

Alternating Smooth and Nonsmooth Bifurcations in a Discontinuous Linear-Power Map

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Abstract We study the dynamics of a one dimensional discontinuous linear-power map. It has a vertical asymptote giving rise to new kinds of border collision bifurcations. We explain the peculiar periods of attracting cycles, appearing due to cascades of alternating smooth and nonsmooth bifurcations. Robust unbounded chaotic attractors are also described.

1 Introduction

The large number of applied models characterized by sharp switching between different states are ultimately described by nonsmooth systems. One of the efficient methods to investigate the dynamics of such systems is related to the construction of a first return map on some Poincaré section of the phase space, leading to piecewise smooth (PWS for short) maps, continuous or discontinuous. In particular, in engineering the well known Nordmark systems associated with grazing bifurcations (see [5, 6]) have been studied using PWS return maps with power function nonlinearities. The present study deals with a particular case of such a map. It is defined by two functions, $f_L(x)$ and $f_R(x)$, as follows:

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$$f: x \mapsto f(x) = \begin{cases} f_L(x) = ax - 1 & \text{if } x \leq 0, \\ f_R(x) = bx^{-\gamma} - 1 & \text{if } x > 0, \end{cases} \quad (1)$$

where a, b and γ are real parameters, $\gamma > 0$.

The PWS map (1) for $\gamma < 0$ corresponding to the continuous case, has been considered by many authors; see, e.g., di-Bernardo–Budd–Champneys–Kowalczyk [1]. One of the characteristic features of map (1) is the occurrence of border collision bifurcations (BCB for short). This term denotes the collision of an invariant set, typically a periodic point, with a border at which the system function changes its definition. In the study of one dimensional *continuous* PWS maps, the skew tent map is used as a border collision (BC) normal form, which is a powerful analytical tool to determine the effect of the BC of a cycle of any period; see Sushko–Avrutin–Gardini [7] for a survey.

Besides $\gamma < 0$, also the case $\gamma > 0$ in which map (1) is a *discontinuous* map with a vertical asymptote at $x = 0$, has been recently analyzed, mainly related to the case of free terms equal to $+1$; see [3, 4]. In the present work we consider the discontinuous map (1) and explain why peculiar cascades of alternating smooth and nonsmooth bifurcations are observed, leading to the appearance of attracting cycles of periods $p_{i+1} = 2p_i$ for odd i and $p_{i+1} = 2p_i - 1$ for even i , where $p_0 = n, n \geq 3$, is the period of an attracting cycle whose flip bifurcation (denoted *S-flip*) initiate the cascade. We show also that map (1) can possess unbounded chaotic attractors which are robust in some parameter regions and not robust in others. For all the proofs we refer to Gardini–Makrooni–Sushko [2].

A typical view of the bifurcation structure of the parameter space of f is presented in Fig. 1. In the following, after some preliminaries, we discuss first the parameter range $a < 0, b < 0$, associated with *invertible* map and, thus, with more simple

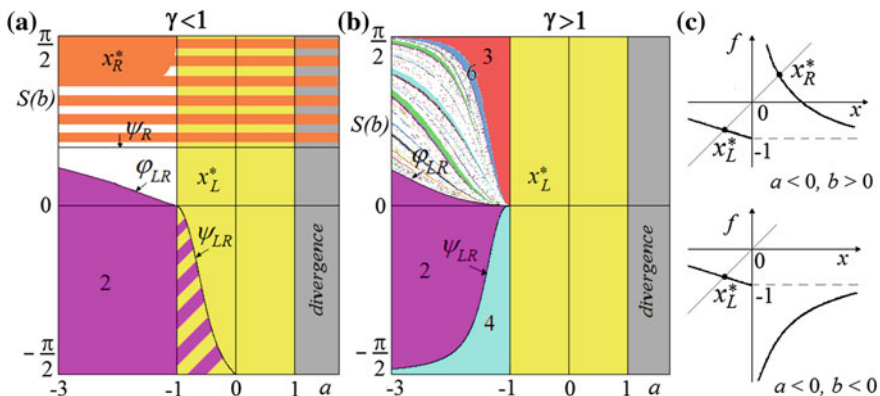


Fig. 1 2D bifurcation diagrams of f in the $(a, S(b))$ -parameter plane, where $S(b) = \arctan(b)$, for $0 < \gamma < 1$ in (a) and $\gamma > 1$ in (b); striped regions are related to coexistence, colored regions to attracting cycles of different periods, uncolored region to higher periodicity or chaotic attractors, grey region to divergence. In (c) examples of map f are shown

dynamics. Then we consider the range $a < 0, b > 0$, when f is *noninvertible*, which we split in two subcases, for $\gamma > 1$ and $0 < \gamma < 1$ related to quite different dynamics.

2 Preliminaries. Invertible Case: $b < 0$

The two partitions of the definition range of the map f are denoted as $I_L = (-\infty, 0]$ and $I_R = (0, +\infty)$. To denote an n -cycle $\{x_i\}_{i=0}^{n-1}$ of the map f , we use its symbolic representation, associating the symbol L with $x_i \in I_L$, and R with $x_i \in I_R$. The fixed points of f are denoted as $x = x_L^* = 1/(a - 1) \in I_L$, and $x = x_R^* \in I_R$. The fixed point $x = x_L^*$ obviously exists for $a < 1$, being attracting for $|a| < 1$; at $a = -1$ it undergoes a degenerate flip bifurcation (DFB for short). The fixed point $x = x_R^*$ exists for $b > 0$; it is repelling for $\gamma \geq 1$, while for $0 < \gamma < 1$ it undergoes a subcritical flip bifurcation at

$$b = \frac{\gamma^\gamma}{(1 - \gamma)^{\gamma+1}} =: \psi_R,$$

being repelling for $0 < b \leq \psi_R$ and attracting for $b > \psi_R$; see Fig. 1. The following propositions summarise the dynamics of f in the invertible case; see also Fig. 2.

Proposition 1 (Flip bifurcations of the 2-cycle LR) *Let $a < 0, b < 0$, and $\gamma > 0$. Then the map f given in (1) has a unique 2-cycle $\{x_0, x_1\}$ with $x_0 < 1/a < 0, x_1 > 0$, which undergoes a flip bifurcation at*

$$b = \frac{1}{a\gamma} \left(\gamma \frac{a+1}{1-\gamma} \right)^{\gamma+1} =: \psi_{LR}$$

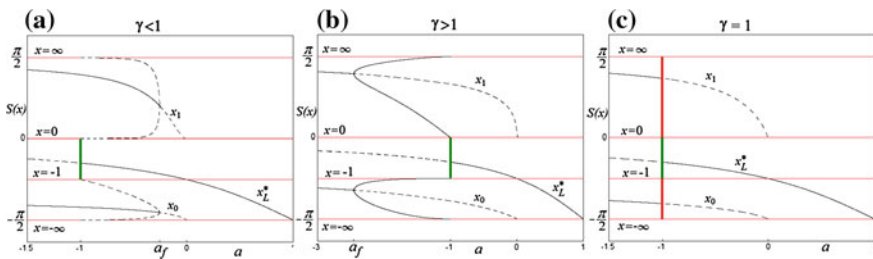


Fig. 2 Bifurcation diagram a vs $S(x)$ at $b = -5$, where $S(x) = \arctan(x)$. In (a) $\gamma = 0.5$, in (b) $\gamma = 2$ and in (c) $\gamma = 1$, associated with the subcritical, supercritical and degenerate flip bifurcations of the 2-cycle LR , respectively. For $a = 1$ related to DFB of x_L^* the interval $[-1, 0] \setminus x_L^*$ is filled by 2-periodic points; additionally, for $\gamma = 1$ intervals $(-\infty, -1] \setminus x_0$ and $[0, +\infty) \setminus x_1$ are filled by 4-periodic points, being associated with DFB of the 2-cycle. The value a_f is obtained by substituting $b = -5$ and the related value of γ to $b = \psi_{LR}$; see Proposition 1

for $\gamma \neq 1$, and $a = -1$ for $\gamma = 1$. This flip bifurcation is subcritical for $0 < \gamma < 1$ and the 2-cycle is attracting for $b < \psi_{LR}$; supercritical for $\gamma > 1$ and the 2-cycle is attracting for $b \geq \psi_{LR}$; degenerate for $\gamma = 1$, and the 2-cycle is attracting for $a < -1$. Moreover, as $a \rightarrow 0_-$, the repelling 2-cycle LR disappears by a nonregular BCB at $a = 0$: $\{x_0, x_1\}|_{a=0} = \{-\infty, 0\}$.

Proposition 2 (BCB of the 4-cycle) *Let $a < 0$, $b < 0$, and $\gamma > 0$. Then, for the map f given in (1) at $a = -1$, a nonregular BCB of a 4-cycle $(LR)^2$ occurs such that for $0 < \gamma < 1$ a repelling 4-cycle disappears as $a \rightarrow -1_+$; for $\gamma > 1$ an attracting 4-cycle disappears as $a \rightarrow -1_-$; the points of the 4-cycle $(LR)^2$ at $a = -1$ are $\{0, -\infty, +\infty, -1\}$.*

3 Noninvertible Case: $b > 0$, $\gamma > 1$ (Periodicity Regions)

Proposition 3 (map g) *Let $b > 0$ and $\gamma > 0$. Then the dynamics of the map f given in (1) are in one-to-one correspondence with the dynamics of the map g defined by three functions, $f_L(x)$, $f_M(x) = f_R^2(x)$ and $f_R(x)$, as follows:*

$$g: x \mapsto g(x) = \begin{cases} f_L(x) = ax - 1 & \text{if } x \leq 0, \\ f_M(x) = \frac{b}{(bx^{-\gamma} - 1)^\gamma} - 1 & \text{if } 0 < x < f_R^{-1}(0) = b^{\frac{1}{\gamma}}, \\ f_R(x) = bx^{-\gamma} - 1 & \text{if } x \geq f_R^{-1}(0), \end{cases} \quad (2)$$

which is continuous at $x = 0$, with $g(0) = -1$, and discontinuous at $x = f_R^{-1}(0)$ with $\lim_{x \rightarrow f_R^{-1}(0)_-} g(x) = +\infty$ and $\lim_{x \rightarrow f_R^{-1}(0)_+} g(x) = 0$.

The auxiliary map g is of help to study the dynamics of f in the noninvertible case:

Proposition 4 *Let $b > 0$ and $\gamma > 1$ be fixed, and $a \in (a_R^h, -1)$, where $a = a_R^h$ satisfies the condition $b = -a(-a - 1)^\gamma$ of the first homoclinic bifurcation of x_R^* . Then,*

- (i) *any fold bifurcation (either fold-BCB or S-fold) of the map g given in (2) is associated with the appearance of a pair of cycles for map f , one attracting and one repelling, whose periods differ by 1 (say n and $n - 1$);*
- (ii) *let $x = x^* > 0$ be a periodic point, closest to $x = 0$, of an attracting n -cycle of f with a negative eigenvalue, which attracts all points of the interval $(0, x^*)$, and let $R^2\sigma_0$ be the symbolic sequence of this cycle (here, σ_0 stands for the remaining symbolic sequence, necessarily starting with L). Then, decreasing a it is observed a cascade of alternating S-flip bifurcations and BCs leading to cycles whose symbolic sequences can be written as $R^2\sigma_k$, where $\sigma_k = \sigma_{k-1}T\sigma_{k-1}$, $k = 1, 2, \dots$, with the alternating symbols $T = R^2$ and $T = L$. The symbol $T = R^2$ corresponds to an S-flip bifurcation, so that an attracting m -cycle in this cascade is followed by an attracting $2m$ -cycle, while the symbol $T = L$ is*

associated with a BC and the m -cycle is followed by an attracting $(2m - 1)$ -cycle.

For example, starting from an attracting 3-cycle, for decreasing a one observes a cascade of bifurcations leading to appearance of attracting cycles of periods 6, 11, 22, 43, ...

4 Noninvertible Case: $b > 0, 0 < \gamma < 1$ (Dominant Chaos)

Proposition 5 (robust unbounded chaos) *Let $0 < \gamma < 1$ and $0 < b \leq \psi_R$. Then,*

(i) *for $a_L^h < a < -1$, the map f has an unbounded chaotic attractor consisting of intervals $[-1, f_R^2(-a - 1)]$, $[f_L \circ f_R^2(-a - 1), -a - 1]$ and $[f_R(-a - 1), +\infty)$; here, $a = a_L^h$ satisfies the condition of the first homoclinic bifurcation of x_L^* defined by*

$$\frac{b}{(b(-a - 1)^{-\gamma} - 1)^\gamma} = \frac{a}{a - 1};$$

(ii) *for $a_R^h < a \leq a_L^h$ the map f has an unbounded chaotic attractor consisting of intervals $[-1, -a - 1]$ and $[f_R(-a - 1), +\infty)$;*

(iii) *for $a_{LR} < a \leq a_R^h$, depending on the value of b , the map f may have the attracting unbounded chaotic interval $[-1, +\infty)$; here, $a = a_{LR}$ satisfies the condition*

$$b = -\frac{1}{a^\gamma} \left(-\gamma \frac{a + 1}{\gamma + 1} \right)^{\gamma + 1}$$

of the fold bifurcation of the 2-cycle LR (indicated in Fig. 1 as ϕ_{LR});

(iv) *for $a < a_{LR}$ almost all the trajectories converge to the attracting 2-cycle born crossing the curve ϕ_{RL} , and a chaotic repeller exists.*

Proposition 6 *Let $0 < \gamma < 1$ and $b > \psi_R$. Then,*

(i) *for $a_L^h < a < -1$, the attracting fixed point x_R^* of the map f coexists with a chaotic attractor in the unbounded invariant absorbing intervals*

$$[-1, f_R^2(-a - 1)] \cup [f_L \circ f_R^2(-a - 1), -a - 1] \cup [f_R(-a - 1), +\infty);$$

(ii) *for $a_{RR}^h < a \leq a_L^h$, the attracting fixed point x_R^* of the map f coexists with a chaotic attractor in the unbounded invariant absorbing intervals $[-1, -a - 1] \cup [f_R(-a - 1), +\infty)$; here, $a = a_{RR}^h$ is related to the first homoclinic bifurcation of the repelling 2-cycle;*

(iii) *for $a_{LR} < a \leq a_{RR}^h$, depending on the value of b , the fixed point x_R^* may be the unique attractor of the map f , and a chaotic repeller exists;*

(iv) *for $a < a_{LR}$, the attracting fixed point x_R^* coexists with an attracting 2-cycle born due to an S-fold crossing the curve ϕ_{RL} , and a chaotic repeller exists.*

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