Chapter 9
On the Geometric Brownian Motion assumption for financial time series

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Abstract The Geometric Brownian Motion type process is commonly used to describe stock price movements and is basic for many option pricing models. In this paper a new methodology for recognizing Brownian functionals is applied to financial datasets in order to evaluate the compatibility between real financial data and the above modeling assumption. The method rests on using the volumetric term which appears in the factorization of the small–ball probability of a random curve.

9.1 Introduction

Modeling stock prices represents an important task in finance since this is the starting point for evaluating derivatives and other contracts, which have these prices as underlying. The most famous approach, dating back to Black and Scholes [1], states that the dynamic of prices behaves as a Geometric Brownian motion (GBM), with constant coefficients of drift and volatility. In the time, many variants have succeeded (see [5] for a review): in general, in such literature, it is common to assume that the prices follow a GBM, with drift and volatility which evolve during the time.

The problem to verify the compatibility of observed data with the GBM assumption is still an open problem: only indirect empirical evidences have been provided to

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G. Aneiros et al. (eds.), *Functional Statistics and Related Fields*,
Contributions to Statistics, DOI 10.1007/978-3-319-55846-2_9

support that modeling (for instance, by testing marginal Gaussianity, serial correlation of increments, and so on; see among many others, [7] and [8]).

In [2] a new approach to explore the nature of functional data has been introduced and discussed: starting from the possibility to factorize the small–ball probability of a random curve in a spatial factor and a volumetric one, the authors use this latter as the leading term to characterize the nature of the underlying process which has generated the observed curves.

The aim of this paper is to apply such new methodology to financial time series in order to verify the compatibility with the GBM assumption: after introducing in Section 9.2 the notation and summarize some important steps of the methodology introduced in [2], in Section 9.3 an application to real datasets is provided and the main results are illustrated.

9.2 Recognizing some Brownian functionals

Consider a random element X defined on a suitable probability space and mapping in $\mathscr{L}^2_{[0,1]}$, the space of square integrable functions on $[0,1]$, equipped with its natural inner product $\langle g,h\rangle = \int_0^1 g(t)\,h(t)\,dt$, and the induced norm $\|g\|^2 = \langle g,g\rangle$, $g,h \in \mathscr{L}^2_{[0,1]}$. In order to characterize the probability distribution of X, it is useful to known the behaviour of the small-ball probability of X, that is $P(\|X-x\| < \varepsilon)$, as ε tends to zero. The results on this topic available in the literature concern essentially some special classes of Gaussian processes and are presented in the form

$$\mathbb{P}(\|X-x\| < \varepsilon) \sim \psi(x)\,\phi(\varepsilon) \qquad \text{as } \varepsilon \to 0,$$

where $\psi(x)$ is a positive constant depending on x, which plays the role of the surrogate density of X, and $\phi(\varepsilon)$ representing the volumetric term independent on x. For some processes, the latter can be asymptotically approximated by $\varepsilon^{\alpha}\exp\left(-\gamma\varepsilon^{-\beta}\right)$, with α, β and γ non–negative constants; in particular, it is known that when X is a Brownian bridge process, $\alpha = 0$, $\gamma = 1/8$ and $\beta = 2$ (see [6]).

Suppose now to dispose of a sample $\{X_i, i = 1,\ldots,n\}$ of i.i.d. copies of X, from which one can obtain an estimate $\widehat{\phi}(\varepsilon)$ of $\phi(\varepsilon)$. The comparison of $\widehat{\phi}(\varepsilon)$ and $\phi(\varepsilon)$ by a suitable dissimilarity measure allows to evaluate the parameters involved, as the ones which minimize that dissimilarity. This idea has been developed in [2] where the goodness of the approach is shown by a simulations study.

In particular, if one assumes that the sample comes from a Brownian bridge, once $\widehat{\phi}(\varepsilon)$ is computed for $\varepsilon \in \mathscr{E}$, a suitable subset of $\mathbb{R}^+$, it is possible to estimate β as the minimizer $\widehat{\beta}$ of the *centered cosine dissimilarity (ccd)* between $\log\widehat{\phi}(\varepsilon)$ and $\varepsilon^{-\beta}$ over $\mathscr{E}$, where the ccd between g and h is defined by $1 - \langle g^\star, h^\star\rangle\left(\|g^\star\|\,\|h^\star\|\right)^{-1}$, with $g^\star = g - \int g(t)\,dt$ and all integrals are computed on $\mathscr{E}$. If $\widehat{\beta}$ is close to 2, then this represents an empirical evidence in favour of the correctness of assumption that the process is a Brownian bridge, whereas if $\widehat{\beta}$ is far from 2, there is not compatibility with that specification. To have an idea about the variability of $\widehat{\beta}$, one can refer to confidence intervals estimated, by means of Monte Carlo approach, in [2] for various sample sizes.

The latter approach can be extended directly to other Gaussian processes which are functionals of the Browian bridge (as, for instance, the Wiener process and the Geometric Brownian Motion): in fact, it is enough to apply a suitable tranformation to the data to obtain a Brownian bridge.

9.3 Analysis of financial time series

In this Section we apply the methodology illustrated above to some time series of stock prices, in order to evaluate the compatibility of the data with the assumption that the ones come from a Geometric Brownian Motion.

To do this, we concentrate our attention on four well known financial indexes, used as underlying for a lot of derivatives, futures and other contracts: Dow Jones Industrial Average, NASDAQ composite, NIKKEI 225 and S&P 500.

Time series consist in daily closing prices from 12 March 1985 to 1 December 2016 for Dow Jones, from 5 June 1984 to 1 December 2016 for NIKKEI index, and from 13 April 1977 to 1 December 2016 for NASDAQ and S&P 500 indexes. Hence, overall we have 8 thousand daily prices for Dow Jones and NIKKEI indexes, and 10 thousand for NASDAQ and S&P 500. The plots of these time series are reported in Figure 9.1.

9.3.1 Modeling

Denote by $S(t)$ the price of a stock observed at time t. From the time series $\{S(t_j), j = 1, \ldots, N\}$ it is possible to build a sample of n discretized functional data X_i by dividing the interval $\mathscr{T} = [t_1, t_N]$ in n disjoint intervals $\mathscr{T}_i$ with constant width τ (positive and integer, so that $N = n\tau$) and cutting the whole trajectory as follows:

$$X_i(t_j) = S((i-1)\tau + t_j) \qquad t_j \in [0, \tau),\ i = 1, \ldots, n.$$

Accordingly with the financial literature (see e.g. [3], [5]) we assume that the underlying continuous process, from which data come, follows the GBM model:

$$X_i(t) = X_i(0) \exp\left\{ \left(\mu_i - \frac{1}{2}\sigma_i^2 \right) t + \sigma_i W(t) \right\} \qquad t \in [0, \tau)$$

where μ_i and σ_i are the specific drift term and the specific volatility rate of the period $\mathscr{T}_i$ and $W(t)$ is a Wiener process. In this way, we take into account the fact that volatility rate vary with time over $\mathscr{T}$, but can be considered constant over suitable subintervals.

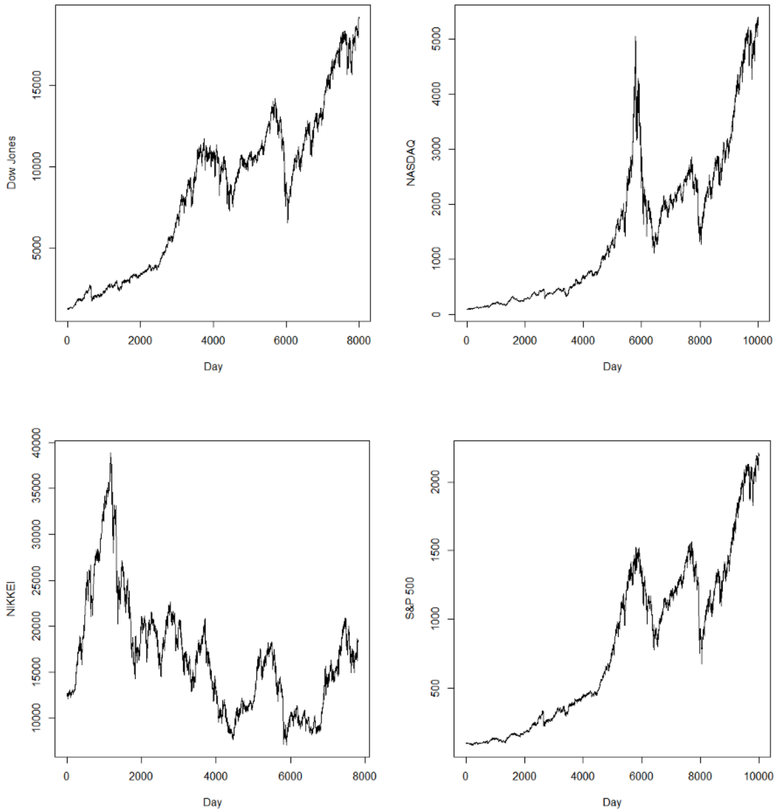

Fig. 9.1: Time series of daily prices of Dow Jones (left, top), NASDAQ (right, top), NIKKEI (left, bottom) and S&P 500 (rigth, bottom) indexes.

Since

$$W(t) = \left[\log\left(X_i(t)/X_i(0)\right) - \left(\mu_i - \sigma_i^2/2\right)t\right]/\sigma_i \tag{9.1}$$

and $W(t) - tW(\tau)$ is a Brownian Bridge, the methodology illustrated in Section 9.2 applies to verify the compatibility of data with the assumption.

9.3.2 Estimates and main results

The first step to operationalize the methodology is to cut the time series in order to obtain the samples of functional data: we decided to divide the whole intervals in subintervals of d days each one, with $d = 25, 50, 80, 100$ in order to evaluate the

Table 9.1: Estimated values of β for the the different stock prices indexes varying d. Into brackets the sample size.

d	Dow Jones	NASDAQ	NIKKEI	S&P 500
100	2.18 [159]	1.62 [199]	2.16 [159]	2.14 [199]
80	2.24 [199]	1.68 [249]	2.24 [199]	2.22 [249]
50	2.20 [160]	1.80 [200]	2.14 [160]	2.24 [200]
25	2.16 [320]	1.88 [400]	2.08 [320]	2.12 [400]

results with respect to the cutting criterion. In this way we should obtain samples of $n = 10000/d$ curves for NASDAQ and S&P 500 and of $n = 8000/d$ curves for Dow Jones and NIKKEI. Since, the larger d is, the smaller n is, for $d = 100, 80$ we add to the sample some curves built with the same cutting criterion but with starting point shifted by $d/2$ ahead. In this way we guarantee more accurate estimations.

For each sample the terms μ_i and σ_i are estimated from each curve by using the maximum likelihood estimates as follows:

$$\widehat{\mu}_i = \frac{1}{d}\sum_{j=1}^{d} X_i(t_j) \qquad \text{and} \qquad \widehat{\sigma}_i^2 = \frac{1}{d}\sum_{j=1}^{d}\left(X_i(t_j) - \widehat{\mu}_i\right)^2 .$$

These values are used to transform the samples by means of (9.1). To the sake of illustration, the samples of curves when $d = 80$ are drawn in Figure 9.2.

For each case (varying d and the stock index), the volumetric part $\phi(\varepsilon)$ is estimated using the k–NN approach in [4]: here we used the box kernel (see Corollary 5.2 in [4]), and the number of neighbours equals the integer part of $n/2$. Finally, by means of the method described in Section 9.2, we get estimates of β which are reported in Table 9.1.

It is worth to noticing that $\widehat{\beta}$ is strictly positive: we can deduce that the log–volumetric part $\log(\phi)$ is proportional to $\varepsilon^{-\beta}$. Moreover, all the values are quite close to 2, that corresponds to the most of the Brownian functionals, and the smaller the length d of $\mathscr{T}_i$ is, the closer $\widehat{\beta}$ is to 2; this turns out to be coherent with the literature: the BGM can be used as a model for stock prices whenever the observation window is short enough to assume that drift and volatility terms are constant over that period. Indeed, if one has high–frequency data instead of daily ones (for instance observed at each 5 minutes), the effect of non–constant drift and volatility would be further relaxed and $\widehat{\beta}$ is very closed to 2. This is shown empirically in [2].

Acknowledgements E. Bongiorno and A. Goia are members of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

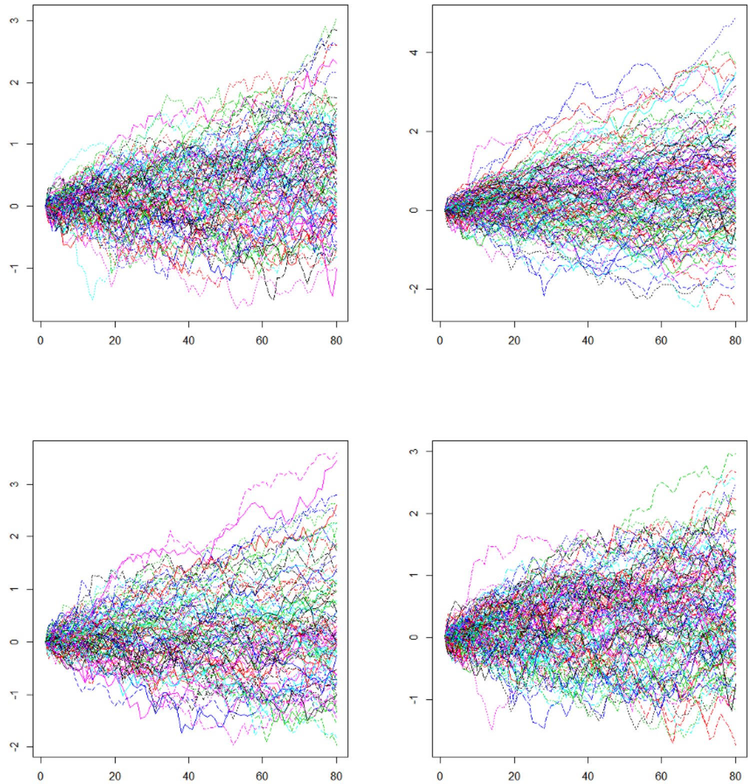

Fig. 9.2: The samples of functional data, derived from original time series, for stock prices Dow Jones (left, top), NASDAQ (rigth, top), NIKKEI (left, bottom) and S&P 500 (right, bottom) indexes, when $d = 80$.

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