Chapter 16
Questions on the Classical Schemes of Inference

In the classical calculus with precise concepts, some schemes of deductive inference are used such as the modus ponens (MP), and the modus tollens (MT), but also the so-called disjunctive mode, among others. They are instances of what is often known as the Aristotelian logical forms, and are of some interest for the mechanizing of formal deduction.

In what follows a scrutiny of the validity of these schemes is conducted for first certifying them in Boolean algebras, second (in Chap. 5) to know which conditions can hold in a basic fuzzy algebra (BAF), and finally what can be said about their validity in the general case of ordinary reasoning under the model of natural inference. Of course, several laws should be applied for proving such schemes in the classical calculus, laws that, in general, cannot always be presumed in ordinary reasoning.

16.1. Concerning the schemes of modus ponens and modus tollens, respectively,

$$p, p < q : q, \quad \text{and} \quad q', p < q : p',$$

they can be posed in three not properly coincidental forms; the first is purely algebraic, the second is tautological, and the third concerns truth values for modus ponens. They are the following.

- p and $p \to q = p' + q$, imply q;
- $p = 1$, and $p \to q = p' + q = 1$, imply $q = 1$,
  and
- $t(p) = t(p \to q) = t(p' + q) = 1$, imply $t(q) = 1$.

The first is formally proven by: $p \cdot (p' + q) = p \cdot q \leq q$; the second follows immediately because $p = 1$, means $1 = p' + q = 0 + q = q$; and, concerning the third $t(p) = 1$ implies $1 = t(p' + q) = \max(1 - t(p), t(q)) = \max(0, t(q)) = t(q)$.

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Hence, the Boolean model actually certifies that the three versions of the scheme hold.

It should be noted that MP holds universally in the former model of ordinary reasoning, provided the "conditional statement" $p < q$, once p is known, would effectively allow a "movement" up to q. Concerning MT, once q' is known, and because $p < q$ implies $q' < p'$, if p' can be effectively reached, MT also holds in the general model. In it, MT is a consequence of MP (q': $p < q \Rightarrow q'$: $q' < p'$:: q'), although the reciprocal cannot always be stated.

Regarding the Boolean case, the three previous forms also hold for MT because $p \to q = q' \to p'$; MT is only equivalent to MP in the Boolean framework.

Regarding the Boolean case with $p \to q$ expressed in conjunctive form $p \cdot$ q, note that MP always holds in all lattices because $(p \cdot (p \to q)) = p \cdot (p \cdot q) = p \cdot q \leq q)$, but MT cannot hold in Boolean algebras because $q' \cdot (p \cdot q) = 0$. It should be pointed out that, were the lattice a De Morgan algebra, then $q' \cdot (p \cdot q) = p \cdot (q \cdot q')$ would not always be 0, hence $q' \cdot (p \to q) = q' \cdot (p \cdot q) = p \cdot (q \cdot q') = q \cdot (p \cdot q') \leq q$, and MT holds for the non-Boolean elements. Thus, the validity of MT not only depends on the laws of the corresponding algebraic structure, but also in how the conditional $p \to q$ is expressed.

16.2. Consider the disjunctive scheme:

$$\text{If} \quad p' \quad \text{and} \quad p+q, \text{then} \quad q.$$

In fact, it follows from $p' \cdot (p + q) = p' \cdot q \leq q$, and q can be concluded.

Another form of posing this scheme is:

$$p' = 1, \quad p+q = 1, \quad q = 1$$

by presuming that both p' and $p + q$ are tautologies. Thus, $p = 0$ and $1 = p + q = 0 + q = q$.

Finally, the last form is with just truth values:

$$t(p') = 1, \quad t(p+q) = 1, \quad t(q) = 1$$

where neither p', nor $p + q$, are necessarily tautologies but have truth value one. In this case, $t(p') = 1 - t(p)$ implies $t(p) = 0$ and $1 = t(p + q) = \max(t(p), t(q)) = t(q)$. Thus, $t(q) = 1$ is concluded.

Under the three forms of posing the question, q is concluded, and the Boolean model certifies a scheme that has been accepted from very old times in precise deductive reasoning.

Concerning its possible universal validity in the general model, such as those of the schemes MP, MT, once $q < p + q$ is accepted, it follows that $(p + q)' < q'$; thus once q' is obtained and provided the negation of q were intuitionistic, $(q')' < q$ (or strong), q is forward reached from the negation of q'. Hence, with the conditions that the negation reverses the inferential relation < and it is intuitionistic or strong in particular, the disjunctive scheme holds universally.

16.3. Regarding the scheme of proving by reduction to absurdity,

$$q, p < q' : p',$$

expressed in its three versions in a Boolean algebra,

- $q \cdot (p' + q') = q \cdot p' \leq p'$;
- $q = 1,\ 1 = p' + q' = p' + 0 = p'$, and
- $1 = t(q) = t(p' + q') = \max(t(p'), 1 - t(q)) = \max(t(p'), 0) = t(p')$,

is also certified in the Boolean model. Note that it does not hold in a lattice whatsoever with $p \to q = p \cdot q$, inasmuch as $q \cdot (p \to q') = q \cdot (p \cdot q') = 0$.

Regarding the general model, provided the negation of q were weak, or strong in particular, that is, it would verify $q < (q')'$, because from $p < q'$ follows $(q')' < p'$; then and provided the negation reverses <, and the triplet $(q, (q')', p')$ is transitive, it would result in $q < p'$. Thus, under the conditions of reversing <, weak or strong negation, and transitivity, *reductio ad absurdum*, holds universally; but, provided one of these conditions were to fail, the question would remain open. Reduction to absurdity is risky when <- transitivity fails.

16.4. The so-called scheme of resolution,

$$p' < q, q < s : p + s,$$

algebraically follows from

$$(p + q) \cdot (q' + s) = p \cdot q' + p \cdot s + q \cdot s \leq p + s + s = p + s,$$

with the distributive law playing a pivotal role.

In the case of tautologies, $p + q = q' + s = 1$, it also follows that $1 \cdot 1 = 1$ $p + s$, or $p + s = 1$.

With truth values, $t(p + q) = t(q' + s) = 1$, or $\max(t(p), t(q)) = 1$, and $\max(1 - t(q), t(s)) = 1$. The first implies either $t(p) = 1$, or $t(q) = 1$. If $t(q) = 1$, the second shows $t(s) = 1$; if $t(p) = 1$, then $t(q)$ can be either 0 or 1, and if $t(q) = 0$, $t(s)$ is whatever 0 or 1. In conclusion $t(p + s) = \max(t(p), t(s))$ always equals 1.

Note that this scheme holds in all lattices when $p \to q = p \cdot q$, because $(p' \to q) \cdot (q \to s) = p' \cdot q \cdot s \leq s \leq p + s$.

Concerning its universal validity in the general model, note that if the triplet (p', q, s) is transitive, it follows $p' < s$, and once $s < p + s$ is accepted, provided the triplet $(p', s, p + s)$ were also transitive, it would be concluded that $p + s$. Hence, the scheme of resolution could be stated in plain reasoning under the transitive law for <, but not without it.

16.5. The "constructive dilemma" is the scheme,

$$p + q, p < r, q < s : r + s,$$

that, in the case of tautologies and because it is $p \leq q \Leftrightarrow p' + q = 1$, follows immediately from $p \leq r, q \leq s \Rightarrow p + q \leq r + s$.

Algebraically, it also follows from

$$(p + q) \cdot (p' + r) \cdot (q' + s) = p \cdot q' \cdot r + p \cdot r \cdot s + q \cdot p' \cdot s + q \cdot r \cdot s$$
$$\leq r + r + s + s = r + s.$$

With truth values,

$t(p + q) = t(p' + r) = t(q' + s) = 1$, and,

- if $t(p) = 1$, follows $t(r) = 1$;
- if $t(p) = 0$, follows $t(s) = 1$;

thus, in both cases, it is $t(r + s) = \max(t(r), t(s)) = 1$.

Concerning its universal validity in the general model, once transitivity is presumed, it is $p < r + s$ and $q < r + s$, and once it is accepted that $p < p + q$ ($q < p + q$), it is a backward path from $p + q$ to $p(q)$, and a forward one from p (q) to $r + s$. Hence under transitivity, $r + s$ can be reached from $p + q$ with combined forward–backward "movements". In plain reasoning the constructive dilemma can fail if transitivity fails.

16.6. The scheme

$$p' + q', r < p, s < q : r' + s',$$

is called the "destructive dilemma", and the proofs of it in a Boolean algebra are obtained analogously to those of the constructive dilemma.

Concerning its universal validity, because it follows that $p' < r'$ and $q' < r'$, it also follows that $p' < r' + s'$ and $q' < r' + s'$, provided the transitive law were to hold. Hence, from $p' < p' + q'$ ($q' < p' + q'$), it is a backward path from $p' + q'$ up to $p'(q')$, and a forward path from $p'(q')$ up to $r' + s'$, that allows reaching $r' + s'$. In any case, $r' + s'$ can be reached from $p' + q'$. Under transitivity, it holds under a combined backward-forward movement, but it can fail with a lack of transitivity.

16.7. The four most well-known schemes of deductive reasoning, the ancient "modus" of the old logic, are the following.

- MP, Modus ponendo ponens, shortened to modus ponens
- MT, Modus tollendo tollens, shortened to modus tollens
- MPT, Modus ponendo tollens, shortened to disjunctive scheme,
- MTP, Modus tollendo ponens, $(p \cdot q)', q : p'$.

Of them, it only lacks reviewing MTP in Boolean algebras, and reflecting on the possibility of its universal validity. The first is immediate after proving what seems was unknown in Middle Ages logic: That the four "modi" are equivalent in a Boolean framework to $p \rightarrow q = p' + q$. Such proof is as follows.

- $p \rightarrow q = q' \rightarrow p'$, shows MP ⇔ MT.
- $p + q = p' \rightarrow q$, shows MT ⇔ MPT
- $(p \cdot q)' = p' + q' = q \rightarrow p'$, shows MP ⇔ MTP.

Hence, although MTP can be directly proven algebraically by the Boolean calculation $(p \cdot q)' \cdot q = (p' + q') \cdot q = p' \cdot q \leq p'$, its equivalence with MP allows avoiding any additional consideration.

Regarding MTP′s possible universal validity in the general model, it is obvious that it cannot be certified by the last considerations, and, in particular, due to the Boolean identification of $p < q$ with $p' + q$. Nevertheless, as shown in the dilemmas, the following can just be said. Because $p \cdot q < p$ implies $p' < (p \cdot q)'$, provided it were accepted that this also implies $q \cdot p' < q \cdot (p \cdot q)'$, it would then be clear that, once the data $q \cdot (p \cdot q)'$ are known, $q \cdot p'$ can be backward reached, and that from $q \cdot p' < p'$, p' is finally forward reached. Hence, provided the law of monotony $a < b \Rightarrow c \cdot a < c \cdot b$ for any c, were accepted, p' could be concluded after a backward deduction up to $q \cdot p'$, and a forward one up to p'. In any case, such a law of monotony does not seem to be a bizarre one for ordinary reasoning or, at least, for some parts of it, and what remains an open question is to know on which weak suppositions the four modi can be equivalent in the general model of ordinary reasoning.

Let's recall that the Latin words *ponendo* and *tollens*, mean "placing" and "suppressing," respectively. For instance, in the MTP it refers to reaching truth by first suppressing it, and secondly, placing it; in MPP = MP, it refers to reaching truth by first placing it, and secondly also placing it. In the naïve symbolic representation managed here, placing corresponds to doing a forward movement, and suppressing to a backward one. The old terminology still keeps some significance, and it is credible that in Middle Age's scholastics the modi were seen in a form that the new general model reproduces in different and symbolic terms.

In the old scholastic logic, the four modi were not seen as equivalent. Because these modi are not known to be equivalent for all kinds of ordinary reasoning, neither Boolean algebras, nor non-Boolean De Morgan algebras, nor non-Boolean orthomodular lattices, nor BAF, would be possible models for the totality of ordinary reasoning, but only for some and perhaps very small parts of it. It is clear that a more general mathematical framework is necessary for a formal global study of ordinary reasoning.

16.8. What has been presented in this section only refers to deduction, but what about conjecturing and refuting by means of the classical schemes? Something was further advanced on MP and MT, but it still lacks asking for the other schemes. Is there some actual possibility for using them to refute or to conjecture? For instance, MT and MTP are actually ways for deductively refuting p, and MP and MPT for deductively proving p. What can be said about conjecturing or refuting with the schemes of resolution and MPT?

For instance, can the data in MPT serve p' and $p + q$ to refute q? In the Boolean framework it is equivalent to satisfy the inequality $p' \cdot (p + q) = p' \cdot q \leq q'$, implying $p' \cdot q = 0$, forcing $q = q \cdot p + q \cdot p' = q \cdot p$; that is, $q \leq p$ or $q' + p = 1$: $q \to p$ should be a tautology.

Can these data serve to reach q as a type-one speculation? This is equivalent to both $p' \cdot (p + q)$ NC q, and $q' \leq p' \cdot (p + q)$, but the first is impossible because it is $p' \cdot (p + q) = p' \cdot q \leq q$, and the second would imply $q' \leq p' \cdot q$, meaning $q' = 0$, or $q = 1$. No type-one speculation is available from $\{p', p + q\}$.

Can q be a hypothesis? Can it be $q \leq p' \cdot (p + q) = p' \cdot q$? It would imply $q = p' \cdot q \Leftrightarrow q \leq p'$ that q is contradictory to p and that p already refutes q.

The scheme of reduction to absurdity actually refutes p. Can it serve to say that p' is a hypothesis for the data q and $p < q'$? It would require $p' \leq q \cdot (p' + q') = q \cdot p'$ $\Leftrightarrow p' = \mathrm{q} \cdot p' \Leftrightarrow p' \leq q \Leftrightarrow p' \leq (q')'$ that p' and q' should be contradictory and that q' already refutes p'. For serving to conjecture p, it should be $p' \cdot q$ NC p, something that is actually possible provided $p \neq 0$ and $p' \cdot q \neq 0$, because if it were $p \leq p' \cdot q$ it would imply $p = 0$, and if it were $p' \cdot q \leq \mathrm{p}$ it would imply $p' \cdot q = 0$. Thus, for being a type-one speculation it should be $p' \leq p' \cdot q \Leftrightarrow p' = p' \cdot q \Leftrightarrow p' \leq q \Leftrightarrow q' \leq (p')'$, q' and p' should be contradictory; p' should already refute q'.

Can the scheme of resolution, whose data are $p' < q$ and $q < s$ serve for refuting $p + s$? It should be $(p + q) \cdot (q' + s) = p \cdot q' + p \cdot s + q \cdot s \leq (p + s)' = p' \cdot s'$, implying $p + s = 1$.

- Can it serve for conjecturing $p + s$? The answer is no, because the former expression is, obviously, less than or equal to $p + s$.
- Can $p + s$ be a hypothesis? For it, the former union should equal $p + s$: $p \cdot q' + (p + q) \cdot s = p + s$, a Boolean equation whose solution could enlighten a possible answer. For instance, from it follows $p \cdot q' \cdot s' = p \cdot s' \Leftrightarrow p \cdot s' \leq q' \Leftrightarrow q \leq p' + s$, meaning that the possible involved triplets (p, q, s) should be searched between those verifying $q \leq p' + s$, and so on.

16.9. Although the typical scholarly proofs by means of truth-tables hides them, the rules for deducing consequences can be translated into the Boolean formal model by means of equations and inequalities. It also happens with hypotheses and, in some cases, with type-one speculations, but never with type-two speculations just characterized by the lack of comparability with the reasoning's premises and its negation.

This does not mean, nevertheless, that these speculations are never accessible step by step through some forward and backward deductive paths; perhaps some of those speculations could be reached by means of a system of inequalities mixing forward ($\leq$), and backward ($\geq$) paths.

For instance, in a finite Boolean algebra with five atoms a, b, c, d, e, taking $a + c$ as the résumé of the premises, $b + d$ is neither below nor after $a + c$, and is $(a + c)$ NC $(b + d)$, and $(a + c)' = b + d + e \geq b + d$; hence $b + d$ is a type-one speculation of $a + c$. Nevertheless, because $a + c \leq a + c + b + d \geq b + d$, $b + d$ is reachable from $a + c$ by means of a first forward movement up to $a + c + b + d$, followed by a backward one to $b + d$. It analogously happens with $a + c$ and $b + c$, for which it is $(a + c)$ NC $(b + c)$, $a + c$ NC $(b + c)' = a + d + e$, and hence $b + c$ is a type-two speculation of $a + c$; but because $c \leq a + c$, $c \leq b + c$, the speculation is reached by a backward movement up to c followed by a forward one up to $b + c$.

Characterizing speculations that can be reached by a sequence of backward and forward movements, that is, that are algorithmically reachable step by step, is an open question surely dependent on the formal framework; anyway, those that are not reachable are the properly inductive or creative speculations.

Another open topic, perhaps related to this last, refers to obtaining a definition of the heuristics used in artificial intelligence programs, and for which it is needed to know something previously on a searched conclusion.

Because (deductive) logic cannot be seen as a subject only concerning the preservation of Aristotelian logical forms, nor as only doing reasoning by using them, it is important to study whether there are other forms that, even possibly of an approximate character, can be useful for not only doing deductive reasoning. Its existence can be, eventually, of relevance for the computer mechanization of ordinary reasoning.

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