Chapter 6
A time-dependent PDE regularization to model functional data defined over spatio-temporal domains

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Abstract We propose a method for the analysis of functional data defined over spatio-temporal domains when prior knowledge on the phenomenon under study is available. The model is based on regression with Partial Differential Equations (PDE) penalization. The PDE formalizes the information on the phenomenon and models the regularity of the field in space and time.

6.1 Space-Time Regression with PDE Penalization

We propose a new method for the analysis of functional data defined over spatio-temporal domains. These data can be interpreted as time evolving surfaces or spatially dependent curves. The proposed method is based on regression with differential regularization and extends the models proposed in [16, 10, 3, 4]. We are in particular interested to the case when prior knowledge on the phenomenon under study is available. Analogously to [4], the prior knowledge is described in terms of a PDE. But, with respect to [4], we here also include the temporal dimension, considering

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a time-dependent PDE that jointly models the spatial and temporal variation of the phenomenon.

Recently, various methods have been proposed to deal with spatially dependent functional data [12]. Starting from the pioneering work in [9], kriging prediction models for stationary spatial functional data have been developed in [7, 15, 8]. Universal kriging approaches have been presented in [6, 13, 14] and kriging with external drift in [10]. Moreover, methods based on differential regularizations has been recently developed in [2, 11, 5, 1]. In these works, the authors consider two roughness penalties that account separately for the regularity of the field in space and in time, using a tensor product approach. Differently from the latter works, we here use a unique regularizing term that jointly model the space-time variation of the phenomenon, on the base of problem-specific prior information. Specifically, the regularization involves the misfit of a time-dependent parabolic PDE modeling the phenomenon behavior, $\dot{f}+Lf=u$, where $\dot{f}$ is the time derivative of the spatio-temporal function f, and L is a differential operator in space. We consider various samplings designs, including geo-statistical and areal data. We show that the corresponding estimation problems are well posed and can be discretized in space by means of the Finite Element method, similarly to [16, 10, 3, 4], and in time by means of the Finite Difference method. The model can handle data distributed over domains having complex shapes such as domains with strong concavities and holes. Moreover, various types of boundary conditions can be considered, with a very flexible modeling of the behavior of the spatio-temporal field and the boundaries of the domain of interest.

6.2 Motivating application

As a motivating example, we want to study the blood flow velocity field in the common carotid artery, using data from Echo-Color Doppler (ECD). These data were analyzed for a single time instant, the systolic peak, in [4]. The echo doppler scan provides a time-dependent measure of the velocities of blood flow particles sampled within a bean in an artery section. Figure 6.1 shows the ECD signal registered in a centrally located bean at the cross-section of the common carotid artery located 2 cm before the artery bifurcation. The lower part of the ECD image displays the acquired velocity signal during the time lapse of about three heart beats. This signal represents the time-evolving histogram of the measured velocities in the beam: the gray-scaled intensity of pixels is proportional to the number of blood-cells in the beam moving at a certain velocity for any fixed time. Starting from the ECD signals over multiple beans, we would like to reconstruct the time-varying mean velocity of the blood-flow over the whole carotid section. A fundamental constraint in this application is given by the so-called no-slip boundary conditions; indeed the physics of the problem implies that, because of the friction between blood cells and arterial wall, blood-flow velocity is zero at the arterial wall. Including the prior knowledge about the blood fluid dynamics and appropriate boundaries conditions is in this context fundamental to achieve meaningful and physiological estimates.

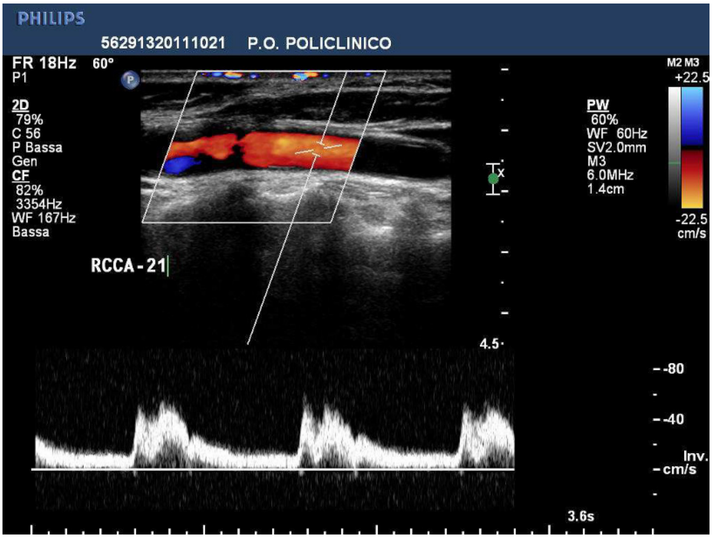

Fig. 6.1: ECD image corresponding to the central point of the carotid section located 2 cm before the carotid bifurcation.

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