoline-Mixture Problem

A gasoline refiner needs to produce a cost-minimizing blend of ethanol and traditional gasoline. The blend needs to have at least 65% burning efficiency and a pollution level no greater than 85%. The burning efficiency, pollution level, and per-ton cost of ethanol and traditional gasoline are given in Table 2.4.

Table 2.4 Burning efficiency, pollution level, and per-ton cost of ethanol and traditional gasoline in the Gasoline-Mixture Problem

Product	Efficiency [%]	Pollution [%]	Cost [$/ton]
Gasoline	70	90	200
Ethanol	60	80	220

To formulate this problem, we model the refiner as determining a least-cost mixture of gasoline and ethanol to produce one ton of blend. The resulting optimal mixture can then be scaled up or down by the refiner depending on how much blend it actually wants to produce. There are two decision variables in this problem—x_1 and x_2 denote how many tons of gasoline and ethanol are used in the blend, respectively.

The objective is to minimize the cost of the blend:

$$\min_x \ 200x_1 + 220x_2.$$

There are three sets of problem constraints. The first ensures that the blend meets the minimum burning efficiency level:

$$0.7x_1 + 0.6x_2 \geq 0.65,$$

and the maximum pollution level:

$$0.9x_1 + 0.8x_2 \leq 0.85.$$

Next we must ensure that we produce one ton of the blend:

$$x_1 + x_2 = 1.$$

Finally, the decision variables must be non-negative, because it is physically impossible to have negative tons of ethanol or gasoline in the blend:

$$x_1, x_2 \geq 0.$$

Putting all of the problem elements together, the LPP can be written as:

$$\min_{x} z = 200x_1 + 220x_2$$
$$\text{s.t. } 0.7x_1 + 0.6x_2 \geq 0.65$$
$$0.9x_1 + 0.8x_2 \leq 0.85$$
$$x_1 + x_2 = 1$$
$$x_1, x_2 \geq 0.$$

The Gasoline-Mixture Problem is a simplified example of a commodity-mixing problem.

2.1.4 Electricity-Dispatch Problem

The electric power network in Figure 2.2 includes two production plants, at nodes 1 and 2, and demand at node 3. The production plants at nodes 1 and 2 have production capacities of 6 and 8 units, respectively, and their per-unit production costs are \$1 and \$2, respectively. There is demand for 10 units of energy at node 3.

Fig. 2.2 Electric power network in the Electricity-Dispatch Problem
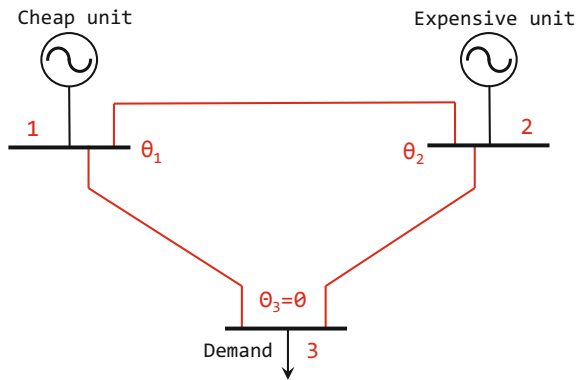
The operation of the network is governed by differences in the electrical heights of the three nodes. More specifically, the flow of electricity through any line is proportional to the difference of electrical heights of the initial and final nodes of the line. This means that the amount of energy produced at node 1 is equal to the difference between the electrical heights of nodes 1 and 2 plus the difference between the electrical heights of nodes 1 and 3. Electricity produced at node 2 is similarly equal to the difference between the electrical heights of nodes 2 and 1 plus the difference between the electrical heights of nodes 2 and 3. Finally, the electricity consumed at node 3 is defined as the difference between the electrical-heights of nodes 1 and 3 plus the difference between the electrical heights of nodes 2 and 3 (the

electrical-height differences are opposite to those for nodes 1 and 2 because energy is consumed at node 3 as opposed to being produced).

The network operator seeks to produce electricity at the plants and operate the network in such a way to serve the demand at node 3 at minimum cost.

This problem has five decision variables. We let x_1 and x_2 denote the units of electricity produced at nodes 1 and 2, respectively. We also let θ_1, θ_2, and θ_3 denote the electrical heights of the three nodes.

The objective is to minimize total production cost:

$$\min_{x,\theta}\ 1x_1 + 2x_2.$$

There are three sets of problem constraints. The first defines the amount produced and consumed at each node in terms of the electrical-height differences. The node 1 constraint is:

$$x_1 = (\theta_1 - \theta_2) + (\theta_1 - \theta_3),$$

the node 2 constraint is:

$$x_2 = (\theta_2 - \theta_1) + (\theta_2 - \theta_3),$$

and the node 3 constraint is:

$$10 = (\theta_1 - \theta_3) + (\theta_2 - \theta_3).$$

As noted above, the electrical-height differences defining consumption at node 3 is opposite to the height differences defining production at nodes 1 and 2. The second set of constraints imposes the production limits at the two nodes:

$$x_1 \le 6,$$

and:

$$x_2 \le 8.$$

We finally need non-negativity for the production variables *only*:

$$x_1, x_2 \ge 0.$$

The θ's can take negative values, because they are used to define the relative electrical heights of the three nodes.

This LPP can be slightly simplified. This is because the production and consumption levels at the three nodes are defined in terms of electrical height differences. As such, we can arbitrarily fix one of the three θ's and only keep the other two as variables. If we fix $\theta_3 = 0$, the LPP is further simplified to:

$$\min_{x,\theta} z = x_1 + 2x_2$$

$$\text{s.t. } x_1 = 2\theta_1 - \theta_2$$

$$x_2 = 2\theta_2 - \theta_1$$

$$10 = \theta_1 + \theta_2$$

$$x_1 \leq 6$$

$$x_2 \leq 8$$

$$x_1, x_2 \geq 0.$$

The Electricity-Dispatch Problem is a simplified instance of a scheduling problem with network constraints.

2.2 Forms of Linear Optimization Problems

As noted before, linear optimization problems vary in terms of a number of attributes. The objective can either be a minimization or maximization. Moreover, it can include a mixture of less-than-or-equal-to, greater-than-or-equal-to, or equality constraints. In this section we first give the general form of a linear optimization problem. We then discuss two important special forms that any linear optimization problem can be converted to: standard and canonical forms. These forms are used later to solve and study algebraic features of linear optimization problems. In addition to introducing these special forms, we discuss how to convert any generic linear optimization problem into these two forms.

2.2.1 General Form of Linear Optimization Problems

As noted in Section 2.1.1, linear optimization problems have a very important defining feature. This feature is that the objective function and all of the constraints are linear in all of the decision variables. Because LPPs have this special feature, we can write them generically as:

$$\min_{x_1,\ldots,x_n} \sum_{i=1}^{n} c_i x_i \tag{2.10}$$

$$\text{s.t. } \sum_{i=1}^{n} A_{j,i}^e x_i = b_j^e, \qquad \forall\, j = 1, \ldots, m_e \tag{2.11}$$

$$\sum_{i=1}^{n} A_{j,i}^g x_i \geq b_j^g, \qquad \forall\, j = 1, \ldots, m_g \tag{2.12}$$

$$\sum_{i=1}^{n} A_{j,i}^{l} x_i \leq b_j^{l}, \qquad\qquad \forall \, j = 1, \ldots, m_l. \qquad (2.13)$$

This generic problem has n decision variables: $x_1, \ldots, x_n$. The c_i's in the objective function and the $A_{j,i}^{e}$'s, $A_{j,i}^{g}$'s, $A_{j,i}^{l}$'s, b_j^{e}'s, b_j^{g}'s, and b_j^{l}'s in the constraints are constants. Thus the objective function and the constraints are linear in the decision variables, because each decision variable is multiplied by a constant coefficient in the objective function and in each constraint and those products are summed together.

Although this generic LPP is written as a minimization problem, we could have just as easily written it as a maximization problem. This generic problem has m_e equality, m_g greater-than-or-equal-to, and m_l less-than-or-equal-to constraints. This means that there are $m = m_e + m_g + m_l$ constraints in total. An LPP does not have to include all of the three types of constraints. For instance, the Electricity-Production Problem, which is introduced in Section 2.1.1, does not have any equality constraints, meaning that $m_e = 0$ for that particular problem.

As noted in Section 2.1.1, LPPs can only include weak inequalities. Strict inequalities cannot be used, because they typically raise technical issues. If a problem calls for the use of a strict inequality, for instance of the form:

$$\sum_{i=1}^{n} A_{j,i} y_i > b_j,$$

this can be approximated by introducing a sufficiently small positive constant, ε_j, and replacing the strict inequality with a weak inequality of the form:

$$\sum_{i=1}^{n} A_{j,i} y_i \geq b_j + \varepsilon_j.$$

Oftentimes, the physical properties of the system being modeled may allow for such a value of ε_j to be chosen (we discuss one example of such a physical property in Section 2.1.1). If not, then one must simply choose a very small value for ε_j to ensure that the final problem solution is not drastically affected by it. A strictly less-than inequality of the form:

$$\sum_{i=1}^{n} A_{j,i} y_i < b_j,$$

can be similarly approximated by replacing it with a weak inequality of the form:

$$\sum_{i=1}^{n} A_{j,i} y_i \leq b_j - \varepsilon_j,$$

where ε_j is again a sufficiently small positive constant.

2.2.2 Standard and Canonical Forms of Linear Optimization Problems

Although any LPP can be written in the generic form just introduced, we occasionally want to write a problem in one of two more tailored forms. These forms—the so-called standard and canonical forms—are used because they make solving or analyzing a linear optimization problem more straightforward. We introduce each of these forms in turn and then discuss how to convert any generic LPP into them.

2.2.2.1 Standard Form of Linear Optimization Problems

The **standard form** of a linear optimization problem has three defining features. First, the objective function is a minimization. The other two properties are that the problem has two types of constraints. The first types are non-negativity constraints, which require *all* of the decision variables to be non-negative. The other types are **structural constraints**, which are any constraints other than non-negativity constraints. All of the structural constraints of a linear optimization problem in standard form must be equality constraints.

Converting a generic linear optimization problem to standard form requires several steps. We begin with the decision variables. Any decision variables that are non-negative in the original generic problem are already in the correct form for the standard form of the problem. If a decision variable has a non-positivity restriction of the form:

$$y \leq 0,$$

it can be replaced throughout the LPP with a new variable, $\tilde{y}$, which is defined as:

$$\tilde{y} = -y.$$

Clearly, the new variable would have a non-negativity restriction, because the original non-positivity constraint:

$$y \leq 0,$$

could be rewritten as:

$$-y \geq 0,$$

by multiplying the constraint through by -1. We would then substitute $\tilde{y}$ for $-y$ in the left-hand side of this constraint, which gives:

$$\tilde{y} \geq 0.$$

If a variable is unrestricted in sign, a similar type of substitution can be done. More specifically, suppose a variable, y, in a generic LPP is unrestricted in sign. We

can then introduce two new non-negative variables, y^- and y^+, and define them as:

$$y^+ - y^- = y.$$

We then substitute y with $y^+ - y^-$ throughout the LPP and also add two non-negativity constraints:

$$y^-, y^+ \geq 0.$$

Note that because y is defined as the difference between two non-negative variables, y can be made positive or negative depending on which of y^- or y^+ is bigger (or, if we want $y = 0$, we would have $y^- = y^+$).

After all of the variables have been made non-negative, we next turn our attention to the structural constraints. If a structural constraint in a generic LPP is an equality, then it is already in the correct format for the standard form LPP and no further work is needed. If, however, we have a less-than-or-equal-to constraint of the form:

$$\sum_{i=1}^{n} A_{j,i}^{l} x_i \leq b_j^{l},$$

we can convert this to an equality constraint by introducing a non-negative **slack variable**, which we will denote as s_j. With this slack variable, we can replace the less-than-or-equal-to constraint with the equivalent equality constraint:

$$\sum_{i=1}^{n} A_{j,i}^{l} x_i + s_j = b_j^{l},$$

and also add the non-negativity constraint:

$$s_j \geq 0.$$

A greater-than-or-equal-to constraint of the form:

$$\sum_{i=1}^{n} A_{j,i}^{g} x_i \geq b_j^{g},$$

can be similarly converted to an equality constraint by introducing a non-negative **surplus variable**, which we denote r_j. With this surplus variable, we can replace the greater-than-or-equal-to constraint with the equivalent equality constraint:

$$\sum_{i=1}^{n} A_{j,i}^{g} x_i - r_j = b_j^{g}.$$

We must also add the non-negativity constraint:

$$r_j \geq 0.$$

The slack and surplus variables introduced to convert inequalities into equalities can be interpreted as measuring the difference between the left- and right-hand sides of the original inequality constraints.

The final step to convert a generic LPP to standard form is to ensure that the objective function is a minimization. If the objective of the generic problem is a minimization, then no further work is needed. Otherwise, if the objective is maximization, it can be converted by multiplying the objective through by -1.

We demonstrate the use of these steps to convert a generic LPP into standard form with the following example.

Example 2.1   Consider the following LPP:

$$\max_x \; 3x_1 + 5x_2 - 3x_3 + 1.3x_4 - x_5$$
$$\text{s.t.} \; x_1 + x_2 - 4x_4 \leq 10$$
$$x_2 - 0.5x_3 + x_5 = -1$$
$$x_3 \geq 5$$
$$x_1, x_2 \geq 0$$
$$x_4 \leq 0.$$

To convert this generic LPP into standard form, we begin by first noting that both x_1 and x_2 are non-negative, thus no substitutions have to be made for these variables. The variable x_4 is non-positive, thus we define a new variable, $\tilde{x}_4 = -x_4$. Substituting $\tilde{x}_4$ for x_4 in the LPP gives:

$$\max_x \; 3x_1 + 5x_2 - 3x_3 - 1.3\tilde{x}_4 - x_5$$
$$\text{s.t.} \; x_1 + x_2 + 4\tilde{x}_4 \leq 10$$
$$x_2 - 0.5x_3 + x_5 = -1$$
$$x_3 \geq 5$$
$$x_1, x_2, \tilde{x}_4 \geq 0.$$

The signs of the coefficients in the objective function and first constraint on $\tilde{x}_4$ have been changed, because we have defined $\tilde{x}_4$ as being equal to $-x_4$. Next, we note that because x_3 and x_5 are unrestricted in sign, we must introduce four new non-negative variables, x_3^-, x_3^+, x_5^-, and x_3^+, and define them as:

$$x_3^+ - x_3^- = x_3,$$

and:

$$x_5^+ - x_5^- = x_5.$$

We make these substitutions for x_3 and x_5 and add the non-negativity constraints, which gives:

$$\max_{x} \; 3x_1 + 5x_2 - 3(x_3^+ - x_3^-) - 1.3\tilde{x}_4 - (x_5^+ - x_5^-)$$

$$\text{s.t.} \; x_1 + x_2 + 4\tilde{x}_4 \leq 10$$
$$x_2 - 0.5(x_3^+ - x_3^-) + (x_5^+ - x_5^-) = -1$$
$$(x_3^+ - x_3^-) \geq 5$$
$$x_1, x_2, x_3^-, x_3^+, \tilde{x}_4, x_5^-, x_5^+ \geq 0.$$

Next, we must add a non-negative slack and subtract a non-negative surplus variable, which we call s_1 and r_1, to and from structural constraints 1 and 3, respectively. This gives:

$$\max_{x,s,r} \; 3x_1 + 5x_2 - 3x_3^+ + 3x_3^- - 1.3\tilde{x}_4 - x_5^+ + x_5^-$$

$$\text{s.t.} \; x_1 + x_2 + 4\tilde{x}_4 + s_1 = 10$$
$$x_2 - 0.5x_3^+ + 0.5x_3^- + x_5^+ - x_5^- = -1$$
$$x_3^+ - x_3^- - r_1 = 5$$
$$x_1, x_2, x_3^-, x_3^+, \tilde{x}_4, x_5^-, x_5^+, s_1, r_1 \geq 0.$$

Finally, we convert the objective function to a minimization, by multiplying it through by -1, giving:

$$\min_{x,s,r} \; -3x_1 - 5x_2 + 3x_3^+ - 3x_3^- + 1.3\tilde{x}_4 + x_5^+ - x_5^-$$

$$\text{s.t.} \; x_1 + x_2 + 4\tilde{x}_4 + s_1 = 10$$
$$x_2 - 0.5x_3^+ + 0.5x_3^- + x_5^+ - x_5^- = -1$$
$$x_3^+ - x_3^- - r_1 = 5$$
$$x_1, x_2, x_3^-, x_3^+, \tilde{x}_4, x_5^-, x_5^+, s_1, r_1 \geq 0,$$

which is the standard form of our starting LPP.

$\square$

Example 2.2 Consider the Gasoline-Mixture Problem, which is introduced in Section 2.1.3. This problem is formulated generically as:

$$\min_{x} \; 200x_1 + 220x_2$$

$$\text{s.t.} \; 0.7x_1 + 0.6x_2 \geq 0.65$$
$$0.9x_1 + 0.8x_2 \leq 0.85$$
$$x_1 + x_2 = 1$$
$$x_1, x_2 \geq 0.$$

To convert this to standard form, we simply need to introduce one non-negative surplus variable, r_1, and a non-negative slack variable, s_1. The standard form of the LPP would then be:

$$\min_{x,r,s} 200x_1 + 220x_2$$

$$\text{s.t. } 0.7x_1 + 0.6x_2 - r_1 = 0.65$$

$$0.9x_1 + 0.8x_2 + s_1 = 0.85$$

$$x_1 + x_2 = 1$$

$$x_1, x_2, r_1, s_1 \geq 0. \qquad \square$$

An LPP in standard form can be generically written as:

$$\min_{x_1,\ldots,x_n} \sum_{i=1}^{n} c_i x_i$$

$$\text{s.t. } \sum_{i=1}^{n} A_{j,i} x_i = b_j, \qquad\qquad \forall\, j = 1, \ldots, m$$

$$x_i \geq 0, \qquad\qquad \forall\, i = 1, \ldots, n.$$

We can write the generic standard form even more compactly. This is done by first defining a vector of objective-function coefficients:

$$c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix},$$

a vector of decision variables:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

a matrix of constraint coefficients:

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix},$$

and a vector of constraint right-hand-side constants:

$$b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

The standard-form LPP is then compactly written as:

$$\min_x c^\top x \tag{2.14}$$

$$\text{s.t.} \ Ax = b \tag{2.15}$$

$$x \geq 0. \tag{2.16}$$

2.2.2.2 Canonical Form of Linear Optimization Problems

The **canonical form** of a linear optimization problem has: (i) an objective function that is a minimization, (ii) greater-than-or-equal-to structural constraints, and (iii) non-negative decision variables. Converting a generic linear optimization problem to standard form requires several steps. Ensuring that the decision variables are non-negative and that the objective is a minimization are handled in the same manner as they are in converting an LPP to standard form.

As for the structural constraints, any constraints that are greater-than-or-equal-to require no further work. A less-than-or-equal-to constraint of the form:

$$\sum_{i=1}^n A_{j,i}^l x_i \leq b_j^l,$$

can be converted to a greater-than-or-equal-to constraint by multiplying both sides by -1. This converts the constraint to:

$$-\sum_{i=1}^n A_{j,i}^l x_i \geq -b_j^l.$$

Finally, an equality constraint of the form:

$$\sum_{i=1}^n A_{j,i}^e x_i = b_j^e,$$

can be replaced by two inequalities of the form:

$$\sum_{i=1}^n A_{j,i}^e x_i \leq b_j^e,$$

and:

$$\sum_{i=1}^{n} A_{j,i}^{e} x_i \geq b_j^e.$$

We can then convert the first inequality constraint into a greater-than-or-equal-to constraint by multiplying both sides by -1. Thus, the equality constraint is replaced with:

$$-\sum_{i=1}^{n} A_{j,i}^{e} x_i \geq -b_j^e,$$

and:

$$\sum_{i=1}^{n} A_{j,i}^{e} x_i \geq b_j^e.$$

It should be noted that this transformation of an equality constraint when converting to canonical form can create numerical issues when solving the LPP. The reason for this is that the two inequalities 'fight' one another to bring the solution to its corresponding 'side' of b_j^e. Because a feasible solution is right in the middle, not in either of these two sides, this 'fight' may result in a sluggish back-and-forth progression to the middle, where a feasible solution lies.

Example 2.3 Recall the generic LPP, which is introduced in Example 2.1:

$$\max_{x} \ 3x_1 + 5x_2 - 3x_3 + 1.3x_4 - x_5$$
$$\text{s.t.} \ \ x_1 + x_2 - 4x_4 \leq 10$$
$$x_2 - 0.5x_3 + x_5 = -1$$
$$x_3 \geq 5$$
$$x_1, x_2 \geq 0$$
$$x_4 \leq 0.$$

To convert this LPP to canonical form, we first undertake the same steps to have all of the decision variables non-negative, which gives:

$$\max_{x} \ 3x_1 + 5x_2 - 3x_3^+ + 3x_3^- - 1.3\tilde{x}_4 - x_5^+ + x_5^-$$
$$\text{s.t.} \ \ x_1 + x_2 + 4\tilde{x}_4 \leq 10$$
$$x_2 - 0.5x_3^+ + 0.5x_3^- + x_5^+ - x_5^- = -1$$
$$x_3^+ - x_3^- \geq 5$$
$$x_1, x_2, x_3^-, x_3^+, \tilde{x}_4, x_5^-, x_5^+ \geq 0.$$

We next convert the first structural inequality into a greater-than-or-equal-to by multiplying both sides by -1, which gives:

$$\max_{x} \; 3x_1 + 5x_2 - 3x_3^+ + 3x_3^- - 1.3\tilde{x}_4 - x_5^+ + x_5^-$$

$$\text{s.t.} \quad -x_1 - x_2 - 4\tilde{x}_4 \geq -10$$
$$x_2 - 0.5x_3^+ + 0.5x_3^- + x_5^+ - x_5^- = -1$$
$$x_3^+ - x_3^- \geq 5$$
$$x_1, x_2, x_3^-, x_3^+, \tilde{x}_4, x_5^-, x_5^+ \geq 0.$$

We then replace the second structural constraint, which is an equality, with two inequalities, giving:

$$\max_{x} \; 3x_1 + 5x_2 - 3x_3^+ + 3x_3^- - 1.3\tilde{x}_4 - x_5^+ + x_5^-$$

$$\text{s.t.} \quad -x_1 - x_2 - 4\tilde{x}_4 \geq -10$$
$$x_2 - 0.5x_3^+ + 0.5x_3^- + x_5^+ - x_5^- \leq -1$$
$$x_2 - 0.5x_3^+ + 0.5x_3^- + x_5^+ - x_5^- \geq -1$$
$$x_3^+ - x_3^- \geq 5$$
$$x_1, x_2, x_3^-, x_3^+, \tilde{x}_4, x_5^-, x_5^+ \geq 0.$$

We convert the first of these two into a greater-than-or-equal-to by multiplying it through by -1, giving:

$$\max_{x} \; 3x_1 + 5x_2 - 3x_3^+ + 3x_3^- - 1.3\tilde{x}_4 - x_5^+ + x_5^-$$

$$\text{s.t.} \quad -x_1 - x_2 - 4\tilde{x}_4 \geq -10$$
$$-x_2 + 0.5x_3^+ - 0.5x_3^- - x_5^+ + x_5^- \geq 1$$
$$x_2 - 0.5x_3^+ + 0.5x_3^- + x_5^+ - x_5^- \geq -1$$
$$x_3^+ - x_3^- \geq 5$$
$$x_1, x_2, x_3^-, x_3^+, \tilde{x}_4, x_5^-, x_5^+ \geq 0.$$

We finally convert the objective function into a minimization by multiplying through by -1, which gives:

$$\min_{x} \; -3x_1 - 5x_2 + 3x_3^+ - 3x_3^- + 1.3\tilde{x}_4 + x_5^+ - x_5^-$$

$$\text{s.t.} \quad -x_1 - x_2 - 4\tilde{x}_4 \geq -10$$
$$-x_2 + 0.5x_3^+ - 0.5x_3^- - x_5^+ + x_5^- \geq 1$$
$$x_2 - 0.5x_3^+ + 0.5x_3^- + x_5^+ - x_5^- \geq -1$$
$$x_3^+ - x_3^- \geq 5$$
$$x_1, x_2, x_3^-, x_3^+, \tilde{x}_4, x_5^-, x_5^+ \geq 0,$$

which is the canonical form of the starting LPP.

□

Example 2.4 Consider the Electricity-Production Problem, which is introduced in Section 2.1.1. The generic formulation of this problem is:

$$\max_{x_1, x_2} x_1 + x_2$$

$$\text{s.t.} \quad \frac{2}{3}x_1 + x_2 \le 18$$

$$2x_1 + x_2 \ge 8$$

$$x_1 \le 12$$

$$x_2 \le 16$$

$$x_1, x_2 \ge 0.$$

To convert this problem to canonical form, we must multiply the objective function and the first, third, and fourth structural constraints through by -1. This gives:

$$\min_{x_1, x_2} -x_1 - x_2$$

$$\text{s.t.} \quad -\frac{2}{3}x_1 - x_2 \ge -18$$

$$2x_1 + x_2 \ge 8$$

$$-x_1 \ge -12$$

$$-x_2 \ge -16$$

$$x_1, x_2 \ge 0,$$

as the canonical form.

$\square$

Example 2.5 Consider the Natural Gas-Transportation Problem, which is introduced in Section 2.1.2. This problem is generically formulated as:

$$\min_{x} z = 5x_{1,1} + 4x_{1,2} + 3x_{2,1} + 6x_{2,2}$$

$$\text{s.t.} \quad x_{1,1} + x_{1,2} \le 7$$

$$x_{2,1} + x_{2,2} \le 12$$

$$x_{1,1} + x_{2,1} = 10$$

$$x_{1,2} + x_{2,2} = 8$$

$$x_{i,j} \ge 0, \forall i = 1, 2; j = 1, 2.$$

To convert this to canonical form, both sides of the first two inequalities must be multiplied by -1. Moreover, the two equality constraints must be replaced with two inequalities, one of each of which is multiplied by -1 to convert all of the structural constraints into greater-than-or-equal-to constraints. This gives:

$$\min_x z = 5x_{1,1} + 4x_{1,2} + 3x_{2,1} + 6x_{2,2}$$

$$\text{s.t.} \; -x_{1,1} - x_{1,2} \geq -7$$
$$-x_{2,1} - x_{2,2} \geq -12$$
$$-x_{1,1} - x_{2,1} \geq -10$$
$$x_{1,1} + x_{2,1} \geq 10$$
$$-x_{1,2} - x_{2,2} \geq -8$$
$$x_{1,2} + x_{2,2} \geq 8$$
$$x_{i,j} \geq 0, \forall \, i = 1, 2; \, j = 1, 2,$$

as the canonical form of this LPP.

$\square$

The canonical form of an LPP can be written generically as:

$$\min_{x_1,\ldots,x_n} \sum_{i=1}^{n} c_i x_i$$

$$\text{s.t.} \; \sum_{i=1}^{n} A_{j,i} x_i \geq b_j, \qquad\qquad \forall \, j = 1, \ldots, m$$

$$x_i \geq 0, \qquad\qquad \forall \, i = 1, \ldots, n.$$

This can also be written more compactly as:

$$\min_x c^\top x \qquad\qquad (2.17)$$

$$\text{s.t.} \; Ax \geq b \qquad\qquad (2.18)$$

$$x \geq 0, \qquad\qquad (2.19)$$

where c, x, A, and b maintain the same definitions as in the compact standard-form LPP, given by (2.14)–(2.16).

2.3 Basic Feasible Solutions and Optimality

This section provides both geometric and algebraic analyses of the feasible region and objective function of linear optimization problems. Based on the geometric analysis, we draw some conclusions regarding the geometrical properties of an optimal solution of a linear optimization problem. We then use an algebraic analysis of the constraints of a linear optimization problem to determine a way to characterize points that may be optimal solutions of an LPP. This algebraic analysis is the backbone of the algorithm used to solve LPPs, which is later introduced in Section 2.5.

2.3.1 Geometric View of Linear Optimization Problems

Recall the Electricity-Production Problem, which is introduced in Section 2.1.1. This problem is formulated as:

$$\max_{x_1,x_2} \ z = x_1 + x_2$$

$$\text{s.t.} \quad \frac{2}{3}x_1 + x_2 \leq 18$$

$$2x_1 + x_2 \geq 8$$

$$x_1 \leq 12$$

$$x_2 \leq 16$$

$$x_1, x_2 \geq 0.$$

Figure 2.1 shows the feasible region of this problem and the contour plot of its objective function.

From the discussion in Section 2.1.1, we know that linear optimization problems have two important geometric properties, which are due to the linearity of their constraints and objective function. The first is that the feasible region of a linear optimization problem is always a polytope, which is the multidimensional analogue of a polygon. We also see from Figure 2.1 that the contour plot of the objective function of an LPP is always a set of parallel hyperplanes, which are the multidimensional analogue of lines. Moreover, the objective function is always increasing or decreasing in the same direction, which is perpendicular to the contours.

This latter geometric property of LPPs, that the contours are parallel and always increasing or decreasing in the same direction, implies that we find an optimal solution by moving as far as possible within the feasible region until hitting a boundary. Put another way, we always find an optimal solution to a linear optimization problem on the boundary of its feasible region. The first geometric property of LPPs, that the feasible region is a polytope, allows us to make an even stronger statement about their optimal solutions. Because the feasible set of an LPP is a polytope, we can always find a vertex or extreme point of the feasible region that is optimal. This is, indeed, one way of stating the fundamental theorem of linear optimization. Figure 2.3 shows the feasible region of the Electricity-Production Problem and identifies its extreme points. We know from the discussion in Section 2.1.1 that $(x_1^*, x_2^*) = (12, 10)$ is the optimal extreme point of this problem.

Fig. 2.3 Feasible region of
Electricity-Production
Problem and its extreme
points
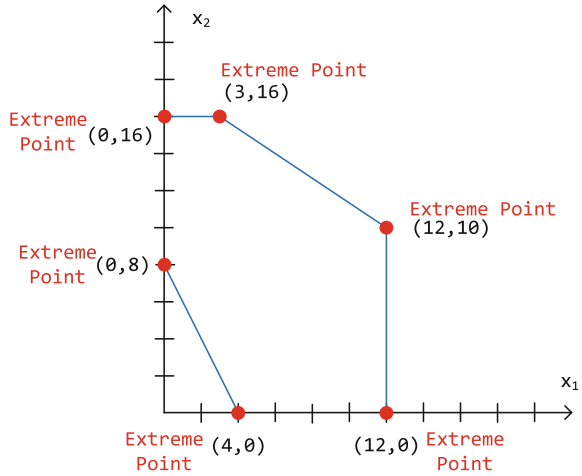
There are also some 'pathological' cases that may arise with an LPP. The first is
that all of the points along a line or hyperplane defining a boundary of the feasible are
optimal solutions. This occurs if the contour lines of the objective function are parallel
to that side of the polytope. We call this a case of **multiple optimal solutions**. To see
how this happens, suppose that the objective function of the Electricity-Production
Problem is changed to:

$$\max_{x_1, x_2} \ z = \frac{2}{3}x_1 + x_2.$$

Figure 2.4 shows the contour plot of the objective function in this case. Note that all
of the points highlighted in purple are now optimal solutions of the LPP.

Fig. 2.4 Geometrical
representation of the
Electricity-Production
Problem with $\frac{2}{3}x_1 + x_2$ as its
objective function
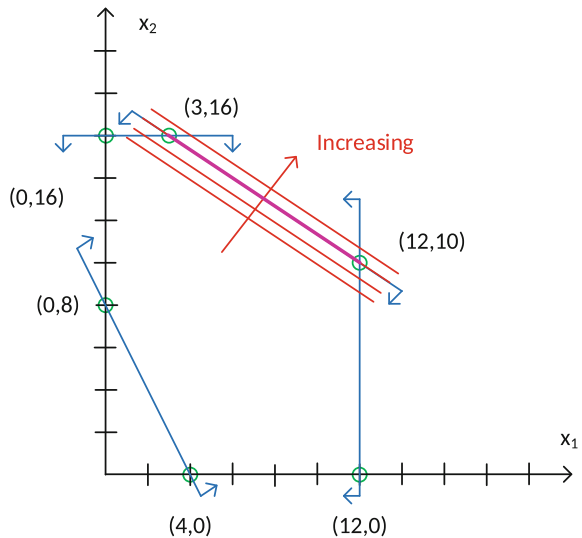
Another issue arises if the feasible region is not bounded. Recall from the discussion in Section 2.1.1 that the feasible region of an optimization problem does not necessarily have to be bounded. The feasible region of the Electricity-Production Problem is bounded, as illustrated in Figure 2.1. A linear optimization problem with a bounded feasible region is guaranteed to have an optimal solution. Otherwise, if the feasible region is **unbounded**, the problem may have an optimal solution or it may be possible to have the objective increase or decrease without limit.

To understand how an LPP with an unbounded feasible region may have an optimal solution, suppose that we remove constraints (2.3) and (2.6) from the Electricity-Production Problem. The LPP would then be:

$$\max_{x_1, x_2} \ z = x_1 + x_2$$
$$\text{s.t.} \ \frac{2}{3}x_1 + x_2 \leq 18$$
$$x_1 \leq 12$$
$$x_2 \leq 16.$$

Figure 2.5 shows the feasible region of the new LPP, which is indeed unbounded (we can make both x_1 and x_2 go to $-\infty$ without violating any of the constraints). Note, however, that the same point, $(x_1^*, x_2^*) = (12, 10)$, that is optimal in the original LPP is optimal in the new problem as well. This is because the side of the polytope that is unbounded is not the side in which the objective improves.

**Fig. 2.5** Geometrical representation of the Electricity-Production Problem with constraints (2.3) and (2.6) removed
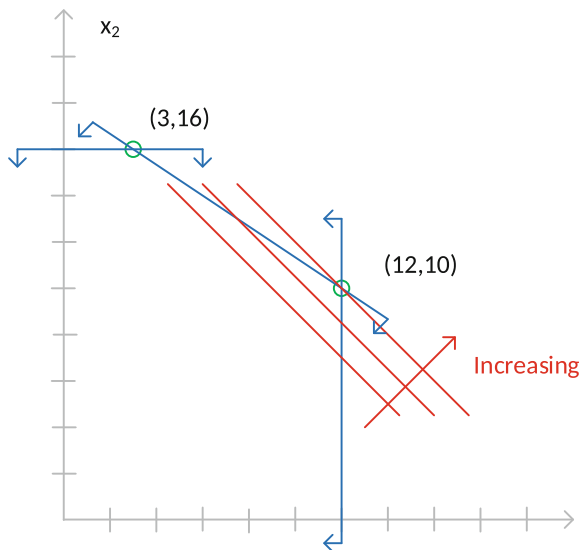
Consider, as an opposite example, if constraints (2.2) and (2.4) are removed from the Electricity-Production Problem. Our LPP would then be:

$$\max_{x_1, x_2} z = x_1 + x_2$$

$$\text{s.t.} \quad 2x_1 + x_2 \geq 8$$

$$x_2 \leq 16$$

$$x_1, x_2 \geq 0.$$

Figure 2.6 shows the feasible region of this LPP, which is also unbounded. Note that there is an important distinction between this LPP and that shown in Figure 2.5. The new LPP no longer has an optimal solution, because the objective function can be made arbitrarily large without violating any of the constraints (we can make x_1 go to $+\infty$ without violating any constraints, and doing so makes the objective function go to $+\infty$). This LPP is said to be **unbounded**.

An unbounded optimization problem is said to have an optimal objective function value of either $-\infty$ or $+\infty$ (depending on whether the problem is a minimization or maximization). Unbounded optimization problems are uncommon in practice, because the physical and economic worlds are bounded. Thus, we do not study unbounded problems in much detail, although we do discuss in Section 2.5.6 how to determine analytically (as opposed to graphically) if an LPP is unbounded.

Fig. 2.6 Geometrical representation of the Electricity-Production Problem with constraints (2.2) and (2.4) removed
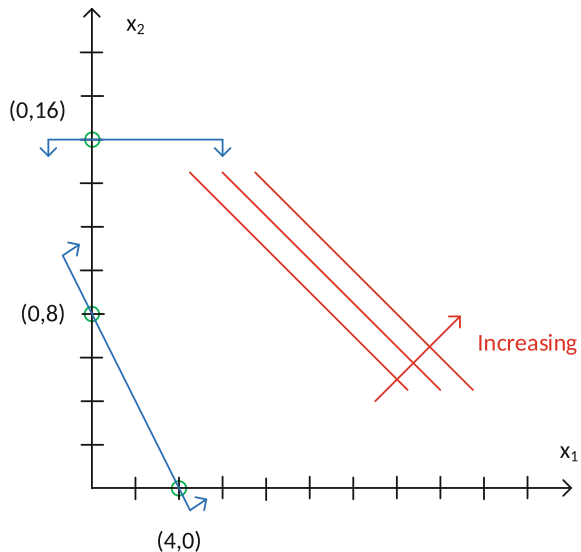
Finally, we should stress that there are two notions of boundedness and unboundedness in the context of optimization. One is whether the feasible region of an LPP is bounded or unbounded. This is a property of the constraints. The second is whether

the problem is bounded or unbounded. An LPP must have an unbounded feasible region for it to be unbounded. Moreover, the objective function must improve (either increase or decrease, depending on whether we are considering a maximization or minimization problem) in the direction that the feasible region is unbounded.

The final pathological case is one in which the feasible region is empty. In such a case there are no feasible solutions that satisfy all of the constraints and we say that such a problem is **infeasible**. To illustrate how a problem can be infeasible, suppose that the Electricity-Production Problem is changed to:

$$\max_{x_1, x_2} z = x_1 + x_2$$

$$\text{s.t. } \frac{2}{3}x_1 + x_2 \geq 18$$

$$2x_1 + x_2 \leq 8$$

$$x_1 \leq 12$$

$$x_2 \leq 16$$

$$x_1, x_2 \geq 0.$$

Figure 2.7 shows the feasible region of this LPP, which is indeed empty. It is empty because there are no points that simultaneously satisfy the $\frac{2}{3}x_1 + x_2 \geq 18$, $2x_1 + x_2 \leq 8$, and $x_1 \geq 0$ constraints. These three constraints conflict with one another.

Fig. 2.7 Geometrical representation of the infeasible variant of the Electricity-Production Problem
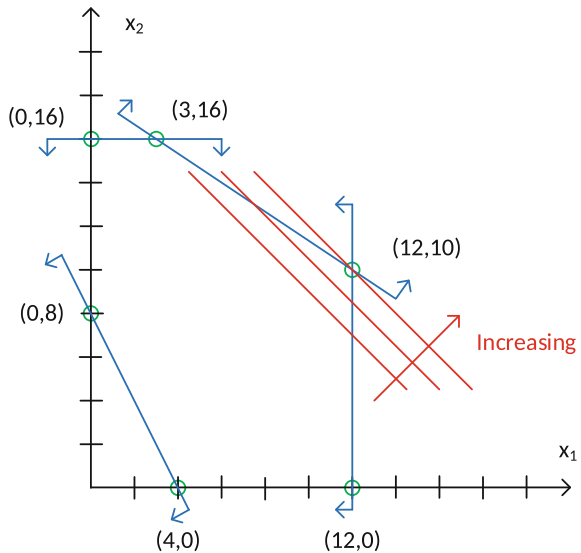
In practice, an infeasible LPP may indicate that there are problem constraints that are not properly specified. This is because the physical world is (normally) feasible. For this reason, we do not pay particular attention to infeasible problems.

An exception to this is if a hypothetical system is being modeled. For instance, suppose that a system is being designed to meet certain criteria. If the resulting model is infeasible, this may be an indication that the system cannot feasibly meet the design criteria specified in the model constraints.

It should, finally, be noted that infeasibility is not caused by a single constraint. Rather, it is caused by two or more constraints that conflict with each other. Thus, when diagnosing the cause of infeasibility of a model, one must identify two or more conflicting constraints.

2.3.2   Algebraic View of Linear Optimization Problems

We now focus our attention on bounded and feasible linear optimization problems. Based on our discussion in Section 2.3.1, we note that every bounded and feasible linear optimization problem has an extreme point that is an optimal solution. Thus, we now work on determining if we can characterize extreme points algebraically, by analyzing the constraints of the LPP.

To gain this insight, we transform the Electricity-Production Problem, which is introduced in Section 2.1.1, into standard form, which is:

$$\min_{x} \; z = -x_1 - x_2$$
$$\text{s.t.} \;\; \frac{2}{3}x_1 + x_2 + x_3 = 18$$
$$2x_1 + x_2 - x_4 = 8$$
$$x_1 + x_5 = 12$$
$$x_2 + x_6 = 16$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0.$$

The standard form-version of this problem can be written more compactly in matrix form as:

$$\min_{x} \; z = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \tag{2.20}$$

$$\text{s.t.} \quad \begin{bmatrix} 2/3 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \\ 12 \\ 16 \end{pmatrix} \quad (2.21)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (2.22)$$

One way to generate candidate points that may be feasible optimal solutions of the LPP is to focus on the algebraic properties of structural equality constraints (2.21). More specifically, we see that this is a system of four equations with six variables that we are solving for. This means that if we fix the values of $6 - 4 = 2$ variables, we can solve for the remaining variables using the structural equality constraints. Once we have solved the structural equality constraints, we then verify whether the resulting values for x are all non-negative, which is the other constraint in the standard-form LPP.

To make the algebra (*i.e.*, the solution of the structural equalities) easier, we fix the $6 - 4 = 2$ variables equal to zero. Solutions that have this structure (*i.e.*, setting a subset of variables equal to zero and solving for the remaining variables using the equality constraints) are called **basic solutions**. A solution that has this structure and also satisfies the non-negativity constraint is called a **basic feasible solution**.

To illustrate how we find basic solutions, let us take the case in which we set x_1 and x_2 equal to zero and solve for the remaining variables using the structural equality constraints. In this case, constraint (2.21) becomes:

$$\begin{bmatrix} 2/3 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 0 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \\ 12 \\ 16 \end{pmatrix}. \quad (2.23)$$

Note, however, that because x_1 and x_2 are set equal to zero in equation (2.23), we can actually ignore the first two columns of the matrix on the left-hand-side of the equality. This is because all of the entries in those columns are multiplied by zero. Thus, equation (2.23) can be further simplified to:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \\ 12 \\ 16 \end{pmatrix},$$

which has the solution:

$$(x_3, x_4, x_5, x_6) = (18, -8, 12, 16).$$

This means that we have found a solution to the structural equality constraints, which is:

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 18, -8, 12, 16).$$

Because we found the values for x by first setting a subset of them equal to zero and solving for the remainder in the equality constraints, this is a basic solution. Note that because $x_4 = -8$ is not non-negative, this is not a basic feasible solution but rather a **basic infeasible solution**.

It is worth noting that whenever we find a basic solution, we solve a square system of equations given by the structural equality constraints. This is because we can neglect the columns of the coefficient matrix that correspond to the variables fixed equal to zero. For instance, if we fix x_5 and x_6 equal to zero in the standard-form version of the Electricity-Production Problem, structural equality constraint (2.21) becomes:

$$\begin{bmatrix} 2/3 & 1 & 1 & 0 \\ 2 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \\ 12 \\ 16 \end{pmatrix}.$$

As a matter of terminology, the variables that are fixed equal to zero when solving for a basic solution are called **non-basic variables**. The other variables, which are solved for using the structural equality constraints, are called **basic variables**.

The number of basic variables that an LPP has is determined by the number of structural equality constraints and the variables that its standard form has. This is because some subset of the variables is set equal to zero to find a basic solution. The standard form of the Electricity-Production Problem has six variables and two of them must be chosen to be set equal to zero. This means that the Electricity-Production Problem has:

$$\binom{6}{2} = \frac{6!}{2!(6-2)!} = 15,$$

basic solutions.

Table 2.5 lists the 15 basic solutions of the Electricity-Production Problem. Each solution is characterized (in the second column of the table) by which variables are basic. The third column of the table gives the values for the basic variables, which are found by setting the non-basic variables equal to zero and solving the structural equality constraints. Two of the basic solutions, numbers 7 and 12, are listed as singular. What this means is that when the non-basic variables are fixed equal to zero, the structural equality constraints do not have a solution. Put another way, when we select the subset of columns of the coefficient matrix that defines the structural constraints, that submatix is singular. We discuss the geometric interpretation of these types of basic solutions later.

Table 2.5 Basic solutions of the Electricity-Production Problem

Solution #	Basic Variables	Basic-Variable Values	Objective-Function Value	x_1	x_2
1	1, 2, 3, 4	12, 16, −6, 32	basic infeasible solution		
2	1, 2, 3, 5	−4, 16, 14/3, 16	basic infeasible solution		
3	1, 2, 3, 6	12, −16, 26, 32	basic infeasible solution		
4	1, 2, 4, 5	3, 16, 14, 9	−19	3	16
5	1, 2, 4, 6	12, 10, 26, 6	−22	12	10
6	1, 2, 5, 6	−15/2, 23, 39/2, −7	basic infeasible solution		
7	1, 3, 4, 5	singular			
8	1, 3, 4, 6	12, 10, 16, 16	−12	12	0
9	1, 3, 5, 6	4, 46/3, 8, 16	−4	4	0
10	1, 4, 5, 6	27, 46, −15, 16	basic infeasible solution		
11	2, 3, 4, 5	16, 2, 8, 12	−16	0	16
12	2, 3, 4, 6	singular			
13	2, 3, 5, 6	8, 10, 12, 8	−8	0	8
14	2, 4, 5, 6	18, 10, 12, −2	basic infeasible solution		
15	3, 4, 5, 6	18, −8, 12, 16	basic infeasible solution		

Table 2.5 also shows that some of the basic solutions, specifically solutions 1 through 3, 6, 10, 14, and 15, are basic infeasible solutions. This is because at least one of the basic variables turns out to have a negative value when the structural equality constraints are solved. For the remaining six basic feasible solutions (*i.e.*, those that are neither singular nor a basic infeasible solution), the last three columns of the table provides the objective-function value and the values of the two variables in the original generic formulation of the problem, x_1 and x_2.

Note that because the standard-form LPP is a minimization problem, basic feasible solution number 5 is the best one from among the six basic feasible solutions found. Moreover, this solution corresponds to the optimal solution that is found graphically in Section 2.1.1. It gives the same objective-function value (when we take into account the fact that the objective function is multiplied by −1 to convert it into a minimization) and the values for the decision variables are the same as well.

Inspecting the six basic feasible solutions in Table 2.5 reveals that they correspond to the extreme points of the feasible region in Figure 2.1. Figure 2.8 shows the feasible region of the Electricity-Production Problem only (*i.e.*, without the contour plot of the objective function) and the six basic feasible solutions that are listed in Table 2.5.
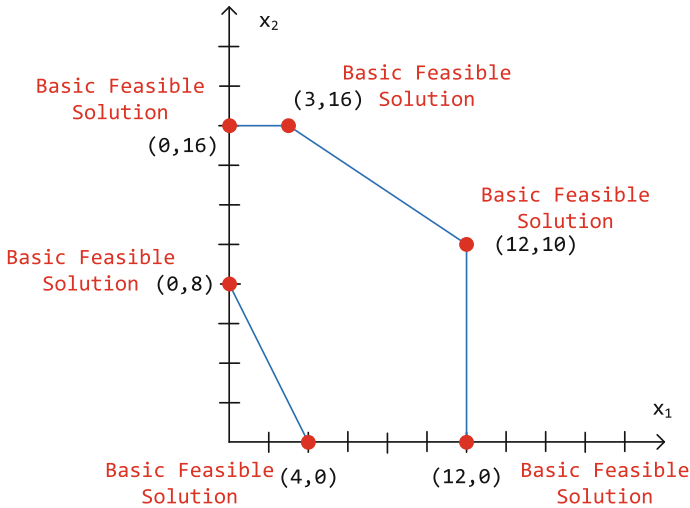

Fig. 2.8 Basic feasible solutions of Electricity-Production Problem

This is a fundamental property of linear optimization: each extreme point of the polytope is a basic feasible solution and each basic feasible solution is an extreme point of the polytope. Proving this property is beyond the scope of this book, and more advanced texts [1] provide the formal proof. The important takeaway from this observation is that if an LPP has an optimal solution, there must be a basic feasible solution that is optimal. This is because the shape of the feasible region and contour plot of the objective imply that there must be an optimal extreme point. Thus, a possible approach to solving an LPP is to enumerate all of the basic feasible solutions and select the one that provides the best objective-function value. This can be quite cumbersome, however, because the number of basic feasible solutions grows exponentially in the problem size (*i.e.*, number of constraints and variables). A more efficient way to solve an LPP is to find a starting basic feasible solution. From this starting point, we then look to see if there is a basic feasible solution next to it that improves the objective function. If not, the basic feasible solution we are currently at is optimal. If so, we move to that basic feasible solution and repeat the process (*i.e.*, determine if there is another basic feasible solution next to the new one that improves the objective). This process is done iteratively until we arrive at a basic feasible solution where the objective cannot be improved. This algorithm, known as the Simplex method, is a standard technique for solving LPPs and is fully detailed in Section 2.5.

Table 2.5 also lists seven basic infeasible solutions, which are shown in Figure 2.9. Basic infeasible solutions are found in the same way as basic feasible solutions— they are 'corners' of the feasible region that are defined by the intersection of the boundaries of two of the linear constraints. However, the intersection point found is infeasible, because it violates at least one other constraint. For instance, the point $(x_1, x_2) = (0, 0)$ is found by intersecting the boundaries of $x_1 \geq 0$ and $x_2 \geq 0$ constraints. However, this point violates the $2x_1 + x_2 \geq 8$ constraint and is, thus, infeasible.

Fig. 2.9 Basic infeasible solutions of Electricity-Production Problem
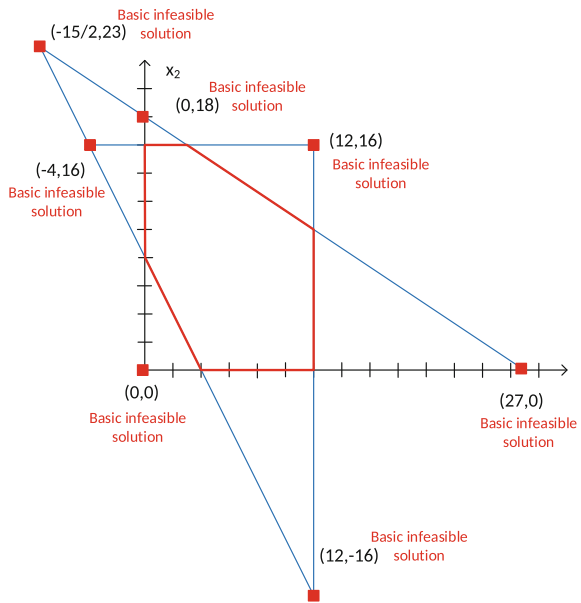
We can also 'visualize' the two basic solutions in Table 2.5 that are labeled as 'singular.' These basic solutions are defined by trying to intersect constraint boundaries that do not actually intersect. For instance, solution number 7 corresponds to the intersection between the boundaries of the $x_2 \geq 0$ and $x_2 \leq 16$ constraints. However, the boundaries of these two constraints are parallel to one another, meaning that there is no basic solution at their intersection. The other singular basic solution corresponds to intersecting the boundaries of the $x_1 \geq 0$ and $x_1 \leq 12$ constraints.

2.4 A Clever Partition

This section establishes three important foundations of the Simplex method. First, we derive expressions that allow us to find the objective-function and basic-variable values of an LPP in terms of the values of the non-basic variables. These expressions

are useful when we want to determine in the Simplex method whether a given basic solution is optimal or not. If it is not, these expressions are useful for finding a new basic solution. We then introduce a tabular format to efficiently organize all of the calculations associated with a given basic solution. Finally, we discuss the algebraic steps used to move from one basic solution to another in the Simplex method.

2.4.1 The Partition

Recall that a basic solution is found by setting one set of variables (the non-basic variables) equal to zero and solving for the others (the basic variables) using the structural equality constraints. For this reason, it is often useful to partition the variables, the coefficients from the left-hand side of the equality constraints, and the objective-function coefficients between the basic and non-basic variables. Moreover, it is useful for developing the Simplex method to derive expressions that give us the values of the objective function and the basic variables in terms of the values of the non-basic variables.

We begin by first examining the structural equality constraints of a standard-form LPP, which can be written as:

$$Ax = b,$$

where A is an $m \times n$ coefficient matrix and b is an m-dimensional vector of constraint right-hand-side constants (*cf.* Equation (2.15) in Section 2.2.2.1). The order that the variables are listed in the x vector is arbitrary. Thus, we can write the x vector as:

$$x = \begin{pmatrix} x_B \\ x_N \end{pmatrix},$$

where x_B are the basic variables and x_N the non-basic variables. If we reorder the x's in this way, then the columns of the A matrix must also be reordered. We do this by writing A as:

$$A = \begin{bmatrix} B & N \end{bmatrix},$$

where B is a submatrix with the constraint coefficients on the basic variables and N is a submatrix with the constraint coefficients on the non-basic variables. We know that x_N is an $(n - m)$-dimensional vector (if there are m structural equality constraints and n variables in the standard-form LPP, we must set $(n - m)$ non-basic variables equal to zero). Thus, B is an $m \times m$ matrix and N is an $m \times (n - m)$ matrix.

Recall from the discussion in Section 2.3.2 and the derivation of the basic solutions for the Electricity-Production Problem in particular, that basic solutions are found by setting by non-basic variables equal to zero. When we do this, we can ignore the columns of the A matrix that are associated with the non-basic variables and solve for the basic variables. This means that we find the basic variables by solving $Bx_B = b$,

which gives $x_B = B^{-1}b$. Thus, whenever we find a basic solution, the B matrix must be full-rank.

Based on these observations, we now know that the structural equality constraints (2.15) can be written as:

$$[B \; N]\begin{pmatrix} x_B \\ x_N \end{pmatrix} = b,$$

or as:

$$Bx_B + Nx_N = b.$$

Because B is full-rank, we can solve for the basic variables in terms of the non-basic variables, giving:

$$\begin{aligned} x_B &= B^{-1}b - B^{-1}Nx_N \\ &= \tilde{b} + \tilde{N}x_N, \end{aligned} \tag{2.24}$$

where $\tilde{b} = B^{-1}b$ and $\tilde{N} = -B^{-1}N$. As a matter of terminology, when solving for a basic solution in this manner the submatrix B is called the **basis**. It is also common to say that basic variables are in the basis, while non-basic variables are said to not be in the basis.

We can also express the objective function in terms of the non-basic variables. To do this, we first note that the standard-form objective function:

$$z = c^\top x,$$

can be written as:

$$z = [c_B \; c_N]\begin{pmatrix} x_B \\ x_N \end{pmatrix},$$

where c_B and c_N are vectors with the objective-function coefficients in c reordered in the same way that the x vector is reordered into x_B and x_N. Using Equation (2.24) we can write this as:

$$\begin{aligned} z &= [c_B \; c_N]\begin{pmatrix} x_B \\ x_N \end{pmatrix} \\ &= c_B^\top x_B + c_N^\top x_N \\ &= c_B^\top \cdot (\tilde{b} + \tilde{N}x_N) + c_N^\top x_N \\ &= c_B^\top \tilde{b} + (c_B^\top \tilde{N} + c_N^\top)x_N \\ &= \tilde{c}_0 + \tilde{c}^\top x_N, \end{aligned} \tag{2.25}$$

where $\tilde{c}_0 = c_B^\top \tilde{b}$ and $\tilde{c}^\top = c_B^\top \tilde{N} + c_N^\top$.

Finally, we can arrange the objective function and structural equality constraints of a standard-form LPP in matrix form, which gives:

$$\begin{pmatrix} z \\ x_B \end{pmatrix} = \begin{bmatrix} \tilde{c}_0 & \tilde{c}^\top \\ \tilde{b} & \tilde{N} \end{bmatrix} \begin{pmatrix} 1 \\ x_N \end{pmatrix}.$$

This can be written more explicitly as:

$$\begin{pmatrix} z \\ x_{B,1} \\ \vdots \\ x_{B,r} \\ \vdots \\ x_{B,m} \end{pmatrix} = \begin{bmatrix} \tilde{c}_0 & \tilde{c}_1 & \cdots & \tilde{c}_s & \cdots & \tilde{c}_{n-m} \\ \tilde{b}_1 & \tilde{N}_{1,1} & \cdots & \tilde{N}_{1,s} & \cdots & \tilde{N}_{1,n-m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{b}_r & \tilde{N}_{r,1} & \cdots & \tilde{N}_{r,s} & \cdots & \tilde{N}_{r,n-m} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \tilde{b}_m & \tilde{N}_{m,1} & \cdots & \tilde{N}_{m,s} & \cdots & \tilde{N}_{m,n-m} \end{bmatrix} \begin{pmatrix} 1 \\ x_{N,1} \\ \vdots \\ x_{N,s} \\ \vdots \\ x_{N,n-m} \end{pmatrix}. \tag{2.26}$$

These matrix expressions, which give the objective-function and basic-variable values in terms of the non-basic-variable values, form the algebraic backbone of the Simplex method, which is developed in Section 2.5.

Example 2.6 Consider the Electricity-Production Problem, which is introduced in Section 2.1.1. When converted to standard form, we can write the objective-function coefficients as:

$$c^\top = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the constraint coefficients as:

$$A = \begin{bmatrix} 2/3 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and the constraint right-hand-side constants as:

$$b^\top = \begin{pmatrix} 18 & 8 & 12 & 16 \end{pmatrix}.$$

If we let $x_B = (x_1, x_2, x_4, x_5)$ and $x_N = (x_3, x_6)$, then we would have:

$$c_B^\top = \begin{pmatrix} -1 & -1 & 0 & 0 \end{pmatrix},$$

$$c_N^\top = \begin{pmatrix} 0 & 0 \end{pmatrix},$$

$$B = \begin{bmatrix} 2/3 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and:

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using the definitions above, and Equations (2.24) and (2.25) in particular, we have:

$$\tilde{b} = B^{-1}b = \begin{bmatrix} 2/3 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} \begin{pmatrix} 18 \\ 8 \\ 12 \\ 16 \end{pmatrix} = \begin{pmatrix} 3 \\ 16 \\ 14 \\ 9 \end{pmatrix},$$

$$\tilde{N} = -B^{-1}N = \begin{bmatrix} 2/3 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -3/2 & 3/2 \\ 0 & -1 \\ -3 & 2 \\ 3/2 & -3/2 \end{bmatrix},$$

$$\tilde{c}_0 = c_B^\top \tilde{b} = \begin{pmatrix} -1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 16 \\ 14 \\ 9 \end{pmatrix} = -19,$$

and:

$$\tilde{c}^\top = c_B^\top \tilde{N} + c_N^\top = \begin{pmatrix} -1 & -1 & 0 & 0 \end{pmatrix} \begin{bmatrix} -3/2 & 3/2 \\ 0 & -1 \\ -3 & 2 \\ 3/2 & -3/2 \end{bmatrix} + \begin{pmatrix} 0 & 0 \end{pmatrix} = \begin{pmatrix} 3/2 & -1/2 \end{pmatrix}.$$

These simple matrix operations can be effortlessly carried out using the public-domain Octave software package [6] or the MATLAB commercial software package [10].

□

2.4.2 The Tableau

Matrix expression (2.26) forms the backbone of the Simplex method. This expression can be even more compactly arranged in what is called a **tableau**, which takes the general form shown in Table 2.6.

Table 2.6 General form of tableau

	1	$x_{N,1}$	$\cdots$	$x_{N,s}$	$\cdots$	$x_{N,n-m}$	
z	$\tilde{c}_0$	$\tilde{c}_1$	$\cdots$	$\tilde{c}_s$	$\cdots$	$\tilde{c}_{n-m}$	$\Longleftarrow$ objective-function block
$x_{B,1}$	$\tilde{b}_1$	$\tilde{N}_{1,1}$	$\cdots$	$\tilde{N}_{1,s}$	$\cdots$	$\tilde{N}_{1,n-m}$	
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	
$x_{B,r}$	$\tilde{b}_r$	$\tilde{N}_{r,1}$	$\cdots$	$\tilde{N}_{r,s}$	$\cdots$	$\tilde{N}_{r,n-m}$	$\Longleftarrow$ basic-variable block
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	
$x_{B,m}$	$\tilde{b}_m$	$\tilde{N}_{m,1}$	$\cdots$	$\tilde{N}_{m,s}$	$\cdots$	$\tilde{N}_{m,n-m}$	

The top row of the table is annotated $\Longleftarrow$ non-basic-variable block.

Table 2.6 identifies three blocks of rows in the tableau. Comparing these three blocks to Equation (2.26) provides some insight into the structure of the tableau. First, the bottom 'basic-variable block' of the tableau is associated with all but the first row of Equation (2.26). These rows of Equation (2.26) each have one basic variable on their left-hand sides, which are in the first column of the basic-variable block of the tableau. Moreover, the right-hand sides of these rows of Equation (2.26) have $\tilde{b}$ and $\tilde{N}$ terms, all of which are in the second and remaining columns of the basic-variable block of the tableau.

Next, the 'objective-function block' of the tableau corresponds to the first row of Equation (2.26). It contains z, which is on the left-hand side of the first row of Equation (2.26), in its first column and the $\tilde{c}$ terms in the remaining columns. Finally, inspecting the 'non-basic-variable block' of the tableau reveals that each column in the tableau is associated with a non-basic variable. These non-basic variables appear in the vector on the right-hand side of Equation (2.26).

It is important to stress that each basic solution has a tableau associated with it. That is because once the variables are partitioned into basic and non-basic variables, that partition determines the entries that label each row and column of the tableau. Moreover, the values of $\tilde{b}$, $\tilde{c}$, and $\tilde{N}$ that go in the tableau are also determined by which columns of the A matrix are put into the B and N submatrices.

The tableau also allows us to easily find a basic variable by setting all of the non-basic variables equal to zero (as noted in Sections 2.3.2 and 2.4.1). If we do this, we obtain basic-variable values:

$$x_B = \tilde{b},$$

and the objective-function value:

$$z = \tilde{c}_0.$$

These values for the basic variables and objective function follow immediately from the discussions and algebraic manipulations of the LPP carried out in Sections 2.3.2 and 2.4.1.

Example 2.7 Recall Example 2.6. If we let $x_B = (x_1, x_2, x_4, x_5)$ and $x_N = (x_3, x_6)$, then we would have:

$$\tilde{b} = B^{-1}b = \begin{pmatrix} 3 \\ 16 \\ 14 \\ 9 \end{pmatrix},$$

$$\tilde{N} = -B^{-1}N = \begin{bmatrix} -3/2 & 3/2 \\ 0 & -1 \\ -3 & 2 \\ 3/2 & -3/2 \end{bmatrix},$$

$$\tilde{c}_0 = c_B^\top \tilde{b} = -19,$$

and:

$$\tilde{c}^\top = c_B^\top \tilde{N} + c_N^\top = \begin{pmatrix} 3/2 & -1/2 \end{pmatrix}.$$

The tableau associated with this basic solution is shown in Table 2.7. The 'B' and 'N' subscripts on the basic and non-basic variables have been omitted, because it is clear from the way that the tableau is arranged that x_1, x_2, x_4, and x_5 are basic variables while x_3 and x_6 are non-basic variables.

Table 2.7 Tableau for Example 2.7

	1	x_3	x_6
z	-19	$3/2$	$-1/2$
x_1	3	$-3/2$	$3/2$
x_2	16	0	-1
x_4	14	-3	2
x_5	9	$3/2$	$-3/2$

We can also determine the basic-variable and objective-function values when the non-basic variables are fixed equal to zero. The basic variables take on the values given in the second column of the tableau that are next to each basic variable (*i.e.*, $(x_1, x_2, x_4, x_5) = (3, 16, 14, 9)$ when the non-basic variables are set equal to zero) and the objective function takes on the value next to it in the second column of the tableau (*i.e.*, $z = -19$).   □

2.4.3 Pivoting

One way of characterizing a basic solution is by which variables are basic variables (we also say that such variables are in the basis) and which variables are non-basic variables (we say that such variables are not in the basis). One of the features of the Simplex method is that the basic solutions found in each successive iteration differ by one variable. That is to say, if we compare the basic solutions found in two successive iterations, one of the basic variables in the first basic solution will be a non-basic variable in the second solution and one of the non-basic variables in the first solution will be a basic variable in the second solution.

Section 2.4.1 shows how the basic-variable and objective-function values can be expressed in terms of the values of the non-basic variables. Section 2.4.2 further shows how this information can be compactly written in tableau form. As successive iterations of the Simplex method are carried out, the tableau must be updated to reflect the fact that we move from one basic solution to another. Of course, the tableau could be updated in each iteration simply by redefining B, N, c_B, and c_N and applying Equations (2.24) and (2.25) to compute $\tilde{b}$, $\tilde{N}$, and $\tilde{c}$. This can be very computationally expensive, however, especially if the Simplex method is being applied by hand.

There is, however, a shortcut to update the tableau, which is called **pivoting**. Pivoting relies on the property of successive basic solutions found in the Simplex method that they differ only in that one basic variable leaves that basis and one non-basic variable enters the basis. To demonstrate the pivoting operation, suppose that we are currently at a basic solution and would like to move to a new basic solution. In the new basic solution there is a basic variable, $x_{B,r}$, that leaves the basis and a non-basic variable, $x_{N,s}$, that enters the basis. Table 2.8 shows the initial tableau before the pivoting operation. For notational convenience, we omit the 'B,' and 'N' subscripts on the variables exiting and entering the basis. Thus, these two variables are labeled x_r and x_s in the tableau.

Table 2.8 Initial tableau before pivoting

	1	$x_{N,1}$	$\cdots$	x_s	$\cdots$	$x_{N,n-m}$
z	$\tilde{c}_0$	$\tilde{c}_1$	$\cdots$	$\tilde{c}_s$	$\cdots$	$\tilde{c}_{n-m}$
$x_{B,1}$	$\tilde{b}_1$	$\tilde{N}_{1,1}$	$\cdots$	$\tilde{N}_{1,s}$	$\cdots$	$\tilde{N}_{1,n-m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
x_r	$\tilde{b}_r$	$\tilde{N}_{r,1}$	$\cdots$	$\tilde{N}_{r,s}$	$\cdots$	$\tilde{N}_{r,n-m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$x_{B,m}$	$\tilde{b}_m$	$\tilde{N}_{m,1}$	$\cdots$	$\tilde{N}_{m,s}$	$\cdots$	$\tilde{N}_{m,n-m}$

To derive the pivoting operation, we first write the row associated with x_r in Table 2.8 explicitly as:

$$x_r = \tilde{b}_r + \tilde{N}_{r,1}x_{N,1} + \cdots + \tilde{N}_{r,s}x_s + \cdots + \tilde{N}_{r,n-m}x_{N,n-m}. \qquad (2.27)$$

We then manipulate Equation (2.24) to express x_s as a function of x_r and the remaining non-basic variables:

$$x_s = -\frac{\tilde{b}_r}{\tilde{N}_{r,s}} - \frac{\tilde{N}_{r,1}}{\tilde{N}_{r,s}}x_{N1} - \cdots + \frac{1}{\tilde{N}_{r,s}}x_r - \cdots - \frac{\tilde{N}_{r,n-m}}{\tilde{N}_{r,s}}x_{N,n-m}. \tag{2.28}$$

Next, we write the objective-function row of the tableau in Table 2.8 explicitly as:

$$z = \tilde{c}_0 + \tilde{c}_1 x_{N,1} + \cdots + \tilde{c}_s x_s + \cdots + \tilde{c}_{n-m} x_{N,n-m}. \tag{2.29}$$

We then use Equation (2.28) to substitute for x_s in Equation (2.29), which gives:

$$z = \left(\tilde{c}_0 - \frac{\tilde{c}_s}{\tilde{N}_{r,s}}\tilde{b}_r\right) + \left(\tilde{c}_1 - \tilde{c}_s\frac{\tilde{N}_{r,1}}{\tilde{N}_{r,s}}\right)x_{N,1} + \cdots + \frac{\tilde{c}_s}{\tilde{N}_{r,s}}x_r \tag{2.30}$$

$$+ \cdots + \left(\tilde{c}_{n-m} - \tilde{c}_s\frac{\tilde{N}_{r,n-m}}{\tilde{N}_{r,s}}\right)x_{N,n-m}.$$

Next we write the row associated with $x_{B,1}$ in Table 2.8 as:

$$x_{B,1} = \tilde{b}_1 + \tilde{N}_{1,1}x_{N,1} + \cdots + \tilde{N}_{1,s}x_s + \cdots + \tilde{N}_{1,n-m}x_{N,n-m}.$$

If we use Equation (2.28) to substitute for x_s in this expression we have:

$$x_{B,1} = \left(\tilde{b}_1 - \frac{\tilde{N}_{1,s}}{\tilde{N}_{r,s}}\tilde{b}_r\right) + \left(\tilde{N}_{1,1} - \tilde{N}_{1,s}\frac{\tilde{N}_{r,1}}{\tilde{N}_{r,s}}\right)x_{N,1} + \cdots + \frac{\tilde{N}_{1,s}}{\tilde{N}_{r,s}}x_r \tag{2.31}$$

$$+ \cdots + \left(\tilde{N}_{1,n-m} - \tilde{N}_{1,s}\frac{\tilde{N}_{r,n-m}}{\tilde{N}_{r,s}}\right)x_{N,n-m}.$$

A similar manipulation of the row associated with $x_{B,m}$ in Table 2.8 yields:

$$x_{Bm} = \left(\tilde{b}_m - \frac{\tilde{N}_{m,s}}{\tilde{N}_{r,s}}\tilde{b}_r\right) + \left(\tilde{N}_{m,1} - \tilde{N}_{m,s}\frac{\tilde{N}_{r,1}}{\tilde{N}_{r,s}}\right)x_{N,1} + \cdots + \frac{\tilde{N}_{m,s}}{\tilde{N}_{r,s}}x_r \tag{2.32}$$

$$+ \cdots + \left(\tilde{N}_{m,n-m} - \tilde{N}_{m,s}\frac{\tilde{N}_{r,n-m}}{\tilde{N}_{r,s}}\right)x_{N,n-m}. \tag{2.33}$$

Using Equations (2.28), (2.30), (2.31), and (2.32), which express x_s, z, $x_{B,1}$, and $x_{B,m}$ in terms of x_r, we can update the tableau to that given in Table 2.9.

Table 2.9 Updated tableau after pivoting

	1	$x_{N,1}$	$\cdots$	x_r	$\cdots$	$x_{N,n-m}$
z	$\tilde{c}_0 - \frac{\tilde{c}_s}{\tilde{N}_{r,s}}\tilde{b}_r$	$\tilde{c}_1 - \tilde{c}_s \frac{\tilde{N}_{r,1}}{\tilde{N}_{r,s}}$	$\cdots$	$\frac{\tilde{c}_s}{\tilde{N}_{r,s}}$	$\cdots$	$\tilde{c}_{n-m} - \tilde{c}_s \frac{\tilde{N}_{r,n-m}}{\tilde{N}_{r,s}}$
$x_{B,1}$	$\tilde{b}_1 - \frac{\tilde{N}_{1,s}}{\tilde{N}_{r,s}}\tilde{b}_r$	$\tilde{N}_{1,1} - \tilde{N}_{1,s}\frac{\tilde{N}_{r,1}}{\tilde{N}_{r,s}}$	$\cdots$	$\frac{\tilde{N}_{1,s}}{\tilde{N}_{r,s}}$	$\cdots$	$\tilde{N}_{1,n-m} - \tilde{N}_{1,s}\frac{\tilde{N}_{r,n-m}}{\tilde{N}_{r,s}}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
x_s	$-\frac{\tilde{b}_r}{\tilde{N}_{r,s}}$	$-\frac{\tilde{N}_{r,1}}{\tilde{N}_{r,s}}$	$\cdots$	$\frac{1}{\tilde{N}_{r,s}}$	$\cdots$	$-\frac{\tilde{N}_{r,n-m}}{\tilde{N}_{r,s}}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$x_{B,m}$	$\tilde{b}_m - \frac{\tilde{N}_{m,s}}{\tilde{N}_{r,s}}\tilde{b}_r$	$\tilde{N}_{m,1} - \tilde{N}_{m,s}\frac{\tilde{N}_{r,1}}{\tilde{N}_{r,s}}$	$\cdots$	$\frac{\tilde{N}_{m,s}}{\tilde{N}_{r,s}}$	$\cdots$	$\tilde{N}_{m,n-m} - \tilde{N}_{m,s}\frac{\tilde{N}_{r,n-m}}{\tilde{N}_{r,s}}$

Example 2.8 Recall Example 2.7 and that if we let $x_B = (x_1, x_2, x_4, x_5)$ and $x_N = (x_3, x_6)$ then the tableau is given by Table 2.7. Let us conduct one pivot operation in which x_5 leaves the basis and x_6 enters it. From the x_5 row of Table 2.7 we have:

$$x_5 = 9 + \frac{3}{2}x_3 - \frac{3}{2}x_6.$$

This can be rewritten as:

$$x_6 = 6 + x_3 - \frac{1}{2}x_5. \tag{2.34}$$

Substituting Equation (2.34) into the objective-function row of Table 2.7:

$$z = -19 + \frac{3}{2}x_3 - \frac{1}{2}x_6,$$

gives:

$$z = -22 + x_3 + \frac{1}{3}x_5. \tag{2.35}$$

We next consider the rows of Table 2.7, which are:

$$x_1 = 3 - \frac{3}{2}x_3 + \frac{3}{2}x_6,$$

$$x_2 = 16 - x_6,$$

and:

$$x_4 = 14 - 3x_3 + 2x_6.$$

Substituting Equation (2.34) into these three equations gives:

$$x_1 = 12 - x_5, \tag{2.36}$$

$$x_2 = 10 - x_3 + \frac{2}{3}x_5, \tag{2.37}$$

and:

$$x_4 = 26 - x_3 - \frac{4}{3}x_5. \tag{2.38}$$

Substituting Equations (2.34) through (2.38) into Table 2.7 gives an updated tableau after the pivot operation, which is given in Table 2.10.

Table 2.10 Updated tableau after conducting pivot operation in Example 2.8

	1	x_3	x_5
z	−22	1	1/3
x_1	12	0	−1
x_2	10	−1	2/3
x_4	26	−1	−4/3
x_6	6	1	−1/2

It is finally worth noting that we have performed a step-by-step pivot operation using the expressions derived above. One could also obtain the new tableau directly by using the expressions in Table 2.9. Doing so yields the same tableau. □

2.5 The Simplex Method

The Simplex method is the most commonly used approach to solving LPPs. At its heart, the Simplex method relies on two important properties of linear optimization problems. First, as noted in Sections 2.1.1 and 2.3.1, if an LPP has an optimal solution, then there must be at least one extreme point of the feasible set that is optimal. Secondly, as discussed in Section 2.3.2, there is a one-to-one correspondence between extreme points of the feasible set and basic feasible solutions. That is to say, each basic feasible solution corresponds to an extreme point of the feasible set and each extreme point of the feasible set corresponds to a basic feasible solution.

Building off of these two properties, the Simplex method solves an LPP by following two major steps. First, it works to find a starting basic feasible solution. Once it has found a basic feasible solution, the Simplex method iteratively determines if there is another corner of the feasible region next to the corner that the current basic feasible solution corresponds to that gives a better objective-function value. If there is, the Simplex method moves to this new basic feasible solution. Otherwise, the algorithm terminates because the basic feasible solution it is currently at is optimal.

In the following sections we describe each of these steps of the Simplex method in turn. We then provide an overview of the entire algorithm. We finally discuss

some technical issues, such as guaranteeing that the Simplex method terminates and detecting if an LPP is unbounded, infeasible, or has multiple optimal solutions.

2.5.1 Finding an Initial Basic Feasible Solution

We know from the discussions in Sections 2.3.2 and 2.4.1 that finding a basic solution is relatively straightforward. All we must do is partition the variables into basic and non-basic variables, x_B and x_N, respectively. Once we have done this we determine B, N, c_B, and c_N and then compute:

$$\tilde{b} = B^{-1}b,$$

$$\tilde{N} = -B^{-1}N,$$

$$\tilde{c}_0 = c_B^\top \tilde{b},$$

and:

$$\tilde{c}^\top = c_B^\top \tilde{N} + c_N^\top.$$

To the extent possible, one can choose x_B in a way such that the B matrix is relatively easy to invert.

After these calculations are done, we can put them into a tableau, such as the one given in Table 2.11. We know from the discussion in Section 2.4.2 that for the chosen basic solution, the values of the basic variables can be easily read from the tableau as the value of $\tilde{b}$. We further know that if $\tilde{b} \geq 0$, then the basic solution that we have found is a basic feasible solution and no further work must be done (*i.e.*, we can proceed to the next step of the Simplex method, which is discussed in Section 2.5.2). Otherwise, we must conduct what is called a regularization step.

Table 2.11 The tableau for a starting basic solution

	1	$x_{N,1}$	$\cdots$	$x_{N,s}$	$\cdots$	$x_{N,n-m}$
z	$\tilde{c}_0$	$\tilde{c}_1$	$\cdots$	$\tilde{c}_s$	$\cdots$	$\tilde{c}_{n-m}$
$x_{B,1}$	$\tilde{b}_1$	$\tilde{N}_{1,1}$	$\cdots$	$\tilde{N}_{1,s}$	$\cdots$	$\tilde{N}_{1,n-m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$x_{B,r}$	$\tilde{b}_r$	$\tilde{N}_{r,1}$	$\cdots$	$\tilde{N}_{r,s}$	$\cdots$	$\tilde{N}_{r,n-m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$x_{B,m}$	$\tilde{b}_m$	$\tilde{N}_{m,1}$	$\cdots$	$\tilde{N}_{m,s}$	$\cdots$	$\tilde{N}_{m,n-m}$

In the regularization step we add one new column to the tableau, which is highlighted in boldface in Table 2.12. This added column has a new non-basic variable, which we call $x_{N,n-m+1}$, all ones in the basic-variable rows, and a value of K in the

objective-function row. The value of K is chosen to be larger than any of the other existing values in the objective-function row of the tableau. We next conduct one pivot operation in which the added variable, $x_{N,n-m+1}$, becomes a basic variable. The basic variable that exits the basis is the one that has the smallest or most negative $\tilde{b}$ value. That is, the variable that exits the basis corresponds to:

$$\min \left\{\tilde{b}_1, \ldots, \tilde{b}_r, \ldots, \tilde{b}_m\right\}.$$

Table 2.12 The tableau after the regularization step

	1	$x_{N,1}$	$\cdots$	$x_{N,s}$	$\cdots$	$x_{N,n-m}$	$x_{N,n-m+1}$
z	$\tilde{c}_0$	$\tilde{c}_1$	$\cdots$	$\tilde{c}_s$	$\cdots$	$\tilde{c}_{n-m}$	K
$x_{B,1}$	$\tilde{b}_1$	$\tilde{N}_{1,1}$	$\cdots$	$\tilde{N}_{1,s}$	$\cdots$	$\tilde{N}_{1,n-m}$	1
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$x_{B,r}$	$\tilde{b}_r$	$\tilde{N}_{r,1}$	$\cdots$	$\tilde{N}_{r,s}$	$\cdots$	$\tilde{N}_{r,n-m}$	1
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
$x_{B,m}$	$\tilde{b}_m$	$\tilde{N}_{m,1}$	$\cdots$	$\tilde{N}_{m,s}$	$\cdots$	$\tilde{N}_{m,n-m}$	1

After this pivoting operation is conducted, we can guarantee that we have a basic feasible solution (*i.e.*, all of the $\tilde{b}$'s are non-negative after the tableau is updated). We now show this formally.

Regularization Property: After conducting the regularization step and a pivot operation, the updated $\tilde{b}$ will be non-negative.

To show this, suppose that we denote the basic variable that will be exiting the basis as $x_{B,r}$. The basic-variable rows of the tableau in Table 2.12 can be written as:

$$x_{B,1} = \tilde{b}_1 + \tilde{N}_{1,1}x_{N,1} + \cdots + \tilde{N}_{1,s}x_{N,s} + \cdots \tilde{N}_{1,n-m}x_{N,n-m} + x_{N,n-m+1},$$

$$\vdots$$

$$x_{B,r} = \tilde{b}_r + \tilde{N}_{r,1}x_{N,1} + \cdots + \tilde{N}_{r,s}x_{N,s} + \cdots \tilde{N}_{r,n-m}x_{N,n-m} + x_{N,n-m+1},$$

$$\vdots$$

$$x_{B,m} = \tilde{b}_m + \tilde{N}_{m,1}x_{N,1} + \cdots + \tilde{N}_{m,s}x_{N,s} + \cdots \tilde{N}_{m,n-m}x_{N,n-m} + x_{N,n-m+1}.$$

After the pivot operation $x_{N,1}, \ldots, x_{N,n-m}$ will remain non-basic variables that are fixed equal to zero, thus these equations can be simplified to:

$$x_{B,1} = \tilde{b}_1 + x_{N,n-m+1}, \qquad (2.39)$$

$$\vdots$$

$$x_{B,r} = \tilde{b}_r + x_{N,n-m+1}, \qquad (2.40)$$

$$\vdots$$

$$x_{B,m} = \tilde{b}_m + x_{N,n-m+1}. \qquad (2.41)$$

From Equation (2.40) we have the value that $x_{N,n-m+1}$ takes when it becomes a basic variable as:

$$x_{N,n-m+1} = x_{B,r} - \tilde{b}_r = -\tilde{b}_r,$$

where the second equality follows because we know $x_{B,r}$ is becoming a non-basic variable after the pivot operation is completed. Because r is chosen such that $\tilde{b}_r < 0$, we know that $x_{N,n-m+1} > 0$ after this pivot operation is completed.

Moreover, if we substitute this value of $x_{N,n-m+1}$ into the remainder of Equations (2.39) through (2.41) then we have:

$$x_{B,1} = \tilde{b}_1 - \tilde{b}_r,$$

$$\vdots$$

$$x_{B,m} = \tilde{b}_m - \tilde{b}_r.$$

Note, however, that because r is chosen such that it gives the most negative value of $\tilde{b}$, the right-hand sides of all of these equations are non-negative. Thus, our new basic solution is guaranteed to be feasible.

This Regularization Property implies that for any LPP we must do at most one regularization step to find a starting basic feasible solution. The idea of the regularization step is that we add a new **artificial variable** to the LPP and set its value in a way that all of the variables take on non-negative values. Of course, adding this new variable is 'cheating' in the sense that it is not a variable of the original LPP. Thus, adding the artificial variable changes the problem's feasible region.

The value of K in the new tableau is intended to take care of this. As we see in Section 2.5.2, the Simplex method determines whether the current basic feasible solution is optimal by examining the values in objective-function row of the tableau.

The high value of K works to add a penalty to the objective function for allowing the artificial variable to take on a value greater than zero (it should be easy to convince yourself that if the artificial variable is equal to zero in a basic feasible solution, then that solution is feasible in the original LPP without the artificial variable added). The high value for K will have the Simplex method try to reduce the value of the artificial variable to zero. Once the simplex method drives the value of the artificial variable to zero, then we have found a basic feasible solution that is feasible in the original LPP without the artificial variable added. Once the value of the artificial variable has been driven to zero, it can be removed from the tableau and the Simplex method can be further applied without that variable in the problem.

It should be further noted that in some circumstances, an artificial variable may not need to be added to conduct the regularization step. This would be the case if the starting tableau already has a non-basic variable with a column of ones in the basic-variable rows. If so, one can conduct a pivot operation in which this non-basic variable enters the basis and the basic variable with the most negative $\tilde{b}$ value exits to obtain a starting basic feasible solution.

2.5.2 Moving Between Basic Feasible Solutions

The main optimization step of the Simplex method checks to see whether the current basic feasible solution is optimal or not. If it is optimal, then the method terminates. Otherwise, the Simplex method moves to a new basic feasible solution. To determine whether the current basic feasible solution is optimal or not, we examine the objective-function row of the tableau.

Recall from Equation (2.25) that the objective-function row of the tableau can be written as:

$$z = \tilde{c}_0 + \tilde{c}^\top x_N,$$

which expresses the objective function value of the LPP in terms of the values of the non-basic variables. If any of the elements of $\tilde{c}$ are negative, this implies that increasing the value of the corresponding non-basic variable from zero to some positive value improves (decreases) the objective function. Thus, the Simplex method determines whether the current basic feasible solution is optimal or not by checking the signs of the $\tilde{c}$ values in the tableau. If they are all non-negative, then the current solution is optimal and the algorithm terminates. Otherwise, if at least one of the values is negative, then the current solution is not optimal.

In this latter case that one or more of the $\tilde{c}$'s is negative, one of the non-basic variables with a negative $\tilde{c}$ is chosen to enter the basis. Any non-basic variable with a negative $\tilde{c}$ can be chosen to enter the basis. However, in practice it is common to choose the non-basic variable with the most negative $\tilde{c}$. This is because each unit increase in the value of the non-basic variable with the most negative $\tilde{c}$ gives the greatest objective-function decrease. We let s denote the index of the non-basic variable that enters the basis.

The next step of the Simplex method is to determine which basic variable exits the basis when $x_{N,s}$ enters it. To determine this, we note that after we swap $x_{N,s}$ for whatever basic variable exits the basis, we want to ensure that we are still at a basic feasible solution. This means that we want to ensure that the basic variables all have non-negative values after the variables are swapped. To ensure this, we examine the problem tableau, which is shown in Table 2.13.

Table 2.13 The tableau for the current basic feasible solution

	1	$x_{N,1}$	$\cdots$	$x_{N,s}$	$\cdots$	$x_{N,n-m}$
z	$\tilde{c}_0$	$\tilde{c}_1$	$\cdots$	$\tilde{c}_s$	$\cdots$	$\tilde{c}_{n-m}$
$x_{B,1}$	$\tilde{b}_1$	$\tilde{N}_{1,1}$	$\cdots$	$\tilde{N}_{1,s}$	$\cdots$	$\tilde{N}_{1,n-m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$x_{B,r}$	$\tilde{b}_r$	$\tilde{N}_{r,1}$	$\cdots$	$\tilde{N}_{r,s}$	$\cdots$	$\tilde{N}_{r,n-m}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$
$x_{B,m}$	$\tilde{b}_m$	$\tilde{N}_{m,1}$	$\cdots$	$\tilde{N}_{m,s}$	$\cdots$	$\tilde{N}_{m,n-m}$

The basic-variable rows of the tableau can be expanded as:

$$x_{B,1} = \tilde{b}_1 + \tilde{N}_{1,1}x_{N,1} + \cdots + \tilde{N}_{1,s}x_{N,s} + \cdots \tilde{N}_{1,n-m}x_{N,n-m},$$

$$\vdots$$

$$x_{B,r} = \tilde{b}_r + \tilde{N}_{r,1}x_{N,1} + \cdots + \tilde{N}_{r,s}x_{N,s} + \cdots \tilde{N}_{r,n-m}x_{N,n-m},$$

$$\vdots$$

$$x_{B,m} = \tilde{b}_m + \tilde{N}_{m,1}x_{N,1} + \cdots + \tilde{N}_{m,s}x_{N,s} + \cdots \tilde{N}_{m,n-m}x_{N,n-m}.$$

These equations define the value of the basic variables in terms of the values of the non-basic variables. These equations simplify to:

$$x_{B,1} = \tilde{b}_1 + \tilde{N}_{1,s}x_{N,s}, \tag{2.42}$$

$$\vdots$$

$$x_{B,r} = \tilde{b}_r + \tilde{N}_{r,s}x_{N,s}, \tag{2.43}$$

$$\vdots$$

$$x_{B,m} = \tilde{b}_m + \tilde{N}_{m,s}x_{N,s}, \tag{2.44}$$

because after $x_{N,s}$ enters the basis the other variables that are non-basic at the current basic feasible solution remain non-basic. Inspecting Equations (2.42)–(2.44), we see that increasing the value of $x_{N,s}$ has two possible effects. A basic variable that has a negative $\tilde{N}$ coefficient on $x_{N,s}$ in the equation defining its value decreases as $x_{N,s}$ increases. On the other hand, a basic variable that has a zero or positive $\tilde{N}$ coefficient on $x_{N,s}$ remains the same or increases as $x_{N,s}$ increases. Thus, to ensure that our new basic solution is feasible, we only need to concern ourselves with basic variables that have a negative $\tilde{N}$ coefficient in the tableau (because we do not want any of the basic variables to become negative at our new basic solution).

This means that we can restrict attention to the subset of Equations (2.42)–(2.44) that have a negative $\tilde{N}$ coefficient on $x_{N,s}$. We write these equations as:

$$x_{B,1} = \tilde{b}_1 + \tilde{N}_{1,s} x_{N,s},$$

$$\vdots$$

$$x_{B,r} = \tilde{b}_r + \tilde{N}_{r,s} x_{N,s},$$

$$\vdots$$

$$x_{B,m'} = \tilde{b}_{m'} + \tilde{N}_{m',s} x_{N,s},$$

where we let $x_{B,1}, \ldots, x_{B,m'}$ be the subset of basic variables that have negative $\tilde{N}$ coefficients on $x_{N,s}$. We want all of our basic variables to be non-negative when we increase the value of $x_{N,s}$, which we can write as:

$$x_{B,1} = \tilde{b}_1 + \tilde{N}_{1,s} x_{N,s} \geq 0,$$

$$\vdots$$

$$x_{B,r} = \tilde{b}_r + \tilde{N}_{t,s} x_{N,s} \geq 0,$$

$$\vdots$$

$$x_{B,m'} = \tilde{b}_{m'} + \tilde{N}_{m',s} x_{N,s} \geq 0.$$

Subtracting $\tilde{b}$ from both sides of each inequality and dividing both sides of each by $\tilde{N}$ gives:

$$x_{N,s} \leq -\frac{\tilde{b}_1}{\tilde{N}_{1,s}},$$

$$\vdots$$

$$x_{N,s} \leq -\frac{\tilde{b}_r}{\tilde{N}_{r,s}},$$

$$\vdots$$

$$x_{N,s} \leq -\frac{\tilde{b}_{m'}}{\tilde{N}_{m',s}},$$

where the directions of the inequalities change because we are focusing on basic variables that have a negative $\tilde{N}$ coefficient on $x_{N,s}$. Also note that because we are only examining basic variables that have a negative $\tilde{N}$ coefficient on $x_{N,s}$, all of the ratios, $-\tilde{b}_1/\tilde{N}_{1,s}, \ldots, -\tilde{b}_{m'}/\tilde{N}_{m',s}$, are positive.

Taken together, these inequalities imply that the largest $x_{N,s}$ can be made without causing any of the basic variables to become negative is:

$$x_{N,s} = \min\left\{ -\frac{\tilde{b}_1}{\tilde{N}_{1,s}}, \ldots, -\frac{\tilde{b}_r}{\tilde{N}_{r,s}}, \ldots, -\frac{\tilde{b}_{m'}}{\tilde{N}_{m',s}} \right\}.$$

Because we restricted our attention to basic variables that have a negative $\tilde{N}$ coefficient on $x_{N,s}$ in the tableau, we can also write this maximum value that $x_{N,s}$ can take as:

$$x_{N,s} = \min_{i=1,\ldots,m:\tilde{N}_{i,s}<0}\left\{ -\frac{\tilde{b}_i}{\tilde{N}_{i,s}} \right\}. \tag{2.45}$$

If we define r as the index of the basic variable that satisfies condition (2.45) then we know that when we increase $x_{N,s}$ to:

$$x_{N,s} = -\frac{\tilde{b}_r}{\tilde{N}_{r,s}},$$

$x_{B,r}$ becomes equal to zero. This means that $x_{B,r}$ becomes the new non-basic variable when $x_{N,s}$ becomes a basic variable.

Once the basic variable that enters the basis, $x_{N,s}$, and the non-basic variable that exits the basis, $x_{B,r}$, are identified, a pivot operation (*cf.* Section 2.4.3) is conducted and the tableau is updated. The process outlined in the current section to determine if the new basic feasible solution found after the pivoting operation is optimal or not is then applied to the updated tableau. If the updated tableau (specifically, the values in the objective-function row) indicates that the new basic feasible solution is optimal, then the Simplex method terminates. Otherwise, a non-basic variable is chosen to enter the basis and the ratio test shown in Equation (2.45) is conducted to

determine which basic variable exits the basis. This process is repeated iteratively until the Simplex method terminates.

2.5.3 Simplex Method Algorithm

We now provide a more general outline of how to apply the Simplex method to solve any linear optimization problem. We assume that the problem has been converted to standard form and that we have chosen a starting set of basic and non-basic variables. Note, however, that the basic and non-basic variables chosen do not necessarily have to give us a basic feasible solution. If they do not, we conduct the regularization step to make the basic-variable values all non-negative. Otherwise, we skip the regularization step and proceed to conducting Simplex iterations to move between basic feasible solutions while improving the objective function.

The following algorithm outlines the major steps of the Simplex method. We begin in Step 2 by computing the starting tableau, based on the chosen partition of the variables into basic and non-basic variables. In Step 3 we determine if the regularization step is needed. Recall that if $\tilde{b} \geq 0$, then our starting basic solution is also a basic feasible solution and regularization is not needed. Otherwise, if at least one component of $\tilde{b}$ is negative, regularization must be conducted. Regularization consists of first adding an artificial variable to the tableau in Step 4. We then select which basic variable exits the basis in Step 5 and conduct a pivot operation in Step 6. Recall from the discussion in Section 2.5.1 that after this one regularization step, the new $\tilde{b}$ vector is guaranteed to be non-negative and no further regularization steps are needed.

Simplex Method Algorithm

1: **procedure** SIMPLEX METHOD
2:     Compute $\tilde{b} \leftarrow B^{-1}b$, $\tilde{N} \leftarrow -B^{-1}N$, $\tilde{c}_0 \leftarrow c_B^\top \tilde{b}$, $\tilde{c}^\top \leftarrow c_B^\top \tilde{N} + c_N^\top$
3:     **if** $\tilde{b} \ngeq 0$ **then**
4:         Add non-basic variable, $x_{N,n-m+1}$, with ones in basic-variable rows and K larger than all other $\tilde{c}$'s in objective-function row of tableau
5:         $r \leftarrow \arg\min_i\{\tilde{b}_i\}$
6:         Conduct a pivot in which $x_{N,n-m+1}$ enters the basis and $x_{B,r}$ exits
7:     **end if**
8:     **while** $\tilde{c} \ngeq 0$ **do**
9:         Select a non-basic variable, $_{N,s}$, with $\tilde{c}_s < 0$ to enter the basis
10:        Select a basic variable, $x_{B,r}$, with $r = \arg\min_{i:\tilde{N}_{i,s}<0} -\tilde{b}_i/\tilde{N}_{i,s}$ to exit the basis
11:        Conduct a pivot in which $x_{N,s}$ enters the basis and $x_{B,r}$ exits
12:     **end while**
13: **end procedure**

The main Simplex iteration takes place in Steps 8 through 12. We first determine in Step 8 whether we are currently at an optimal basic feasible solution. If the $\tilde{c}$ vector is non-negative, this means that we cannot improve the objective function by increasing the values of any of the non-basic variables. Thus, the current basic feasible solution is optimal. This means that Step 8 constitutes the termination criterion of the Simplex method—we conduct iterations until $\tilde{c} \geq 0$.

If at least one component of $\tilde{c}$ is negative, then the objective-function value can be improved by increasing the value of the corresponding non-basic variable. This means that the current basic feasible solution is not optimal. In this case, one of the non-basic variables with a negative $\tilde{c}$ coefficient is chosen to enter the basis (Step 9). In Step 10 the ratio test outlined in Equation (2.45) is conducted to determine which basic variable exits the basis. A pivot operation is then conducted to update the tableau in Step 11. After the tableau is updated we return to Step 8 to determine if the new basic feasible solution is optimal. If it is, the algorithm terminates, otherwise, the algorithm continues.

Example 2.9 Consider the standard form-version of the Electricity-Production Problem, which is introduced in Section 2.1.1 in matrix form. This matrix form is given by (2.20)–(2.22). Taking $x_B = (x_3, x_4, x_5, x_6)$ and $x_N = (x_1, x_2)$, we use the Simplex method to solve this LPP.

Using this starting partition of the variables into basic and non-basic variables, we can define:

$$c_B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$c_N = \begin{pmatrix} -1 \\ -1 \end{pmatrix},$$

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

and:

$$N = \begin{bmatrix} 2/3 & 1 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using Equations (2.24) and (2.25) we have:

$$\tilde{b} = B^{-1}b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{pmatrix} 18 \\ 8 \\ 12 \\ 16 \end{pmatrix} = \begin{pmatrix} 18 \\ -8 \\ 12 \\ 16 \end{pmatrix},$$

$$\tilde{N} = -B^{-1}N = -\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2/3 & 1 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2/3 & -1 \\ 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$\tilde{c}_0 = c_B^\top \tilde{b} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 18 \\ -8 \\ 12 \\ 16 \end{pmatrix} = 0,$$

and:

$$\tilde{c}^\top = c_B^\top \tilde{N} + c_N^\top = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \begin{bmatrix} 2/3 & 1 \\ 2 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{pmatrix} -1 & -1 \end{pmatrix} = \begin{pmatrix} -1 & -1 \end{pmatrix}.$$

The starting tableau corresponding to this basic solution is shown in Table 2.14, where the 'B' and 'N' subscripts on the basic and non-basic variables are omitted. The starting basic solution has x_1 and x_2 (the variables in the original formulation given in Section 2.1.1) equal to zero and an objective-function value of zero. This starting basic solution is infeasible, however, because $\tilde{b}_4 = -8$ is negative, meaning that $x_4 = -8$. Figure 2.10 shows the feasible region of the LPP and the starting solution, further illustrating that the starting basic solution is infeasible.

Table 2.14 Starting tableau for Example 2.9

	1	x_1	x_2
z	0	-1	-1
x_3	18	$-2/3$	-1
x_4	-8	2	1
x_5	12	-1	0
x_6	16	0	-1

Fig. 2.10 Feasible region of
the Electricity-Production
Problem in Example 2.9 and
starting basic solution
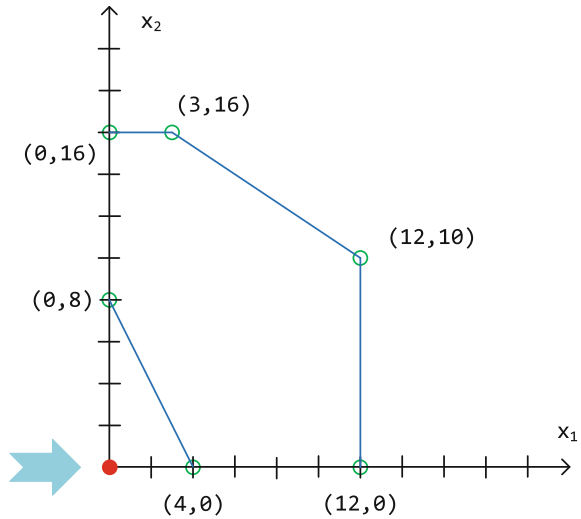
Because $\tilde{b}$ is not non-negative, a regularization step is needed at this point. To
do this, we add a non-basic variable, which we denote x_7, and assign a value of
$K = 2$, which is greater than all of the other values in the objective-function row.
This selection of $K = 2$ is arbitrary. The regularization step can be conducted with
any $K > -1$. At this point, the tableau is updated to that shown in Table 2.15. We
next conduct a pivot operation, in which x_7 enters the basis. The basic variable that
exits the basis is the one with index corresponding to:

$$\min \{\tilde{b}_3, \tilde{b}_4, \tilde{b}_5, \tilde{b}_6\} = \min \{18, -8, 12, 16\},$$

which is x_4.

Table 2.15 Tableau for
Example 2.9 after the
artificial variable, x_7, is added

	1	x_1	x_2	x_7
z	0	-1	-1	2
x_3	18	$-2/3$	-1	1
x_4	-8	2	1	1
x_5	12	-1	0	1
x_6	16	0	-1	1

Swapping x_7 and x_4 through a pivot operation gives the tableau shown in
Table 2.16. Note that after conducting this regularization step, our new basic solution
gives $(x_1, x_2) = (0, 0)$, which is infeasible in the original problem. This is consistent
with our intuition in Section 2.5.1. Adding an artificial variable to an LPP is 'cheat-
ing' in the sense that we have added a new variable to find a starting basic solution in

which all of the basic variables are non-negative. It is only after conducting Simplex iterations and (hopefully) driving the artificial variable, x_7, down to zero that we find a basic solution that is feasible in the original LPP. We will see this happen as we proceed with solving the problem.

Table 2.16 Tableau for Example 2.9 after regularization step is complete

		1	x_1	x_2	x_4
z		16	-5	-3	2
x_3		26	$-8/3$	-2	1
x_7		8	-2	-1	1
x_5		20	-3	-1	1
x_6		24	-2	-2	1

Now that we have a starting basic solution with non-negative values for the basic variables, we determine if the solution is optimal. This is done by examining the objective-function row of the tableau in Table 2.16. Seeing that both $\tilde{c}_1$ and $\tilde{c}_2$ are negative, we know that increasing either from zero will improve (decrease) the objective function. Because x_1 has a more negative objective-function coefficient, we chose x_1 to enter the basis. Note, however, that we could choose x_2 to enter the basis at the current iteration instead. The final solution that we find after finishing the Simplex method will be optimal regardless of the variable chosen to enter the basis. To determine the basic variable that exits the basis, we compute the ratios between $\tilde{b}$ and the column of negative $\tilde{N}$'s below x_1 in the tableau in Table 2.16. The basic variable to exit the basis is the one with index corresponding to:

$$\min_{\tilde{N}_{1,3}, \tilde{N}_{1,7}, \tilde{N}_{1,5}, \tilde{N}_{1,6} < 0} \left\{ -\frac{\tilde{b}_3}{\tilde{N}_{1,3}}, -\frac{\tilde{b}_7}{\tilde{N}_{1,7}}, -\frac{\tilde{b}_5}{\tilde{N}_{1,5}}, -\frac{\tilde{b}_6}{\tilde{N}_{1,6}} \right\} = \min \left\{ \frac{39}{4}, 4, \frac{20}{3}, 12 \right\},$$

which is x_7.

Thus, we conduct a pivot operation to swap x_1 and x_7, which gives the tableau shown in Table 2.17. This tableau gives the basic solution $(x_1, x_2) = (4, 0)$, which is feasible in the original problem. This can be verified by substituting these values of x_1 and x_2 into the original formulation given in Section 2.1.1. It can also be verified by observing that at this basic feasible solution we have the artificial variable x_7 equal to zero and $\tilde{b} \geq 0$. This means that the variable values satisfy the constraints of the original problem without needing the artificial variable any longer. Indeed, now that the artificial variable is equal to zero, we could drop it and its column from the tableau, as it will never again enter the basis. This is because we chose K in a way to ensure that the objective-function coefficient on the artificial variable never becomes negative again. The tableau also tells us that this basic feasible solution gives an objective function value of -4. Figure 2.11 shows the feasible region of the problem and our new basic solution after the first Simplex iteration, also illustrating that this solution is feasible.

Table 2.17 Tableau for
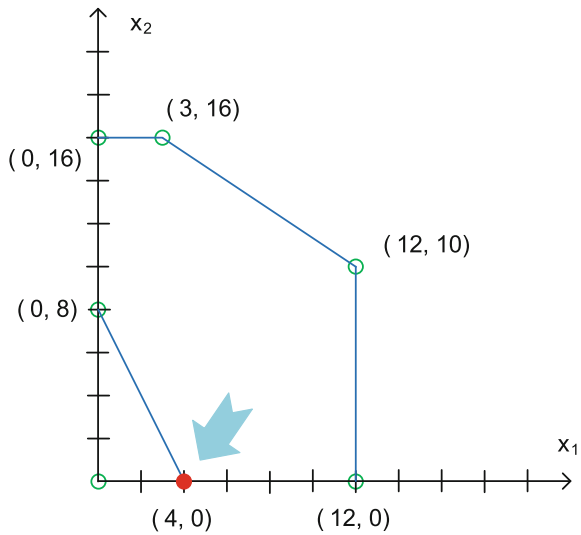Example 2.9 after completing
one Simplex iteration

	1	x_7	x_2	x_4
z	-4	5/2	$-1/2$	$-1/2$
x_3	46/3	4/3	$-2/3$	$-1/3$
x_1	4	$-1/2$	$-1/2$	1/2
x_5	8	3/2	1/2	$-1/2$
x_6	16	1	-1	0

We now proceed by conducting another Simplex iteration using the tableau in
Table 2.17. We first note that because the objective-function row of the tableau has
negative values in it, the current basic feasible solution is not optimal. Increasing
the values of either of x_2 or x_4 from zero improves the objective-function value.
Moreover, both x_2 and x_4 have the same value in the objective-function row, thus we
can arbitrarily choose either to enter the basis. We choose x_4 here. We next conduct
the ratio test to determine which basic variable exits the basis. This will be the variable
with index corresponding to:

$$\min_{\tilde{N}_{4,3}, \tilde{N}_{4,1}, \tilde{N}_{4,5}, \tilde{N}_{4,6} < 0} \left\{ -\frac{\tilde{b}_3}{\tilde{N}_{4,3}}, -\frac{\tilde{b}_1}{\tilde{N}_{4,1}}, -\frac{\tilde{b}_5}{\tilde{N}_{4,5}}, -\frac{\tilde{b}_6}{\tilde{N}_{4,6}} \right\} = \min \{46, /, 18, /\},$$

where the slashes on the right-hand side of the equality indicate values of $\tilde{N}$ that are
non-negative, and are, thus, excluded from consideration. Based on this test, x_5 is
the variable to exit the basis. We then conduct a pivot operation to update the tableau
to that shown in Table 2.18. Our new basic feasible solution has $(x_1, x_2) = (12, 0)$

Fig. 2.11 Feasible region of
the Electricity-Production
Problem in Example 2.9 and
basic feasible solution after
completing one Simplex
iteration

and gives an objective-function value of -12. Figure 2.12 shows the feasible region of the problem and the new basic feasible solution.

Table 2.18 Tableau for Example 2.9 after completing two Simplex iterations

	1	x_7	x_2	x_5
z	-12	1	-1	1
x_3	10	1/3	-1	2/3
x_1	12	1	0	-1
x_4	16	3	1	-2
x_6	16	1	-1	0

We again conduct another Simplex iteration using the tableau in Table 2.18. We note that the objective-function row is not non-negative and that increasing the value of x_2 from zero would improve the objective function. We next conduct the ratio test to determine which basic variable leaves the basis when x_2 enters it. The variable to exit the basis has index that corresponds to:

$$\min_{\tilde{N}_{2,3},\tilde{N}_{2,1},\tilde{N}_{2,4},\tilde{N}_{2,6}<0} \left\{ -\frac{\tilde{b}_3}{\tilde{N}_{2,3}}, -\frac{\tilde{b}_1}{\tilde{N}_{2,1}}, -\frac{\tilde{b}_4}{\tilde{N}_{2,3}}, -\frac{\tilde{b}_6}{\tilde{N}_{2,6}} \right\} = \min\{10, /, /, 16\},$$

which is x_3. We conduct a pivot operation to swap x_2 and x_3, which gives the updated tableau in Table 2.19. Our new basic feasible solution has $(x_1, x_2) = (12, 10)$ and an objective-function value of -22. Figure 2.13 shows the feasible region of the LPP and the new basic feasible solution found.

Fig. 2.12 Feasible region of the Electricity-Production Problem in Example 2.9 and basic feasible solution after completing two Simplex iterations
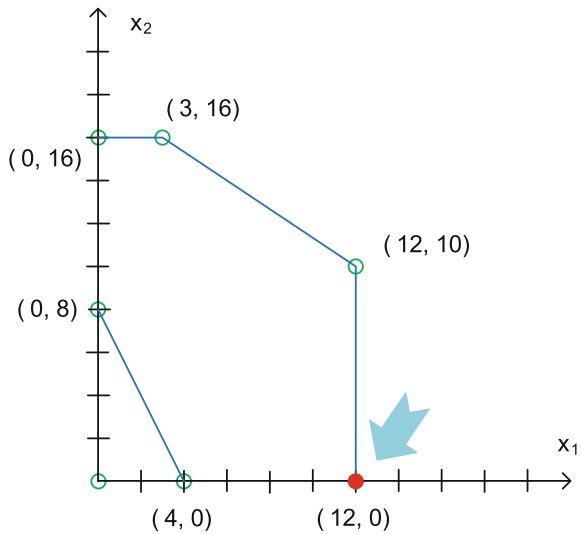
Table 2.19 Tableau for Example 2.9 after completing three Simplex iterations

		1	x_7	x_3	x_5
	z	-22	2/3	1	1/3
	x_2	10	1/3	-1	2/3
	x_1	12	1	0	-1
	x_4	26	10/3	-1	$-4/3$
	x_6	6	2/3	1	$-2/3$

If we proceed to conduct an additional Simplex iteration using the tableau in Table 2.19, we find that the Simplex method terminates. This is because we now have $\tilde{c} \geq 0$ in the objective-function row, meaning that we cannot improve on the current solution. The point $(x_1, x_2) = (12, 10)$ is the same optimal solution to this problem found in Sections 2.1.1, 2.3.1, and 2.3.2. The tableau gives an optimal objective-function value of -22. However, recall that the problem was converted from a maximization to a minimization to put it into standard form. When the objective is converted back to a maximization, the objective-function value becomes 22, which is consistent with the discussion in Sections 2.1.1, 2.3.1, and 2.3.2.

Figure 2.14 shows the sequence of points that the Simplex method goes through to get from the starting basic solution, $(x_1, x_2) = (0, 0)$, to the final optimal basic feasible solution, $(x_1, x_2) = (12, 10)$. □

Fig. 2.13 Feasible region of the Electricity-Production Problem in Example 2.9 and basic feasible solution after completing three Simplex iterations
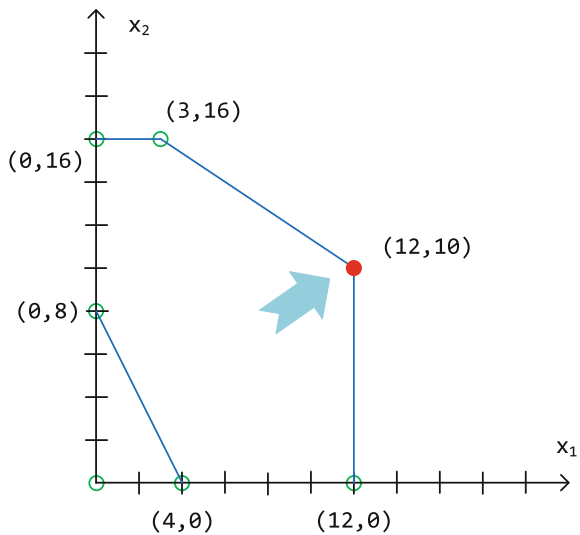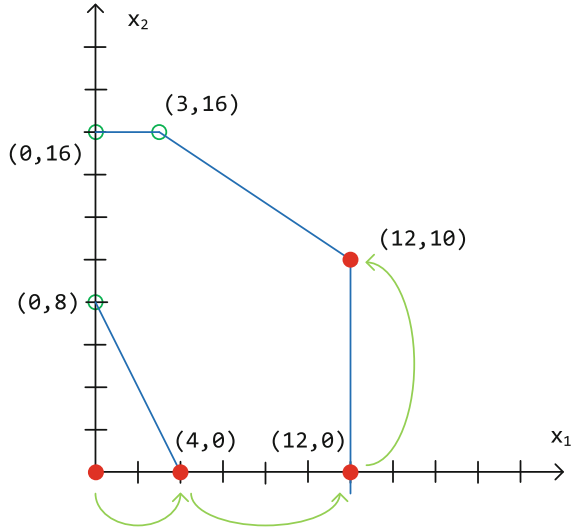
Fig. 2.14 Feasible region of the Electricity-Production Problem in Example 2.9 and the sequence of points that the Simplex method goes through

2.5.4 Convergence of the Simplex Method

The Simplex method is an iterative algorithm. As such, an important question is whether it is guaranteed to converge. That is to say, are we guaranteed to eventually find a basic feasible solution at which $\tilde{c} \geq 0$, which allows the Simplex method to terminate? If not, it is possible that we can get stuck in Steps 8 through 12 of the Simplex algorithm outlined in Section 2.5.3 without ever terminating.

To answer this question, we note that the Simplex method solves a linear optimization problem by going through extreme points of the feasible region. Because a linear optimization problem has a finite number of extreme points, this implies that the algorithm should eventually terminate. There is one added wrinkle to this, however. Certain problems can have multiple basic feasible solutions corresponding to a single extreme point. These occur because of what is called **degeneracy**. A degenerate basic solution is one in which one or more basic variables take on a value of zero when we solve for them using the structural equality constraints. Degenerate basic solutions normally arise because there are extra redundant constraints at an extreme point of the feasible region.

The difficulty that degeneracy raises is that the Simplex method may get 'stuck' at an extreme point by cycling through the same set of basic feasible solutions corresponding to that extreme point without ever moving. There is, however, a very easy way to ensure that the Simplex method does not get stuck at a degenerate extreme point. This is done by choosing the variable that enters the basis at each Simplex iterations based on their index number. That is to say, if both $\tilde{c}_i$ and $\tilde{c}_j$ are negative at a given Simplex iteration, then choose whichever has the smaller index (*i.e.*, the smaller of i or j) to be the variable entering the basis. One can show that using this

rule to select the entering variable guarantees that the Simplex method leaves every degenerate extreme point after a finite number of Simplex iterations [9].

In practice, the Simplex method is not applied this way. Rather, the entering variable is chosen on the basis of which one has the most negative $\tilde{c}$. If the Simplex method spends multiple iterations at the same extreme point (suggesting a degeneracy problem), then the selection rule based on the index number is used instead until the Simplex method moves to a different extreme point.

2.5.5 Detecting Infeasible Linear Optimization Problems

The Simplex method detects that a linear optimization problem is infeasible based on the final value, after the Simplex algorithm terminates, of the artificial variable added in the regularization step. If the Simplex method terminates (*i.e.*, if $\tilde{c} \geq 0$) and the artificial variable has a non-zero value, this means that the starting linear optimization problem is infeasible.

To understand why, first note that if a linear optimization problem is infeasible, then the regularization step must be done at the beginning of the Simplex method. This is because for any starting partition of the variables into basic and non-basic variables, the resulting basic solution must be infeasible. Otherwise, if we can find a basic solution that is feasible then the linear optimization problem cannot be infeasible.

Next, recall that when the artificial variable is added in the regularization step, a comparatively high value of K is put in the objective-function row of the tableau. The purpose of K, as discussed in Section 2.5.1, is to make the objective function (which we seek to minimize) larger if the artificial variable takes on a positive value. Indeed, by making K larger than all of the other values in the objective-function row of the tableau, the cost on the artificial variable is higher than all of the other variables and the Simplex method seeks to make the artificial variable as small as possible. If the Simplex method terminates but the artificial variable is still positive, that means it is impossible to satisfy the constraints of the original problem without having the artificial variable allow for constraint violations. Thus, the original problem must be infeasible. Otherwise, if the Simplex method terminates and the artificial variable is equal to zero, the original problem is feasible.

2.5.6 Detecting Unbounded Linear Optimization Problems

The Simplex method detects that a linear optimization problem is unbounded through the ratio test conducted in Step 10 of the Simplex Method Algorithm, which is outlined in Section 2.5.3. Recall that the purpose of the ratio test is to determine how large the non-basic variable entering the basis can be made before causing one of the basic variables to become negative. If there is no such restriction, then the Simplex method would make the entering variable infinitely large because the negative $\tilde{c}$ value in the objective-function row of the tableau means that doing so would make the objective function go to $-\infty$.

We know that if at least one of the $\tilde{N}$ values in the column underneath the entering variable in the tableau is negative, then the ratio test gives a limit on how much the entering variable can increase. Otherwise, if all of the $\tilde{N}$ values are non-negative, then the ratio test allows the entering variable to become as large as possible. Thus, if at any point in the Simplex method there is an entering variable without any negative $\tilde{N}$ values in the tableau, the problem is known to be unbounded.

2.5.7 Detecting Multiple Optima

The Simplex method detects multiple optimal solutions based on the objective-function row of the final tableau. If all of the values in the objective-function row of the final tableau are strictly positive, this means that the optimal solution found is a unique optimal solution. Otherwise, if there are any zero values in the objective-function row of the final tableau, this means that there are multiple optimal solutions. The reason for this is that a zero in the objective-function row of the final tableau means that a non-basic variable can enter the basis without changing the objective-function value at all. Thus, there are additional optimal solutions in which non-basic variables with a zero in the objective-function row take on positive values (keeping in mind that the basic-variable values would have to be recomputed).

2.6 Sensitivity Analysis

The subject of **sensitivity analysis** answers the question of what effect changing a linear optimization problem has on the resulting optimal solution. Of course, one way to answer this question is to change a given problem, solve the new problem, and examine any resulting changes in the solution. This can be quite cumbersome and time-consuming, however. A large-scale linear optimization problem with millions of variables could take several hours to solve. Having to re-solve multiple versions of the same basic problem with different data can be impractical. Sensitivity analysis answers this question by using information from the optimal solution and the final tableau after applying the Simplex method.

Throughout this discussion we assume that we have a linear optimization problem that is already in standard form, which can be generically written as:

$$\min_{x} c^{\top} x \tag{2.46}$$

$$\text{s.t. } Ax = b \tag{2.47}$$

$$x \geq 0. \tag{2.48}$$

We also assume that we have solved the problem using the Simplex method and have an optimal set of decision-variable values, x^*. More specifically, we assume that we

have an optimal basic feasible solution, meaning that x^* is partitioned into basic variables, x_B^*, and non-basic variables, $x_N^* = 0$. We, thus, also have the objective-function-coefficient vector partitioned into c_B and c_N and the A matrix partitioned into submatrices, B and N.

We use sensitivity analysis to examine the effect of three different types of changes: (i) changing the constants on the right-hand sides of the constraints (*i.e.*, the b vector in the structural equality constraints), (ii) changing the objective-function coefficients (*i.e.*, the c vector), and (iii) changing the coefficients on the left-hand sides of the constraints (*i.e.*, the A matrix).

2.6.1   Changing the Right-Hand Sides of the Constraints

We begin by considering the case in which the b vector is changed. To do this, recall that once we solve the original LPP, given by (2.46)–(2.48), the basic variable values are given by:

$$x_B^* = \tilde{b} = B^{-1}b.$$

Let us next examine the effect of changing the structural equality constraints from (2.47) to:

$$Ax = b + \Delta b,$$

on the basic feasible solution, x^*, that is optimal in the original problem. From (2.24) and (2.25) we have:

$$\hat{b} = B^{-1}(b + \Delta b),$$

$$\hat{N} = -B^{-1}N,$$

$$\hat{c}_0 = c_B^\top \hat{b},$$

and:

$$\hat{c}^\top = c_B^\top \hat{N} + c_N^\top,$$

where $\hat{b}$, $\hat{N}$, $\hat{c}_0$, and $\hat{c}$ denote the new values of the terms in the final tableau after the right-hand sides of the structural equality constraints are changed. Note that changing the right-hand sides of the structural equality constraints only changes the values of $\hat{b}$ and $\hat{c}_0$ in the final tableau. We have that $\hat{N} = \tilde{N}$ and $\hat{c} = \tilde{c}$.

From this observation we can draw the following important insight. If $\hat{b} \geq 0$, then the partition of x^* into basic and non-basic variables is still feasible when the equality constraints are changed. Moreover, the values in the objective-function row of the final tableau, $\hat{c} = \tilde{c}$, are not affected by the change in the right-hand side of the equality constraints. This means that if $\hat{b} \geq 0$ the partition of x^* into basic and

non-basic variables is not only feasible but also optimal after the equality constraints are changed.

We can use this insight to first determine how much the right-hand side of the equality constraints can be changed before the partition of x^* into basic and non-basic variables becomes infeasible. We determine this bound from the requirement that $\hat{b} \geq 0$ as follows:

$$\hat{b} \geq 0$$
$$B^{-1} \cdot (b + \Delta b) \geq 0$$
$$B^{-1} b \geq -B^{-1} \Delta b$$
$$\tilde{b} \geq -B^{-1} \Delta b. \tag{2.49}$$

If Δb satisfies (2.49), then the optimal basis remains unchanged by the changes in the right-hand side of the equality constraints. Although the basis remains the same if (2.49) is satisfied, the values of the basic variables change. Specifically, we can compute the new values of the basic variables, $\hat{x}_B$, as:

$$\hat{x}_B = \hat{b} = B^{-1} \cdot (b + \Delta b) = B^{-1} b + B^{-1} \Delta b = \tilde{b} + B^{-1} \Delta b. \tag{2.50}$$

Equation (2.50) gives us an exact expression for how much the values of the basic variables change as a result of changing the right-hand sides of the equalities. Specifically, this change is $B^{-1} \Delta b$.

The next question is how much of an effect these changes in the values of the basic variables have on the objective-function value. From (2.25) we can write the objective-function value as:

$$z = c_B^\top \hat{x}_B + c_N^\top \hat{x}_N.$$

where $\hat{x}_N$ are the new non-basic variable values. We know, however, that the non-basic variables will still equal zero after the right-hand sides of the equality constraints are changed. Thus, using (2.50) we can write the objective-function value as:

$$z = c_B^\top \cdot (\tilde{b} + B^{-1} \Delta b)$$
$$= c_B^\top \tilde{b} + c_B^\top B^{-1} \Delta b$$
$$= z^* + \lambda^\top \Delta b, \tag{2.51}$$

where we define $\lambda^\top = c_B^\top B^{-1}$. The vector λ, which is called the **sensitivity vector**, gives the change in the optimal objective-function value of an LPP resulting from a sufficiently small change in the right-hand side of a structural equality constraint.

If Δb does not satisfy (2.49), then the basis that is optimal for the original LPP is no longer feasible after the constraints are changed. In such a case, one cannot directly compute the effect of changing the constraints on the solution. Rather, additional Simplex iterations must be conducted to find a new basis that is feasible and optimal. Note that when conducting the additional Simplex iterations, one can start with the

final basis from solving the original LPP and conduct a regularization step to begin the additional Simplex iterations. Doing so can often decrease the number of Simplex iterations that must be conducted.

Example 2.10 Consider the standard form-version of the Electricity-Production Problem, which is introduced in Section 2.1.1. In Example 2.9 we find that the optimal solution has basis $x_B = (x_2, x_1, x_4, x_6)$ and $x_N = (x_3, x_5)$. We are excluding x_7 from the vector of non-basic variables, because this is an artificial variable added in the regularization step. However, one can list x_7 as a non-basic variable without affecting any of the results in this example. The optimal solution also has:

$$B = \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

$$\tilde{b} = \begin{pmatrix} 10 \\ 12 \\ 26 \\ 6 \end{pmatrix},$$

and:

$$c_B = \begin{pmatrix} -1 & -1 & 0 & 0 \end{pmatrix}.$$

We can compute the sensitivity vector as:

$$\lambda^\top = c_B^\top B^{-1} = \begin{pmatrix} -1 & 0 & -1/3 & 0 \end{pmatrix}.$$

Suppose that the structural equality constraints of the problem are changed from:

$$\begin{bmatrix} 2/3 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \\ 12 \\ 16 \end{pmatrix},$$

to:

$$\begin{bmatrix} 2/3 & 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 18 \\ 8 \\ 12 \\ 16 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix}.$$

We first determine whether the basis that is optimal in the original problem is still feasible after the constraints are changed. From (2.49) we know that the basis is still feasible if:

$$\tilde{b} \geq -B^{-1}\Delta b,$$

or if:

$$\begin{pmatrix} 10 \\ 12 \\ 26 \\ 6 \end{pmatrix} \geq - \begin{bmatrix} 1 & 2/3 & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{pmatrix} 2 \\ -1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -3 \\ 2 \end{pmatrix}.$$

Seeing that it is, we can next determine the effect of the change in the constraints on the optimal objective-function value using the sensitivity vector as:

$$\lambda^{\top}\Delta b = -2.$$

$\square$

2.6.2 Changing the Objective-Function Coefficients

We next consider the case in which the c vector is changed. More specifically, let us suppose that the objective function changes from (2.46) to:

$$(c + \Delta c)^{\top}x.$$

We analyze the effect of this change following the same line of reasoning used to analyze changes in the b vector in Section 2.6.1. That is to say, we examine what effect it has on x^*, the optimal basic feasible solution of the original problem. From (2.24) and (2.25) we have that:

$$\hat{b} = B^{-1}b,$$

$$\hat{N} = -B^{-1}N,$$

$$\hat{c}_0 = (c_B + \Delta c_B)^{\top}\hat{b},$$

and:

$$\hat{c}^{\top} = (c_B + \Delta c_B)^{\top}\hat{N} + (c_N + \Delta c_N)^{\top},$$

where $\hat{b}$, $\hat{N}$, $\hat{c}_0$, and $\hat{c}$ denote the new values of the terms in the final tableau after the c vector is changed and Δc_B and Δc_N partition the Δc vector according to the final partition of basic and non-basic variables. Note that changing the c vector only changes the values of $\hat{c}_0$ and $\hat{c}$ and that we have $\hat{b} = \tilde{b}$ and $\hat{N} = \tilde{N}$.

From this observation we can conclude that when the c vector is changed, the optimal basic feasible solution of the original problem, x^*, is still feasible in the new

problem. The only question is whether this basic feasible solution remains optimal after the objective function is changed. We know that x^* remains optimal if $\hat{c} \geq 0$. From this, we can derive the following bound:

$$
\begin{aligned}
\hat{c} &\geq 0 \\
(c_B + \Delta c_B)^\top \hat{N} + (c_N + \Delta c_N)^\top &\geq 0 \\
(c_B + \Delta c_B)^\top \tilde{N} + (c_N + \Delta c_N)^\top &\geq 0 \\
c_B^\top \tilde{N} + c_N^\top + \Delta c_B^\top \tilde{N} + \Delta c_N^\top &\geq 0 \\
\tilde{c}^\top + \Delta c_B^\top \tilde{N} + \Delta c_N^\top &\geq 0 \\
\Delta c_B^\top \tilde{N} + \Delta c_N^\top &\geq -\tilde{c}^\top,
\end{aligned}
\tag{2.52}
$$

on how much c can change and have x^* remain an optimal basic feasible solution. If (2.52) is satisfied, then x^* remains an optimal basic feasible solution. Of course, the objective-function value changes as a result of the c vector changing. However, it is straightforward to compute the new objective-function value as:

$$
\begin{aligned}
z &= (c_B + \Delta c_B)^\top x_B^* + (c_N + \Delta c_N)^\top x_N^* \\
&= (c_B + \Delta c_B)^\top x_B^*,
\end{aligned}
$$

because $x_N^* = 0$. This then simplifies to:

$$
z = z^* + \Delta c_B^\top x_B^*.
\tag{2.53}
$$

Thus, if Δc satisfies (2.52), the impact on the optimal objective-function value of changing the objective-function coefficient of a basic variable is equal to the basic variable's optimal value itself. Changing the objective-function value of a non-basic variable has no impact on the optimal objective-function value.

On the other hand, if (2.52) is not satisfied, then x^* is a basic feasible solution of the changed LPP but is not optimal in the new problem. As such, additional Simplex iterations would have to be conducted to find a new optimal basic feasible solution.

Example 2.11 Consider the standard form-version of the Electricity-Production Problem, which is introduced in Section 2.1.1. In Example 2.9 we find that:

$$
\tilde{c} = \begin{pmatrix} 1 \\ 5/3 \end{pmatrix},
$$

and:

$$
\tilde{N} = \begin{bmatrix} -1 & -2/3 \\ 0 & -1 \\ -1 & -8/3 \\ 1 & 2/3 \end{bmatrix}.
$$

Suppose we wish to change c_1 from -1 to -0.9. From (2.52) we know that x^* will remain an optimal solution so long as:

$$\Delta c_B^\top \tilde{N} + \Delta c_N^\top \geq -\tilde{c}^\top,$$

or so long as:

$$\begin{pmatrix} 0 & 0.1 & 0 & 0 \end{pmatrix} \begin{bmatrix} -1 & -2/3 \\ 0 & -1 \\ -1 & -8/3 \\ 1 & 2/3 \end{bmatrix} \geq - \begin{pmatrix} 1 & 5/3 \end{pmatrix},$$

or so long as:

$$\begin{pmatrix} 0 & -0.1 \end{pmatrix} \geq \begin{pmatrix} -1 & -5/3 \end{pmatrix},$$

which holds true. Thus, based on (2.53) we know that the new objective-function value after this change is given by:

$$\Delta c_B^\top x_B^* = \begin{pmatrix} 0 & 0.1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 10 \\ 12 \\ 26 \\ 6 \end{pmatrix} = 1.2.$$

$\square$

2.6.3 Changing the Left-Hand-Side Coefficients of the Constraints

We finally examine the effect of changing the coefficients multiplying the variables on the left-hand side of the structural equality constraints. More specifically, we examine the effect of changing (2.47) to:

$$(A + \Delta A)x = b.$$

We begin by partitioning ΔA into:

$$\Delta A = \begin{bmatrix} \Delta B & \Delta N \end{bmatrix}.$$

Using this and (2.24) and (2.25) we have that:

$$\hat{b} = (B + \Delta B)^{-1} b,$$

$$\hat{N} = -(B + \Delta B)^{-1}(N + \Delta N),$$

$$\hat{c}_0 = c_B^\top \hat{b},$$

and:

$$\hat{c}^\top = c_B^\top \hat{N} + c_N^\top,$$

where $\hat{b}$, $\hat{N}$, $\hat{c}_0$, and $\hat{c}$ denote the new values of the terms in the final tableau after the A matrix is changed.

We can draw two conclusions from these expressions. The first is that if ΔA changes values in the B matrix, then all of $\hat{b}$, $\hat{N}$, $\hat{c}_0$, and $\hat{c}$ are changed in the final tableau. Moreover, it is not straightforward to derive a bound on how large or small ΔB can be before x^* is no longer a feasible or optimal basic solution. This is because ΔB appears in matrix inversions in the expressions giving $\hat{b}$ and $\hat{c}$. Thus, we do not provide any such bounds on ΔB (because they are difficult to derive and work with).

We can, however, derive such bounds in the case in which ΔA changes values in the N matrix *only*. To do so, we note that if $\Delta B = 0$, then from (2.24) and (2.25) we have:

$$\hat{b} = B^{-1}b,$$

$$\hat{N} = -B^{-1} \cdot (N + \Delta N),$$

$$\hat{c}_0 = c_B^\top \hat{b},$$

and:

$$\hat{c}^\top = c_B^\top \hat{N} + c_N^\top.$$

Thus, we see that if the N matrix is changed, this only changes the values of $\hat{N}$ and $\hat{c}$ and that we have $\hat{b} = \tilde{b}$ and $\hat{c}_0 = \tilde{c}_0$. Thus, we know that x^* remains a basic feasible solution after the N matrix is changed. The only question is whether x^* remains an optimal basic feasible solution. We can derive the bound:

$$
\begin{aligned}
\hat{c} &\geq 0 \\
-c_B^\top B^{-1} \cdot (N + \Delta N) + c_N^\top &\geq 0 \\
-c_B^\top B^{-1} N - c_B^\top B^{-1} \Delta N + c_N^\top &\geq 0 \\
c_B^\top \tilde{N} + c_N^\top - c_B^\top B^{-1} \Delta N &\geq 0 \\
\tilde{c}^\top - c_B^\top B^{-1} \Delta N &\geq 0 \\
\tilde{c}^\top &\geq c_B^\top B^{-1} \Delta N, \quad\quad (2.54)
\end{aligned}
$$

on ΔN, which ensures that x^* remains optimal. If (2.54) is satisfied, then x^* remains an optimal basic feasible solution. If so, we can determine the new objective-function value from (2.25) as:

$$
\begin{aligned}
z &= \hat{c}_0 + \hat{c}^\top x_N^* \\
&= \tilde{c}_0, \quad\quad (2.55)
\end{aligned}
$$

because we have that $\hat{c}_0 = \tilde{c}_0$ and $x_N^* = 0$. Thus, if (2.54) is satisfied, there is no change in the optimal objective-function value. Otherwise, if ΔN does not satisfy (2.54), then x^* is no longer an optimal basic feasible solution and additional Simplex iterations must be conducted to find a new optimal basic solution.

We finally conclude this discussion by noting that although we cannot derive a bound on the allowable change in the B matrix, we can approximate changes in the optimal objective-function value from changing the B matrix. We do this by supposing that one element of the A matrix, $A_{i,j}$, is changed to $A_{i,j} + \Delta A_{i,j}$. When this change is made, the ith structural equality constraint changes from:

$$A_{i,1}x_1^* + \cdots + A_{i,j-1}x_{j-1}^* + A_{i,j}x_j^* + A_{i,j+1}x_{j+1}^* + \cdots + A_{i,n}x_n^* = b_i,$$

to:

$$A_{i,1}x_1^* + \cdots + A_{i,j-1}x_{j-1}^* + (A_{i,j} + \Delta A_{i,j})x_j^* + A_{i,j+1}x_{j+1}^* + \cdots + A_{i,n}x_n^* = b_i.$$

The changed ith constraint can be rewritten as:

$$A_{i,1}x_1^* + \cdots + A_{i,j-1}x_{j-1}^* + A_{i,j}x_j^* + A_{i,j+1}x_{j+1}^* + \cdots + A_{i,n}x_n^* = b_i - \Delta A_{i,j}x_j^*.$$

Thus, one can view changing the A matrix as changing the right-hand side of the ith constraint by $-\Delta A_{i,j}x_j^*$. One can then use the results of Section 2.6.1 to estimate the change in the optimal objective-function value as:

$$\Delta z^* \approx -\lambda_i \Delta A_{i,j} x_j^*, \tag{2.56}$$

where λ_i is the ith component of the sensitivity vector derived in (2.51). Note that if x_j is a non-basic variable, then we have that the change in the optimal objective-function value is:

$$-\lambda_i \Delta A_{i,j} x_j^* = 0,$$

which is what is found in analyzing the case of changes in the N matrix in (2.55). However, Equation (2.56) can also be applied to approximate the change in the optimal objective-function value when changes are made to the B matrix. It is important to note, however, that (2.56) only approximates such changes (whereas all of the other sensitivity expressions are exact).

Example 2.12 Consider the standard form-version of the Electricity-Production Problem, which is introduced in Section 2.1.1, and suppose that $A_{1,2}$ changes from 1 to 1.01. Because this is making a change to a coefficient in the B matrix, we cannot exactly estimate the effect of this change on the objective function. We can, however, approximate this using (2.56) as:

$$-\lambda_1 \Delta A_{1,2} x_2^* = -1 \cdot (-0.01) \cdot 10 = 0.1.$$

To see that this is only an approximation, we can solve the problem with $A_{1,2} = 1.01$, which gives an optimal objective-function value of -21.900990. The exact change in the objective-function value is thus:

$$-21.900990 - (-22) = 0.099010 \approx 0.1.$$

$\square$

2.7 Duality Theory

Every linear optimization problem has an associated optimization problem called its **dual problem**. We show in this section that the starting linear optimization problem, which in the context of duality theory is called the **primal problem**, and its associated dual problem have some very important relationships. These relationships include the underlying structure of the primal and dual problems (*i.e.*, the objective function, constraints, and variables of the two problems) and properties of solutions of the two problems.

2.7.1 Symmetric Duals

We begin by looking at the simple case of a primal problem that is in canonical form. Recall from Section 2.2.2.2 that a generic primal problem in canonical form can be written compactly as:

$$\min_{x} z_P = c^\top x \tag{2.57}$$

$$\text{s.t. } Ax \geq b \tag{2.58}$$

$$x \geq 0. \tag{2.59}$$

The dual problem associated with this primal problem is:

$$\max_{y} z_D = b^\top y \tag{2.60}$$

$$\text{s.t. } A^\top y \leq c \tag{2.61}$$

$$y \geq 0. \tag{2.62}$$

As indicated above, we can note some relationships in the structures of the primal and dual problems. The vector of objective-function coefficients in the primal problem, c, is the vector of the right-hand side constants in the structural constraints in the dual problem. The vector of the right-hand side constants in the structural constraints in the primal problem, b, is the vector of objective-function coefficients in the dual problem. We also see that the coefficients multiplying the variables in the

left-hand sides of the constraints (which are in the A matrix) are the same in the dual problem as in the primal, except that the coefficient matrix is transposed in the dual. We finally note that whereas the primal problem is a minimization, the dual problem is a maximization.

We can also observe a one-to-one relationship between the variables and constraints in the primal and dual problems. Recall that for a generic primal problem in canonical form, A is an $m \times n$ matrix. This means that the primal problem has m structural constraints and n variables (*i.e.*, $x \in \mathbb{R}^n$). Because $A^\top$ is an $n \times m$ matrix, this means that the dual problem has n structural constraints and m variables (*i.e.*, $y \in \mathbb{R}^m$). Thus, we can say that each primal constraint has an associated dual variable and each primal variable has an associated dual constraint.

We can show a further symmetry between the primal and dual problems. By this we mean that each dual constraint has an associated primal variable and each dual variable has an associated primal constraint. This is because if we start with a primal problem in canonical form and find the dual of its dual problem, we get back the original primal problem. To see this, we note from the discussion above that if we have the primal problem:

$$\min_x c^\top x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0,$$

its dual problem is:

$$\max_y b^\top y$$
$$\text{s.t. } A^\top y \leq c$$
$$y \geq 0.$$

If we want to find the dual of the dual problem, we must first convert it to canonical form, which is:

$$\min_y -b^\top y$$
$$\text{s.t. } -A^\top y \geq -c$$
$$y \geq 0.$$

The dual of this problem is:

$$\max_w -c^\top w$$
$$\text{s.t. } (-A^\top)^\top w \leq -b$$
$$w \geq 0,$$

where we let w be a vector of variables. This problem can be rewritten as:

$$\min_{w} c^{\top} w$$
$$\text{s.t. } Aw \geq b$$
$$w \geq 0,$$

which is identical to the starting problem, except that the variables are now called w instead of x.

Example 2.13 Consider the Electricity-Production Problem, which is introduced in Section 2.1.1. In Example 2.4 we show the canonical form of this problem to be:

$$\min_{x} z_P = 5x_{1,1} + 4x_{1,2} + 3x_{2,1} + 6x_{2,2}$$
$$\text{s.t. } -x_{1,1} - x_{1,2} \geq -7$$
$$-x_{2,1} - x_{2,2} \geq -12$$
$$-x_{1,1} - x_{2,1} \geq -10$$
$$x_{1,1} + x_{2,1} \geq 10$$
$$-x_{1,2} - x_{2,2} \geq -8$$
$$x_{1,2} + x_{2,2} \geq 8$$
$$x_{i,j} \geq 0, \forall\, i = 1, 2;\, j = 1, 2.$$

This can be written as:

$$\min_{x} z_P = c^{\top} x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0,$$

where:

$$x = \begin{pmatrix} x_{1,1} \\ x_{1,2} \\ x_{2,1} \\ x_{2,2} \end{pmatrix},$$

$$c = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 6 \end{pmatrix},$$

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

and:

$$b = \begin{pmatrix} -7 \\ -12 \\ -10 \\ 10 \\ -8 \\ 8 \end{pmatrix}.$$

The dual problem is:

$$\max_{y} z_D = b^\top y$$
$$\text{s.t. } A^\top y \le c$$
$$y \ge 0,$$

which can be expanded out to:

$$\max_{y} z_D = \begin{pmatrix} -7 & -12 & -10 & 10 & -8 & 8 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix}$$

$$\text{s.t. } \begin{bmatrix} -1 & 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} \le \begin{pmatrix} 5 \\ 4 \\ 3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{pmatrix} \ge \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

or, by simplifying, to:

$$\max_{y} z_D = -7y_1 - 12y_2 - 10y_3 + 10y_4 - 8y_5 + 8y_6$$

$$\text{s.t.} \quad -y_1 - y_3 + y_4 \leq 5$$
$$-y_1 - y_5 + y_6 \leq 4$$
$$-y_2 - y_3 + y_4 \leq 3$$
$$-y_2 - y_5 \leq 6$$
$$y_1, y_2, y_3, y_4, y_5, y_6 \geq 0.$$

$\square$

2.7.2 Asymmetrical Duals

Now consider a generic LPP in standard form:

$$\min_{x} z_P = c^\top x$$
$$\text{s.t.} \quad Ax = b$$
$$x \geq 0.$$

To find the dual of this problem, we can convert it to canonical form, by replacing the structural equality constraints with the pair of inequalities:

$$\min_{x} z_P = c^\top x$$
$$\text{s.t.} \quad Ax \geq b$$
$$Ax \leq b$$
$$x \geq 0,$$

and then converting the less-than-or-equal-to constraint into a greater-than-or-equal-to constraint:

$$\min_{x} z_P = c^\top x$$
$$\text{s.t.} \quad Ax \geq b$$
$$-Ax \geq -b$$
$$x \geq 0.$$

If we let u and v denote the dual variables associated with the two structural inequality constraints, the dual of this problem is then:

$$\max_{u,v} z_D = b^\top u - b^\top v$$

$$\text{s.t. } A^\top u - A^\top v \leq c$$

$$u, v \geq 0.$$

If we define y as the difference between the dual-variable vectors:

$$y = u - v,$$

then the dual can be written as:

$$\max_{y} z_D = b^\top y$$

$$\text{s.t. } A^\top y \leq c,$$

where there is no sign restriction on y because it is the difference of two non-negative vectors. This analysis of a primal problem with equality constraints gives an important conclusion, which is that the dual variable associated with an equality constraint has no sign restriction.

2.7.3 Duality Conversion Rules

Example 2.13 demonstrates a straightforward, but often tedious way of finding the dual of any linear optimization problem. This method is to first convert the problem into canonical form, using the rules outlined in Section 2.2.2.2. Then, one can use the symmetric dual described in Section 2.7.1 to find the dual of the starting primal problem. While straightforward, this is cumbersome, as it requires the added step of first converting the primal problem to canonical form. Here we outline a standard set of rules that allow us to directly find a dual problem without first converting the primal problem to canonical form. We demonstrate these rules using the problem introduced in Example 2.1, which is:

$$\max_{x} \ 3x_1 + 5x_2 - 3x_3 + 1.3x_4 - x_5$$

$$\text{s.t. } x_1 + x_2 - 4x_4 \leq 10$$

$$x_2 - 0.5x_3 + x_5 = -1$$

$$x_3 \geq 5$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_4 \leq 0.$$

We then demonstrate that the dual we find using the rules outlined is equivalent to the dual we would obtain if we first convert the primal problem to canonical form and use the symmetric dual introduced in Section 2.7.1.

2.7.3.1   Step 1: Determine Number of Dual Variables

The first step is to determine the number of dual variables, which is based on the number of structural constraints in the primal problem. Our example problem has three structural constraints, which are:

$$x_1 + x_2 - 4x_4 \leq 10$$

$$x_2 - 0.5x_3 + x_5 = -1$$

$$x_3 \geq 5.$$

The three other constraints:

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_4 \leq 0,$$

are not structural constraints but are rather non-negativity and non-positivity restrictions. These are handled differently than structural constraints when writing the dual problem. Because there are three structural constraints, there will be three dual variables, which we refer to as y_1, y_2, and y_3. To help in the subsequent steps of writing the dual problem, we write each dual variable next to its associated primal constraint as follows:

$$\begin{aligned}
\max_{x} \quad & 3x_1 + 5x_2 - 3x_3 + 1.3x_4 - x_5 \\
\text{s.t.} \quad & x_1 + x_2 - 4x_4 \leq 10 & (y_1) \\
& x_2 - 0.5x_3 + x_5 = -1 & (y_2) \\
& x_3 \geq 5 & (y_3) \\
& x_1 \geq 0 \\
& x_2 \geq 0 \\
& x_4 \leq 0.
\end{aligned}$$

2.7.3.2   Step 2: Determine Objective of the Dual Problem

The next step is to determine the objective function of the dual problem. First, the direction of the optimization of the dual problem is always opposite to that of the primal problem. Because the primal problem in our example is a maximization, the dual will be a minimization. Next, to determine the objective function itself, we multiply each dual variable with the constant on the right-hand side of its associated primal constraint and sum the products. Thus, in our example we multiply y_1 by 10, y_2 by -1 and y_3 by 5 giving:

$$\min_{y}\ 10y_1 - y_2 + 5y_3.$$

2.7.3.3   Step 3: Determine Number of Structural Constraints in the Dual Problem

There is one structural constraint in the dual problem for each primal variable. Moreover, there is a one-to-one correspondence between dual constraints and primal variables. Thus, just as we associate dual variables with primal constraints in Step 1, we associate primal variables with dual constraints here. Our example problem has five primal variables. Thus, the dual problem will have five structural constraints, and we can associate the constraints and variables as follows:

$$\min_{y}\ 10y_1 - y_2 + 5y_3$$

$$\text{s.t.} \hspace{8cm} (x_1)$$
$$(x_2)$$
$$(x_3)$$
$$(x_4)$$
$$(x_5)$$

2.7.3.4   Step 4: Determine Right-Hand Sides of Structural Constraints in the Dual Problem

We next determine the right-hand side of each structural constraint in the dual problem. These are given by the coefficient multiplying the associated primal variable in the objective function of the primal problem. In our example, the five structural constraints in the dual problem correspond to x_1, x_2, x_3, x_4, and x_5, respectively, which have objective-function coefficients in the primal problem of 3, 5, -3, 1.3, and -1, respectively. Thus, these will be the right-hand sides of the associated structural constraints in the dual:

$$\min_{y} \; 10y_1 - y_2 + 5y_3$$

$$
\begin{array}{llr}
\text{s.t.} & 3 & (x_1) \\
& \quad\quad 5 & (x_2) \\
& \quad\quad\quad -3 & (x_3) \\
& \quad\quad 1.3 & (x_4) \\
& \quad\quad -1. & (x_5)
\end{array}
$$

2.7.3.5 Step 5: Determine Left-Hand Sides of Structural Constraints in the Dual Problem

The left-hand sides of the structural constraints in the dual problem have the dual variables multiplied by the transpose of the coefficient matrix defining the structural constraints in the primal problem. The result of this is that the coefficient multiplying the primal variable associated with each structural dual constraint is multiplied by the dual variable associated with the structural primal constraint.

To illustrate this rule, take the first structural dual constraint, which is associated with x_1. If we examine the structural constraints in the primal problem, we see that x_1 has coefficients of 1, 0, and 0 in each of the structural constraints in the primal problem. Moreover, these three primal constraints are associated with dual variables y_1, y_2, and y_3. Multiplying these coefficients with the associated dual variables and summing the products gives:

$$1y_1 + 0y_2 + 0y_3,$$

as the left-hand side of the first structural constraint in the dual problem. Next, take the second structural dual constraint, which is associated with x_2. The primal variable x_2 has coefficients 1, 1, and 0 in the three structural constraints of the primal, which are associated with dual variables y_1, y_2, and y_3. Thus, the left-hand side of the second structural constraint of the dual problem is:

$$1y_1 + 1y_2 + 0y_3.$$

Repeating this process three more times gives the following left-hand sides of the structural constraints in the dual problem:

$$\min_{y} \; 10y_1 - y_2 + 5y_3$$

$$
\begin{array}{llr}
\text{s.t.} & y_1 \quad 3 & (x_1) \\
& y_1 + y_2 \quad 5 & (x_2) \\
& -0.5y_2 + y_3 \quad -3 & (x_3) \\
& -4y_1 \quad 1.3 & (x_4) \\
& y_2 \quad -1. & (x_5)
\end{array}
$$

2.7.3.6   Step 6: Determine the Types of Structural Constraints in the Dual Problem

The types of structural constraints in the dual problem (*i.e.*, greater-than-or-equal-to, less-than-or-equal-to, or equality constraints) is determined by the sign restrictions on the associated primal variables. To determine the type of each constraint, we refer to the symmetric duals introduced in Section 2.7.1.

Recall that the canonical primal:

$$\min_{x} c^\top x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0,$$

has:

$$\max_{y} b^\top y$$
$$\text{s.t. } A^\top y \leq c$$
$$y \geq 0,$$

as its dual. Recall, also, that there is a further symmetry between these two problems, in that the dual of the dual problem is the original primal problem. Because of this, we begin by first determining which of these two canonical problems matches the type of problem that our primal is. In our example, the primal is a maximization problem. Thus, we examine the problem:

$$\max_{y} b^\top y$$
$$\text{s.t. } A^\top y \leq c$$
$$y \geq 0.$$

Specifically, we notice that each variable in this problem is non-negative and that the associated structural constraints in the problem:

$$\min_{x} c^\top x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0,$$

are greater-than-or-equal-to inequalities. Thus, in our example, each structural constraint in the dual that is associated with a primal variable that is non-negative in the primal problem will be a greater-than-or-equal-to constraint. Specifically, in our example, x_1 and x_2 are non-negative, and so their associated dual constraints will be greater-than-or-equal-to inequalities, as follows:

$$\min_y \; 10y_1 - y_2 + 5y_3$$

$$\text{s.t.} \quad y_1 \geq 3 \qquad\qquad\qquad\qquad\qquad (x_1)$$
$$\qquad\quad y_1 + y_2 \geq 5 \qquad\qquad\qquad\quad (x_2)$$
$$\qquad\quad -0.5y_2 + y_3 \quad -3 \qquad\qquad (x_3)$$
$$\qquad\quad -4y_1 \quad 1.3 \qquad\qquad\qquad\; (x_4)$$
$$\qquad\quad y_2 \quad -1. \qquad\qquad\qquad\qquad (x_5)$$

Next, we examine the variable x_4 in our primal problem. Notice that the sign restriction on this variable is opposite sign restriction (2.62) in the canonical maximization problem. Thus, its associated dual structural constraint will have the opposite direction of the canonical minimization problem (*i.e.*, opposite greater-than-or-equal-to constraint (2.58) in the canonical minimization problem). This means that the dual constraint associated with x_4 will be a less-than-or-equal-to constraint, as follows:

$$\min_y \; 10y_1 - y_2 + 5y_3$$

$$\text{s.t.} \quad y_1 \geq 3 \qquad\qquad\qquad\qquad\qquad (x_1)$$
$$\qquad\quad y_1 + y_2 \geq 5 \qquad\qquad\qquad\quad (x_2)$$
$$\qquad\quad -0.5y_2 + y_3 \quad -3 \qquad\qquad (x_3)$$
$$\qquad\quad -4y_1 \leq 1.3 \qquad\qquad\qquad\; (x_4)$$
$$\qquad\quad y_2 \quad -1. \qquad\qquad\qquad\qquad (x_5)$$

Finally, for the variables x_3 and x_5, which are unrestricted in sign in the primal problem, their associated structural constraints in the dual problem are equalities, as follows:

$$\min_y \; 10y_1 - y_2 + 5y_3$$

$$\text{s.t.} \quad y_1 \geq 3 \qquad\qquad\qquad\qquad\qquad (x_1)$$
$$\qquad\quad y_1 + y_2 \geq 5 \qquad\qquad\qquad\quad (x_2)$$
$$\qquad\quad -0.5y_2 + y_3 = -3 \qquad\qquad (x_3)$$
$$\qquad\quad -4y_1 \leq 1.3 \qquad\qquad\qquad\; (x_4)$$
$$\qquad\quad y_2 = -1. \qquad\qquad\qquad\qquad (x_5)$$

When finding the dual of a minimization problem, one would reverse the roles of the two canonical problems (*i.e.*, the roles of problems (2.60)–(2.62) and (2.57)–(2.59)). Specifically, one would have less-than-or-equal-to structural inequality constraints in the dual for primal variables that are non-negative, greater-than-or-equal-to inequalities for primal variables that are non-positive, and equality constraints for primal variables that are unrestricted in sign.

2.7.3.7   Step 7: Determine Sign Restrictions on the Dual Variables

The final step is to determine sign restrictions on the dual variables. The sign restrictions depend on the type of structural constraints in the primal problem. The method of determining the sign restrictions is analogous to determining the types of structural constraints in the dual problem in Step 6. We again rely on the two symmetric dual problems introduced in Section 2.7.1.

Because the primal problem in our example is a maximization, we focus on the canonical problem:

$$\max_{y} b^\top y$$
$$\text{s.t. } A^\top y \leq c$$
$$y \geq 0,$$

and its dual:

$$\min_{x} c^\top x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0.$$

Specifically, we begin by noting that the first primal constraint:

$$x_1 + x_2 - 4x_4 \leq 10,$$

is consistent with constraint (2.61) in the canonical maximization problem. Thus, the dual variable associated with this constraint, which is y_1, is non-negative, which is consistent with sign restriction (2.59) in the canonical minimization problem. We next examine the third primal structural constraint:

$$x_3 \geq 5.$$

This is the opposite type of constraint compared to constraint (2.61) in the canonical maximization problem. Thus, its associated dual variable, which is y_3, will have a sign restriction that is opposite to non-negativity constraint (2.59) in the canonical minimization problem. Finally, we note that the second structural constraint in the primal problem is an equality. As such, its associated dual variable, which is y_2, is unrestricted in sign. Taking these sign restrictions together, the dual problem is:

$$\min_{y} \ 10y_1 - y_2 + 5y_3$$

$$\begin{aligned}
\text{s.t.} \quad & y_1 \geq 3 & (x_1) \\
& y_1 + y_2 \geq 5 & (x_2) \\
& -0.5y_2 + y_3 = -3 & (x_3) \\
& -4y_1 \leq 1.3 & (x_4) \\
& y_2 = -1 & (x_5) \\
& y_1 \geq 0 \\
& y_3 \leq 0.
\end{aligned}$$

One can verify these rules by taking different starting primal problems, converting them to canonical form, and examining the resulting dual. We now show, in the following example, that the dual problem found by applying these rules is identical to what would be obtained if the primal problem is first converted to canonical form.

Example 2.14 Consider the linear optimization problem:

$$\max_{x} \ 3x_1 + 5x_2 - 3x_3 + 1.3x_4 - x_5$$

$$\begin{aligned}
\text{s.t.} \quad & x_1 + x_2 - 4x_4 \leq 10 \\
& x_2 - 0.5x_3 + x_5 = -1 \\
& x_3 \geq 5 \\
& x_1, x_2 \geq 0 \\
& x_4 \leq 0,
\end{aligned}$$

which is introduced in Example 2.1. In Example 2.3 we find that the canonical form of this problem is:

$$\min_{x} \ -3x_1 - 5x_2 + 3x_3^+ - 3x_3^- + 1.3\tilde{x}_4 + x_5^+ - x_5^-$$

$$\begin{aligned}
\text{s.t.} \quad & -x_1 - x_2 - 4\tilde{x}_4 \geq -10 \\
& -x_2 + 0.5x_3^+ - 0.5x_3^- - x_5^+ + x_5^- \geq 1 \\
& x_2 - 0.5x_3^+ + 0.5x_3^- + x_5^+ - x_5^- \geq -1 \\
& x_3^+ - x_3^- \geq 5 \\
& x_1, x_2, x_3^-, x_3^+, \tilde{x}_4, x_5^-, x_5^+ \geq 0,
\end{aligned}$$

which can be written as:

$$\min_{x} \ c^\top x$$

$$\begin{aligned}
\text{s.t.} \quad & Ax \geq b \\
& x \geq 0,
\end{aligned}$$

where we have:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3^+ \\ x_3^- \\ \tilde{x}_4 \\ x_5^+ \\ x_5^- \end{pmatrix},$$

$$c = \begin{pmatrix} -3 \\ -5 \\ 3 \\ -3 \\ 1.3 \\ 1 \\ -1 \end{pmatrix},$$

$$A = \begin{bmatrix} -1 & -1 & 0 & 0 & -4 & 0 & 0 \\ 0 & -1 & 0.5 & -0.5 & 0 & -1 & 1 \\ 0 & 1 & -0.5 & 0.5 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \end{bmatrix},$$

and:

$$b = \begin{pmatrix} -10 \\ 1 \\ -1 \\ 5 \end{pmatrix}.$$

Thus, the dual problem is:

$$\max_{w} \begin{pmatrix} -10 & 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$$

$$\text{s.t.} \quad \begin{bmatrix} -1 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0.5 & -0.5 & 1 \\ 0 & -0.5 & 0.5 & -1 \\ -4 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \leq \begin{pmatrix} -3 \\ -5 \\ 3 \\ -3 \\ 1.3 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

which simplifies to:

$$\max_{w} \; -10w_1 + w_2 - w_3 + 5w_4$$
$$\text{s.t.} \; -w_1 \leq -3$$
$$-w_1 - w_2 + w_3 \leq -5$$
$$0.5w_2 - 0.5w_3 + w_4 \leq 3$$
$$-0.5w_2 + 0.5w_3 - w_4 \leq -3$$
$$-4w_1 \leq 1.3$$
$$-w_2 + w_3 \leq 1$$
$$w_2 - w_3 \leq -1$$
$$w_1, w_2, w_3, w_4 \geq 0.$$

Note that if we change the direction of optimization, this problem becomes:

$$\min_{w} \; 10w_1 - w_2 + w_3 - 5w_4$$
$$\text{s.t.} \; -w_1 \leq -3$$
$$-w_1 - w_2 + w_3 \leq -5$$
$$0.5w_2 - 0.5w_3 + w_4 \leq 3$$
$$-0.5w_2 + 0.5w_3 - w_4 \leq -3$$
$$-4w_1 \leq 1.3$$
$$-w_2 + w_3 \leq 1$$
$$w_2 - w_3 \leq -1$$
$$w_1, w_2, w_3, w_4 \geq 0.$$

Next, note that if we define the variables $y_1 = w_1$, $y_2 = w_2 - w_3$, and $y_3 = -w_4$, this problem can be written as:

$$\min_{y} \; 10y_1 - y_2 + 5y_3$$
$$\text{s.t.} \; -y_1 \leq -3$$
$$-y_1 - y_2 \leq -5$$
$$0.5y_2 - y_3 \leq 3 \tag{2.63}$$
$$-0.5y_2 + y_3 \leq -3 \tag{2.64}$$
$$-4y_1 \leq 1.3$$

$$-y_2 \leq 1 \tag{2.65}$$
$$y_2 \leq -1 \tag{2.66}$$
$$y_1 \geq 0$$
$$y_3 \leq 0.$$

Note that because $y_1 = w_1$ and $w_1 \geq 0$ we have $y_1 \geq 0$. Similarly, $y_3 = -w_4$ and $w_4 \geq 0$, implying that $y_3 \leq 0$. Because y_2 is defined as the difference between two non-negative variables, it can be either positive or negative in sign.

We next note that (2.63) and (2.64) can be rewritten as:

$$-0.5w_2 + 0.5w_3 + w_4 \geq -3,$$

and:

$$-0.5w_2 + 0.5w_3 + w_4 \leq -3,$$

which together imply:

$$-0.5w_2 + 0.5w_3 + w_4 = -3.$$

Similarly, (2.65) and (2.66) imply:

$$y_2 = -1.$$

Making these substitutions and multiplying some of the other structural inequalities through by -1 gives the following dual problem:

$$\min_{y} 10y_1 - y_2 + 5y_3$$
$$\text{s.t. } y_1 \geq 3$$
$$y_1 + y_2 \geq 5$$
$$-0.5y_2 + y_3 = -3$$
$$-4y_1 \leq 1.3$$
$$y_2 = -1$$
$$y_1 \geq 0$$
$$y_3 \leq 0,$$

which is identical to the dual problem found directly by applying the conversion rules.

$\square$

Example 2.15 Consider the Gasoline-Mixture Problem, which is introduced in Section 2.1.3. The primal problem is formulated as:

$$\min_{x} 200x_1 + 220x_2$$

$$\text{s.t.}\ \ 0.7x_1 + 0.6x_2 \geq 0.65$$

$$0.9x_1 + 0.8x_2 \leq 0.85$$

$$x_1 + x_2 = 1$$

$$x_1, x_2 \geq 0.$$

To find the dual of this problem directly, we first observe that this problem has three structural constraints, thus the dual will have three variables, which we call y_1, y_2, and y_3. We associate these dual variables with the primal constraints as follows:

$$\min_{x} 200x_1 + 220x_2$$

$$\begin{aligned}
\text{s.t.}\ \ & 0.7x_1 + 0.6x_2 \geq 0.65 && (y_1) \\
& 0.9x_1 + 0.8x_2 \leq 0.85 && (y_2) \\
& x_1 + x_2 = 1 && (y_3) \\
& x_1, x_2 \geq 0.
\end{aligned}$$

We next determine the objective function of the dual problem, which is:

$$\max_{y}\ \ 0.65y_1 + 0.85y_2 + y_3,$$

based on the direction of optimization of the primal problem and the right-hand side constants of the structural constraints. We also know that because the primal problem has two variables the dual problem has two structural constraints, each of which we associate with a primal variable as follows:

$$\max_{y} 0.65y_1 + 0.85y_2 + y_3$$

$$\begin{aligned}
\text{s.t.}\ \ && (x_1) \\
&& (x_2)
\end{aligned}$$

We next determine the right-hand sides of the structural constraints in the dual problem based on the objective-function coefficients in the primal problem, which gives:

$$\max_{y} 0.65y_1 + 0.85y_2 + y_3$$

$$\begin{aligned}
\text{s.t.}\ \ & 200 && (x_1) \\
& 220. && (x_2)
\end{aligned}$$

Next we determine the left-hand sides of the structural constraints in the dual problem using the coefficients on the left-hand sides of the structural constraints of the primal problem, giving:

$$\max_{y} 0.65y_1 + 0.85y_2 + y_3$$

$$\text{s.t. } 0.7y_1 + 0.9y_2 + y_3 \quad 200 \qquad (x_1)$$

$$0.6y_1 + 0.8y_2 + y_3 \quad 220. \qquad (x_2)$$

Next, to determine the types of structural constraints in the dual problem, we note that the primal problem is a minimization problem and each of x_1 and x_2 are non-negative. Based on the canonical minimization problem given by (2.57)–(2.59) and its dual, we observe that a minimization problem with non-negative constraints has a dual problem with less-than-or-equal-to structural constraints. Thus, both of the structural constraints will be less-than-or-equal-to constraints, which gives:

$$\max_{y} 0.65y_1 + 0.85y_2 + y_3$$

$$\text{s.t. } 0.7y_1 + 0.9y_2 + y_3 \leq 200 \qquad (x_1)$$

$$0.6y_1 + 0.8y_2 + y_3 \leq 220. \qquad (x_2)$$

Finally, we must determine the sign restrictions on the dual variables. Again, we examine the canonical minimization problem given by (2.57)–(2.59) and its dual. Note that the first primal constraint in the Gasoline-Mixture Problem is a greater-than-or-equal-to constraint, which is consistent with (2.58). Thus, the associated dual variable, y_1, is non-negative. Because the second primal constraint is inconsistent with (2.58), the associated variable, y_2, is non-positive. The third primal constraint is an equality, meaning that the associated variable, y_3, is unrestricted in sign. Thus, the dual problem is:

$$\max_{y} 0.65y_1 + 0.85y_2 + y_3$$

$$\text{s.t. } 0.7y_1 + 0.9y_2 + y_3 \leq 200 \qquad (x_1)$$

$$0.6y_1 + 0.8y_2 + y_3 \leq 220 \qquad (x_2)$$

$$y_1 \geq 0$$

$$y_2 \leq 0.$$

□

2.7.3.8 Summary of Duality Conversion Rules

From the discussion in this section, we can summarize the following duality conversion rules. We use the terms 'primal' and 'dual' in these rules, however the two words can be interchanged. This is because there is a symmetry between primal and dual problems (*i.e.*, the primal problem is the dual of the dual).

For a primal problem that is a minimization we have that:

- the dual problem is a maximization;
- dual variables associated with equality constraints in the primal problem are unrestricted in sign;
- dual variables associated with greater-than-or-equal-to constraints in the primal problem are non-negative;
- dual variables associated with less-than-or-equal-to constraints in the primal problem are non-positive;
- dual constraints associated with non-negative primal variables are less-than-or-equal-to inequalities;
- dual constraints associated with non-positive primal variables are greater-than-or-equal-to inequalities; and
- dual constraints associated with primal variables that are unrestricted in sign are equalities.

For a dual problem that is a maximization we have that:

- the primal problem is a minimization;
- primal variables associated with equality constraints in the dual problem are unrestricted in sign;
- primal variables associated with less-than-or-equal-to constraints in the dual problem are non-negative;
- primal variables associated with greater-than-or-equal-to constraints in the dual problem are non-positive;
- primal constraints associated with non-negative dual variables are greater-than-or-equal-to inequalities;
- primal constraints associated with non-positive dual variables are less-than-or-equal-to inequalities; and
- primal constraints associated with dual variables that are unrestricted in sign are equalities.

2.7.4 Weak- and Strong-Duality Properties

We saw in Section 2.7.1 that any linear optimization problem and its dual have structural relationships. We now show some further relationships between a primal problem and its dual, focusing specifically on the objective-function values of points that are feasible in a primal problem and its dual. We show these relationships for the special case of the symmetric duals introduced in Section 2.7.1. However, these results extend to any primal problem (even if it is not in canonical form) [9]. This is because any primal problem can be converted to canonical form and then its symmetric dual can be found.

Weak-Duality Property: Consider the primal problem:

$$\min_{x} c^\top x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0,$$

and its dual:

$$\max_{y} b^\top y$$
$$\text{s.t. } A^\top y \leq c$$
$$y \geq 0.$$

If x is feasible in the primal problem and y is feasible in the dual problem, then we have that:

$$c^\top x \geq b^\top y.$$

To show this we first note that if x is feasible in the primal problem then we must have:

$$b \leq Ax. \tag{2.67}$$

Next, we know that if y is feasible in the dual problem, we must have $y \geq 0$. Thus, multiplying both sides of (2.67) by y gives:

$$y^\top b \leq y^\top Ax. \tag{2.68}$$

We also know that if y is feasible in the dual problem, then:

$$A^\top y \leq c,$$

which can be rewritten as:

$$y^\top A \leq c^\top,$$

by transposing the expressions on both sides of the inequality. Combining this inequality with (2.68) gives:

$$y^\top b \leq y^\top Ax \leq c^\top x,$$

or simply:

$$y^\top b \leq c^\top x,$$

which is the weak-duality inequality.

The weak-duality property says that the objective-function value of any point that is feasible in the dual problem provides a bound on the objective-function value of any point that is feasible in the primal problem and *vice versa*. One can take this weak-duality relationship further. That is because any point that is feasible in the dual problem provides a bound on the objective-function of the *optimal* objective-function value of the primal problem. For a given pair of feasible solutions to the primal and dual problems, the difference:

$$c^\top x - y^\top b,$$

is called the **duality gap**. One way that the weak-duality property can be used is to determine how close to optimal a feasible solution to the primal problem is. One can estimate this by finding a solution that is feasible in the dual problem and computing the duality gap.

As noted above, the weak-duality property applies to any primal problem and its dual (not only to a primal problem in canonical form). However, the direction of the inequality between the objective functions of the two problems may change, depending on the directions of the inequalities, signs of the variables, and the direction of optimization in the primal and dual problems. Thus, for the purposes of applying the weak-duality property, it is often beneficial to convert the primal problem into canonical form.

Primal and dual problems have a further related property, known as the strong-duality equality.

Strong-Duality Property: Consider the primal problem:

$$\min_{x} c^\top x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0,$$

and its dual:

$$\max_{y} b^\top y$$
$$\text{s.t. } A^\top y \leq c$$
$$y \geq 0.$$

If x^* is optimal in the primal problem and y^* is optimal in the dual problem, then we have that:
$$c^\top x^* = b^\top y^*.$$

The strong-duality equality says that the objective-function value of the dual problem evaluated at an optimal solution is equal to the optimal primal objective-function value. Luenberger and Ye [9] provide a formal proof of this so-called strong-duality theorem. In some sense, the primal problem pushes its objective-function value down (because it is a minimization problem) while the dual problem pushes its objective-function value up (as it is a maximization problem). The two problems 'meet' at an optimal solution where they have the same objective-function value. Figure 2.15 illustrates this concept by showing the duality gap for a given pair of solutions that are feasible in the primal and dual problems (in the left-hand side of the figure) and the gap being reduced to zero as the two problems are solved to optimality (in the right-hand side of the figure).
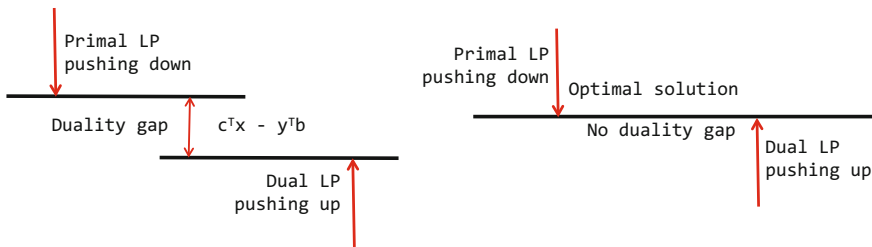

Fig. 2.15 Illustration of the duality gap being reduced as the primal and dual problems are solved to optimality

The strong-duality equality also gives us another way to solve a linear optimization problem. As an illustrative example, consider a primal problem in canonical form:

$$\min_{x} c^\top x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0,$$

and its dual:

$$\max_{y} b^\top y$$
$$\text{s.t. } A^\top y \leq c$$
$$y \geq 0.$$

Instead of solving the primal problem directly, one can solve the following system of equalities and inequalities:

$$Ax \geq b$$

$$A^\top y \leq c$$

$$c^\top x = b^\top y$$

$$x \geq 0$$

$$y \geq 0,$$

for x and y. This system of equalities and inequalities consist of the constraints of the primal problem (*i.e.*, $Ax \geq b$ and $x \geq 0$), the constraints of the dual problem (*i.e.*, $A^\top y \leq c$ and $y \geq 0$), and the strong-duality equality (*i.e.*, $c^\top x = b^\top y$). If we find a pair of vectors, x^* and y^*, that satisfy all of these conditions, then from the strong-duality property we know that x^* is optimal in the primal problem and y^* is optimal is the dual problem.

These strong-duality properties (including the alternate method of solving a linear optimization problem) apply to any primal problem and its dual (regardless of whether the primal problem is in canonical form). This is because the strong-duality property is an equality, meaning that the result is the same regardless of the form of the primal problem. This is demonstrated in the following example.

Example 2.16 Consider the Electricity-Production Problem, which is introduced in Section 2.1.1. This problem is formulated as:

$$\max_{x} x_1 + x_2 \tag{2.69}$$

$$\text{s.t. } \frac{2}{3}x_1 + x_2 \leq 18 \tag{2.70}$$

$$2x_1 + x_2 \geq 8 \tag{2.71}$$

$$x_1 \leq 12 \tag{2.72}$$

$$x_2 \leq 16 \tag{2.73}$$

$$x_1, x_2 \geq 0. \tag{2.74}$$

Taking this is as the primal problem, its dual is:

$$\min_{y} 18y_1 + 8y_2 + 12y_3 + 16y_4 \tag{2.75}$$

$$\text{s.t. } \frac{2}{3}y_1 + 2y_2 + y_3 \geq 1 \tag{2.76}$$

$$y_1 + y_2 + y_4 \geq 1 \tag{2.77}$$

$$y_1, y_3, y_4 \geq 0 \tag{2.78}$$

$$y_2 \leq 0. \tag{2.79}$$

We can also find the dual of this problem by first converting it to canonical form, which is:

$$\min_x \; -x_1 - x_2 \tag{2.80}$$

$$\text{s.t.} \quad -\frac{2}{3}x_1 - x_2 \geq -18 \tag{2.81}$$

$$2x_1 + x_2 \geq 8 \tag{2.82}$$

$$-x_1 \geq -12 \tag{2.83}$$

$$-x_2 \geq -16 \tag{2.84}$$

$$x_1, x_2 \geq 0, \tag{2.85}$$

and has the dual problem:

$$\max_y \; -18y_1 + 8y_2 - 12y_3 - 16y_4 \tag{2.86}$$

$$\text{s.t.} \quad -\frac{2}{3}y_1 + 2y_2 - y_3 \leq -1 \tag{2.87}$$

$$-y_1 + y_2 - y_4 \leq -1 \tag{2.88}$$

$$y_1, y_2, y_3, y_4 \geq 0. \tag{2.89}$$

Consider the solution $(x_1, x_2) = (4, 0)$, which is feasible but not optimal in the canonical-form primal problem, which is given by (2.80)–(2.85). Substituting this solution into objective function (2.80) gives a value of -4. Next, consider the solution $(y_1, y_2, y_3, y_4) = (1.5, 0, 0, 0)$, which is feasible but not optimal in the symmetric dual problem, which is given by (2.86)–(2.89). Substituting this solution into objective function (2.86) gives a value of -27. We see that the dual objective-function value is less than the primal value, verifying that the weak-duality inequality is satisfied.

These solutions are also feasible in the original primal problem, given by (2.69)–(2.74), and its dual, given by (2.75)–(2.79). However, substituting these solutions gives a primal objective-function value of 4 and a dual value of 27. We see that in this case the weak-duality inequality is reversed, because the original primal problem is not in canonical form.

Next, consider the primal solution $(x_1^*, x_2^*) = (12, 10)$ and the dual solution $(y_1^*, y_2^*, y_3^*, y_4^*) = (1, 0, 1/3, 0)$. It is straightforward to verify that these solutions are feasible in both forms of the primal and dual problems. Moreover, note that if we substitute x^* into primal objective function (2.69) we obtain a value of 22. Similarly, substituting y^* into (2.75) yields a dual objective-function value of 22. Thus, by the strong-duality property, we know that x^* and y^* are optimal in their respective problems.

We also obtain the same results from examining the objective functions of the canonical-form primal and its dual, except that the signs are reversed. More specifically, substituting x^* into (2.80) yields a value of -22 and substituting y^* into (2.86) also gives -22. $\qquad\qquad\square$

2.7.5 *Duality and Sensitivity*

In this section we show yet another important relationship between a primal problem and its dual, specifically focusing on the relationship between dual variables and the sensitivity vector of the primal problem. We assume that we are given a linear optimization problem in standard form:

$$\min_{x} c^\top x$$
$$\text{s.t. } Ax = b$$
$$x \geq 0.$$

Moreover, we assume that we have an optimal basic feasible solution x^*, which is partitioned into basic and nonbasic variables, $x^* = (x_B^*, x_N^*)$. Thus, we also have the objective-function and constraint coefficients partitioned as $c = (c_B, c_N)$ and $A = [B, N]$. Because the nonbasic variables are equal to zero, the objective function can be written as:

$$c^\top x^* = c_B^\top x_B^*.$$

We also know that the basic-variable vector can be written as:

$$x_B^* = B^{-1}b,$$

thus we can write the optimal objective-function value as:

$$c^\top x^* = c_B^\top x_B^* = c_B^\top B^{-1}b. \tag{2.90}$$

At the same time, the strong-duality equality gives us:

$$c^\top x^* = b^\top y^*, \tag{2.91}$$

where y^* is a dual-optimal solution. Combining (2.90) and (2.91) gives:

$$y^{*\top}b = c_B^\top B^{-1}b,$$

or:

$$y^{*\top} = c_B^\top B^{-1},$$

which provides a convenient way of computing dual-variable values from the final tableau after solving the primal problem using the Simplex method. We next recall sensitivity result (2.51):

$$c_B^\top B^{-1}\Delta b = \lambda^\top \Delta b,$$

which implies:

$$y^{*\top} = c_B^\top B^{-1} = \lambda^\top,$$

or that the sensitivity vector is equal to the dual variables.

2.7.6 Complementary Slackness

In this section we show a final relationship between a primal problem and its dual. We do not prove this relationship formally, but rather rely on the interpretation of dual variables as providing sensitivity information derived in Section 2.7.5. We, again, consider the case of a primal problem in canonical form:

$$\min_x c^\top x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0,$$

and its symmetric dual:

$$\max_y b^\top y$$
$$\text{s.t. } A^\top y \leq c$$
$$y \geq 0.$$

The discussion in Section 2.7.5 leads to the conclusion that the dual variables, y, provide sensitivity information for the structural constraints in the primal problem. Moreover, we know from Section 2.7.1 that the primal problem is the dual of the dual. Thus, we can also conclude that the primal variables, x, provide sensitivity information for the structural constraints in the dual problem.

Before proceeding, we first define what it means for an inequality constraint in a linear optimization to be binding as opposed to non-binding. For this discussion, let us consider the jth structural constraint in the primal problem, which can be written as:

$$A_{j,.}x \geq b_j,$$

where $A_{j,.}$ is the jth row of the A matrix and b_j the jth element of the b vector. Note, however, that these definitions can be applied to any inequality constraint (of any direction) in any problem. This constraint is said to be **non-binding** at the point $\hat{x}$ if:

$$A_{j,.}\hat{x} > b_j.$$

That is to say, the inequality constraint is non-binding if there is a difference or slack between the left- and right-hand sides of the constraint when we substitute the values of $\hat{x}$ into the constraint. Conversely, we say that this constraint is **binding** at the point $\hat{x}$ if:

$$A_{j,.}\hat{x} = b_j.$$

The constraint is binding if there is no difference or slack between its two sides at $\hat{x}$.

Now that we have these two definitions, let us consider the effect of changing the right-hand side of the jth structural constraint in the primal problem, which is:

$$A_{j,.}x \geq b_j,$$

to:

$$A_{j,.}x \geq b_j + \Delta b_j,$$

where Δb_j is sufficiently close to zero to satisfy condition (2.49). We know from Sections 2.6 and 2.7.5 that the change in the primal objective-function value can be computed as:

$$\Delta b_j y_j^*,$$

where y_j^* is the optimal dual-variable value associated with the jth primal structural constraint.

Let us next intuitively determine what the effect of changing this constraint is. First, consider the case in which the original constraint:

$$A_{j,.}x \geq b_j,$$

is not binding. In this case, we can conclude that changing the constraint to:

$$A_{j,.}x \geq b_j + \Delta b_j,$$

will have no effect on the optimal solution, because the constraint is already slack at the point x^*. If changing the constraint causes the feasible region to decrease in size, x^* will still be feasible so long as $|\Delta b_j|$ is not too large. Similarly, if changing the constraint causes the feasible region to increase in size, x^* will still remain optimal because loosening the jth constraint should not result in the optimal solution changing. Based on this intuition, we conclude that if the jth constraint is non-binding then $\Delta b_j y_j^* = 0$ (because the optimal solution does not change) and, thus, $y_j^* = 0$.

We can also draw the converse conclusion for the case of a binding constraint. If the jth constraint is binding, then changing the constraint to:

$$A_{j,.}x \geq b_j + \Delta b_j,$$

may have an effect on the optimal solution. This is because if changing the constraint causes the feasible region to decrease in size, x^* may change causing the primal objective-function value to increase (get worse). If changing the constraint causes the feasible region to increase in size, x^* may change causing the primal objective-function value to get better. Thus, we conclude that if the jth constraint is binding then $\Delta b_j y_j^*$ can be non-zero, meaning that y_j^* may be non-zero.

These conclusions give what is known as the **complementary-slackness** condition between the primal constraints and their associated dual variables. The Primal Complementary-Slackness Property is written explicitly in the following.

Primal Complementary-Slackness Property: Consider the primal problem:

$$\min_x c^\top x$$

$$\text{s.t. } Ax \geq b$$

$$x \geq 0,$$

and its dual:

$$\max_y b^\top y$$

$$\text{s.t. } A^\top y \leq c$$

$$y \geq 0.$$

If x^* is optimal in the primal problem and y^* is optimal in the dual problem, then for each primal constraint, $j = 1, \ldots, m$, we have that either:

1. $A_{j,.} x^* = b_j$,
2. $y_j^* = 0$, or
3. both.

The third case in the Primal Complementary-Slackness Property implies some level of redundancy in the constraints. This is because in Case 3 there is at least one inequality constraint that is binding, but which has a sensitivity value of zero.

This complementary-slackness property can also be written more compactly as:

$$(A_{j,.} x^* - b_j) y_j^* = 0, \forall j = 1, \ldots, m.$$

This is because for the product, $(A_{j,.} x^* - b_j) y_j^*$, to equal zero for a given j we must either have $(A_{j,.} x^* - b_j) = 0$, which is the first complementary slackness condition, or $y_j^* = 0$, which is the second. Indeed, instead of writing:

$$(A_{j,.} x^* - b_j) y_j^* = 0,$$

for each j, one can also write the complementary slackness condition even more compactly as:

$$(Ax^* - b)^\top y = 0,$$

which enforces complementary slackness between all of the primal constraints and dual variables in a single equation.

We can carry out a similar analysis of changing the right-hand sides of the dual constraints, which gives rise to the following Dual Complementary-Slackness Property.

Dual Complementary-Slackness Property: Consider the primal problem:

$$\min_{x} c^\top x$$
$$\text{s.t. } Ax \geq b$$
$$x \geq 0,$$

and its dual:

$$\max_{y} b^\top y$$
$$\text{s.t. } A^\top y \leq c$$
$$y \geq 0.$$

If x^* is optimal in the primal problem and y^* is optimal in the dual problem, then for each dual constraint, $i = 1, \ldots, n$, we have that either:

1. $A_{\cdot,i}^\top y^* = c_i$,
2. $x_i^* = 0$, or
3. both;

where $A_{\cdot,i}$ is the ith column of the A matrix.

As with the Primal Complementary-Slackness Property, the third case in the Dual Complementary-Slackness Property also implies some level of redundancy in the constraints. This is because in Case 3 there is at least one inequality constraint that is binding, but which has a sensitivity value of zero.

As with the Primal Complementary-Slackness Property, the Dual Complementary-Slackness Property can be written more compactly as:

$$(A_{\cdot,i}^\top y^* - c_i)x_i^* = 0, \forall i = 1, \ldots, n;$$

or as:

$$(A^\top y^* - c)^\top x^* = 0.$$

The Primal and Dual Complementary-Slackness Properties can be extended to structural constraints that are different from the canonical form. The same result is obtained, if there is an inequality in one of the primal or dual problem either it must be binding or the associated variable in the other problem must equal zero. One can also write complementary slackness involving equality constraints, however these conditions do not provide any useful information. This is because an equality

constraint must always be binding. Thus, the variable associated with an equality constraint can take on any value (*i.e.*, it is never restricted to equal zero).

One of the benefits of the complementary-condition properties is that it provides another means to recover an optimal solution to either of the primal or dual problem from an optimal solution to the other. This is demonstrated in the following example.

Example 2.17 Consider the Electricity-Production Problem, which is introduced in Section 2.1.1. This problem is formulated as:

$$\max_{x} x_1 + x_2$$

$$\text{s.t. } \frac{2}{3}x_1 + x_2 \leq 18 \qquad\qquad (y_1)$$

$$2x_1 + x_2 \geq 8 \qquad\qquad (y_2)$$

$$x_1 \leq 12 \qquad\qquad (y_3)$$

$$x_2 \leq 16 \qquad\qquad (y_4)$$

$$x_1, x_2 \geq 0,$$

and its dual is:

$$\min_{y} 18y_1 + 8y_2 + 12y_3 + 16y_4$$

$$\text{s.t. } \frac{2}{3}y_1 + 2y_2 + y_3 \geq 1 \qquad\qquad (x_1)$$

$$y_1 + y_2 + y_4 \geq 1 \qquad\qquad (x_2)$$

$$y_1, y_3, y_4 \geq 0$$

$$y_2 \leq 0,$$

where the variable associations are denoted in the parentheses. We know that $(x_1^*, x_2^*) = (12, 10)$ is optimal in the primal problem.

Substituting these values into the primal constraints, we see that the second and fourth one are non-binding. Thus, the Primal Complementary-Slackness Property tells us that $y_2^* = 0$ and $y_4^* = 0$. Furthermore, because both x_1^* and x_2^* are non-zero, we know that their associated dual constraints must be binding (*i.e.*, we can write them as equalities). Doing so and substituting in the values found for $y_2^* = 0$ and $y_4^* = 0$ gives:

$$\frac{2}{3}y_1 + y_3 = 1$$

$$y_1 = 1,$$

which have: $y_1^* = 1$ and $y_3^* = 1/3$ as solutions. This dual solution, $(y_1^*, y_2^*, y_3^*, y_4^*) = (1, 0, 1/3, 0)$ coincides with the dual-optimal solution found in Example 2.16. $\square$

2.8 Final Remarks

This chapter introduces the Simplex method, which was initially proposed by Dantzig [5], as an algorithm to solve linear optimization problems. The Simplex method works by navigating around the boundary of the polytope that describes the feasible region of the problem, jumping from one extreme point (or basic feasible solution) to another until reaching an optimal corner. An obvious question that this raises is whether it would be beneficial to navigate through the interior of the polytope instead of around its exterior. For extremely large problems, this may be the case. Interested readers are referred to more advanced textbooks, which introduce such **interior-point** algorithms [11]. Additional information on LPPs, their formulation, properties, and solutions algorithms, can be found in a number of other advanced textbooks [1, 2, 9, 12]. Modeling issues are treated extensively by Castillo *et al.* [4].

2.9 GAMS Codes

This final section provides GAMS [3] codes for the main problems considered in this chapter. GAMS can use a variety of different software packages, among them CPLEX [8] and GUROBI [7], to actually solve an LPP.

2.9.1 Electricity-Production Problem

The Electricity-Production Problem, which is introduced in Section 2.1.1, has the following GAMS formulation:

```
1  variable z;
2  positive variables x1, x2;
3  equations of, eq1, eq2, eq3, eq4;
4  of .. z =e= x1+x2;
5  eq1 .. (2/3)*x1+x2 =l= 18;
6  eq2 .. 2*x1+x2 =g= 8;
7  eq3 .. x1 =l= 12;
8  eq4 .. x2 =l= 16;
9  model ep /all/;
10 solve ep using lp maximizing z;
```

Lines 1 and 2 are variable declarations, Line 3 gives names to the equations (*i.e.*, the objective function, equalities, and inequalities) of the model, and Lines 5–8 define these equations (*i.e.*, the objective function and constraints). The double-dot separates the name of an equation from its definition. "=e=" indicates an equality, "=l=" a less-than-or-equal-to inequality, and "=g=" a greater-than-or-equal-to inequality. Line 9 gives a name to the model and indicates that all equations should be

considered. Finally, Line 10 directs GAMS to solve the problem using an LP solver while minimizing *z*. All lines end with a semicolon.

The GAMS output that provides information about the optimal solution is:

	LOWER	LEVEL	UPPER	MARGINAL
1				
3 ---- VAR z	**-INF**	22.000	**+INF**	.
4 ---- VAR x1	.	12.000	**+INF**	.
5 ---- VAR x2	.	10.000	**+INF**	.

2.9.2 Natural Gas-Transportation Problem

The Natural Gas-Transportation Problem, which is introduced in Section 2.1.2, has the following GAMS formulation:

```
1  variable z;
2  positive variables x11, x12, x21, x22;
3  equations of, s1, s2, d1, d2;
4  of .. z =e= 5*x11+4*x12+3*x21+6*x22;
5  s1 .. x11+x12 =l= 7;
6  s2 .. x21+x22 =l= 12;
7  d1 .. x11+x21 =e= 10;
8  d2 .. x12+x22 =e= 8;
9  model ng /all/;
10 solve ng using lp minimizing z;
```

Lines 1 and 2 declare variables, Line 3 gives names to the model equations, Line 4 defines the objective function, Lines 5–8 specify the constraints, Line 9 defines the model, and Line 10 directs GAMS to solve it.

The GAMS output that provides information about the optimal solution is:

	LOWER	LEVEL	UPPER	MARGINAL
1				
3 ---- VAR z	**-INF**	64.000	**+INF**	.
4 ---- VAR x11	.	.	**+INF**	4.000
5 ---- VAR x12	.	7.000	**+INF**	.
6 ---- VAR x21	.	10.000	**+INF**	.
7 ---- VAR x22	.	1.000	**+INF**	.

2.9.3 Gasoline-Mixture Problem

The Gasoline-Mixture Problem, which is introduced in Section 2.1.3, has the following GAMS formulation:

```
1  variable z;
2  positive variables x1, x2;
3  equations of, eq1, eq2, eq3;
4  of .. z =e= 200*x1+220*x2;
5  eq1 .. 0.7*x1+0.6*x2 =g= 0.65;
6  eq2 .. 0.9*x1+0.8*x2 =l= 0.85;
7  eq3 .. x1+x2=e=1;
8  model mg /all/;
9  solve mg using lp minimizing z;
```

Lines 1 and 2 declare variables, Line 3 gives names to the model equations, Line 4 defines the objective function, Lines 5–7 specify the constraints, Line 8 defines the model, and Line 9 directs GAMS to solve it.

The GAMS output that provides information about the optimal solution is:

```
1                      LOWER     LEVEL      UPPER      MARGINAL

3  ---- VAR z          -INF    210.000     +INF          .
4  ---- VAR x1          .         0.500    +INF          .
5  ---- VAR x2          .         0.500    +INF          .
```

2.9.4 Electricity-Dispatch Problem

The Electricity-Dispatch Problem, which is introduced in Section 2.1.4, has the following GAMS formulation:

```
1  variable z, theta1, theta2;
2  positive variables x1, x2;
3  equations of, ba1, ba2, ba3, bo1, bo2;
4  of .. z =e= x1+2*x2;
5  ba1 .. x1 =e= (theta1-theta2)+(theta1-0);
6  ba2 .. x2 =e= (theta2-theta1)+(theta2-0);
7  ba3 .. 10 =e= (theta1-0) +(theta2-0);
8  bo1 .. x1 =l= 6;
9  bo2 .. x2 =l= 8;
10 model ed /all/;
11 solve ed using lp minimizing z;
```

Lines 1 and 2 declare variables, Line 3 gives names to the equations of the model, Line 4 defines the objective function, Lines 5–9 specify the constraints, Line 10 defines the model, and Line 11 directs GAMS to solve it.

The GAMS output that provides information about the optimal solution is:

		LOWER	LEVEL	UPPER	MARGINAL
1					
3	---- VAR z	-INF	14.000	+INF	.
4	---- VAR theta1	-INF	5.333	+INF	.
5	---- VAR theta2	-INF	4.667	+INF	.
6	---- VAR x1	.	6.000	+INF	.
7	---- VAR x2	.	4.000	+INF	.

2.10 Exercises

2.1 Jose builds electrical cable using two types of metallic alloys. Alloy 1 is 55% aluminum and 45% copper, while alloy 2 is 75% aluminum and 25% copper. Market prices for alloys 1 and 2 are $5 and $4 per ton, respectively. Formulate a linear optimization problem to determine the cost-minimizing quantities of the two alloys that Jose should use to produce 1 ton of cable that is at least 30% copper.

2.2 Transform the linear optimization problem:

$$\max_{x_1,x_2} z = x_1 + 2x_2$$
$$\text{s.t. } 2x_1 + x_2 \leq 12$$
$$x_1 - x_2 \geq 2$$
$$x_1, x_2 \geq 0,$$

to standard and canonical forms.

2.3 Consider the tableau shown in Table 2.20. Conduct a regularization step and pivot operation to make the $\tilde{b}$ vector non-negative.

Table 2.20 Tableau for Exercise 2.3

	1	x_1	x_2
z	0	-1	-1
x_3	12	-2	-1
x_4	-3	1	0

2.4 Consider the tableau shown in Table 2.21. Conduct Simplex iterations to solve the associated linear optimization problem. What are the optimal solution found and sensitivity vector?

Table 2.21 Tableau for
Exercise 2.4

	1	x_5	x_2	x_4
z	−3	21	−1	−1/2
x_3	6	3	−1	−2
x_1	3	−1	0	1

2.5 Find the dual of the primal problem:

$$\min_{x_1, x_2} z_P = 3x_1 + 4x_2$$

$$\text{s.t. } 2x_1 + 3x_2 \geq 4$$

$$3x_1 + 4x_2 \leq 10$$

$$x_1 + x_2 = 5$$

$$x_1 \geq 0$$

$$x_2 \leq 0.$$

2.6 The optimal solution to the primal problem in Exercise 2.5 is $(x_1^*, x_2^*) =$ $(14/3, 8/3)$. The optimal solution to the dual problem has 1 as the value of the dual variable associated with the first primal constraint and −1 as the value of the dual variable associated with the second. Using this information, answer the following three questions.

1. What is the change in the primal objective-function value if the right-hand side of the first primal constraint changes to 12.1?
2. What is the change in the primal objective-function value if the right-hand side of the second primal constraint changes to 1.9?
3. What is the change in the primal objective-function value if the right-hand side of the first primal constraint changes to 12.1 and the right-hand side of the second primal constraint changes to 1.9?

2.7 Write a GAMS code for the model formulated in Exercise 2.1.

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