Chapter 6
The Rainbow of Mathematics—Teaching the Complete Spectrum and the Role Mathematics Competitions Can Play

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Abstract Although it is clear to all of us with some stake in the teaching of mathematics, that it is an important, valuable and fascinating pursuit, there does not seem to be any real agreement concerning where its central value lies with respect to what is taught in school. The core values of the subject present themselves differently to teachers, math education researchers, professional mathematicians and engineers, and this fact makes it difficult to speak with a common vocabulary about what should be taught and how it should be taught. In this paper, a model for the various aspects of mathematics, ranging from “recreational” through “school” to “applied” is presented, and the role of mathematics competitions in the continuum of this model is discussed. The various points raised in this model are then illustrated by a concrete example.

Keywords Mathematics competitions · Secondary schools · Recreational mathematics · Applications of mathematics · History of mathematics

6.1 Introduction

When people from heterogeneous backgrounds get together to think about the role of mathematics in schools, it is important to have some kind of common starting point for the discussion. As things stand, it has been my experience that such a common starting point does not necessarily exist.

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(Note that much of what is being said in this paper is derived from my personal experience. I am therefore taking the liberty of putting some things in the first person. I am fully aware that this goes against common practice in such papers, but it is my hope that it will be helpful for further discussion if my personal opinions and experiences are clearly recognizable as such.)

In dealing with people involved with the didactics of mathematics and curriculum development in Austria, there is currently a wide consensus to the effect that the important central aspect of school mathematics lies more or less exclusively in the application of mathematics to the "real world" (whatever that may be; a point I will be getting back to in a moment). It is taken as given to this group, that "pure" math is not really worthy of deep consideration in the classroom, other than what is needed to be able to deal with the most elementary of everyday applications. From this, a commonly derived opinion states that any calculations or algorithmic aspects of mathematics in the school context can and should be left completely to calculators or computers, and not be done by actual human thought at all, except in the most trivial of circumstances.

Speaking to people in the math competition community on the other hand, we have an utterly different view of what is important (and fundamental) in mathematics. While there is certainly also disagreement within this group, it is clear for people involved in competitions that the fundamentals of mathematics are represented by that part which is commonly called "elementary" mathematics. The term "advanced elementary mathematics" is often bandied about among the members of this community, despite the fact that the concept is utterly foreign to the application-oriented group. Members of this group also tend to be in full agreement that enjoyment of the study of mathematics is of central importance. The internal disagreement here often manifests itself with respect to the question of whether or not the specific skills obtained in preparing for competitions will transfer to actual research mathematics. There is, however, definitely agreement concerning the fact that subjects in pure mathematics, which for the purposes of mathematics competitions normally include combinatorics, Euclidean geometry, algebra and number theory, are the most important things for students to learn about and study in order to form a useful base of mathematical knowledge and competence. The logical skills acquired in the somewhat deeper study of these elementary topics are considered most vital in students' academic development.

Finally, speaking to teachers at the university level, the expectations of math skills that students should bring along from secondary school are different again. Furthermore, they are quite dependent on the specific academic discipline. Students in economics are expected to have quite deep knowledge of statistical methods, for example, while students in the natural sciences or engineering are expected to have some knowledge of things like differential equations or matrices; topics that go beyond what both of the previously mentioned groups generally consider fundamental.

Obviously, we are dealing with a huge disconnect here. The object of this paper is to shed some light on this disconnect, and to offer a fresh perspective. My hope is that this will make it possible to reflect better on the somewhat contradictory viewpoints held dear by the various groups of players in this corner of academia, and to ultimately improve the discussion to the benefit of the students in our secondary schools. It is my firm belief that the viewpoint offered by the universe of mathematics competitions has a great deal to offer in this respect.

In order to find a common denominator for fruitful discussion, we first need to achieve some basic agreement on what it is exactly that we are trying to decide. We must therefore find common answers, acceptable to all, to some quite fundamental questions.

The first of these is existential. Why do we think that mathematics is an important subject in school? What are our fundamental reasons for teaching mathematics as a core discipline in secondary education? As mentioned, the answer to this question seems to depend greatly on the circumstances of the person formulating an answer, and it seems clear that the concerns of each of these groups should addressed seriously.

A second important question to be answered in this context has to do with methodology. How do we best get students interested in the type of mathematics we want them to learn? Answering this question depends to a great extent on the individual tastes of the students in question. Different students have utterly different ideas of what is interesting and what is not. Relating my own personal experience in this matter, I can certainly state that my own interests have always been defined by pure mathematics, and geometry in particular. On the other hand, I have good friends, who also happen to be mathematicians, whose interests lie almost purely in applications, and their original impetus for becoming mathematicians was not derived from interest in pure math at all. For them, the gateway into mathematical research resulted from the applications first and foremost, and the idea of discovering mathematical ideas was always totally dependent on these ideas being useful to solve concrete problems. They might consider my own deep interest in the subject as being not much more than the enjoyment of mathematical puzzles, and not really worthy of total academic commitment. (Of course, since they are my friends, they are willing to allow me this luxury.)

The third problem to be addressed is purpose oriented. What are we ultimately preparing students for in their mathematics classes? To which extent are we teaching them mathematics for their own enjoyment? To which extent is this even appropriate? Are we teaching them primarily to prepare for a specific role in society? Are we primarily training their capacity for systematic logical thought? Are we preparing them for university entrance, for mathematical capability that will allow them to study technical subjects, natural sciences, or finance? Do we want to prepare them in a deep manner for what is known in German as "Allgemeinbildung"? (The term is, of course, derived from the Humboldtian ideal of higher education. The concept does not translate very easily into English, and has, in fact, mutated a bit in common understanding over the decades. It certainly goes quite a bit beyond the literal translation of "general knowledge". Some thoughts on this can be found, for instance, in Skovsmose 1994.)

6.2 Defining the Rainbow

In order to illustrate some of the ideas in this paper, let us take a look at the following picture (Fig. 6.1).

We first note that the central shaded block is composed of three sections, carrying the labels Recreational Mathematics, School Mathematics, and Applications of Mathematics. Above these, there floats a cloud containing the word History, and underneath, we see a box containing some tools alongside the word Didactics. We can often find a rainbow underneath a cloudy sky, and it is certainly possible to consider the three sections in the center as aspects in a continuous rainbow, just as the full spectrum of a rainbow can be represented in a basic way by red—yellow—blue. (Unfortunately, we will have to make do with grayscale representation here, but we can let our imaginations fill in the colors.) So, what could I mean by this in the context of learning mathematics?

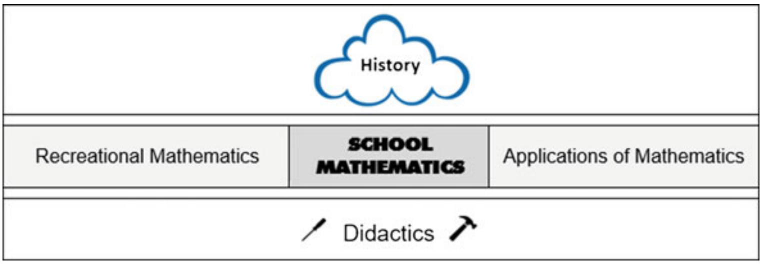

Fig. 6.1 .

Having the box with school mathematics in the center of the diagram (in all capitals for extra emphasis) is meant to illustrate the fact that we are talking about the teaching of mathematics as the central core of our discourse. We are debating mathematical subjects that can and should be talked about in the school context as well as methods that can best be used to engage the interests of students in them. The box with Recreational Mathematics on the left is meant to illustrate the aspects of mathematics that are done primarily for fun. Of course, it is quite possible that there are aspects of mathematics taught in school that students can find quite enjoyable. In fact, if the teaching process is to be successful, we would hope that such topics would be quite common. There are many aspects of so-called recreational mathematics that are not normally dealt with in school. (A very common example of such an aspect is the daily newspaper Sudoku that many people cannot imagine living without. Sudokus are certainly not commonly taught in school, but this is a perfect example of a mathematical topic that many people happily spend their leisure time on, without thought to any external usefulness. We shall be discussing the meaning of Sudoku in this context in greater depth later on.) Still, in an emotionally positive learning environment, we would hope that aspects from this side would spill over into the center.

On the right, we have a box labelled Applications of Mathematics. Many topics commonly covered in school mathematics are taught with a view toward practical applications either in everyday life (as is the case for percentages, for instance), or as a necessary base for higher level applications as can occur in scientific, technical or economic applications. As was the case on the left side however, there are many applications that are certainly never taught in school. Again, we would hope that some ideas from this side seep into the central core of school mathematics, even if higher level applications are almost certainly too sophisticated for consideration at a secondary level.

History hovering above the central boxes is meant to symbolize the fact that all mathematical ideas have a past, and this past can and should have a presence in school, at least up to a certain extent. Some mathematical concepts had their historical start in physical applications (think of differential calculus, for example), while some that may seem very applied from a modern standpoint may have originated in a recreational context (like probability theory, which started from considerations of gambling games). An awareness of this overarching historical aspect of a topic can and should make it easier for the learner to grasp the context of what is being learned. Furthermore, we can hope that an understanding of the historical context of a topic can give many students the necessary motivation toward grappling with its intricacies.

Finally, having the tool-box (represented by the hammer and screwdriver) of practical didactics as the underlying foundation is meant to represent the idea that the entire building of the academic discipline Mathematics rests upon the nuts and bolts of how it is taught. (Sorry about the mixed metaphors. Maybe we need to think of the rainbow as being painted on the side of a grand building.)

In the sections that follow, I will attempt to be a bit more precise about how this model of thinking about mathematical ideas can be useful in thinking about the learning process. More specifically, I will attempt to place mathematics competitions in their appropriate slot in this framework, and illustrate how they can show the path to a more fruitful synthesis of mathematics for enjoyment and useful application. I hope to be able to give a good argument in favor of using mathematics competitions as a tool both for popularizing mathematics as a discipline, and for preparing students for many important aspects that relate to the reason we have the subject in such prominence in the school curriculum.

6.3 Math Is Fun

We are so used to the popular notion of mathematics being called a dry, boring and incomprehensible pursuit in popular discourse that a lot of people outside the math community cannot even conceive of the truth of this heading. But, as we in the community all know, math is indeed fun. And this "fun" aspect of the subject can manifest itself in many different ways.

Why is there even such a thing as the abstract concept of Mathematics? Human nature is such that people have been fascinated by the process of abstraction for at least as long as there has been language. Discovering the fact that there is something highly elementary in the connections between utterly disparate objects exhibiting common traits that can be given a name, like "three" (the leaves on a stalk of clover, or the corners of a triangle, or the more abstract concept of past, present and future) or "circle" (the shape of the sun or the moon in the sky, or a ripple on the surface of a pond when a pebble is tossed in, or the shape you can draw with a stick in the sand by holding one end steady and moving the other) is, simply put, fascinating. And discovering that there are properties that can be found from the definition of such a concept that then turn out to be common to all objects fitting the definition is certainly something wonderful. Realizing this leads us to develop methods of finding such commonalities, resulting in concepts like counting, calculation, axioms and proof. Falling prey to the fascination of such intellectual pursuits is one way in which Math Is Fun.

Another way is well known to all ardent puzzle solvers. There are logical processes involved in solving anything from brainteasers and cryptic crosswords to hidokus and Rubik's Cubes. At first glance, the puzzles seem to be indecipherable, but step-by-step application of logical thought, sometimes combined with some trial and error, lets us inch ever nearer to a solution. Finally, after some effort, the solution presents itself. In a good puzzle, the fact that the result has been found is then completely obvious; there is no doubt that we have succeeded. Most important, a feeling of deep satisfaction results from having found the solution, by application of our own wits, to something that seemed incomprehensible at first glance, but is now utterly clear. This is another way, readily appreciated by any mathematical researcher, of course, in which Math Is Fun.

Another path to enjoyment of mathematics comes from deeper understanding of ways in which mathematical methods allow us to comprehend complex systems. A fine example of this path is the one followed by people involved in high-level financial transactions. The complex mathematical structures that they use make it possible for them to play their high stakes games, and it goes without saying that they have found for themselves a completely different way in which Math Is Fun.

Finally, for some people, simple mental calculation is enjoyable enough, and they may go so far as to cultivate arcane skills involving such things as mental division of huge numbers, memorization of the decimal digits of pi to an incredible number of places or the capacity to manoeuvre freely through hyper-cube cells in four-dimensional space in their minds. Not everyone can appreciate this type of entertainment, but to those who can, they are manifestly yet another way in which Math Is Fun.

Of course, any number of collections of mathematical puzzles is available on the book market, mechanical puzzles are readily available for purchase, and so on. It seems clear that a lot of people are actually quite aware of the fact that mathematics is, indeed, fun.

If there are so many ways in which pure enjoyment of mathematics is possible, isn't it unfortunate that so many people pass through the school system without being able to enjoy the subject in any such a way? In school, we as a society want to help our children speed up the process of abstraction, and expose them to as much as possible of the wealth of knowledge humanity has developed over the millennia. During the course of this process, we present a great deal of that knowledge in a pre-processed way, reducing the elements of individual discovery to a minimum. Of course, this is with good reason. It took humanity many generations to reach the level of sophistication we have now, and it would not be feasible to expect every youngster to figure everything out on his or her own. After all, it took the wisest brains of many generations to come up with what we, as a society, know now. Unfortunately, the accelerated processes typically used in school tend to suck much of the entertainment out of the subject.

Even knowing this, a lot of mathematics remains enjoyable. Sometimes, we may not realize that we are doing mathematics while we are playing with it. Not everyone solving a newspaper number puzzle is cognizant of doing mathematics. Nor did every participant in the great puzzle crazes of the past decades, from the 15-puzzle through Instant Insanity and Soma to the Rubik's Cube necessarily think of their hobbies as intrinsically mathematical, even though they obviously were. It seems clear that any way to introduce this type of enjoyment to the learning process must be advantageous.

Some mathematics competitions offer puzzle problems that give a large number of contestants the opportunity to have some mathematical fun of this type, and the millions of competitors taking part in competitions like the Mathematical Kangaroo, the American Mathematics Competition, or the Australian Mathematics Competitions (just to name a few) show that the enjoyment to be derived from thinking about such questions is well known to many. So, here we have an obvious way in which the math competitions scene is helping to achieve the goals we aim for in regular mathematics education. Helping students to see how enjoyable it can be to solve mathematical problems/puzzles (the distinction becomes quite blurry at times) gives them the impetus to delve deeper into the subject.

Here is the first facet of the Rainbow. One big reason for us to do mathematics is simply because it is fascinating and because it is enjoyable. Next, let us have a look at the opposite end of the spectrum; the other reason we should all be able to agree on for why mathematics is such an important discipline.

6.4 Math Is Useful

On the opposite side of the spectrum of mathematics, we have the utility of mathematical abstraction combined with practical calculation that makes mathematics so useful. Of course, this is also a reason why many people are fascinated by mathematics as a discipline in the first place. Many, for whom mathematics may not have held any particularly high level of fascination in school, become quite enamoured of the pursuit because of the surprising connections it helps to uncover in practical applications. This can be derived from physics, applications in engineering, financial transactions or any number of other things.

Unfortunately even research mathematicians cannot always agree on what exactly is meant by “useful”. As was already pointed out, pure mathematicians have a quite different point of view from applied mathematicians, and therefore often find different areas of elementary mathematics to be of elementary importance to their work. Nevertheless, all can agree that things can and should be taught in school because they are, simply, useful. And in any case, the fact remains that mathematics is in some way intrinsic to most any abstract discipline.

For many people, the day-to-day practical aspect of the subject is the central, and perhaps only, justification for its inclusion in the school curriculum in a central role. This is certainly currently the case in the Austrian school system, which I know best from practical experience, and I shall elaborate a bit on in the next section. In my opinion, it is however quite unfortunate if this is considered to be the sole defining justification for the subject. It does seem clear that the things we teach our students in school should have some connections to future applications, of course, but this statement can be interpreted in different ways. We can all agree that school should certainly convey the capability for dealing with everyday calculation to all students. They should learn how to deal with cash transactions in the course of making their daily grocery purchases, calculating the savings involved when something is advertised as being on sale at 10% off, or figuring out how many cans of hi-gloss are required to repaint the garage, and we are certainly all in agreement that the basic intellectual tools needed to solve such problems should be acquired in school.

From the standpoint of preparing secondary school students for the tertiary level, however, there does not seem to be so much common ground. Most would agree that there is a certain amount of higher level mathematics that must be taught in an effective manner, but what does this include? If we want to prepare our high school graduates for studies in the sciences or engineering, we will want them to have some accessible fundamental knowledge of real functions and calculus, algebraic manipulation of polynomials and solving equations, and so on. If we are worried about preparing them for the necessities of anything involving statistical analysis, like medicine, economics or the social sciences, we will want them to have some skills in interpreting statistical tests and working with random distributions. If we are worried about training future mathematicians and computer programmers, we will want them to have some understanding of mathematical proof and algorithms. Or, in the extreme, we can take the position that we do not want to train our students to understand deeply any of this, arguing that they can pick up the necessary knowledge at the tertiary level, and limiting what is taught in secondary school to what is needed for "communication with experts" (see Fischer 2001). This is the current basis for the Austrian school system, and in my opinion this is not at all sufficient.

6.5 Math in School. Connecting the Fun and the Usefulness

I would argue that all aspects of mathematics should be included in an ideal secondary curriculum. In order to keep all students interested and motivated, there should be aspects of recreational mathematics, applications of mathematics, and the history of the subject represented in the classroom. Graduates of our schools should have a reasonably developed feel for numbers, shapes, data and functions. They should understand the value of proof in an axiomatic system and be somewhat schooled in abstract thought. There should be room for the many fascinating aspects and the many uses of the subject, as well as aiming toward achieving the ideal of educated people having a well-grounded understanding of the subject.

Depending on their own point of view, many people think that only one or the other of these aspects is appropriate for schools to worry about. Limiting mathematics in school to practical applicability, however, leaves no room at all for recreational aspects or for the development of pure mathematics as a scientific discipline. Also, the reality of schools often does not allow any kind of deeper insight or any kind of enjoyable work with mathematics because the available time must be used to prepare students for specific types of central exams, which typically test only the ability to deal with highly specific problem formats. This is not good. A good school system will not put undue emphasis on simple calculation, nor will it force the majority of available classroom time to be spent on the study of specific test formats. A good class is one in which the students' minds are challenged in many different ways and in which their individual preferences and interests can find a home, whatever they may be.

Taking a closer look now at the current state of the Austrian school system, we see that there has recently been a shift completely away from anything involving operative mathematics in the secondary schools, and oriented strictly toward applicability.

The opinion of some mathematics educators who feel that all mathematics taught in school should be introduced through "real world" applications now completely dominates the discussion, even if many teachers put up quite a bit of resistance in their classes. (It is worth noting that there is a good reason for the quotation marks here. What is considered the "real world" in mathematical texts is, of necessity, always a stark simplification of reality, with a strong element of pre-digestion having been introduced by the problem authors. The real "real world" is invariably more complex than the highly simplified mathematical models used in the school situation would generally suggest.)

The pure enjoyment of mathematical pursuit is thrown out the window in this educational model, as is the value of abstract thought in a liberal arts education. Both aspects are sacrificed at the altar of applicability.

Furthermore, centralized testing has led to complete dominance of the teaching-to-the-test phenomenon, to the detriment of all else. One can only hope that this state of affairs, which has only come into full force in the last few years, will soon pass, but the plan to move to stronger inclusion of technological aids in mathematics instruction (graphing calculators, CAS and spreadsheets) unfortunately suggests that things will get worse before they get better.

This unfortunate development resulted from an attempt to improve mathematics teaching, of course. Comparing any current textbook to one used, say, in the 1960s, gives an excellent view of what has happened. It is certainly true that there was formerly far too much emphasis placed on calculation for its own sake. Looking at the old textbooks, we find any number of difficult problems involving simplification of quite involved term expressions, for example, and such things can no longer be found in current textbooks. The argument given for the change was that students did not actually gain any real understanding of what they were calculating, and there is a great deal to be said for this. Unfortunately, in the process of reducing this type of rote learning, some topics were eliminated completely, despite the fact that the fascination emanating from them can certainly help a great deal in giving students the motivation to learn.

Different people have different tastes, and while some are readily motivated by pure abstraction and others by the wish to come to grips with practical matters, there cannot be one singular path to motivation equally applicable to all learners. Surely the aim of teaching is to optimize the motivation to learn for as many students as possible, in order to maximize the amount of knowledge students can absorb and develop. Since students can be motivated by quite disparate pathways to such knowledge, it seems quite obvious that all such paths should be reasonably represented.

6.6 Mathematics Competitions: Great at Connecting

One of the main points I would like to get across with this paper is the idea that mathematics competitions are uniquely suited to getting many (though, of course, not all) students more deeply and more actively engaged in mathematical pursuits. Parts of this argument have already been hinted at, but in this section, I would like to present it in a more structured way.

When students get hooked on mathematics competitions, this means that they have developed a feeling for the fascination of problem-solving on an abstract level. Finding solutions to competition problems of progressively higher levels of difficulty leads them on a journey to discovering and writing proofs, and with this they are really learning to be active mathematicians themselves. Compared to what they are confronted with in "regular" math classes, there are some specific qualities to the style of mathematics they encounter in the competitions world.

First of all, there is the feeling of accomplishment that comes from solving a competition problem. This is the same feeling one gets from successfully solving a puzzle or from proving a theorem, but in the context of a competition, it can be reinforced by the fact that points are awarded, and the student may have achieved something that others writing the competition have not. Regular classroom mathematics tends to negatively reinforce not being able to solve a problem (which might even result in failing a test) rather than positively reinforcing the solution of a problem that can be considered at the outset as being optional. It goes without saying that positive reinforcement of this type is preferable from a psychological viewpoint. This positive reinforcement then usually transfers quite well to regular classroom work. (This last claim is what I see quite commonly in my own classrooms, but I am sure that anyone working both directly with students in competition preparation and in a regular classroom setting will agree.)

Essentially, this is part of the argument in favor of using recreational mathematics to get students more actively involved in their classrooms. In the Rainbow, this means that the left-hand box positively influences the central box. The implication is that the participation of students in competitions is therefore quite useful as part of the underlying Didactics tool-box.

Another strong influence of math competitions lies in getting the students to accept the need for logical rigor in their work. If any of their calculations or proofs is logically incomplete, they will simply not score full points, even if they have understood all of the essential parts of an argument. This is disappointing for a student who has become used to the feeling of success that comes with solving a problem. Again, the positive reinforcement that then comes with understanding the need for a logically complete argument in order to get full points in a competition is much better than the negative reinforcement of just being criticized for something incomplete.

While this applies to any kind of mathematical argument, including simple computation, it is especially true for learning to understand the meaning of the axiomatic method in producing proof. Learning this in a normal classroom is quite abstract and involved. In the context of a competition, however, it is very natural (although perhaps not really any easier). It is obvious to all competitors that an argument must be complete if a student hopes to receive full points. It is quite easy to accept this in the context of a competition, as a competitor's more complete argument will obviously be better than mine, if mine is missing some salient points.

For the purpose of learning the axiomatic method and the concept of what constitutes complete proof, classic topics are certainly the best. There is an obvious historical reason why the classic Euclidean topics of geometry and number theory/arithmetic are the areas in which the axiomatic method was developed, and this is certainly also the reason why there is still a wide international consensus that these topics should be included in a central role in competitions. The historical argument is quite strong, not just for intrinsically historical reasons, but because historical development in this area happened for a reason. These topics are basic to human abstract thought, and taking this route during the learning process is as basic and reasonable now as it ever was.

Starting on the right-hand side of the Rainbow, it can also be argued that a similar path from the Applications box is offered by classes in mathematical modelling. In many places around the world, students especially interested in applied mathematical problems are offered participation in such activities that are also competitions of a sort, even if there are generally no "winners" declared. (I refer here specifically to the model of the "Mathematical Modelling Week" as I know it in Styria, in the south of Austria, as this is the one I am most familiar with. Similar programs are, however, offered in many places.) As a path to applied math at a higher level, high school students are invited to work for a week under the tutelage of professional mathematicians on the modelling of some applied problem. These can be from physics, medicine, economics, or any number of other areas, but generally they will be derived from the research specialties of the tutors. While these are not competitions in the traditional sense since there are no winners, it can be argued that all participants in these workshops are "winners" by virtue of their completion of the tasks at hand, and there are simply no "losers". Psychologically, this is certainly a good thing. Otherwise, I would argue that the net positive results of such an activity are the same as those in a more typical competition format. Participants derive the same sense of accomplishment in finding a path toward solving a problem that they could not initially deal with. Through diligent application of logic, they finally arrive at a result that they have every right to be proud of, yielding a strong positive reinforcement.

This can be seen as giving added value to the middle box in the Rainbow from both sides. The problems in such modelling projects can be considered as both Applied and Recreational, at least from the point of view of the active participants.

All told, the argument in favor of mathematical competitions of all types in reinforcing the path to a deeper understanding of mathematics among interested participants is quite strong.

6.7 History on Top; Didactics on the Bottom

Returning briefly now to the picture of the Rainbow (Fig. 6.2), we can concern ourselves a bit more with the top and bottom bars.

The underlying bar labelled "Didactics" is more or less self-explanatory. In school, everything is built up on a base of teaching methodology, and this base is symbolized here by this one term. It includes matters of curriculum, textbooks and worksheets, classroom organization, and so on, and is symbolized here by very elementary tools, namely a screwdriver and a hammer. No matter what we decide to teach in school, we must certainly worry about how we are going to go about teaching it—the nuts and bolts of work in the classroom.

Perhaps a bit more explanation is required for the History cloud. Its floating above all else is meant to imply the fact that all areas of mathematical thought not only have a genesis, but that this genesis is an important intrinsic part of the area.

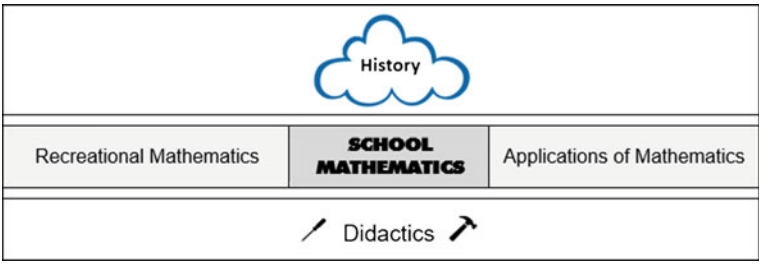

Fig. 6.2 .

No part of mathematics starts in school. Everything starts either as a game like statistics or as an application for further development of something that already existed. Much mathematics is derived from axiomatic interpretation of some aspects of real life. Mathematics is in its core abstraction.

Let us take probability theory as an example. The roots of what we now think of as probability reach back to the 17th century. Some of the biggest thinkers of the day (Cardano, Fermat, Huygens, Pascal) were thinking about games of chance, and the likelihood of winning and losing. While such considerations can certainly have very practical applications for some people, there is an argument to be made for placing these considerations firmly in the realm of recreational mathematics. Throwing dice, flipping coins or playing card games are certainly recreations for all but the most hard-core professional gambler. From this beginning, however, there arose an elaborate theory with applications in such disparate areas as medicine, finance and opinion research.

As has already been alluded to, there are at least two strong arguments to be made for the inclusion of at least some of the history of such a discipline in its teaching.

For one, there is the motivational argument. Getting students interested in a topic gets them invested in the learning process, and the consideration of the historical process that led to the development of a topic can help get students interested in the topic for the same reasons that the scientists that originally developed the theories were interested in them. This is completely independent of the question of applicability of the whole logical structure once it has been developed. The a posteriori uses of a mathematical method are generally not clear at the historical outset of its development.

Furthermore, there is also the methodological argument that a topic can be better understood if it is learned at least in part by following the train of thought that led historically to our current understanding of it. Skipping over the history by reducing mathematics to a system of definition-theorem-proof (which certainly has its place in the university) deprives the student of an important level of understanding.

6.8 An Example from the Rainbow: Sudoku to Graph Coloring

Let us now take a look at a specific topic, how various aspects of it are represented in the different boxes of the Rainbow, and the role that mathematics competitions can play in developing understanding of it.

		1				7		4
	6							
3			8	4	5			
								2
	9	2		7	6	4	5	
8								
			5	9	2			7
							8	
4		7				1		

Fig. 6.3 .

8a: A Popular Pastime: the Daily Sudoku
In the last ten years or so, sudokus have assumed a prominent place in the public consciousness by their ubiquity in the daily papers and in puzzle books available at any book store or news agents'. As is well known, the idea is to fill in a grid of numbers satisfying certain constraints. In a classic sudoku, the numbers from 1 through 9 must be placed in each row and in each column of a 9×9 square grid, and each number must be present in each of the nine 3×3 squares the 9×9 square is composed of. Several numbers are already given in the grid, and the point of the puzzle is to find the unique way to fill in the rest. An example of such a problem grid is shown in Fig. 6.3.

There is no doubt that this is an incredibly popular pastime, and the fact that there is at least a bit of mathematical content involved is already obvious from the fact that numbers are used in the squares. There are many related puzzle types that have found their way into some daily papers and the public consciousness along with them, like Kakuro, Hidoku, Fillomino, and so on.

The reason that such puzzles are so popular lies in the fact that solving them gives the solver a distinct feeling of accomplishment. While we are aware of the fact that we are doing something that isn't really of any immediate use to us (or anyone else for that matter), there is an intrinsic joy in finding the solution. This is the basis for all so-called "recreational mathematics". If the only argument for doing it lies in the recreational aspect, the external value of the actual mathematical content becomes completely irrelevant for the time we spend on the problem.

Of course, this is an aspect of competition mathematics. When students are solving problems in a competition, they are not worried about applicability. They are simply solving the problems for their own sake. The problems themselves are considered interesting, independent of any meaning they may take on in the "real world", and finding the solution (and then possibly being awarded points for it) is the reward they are seeking.

Notably, this is also often the main motivation behind more serious mathematical research. Certainly, some research problems must just be solved in order for a specific application to work, or to guarantee funding for yet another financial period in some research institution. In general, however, anyone involved in any reasonably abstract mathematical research is searching for the solutions because of an intrinsic interest in the problem itself and the deep sense of achievement that comes with finding a solution to a difficult problem.

8b: Mathematical Research and Applications related to Sudoku

Starting from the highly elementary content of Sudokus, there are several different directions our thoughts can take in order to derive mathematical research problems.

Perhaps the most obvious concerns itself with the internal mathematics of the puzzles themselves. There are many questions that can be posed concerning the statement of a sudoku problem or its solution. Some of these are the following:

- What is the smallest number of numbers that can be given in a sudoku grid, such that the solution is unique?
- What is the largest number of numbers that can be given in a sudoku grid, such that the solution is not unique?
- How many minimal sudokus exist? (A "minimal" sudoku is one in which the solution is unique from the given numbers, but in which no given number can be deleted with the resulting sudoku remaining unique.)

Such questions are the focus of a certain strand of mathematical research, and some prove much easier to answer than others. (Interested readers are invited to find out the current state of knowledge concerning such questions by checking Wikipedia (https://en.wikipedia.org/wiki/Mathematics_of_Sudoku) or other easily accessible internet sources.) Solving this type of problem, however, does not stray far from the mathematical content of the Sudokus themselves.

Taking a closer look at the sudoku concept, we see that there is another path to abstraction we can take, that will lead us right into the heart of research mathematics.

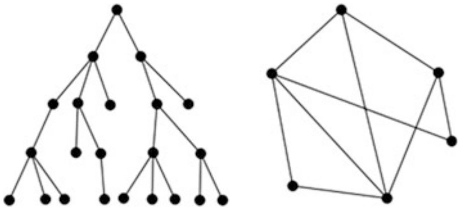

Fig. 6.4 .

As it turns out, it is quite straight-forward to express the solution of a sudoku as a graph coloring problem, and this idea connects the popular puzzle both to cutting edge research in abstract mathematics and to real-world mathematical applications. So, what do we mean when we say that solving a sudoku is equivalent to solving a graph coloring problem?

In mathematics, a graph is, of course, a structure composed of points (or vertices), that are joined by lines (or edges). These are commonly represented by pictures like the ones in Fig. 6.4:

We can consider a sudoku, composed of 81 cells in a 9×9 square grid, to be represented by a graph with 81 vertices. Each vertex is to be colored with one of 9 colors, corresponding to the numbers 1 through 9. Some of these colors are given, with the rest to be determined.

The nine cells in a common row (or a common column or a common 3×3 square) can be thought of as being joined pairwise by an edge. Solving the sudoku then amounts to finding a coloring of the graph with the nine colors, such that no two vertices with the same color are joined by a common edge.

When thought of in this way, it becomes clear that our daily newspaper sudoku is completely equivalent to a seemingly much more abstract problem. With this, we are already firmly in the middle of a practical research topic. The puzzle, considered purely for the sake of the enjoyment of finding its solution, has led us directly into the world of mathematical applications. Now that we understand this, we can strip away the camouflage and take a look at where graph coloring can lead in mathematical research.

First, let us consider a practical application of graph coloring, namely the problem of job scheduling.

Let us assume that we have a certain number of jobs that need to be done in some order. Certain of these jobs may be in conflict with each other, i.e. there may be some reason why they cannot be dealt with simultaneously. (For instance, the same person may be required to fullfil two tasks or the same machine may be needed for two distinct steps in production.) It is possible to represent the scheduling problem by drawing vertices of a graph corresponding to the jobs. Any two jobs that conflict with another can then be joined by an edge. The smallest number of colors with which it is possible to color the vertices of the graph without like-colored vertices ever being joined by a common edge then gives us information on the most efficient way for the jobs to be scheduled. This model can translate not only to concrete "jobs" that need to be done by people, but also to organizational problems ranging from the assignment of vehicles to individual trips for a delivery company to the assignment of frequencies to terrestrial television broadcasters in geographically conflicting areas.

Next, let us have a look at a more theoretical graph coloring problem that happens to be right at the cutting edge of modern mathematical research, namely the question of the chromatic number of the plane, also known as the Hadwiger-Nelson problem.

The problem can be stated in the following way. What is the smallest number of colors with which it is possible to color the points of the plane in such a way that no two points at unit distance have the same color?

Much has been written about this problem (see, for instance (Soifer 2008)), but despite more than half a century of intense research, the problem has not yet been solved. In fact, as easy as the problem is to state and understand, it is one of those intractable mathematical questions that are really devilishly difficult to grasp. It may well be that the problem cannot even be completely solved without making some non-standard assumptions, like the validity of the Axiom of Choice. It is relatively straight forward to show that the number in question must be larger than 3 and it can also be shown that it must be smaller than 8, but values of 4, 5, 6 or 7 are still possible.

A famous coloring problem of a related type, located somewhere on the spectrum between purely theoretical and practical, is the four-color map problem. For many years, there existed a conjecture, since famously proven with the help of computer-based methods, that any map in the plane can be colored by at most four colors in such a way that no two countries sharing a common border have the same color.

We see that the same sudokus that we know so well from purely recreational mathematics are related quite directly to problems both in concrete applications of mathematics and in high-level research in pure mathematics.

8c: Sudoku, Graphs and Coloring in School.

Neither sudokus nor graph theory are a standard school topic in most countries. Recently, many schools have taken to using something closely related to sudoku in order to give students an opportunity to practice mental calculation, namely kenken. (Note that KenKen is a registered trademark. Interested readers can find a large number of such problems at (http://www.kenkenpuzzle.com). The puzzles are sometimes also referred to under other names, such as Kendoku.)

For those not familiar with kenken, a brief introduction seems in order. As is the case for sudoku, a kenken puzzle is a square grid, and the goal of the puzzle is to place numbers in the grid in such a way that none of the numbers repeat in any column or row. If the size of the grid is nxn, the numbers from 1 through n are to be placed in the cells of the grid. Unlike sudokus, however, no digits are given in advance. Instead, certain areas are given, in which the numbers can be combined by addition, subtraction, multiplication or division with some given result. For instance, if two cells are joined to a 2 × 1 rectangle with the symbol "4 + ", this means that the two cells are to contain two different digits with the sum 4, and therefore one must contain the digit 3, and the other the digit 1. In some cases, there is more than one combination possible, as for instance for "2-". This could be the result of 3–1, 4–2, 5–3, and so on. Furthermore, if a single cell contains only one number without an operation, this number can be considered as given in that cell. An example of such a puzzle is shown in Fig. 6.5.

However, use of these puzzles in the classroom is not normally a path to understanding about graph coloring. The didactic idea behind the use of this in the classroom is for the students to get a better feel for number combinations in simple elementary calculations, and kenken gives an amusing context to such calculations.

Simple graph theoretical ideas are, however, also often championed for inclusion in the school curriculum (see, for instance Smithers 2005), especially in schools that are preparing students for computer programming. Most school systems, however, do not currently include this subject in their curricula. Students preparing for mathematical olympiads do, however, routinely deal with elementary graph theoretical ideas, as this is a common topic of olympiad problems in the so-called Combinatorics category. An example of such a problem will be given in 8e.

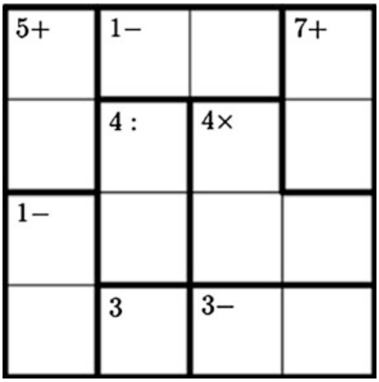

Fig. 6.5 .

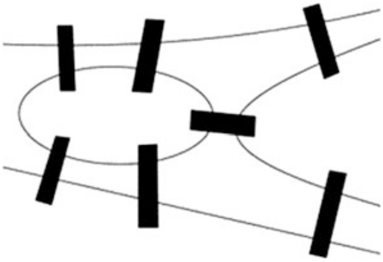

Fig. 6.6 .

A	B	C
B	C	A
C	A	B

Fig. 6.7 .

8d: History and Didactics: Graph Theory and more

If any graph theoretical ideas make it to the classroom at all, a likely candidate for inclusion is the classic Königsberg Bridge problem of Leonhard Euler (1707–1783). This problem, asking whether it is possible to cross each of the seven bridges in old-time Königsberg exactly once in one walking tour of the town, which straddles a river with islands as shown in Fig. 6.6 is the starting point of modern graph theory.

Students may not know anything about the history of the city of Königsberg (now the Russian city of Kaliningrad), but the question is a very practical one that can be readily understood. Also, its solution can be developed by simple logic, without resorting to any high-level mathematical tools. Giving some historical context can certainly make the topic more interesting for many students, and this is also a good excuse to name-drop Leonhard Euler in class.

Another interesting historical sidebar that might be mentioned in this context, is the Latin Square. A Latin Square is an nxn array, with n symbols written in the cells in such a way that each of the n symbols is represented once in every row and in every column of the array. This is also a subject studied by Euler, and the name is derived from his work, in which he used Latin letters as his symbols. A 3×3 example, such as could be found there, is shown in Fig. 6.7.

A Sudoku is, of course, a Latin square with some special restrictions, in which the symbols are digits. These mathematical objects have been studied to quite some extent since the 18th century. The idea behind them is closely related to (but not to be confused with) the idea behind the so-called Magic

Squares, in which the sums of numbers in all rows and columns (and often also diagonals) are equal.

Both these topics are typically seen as purely recreational, but as shown here, they are at the very foundation of an important section of mathematics that ranges through the whole rainbow, from recreational to applicable.

8e: An example of a graph coloring problem from an international competition.

An example of a nice competition problem concerning graph coloring is the following problem from the International Tournament of the Towns (Spring 1990, Senior O level):

(a) Some vertices of a dodecahedron are to be marked so that each face contains a marked vertex. What is the smallest number of marked vertices for which this possible?
(b) Answer the same question, but for an icosahedron.

(Recall that a dodecahedron has 12 pentagonal faces which meet in threes at each vertex, while an icosahedron has 20 triangular faces which meet in fives at each vertex).

In order for a student to solve this problem successfully, it is helpful to realize that it is indeed a graph coloring problem. The vertices of the polyhedron being considered can be thought of as the vertices of graphs, and the edges of the polyhedron as edges of these graphs. Of course, this is a three dimensional concept, but the graphs in 3-space can be projected onto a plane (for instance, from a point on the circumscribed sphere of the polyhedron onto the tangent plane diametrically opposite to this point), resulting in corresponding plane graphs with completely analogous properties. Since we then wish to "mark" vertices, we can think of this as coloring all the vertices of the graph with two colors, say black and white, with black corresponding to "marked" vertices and white to "non-marked" vertices.

The solution to part (a) is then quite simple. Since each vertex lies on three faces of the dodecahedron, marking any vertex gives three faces a marked vertex. Since there are 12 faces, we must certainly mark at least $12:3 = 4$ vertices. We can see in Fig. 6.8 (a graph representing the dodecahedron's vertices and edges), that a marking of four vertices (represented by the full points) is indeed possible, such that each face has a marked vertex.

Part b is a bit more sophisticated. We can see in Fig. 6.9 that a marking of six vertices such that each face has a marked vertex is possible.

It remains to be shown that such a marking of five (or less) vertices is not possible. We can prove this by contradiction.

Let us assume that it is possible to mark five vertices in such a way that each face has a marked vertex. We consider the graph (as shown above) and delete all edges with the exception of those joining two marked vertices,

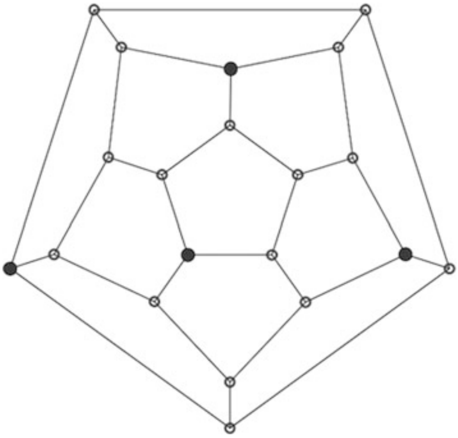

Fig. 6.8 .

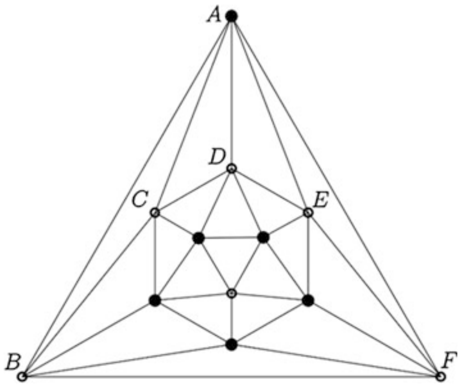

Fig. 6.9 .

and consider the number of components of the resulting graph. (Recall that a "component" of a graph consists of a subset of the vertices, connected by edges of the graph.) In any of these components, a first marked vertex contributes to 5 faces, but any succeeding vertex in this component can only contribute to at most 3 further faces that do not yet have a marked vertex. If there are at most 5 marked vertices and at most two components, the marked vertices can contribute to at most $5 + 5 + 3 + 3 + 3 = 19$ faces. We see that the graph must consist of at least three components. At least one of these components must then consist of only one marked vertex. Let us assume that this is vertex A in the figure above. This means that none of the vertices B, C, D, E and F is marked, and four of the remaining vertices must be marked. This is not possible, however, since these four would then certainly all be in the same component, in contradiction to the assumption that they contribute to faces in at least two separate components. We see that at least six vertices must be marked, as claimed, thus finishing the proof.

6.9 Conclusion

Mathematical instruction should include all aspects of the subject and engage students in whatever way they can be led to be interested in the subject. This is different for each person. Some will be excited by abstract math problems independent of any applications in the real world. This includes mathematical puzzles, mathematical games or individual pure math research. Others will be excited by the opportunity for applications, for instance in physics or other areas.

In this paper, I have attempted to argue that a complete treatment of any mathematical topic in school should include aspects from the complete Rainbow of Mathematics, in order to help every student of the subject find something suited to her or his tastes. A mathematical topic can be introduced starting from most any mathematical problem, be it a number puzzle (number theory, coding), a triangle problem (olympiad geometry, school trigonometry, land surveying) or a practical application. I have also tried to argue the fact that the world of mathematical competitions offers a strong tool, independent of where a student hops on board the math train.

Let us briefly return to the fundamental questions on the value of mathematics as a core subject in secondary education as posed in the introduction. Here are some answers I believe we could and should all agree on, considering all that I have presented here.

Question 1: Why do we think that mathematics is an important subject in school?
It seems clear to me that there are essentially three equally valid answers to this question.

First of all, mathematics is necessary for many things. Some elementary things, like basic number skills, are obvious prerequisites to life in a modern industrial-technological world. Other things are not of such import to everyone, but since school is meant to prepare students for their future professions and for their tertiary studies, a great deal of mathematical knowledge must be at their disposal when they leave high school, simply to prepare them for this. This is the practical argument.

Secondly, mathematics is interesting and enjoyable. This is true in many ways. Logical abstraction is a fundamental human thought process that has fascinated humanity for EONS. Individual mathematical problems are often interesting for their own sake, and finding their solutions is an enjoyable process. Students should certainly be offered the opportunity to experience this enjoyment for themselves. Mathematics competitions can play a large role in this, even if not every individual enjoys them in the same manner. Still, math is important because math is fun. This is the recreational argument.

Finally, studying mathematics schools abstract rational thought. Ideally, this should be true of most subjects in school, but the abstract world of pure mathematics is certainly the optimal ecosystem for such things to flourish. This is the abstract argument.

It is my firm belief that all three arguments are legitimate and strong, and that the various aspects of mathematics must therefore all be strongly represented in any complete curriculum.

Question 2: How do we best get students interested in the type of mathematics we want them to learn?
The answer to this question is, of course, dependent on the individual student's interests. Some students will be drawn in by the mathematical abstractions themselves. For some, the most interesting aspect will lie in potential applications. For yet others, it may be the historical context, the development of human thought through the generations. And for some, it may simply all be a game, and playing around with puzzles will prove the best path to the subject. All of these gateways are perfectly legitimate, and it seems clear to me that optimal teaching practices must offer at least a little bit of everything.

Question 3: What are we ultimately preparing students for in their mathematics classes?
Again, my answer to this question must necessarily be quite wide. We certainly want students to enjoy mathematics. Whether this is the most important aspect, or even important at all, will be up to individual teachers to decide. To my way of thinking, this is the base of all else, and students who do not have at least a semblance of enjoyment in their class work cannot be expected to fully appreciate the subject.

We are certainly teaching our students to prepare them for their future roles in society. This aspect cannot and should not be ignored. In this context, we must also prepare them for university. The tertiary institutions cannot be expected to start from scratch; human brains must have some developed mathematical competence by the time puberty is over, otherwise it is too late.

I would also argue that we should be training students' capacity for systematic logical thought and offering them as much general knowledge (here is that pesky concept of "Allgemeinbildung" again) as possible. If this is not to be imparted in the schools, then where?

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