Chapter 10
Commutator of projectors and of unitary operators

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Abstract We define and study the concept of commutator for two projectors, for a projector and a unitary operator, and for two unitary operators. Then we state several properties of these commutators. We recall that projectors and unitary operators are linked with the spectral elements of stationary processes. We establish relations between these commutators and some other tools related to the proximity between processes.

10.1 Introduction

This work relates to the field of the operatorial domain, dealing with projectors and unitary operators. These operators take a large place in the statistics of stationary processes. For example, the shift operator is a unitary operator, and a unitary operator is a linear combination of projectors. We define and study the concepts of commutator for two projectors, for a projector and a unitary operator, and for two unitary operators. These concepts are developped in the Hilbertian frame, and when the $\mathbb{C}-$Hilbert space H is of the type $L^2(\Omega,\mathscr{A},P)$, our results apply to stationary processes. The commutativity of two stationary processes is a generalization of the notion of stationary correlation. When there is not a complete commutativity, we may extend the notion of commutativity, asking how to retrieve the part of each process which commutes. The commutator proposes an answer to this question. We recall that the product of convolution of spectral measures, such as defined in Boudou and

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Romain [5], needs an hypothesis of commutativity. Our work uses tools defined in Boudou and Viguier-Pla [8], such as the $r-$convergence and a distance-like measure of the gap between projectors.

Obviously, the commutator of two projectors is linked with the canonical analysis of the spaces generated by these projectors. When these spaces are complex, the practical interest of it is the domain of stationary processes, as seen above. Several authors work on spectral elements of processes, as, for example, in the large deviation field (Gamboa and Rouault [11]), the autoregressive processes (Bosq [2]), and for reduction of dimension (Brillinger [9], Boudou [3]). The joint study of two processes may lead to the comparison of these processes, by the way of the commutators. When these spaces are real, applications may be forseen by the search of common and specific subspaces of two spaces. Such problematics have been largely developped with other tools, as, for example, in the works of Flury and Gautschi [10], Benko and Kneip [1], or Viguier-Pla [12].

10.2 Prerequisites, recalls and notation

In this text, H is a $\mathbb{C}-$Hilbert space, (E,ξ) a measurable space, and $\mathscr{B}$ the Borel $\sigma-$field of $\Pi=[-\pi;\pi[$. We note S the measurable application $(\lambda,\lambda')\in\Pi\times\Pi\mapsto\lambda+\lambda'-2\pi\left[\frac{\lambda+\lambda'+\pi}{2\pi}\right]\in\Pi$, where $[x]$ denotes the integer part of x. When H' is a $\mathbb{C}-$Hilbert, $\mathscr{P}(H')$ denotes the set of the orthogonal projectors of H'.

Let us examine the notions of stationarity and of correlated stationarity.

Definitions 10.2.1. *A series $(X_n)_{n\in\mathbb{Z}}$ of elements of H is said to be stationary when, for any pair (n,m) of elements of $\mathbb{Z}$, we have $<X_n,X_m>=<X_{n-m},X_0>$.*
Two stationary series $(X_n)_{n\in\mathbb{Z}}$ and $(Y_n)_{n\in\mathbb{Z}}$ are said to be stationarily correlated when $<X_n,Y_m>=<X_{n-m},Y_0>$, for any pair (n,m) of elements of $\mathbb{Z}$.

Let us recall the notion of integral with respect to a random measure (r.m.).

Definition 10.2.2. *A r.m. Z is a vector measure, defined on $\mathscr{B}$ and taking values in H, such that $<ZA,ZB>=0$, for any pair (A,B) of disjoint elements of $\mathscr{B}$.*

Proposition 10.2.1. *If Z is a r.m., then the application $\mu_Z:A\in\mathscr{B}\mapsto\|ZA\|^2\in\mathbb{R}^+$ is a bounded measure. There exists one and only one isometry $\mathscr{I}_Z$ from $L^2(\Pi,\mathscr{B},\mu_Z)$ onto $H_Z=\overline{vect}\{ZA;A\in\mathscr{B}\}$ which, with A, associates ZA, for any A of $\mathscr{B}$.*

Definition 10.2.3. *If Z is a r.m., for any φ of $L^2(\Pi,\mathscr{B},\mu_Z)$, $\mathscr{I}_Z\varphi$ is named integral of φ with respect to the r.m. Z, and is denoted $\int\varphi dZ$.*

Let us now examine the association "stationary series$-$r.m.".

Proposition 10.2.2. *If Z is a r.m., then $(\int e^{i.n}dZ)_{n\in\mathbb{Z}}$ is a stationary series. Conversely, with any stationary series $(X_n)_{n\in\mathbb{Z}}$ of elements of H, we can associate a r.m. Z, and only one, defined on $\mathscr{B}$, taking values in H, such that $X_n=\int e^{i.n}dZ$, for any n of $\mathbb{Z}$.*

A spectral measure (s.m.) is a projector-valued application, defined on a $\sigma-$field, as any measure.

Definition 10.2.4. *A spectral measure (s.m.) $\mathscr{E}$, on ξ for H, is an application from ξ in $\mathscr{P}(H)$ such that*
i) $\mathscr{E}E = I_H$,
ii) $\mathscr{E}(A \cup B) = \mathscr{E}A + \mathscr{E}B$, for any pair (A,B) of disjoint elements of ξ,
iii) $lim_n \mathscr{E}A_n X = 0$, for any sequence $(A_n)_{n\in\mathbb{N}}$ of elements of ξ which decreasingly converges to $\emptyset$ and for any X of H.

With a s.m., we can associate an infinity of r.m.'s.

Proposition 10.2.3. *If $\mathscr{E}$ is a s.m. on $\mathscr{B}$ for H, then the application $Z^X_{\mathscr{E}} : A \in \mathscr{B} \mapsto \mathscr{E}AX \in H$ is a r.m..*

Just like the product of two probability measures is defined, we can define a product for two s.m.'s, with an hypothesis of commutativity.

Definition 10.2.5. *Two s.m.'s $\mathscr{E}_1$ and $\mathscr{E}_2$ on $\mathscr{B}$ for H commute when $\mathscr{E}_1 A_1 \mathscr{E}_2 A_2 = \mathscr{E}_2 A_2 \mathscr{E}_1 A_1$, for any A_1 and A_2 of $\mathscr{B}$.*

Proposition 10.2.4. *If the s.m.'s $\mathscr{E}_1$ and $\mathscr{E}_2$ commute, then there exists a s.m., and only one, denoted $\mathscr{E}_1 \otimes \mathscr{E}_2$, on $\mathscr{B} \otimes \mathscr{B}$ for H, such that $\mathscr{E}_1 \otimes \mathscr{E}_2 A_1 \times A_2 = \mathscr{E}_1 A_1 \mathscr{E}_2 A_2$, for any pair (A_1, A_2) of elements of $\mathscr{B}$.*

Proposition 10.2.5. *If the s.m.'s $\mathscr{E}_1$ and $\mathscr{E}_2$ commute, then the application $\mathscr{E}_1 * \mathscr{E}_2 : A \in \mathscr{B} \mapsto \mathscr{E}_1 \otimes \mathscr{E}_2 S^{-1} A \in \mathscr{P}(H)$ is a s.m. on $\mathscr{B}$ for H named product of convolution of the s.m.'s $\mathscr{E}_1$ and $\mathscr{E}_2$.*

With any s.m. on $\mathscr{B}$ for H, we can associate a unitary operator (u.o.).

Proposition 10.2.6. *If $\mathscr{E}$ is a s.m. on $\mathscr{B}$ for H, then the application $X \in H \mapsto \int e^{i.1} dZ^X_{\mathscr{E}} \in H$ is a u.o.. Conversely, if U is a u.o. of H, then there exists one, and only one, s.m. $\mathscr{E}$, on $\mathscr{B}$ for H, such that $UX = \int e^{i.1} dZ^X_{\mathscr{E}}$, for any X of H.*

From a u.o., we can generate a family of stationary series.

Proposition 10.2.7. *Assume that U is a u.o. of H of associated s.m. $\mathscr{E}$, then $(U^n X)_{n\in\mathbb{Z}}$ is a stationary series of associated r.m. $Z^X_{\mathscr{E}}$.*

When two o.u.'s commute, we can easily express the s.m. associated with their product.

Proposition 10.2.8. *Two u.o.'s U_1 and U_2, of respective associated s.m.'s $\mathscr{E}_1$ and $\mathscr{E}_2$, commute if and only if $\mathscr{E}_1$ and $\mathscr{E}_2$ commute. In that case, $\mathscr{E}_1 * \mathscr{E}_2$ is the s.m. associated with the u.o. $U_1 U_2$.*

For developments of these notions, the reader can refer to Boudou [4], Boudou and Romain [5], and Boudou and Romain [6].

We will end this section by recalls concerning a relation of partial order defined on $\mathscr{P}(H)$.

Definition 10.2.6. *We say that a projector P is smaller than a projector Q, and we note $P \ll Q$, when $P = PQ = QP$.*

Proposition 10.2.9. *If P and Q are two projectors such that $P \ll Q$, then $\|PX\| \leq \|QX\|$, for any X of H.*

The relation $\ll$ is of partial order, but it has the advantage that any family $\{P_\lambda ; \lambda \in \Lambda\}$ of projectors, finite or not, countable or not, has got a larger minorant, that is a lower bound, denoted $\inf\{P_\lambda;\lambda\in\Lambda\}$, and a smaller majorant, that is an upper bound, denoted $\sup\{P_\lambda;\lambda\in\Lambda\}$. Then we have the following properties.

Proposition 10.2.10. *Let $\{P_\lambda;\lambda\in\Lambda\}$ be a family of projectors. Then*
i) $Im\,inf\{P_\lambda;\lambda\in\Lambda\}=\cap_{\lambda\in\Lambda}Im P_\lambda$;
ii) $(sup\{P_\lambda;\lambda\in\Lambda\})^\perp=inf\{P_\lambda^\perp;\lambda\in\Lambda\}$;
iii) $(inf\{P_\lambda;\lambda\in\Lambda\})^\perp=sup\{P_\lambda^\perp;\lambda\in\Lambda\}$;
iv) if P_1 and P_2 are two projectors which commute, then $inf\{P_1;P_2\}=P_1P_2$.

We can then obtain some properties which are similar to those existing in the classic analysis of sequences.

Definition 10.2.7. *Let $(P_n)_{n\in\mathbb{N}}$ be a sequence of projectors. If $sup\{inf\{P_m;m\geq n\};n\in\mathbb{N}\}=inf\{sup\{P_m;m\geq n\};n\in\mathbb{N}\}$, we say that $(P_n)_{n\in\mathbb{N}}$ $r-$converges to P, and we note it $lim_n^r P_n=P$.*

The $r-$convergence implies the point by point convergence, but the converse is not true.

From the relation $\ll$, we can measure the gap between two projectors.

Definition 10.2.8. *For any (P_1,P_2) of $\mathscr{P}(H)\times\mathscr{P}(H)$, we define*
$d(P_1,P_2)=sup(P_1,P_2)-inf(P_1,P_2)$.

Proposition 10.2.10. *For any (P_1,P_2) of $\mathscr{P}(H)\times\mathscr{P}(H)$, we have $Im(d(P_1,P_2))^\perp=Ker(P_1-P_2)$.*

This notion evoques the notion of distance, however, it is not a distance, as $d(P_1,P_2)$ is a projector. Its interest lies on the following property.

Proposition 10.2.10. *A sequence $(P_n)_{n\in\mathbb{N}}$ of projectors $r-$converges to P if and only if $lim_n^r d(P_n,P)=0$.*

Finally, let us define the notion of maximal equalizator of two u.o.'s.

Definition 10.2.11. *We name maximal equalizator of the u.o.'s U_1 and U_2 the projector, denoted R_{U_1,U_2}, on $\cap_{n\in\mathbb{Z}}Ker(U_1^n-U_2^n)$.*

These notions are developped in Boudou and Viguier-Pla [8].

10.3 Commutator of two projectors

Let us first define this notion of commutator.

Definition 10.3.1. *A projector K is a commutator of the projectors P_1 and P_2 when it commutes with P_1 and P_2 and when $P_1KP_2=P_2KP_1$.*

We can establish the following properties.

Proposition 10.3.1. *Let P_1 and P_2 be two projectors. Then*
i) the upper bound of a family of commutators of the projectors P_1 and P_2 is a commutator of the projectors P_1 and P_2;
ii) 0 is a commutator of the projectors P_1 and P_2;
iii) the upper bound of the family of the commutators of the projectors P_1 and P_2 is

the projector on $Ker(P_1P_2 - P_2P_1)$*, that is* $inf\{P_1, P_2\} + inf\{P_1, P_2^\perp\} + inf\{P_1^\perp, P_2\} + inf\{P_1^\perp, P_2^\perp\}$.

So we have the following definition.

Definition 10.3.2. *Let* P_1 *and* P_2 *be two projectors. We call maximal commutator of the projectors* P_1 *and* P_2 *the projector* C_{P_1,P_2} *on* $Ker(P_1P_2 - P_2P_1)$.

Of course, it is easy to establish that $C_{P_1,P_2} = I$ if and only if P_1 and P_2 commute.

The maximal commutator is a tool for measuring the degree of commutativity of the projectors P_1 and P_2. The larger it is, the larger $\mathrm{Ker}(P_1P_2 - P_2P_1)$ is. It is clear that when X belongs to $\mathrm{Ker}(P_1P_2 - P_2P_1) = \mathrm{Im}\, C_{P_1,P_2}$, $\|C^\perp_{P_1,P_2}X\| = 0$. So what can we speculate when X is close to $\mathrm{Ker}(P_1P_2 - P_2P_1)$, that is when $\|C^\perp_{P_1,P_2}X\|$ is small? We will bring an answer to this question, with the following property.

Proposition 10.3.2. *For any pair* (P_1, P_2) *of projectors, and for any* X *of* H*, we have* $\|P_1P_2X - P_2P_1X\| \leq 2\|C^\perp_{P_1,P_2}X\|$.

So when X is close to $\mathrm{Ker}(P_1P_2 - P_2P_1)$, then P_1P_2X is close to P_2P_1X.

10.4 Commutator of a projector and of a unitary operator

In the same way as we have defined a commutator of two projectors, we can define a commutator of a projector and of a u.o..

Definition 10.4.1. *A projector* K *is a commutator of the projector* P *and of the u.o.* U *when it commutes with* P *and* U*, and when* $PKU = UKP$.

We have got similar properties as those of the previous section.

Proposition 10.4.1. *Let* P *be a projector and* U *a u.o.. We can affirm that*
i) the upper bound of a family of commutators of the projector P *and of the u.o.* U *is a commutator of the projector* P *and of the u.o.* U*;*
ii) 0 *is a commutator of the projector* P *and of the u.o.* U*;*
iii) the upper bound of the family of commutators of the projector P *and of the u.o.* U *is the projector on* $\cap_{n\in\mathbb{Z}} Ker(PU^n - U^nP)$.

The following definition is a consequence of these properties.

Definition 10.4.2. *Let* P *be a projector and* U *a u.o.. We name maximal commutator of the projector* P *and of the u.o.* U*, and we note it* $C_{P,U}$*, the projector on* $\cap_{n\in\mathbb{Z}} Ker(PU^n - U^nP)$.

Of course, it is easy to verify that P and U commute if and only if $C_{P,U} = I$. The association "u.o.-s.m." being biunivocal, all these properties can express by means of the s.m. which is associated with a u.o.. We get then a relation between a commutator of a projector and of a u.o. and a family of commutators of two projectors.

Proposition 10.4.2. *If* P *is a projector and* U *a u.o. of associated s.m.* $\mathscr{E}$*, we can affirm that*
i) a projector K *is a commutator of the projector* P *and of the u.o.* U *if and only if, for any* A *of* $\mathscr{B}$*,* K *is a commutator of the projectors* P *and* $\mathscr{E}A$*;*

ii) $Im C_{P,U} = \cap_{A\in\mathscr{B}} Ker(P\mathscr{E}A - \mathscr{E}AP)$;
iii) $C_{P,U} = inf\{C_{P,\mathscr{E}A}; A \in \mathscr{B}\}$.

The last two points come from the fact that $\cap_{n\in\mathbb{Z}}\text{Ker}(PU^n - U^nP) = \cap_{A\in\mathscr{B}}$ Ker $(P\mathscr{E}A - \mathscr{E}AP)$, and that $\text{Im}\, C_{P,\mathscr{E}A} = \text{Ker}(P\mathscr{E}A - \mathscr{E}AP)$.

If we remark that $\{U^{-n}PU^n; n \in \mathbb{Z}\}$ is a family of projectors and that $\text{Ker}(PU^n - U^nP) = \text{Ker}(U^{-n}PU^n - P) = (\text{Im}\, d(U^{-n}PU^n, P))^{\perp}$, we can give to $C_{P,U}$ an ergodic definition.

Proposition 10.4.3. *For any projector P and for any u.o. U, we have*

$$C_{P,U} = inf\{(d(U^{-n}PU^n, P))^{\perp}; n \in \mathbb{Z}\}.$$

This last result can have the following interpretation. If all the elements of the family $\{U^{-n}PU^n; n \in \mathbb{Z}\}$ are close to P, that is, if for any n of $\mathbb{Z}$, $d(U^{-n}PU^n, P)$ is small, or even more, for any n of $\mathbb{Z}$, $(d(U^{-n}PU^n, P))^{\perp}$ is large, then it is the same for the lower bound $C_{P,U}$. This means that P and U are near to commute.

Proposition 10.4.3 lets us write $d(U^{-n}PU^n, P) \ll C_{P,U}^{\perp}$, so

$$\|(PU^n - U^nP)X\| = \|U^{-n}PU^nX - PX\| \leq 2\|d(U^{-n}PU^n, P)X\| \leq 2\|C_{P,U}^{\perp}X\|,$$

because for any pair of projectors (P, P'), we have $\|PX - P'X\| \leq 2\|d(P, P')X\|$ (cf. Boudou and Viguier-Pla [8]).

Thanks to a similar approach, Propositions 10.3.2 and 10.4.2 let us affirm that

$$\|PZ_{\mathscr{E}}^{X}A - Z_{\mathscr{E}}^{PX}A\| = \|P\mathscr{E}AX - \mathscr{E}APX\| \leq 2\|C_{P,\mathscr{E}A}^{\perp}X\| \leq 2\|C_{P,U}^{\perp}X\|.$$

Then the following stands.

Proposition 10.4.4. *For any projector P and for any u.o. U of associated s.m. $\mathscr{E}$, for any X of H, we have*
i) $\|PU^nX - U^nPX\| \leq 2\|C_{P,U}^{\perp}X\|$, *for any n of* $\mathbb{N}$;
ii) $\|PZ_{\mathscr{E}}^{X}A - Z_{\mathscr{E}}^{PX}A\| \leq 2\|C_{P,U}^{\perp}X\|$, *for any A of* $\mathscr{B}$.

So, if X is close to $\text{Im}\, C_{P,U}$, that is if $\|C_{P,U}^{\perp}X\|$ is small, then the series $(PU^nX)_{n\in\mathbb{Z}}$ is "almost stationary", in such a way it is close to the stationary series $(U^nPX)_{n\in\mathbb{Z}}$. As for the application $P \circ Z_{\mathscr{E}}^{PX}$, it is almost a r.m., close to $Z_{\mathscr{E}}^{PX}$, r.m. associated with the stationary series $(U^nPX)_{n\in\mathbb{Z}}$.

Let us end this section by the resolution of the following problem.

Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary series, of associated r.m. Z, and P be a projector. We wish to define all the stationary series, stationarily correlated with $(X_n)_{n\in\mathbb{Z}}$, included in $\text{Im}\, P$. Such series will be named "solution series". Then we have the following.

Proposition 10.4.5. *If U is a u.o. whose associated s.m. $\mathscr{E}$ is such that $Z_{\mathscr{E}}^{X_0} = Z$, then, for any X of $Im C_{P,U}$, we can affirm that $(U^nPX)_{n\in\mathbb{Z}}$ is a "solution series". Any "solution series" is of this type.*

Let us recall that when Z is a r.m. defined on $\mathscr{B}$, taking values in H, there exists at least one s.m. $\mathscr{E}$ on $\mathscr{B}$ for H such that $Z_{\mathscr{E}}^{X_0} = Z$, where $X_0 = \int e^{i.0} dZ$ (cf. Boudou [4]).

10.5 Commutator of two unitary operators

When two u.o.'s U and V commute, the s.m. which is associated with the u.o. UV is the product of convolution of the s.m.'s respectively associated with U and V. But what happens when $UV \neq VU$? The maximal commutator will bring a partial solution to this question.

Definition 10.5.1. *A projector K is a commutator of the u.o.'s U and V when it commutes with U and V, and when $UKV = VKU$.*

Then we can establish the following properties.

Proposition 10.5.1. *Let U and V be two u.o.'s. We can affirm that*
i) the upper bound of a family of commutators of the u.o.'s U and V is a commutator of the u.o.'s U and V;
ii) 0 is a commutator of the u.o.'s U and V;
iii) the upper bound of the family of commutators of the u.o.'s U and V is the projector on $\cap_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} Ker(U^nV^m - V^mU^n)$.

So we can define the following.

Definition 10.5.2. *Let U and V be two u.o.'s. We name maximal commutator of the u.o.'s U and V, and we note it $C_{U,V}$, the projector on the space $\cap_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} Ker(U^nV^m - V^mU^n)$.*

Of course, it is easy to verify that U and V commute if and only if $C_{U,V} = I$. The reader will notice the similarities between Definitions 10.3.1, 10.4.1 and 10.5.1, between Propositions 10.3.1, 10.4.1 and 10.5.1, and between Definitions 10.3.2, 10.4.2 and 10.5.2.

The commutator of two u.o.'s can be defined from the associated s.m.'s.

Proposition 10.5.2. *If U and V are two u.o.'s of respective associated s.m.'s $\mathscr{E}$ and α, we can affirm that*
i) a projector K is a commutator of U and V if and only if, for any pair (A,B) of elements of $\mathscr{B}$, K is a commutator of the projectors $\mathscr{E}A$ and αB;
ii) $Im C_{U,V} = \cap_{(A,B)\in\mathscr{B}\times\mathscr{B}} Ker(\mathscr{E}A\alpha B - \alpha B\mathscr{E}A)$;
iii) $C_{U,V} = inf\{C_{\mathscr{E}A,\alpha B}; (A,B) \in \mathscr{B}\times\mathscr{B}\} = inf\{C_{\alpha B,U}; B \in \mathscr{B}\}$.

To establish the last two points, we must notice that

$$\cap_{(n,m)\in\mathbb{Z}\times\mathbb{Z}} \mathrm{Ker}(U^nV^m - V^mU^n) = \cap_{(A,B)\in\mathscr{B}\times\mathscr{B}} \mathrm{Ker}(\mathscr{E}A\alpha B - \alpha B\mathscr{E}A).$$

Point *iii)* provides a relation between the three types of maximal commutators which we study. We can also establish a relation between the maximal commutator of two u.o.'s and the maximal equalizator of two u.o.'s.

Proposition 10.5.3. *Let U and V be two u.o.'s. We have $C_{U,V} = inf\{R_{V,U^{-n}VU^n}; n \in \mathbb{Z}\}$.*

For the proof, we have just to notice that

$$\mathrm{Im}\, C_{U,V} = \cap_n \cap_m \mathrm{Ker}(U^nV^m - V^mU^n) = \cap_n \cap_m \mathrm{Ker}(V^m - U^{-n}V^mU^n)$$
$$= \cap_n \cap_m \mathrm{Ker}(V^m - (U^{-n}VU^n)^m) = \cap_n \mathrm{Im}\, R_{V,U^{-n}VU^n} = \mathrm{Im}\inf\{R_{V,U^{-n}VU^n}; n \in \mathbb{Z}\}.$$

We can now approach the questions suggested at the beginning of the section. Let us recall some results that we can find in Boudou and Viguier [8].

Proposition 10.5.4. *Let U and V be two u.o.'s of associated s.m.'s $\mathscr{E}$ and α, and L be the application $X \in ImC_{U,V} \mapsto X \in H$. We have*
i) $L^*(X) = C_{U,V}X$, for any X of H; $L^*L = I_{ImC_{U,V}}$; $LL^* = C_{U,V}$; $L^*C_{U,V} = L^*$; $C_{U,V}L = L$;
ii) $U' = L^*UL$ and $V' = L^*VL$ are u.o.'s of $ImC_{U,V}$;
iii) for any A of $\mathscr{B}$, $\mathscr{E}'A = L^*\mathscr{E}AL$ and $\alpha'A = L^*\alpha AL$ are projectors of $ImC_{U,V}$;
iv) the applications $\mathscr{E}' : A \in \mathscr{B} \mapsto \mathscr{E}'A \in \mathscr{P}(ImC_{U,V})$ and $\alpha' : A \in \mathscr{B} \mapsto \alpha'A \in \mathscr{P}(ImC_{U,V})$ are the s.m.'s respectively associated with the u.o.'s U' and V'.

From the fact that

$$U'V' = L^*ULL^*VL = L^*UC_{U,V}VL = L^*VC_{U,V}UL = L^*VLL^*UL = V'U',$$

we can consider the s.m. $\mathscr{E}' \otimes \alpha'$ on $\mathscr{B} \otimes \mathscr{B}$ for Im$C_{U,V}$ (as the s.m.'s $\mathscr{E}'$ and α' commute). For any pair (A,B) of elements of $\mathscr{B}$, we have

$$\mathscr{E}' \otimes \alpha'(A \times B) = \mathscr{E}'A\alpha'B = \inf\{L^*\mathscr{E}AL, L^*\alpha BL\} = L^*\inf\{\mathscr{E}A, \alpha B\}L.$$

If we notice that $U'V' = L^*UVL = L^*VUL$, we have the following.

Proposition 10.5.5. *There exists one and only one s.m., $\mathscr{E}' \otimes \alpha'$, on $\mathscr{B} \otimes \mathscr{B}$ for $ImC_{U,V}$, such that $\mathscr{E}' \otimes \alpha'(A \times B) = L^*inf\{\mathscr{E}A, \alpha B\}L$, for any pair (A,B) of elements of $\mathscr{B}$. Its image by S is the s.m. associated with the u.o. $L^*UVL = L^*VUL$.*

Of course, when U and V commute, that is when $C_{U,V} = I$, we have $L = L^* = I$, $U' = U$, $V' = V$, $\mathscr{E}' = \mathscr{E}$, $\alpha' = \alpha$ and $\mathscr{E} \otimes \alpha(A \times B) = \inf\{\mathscr{E}A, \alpha B\} = \mathscr{E}A\alpha B$, for any (A,B) of $\mathscr{B} \times \mathscr{B}$.

Finally, we establish a link between the commutator and the equalizator of a same pair (U,V) of u.o.'s.

Proposition 10.5.6. *Let U and V be two u.o.'s, and J be the application $X \in ImR^{\perp} \mapsto X \in H$. Then*
i) J^*UJ and J^*VJ are u.o.'s of $ImR^{\perp}_{U,V}$;
ii) $C_{U,V} = JC_{J^*UJ,J^*VJ}J^* + R_{U,V}$.

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