Applied and Numerical Harmonic Analysis

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Frames and Other Bases in Abstract and Function Spaces

Novel Methods in Harmonic Analysis, Volume 1

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Applied and Numerical Harmonic Analysis

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Editors
Isaac Pesenson
Department of Mathematics
Temple University
Philadelphia, PA, USA

Quoc Thong Le Gia
School of Mathematics and Statistics
University of New South Wales
Sydney, NSW, Australia

Azita Mayeli
Department of Mathematics
The Graduate Center, CUNY
New York, NY, USA

Hrushikesh Mhaskar
Institute of Mathematical Sciences
Claremont Graduate University
Claremont, CA, USA

Ding-Xuan Zhou
Department of Mathematics
City University of Hong Kong
Kowloon Tong, Hong Kong

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ANHA Series Preface

The *Applied and Numerical Harmonic Analysis* (*ANHA*) book series aims to provide the engineering, mathematical, and scientific communities with significant developments in harmonic analysis, ranging from abstract harmonic analysis to basic applications. The title of the series reflects the importance of applications and numerical implementation, but richness and relevance of applications and implementation depend fundamentally on the structure and depth of theoretical underpinnings. Thus, from our point of view, the interleaving of theory and applications and their creative symbiotic evolution is axiomatic.

Harmonic analysis is a wellspring of ideas and applicability that has flourished, developed, and deepened over time within many disciplines and by means of creative cross-fertilization with diverse areas. The intricate and fundamental relationship between harmonic analysis and fields such as signal processing, partial differential equations (PDEs), and image processing is reflected in our state-of-the-art *ANHA* series.

Our vision of modern harmonic analysis includes mathematical areas such as wavelet theory, Banach algebras, classical Fourier analysis, time-frequency analysis, and fractal geometry, as well as the diverse topics that impinge on them.

For example, wavelet theory can be considered an appropriate tool to deal with some basic problems in digital signal processing, speech and image processing, geophysics, pattern recognition, biomedical engineering, and turbulence. These areas implement the latest technology from sampling methods on surfaces to fast algorithms and computer vision methods. The underlying mathematics of wavelet theory depends not only on classical Fourier analysis but also on ideas from abstract harmonic analysis, including von Neumann algebras and the affine group. This leads to a study of the Heisenberg group and its relationship to Gabor systems, and of the metaplectic group for a meaningful interaction of signal decomposition methods. The unifying influence of wavelet theory in the aforementioned topics illustrates the justification for providing a means for centralizing and disseminating information from the broader, but still focused, area of harmonic analysis. This will be a key role of *ANHA*. We intend to publish with the scope and interaction that such a host of issues demands.

Along with our commitment to publish mathematically significant works at the frontiers of harmonic analysis, we have a comparably strong commitment to publish major advances in the following applicable topics in which harmonic analysis plays a substantial role:

Antenna theory	*Prediction theory*
Biomedical signal processing	*Radar applications*
Digital signal processing	*Sampling theory*
Fast algorithms	*Spectral estimation*
Gabor theory and applications	*Speech processing*
Image processing	*Time-frequency and time-scale analysis*
Numerical partial differential equations	*Wavelet theory*

The above point of view for the *ANHA* book series is inspired by the history of Fourier analysis itself, whose tentacles reach into so many fields.

In the last two centuries Fourier analysis has had a major impact on the development of mathematics, on the understanding of many engineering and scientific phenomena, and on the solution of some of the most important problems in mathematics and the sciences. Historically, Fourier series were developed in the analysis of some of the classical PDEs of mathematical physics; these series were used to solve such equations. In order to understand Fourier series and the kinds of solutions they could represent, some of the most basic notions of analysis were defined, e.g., the concept of "function." Since the coefficients of Fourier series are integrals, it is no surprise that Riemann integrals were conceived to deal with uniqueness properties of trigonometric series. Cantor's set theory was also developed because of such uniqueness questions.

A basic problem in Fourier analysis is to show how complicated phenomena, such as sound waves, can be described in terms of elementary harmonics. There are two aspects of this problem: first, to find, or even define properly, the harmonics or spectrum of a given phenomenon, e.g., the spectroscopy problem in optics; second, to determine which phenomena can be constructed from given classes of harmonics, as done, for example, by the mechanical synthesizers in tidal analysis.

Fourier analysis is also the natural setting for many other problems in engineering, mathematics, and the sciences. For example, Wiener's Tauberian theorem in Fourier analysis not only characterizes the behavior of the prime numbers but also provides the proper notion of spectrum for phenomena such as white light; this latter process leads to the Fourier analysis associated with correlation functions in filtering and prediction problems, and these problems, in turn, deal naturally with Hardy spaces in the theory of complex variables.

Nowadays, some of the theory of PDEs has given way to the study of Fourier integral operators. Problems in antenna theory are studied in terms of unimodular trigonometric polynomials. Applications of Fourier analysis abound in signal processing, whether with the fast Fourier transform (FFT), or filter design, or the adaptive modeling inherent in time-frequency-scale methods such as wavelet theory.

The coherent states of mathematical physics are translated and modulated Fourier transforms, and these are used, in conjunction with the uncertainty principle, for dealing with signal reconstruction in communications theory. We are back to the raison d'être of the *ANHA* series!

College Park, MD, USA

John J. Benedetto

Preface

We present the first of two volumes, which are composed of more than 30 articles related to harmonic analysis. Harmonic analysis is a very old topic, which still continues to draw the interest of many mathematicians. Modern research in this area is motivated both by deeper and new theoretical questions and numerous practical applications. These volumes aim to provide a sample of some of these directions. All the authors were selectively invited and comprise both senior and junior mathematicians. We are pleased to have received an unexpectedly enthusiastic response to our invitations.

In response to the number of papers we received, it was suggested by Birkhäuser/Springer to split our book into two volumes. Chapters in each volume are organized into parts according to their topics, and the order of chapters in each part is alphabetical. This first volume, entitled "Frames and Other Bases in Abstract and Function Spaces," consists of 16 chapters. It is quite homogeneous mathematically since every chapter relates to the notion of frames or bases of other types. The introduction to this volume contains some basic notions of the theory of frames and underlines the way the chapters fit into the general theme. The second volume, which is called "Recent Applications of Harmonic Analysis to Function Spaces, Differential Equations, and Data Science," consists of 15 chapters and is very diverse. Its introduction is just a collection of extended abstracts.

We were lucky to receive excellent contributions by the authors, and we enjoyed working with them. We deeply appreciate the generous help of many of our colleagues who were willing to write very professional and honest reviews on submissions to our volumes. We are very thankful to John Benedetto, who is the series editor of the Birkhäuser Applied and Numerical Harmonic Analysis Series, for his constant and friendly support. We appreciate the constant assistance of Birkhäuser/Springer editors Danielle Walker and Benjamin Levitt. We are thankful to Meyer Pesenson and Alexander Powell for their constructive comments regarding introductions. We acknowledge our young colleague Hussein Awala for his help with organizing files and templates.

We hope these volumes will be useful for people working in different fields of harmonic analysis.

Philadelphia, PA, USA — Isaac Pesenson
Sydney, NSW, Australia — Quoc Thong Le Gia
New York, NY, USA — Azita Mayeli
Claremont, CA, USA — Hrushikesh Mhaskar
Kowloon Tong, Hong Kong — Ding-Xuan Zhou

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Part I
Introduction

Frames: Theory and Practice

Isaac Pesenson

Frames in Hilbert spaces were introduced by Duffin and Schaeffer in [1] for reconstruction of bandlimited signals from irregularly spaced sampling points. The precise definition of a frame is the following.

Definition 1. A set of vectors $\{\phi_j\}$ in a Hilbert space H is called a frame in H if there are $0 < A \leq B < \infty$ such that for any $f \in H$

$$A\|f\|_H^2 \leq \sum_j |\langle f, \phi_j \rangle_H|^2 \leq B\|f\|_H^2, \tag{1}$$

where $\langle ., . \rangle_H$ is the inner product in H.

The informal meaning of the frame inequalities (1) is that every vector $f \in H$ can be uniquely reconstructed from a set of "measurements" $\{\langle f, \phi_j \rangle\}$ and this reconstruction is stable.

The ratio A/B of the constants in (1) is called the tightness of the frame. The frame is said to be tight if $A = B$ and it is called Parseval if $A = B = 1$. The frame inequalities (1) imply that the so-called frame operator

$$S : f \to \sum \langle f, \phi_j \rangle \phi_j$$

is bounded, positive-definite, and invertible. As a result one has the following reconstruction formulas:

$$f = \sum \langle f, \phi_j \rangle S^{-1}\phi_j = \sum \langle f, S^{-1}\phi_j \rangle \phi_j,$$

I. Pesenson (✉)
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA
e-mail: pesenson@temple.edu

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where $\{S^{-1}\phi_j\}$ forms another frame with constants B^{-1}, A^{-1}, known as the canonical dual frame. If $\{\phi_j\}$ is a Parseval, frame then it coincides with its canonical dual and one has a remarkable reconstruction formula

$$f = \sum \langle f, \phi_j \rangle \phi_j.$$

Riesz bases and orthonormal bases are examples of frames. A set of vectors $\{\phi_j\}$ in a finite-dimensional Hilbert space is a frame if and only if it is (just) a spanning set. Compared to orthonormal bases, frame bases are generally overcomplete. It turns out that this property is very advantageous for many applications. It makes frame bases a very flexible tool for analysis and syntheses of signals in many branches of information theory.

For example, frames naturally appear in a recently developed theory known as the Dynamical Sampling. This theory is considered in chapter "Dynamical sampling and systems from iterative actions of operators" written by Akram Aldroubi and Armenak Petrosyan. This chapter opens Part II which is called "Frames in abstract spaces." In dynamical sampling, an unknown function f and its future states $A^n f$ are coarsely sampled at each time level n, $0 \leq n < L$, where A is an evolution operator that drives the system. The goal is to recover f from these space-time samples. The authors review some of the recent developments and prove new results concerning frames and Bessel systems generated by iterations of the form $\{A^n g : g \in G,\ n = 0, 1, 2, \ldots\}$, where A is a bounded linear operator on a separable complex Hilbert space H and G is a countable set of vectors in H. For example, one of the main results they state says that if A has a discrete spectrum then every f in H can be recovered from $\{\langle A^n f, g\rangle\}_{g\in G, 0\leq n<L(g)}$ in a stable way if and only if the system $\{(A^*)^n g\}_{g\in G, 0\leq n<L(g)}$ is a frame in H.

Chapter "Optimization methods for frame conditioning and application to graph Laplacian scaling" by Radu Balan, Mathew Begue, Chae Clark, Kasso Okoudjou is motivated by the theory of scalable frames in finite-dimensional Hilbert spaces. As it follows from the very definition tight frames are optimal in the sense that their frame constant A/B is one. This property makes them very attractive for computational (reconstruction) purposes. Scalable frames were introduced to turn a nonoptimal (non-tight) frame into an optimal one, by just rescaling the length of each frame vector. Namely,

Definition 2. A frame $\{\phi_j\}_{j=1}^J$ in a finite-dimensional Hilbert space H is scalable if there exists nonnegative numbers $s_1, \ldots, s_J$ such that $\{s_j\phi_j\}_{j=1}^J$ is a Parseval frame for H.

Authors are interested in computational aspects of constructing scalable or approximately scalable frames by using methods of convex optimization. The chapter also contains an interesting section about scalable frames and optimization methods associated with a combinatorial Laplace operator on a finite graph.

In practice, one can be interested in frames whose elements have some specific properties. In chapter "A Guide to Localized Frames and Applications to Galerkin-like Representations of Operators" by Peter Balazs and Karlheinz Gröchenig, authors consider a general theory of localized frames. Their aim is to formulate an abstract concept of localization that can explain the success of certain structured frames in applications. Originally, a frame $\Phi = \{\phi_j\}_{j=1}^{J}$ was called localized, if its Gramian matrix G with entries $G_{j,k} = \langle \phi_j, \phi_k \rangle$, $1 \leq j, k \leq J$ possesses enough off-diagonal decay. In the further developments powerful methods of Banach algebra theory were used, and nowadays, a frame is called localized, if its Gramian belongs to a solid, inverse-closed Banach algebra of matrices.

The chapter is focusing on the so-called intrinsically localized frames whose definition is the following.

Definition 3. Let $\Phi = \{\phi_j\}, \Psi = \{\psi_k\}$ be two frames in a Hilbert space H and $(G_{\Phi,\Psi})_{j,k} = \langle \phi_j, \psi_k \rangle$, be their cross-Gram matrix. For a solid spectral Banach *-algebra of infinite matrices $\mathscr{A}$, one says that Φ and Ψ are $\mathscr{A}$-localized with respect to each other, if $(G_{\Phi,\Psi})_{j,k} \in \mathscr{A}$.

The chapter also gives a survey on localized frames and the corresponding matrix representation of operators. The authors explain the concept of localization of frames and present many useful properties of these frames. Authors are mainly interested in the theory of the associated coorbit spaces and in the Galerkin discretization of operators with respect to localized frames.

It should be noted that localized frames appear in a number of applications. For example, they allow the sparsification of Fourier integral operators. Also, the evolution operators related to the wave, Schrödinger, heat equation, and other PDEs are examples of operators whose corresponding Gramian matrix is sparse, provided that the right localized frame is chosen.

As it was already mentioned a frame in a finite-dimensional Hilbert space is any set of vectors which spans the entire space. Clearly, even the variety of frames of the same length is “too big.” The goal of the chapter “Computing the Distance Between Frames and Between Subspaces of a Hilbert Space” by Travis Bemrose, Peter G. Casazza, Desai Cheng, John Haas, and Hanh Van Nguyen is to understand some properties of a variety of frames of a fixed length in a finite dimensional Hilbert space. The authors convert a variety of such frames to a metric space and measure distances between frames. There are many ways to convert a variety of frames into a metric space and, accordingly, many notions of a distance between frames.

For example, given an M-dimensional Hilbert space H_M one can fix an orthonormal basis and identify every frame $\Phi = \{\phi_j\}_{j=1}^{N}$ of length N with the matrix

$$\Phi = [\phi_1, \phi_2, \ldots, \phi_N]$$

with columns given by the coordinate representations of the frame vectors. Next, one can use, say, the Frobenius metric for matrices to measure distance between frames which is denoted as d_F. In this chapter authors make a detailed study of six major distance measurements for frames and subspaces: the frame distance d_F, the Gramian distance d_G, the isomorphy distance d_I , the subspace distance d_S, the chordal distance d_C, and the Parseval equivalence class distance d_P. The chapter establishes different relationships between these metrics and provides a number of equivalent ways for computing them.

The chapter “Sigma-Delta quantization for fusion frames and distributed sensor networks” by Jiayi Jiang and Alexander M. Powell contains further development and applications of such notions as Sigma-Delta ($\Sigma\Delta$) quantization and fusion frames. Fusion frames have been used as a natural medium for studying distributed sensor networks. They may be viewed as a vector-valued generalization of classical frames.

Definition 4. By a fusion frame $\{P_{W_j}\}$ one understands a family of orthogonal projectors P_{W_n} of H onto a subspace W_n such that for $0 < A \le B < \infty$ one has for every $f \in H$ the frame-like inequality

$$A\|f\|_H^2 \le \sum_j | \le \left\|P_{W_j} f\right\|_H^2 \le B\|f\|_H^2,$$

To make the continuous measurements $\{P_{W_j} f\}$ “digital friendly” one has to reduce them to discrete or finite. The corresponding step is known as quantization. For $1 \le n \le N$, let $\mathscr{A}_n \subset W_n$ be a finite set, referred to as a quantization alphabet, and let $Q_n : W_n \to \mathscr{A}_n$ be an associated quantizer map with the property that for every $w \in W_n$

$$\|w - Q_n(w)\| = \min_{a \in \mathscr{A}_n} \|w - a\|.$$

The first order fusion frame $\Sigma\Delta$ quantizer takes the fusion frame measurements $y_n = P_{W_n}(f) \in W_n,\ 1 \le n \le N$, and produces the quantized outputs $q_n \in \mathscr{A}_n,\ 1 \le n \le N$, by initializing $u_0 = 0 \in \mathbf{R}^d$ and iterating the following for $n = 1, \ldots, N$

$$q_n = Q_n\left(P_{W_n}(u_{n-1}) + y_n\right)$$

$$u_n = P_{W_n}(u_{n-1}) + y_n - q_n.$$

Authors also consider general order Sigma-Delta quantization algorithms for fusion frames and provide stability results. Finally, they present numerical results and experiments to illustrate the developed theory.

Part III “Space-frequency analysis in function spaces on $\mathbf{R}^n$” begins with chapter “Recent Progress in Shearlet Theory: Systematic Construction of Shearlet Dilation Groups, Characterization of Wavefront Sets, and New Embeddings” by Giovanni S. Alberti, Stephan Dahlke, Filippo De Mari, Ernesto De Vito, and Hartmut Führ. The chapter presents a very general systematic approach to the continuous shearlet transform with applications to microlocal analysis.

Let us recall that a continuous wavelet transform of a function f in $L_2(\mathbf{R})$ is introduced as

$$W_\psi f(a,b) = a^{-1/2} \int_{\mathbf{R}} f(t) \bar{\psi}\left(\frac{t-b}{a}\right) db \frac{da}{a^2}, \quad (a,b) \in (0,\infty) \times \mathbf{R}, \tag{2}$$

where $\bar{\psi}$ is the complex conjugate of a sufficiently nice function $\psi \in L_2(\mathbf{R})$ known as a mother wavelet. The entire one-dimensional theory can be formulated in group-theoretic terms by using the so-called $ax + b$ group. From the very beginning this transform was used to characterize singularities of functions of one variable.

To obtain a shearlet transform in $L_2(\mathbf{R}^n)$ one replaces one-dimensional translation by b with n-dimensional translations and dilation by $a > 0$ with multiplication by an appropriate upper triangular matrix. Chapter 7 contains a deep approach to shearlet dilation groups which unifies the results of a number of recent research papers. The authors apply their results to derive a very elegant characterization of the wavefront set of distributions in $\mathbf{R}^n$. One of the main discoveries made in the paper shows that the property of characterizing the wavefront set is linked to anisotropic scaling which is inherited by the shearlet transform. The paper also contains some results about the embeddings of shearing dilation groups into symplectic groups.

"Chapter Numerical Solution to an Energy Concentration Problem Associated with the Special Affine Fourier Transformation" is written by Amara Ammari, Tahar Moumni, and Ahmed Zayed.

A fundamental problem in signal processing and optics is the problem of identifying bandlimited signals of a given bandwidth whose energy is essentially contained in a prescribed time interval. This concentration of energy problem can be reduced to a problem of finding the largest eigenvalues and associated eigenfunctions of certain two-dimensional integral equations. The authors consider equations of the form

$$\int_T K(x,y)f(y)dy = \lambda f(x), \quad x, y \in \mathbf{R}^2, \tag{3}$$

where T is a unit disk or a standard quadrilateral in $\mathbf{R}^2$. In both cases they reduce the equation (3) to a similar equation in which T is replaced by the square $[0, 1] \times [0, 1]$. The reduction is performed by using the Special Affine Fourier Transformation, a general inhomogeneous lossless linear mapping in phase space that maps the position x and the wave number k into

$$\begin{pmatrix} x' \\ k' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ k \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix}, \quad ad - bc = 1.$$

The authors find numerical solutions of the problem for the square by discretizing it by means of the Gaussian quadrature method in two dimensions. The results for eigenvalues are presented as tables for the absolute values of selected eigenvalues and eigenfunctions are presented by their graphs. The chapter also contains a computation of the so-called generalized prolate spheroidal wave functions.

In the next well-written chapter, the reader can learn a lot about the mathematics of Magnetic Resonance Imaging (MRI). This chapter is called "A frame reconstruction algorithm with applications to magnetic resonance imaging" and it is authored by John J. Benedetto, Alfredo Nava-Tudela, Alexander M. Powell, and Yang Wang.

The main goal of it is to develop an algorithm for a class of signal reconstruction problems, where the efficient noise reduction and stable signal reconstruction are essential. This goal is achieved by combining general theorems (based on fundamental theorems by Beurling and Landau) about irregular sampling of Paley-Wiener functions in two variables with methods of finite frames in Hilbert spaces. In fact, they further develop fast spectral data acquisition (this is especially important for MRI) in the two-dimensional spectral domain by performing a one-dimensional irregular sampling along interleaving Archimedean spirals which densely cover $\mathbf{R}^2$ and then present an approximate image reconstruction method in terms of specific finite frames adapted to this sampling method.

Chapter "Frame properties of shifts of prolate and bandpass prolate functions" by Jeffrey A. Hogan and Joseph D. Lakey is devoted to specific frames in Paley-Wiener spaces

$$PW_{[-\Omega/2,\Omega/2]}(\mathbf{R}) = PW_{\Omega}(\mathbf{R})$$

of functions bandlimited to $[-\Omega/2, \Omega/2]$, $\Omega > 0$, and in spaces of bandpass-limited functions

$$PW_{[-\Omega/2,-\Omega'/2]\setminus[\Omega'/2,\Omega/2]}(\mathbf{R}) = PW_{\Omega',\Omega}(\mathbf{R}),\ \ 0 < \Omega^{'} < \Omega.$$

They begin by considering the operator $P_{\Omega}Q_T : PW_{\Omega}(\mathbf{R}) \to PW_{\Omega}(\mathbf{R})$ where Q_T multiplies $f \in L_2(\mathbf{R})$ by the characteristic function of the time interval $[-T, T]$ and $P_{\Omega} = \mathscr{F}^{-1}Q_{\Omega/2}\mathscr{F}$. The space $PW_{\Omega}(\mathbf{R})$ has an orthonormal basis composed of eigenfunctions of the compact operator $P_{\Omega}Q_1$. These functions are known as prolate spheroidal wave functions. Note that this orthonormal basis contains eigenfunctions from every eigenspace of $P_{\Omega}Q_1$ and there are infinitely many such eigenspaces. The authors construct a frame in $PW_{\Omega}(\mathbf{R})$ which involves only a finite number $P \in \mathbf{N}$ of eigenfunctions $\{\psi_n\}_{n=0}^{P}$ which correspond to highest eigenvalues and the "missing" eigenspaces are "compensated" by translations of these functions $\psi_n(\cdot - \alpha k)$, $n = 0, \dots P-1, k \in \mathbf{Z}$. It is worth mentioning that the eigenfunctions ψ_n which correspond to highest eigenvalues are the Ω-bandlimited functions whose essential support is in $[-1, 1]$.

In chapter "Fast Fourier Transforms for Spherical Gauss-Laguerre Basis Functions" authors Jürgen Prestin and Christian Wülker consider the space $\mathbf{H}$ of all Lebesgue measurable functions $f : \mathbf{R}^3 \to \mathbf{C}$ equipped with the norm

$$\|f\|_{\mathbf{H}} = \left(\int_{\mathbf{R}^3} |f(x)|^2 \exp(-|x|^2) dx \right)^{1/2}.$$

By the Fourier analysis in $\mathbf{H}$ they understand the analysis and synthesis with respect to a special orthonormal basis constructed using both the spherical harmonics and generalized Laguerre polynomials. This basis in $\mathbf{H}$ is called the Spherical Gauss-Laguerre (SGL) basis. A function in $\mathbf{H}$ is called bandlimited if it is a finite linear combination of the SGL basis functions. The goal is to develop a kind of Fast Fourier Transform in the space $\mathbf{H}$.

The authors propose fast and reliable algorithms for the computation and evaluation of expansions of bandlimited functions into Spherical Gauss-Laguerre (SGL) basis.

By separation of variables the authors construct explicit quadrature rules for the SGL basis functions from the well-known quadrature rules. These new quadrature rules enable the reconstruction of the SGL-Fourier coefficients of a bandlimited function from a finite number of its samples. For a function of bandlimit B, a naive implementation of such discrete SGL-Fourier transform (DSGFT), as well as its evaluation at the quadrature points, requires $O(B^7)$ arithmetic operations. By exploiting the special structure of the SGL basis together with the use of a fast spherical Fourier transforms the authors present algorithms with complexity $O(B^5)$. For bandlimits up to $B = 64$ numerical results show the reliability and the performance of the proposed algorithms. Finally, the authors discuss further research, such as the development of algorithms with complexity $O(B^3 \log_2 B)$ and algorithms which can work with scattered data. It is worth mentioning that SGL basis function expansions have a number of applications for instance, in biomolecular dynamic simulations.

Chapter "Multiscale Radial Basis Functions" by Holger Wendland is a review of the recent work of the author and his co-workers on the development of multiscale methods by using compactly supported radial basis functions. A radial function is a function $\Phi : \mathbf{R}^d \to \mathbf{R}$ of the form $\Phi(x) = \phi(\|x\|_2), x \in \mathbf{R}^d$, where $\phi : [0, \infty) \to \mathbf{R}$ is a univariate function and $\|x\|_2 = (x_1^2 + \ldots + x_d^2)^{1/2}, \; x \in \mathbf{R}^d$. In most of the paper the author is dealing with such Φ whose support is in the unit disc. For such functions he constructs scaled versions

$$\Phi_\delta(x) = \delta^{-d}\Phi(x/\delta), \quad \delta > 0, \; x \in \mathbf{R}^d.$$

By using scaling and shifts the author defines basic approximation spaces of radial functions. In the first part of the paper the author shows how to use radial basis functions (RBF) to reconstruct target functions from scattered data $X = \{x_1, \ldots, x_N\} \in \Omega \subset \mathbf{R}^d$. All the results involve the main characteristics of scattered data: the mesh norm and the separation distance.

The primary focus of the chapter is on the multiscale method of approximation by compactly supported RBF. The main conclusion here is that convergence of a compactly supported sequence of RBF to a target function can fail if the mesh norm and the support parameter do not scale in the right way. One of the main results of the paper contains conditions for how the mesh parameter and the associated scale parameter should decrease together to guarantee linear convergence of the multilevel method. The chapter also contains a lot of useful information about the so-called native spaces, Wendland functions, and the cost of suggested algorithms.

Part IV of this volume is called "Frames in spaces of functions on manifolds and groups" and it starts with a chapter "Orthogonal wavelet frames on manifolds based on conformal mappings" by Swanhild Bernstein and Paul Keydel. In this chapter the authors construct a continuous wavelet transform, discrete wavelet bases and frames in L_2-spaces on Riemannian manifolds. Their main focus is on two-dimensional manifolds. By using a known fact that every 2-dimensional Riemannian manifold $\mathbf{M}$ is conformally flat (curvature free, looks like $\mathbf{R}^2$) they use conformal maps between $\mathbf{M}$ and $\mathbf{R}^2$ to transfer continuous wavelet transform, multiresolution analysis, and frames from $L_2(\mathbf{R}^2)$ to $L_2(\mathbf{M})$. In particular, a multiresolution analysis they construct in $L_2(\mathbf{M})$ leads to an orthonormal basis in $L_2(\mathbf{M})$. The authors are able to transfer continuous and discrete wavelet transforms from $\mathbf{R}^2$ to a manifold in the case when a corresponding conformal map has no singular points. In general, only frames can be transferred. The results are illustrated by considering examples of 2-dimensional surfaces of revolution.

In chapter “Quasi Monte Carlo integration and kernel-based function approximation on Grassmannians” its authors Anna Breger, Martin Ehler, and Manuel Graf are concerned with numerical integration on an important class of homogeneous manifolds known as Grassman manifolds. Approximation and numerical integration on Euclidean spaces have been central problems in numerical analysis for a long time. Approximation, cubature formulas, and interpolation on spheres are also not new subjects and can be traced back to the classical papers by I. J. Schoenberg and S. L. Sobolev. In recent years numerical integration and function approximation and interpolation on compact Riemannian manifolds based on eigenfunctions of the Laplace-Beltrami operator have been widely researched. A few years ago the so-called Quasi Monte Carlo (QMC) numerical integration on spheres and general compact Riemannian manifolds has been introduced and studied. In particular, in the case of spheres a number of numerical experiments were performed to verify theoretical claims. The major aim of the present paper is to provide numerical experiments for the QMC scheme when the manifold is the Grassmannian $\mathcal{G}_{k,d}$, i.e., the collection of k-dimensional subspaces in $\mathbf{R}^d$, naturally identified with the collection of rank-k orthogonal projectors in $\mathbf{R}^d$. To perform this very laborious task, the authors explored the observation that instead of constructing cubature formulas in the subspaces of eigenfunctions $\Pi_t\left(\mathcal{G}_{k,d}\right)$, $t \geq 0$, of the Laplace-Beltrami operator on $\mathcal{G}_{k,d}$ it is easier to construct them in a relevant subspace of polynomials in $\mathbf{R}^d$ restricted to $\mathcal{G}_{k,d}$. Authors do not explicitly mention frames in the entire paper, however, the fact that they consider positive cubature formulas which are exact on the subspaces of eigenfunctions $\Pi_t\left(\mathcal{G}_{k,d}\right)$, $t \geq 0$, means (in a different language) that they are talking about frames in these subspaces.

In chapter “Construction of Multiresolution Analysis Based on Localized Reproducing Kernels” its authors K. Nowak and M. Pap survey specific methods of extending the classical multiresolution analysis in the space $L_2(\mathbf{R})$ to function spaces on manifolds which do not have regular translations and dilations. A multiresolution analysis in $L_2(\mathbf{R})$ is a framework which allows for construction of orthonormal bases generated by a single function. It leads to a very special orthonormal basis in $L_2(\mathbf{R})$ which is comprised of the discrete translates and dilates of a single admissible function ψ:

$$\psi_{n,k}(x) = 2^{-n/2}\psi(2^{-n}x - k).$$

On manifolds other than $\mathbf{R}$ there is usually a lack o translations and dilations which makes very difficult to generalize ideas of the classical multiresolution analysis.

In this survey the Hardy space and the Bergman space on the unit disc are considered. It is observed that there are two subgroups of the Blaschke group one of which can be treated as a group of generalized translations and another one as a group of generalized dilations in the corresponding Hardy and Bergman spaces. They construct an analog of the multiresolution analysis by proper discretization of two special voice transforms of the Blaschke group. The constructions make use of the localization properties of the reproducing kernels restricted to appropriately selected sets of sampling points. It leads to orthonormal bases comprised of generalized translations and dilations of a single function in the Hardy and Bergman spaces on the unit disk.

Vignon S. Oussa in his chapter "Regular Sampling on Metabelian Nilpotent Lie Groups: The Multiplicity-Free Case" studies questions which relate to extension of Whittaker-Kotel'nikov-Shannon sampling and interpolation of properly defined bandlimited functions on a class of nilpotent Lie groups. His results generalize a number of facts which were known for the Heisenberg group.

Let N be a simply connected nilpotent Lie group with Lie algebra $\mathbf{n}$. If $\mathbf{n}$ has rational structure constants, then there is a strong Malcev basis $X_1, X_2, \ldots, X_k$ of $\mathbf{n}$ such that $\Gamma = \exp \mathbf{Z}X_1 \exp \mathbf{Z}X_2 \ldots \exp \mathbf{Z}X_k$ is a discrete subgroup of N. One says that a subspace of continuous functions $\mathbf{H} \subset L_2(N)$ is a sampling set with respect to Γ if

1. there exists a constant $c_{\mathbf{H}}$ such that for all $f \in \mathbf{H}$

$$\|f\|^2_{L_2(\mathbf{M})} = c_{\mathbf{H}} \sum_{\gamma \in \Gamma} |f(\gamma)|^2 \tag{4}$$

2. there exists a function $s \in \mathbf{H}$ such that

$$f(x) = \sum_{\gamma \in \Gamma} f(\gamma) s\left(\gamma^{-1} x\right).$$

Also, if $\mathbf{H}$ is a sampling space with respect to Γ and if the restriction mapping

$$f \to f|_{\Gamma} \in l_2(\Gamma), \quad f \in \mathbf{H},$$

is surjective, then we say that $\mathbf{H}$ has the interpolation property with respect to Γ.

It is natural to look for subspaces of $L_2(N)$ that are sampling and also interpolation spaces with respect to Γ. In the case when N is a semi-direct product of two abelian groups, the author uses the group Fourier transform to describe a family of sampling spaces via a natural sufficient condition. In this family of sampling spaces the author is able to identify those that are interpolation spaces. It should be noticed, that every equality like (4) can be interpreted in terms of tight frames in the corresponding space $\mathbf{H}$.

Chapter "Parseval space-frequency localized frames on sub-Riemannian compact homogeneous manifolds" is written by Isaac Pesenson. The objective of this chapter is to describe a construction of Parseval bandlimited and localized frames in L_2-spaces on sub-Riemannian compact homogeneous manifolds.

The chapter begins with a brief review of some results about Parseval bandlimited and localized frames in $L_2(\mathbf{M})$, where $\mathbf{M}$ is a compact homogeneous manifold equipped with a natural Riemannian metric. Then author is using a sub-Riemannian structure on the two-dimensional standard unit sphere $\mathbf{S}^2$ to explain the main differences between Riemannian and sub-Riemannian settings. Each of these structures is associated with a second-order differential operator which arises from a metric. The major difference between these operators is that in the case of Riemannian metric the operator is elliptic (the Laplace-Beltrami operator $\mathbf{L}$) and in the sub-Riemannian case it is not (the sub-Laplacian $\mathscr{L}$). As a result, the corresponding Sobolev spaces which are introduced as domains of powers of these operators are quite different. In the elliptic case one obtains the regular Sobolev spaces and in sub-elliptic one obtains function spaces (sub-elliptic Sobolev spaces) in which functions have variable smoothness (compared to regular (elliptic) Sobolev smoothness). The author describes a class of sub-Riemannian structures on compact homogeneous manifolds and a construction of Parseval bandlimited (with respect to a sub-Laplacian) and localized (with respect to a corresponding sub-Riemann metric) frames.

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Part II
Frames in Abstract Spaces

Dynamical Sampling and Systems from Iterative Actions of Operators

Akram Aldroubi and Armenak Petrosyan

1 Introduction

The typical dynamical sampling problem is finding spatial positions $X = \{x_i \in \mathbb{R}^d : i \in I\}$ that allow the reconstruction of an unknown function $f \in \mathscr{H} \subset L^2\left(\mathbb{R}^d\right)$ from samples of the function at spatial positions $x_i \in X$ and subsequent samples of the functions $A^n f$, $n = 0, \cdots, L$, where A is an evolution operator and n represents time. For example, f can be the temperature at time $n = 0$, A the heat evolution operator, and $A^n f$ the temperature at time n. The problem is then to find spatial sampling positions $X \subset \mathbb{R}^d$, and end time L, that allow the determination of the initial temperature f from samples $\{f|_X, (Af)|_X, \cdots, (A^L f)|_X\}$. For the heat evolution operator, the problem has been considered by Vetterli and Lu [20, 21] and inspired our research in dynamical sampling, see, e.g., [3–6].

1.1 The Dynamical Sampling Problem

Let $\mathscr{H}$ be a separable (complex) Hilbert space, $f \in \mathscr{H}$ be an unknown vector, and $f_n \in \mathscr{H}$ be the state of the system at time n. We assume

$$f_0 = f, \;\; f_n = Af_{n-1} = A^n f$$

A. Aldroubi (✉) • A. Petrosyan
Vanderbilt University, Nashville, TN 37240, USA
e-mail: akram.aldroubi@vanderbilt.edu; armenak.petrosyan@vanderbilt.edu

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where A is a known bounded operator on $\mathscr{H}$. Given the measurements

$$\langle A^n f, g\rangle \text{ for } 0 \le n < L(g), \quad g \in G \tag{1}$$

where G is a countable subset of $\mathscr{H}$ and $L : G \mapsto \mathbb{N} \cup \{\infty\}$ is a function, the dynamical sampling problem is to recover the vector $f \in \mathscr{H}$ from the measurements (1). It is important that the recovery of f be robust to noise. Thus, we also require that the sampling data allow the recovery of f in a stable way. Equivalently, for any $f \in \mathscr{H}$, the samples must satisfy the stability condition

$$C_1\|f\|^2 \le \sum_{g\in G}\sum_{n=0}^{L(g)} | < A^n f, g > |^2 \le C_2\|f\|^2,$$

for some $C_1, C_2 > 0$ absolute constants.

A related problem for band-limited signals in $\mathbb{R}^2$ (i.e., the Paley Wiener spaces PW_σ) with time varying sampling locations corresponding to trajectories but time-independent function can be found in [16].

1.2 Dynamical Sampling for Diagonalizable Operators in $l^2(\mathbb{N})$

When the Hilbert space is $\mathscr{H} = l^2(\mathbb{N})$, and when the operator A is equivalent to a diagonal matrix D, i.e., $A^* = B^{-1}DB$ where $D = \sum_j \lambda_j P_j$ is an infinite diagonal matrix, then a characterization of the set of sampling $I \subset \mathbb{N}$ such that any $f \in \mathscr{H}$ can be recovered from the data $Y = \{f(i), (Af)(i), \cdots, (A^{l_i}f)(i) : i \in I\}$ is obtained in [4].

The results are stated in terms of vectors b_i that are the columns of B corresponding to the sampling positions $i \in I$, the projections P_j that are diagonal infinite matrices whose non-zero diagonals are all ones and correspond to the projection on the eigenspace of D associated with λ_j, and the smallest integers l_i such that the sets $\{b_i, Db_i, \dots, D^{l_i}b_i\}$ are minimal [6].

Theorem 1. *Let $A^* = B^{-1}DB$, and let $\{b_i : i \in I\}$ be the column vectors of B whose indices belong to I. Let l_i be the smallest integers such that the set $\{b_i, Db_i, \dots, D^{l_i}b_i\}$ is minimal. Then any vector $f \in l^2(\mathbb{N})$ can be recovered from the samples*

$$Y = \{f(i), Af(i), \dots, A^{l_i}f(i) : i \in I\}$$

if and only if for each j, $\{P_j(b_i) : i \in I\}$ is complete in the range E_j of P_j.

Although Theorem 1 characterizes the sets $I \subset \mathbb{N}$ such that recovery of any $f \in l^2(\mathbb{N})$ is possible, it does not provide conditions for stable recovery, i.e., recovery that is robust to noise. Results on the stable recovery are obtained for the case when

I is finite [6]. Stable recovery is also obtained when $\mathscr{H} = l^2(\mathbb{Z})$, A is a convolution operator, and I is a union of uniform grids [4]. For shift-invariant spaces, and union of uniform grids, stable recovery results can be found in [1]. Obviously, recovery and stable recovery are equivalent in finite dimensional spaces [3]. In [27] the case when the locations of the sampling positions are allowed to change is considered.

1.2.1 Connections with Other Fields and Applications

The dynamical sampling problem has similarity with wavelets [7, 11, 12, 17, 22, 24, 25, 31]. In dynamical sampling an operator A is applied iteratively to the function f producing the functions $f_n = A^n f$. f_n is then, typically, sampled coarsely at each level n. Thus, f cannot be recovered from samples at any single time level. But, similarly to the wavelet transform, the combined data at all time levels is required to reproduce f. However, unlike the wavelet transform, there is a single operator A instead of two complementary operators L (the low pass operator) and H (the high pass operator). Moreover, A is imposed by the constraints of the problem, rather than designed, as in the case of L and H in wavelet theory. Finally, in dynamical sampling, the spatial-sampling grids is not required to be regular.

In inverse problems, given an operator B that represents a physical process, the goal is to recover a function f from the observation Bf. Deconvolution and deblurring are prototypical examples. When B is not bounded below, the problem is considered ill-posed (see, e.g., [23]). The dynamical sampling problem can be viewed as an inverse problem when the operator B is the result of applying the operators $S_{X_0}, S_{X_1}A, S_{X_2}A^2, \dots, S_{X_L}A^L$, where S_{X_l} is the sampling operator at time l on the set X_l, i.e., $B_X = [S_{X_0}, S_{X_1}A, S_{X_2}A^2, \dots, S_{X_L}A^L]^T$. However, unlike the typical inverse problem, in dynamical sampling the goal is to find conditions on L, $\{X_i : i = 0, \dots, L\}$, and A, such that B_X is injective, well conditioned, etc.

The dynamical sampling problem has connections and applications to other areas of mathematics including Banach algebras, C^*-algebras, spectral theory of normal operators, and frame theory [2, 8, 10, 13–15, 26, 32].

Dynamical sampling has potential applications in plenacoustic sampling, on-chip sensing, data center temperature sensing, neuron-imaging, and satellite remote sensing, and more generally to wireless sensor networks (WSN). In wireless sensor networks, measurement devices are distributed to gather information about a physical quantity to be monitored, such as temperature, pressure, or pollution [18, 20, 21, 29, 30]. The goal is to exploit the evolutionary structure and the placement of sensors to reconstruct an unknown field. When it is not possible to place sampling devices at the desired locations (e.g., when there are not enough devices), then the desired information field can be recovered by placing the sensors elsewhere and taking advantage of the evolution process to recover the signals at the relevant locations. Even when the placement of sensors is not constrained, if the cost of a sensor is expensive relative to the cost of activating the sensor, then the relevant information may be recovered with fewer sensors placed judiciously and activated frequently. Super resolution is another application when an evolutionary process acts on the signal of interest.

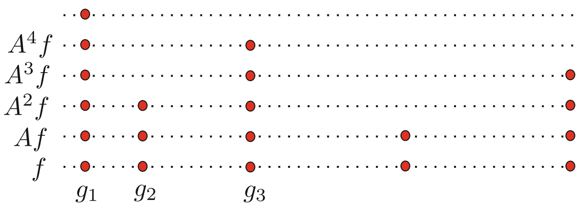

Fig. 1 The dynamical sampling procedure when the sampling happens at different time levels.

1.3 Contribution

In this chapter, we further develop the case of iterative systems generated by the iterative actions of normal operators which was studied in [5, 6]. This is done in Section 2.3. In Section 2.4 we study the case of general iterative systems generated by the iterative actions of operators that are not necessarily normal (Fig. 1).

2 Frame and Bessel Properties of Systems from Iterative Actions of Operators

In this section we review some results from [5, 6] on the iterative actions of normal operators, prove some new results for this case, and generalize several results to the case where the operators are not necessary normal.

2.1 Equivalent Formulation of the Dynamical Sampling Problem

Using the fact that $\langle Af, g\rangle = \langle f, A^*g\rangle$, we get the following equivalent formulation of the dynamical sampling problem described in Section 1.1.

Proposition 1. *1. Any $f \in \mathcal{H}$ can be recovered from $\{\langle A^n f, g\rangle\}_{g\in G, 0\le n<L(g)}$ if and only if the system $\{(A^*)^n g\}_{g\in G, 0\le n<L(g)}$ is complete in $\mathcal{H}$.*
2. Any $f \in \mathcal{H}$ can be recovered from $\{\langle A^n f, g\rangle\}_{g\in G, 0\le n<L(g)}$ in a stable way if and only if the system $\{(A^*)^n g\}_{g\in G, 0\le n<L(g)}$ is a frame in $\mathcal{H}$.

Because of this equivalence, we drop the $*$ and we investigate systems of iterations of the form $\{A^n g\}_{g\in G, 0\le n<L(g)}$, where A is a bounded operator on the Hilbert space $\mathcal{H}$, G is a subset of $\mathcal{H}$, and L is a function from G to the extended set of integers $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$. The goal is then to find conditions on A, G, and L so that $\{A^n g\}_{g\in G, 0\le n<L(g)}$ is complete, Bessel, a basis, Riesz basis, frame, etc. In the remainder of this chapter, we only study the case where $L(g) = \infty$, for each $g \in G$.

2.2 Normal Operators

Theorem 1 as well as most of the results in [6] has been generalized to the case of normal operators in general Hilbert spaces [5], and new results have been obtained. The work relied on the spectral theorem of normal operators in Hilbert spaces (see, e.g., [10] Ch. IX, theorem 10.16, and [9]). Since we will use this theorem again in this work, we state a version of this landmark theorem and give an example to clarify its meaning. In essence, the spectral theorem of normal operators is a way of diagonalizing any normal operator in a separable complex Hilbert space. Using an appropriate unitary transformation U, a normal operator A is equivalent to a multiplication operator $UAU^{-1}\tilde{f} = N_\mu \tilde{f} = z\tilde{f}$ where $\tilde{f}$ is a vector valued function on $\mathbb{C}$, and $N_\mu \tilde{f}(z) = z\tilde{f}(z)$ for every $z \in \mathbb{C}$. Specifically,

Theorem 2 (Spectral theorem with multiplicity). *For any normal operator A on $\mathscr{H}$ there are mutually singular compactly supported Borel measures μ_j, $1 \le j \le \infty$ on $\mathbb{C}$, such that A is equivalent to the operator*

$$N^{(\infty)}_{\mu_\infty} \oplus N_{\mu_1} \oplus N^{(2)}_{\mu_2} \oplus \cdots$$

i.e., there exists a unitary operator U

$$U : \mathscr{H} \to \widetilde{\mathscr{H}} = (L^2(\mu_\infty))^{(\infty)} \oplus L^2(\mu_1) \oplus (L^2(\mu_2))^{(2)} \oplus \cdots$$

such that

$$UAU^{-1} = N_\mu = N^{(\infty)}_{\mu_\infty} \oplus N_{\mu_1} \oplus N^{(2)}_{\mu_2} \oplus \cdots . \tag{2}$$

Moreover, if M is another normal operator with corresponding measures $\nu_\infty, \nu_1, \nu_2, \ldots$ then M is equivalent to A if and only if $[\nu_j] = [\mu_j]$, $j = 1, \ldots, \infty$ (are mutually absolutely continuous).

Since the measures $\{\mu_j : j \in \mathbb{N}^*\}$ are mutually singular, we can define the measure $\mu = \sum_j \mu_j$ on $\mathbb{C}$. A function $\widetilde{g} \in \widetilde{\mathscr{H}}$ is a vector of the form $(\widetilde{g}_j)_{j \in \mathbb{N}^*}$, where $\widetilde{g}_j$ is the restriction of $\widetilde{g}$ to $(L^2(\mu_j))^{(j)}$.

Remark 1. Note that for every $1 \le j < \infty$, $\widetilde{g}_j(z)$ is a finite dimensional vector in $l^2\{1, \ldots, j\}$ and for $j = \infty$, $\widetilde{g}_\infty(z)$ is a vector in $l^2(\mathbb{N})$. In order to simplify notation, we define Ω_j to be the set $\{1, \ldots, j\}$ and Ω_∞ to be the set $\mathbb{N}$. Note that $l^2(\Omega_j) \cong \mathbb{C}^j$, for $j \in \mathbb{N}$, and $l^2(\Omega_\infty) = l^2(\mathbb{N})$. For $j = 0$ we define $l^2(\Omega_0)$ to be the trivial space $\{0\}$.

An example to clarify the use of the theorem above is given below.

Example 1. Let A be the 8×8 diagonal matrix

$$A = \begin{pmatrix} \lambda_1 I_2 & 0 & 0 \\ 0 & \lambda_2 I_3 & 0 \\ 0 & 0 & \lambda_3 I_3 \end{pmatrix}$$

where $\lambda_i \neq \lambda_j$ if $i \neq j$ and I_j denotes the $j \times j$ identity matrix. For this case, the theorem above gives: $\mathscr{H} = (L^2(\mu_2))^{(2)} \oplus (L^2(\mu_3))^{(3)}$, $\mu_2 = \delta_{\lambda_1}$, $\mu_3 = \delta_{\lambda_2} + \delta_{\lambda_3}$, where δ_x is the Dirac measure at x. If $g = (g_1, \dots, g_8)^T$, then $Ug = \tilde{g} = \big(\tilde{g}_j\big)$. In particular, $\widetilde{g}_3(\lambda_2) = \begin{pmatrix} g_3 \\ g_4 \\ g_5 \end{pmatrix}$, $\widetilde{g}_3(\lambda_3) = \begin{pmatrix} g_6 \\ g_7 \\ g_8 \end{pmatrix}$, and $\widetilde{g}_3(z) = \mathbf{0}$ for $z \neq \lambda_2, \lambda_3$ (in fact for $z \neq \lambda_2, \lambda_3, \widetilde{g}_3(z)$ can take any value since the measure μ_3 is concentrated on $\{\lambda_2, \lambda_3\} \subset \mathbb{C}$). We have

$$\begin{aligned}
\langle Uf, Ug\rangle &= \int_{\mathbb{C}} \langle \tilde{f}(z), \tilde{g}(z)\rangle d\mu(z) \\
&= \int_{\mathbb{C}} \langle \tilde{f}_2(z), \tilde{g}_2(z)\rangle d\mu_2(z) + \int_{\mathbb{C}} \langle \tilde{f}_3(z), \tilde{g}_3(z)\rangle d\mu_3(z) \\
&= \langle \tilde{f}_2(\lambda_1), \tilde{g}_2(\lambda_1)\rangle + \langle \tilde{f}_3(\lambda_2), \tilde{g}_3(\lambda_2)\rangle + \langle \tilde{f}_3(\lambda_3), \tilde{g}_3(\lambda_3)\rangle \\
&= \sum_{j=1}^{8} f_j \overline{g}_j = \langle f, g\rangle.
\end{aligned}$$

Since the measures μ_j in Theorem 1 are mutually singular, there are mutually disjoint Borel sets $\{\mathscr{E}_j\}$ such that μ_j is concentrated on $\mathscr{E}_j$ for every $1 \leq j \leq \infty$.

The function $n : \mathbb{C} \to \{1, 2, \dots, \infty\}$ given by

$$n(z) = \begin{cases} j, & z \in \mathscr{E}_j \\ 0, & \textit{otherwise} \end{cases}$$

is called multiplicity function of the operator A. Thus every normal operator is uniquely determined, up to a unitary equivalence, by a pair $(n, [\mu])$ where $[\mu]$ is the class of measures mutually singular with the compactly supported Borel measure μ and $n : \mathbb{C} \to \{1, 2, \dots, \infty\}$ is a μ measurable function.

2.3 Action of Normal Operators via Infinite Iterations

In this section we present results from [5] about a system of infinite iterative action $\{A^n g_i\}_{i \in I,\, n \geq 0}$ of a given normal operator $A \in B(\mathscr{H})$ on a set of vectors $G \subset \mathscr{H}$. Some of the results assume that A is reductive, i.e., every invariant subspace V for A is also invariant for A^*.

Theorem 3. *Let A be a normal operator on a Hilbert space $\mathscr{H}$, and let G be a countable set of vectors in $\mathscr{H}$ such that $\{A^n g\}_{g \in G,\, n \geq 0}$ is complete in $\mathscr{H}$. Let $\mu_\infty, \mu_1, \mu_2, \dots$ be the measures in the representation (2) of the operator A. Then for every $1 \leq j \leq \infty$ and μ_j-a.e. z, the system of vectors $\{\widetilde{g}_j(z)\}_{g \in G}$ is complete in $l^2\{\Omega_j\}$.*

If in addition to being normal, A is also reductive, then $\{A^n g\}_{g\in G,\, n\geq 0}$ *being complete in* $\mathscr{H}$ *is equivalent to* $\{\widetilde{g}_j(z)\}_{g\in G}$ *being complete in* $l^2\{\Omega_j\}$ μ_j*-a.e.* z *for every* $1 \leq j \leq \infty$.

Although, the system of iteration $\{A^n g\}_{g\in G,\, n\geq 0}$ is complete, it is shown in [5] that it cannot be a basis for $\mathscr{H}$. The obstruction is that $\{A^n g\}_{g\in G,\, n\geq 0}$ cannot be minimal and complete at the same time.

Theorem 4. *If A is a normal operator on* $\mathscr{H}$ *then, for any set of vectors* $G \subset \mathscr{H}$, *the system of iterates* $\{A^n g\}_{g\in G, n\geq 0}$ *is not a basis for* $\mathscr{H}$.

Remark 2. The normality assumption on A is essential. For example, if S is the right-shift operator on $\mathscr{H} = l^2(\mathbb{N})$, then $\{S^n e_0\}_{n\geq 0}$ is an orthonormal basis for $\mathscr{H} = l^2(\mathbb{N})$. In fact, it can be shown that, in a Hilbert space, a system of vectors $\mathscr{H}$ $\{T^n g\}_{n\geq 0}$ generated by $T \in B(\mathscr{H})$ and $g \in \mathscr{H}$ is a Riesz basis if and only if is unitarily equivalent to the right-shift operator S in $l^2(\mathbb{N})$ [19].

The fact that, for a normal operator A, $\{A^n g\}_{g\in G,\, n\geq 0}$ cannot be basis is that when it is complete, it must be redundant (since it is not minimal). But it is possible for such a sequence to be a frame. For example, the following theorem characterizes frames generated by the iterative action of diagonalizable normal operators acting on a single vector b [6].

Theorem 5. *Let* $\Lambda = \sum_j \lambda_j P_j$, *acting on* $l^2(\mathbb{N})$, *be such that* P_j *have rank* 1 *for all* $j \in \mathbb{N}$, *and let* $b := \{b(k)\}_{k\in\mathbb{N}} \in l^2(\mathbb{N})$. *Then* $\{\Lambda^l b : l = 0, 1, \dots\}$ *is a frame if and only if*

i) $|\lambda_k| < 1$ *for all* k.
ii) $|\lambda_k| \to 1$.
iii) $\{\lambda_k\}$ *satisfy Carleson's condition*

$$\inf_n \prod_{k\neq n} \frac{|\lambda_n - \lambda_k|}{|1 - \bar{\lambda}_n \lambda_k|} \geq \delta. \tag{3}$$

for some $\delta > 0$.
iv) $b(k) = m_k \sqrt{1 - |\lambda_k|^2}$ *for some sequence* $\{m_k\}$ *satisfying* $0 < C_1 \leq |m_k| \leq C_2 < \infty$.

In the previous theorem, the spectrum lies inside the unit disk D_1. Moreover, the spectrum concentrates near its boundary S_1. These facts can be generalized for normal operators [5].

Theorem 6. *Let A be a normal operator on an infinite dimensional Hilbert space* $\mathscr{H}$ *and G a system of vectors in* $\mathscr{H}$.

1. *If* $\{A^n g\}_{g\in G,\, n\geq 0}$ *is complete in* $\mathscr{H}$ *and for every* $g \in G$ *the system* $\{A^n g\}_{n\geq 0}$ *is Bessel in* $\mathscr{H}$, *then* $\mu\left(D_1^c\right) = 0$ *and* $\mu|_{S_1}$ *is absolutely continuous with respect to arc-length measure (Lebesgue measure) on* S_1.

2. *If* $|G| < \infty$ *and* $\{A^n g\}_{g\in G,\, n\geq 0}$ *satisfy the lower frame bound then, for every* $0 < \epsilon < 1$, $\mu\left(D^c_{1-\epsilon}\right) > 0$, *where* $D_{1-\epsilon}$ *is the closed disc of radius* $1-\epsilon$.

It can be proved that if $\mu\left(D^c_1\right) = 0$ and $\mu|_{S_1}$ is absolutely continuous with respect to arc-length measure on S_1, then there exists a set $G \subset \mathscr{H}$ such that $\{A^n g\}_{g\in G,\, n\geq 0}$ is complete and Bessel system in $\mathscr{H}$. Other developments on this theme can be found in [28].

Corollary 1. *If for a normal operator* $A \in B(\mathscr{H})$ *in an infinite dimensional space* $\mathscr{H}$ *the system of vectors* $\{A^n g\}_{g\in G,\, n\geq 0}$ *with* $|G| < \infty$ *is a frame, then A is unitarily equivalent to an operator* $\Lambda = \sum_j \lambda_j P_j$ *where* P_j *are projections such that* $\dim P_j \leq |G|$. *In particular, if* $|G| = 1$, *then* λ_j *satisfy conditions* $i), ii)$ *in Theorem 5.*

Proof. Define the subspace $\widetilde{V}_\rho$ of $\widetilde{\mathscr{H}}$ to be $\widetilde{V}_\rho = \{\tilde{f} : \operatorname{supp}\tilde{f} \subseteq D_\rho\}$. The restriction of UAU^* to $\widetilde{V}_\rho$ is normal with its spectrum equal to the part of the spectrum of A inside D_ρ. If we iterate the z-multiplication operator on the projections $G_\rho = P_{\widetilde{V}_\rho} G$ of the vectors in G we get a frame for $\widetilde{V}_\rho$ hence, from part (2), $\widetilde{V}_\rho$ must be finite dimensional. That implies the spectrum of A is finite inside every D_ρ with $\rho < 1$. We also know from Part (1) of Theorem 6 that $\mu(D^c_1) = 0$. Furthermore, from Corollary 2 below, $\mu(S_1) = 0$. Thus, UAU^* has the form $\Lambda = \sum_j \lambda_j P_j$. The fact that $\dim P_j \leq |G|$ follows from Theorem 1. The rest follows from Theorem 5. □

2.4 New Results for General Bounded Operators

This section is devoted to the study of the iterative action of general bounded operators in $B(\mathscr{H})$.

Theorem 7. *If for an operator* $A \in B(\mathscr{H})$ *there exists a set of vectors G in* $\mathscr{H}$ *such that* $\{A^n g\}_{g\in G,\, n\geq 0}$ *is a frame in* $\mathscr{H}$, *then for every* $f \in \mathscr{H}$, $(A^*)^n f \to 0$ *as* $n \to \infty$.

Proof. Suppose, for some $\{g\}_{g\in G}$, $\{A^n g\}_{g\in G,\, n\geq 0}$ is a frame with frame bounds B_1 and B_2. Let $f \in \mathscr{H}$. Then for any $m \in \mathbb{Z}$ we have

$$\sum_{g\in G}\sum_{n=0}^{\infty} |\langle (A^*)^m f, A^n g\rangle|^2 = \sum_{g\in G}\sum_{n=0}^{\infty} |\langle f, A^{n+m} g\rangle|^2 = \sum_{g\in G}\sum_{n=m}^{\infty} |\langle f, A^n g\rangle|^2. \tag{4}$$

Since $\sum_{g\in G}\sum_{n=0}^{\infty} |\langle f, A^n g\rangle|^2 \leq B_2\|f\|^2$, we conclude that $\sum_{n=m}^{\infty}\sum_{g\in G} |\langle f, A^n g\rangle|^2 \to 0$ as $m \to \infty$. Thus, from (4), we get that $\sum_{g\in G}\sum_{n=0}^{\infty} |\langle (A^*)^m f, A^n g\rangle|^2 \to 0$ as $m \to \infty$. Using the lower frame inequality, we get

$$B_1\|(A^*)^m f\| \leq \sum_{g\in G}\sum_{n=0}^{\infty} |\langle (A^*)^m f, A^n g\rangle|^2.$$

Since the right side of the inequality tends to zero as m tends to infinity we get that $(A^*)^m f \to 0$ as $m \to \infty$. □

Corollary 2. *For any unitary operator $A : \mathscr{H} \to \mathscr{H}$ and any set of vectors $G \subset \mathscr{H}$, $\{A^n g\}_{g\in G,\, n\geq 0}$ is not a frame in $\mathscr{H}$.*

If for every $f \in \mathscr{H}$, $(A^*)^n f \to 0$ as $n \to \infty$, then we can get the following existence theorem of frames for $\mathscr{H}$ from iterations.

Theorem 8. *If A is a contraction (i.e., $\|A\| \leq 1$), and for every $f \in \mathscr{H}$, $(A^*)^n f \to 0$ as $n \to \infty$, then we can choose $G \subseteq \mathscr{H}$ such that $\{A^n g\}_{g\in G,\, n\geq 0}$ is a tight frame.*

Remark 3. The system we find in this case is not very useful since the initial system G is "too large" (it is complete in $\mathscr{H}$ in some cases). Moreover, the condition $\|A\| \leq 1$ is not necessary for the existence of a frame with iterations. For example, we can take nilpotent operators with large operator norm for which there are frames with iterations.

Proof. Suppose for any $f \in \mathscr{H}$, $(A^*)^n f \to 0$ as $n \to \infty$ and $\|A\| \leq 1$. Let $D = (I - AA^*)^{\frac{1}{2}}$ and $\mathscr{V} = cl(D\mathscr{H})$. Let $\{h\}_{h\in\mathscr{I}}$ be an orthonormal basis for $\mathscr{V}$. Then

$$\begin{aligned}\sum_{n=0}^{m}\sum_{h\in\mathscr{I}} | < f, A^n D h > |^2 &= \sum_{n=0}^{m}\sum_{h\in\mathscr{I}} | < D(A^*)^n f, h > |^2 \\ &= \sum_{n=0}^{m} \|D(A^*)^n f\|^2 \\ &= \sum_{n=0}^{m} < D^2(A^*)^n f, (A^*)^n f > \\ &= \sum_{n=0}^{m} < (I - AA^*)(A^*)^n f, (A^*)^n f > \\ &= \|f\|^2 - \|(A^*)^{m+1} f\|.\end{aligned}$$

Taking limits as $m \to \infty$ and using the fact that $(A^*)^m f \to 0$ we get from the identity above that

$$\sum_{n=0}^{\infty}\sum_{h\in\mathscr{I}} | < f, A^n D h > |^2 = \|f\|^2.$$

Therefore the system of vectors $G = \{g = Dh : h \in \mathscr{I}\}$ is a tight frame for $\mathscr{H}$. □

Theorem 9. *If $\dim \mathscr{H} = \infty$, $|G| < \infty$, and $\{A^n g\}_{g\in G,\, n\geq 0}$ satisfy the lower frame bound, then $\|A\| \geq 1$.*

Proof. Suppose $\|A\| < 1$. Since $\{g\}_{g\in G}$ is finite and $\dim(\mathscr{H}) = \infty$, for any fixed N there exists a vector $f \in \mathscr{H}$ with $\|f\| = 1$ such that $< A^n g, f >= 0$, for every $g \in G$ and $0 \leq n \leq N$. Then

$$\sum_{g\in G}\sum_{n\geq 0} | < A^n g, f > |^2 = \sum_{g\in G}\sum_{n=N}^{\infty} | < A^n g, f > |^2 \leq \sum_{g\in G} \|g\| \sum_{n=N}^{\infty} \|A\|^{2n} \to 0$$

as $N \to \infty$ hence the lower frame bound cannot hold. □

Corollary 3. *Let $\{A^n g\}_{g\in G,\, n\geq 0}$ with $|G| < \infty$ satisfy the lower frame bound. Then for any coinvariant subspace $\mathscr{V} \subset \mathscr{H}$ of A with $\|P_{\mathscr{V}} A P_{\mathscr{V}}\| < 1$ we have that* $\dim(\mathscr{V}) < \infty$.

Proof. $\mathscr{V}$ is coinvariant for A that is equivalent to

$$P_{\mathscr{V}} A = P_{\mathscr{V}} A P_{\mathscr{V}}.$$

It follows that $P_{\mathscr{V}} A^n = P_{\mathscr{V}} A^n P_{\mathscr{V}}$. Hence, if $\{A^n g\}_{g\in G,\, n\geq 0}$ satisfy the lower frame inequality in $\mathscr{H}$, then $\{(P_{\mathscr{V}} A P_{\mathscr{V}})^n g\}_{g\in G,\, n\geq 0}$ also satisfy the lower frame inequality for $\mathscr{V}$ and hence from the previous theorem if $\dim(\mathscr{V}) = \infty$, then $\|P_{\mathscr{V}} A P_{\mathscr{V}}\| \geq 1$. □

3 Related Work and Concluding Remarks

There are several features that are particular to the present work: In the system of iterations $\{(A^*)^n g\}_{g\in G, 0\leq n<L(g)}$ that we considered in this chapter, we let $L(g) = \infty$ for all $g \in G$. This setting implies strong constraints on the spectrum of A when we further require that the system is a Bessel system, a frame, etc. Since in finite dimensional spaces every finite spanning set is a frame, and since for fixed g, if $K > \dim(\mathscr{H})$, then the set $\{(A^*)^n g\}_{g\in G, 0\leq n\leq K}$ is always linearly dependent, it does not make sense to let $L(g) > \dim(\mathscr{H}) + 1$. In fact, the finite dimensional problem has first been studied [3] in which $L(g)$ is a constant for all $g \in G$ and is as small as possible in some sense.

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Optimization Methods for Frame Conditioning and Application to Graph Laplacian Scaling

Radu Balan, Matthew Begué, Chae Clark, and Kasso Okoudjou

1 Introduction

The notion of scalable frame has been investigated in recent years [4, 10, 15, 17], where the focus was more on characterizing frames whose vectors can be rescaled resulting in a tight frame. For completeness, we recall that a set of vectors $F = \{f_i\}_{i=1}^M$ in some (finite dimensional) Hilbert space $\mathscr{H}$ is a frame for $\mathscr{H}$ if there exist two constants $0 < A \leq B < \infty$ such that

$$A\|x\|^2 \leq \sum_{i=i}^{M} |\langle x, f_i \rangle|^2 \leq B\|x\|^2$$

for all $x \in \mathscr{H}$. When $A = B$ the frame is said to be tight and if in addition, $A = B = 1$ it is termed a Parseval frame. When $F = \{f_i\}_{i=1}^M$ is a frame, we shall abuse notations and denote by F again, the $n \times M$ matrix whose i^{th} column is f_i, and where n is the dimension of $\mathscr{H}$. Using this notation, the frame operator is the $n \times n$ matrix $S = FF^*$ where F^* is the adjoint of F. It is a folklore to note that F is a frame if and only if S is a positive definite operator and the optimal lower frame bound, A, coincides with the lowest eigenvalue of S while the optimal upper frame bound, B, equals the largest eigenvalue of S. We refer to [6, 7, 20] for more details on frame theory.

It is apparent that tight frames are optimal frames in the sense that the condition number of their frame operator is 1. We recall that the *condition number* of a matrix A, denoted $\kappa(A)$, is defined as the ratio of the largest singular value and the smallest

R. Balan (✉) • M. Begué • C. Clark • K. Okoudjou
Department of Mathematics, University of Maryland, College Park, MD 20742, USA
e-mail: rvbalan@cscamm.umd.edu; begue@math.umd.edu; cclark18@math.umd.edu; kasso@math.umd.edu

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singular value of A, i.e., $\kappa(A) = \sigma_{\max}(A)/\sigma_{\min}(A)$. By analogy, for a frame in a Hilbert space $\{f_i\}_{i=1}^M \subseteq \mathscr{H}$ with optimal frame bounds A and B, we define the condition number of the frame to be the condition number of its associated frame operator $\kappa(\{f_i\}) := \kappa(S) = B/A$. In particular, if a frame is Parseval, then its condition number equals 1. In fact, a frame is tight if and only if its condition number is 1. Scalable frames were precisely introduced to turn a non-optimal (non-tight) frame into an optimal one, by just rescaling the length of each frame vector. More precisely,

Definition 1 ([16, Definition 2.1]). A frame $\{f_i\}_{i=1}^M$ in some Hilbert space $\mathscr{H}$ is called a *scalable frame* if there exist nonnegative numbers $s_1, \ldots, s_M$ such that $\{s_i f_i\}_{i=1}^M$ is a Parseval frame for $\mathscr{H}$.

It follows from the definition that a frame $\{f_i\}_{i=1}^M$ is scalable if and only if there exist scalars $s_i \geq 0$ so that

$$\kappa\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right) = 1.$$

To date various equivalent characterizations of scalable frames have been proved and attempts to measure how close to scalable a non-scalable frame have been offered [4, 15, 17, 21]. In particular, if a frame is not scalable, then one can naturally measure how "not scalable" the frame is by measuring

$$\min_{s_i \geq 0} \left\| I_n - \sum_{i=1}^M s_i^2 f_i f_i^* \right\|_F, \tag{1}$$

as proposed in [8], where $\|\cdot\|_F$ denotes the Frobenius norm of a matrix. Other measures of scalability were also proposed by the same authors. However, it is not clear that, when a frame is not scalable, an optimal solution to (1) yields a frame $\{s_i f_i\}$ that is as best conditioned as possible. Recently, the relationship between the solution to this problem and the condition number of a frame has been investigated in [5]. In particular, Casazza and Chen show that the problem of minimizing the condition number of a scaled frame

$$\min_{s_i \geq 0} \kappa\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right), \tag{2}$$

is equivalent to solving the minimization problem

$$\min_{s_i \geq 0} \left\| I_n - \sum_{i=1}^M s_i^2 f_i f_i^* \right\|_2, \tag{3}$$

where $\|\cdot\|_2$ is the operator norm of a matrix. Specifically they show that any optimizer of (2) is also an optimizer of (3); vice-versa, any optimizer of (3) minimizes the condition number in (2). Furthermore, they show that the optimal solution to (1) does not even have to be a frame, and so would yield an undefined condition number for the corresponding system.

In this chapter, we consider numerical solutions to the scalability problem. Recall that a frame $F = \{f_i\}_{i=i}^M \subset \mathscr{H}$ is scalable if and only if they exist scalars $\{s_i\}_{i=1}^M \subset [0, \infty)$ such that

$$\sum_{i=1}^{M} s_i^2 f_i f_i = I.$$

Consequently, the condition number of the scaled frame $\tilde{F} = \{s_i f_i\}_{i=i}^M$ is 1. We are thus interested in investigating the solutions to the following three optimization problems:

$$\min_{s_i \geq 0\,,\, s \neq \mathbf{0}} \frac{\lambda_{\max}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right)}{\lambda_{\min}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right)}. \tag{4}$$

$$\min_{\substack{s_i \geq 0\,,\; s \neq \mathbf{0} \\ \sum_{i=1}^M s_i^2 \,\|f_i\|_2^2 = N}} \lambda_{\max}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right) - \lambda_{\min}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right). \tag{5}$$

$$\min_{s_i \geq 0\,,\, s \neq \mathbf{0}} \left\| I_N - \sum_{i=1}^M s_i^2 f_i f_i^* \right\|_F. \tag{6}$$

Our motivation stems from the fact it appears from the existing literature on scalable frames that the set of all such frames is relatively small, e.g., see [17]. As a result, one is interested in scaling a frame in an optimal manner. For example, by minimizing the condition number of the scaled frame (4), or the gap of the spectrum of the scaled frame (5). Furthermore, one can try to find the relationship between the optimal solutions to these two problems with the measures of scalability introduced in [8], of which (1) is a typical example.

In addition, we investigate these optimization problems from a practical point of view: the existence of fast algorithms to produce optimal solutions. As such, we are naturally lead to consider these problems in the context of convex optimization. We recall that in such a setting one wants to solve for $s^* = \arg\min_s f(s)$ for a real convex function $f : X \to \mathbb{R} \cup \{\infty\}$ defined on a convex set X. Using the convexity of f and X it follows that:

1. If s^* is a local minimum of f, then it is a global minimum.
2. The set of all (global) minima is convex.
3. If f is a strictly convex function and a minimum exists, then the minimum is unique.

In addition, the convexity of f and X allows the use of convex analysis to produce fast, efficient algorithmic solvers, we refer to [2] and the references therein for more details.

We point out that (4) is equivalent to (2) simply by the definition of condition number of a frame. However, the condition number function κ is not convex. As such, it is nontrivial to find the optimal solution of (4). However, κ is a quasiconvex function (see [1, Theorem 13.6] for a proof), meaning that its lower level sets form convex sets; that is, the set $\{X : \kappa(X) \leq a\}$ forms a convex set for any real $a \geq 0$. See [12] and the references therein for a survey on some algorithms that can numerically solve certain quasiconvex problems. We refer to [19] for a survey of results on optimizing the condition number. But we note that, while minimizing the condition number κ is not a convex problem, an equivalent convex problem was considered in [18]. For comparison and completeness we state one of the main results of [18]. First, observe that if X is a symmetric positive semidefinite matrix, then its *condition number* is defined as

$$\kappa(X) = \begin{cases} \lambda_{\max}(X)/\lambda_{\min}(X) & \text{if } \lambda_{\min}(X) > 0, \\ \infty & \text{if } \lambda_{\min}(X) = 0 \text{ and } \lambda_{\max}(X) > 0, \\ 0 & \text{if } X \equiv 0. \end{cases}$$

In this setting, it was proved in [18] that the problem of minimizing the condition number is equivalent to solving another problem with convex programming.

Theorem 1 ([18], Theorem 3.1). *Let $\Omega \subseteq \mathscr{S}^N$ be some nonempty closed convex subset of $\mathscr{S}^N$, the space of $N \times N$ symmetric matrices and let $\mathscr{S}_+^N$ be the space of symmetric positive semidefinite $N \times N$ matrices. Then the problem of solving*

$$\kappa^* = \inf\{\kappa(X) : X \in \mathscr{S}_+^N \cap \Omega\}$$

is equivalent to the problem of solving

$$\lambda^* = \inf\{\lambda_{\max}(X) : X \in t\Omega,\ t \geq 0,\ X \succeq I\}, \tag{7}$$

that is, $\lambda^* = \kappa^*$.

The problem described by (7) can be restated as solving for optimal scalars $\{s_i\}$ satisfying

$$\min_{s_i \geq 0\,,\, s \neq \mathbf{0}} \left\{ \lambda_{\max}\left(\sum_{i=1}^{M} s_i^2 f_i f_i^*\right) \,\middle|\, \lambda_{\min}\left(\sum_{i=1}^{M} s_i^2 f_i f_i^*\right) \geq 1 \right\}. \tag{8}$$

Therefore, when we obtain numerical solutions to the condition number problem (4), we actually solve (8) and the theory of [19] guarantees that the optimal solutions to both problems are indeed equal.

Theorem 1 has an intuitive interpretation. Suppose $\kappa(X) = \kappa^*$. Then rescaling X by a positive scalar, t, will also scale its eigenvalues by the same factor $1/t$, thus leaving its condition number, $\kappa(X/t)$, unchanged. Therefore, without loss of generality, we can assume that X is rescaled so that $\lambda_{\min}(X/t) \geq 1$ which is imposed in the last condition of (7). Once we know that $\lambda_{\min}(X/t)$ is at least 1 then minimizing the condition number of X/t is equivalent to minimizing $\lambda_{\max}(X/t)$ so long as $X/t \in \Omega$ which is guaranteed by the first condition in (7).

The goal of this chapter is to investigate the relationship among the solutions to each of the optimization problems (4), (5), and (6). In addition, we shall investigate the behavior of the optimal solution to each of these problems vis-á-vis the projection of a non-scalable frame onto the set of scalable frames. We shall also describe a number of algorithms to solve some of these problems and compare some of the performances of these algorithms. Finally, we shall apply some of the results of frame scalability to the problem of reweighing a graph in such a way that the condition number of the resulting Laplacian is as small as possible. The chapter is organized as follow. In Section 2 we investigate the three problems stated above and compare their solutions, and in Section 3 we consider the application to finite graph reweighing.

2 Non-scalable Frames and Optimally Conditioned Scaled Frames

We begin by showing the relationship between the three formulations of this scalability problem. We shall first show the equivalence of these problems when a frame is exactly scalable, and present toy examples of the different solutions obtained when a frame is only approximately scalable.

Lemma 1. *Let $F = \{f_i\}_{i=1}^M$ be a frame in $\mathbb{R}^N$. Then the following statements are equivalent:*

(a) $F = \{f_i\}_{i=1}^M$ is a scalable frame.
(b) Problem (4) has a global minimum solution, $s^* = \{s_i^*\}$, with objective function value 1.
(c) Problem (5) has a global minimum solution, $s^* = \{s_i^*\}$, with objective function value 0.
(d) Problem (6) has a global minimum solution, $s^* = \{s_i^*\}$, with objective function value 0.

Proof. Assume F is scalable with weights, $\{s_i\}_{i=1}^M$. Then $\widetilde{S} = \sum_{i=1}^M s_i^2 f_i f_i^* = I_N$, and the largest and smallest eigenvalue of the scaled frame operator is 1,

$$\frac{\lambda_{\max}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right)}{\lambda_{\min}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right)} = \frac{\lambda_{\max}\left(\widetilde{S}\right)}{\lambda_{\min}\left(\widetilde{S}\right)} = 1.$$

Assume problem (4) has a global minimum solution, $\{s_i\}_{i=1}^M$. As, $\lambda_{\max} \geq \lambda_{\min}$, the feasible solution must result in $\lambda_{\max} = \lambda_{\min} = A$. Applying this feasible solution as a scaling of F, we have,

$$\widetilde{S} = \sum_{i=1}^M s_i^2 f_i f_i^* = AI_N.$$

By normalizing the feasible solution by the square-root of A, we have the Parseval scaling,

$$\{\tilde{s}_i\}_{i=1}^M = \left\{\frac{1}{\sqrt{A}} s_i\right\}_{i=1}^M.$$

We have just proved that (a) and (b) are equivalent.

Assume F is scalable with weights, $\{s_i\}_{i=1}^M$. Then $\widetilde{S} = \sum_{i=1}^M s_i^2 f_i f_i^* = I_N$, and the difference between the largest and smallest eigenvalue of the scaled frame operator is 0,

$$\lambda_{\max}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right) - \lambda_{\min}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right) = \lambda_{\max}\left(\widetilde{S}\right) - \lambda_{\min}\left(\widetilde{S}\right) = 0.$$

Additionally $N = tr(I_N) = \sum_{i=1}^M s_i^2 \|f_i\|_2^2$ which shows that $\{s_i\}_{i=1}^M$ is a feasible solution for (5).

Assume problem (5) has a global minimum solution, $\{s_i\}_{i=1}^M$. As, $\lambda_{\max} \geq \lambda_{\min}$, the feasible solution must result in $\lambda_{\max} = \lambda_{\min} = A$. Applying this feasible solution as a scaling of F, we have,

$$\widetilde{S} = \sum_{i=1}^M s_i^2 f_i f_i^* = AI_N.$$

But the feasibility condition $\sum_{i=1}^M s_i^2 \|f_i\|_2^2 = N$ implies $N = tr(AI_N)$, hence $A = 1$. We have just proved that (a) and (c) are equivalent.

Assume F is scalable with weights, $\{s_i\}_{i=1}^M$. Then $\widetilde{S} = \sum_{i=1}^M s_i^2 f_i f_i^* = I_N$, and the objective function for (6) attains the global minimum ,

$$\left\| I_N - \sum_{i=1}^M s_i^2 f_i f_i^* \right\|_F = \|I_N - I_N\|_F = 0.$$

Assume problem (6) has a global minimum solution, $\{s_i\}_{i=1}^M$, which occurs when $\left\| I_N - \sum_{i=1}^M s_i^2 f_i f_i^* \right\|_F = 0$. This implies that $\widetilde{S} = \sum_{i=1}^M s_i^2 f_i f_i^* = I_N$, and we have a Parseval scaling. We have just proved that (a) and (d) are equivalent.

Remark 1. Lemma 1 asserts that the problem of finding optimal scalings, $\{s_i\}_{i=1}^M$, for a given scalable frame $F = \{f_i\}_{i=1}^M$ is equivalent to finding the absolute minimums of the following optimization problems:

- $\min_{s_i \geq 0\,,\, s \neq \mathbf{0}} \dfrac{\lambda_{\max}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right)}{\lambda_{\min}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right)}$
- $\min_{\substack{s_i \geq 0\,,\, s \neq \mathbf{0} \\ \sum_{i=1}^M s_i^2 \|f_1\|_2^2 = N}} \lambda_{\max}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right) - \lambda_{\min}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right)$
- $\min_{s_i \geq 0\,,\, s \neq \mathbf{0}} \left\| I_N - \sum_{i=1}^M s_i^2 f_i f_i^* \right\|_F$

Lemma 1 is restrictive in that it requires the frame $F = \{f_i\}_{i=1}^M$ be scalable to state equivalence among problems, but there can be a wide variance in the solutions obtained when the frame is not scalable. Even nearly tight frames vary in initial feasible solutions. We briefly consider ε-tight frames and analyze the distance from the minimum possible objective function value.

Let $F_\varepsilon = \{g_i\}_{i=1}^M$ with $\|g_i\|_2 = 1$ for all i be an ε-tight frame such that,

$$(1-\varepsilon)I_N \preceq \sum_{i=1}^M g_i g_i^* \preceq (1+\varepsilon)I_N.$$

First considering the case in which the frame cannot be conditioned any further, so the optimal scaling weights are $s_i = 1$. Analyzing the solution produced by the three optimization methods, we see the difference in solutions produced.

$$\frac{\lambda_{\max}\left(\sum_{i=1}^M s_i^2 g_i g_i^*\right)}{\lambda_{\min}\left(\sum_{i=1}^M s_i^2 g_i g_i^*\right)} = \frac{\lambda_{\max}\left(\sum_{i=1}^M g_i g_i^*\right)}{\lambda_{\min}\left(\sum_{i=1}^M g_i g_i^*\right)} = \frac{1+\varepsilon}{1-\varepsilon} = 1 + \frac{2\varepsilon}{1-\varepsilon}.$$

$$\lambda_{\max}\left(\sum_{i=1}^M s_i^2 f_i f_i^*\right) - \lambda_{\min}\left(\sum_{i=1}^M s_i^2 g_i g_i^*\right) = (1+\varepsilon) - (1-\varepsilon) = 2\varepsilon$$

$$\lambda_{\max}\left(\sum_{i=1}^M s_i^2 g_i g_i^*\right) = \lambda_{\max}\left(\sum_{i=1}^M g_i g_i^*\right) = 1 + \varepsilon.$$

We lack the information necessary to give exact results for formulation (6), so we instead give an upper bound when $s_i = 1$.

$$\begin{aligned}\left\| I_N - \sum_{i=1}^{M} s_i^2 g_i g_i^* \right\|_F &= \left\| I_N - \sum_{i=1}^{M} g_i g_i^* \right\|_F \\ &\leq \sqrt{N} \left\| I_N - \sum_{i=1}^{M} g_i g_i^* \right\|_2 \\ &\leq \varepsilon\sqrt{N}.\end{aligned}$$

It makes sense that we could enforce this constraint, as we could renormalize the frame elements by the reciprocal of the smallest eigenvalue of the frame operator. It is not true, though, that the scalings produced must be the same. Moreover, when not using the constraint on the smallest eigenvalue, the scalings can vary wildly.

Remark 2. For general frames, the optimization problems (4)–(6) do not produce tight frames. However, they can be solved using special classes of convex optimization algorithms: problems (4) and (5) are solved by *Semi-Definite Programs* (SDP), whereas problem (6) is solved by a Quadratic Program (QP) – see [2] for details on SDPs and QPs. In the following we state these SDPs explicitly.

SDP 1 – Operator Norm Optimization:

$$(t^1, s^{(1)}) = argmin \quad \begin{array}{c} t \\ t, s_1, \ldots, s_M \geq 0 \\ \sum_{i=1}^{M} s_i^2 f_i f_i^* - tI_N - I_N \leq 0 \\ \sum_{i=1}^{M} s_i^2 f_i f_i^* + tI_N - I_N \geq 0 \end{array} \tag{9}$$

This SDP implements the optimization problem (3). In turn, as showed by Cassaza and Chen in [5], the solution to this problem is also an optimizer of the condition number optimization problem (4). Conversely, assume $s^{(*)}$ is a solution of (4). Let $A = \lambda_{min}(\sum_{i=1}^{M} s_i^2 f_i f_i^*)$ and $B = \lambda_{max}(\sum_{i=1}^{M} s_i^2 f_i f_i^*)$. Let $r = \frac{2}{A+B}$. Then $s^{(*)} = (rs_i^2)_{i=1}^{M}$ is a solution of (9) and the optimum value of the optimization criterion is $t^1 = rB - 1 = 1 - rA$.

SDP 2 – Minimum Upper Frame Bound Optimization:

$$(t^2, s^{(2)}) = argmin \quad \begin{array}{c} t \\ t, s_1, \ldots, s_M \geq 0 \\ \sum_{i=1}^{M} s_i^2 f_i f_i^* - I_N \geq 0 \\ \sum_{i=1}^{M} s_i^2 f_i f_i^* - tI_N \leq 0 \end{array} \tag{10}$$

This SDP implements the optimization problem (8) which is as previously discussed, also produces the solution $s^{(2)}$ to (4). Conversely, assume $s^{(*)}$ is a solution of (4). Let $A = \lambda_{min}(\sum_{i=1}^{M} s_i^2 f_i f_i^*)$ and $B = \lambda_{max}(\sum_{i=1}^{M} s_i^2 f_i f_i^*)$. Let $r = \frac{1}{A}$. Then $s^{(*)} = (rs_i^2)_{i=1}^{M}$ is a solution of (10), and the optimum value of the optimization criterion is $t^2 = \frac{B}{A}$.

SDP 3 – Spectral Gap Optimization:

$$(t^3, v^3, s^{(3)}) = argmin_{\substack{t, v, s_1, \ldots, s_M \geq 0 \\ \sum_{i=1}^M s_i^2 f_i f_i^* - tI_N \leq 0 \\ \sum_{i=1}^M s_i^2 f_i f_i^* - vI_N \geq 0 \\ \sum_{i=1}^M s_i \|f_i\|_2^2 = N}} t - v \tag{11}$$

This SDP implements the optimization problem (5). As remarked earlier (5) is not equivalent to any of (3),(4), or (8). A spectral interpretation of these optimization problems is as follows. The SDP 1 (and implicitly (4) and (8)) scales the frame so that the largest and smallest eigenvalues of the scaled frame operator are equidistant and closest to value 1. The SDP 3 scales the frame so that the largest and smallest eigenvalues of the scaled frame operator are closest to one another while the average eigenvalue is set to 1. Equivalently, the solution to SDP 3 also minimizes the following criterion:

$$\frac{\lambda_{max}(\tilde{S}) - \lambda_{min}(\tilde{S})}{\frac{1}{N} tr(\tilde{S})}$$

where $\tilde{S} = \sum_{i=1}^M s_i^2 f_i f_i^*$ is the scaled frame operator.

QP 4 – Frobenius Norm Optimization:

$$s^{(4)} = argmin_{s_1, \ldots, s_M \geq 0} \sum_{i,j=1}^M s_i s_j |\langle f_i, f_j \rangle|^2 - 2 \sum_{i=1}^M s_i^2 \|f_i\|_2^2 + N \tag{12}$$

This QP implements the optimization problem (6).

Example 1. Consider the 5-element frame, $X \subseteq \mathbb{R}^3$, generated such that each coordinate is a random integer from 0 to 5.

$$X = \begin{bmatrix} 2 & 4 & 1 & 4 & 4 \\ 3 & 1 & 2 & 0 & 2 \\ 1 & 4 & 3 & 5 & 2 \end{bmatrix}$$

We then numerically compute X_κ, X_g, X_F, which are the rescaled frames that minimize problems SDP 1, SDP 3, and QP 4, respectively. That is, X_κ is the rescaled frame, $X_\kappa = \{s_i f_i\}$, such that $s^* = \{s_i\}$ is the minimizer to Problem (3), which also minimizes the frame condition number, κ. Similarly, X_g is rescaled to minimize the eigenvalue gap $\lambda_{\max} - \lambda_{\min}$ while the average eigenvalue is 1, and X_F is rescaled to minimize Frobenius distance to the identity matrix.

In our numerical implementation minimizing condition number, we used the CVX toolbox in MATLAB [11] which is a solver for convex optimization problems.

Table 1 Comparisons of extreme eigenvalues, condition number, relative spectral gap, Frobenius distance to identity, and the operator norm distance to identity for the non-scalable frame X and its rescaled versions that minimize Problems (4)–(6).

	$\lambda_{\min}$	$\lambda_{\max}$	κ	$(\lambda_{\max} - \lambda_{\min})/\frac{1}{N}\sum_{i=1}^{N}\lambda_i$	$\|I_3 - \cdot\|_F$	$\|I_3 - \cdot\|_2$
X	4.1658	110.41	26.504	2.5296	109.95	109.41
X_κ	0.1716	1.8284	**10.655**	2.2888	1.4348	**0.8284**
X_g	0.0856	2.3558	27.501	**2.2701**	1.6938	1.3558
X_F	0.01672	1.1989	71.667	2.2903	**1.2048**	0.9832

Let s_κ, s_g, and s_F denote the scaling vectors that determine the frames X_κ, X_g, and X_F, respectively. That is, $X_\kappa = S_\kappa^{1/2}X$ where S_κ is the diagonal matrix with values given by s_κ, and so on. We obtained scalings

$$\begin{aligned} s_\kappa &= [0.0187, \quad 0, \quad 0.0591, \quad 0.0122, \quad 0.0242], \\ s_g &= [0.0875, \quad 0, \quad 0.0398, \quad 0.0297, \quad 0], \\ s_F &= [0.0520, \quad 0, \quad 0.0066, \quad 0.0177, \quad 0]. \end{aligned}$$

The results comparing each of the four frames are summarized in Table 1. Observe that each of the three methods can produce widely varying spectra.

We now demonstrate special conditions in which a frame's condition number can be decreased using matrix perturbation theory.

Lemma 2 (Weyl's Inequality, [23, Corollary 4.9]). *Let A be a Hermitian matrix with real eigenvalues $\{\lambda_i(A)\}_{i=1}^d$ and let B be a Hermitian matrix of the same size as A with eigenvalues $\{\lambda_i(B)\}_{i=1}^d$. Then for any $i = 1, \ldots, d$ we have*

$$\lambda_i(A + B) \in [\lambda_i(A) + \lambda_1(B), \lambda_i(A) + \lambda_d(B)].$$

An immediate corollary of Weyl's inequality tells us that perturbing a matrix by a positive semidefinite matrix will cause the eigenvalues to not decrease.

Corollary 1. *Let A be a Hermitian matrix with real eigenvalues $\{\lambda_i(A)\}_{i=1}^d$ and let $B \succeq 0$ be Hermitian and of the same size of A. Then for any $i = 1, \ldots, d$, we have $\lambda_i(A) \le \lambda_i(A + B)$. The inequality is strict if $B \succ 0$ is positive definite.*

Lemma 3. *Let f be an eigenvector of A with associated eigenvalue λ. Let B be a matrix of the same size as A with the property that $Bf = 0$. Then f is an eigenvector of $A + B$ with eigenvalue λ.*

Lemma 4 ([24, Section 1.3]). *Let A and B be two $N \times N$ Hermitian matrices of same size. Then for any $i = 1, \ldots, N$, the mapping $t \mapsto \lambda_i(A + tB)$ is Lipschitz continuous with Lipschitz constant $\|B\|_2$.*

Corollary 2. *Let A be an $N \times N$ Hermitian matrix with simple spectrum and minimum eigengap $\delta > 0$, i.e.,*

$$\delta = \min_{i \neq j} |\lambda_i - \lambda_j|.$$

Let B be a nonnegative Hermitian matrix of same size as A. Then the mappings $t \mapsto \lambda_i(A + tB)$ are interlacing:

$$\lambda_1(A) \leq \lambda_1(A+tB) \leq \lambda_2(A) \leq \lambda_2(A+tB) \leq \cdots \leq \lambda_{N-1}(A+tB) \leq \lambda_N(A) \leq \lambda_N(A+tB)$$

for $t \in (0, \frac{\delta}{\|B\|_2})$.

The following theorem gives conditions in which we can guarantee that the condition number of frame can be reduced.

Theorem 2. *Let $F = \{f_i\}_{i=1}^m \subseteq \mathbb{C}^d$ be a frame that is not tight and whose frame operator has simple spectrum with minimal eigengap $\delta > 0$. Suppose that there exists some index k such that f_k is orthogonal to the eigenspace corresponding to $\lambda_{\max}(FF^*)$ and not orthogonal to the eigenspace corresponding to $\lambda_{\min}(FF^*)$. Then there exists a rescaled frame $\tilde{F} = \{s_i f_i\}_{i=1}^m$ satisfying $\kappa(\tilde{F}) < \kappa(F)$. In particular, one scaling that decreases the condition number is*

$$s_i = \begin{cases} \frac{m}{m-1+\sqrt{1+\gamma}}, & \text{for } i \neq k \\ \frac{m\sqrt{1+\gamma}}{m-1+\sqrt{1+\gamma}}, & \text{for } i = k \end{cases}$$

for $\gamma \in (0, \delta \,\|f_k\|^{-2})$.

Proof. Let f_k denote the frame element as described in the assumptions in the statement of the theorem. For $\gamma \in (0, \delta)$, consider the frame operator $HH^* = FF^* + \gamma f_k f_k^*$ which corresponds to the rescaled frame of F where each scale $s_i = 1$ except for $s_k = \sqrt{1+\gamma}$. The matrix $f_k f_k^*$ is Hermitian and positive semidefinite so by Corollary 1, we have $\lambda_i(FF^*) \leq \lambda_i(HH^*)$ for every $i = 1, \ldots, N$. Then by Corollary 2, the eigenvalues of the frame operator HH^* satisfy the following interlacing property:

$$\lambda_1(FF^*) \leq \lambda_1(HH^*) \leq \lambda_2(FF^*) \leq \lambda_2(HH^*) \leq \cdots \leq \lambda_N(FF^*) = \lambda_N(HH^*),$$

where the last equality follows from Lemma 3 and the fact that f_k is orthogonal to the eigenspace corresponding to $\lambda_N(FF^*)$.

We can now compute

$$\kappa(FF^*) = \frac{\lambda_N(FF^*)}{\lambda_1(FF^*)} \geq \frac{\lambda_N(HH^*)}{\lambda_1(HH^*)} = \kappa(HH^*).$$

Finally, we renormalize the scales $\{s_i\}$ by the constant factor $m(m-1+\sqrt{1+\gamma})^{-1}$ to preserve the property that $\sum_{i=1}^m s_i = m$. This renormalization scales all eigenvalues by the same factor which leaves the condition number unchanged. The frame

$$\tilde{F} = \frac{m}{m-1+\sqrt{1+\gamma}} H$$

is the frame described in the statement of the theorem, which concludes the proof.

Remark 3. Having discussed the equivalence between the formulations above, we have seen that they do not necessarily produce similar solutions. This brings the question of which formulation we should use in general, to the forefront. One could answer this question by seeking a metric that best describes the distance of a frame to the set of tight frames. This is similar to the Paulsen problem [3], in that, after we have solved one of the formulations above, we produce a scaling and subsequent new frame and wish to determine the distance of this new frame to the canonical Parseval frame associated with our original frame. In [8], the question of distance to Parseval frames was generalized to include frames that could be made tight with a diagonal scaling, resulting in the distance between a frame and the set of scalable frames:

$$d_F = \min_{\Psi \in \mathscr{SC}(M,N)} \|F - \Psi\|_F. \tag{13}$$

However, due to the fact that the topology of the set of scalable frames $\mathscr{SC}(M,N)$ is not yet well-understood, computing d_F is almost impossible for a non-scalable frame. A source of future work involves finding bound on d_F using the optimal solutions to the three problems we stated above to analyze and produce bounds on the minimum distance.

3 Minimizing Condition Number of Graphs

In this section we outline how to apply and generalize the optimization problems from Section 2 in the setting of (finite) graph Laplacians. This task is not as simply as directly applying the condition number minimization problem (4), and the others, with graph Laplacian operators.

Recall that any finite graph has a corresponding positive semidefinite Laplacian matrix with eigenvalues $\{\lambda_k\}_{k=0}^{N-1}$ and eigenvectors $\{f_k\}_{k=0}^{N-1}$. Further any graph has smallest eigenvalue $\lambda = 0$ with multiplicity equal to number of connected components in the graph with eigenvalues equal to constant functions supported on those connected components. Because any Laplacian's smallest eigenvalue equals 0, its condition number $\kappa(L)$ is undefined. For simplicity, let us assume that all graphs in this section are connected and hence $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}$. Suppose we restricted the Laplacian operator to the $(N-1)$-dimensional space spanned by the eigenvectors $f_1, \ldots, f_{N-1}$. Then this new operator, call it L_0, has eigenvalues $\lambda_1, \ldots, \lambda_{N-1}$ which are all strictly positive. Now, $\kappa(L_0)$, the condition number of L_0 is a well-defined number.

Recall that the complete graph on N vertices, K_N, is the most connected a graph on N vertices can be since one can traverse from any two vertices on precisely one edge. It is the only graph that has all nonzero eigenvalues equal, i.e., $\lambda_0 = 0$ and $\lambda_1 = \lambda_2 = \cdots = \lambda_{N-1} = N-1$. This graph achieves the highest possible algebraic connectivity, λ_1, of a graph on N vertices. If we create L_0 by projecting the Laplacian of K_N onto the $N-1$-dimensional space spanned by the eigenvectors corresponding with nonzero eigenvalue, then L_0 equals NI_{N-1}, that is the $(N-1)\times(N-1)$ identity matrix times N.

Lemma 5. *Let G be a connected graph with eigenvalues* $\{\lambda_k\}_{k=0}^{N-1}$ *and eigenvectors* $\{f_k\}_{k=0}^{N-1}$ *of the graph Laplacian L. Let* $\tilde{F} = [f_1 f_2 \cdots f_{N-1}]$ *be the* $N\times(N-1)$ *matrix of eigenvectors excluding the constant vector* f_0*. Then the* $(N-1)\times(N-1)$ *matrix*

$$L_0 = \tilde{F}^* L \tilde{F} \tag{14}$$

has eigenvalues $\{\lambda_k\}_{k=1}^{N-1}$ *and associated orthonormal eigenvectors* $\{\tilde{F}^* f_k\}_{k=1}^{N-1}$.

Proof. We first show that $\{\tilde{F}^* f_k\}_{k=1}^{N-1}$ are eigenvectors to L_0 with eigenvalues λ_k. For any $k = 1,\ldots,N-1$ we have

$$L_0\tilde{F}^* f_k = \tilde{F}^* L\tilde{F}\tilde{F}^* f_k.$$

But since $\tilde{F}$ is an orthonormal basis for the eigenspace that its vectors span, then $\tilde{F}\tilde{F}^*$ is simply the orthogonal projection onto the eigenspace spanned by $\{f_1,\ldots,f_{N-1}\}$. That is, for any vector f, we have $\tilde{F}\tilde{F}^* f = f - \langle f, f_0\rangle f_0$, which is simply the function f minus its mean value. For each $k = 1,\ldots,N-1$, the eigenvectors f_k have zero mean, i.e., $\langle f_k, f_0\rangle = 0$. Hence $\tilde{F}\tilde{F}^* f_k = f_k$ and therefore

$$L_0\tilde{F}^* f_k = \tilde{F}^* L f_k = \tilde{F}^*(\lambda_k f_k) = \lambda_k \tilde{F}^* f_k.$$

The orthonormality of the eigenvectors $\{\tilde{F}^* f_k\}_{k=1}^{N-1}$ follows directly from the orthonormality of $\{f_k\}_{k=0}^{N-1}$ and the computation

$$\langle \tilde{F}^* f_k, \tilde{F}^* f_j\rangle = (\tilde{F}^* f_k)^* \tilde{F}^* f_j = f_k^* \tilde{F}\tilde{F}^* f_j = f_k^* f_j = \delta(k,j).$$

Unlike the Laplacian, the operator in (14) is full rank and its rank equals the rank of the Laplacian. We denote it L_0 because it behaves as the Laplacian after the projection of the function onto the zeroth eigenspace is removed.

For a general finite graph, the Laplacian can be written as the sum of rank-one matrices $L = \sum_{i=1}^m v_i v_i^*$ where v_i is the i'th column in the incidence matrix B associated with the i'th edge in the graph and m is the total number of edges in the graph. Thus, the Laplacian can be formed by the product $L = BB^*$. The columns of the incidence matrix, B, as vectors in $\mathbb{R}^N$ do not form a frame; B has rank $N-1$. However, the restriction B to the $(N-1)$-dimensional space spanned by $f_1,\ldots,f_{N-1}$, call it B_0, is a frame in that space. Then the methods of Section 2 do apply to the frame B_0 with corresponding frame operator $L_0 = B_0 B_0^*$. Therefore the operator L_0 can also be written as one matrix multiplication $L_0 = (\tilde{F}^* B)(\tilde{F}^* B)^*$. For other related results on graphs and frames we refer to [22]. We seek scalars $s_i \geq 1$ so that the rescaled frame $\{s_i \tilde{F}^* v_i\}_{i=1}^m$ is tight or as close to tight as possible. In terms of matrices, we seek a nonnegative diagonal matrix $X = \mathrm{diag}(s_i)$ so that $\tilde{L}_0 := \tilde{F}^* B X^2 B^* \tilde{F}$ has minimal condition number. The resulting graph Laplacian, denoted $\tilde{L}_\kappa = BX^2B^*$, is the operator with minimal condition number, $\tilde{L}_0$, without the projection onto $(N-1)$ eigenspaces, thus acting on the entire N-dimensional space. One can interpret this problem as rescaling weights of graph edges to not only make $\tilde{L}_0$ as close as possible to the $(N-1)$-identity matrix but also make the $N \times N$ Laplacian, $\tilde{L}$, as close as possible to the Laplacian of the complete graph K_N.

We present the pseudocode for the algorithm, GraphCondition, that produces $\tilde{L}_\kappa$, the Laplacian of the graph that minimizes the condition number of L.

> L_κ=GraphCondition(L, F, B)
> where L is the Laplacian matrix of the graph G,
> F is the $N \times N$ eigenvector matrix of L,
> B is the incidence matrix of L.
>
> 1. Set $\tilde{F} = F(:, 2 : N)$.
> 2. Use cvx to solve for X that minimizes $\lambda_{\max}(\tilde{F}^* BX^2B^*\tilde{F})$.
>   subject to: $X \succeq 0$ is diagonal, trace$(X) \geq t \geq 0$, and $\tilde{F}^* BX^2B^*\tilde{F} \succeq I$.
> 3. Create $L_\kappa = BX^2B^*$.

Example 2. We consider the barbell graph G which consists of two complete graphs on 5 vertices that are connected by exactly one edge. The Laplacian for G has eigenvalues $\lambda_1 \approx 0.2984$ and $\lambda_9 \approx 6.7016$, thus giving a condition number of $\kappa(G) \approx 22.45$. We rescaled the edges via the GraphCondition algorithm and obtained a rescaled weighted graph $\tilde{G}_\kappa$ which has eigenvalues $\lambda_1 \approx 0.3900$ and $\lambda_{10} \approx 6.991$, thus giving a condition number $\kappa(\tilde{G}_\kappa) \approx 17.9443$.

Both graphs, G and $\tilde{G}_\kappa$, are shown in Figure 1. The edge bridging the two complete clusters is assigned the highest weight of 1.8473. All other edges emanating from those two vertices are assigned the smallest weights of 0.7389. All other edges not connected to either of the two “bridge” vertices are assigned a weight of 1.1019.

We show in the following example that the scaling coefficients $\{s_i\}_{i=1}^m$ that minimize the condition number of a graph are not necessarily unique.

Example 3. Consider the graph G complete graph on four nodes with the edge $(3, 4)$ removed. Then G was rescaled and conditioned via GraphCondition; both graphs are shown in Figure 2. The original Laplacian, L, and the rescaled conditioned Laplacian, $\tilde{L}_\kappa$, produced by the GraphCondition algorithm are given as

$$L = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{bmatrix}, \quad \tilde{L}_\kappa \approx \begin{bmatrix} 2.8406 & -0.6812 & -1.0797 & -1.0797 \\ -0.6812 & 2.8406 & -1.0797 & -1.0797 \\ -1.0797 & -1.0797 & 2.1594 & 0 \\ -1.0797 & -1.0797 & 0 & 2.1594 \end{bmatrix},$$

with spectra

$$\sigma(L) = \{0, 2, 4, 4\}, \quad \sigma(\tilde{L}_\kappa) = \{0, 2.1594, 3.5218, 4.3188\}.$$

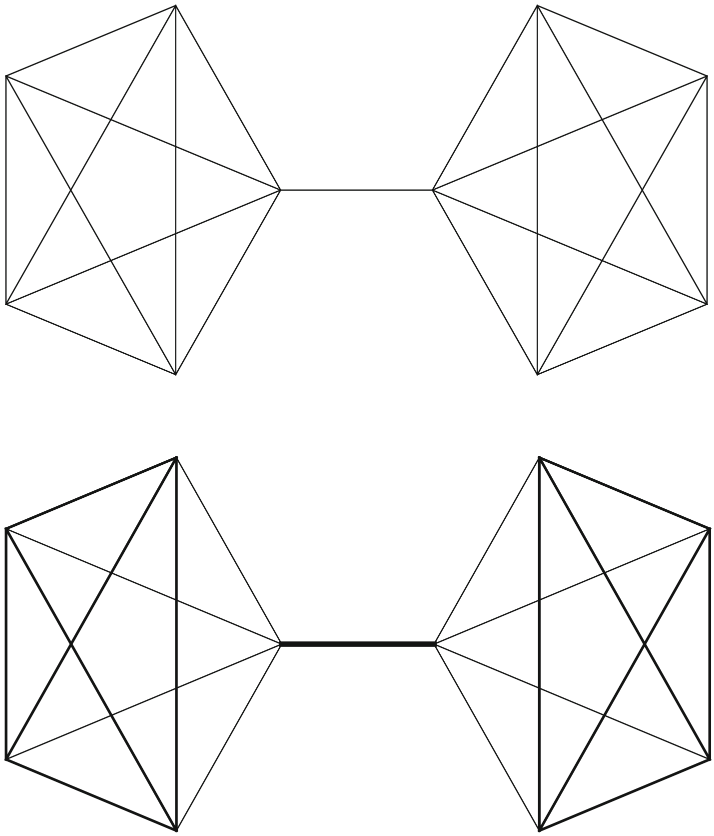


Fig. 1 Top: The barbell graph G. Bottom: The conditioned graph with rescaled weights that minimizes the condition number. The width of the edges is drawn to be proportional to the weight assigned to that edge.

Both Laplacians have a condition number $\kappa(L) = \kappa(\tilde{L}_\kappa) = 2$ which shows that the scaling of edges that minimize condition number are not necessarily unique.

We prove that the GraphCondition algorithm will not disconnect a connected graph.

Proposition 1. *Let $G = G(V, E, \omega)$ be a connected graph and let $\tilde{G}_\kappa = \tilde{G}_\kappa(V, \tilde{E}, \tilde{\omega})$ be the rescaled version of G that minimizes graph condition number. Then $\tilde{G}_\kappa$ is also a connected graph.*

Proof. Let $\kappa_0 := \kappa(G) \geq 1$ and suppose that $\tilde{G}_\kappa$ is disconnected. This implies that $\tilde{G}_\kappa$ has eigenvalue 0 with multiplicity at least 2 (one for each of its connected components). This violates the condition $\tilde{F}^* B X^2 B^* \tilde{F} \succeq I$ in the GraphCondition algorithm, which yields the unique minimizer.

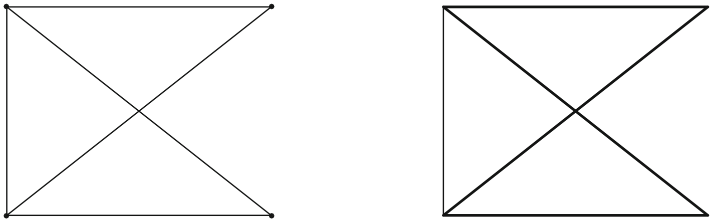
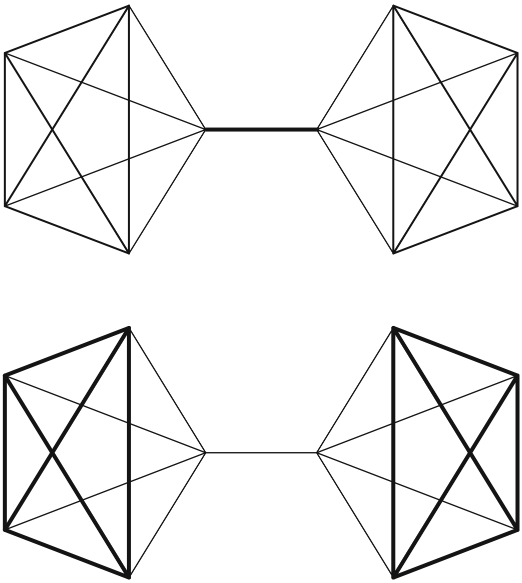


Fig. 2 The unweighted graph G (left) and its rescaled version $\widetilde{G}_\kappa$ (right) yet both graphs have a condition number equal to 2.

Fig. 3 From top to bottom: $\widetilde{G}_\kappa$ and $\widetilde{G}_g$, which minimize the condition number and spectral gap, respectively.



We next consider the analogue of minimizing the spectral gap, $\lambda_{N-1} - \lambda_1$, for graphs. Just as before with condition number, we create the positive definite matrix L_0 and its incidence matrix, B_0, and minimize its spectral gap by the methods in Section 2 to minimize problem (5). We denote the rescaled graph that minimizes the spectral gap by $\widetilde{G}_g$.

Example 4. We present numerical results of each of the graph rescaling techniques for the barbell graph shown in Figure 1. Each of the rescaled graphs is pictured in Figure 3 and numerical data is summarized in Table 2.

As discussed in the motivation of this section, reducing the condition number of a graph makes the graph more "complete," that is, more like the complete graph in terms of its spectrum. Since the algebraic connectivity λ_1 is as great as possible, it is the only graph for which $\lambda_1 = \lambda_{N-1}$, the graph is the most connected a graph can possibly be, and as such the distance between any two points is minimal.

Table 2 Comparison of condition number and spectral gap of the barbell graph, G, shown in Figure 1 and its rescaled versions, respectively.

	λ_1	λ_{N-1}	κ	$\lambda_{N-1} - \lambda_1$
G	0.2984	6.7016	22.4555	6.4031
$\tilde{G}_\kappa$	1.0000	17.9443	**17.9443**	16.9443
$\tilde{G}_g$	0.0504	1.1542	22.8794	**1.1038**

As previously discussed, the effective resistance is a natural metric on graphs and one can compute that for any two distinct vertices, i and j, on the complete graph on N vertices we have

$$\begin{aligned} R(i,j) &= \sum_{k=1}^{N-1} \frac{1}{\lambda_k} (f_k(i) - f_k(j))^2 = \frac{1}{N} \sum_{k=1}^{N-1} (f_k(i) - f_k(j))^2 \\ &= \frac{1}{N}(e_i - e_j)^* F F^* (e_i - e_j) = \frac{1}{N}(e_i - e_j)^*(e_i - e_j) \\ &= \frac{1}{N} \left\| e_i - e_j \right\|^2 = \frac{2}{N}. \end{aligned}$$

Conjecture 1. The process of conditioning a graph reduces the average resistance between any two vertices on the graph.

The intuition behind Conjecture 1 can be motivated by studying the quantity $\sum_{k=1}^{N-1} 1/\lambda_k$. Consider a sequence of positive numbers $\{a_k\}_{k=1}^N$ with average $\bar{a} = 1/N \sum_{k=1}^N a_k$. Then since the function $h(t) = 1/t$ is continuous and convex on the set of positive numbers, it is also midpoint convex on that set, i.e.,

$$\frac{N}{\bar{a}} = Nh(\bar{a}) \leq \sum_{k=1}^{N} h(a_k) = \sum_{k=1}^{N} \frac{1}{a_k}.$$

With this fact, let $\{\lambda_k\}_{k=1}^{N-1}$ denote the eigenvalues of connected graph G and $\{\tilde{\lambda}_k\}_{k=0}^{N-1}$ denote the eigenvalues of the conditioned graph $\tilde{G}_\kappa$, both satisfying $\bar{\lambda} = 1/N \sum_{k=1}^{N-1} \lambda_k = 1/N \sum_{k=1}^{N-1} \tilde{\lambda}_k$. Since $\tilde{G}_\kappa$ is better conditioned than G, then $\left\| \sum_{k=1}^{N-1} \tilde{\lambda}_k - \bar{\lambda} \right\| \leq \left\| \sum_{k=1}^{N-1} \lambda_k - \bar{\lambda} \right\|$. In other words, the eigenvalues $\{\tilde{\lambda}_k\}_{k=1}^{N-1}$ are closer to the average $\bar{\lambda}$ than the eigenvalues $\{\lambda_k\}_{k=1}^{N-1}$ are. Hence

$$\sum_{k=1}^{N-1} \frac{1}{\tilde{\lambda}_k} \leq \sum_{k=1}^{N-1} \frac{1}{\lambda_k}. \tag{15}$$

Equation (15) almost resembles the effective resistance $R(i,j) = \sum_{k=1}^{N-1} 1/\lambda_k (f_k(i) - f_k(j))^2$ except for the term $(f_k(i) - f_k(j))^2$. This term will be difficult to account for since little is known about the eigenvectors of $\tilde{G}_\kappa$. Analysis of eigenvectors of perturbed matrices is a widely open area of research and results are very limited, see [9, 14, 23, 24].

We remark that Conjecture 1 claims that conditioning a graph will reduce the average effective resistance between points; it is not true that the resistance between all points will be reduced. If the weight on edge (i,j) is reduced, then its effective resistance between points i and j is increased. Since we impose that the trace of the Laplacians be preserved, if any edge weights are increased, then by conservation at least one other edge's weight must be decreased. The vertex pairs for those edges will then have an increased effective resistance between them.

While we lack the theoretical justification, numerical simulations support Conjecture 1 and this is a source of future work.

The authors of [13] approach a similar way. They propose using convex optimization to minimize the total effective resistance of the graph,

$$R_{tot} = \sum_{i,j=1}^{N} R(i,j).$$

They show that the optimization problem is related to the problem of reweighting edges to maximize the algebraic connectivity λ_1.

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A Guide to Localized Frames and Applications to Galerkin-Like Representations of Operators

Peter Balazs and Karlheinz Gröchenig

1 Introduction

Localized frames are "good" frames. More precisely, the concept of localized frames was introduced in [45] in an attempt to understand which properties render a frame useful. Whereas an abstract frame can be viewed as a flexible coordinate system for a Hilbert space — and only for one Hilbert space! — localized frames go beyond Hilbert spaces and yield a description and characterization of a whole class of associated Banach spaces. Indeed, the success of structured frames, such as Gabor frames [32], wavelet frames [27], or frames of translates [9], is built on their capacity to describe modulation spaces (Gabor), Besov-Triebel-Lizorkin spaces (wavelet), and their use in sampling theory (translates).

Gabor frames are used for the description and extraction of time-frequency features of a signal. It would be a waste of possibilities to use them to merely determine the L^2-membership of a given function. Likewise wavelets are used to detect edges in an image or compress an image, and not just for the expansion of a function in L^2. In these applications one does not use arbitrary Gabor frames or wavelet frames, but the window and the wavelet are usually carefully designed so as to have some desired time-frequency concentration or a small support and vanishing moments. Thus in such applications the frames come with an additional property, namely some form of *localization*.

P. Balazs (✉)
Acoustics Research Institute, Austrian Academy of Sciences, Wohllebengasse 12-14, Vienna 1040, Austria
e-mail: peter.balazs@oeaw.ac.at

K. Gröchenig
Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, Vienna 1090, Austria
e-mail: karlheinz.groechenig@univie.ac.at

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The general theory of localized frames began as an attempt to formulate an abstract concept of localization that can explain the success of certain structured frames in applications [45]. Roughly speaking, a frame $\Psi = \{\psi_k : k \in K\}$ is called localized, if its Gramian matrix G with entries $G_{k,l} = \langle \psi_l, \psi_k \rangle_{k,l \in K}$ possesses enough off-diagonal decay. In the further developments of the concept powerful methods of Banach algebra theory were used, and nowadays, and also in this survey, we call a frame localized, if its Gramian belongs to a solid, inverse-closed Banach algebra of matrices [35].

Localized frames possess many properties that are not available for general frames.

(i) To every localized frame can be associated a class of Banach spaces, the so-called coorbit spaces [30, 31]. Roughly speaking, the coorbit space $\mathscr{H}_w^p$ contains all elements f such that the sequence $\langle f, \psi_k \rangle_k$ belongs to the weighted ℓ^p-space. For the standard structured frames one obtains well-known and classical families of functions spaces, namely, the modulation spaces are associated to Gabor frames [42], and the Besov spaces are associated to wavelet frames [62, 71]. In this chapter we will explain the construction of the coorbit spaces and derive many of their properties.

(ii) Localized frames possess nice dual frames. Technically speaking, the canonical dual frame possesses the same localization. In fact, this is a fundamental statement about localized frames, and the method of proof (matrix algebras, spectral invariance) has motivated intensive research in Banach algebra theory [48–51].

(iii) Elements in coorbit spaces associated to a localized frame possess good approximation properties [17]. In fact, the results on nonlinear N-term approximations and on the fast convergence of the iterative frame algorithms are based fundamentally on the localization property and do not hold for arbitrary frames.

(iv) Localized frames often possess a characterization "without inequalities" [28, 47, 53]. These results have led to strong results about Gabor frames that have been out of reach with conventional methods.

(v) Every localized frame is a finite union of Riesz sequences [44]. This is a special case of the Feichtinger conjecture and was proved early on with an easy proof, whereas the recent full proof of the Feichtinger conjecture is a monumental result in functional analysis and occupies a completely different mathematical universe [61].

(vi) General frames may be used to describe and discretize operators and operator equations, and thus have led to an important line of frame research in numerical analysis. In the so-called Galerkin approach an operator O is discretized with respect to a frame by associating a matrix M with $M_{k,l} = \langle O\psi_k, \phi_l \rangle$ with respect to given frames Ψ, Φ. The standard discretization uses bases, but recently also frames have been used [3]. The Galerkin approach works particularly well when the corresponding matrix is sparse. The most famous example is the sparsification of singular integral operators by means of wavelet frames (or bases) [15, 63]. This work has led to many adaptive methods [24, 54, 68]. In this regard the time-frequency analysis of pseudodifferential operators by means of Gabor frames is a particularly successful example of the application of localized frames: certain symbol classes containing the Hörmander class $S^0_{0,0}$ can be completely characterized by the off-diagonal decay of the associated matrix [51]. Subsequently Gabor frames were pivotal for the sparsification of Fourier integral operators and certain Schrödinger propagators in the work of the Torino group [18–21] On a more abstract level, localized frames have been heavily used in the adaptive frame methods for the solution of operator equations in [23, 25].

This chapter will offer a survey of localized frames. Of the many possible topics we will focus on the theory of the associated coorbit spaces and on the Galerkin discretization of operators with respect to localized frames. We will mainly explain the abstract theory and focus on the formalism of localized frames. These aspects have not received as much attention as other topics and justify a more thorough treatment. Several results may even claim some novelty, for instance, the inclusion relations of coorbit spaces and the explicit relation between the mapping properties of an operator and of its corresponding matrix seem to be new.

Although the topic of localized frames is eminently applied, we will choose a formalistic point of view and develop and explain the *formalism* of localized frames, their coorbit spaces, and the Galerkin discretization.

The motivation for this formal approach, and for this entire chapter, comes from different readings of the original sources [35, 45] and many ensuing discussions between the authors. One of us (K. G.) claimed that "this is folklore and known," while the other (P. B.) would point out — and rightly so — that the results he needed and wanted to understand in detail were not formulated in the publications. P. B. strongly argued that he needed a general abstract formalism in order to move on to the real applications in acoustic applications as, e.g., in [65]. The result of our discussions is this summary of localized frames with its emphasis on the formalism. We hope that this point of view will also benefit other readers and users of localized frames.

This chapter is organized as follows. In Section 2 we collect some preliminary definitions and notation and then introduce the concept of localization frames. Section 3 is devoted to the study of the associated coorbit spaces and the canonical operators associated to a frame. In Section 4 we describe the Galerkin approach and discuss the formalism of matrix representations of operators with respect to localized frames. We end with a short outlook in Section 5.

2 Preliminaries and Notation

For a standard reference to functional analysis and operator theory, refer, e.g., to [16]. We denote by $\mathfrak{B}(X)$ the Banach algebra of bounded operators on the normed space X. We will write $\|T\|_{X \to Y}$ for the operator norm of a bounded operator $T : X \to Y$, or just $\|T\|$, if the spaces are clear. We will use the same notation for the inner product of a Hilbert space $\langle .,. \rangle_{\mathscr{H}}$ and for the duality relation of two dual sets B, B', $\langle .,. \rangle_{B,B'}$. If there is no chance of confusion, we will just use the notation $\langle .,. \rangle$ for that.

Let $A \in \mathfrak{B}(\mathscr{H}_1, \mathscr{H}_2)$ with closed range. Then there exists a unique bounded operator $A^\dagger : \mathscr{H}_2 \to \mathscr{H}_1$ for which $AA^\dagger f = f, \forall f \in ran(A)$ and $\ker(A^\dagger) = (\mathrm{ran}\,(A))^\perp$. This $A^\dagger$ is called the (Moore-Penrose) *pseudoinverse* of A. See, e.g., [12].

2.1 Sequence Spaces

We use the canonical notation of sequence spaces $\ell^p = \ell^p(K)$ consisting of sequences on a countable and separable index set $K \subseteq \mathbb{R}^d$. By an abuse of notation, but for greater consistency, we define ℓ^0 as those sequences for which $\lim_{k\to\infty} c_k = 0$. (Usually this space is denoted by c_0.) We denote the set of sequence with only finitely many non-zero entries by ℓ^{00} (usually denoted by c_{00}).

A weight is a (strictly) positive sequence $w = (w_k)_{k\in K}$, $w_k > 0$. Then we define the weighted space ℓ^p_w by $c \in \ell^p_w \iff w \cdot c \in \ell^p$ with norm $\|c\|_{\ell^p_w} = \|c \cdot w\|_{\ell^p}$. So for the weighted sequence spaces and $1 < p \leq 2$ and $2 \leq q < \infty$ we get

$$\ell^{00} \subseteq \underbrace{\ell^1_w \subseteq \ell^p_w \subseteq \ell^2_w \subseteq \ell^q_w \subseteq \ell^0_w}_{(*)} \subseteq \ell^\infty_w \tag{1}$$

where the middle $(*)$ is a chain of dense Banach spaces. ℓ^{00} is dense in all ℓ^p_w for $1 \leq p < \infty$ and $p = 0$, and weak-* dense in ℓ^∞_w. Clearly $\ell^0_w = \overline{\ell^{00}}^{\|\cdot\|_{\ell^\infty_w}}$.

For $1 \leq p < \infty$ and $1 = \frac{1}{p} + \frac{1}{q}$ we know that $(\ell^p_w)' \cong \ell^q_{1/w}$ with the duality relation

$$\langle c_k, d_k \rangle_{\ell^p_w, \ell^q_{1/w}} = \left\langle w_k c_k, \frac{1}{w_k} d_k \right\rangle_{l^p, l^q} = \sum_k c_k \overline{d_k}. \tag{2}$$

For $p = \infty$ this is only true in the Köthe dual sense [56]. We also have $(l^0_w)' \cong \ell^1_{1/w}$.

2.2 Frames

A sequence $\Psi = (\psi_k)_{k\in K}$ in a separable Hilbert space $\mathscr{H}$ is a *frame* for $\mathscr{H}$, if there exist positive constants A and B (called lower and upper frame bound, respectively) that satisfy

$$A\|f\|^2 \leq \sum_{k\in K} |\langle f, \psi_k \rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathscr{H}. \tag{3}$$

A frame where the two bounds can be chosen to be equal, i.e., $A = B$, is called *tight*. In the following we will denote by $\Psi = (\psi_k)$ and $\Phi = (\phi_k)$ the corresponding sequences in $\mathscr{H}$.

By $C_\Psi : \mathscr{H} \to \ell^2$ we denote the *analysis operator* defined by $(C_\Psi f)_k = \langle f, \psi_k \rangle$. The adjoint of C_Ψ is the *synthesis operator* $D_\Psi(c_k) = \sum_k c_k \psi_k$. The *frame operator* $S_\Psi = D_\Psi C_\Psi$ can be written as $S_\Psi f = \sum_k \langle f, \psi_k \rangle \psi_k$, it is positive and invertible[1]. By using the *canonical dual frame* $(\tilde{\psi}_k)$, $\tilde{\psi}_k = S^{-1}\psi_k$ for all k, we obtain the reconstruction formula

$$f = \sum_k \langle f, \psi_k \rangle \tilde{\psi}_k = \sum_k \left\langle f, \tilde{\psi}_k \right\rangle \psi_k \quad \text{for all} \quad f \in \mathscr{H} .$$

Any sequence $\Psi_d = (\psi_k^d)$ for which C_{Ψ^d} is bounded on $\mathscr{H}$ and where such a reconstruction holds is called a dual frame.

The *Gram matrix* G_Ψ is defined by $(G_\Psi)_{k,l} = \langle \psi_l, \psi_k \rangle$. This matrix defines an operator on ℓ^2 by matrix multiplication, corresponding to $G_\Psi = C_\Psi D_\Psi$. Similarly we can define the *cross-Gram matrix* $(G_{\Psi,\Phi})_{k,l} = \langle \phi_l, \psi_k \rangle$ for two frames Φ and Ψ. Clearly

$$G_{\Psi,\Phi} c = \sum_l (G_{\Psi,\Phi})_{k,l}\, c_l = \left\langle \sum_l c_l \phi_l, \psi_k \right\rangle = C_\Psi D_\Phi c .$$

If, for the sequence Ψ, there exist constants $A, B > 0$ such that the inequalities

$$A \left\| c \right\|_2^2 \leq \left\| \sum_{k \in K} c_k \psi_k \right\|_{\mathscr{H}}^2 \leq B \left\| c \right\|_2^2$$

are fulfilled, Ψ is called a *Riesz sequence*. If Ψ is complete, it is called a *Riesz basis*.

2.2.1 Banach Frames

The concept of frames can be extended to Banach spaces [10, 14, 41]:

Let X be a Banach space and X_d be a Banach space of scalar sequences. A sequence (ψ_k) in the dual X' is called a X_d-frame for the Banach space X if there exist constants $A, B > 0$ such that

$$A \|f\|_X \leq \| \langle f, \psi_k \rangle_{k \in K} \|_{X_d} \leq B \|f\|_X \quad \text{for all} \quad f \in X. \tag{4}$$

[1]Note that those "frame-related" operators can be defined as possibly unbounded operators for any sequence [6].

An X_d-frame is called a Banach frame with respect to a sequence space X_d, if there exists a bounded reconstruction operator $R : X_d \to X$, such that $R(\psi_k(f)) = f$ for all $f \in X$. If $X_d = \ell^p$ for $1 \leq p \leq \infty$, we speak of p-frames, respectively p-Banach frames. The distinction between X_d-frames and Banach frames will disappear for localized frames. The norm equivalence (4) already implies the existence of a reconstruction operator for X, in this setting.

2.2.2 Gelfand Triples

Let X be a Banach space and $\mathscr{H}$ a Hilbert space. Then the triple $(X, \mathscr{H}, X')$ is called a Banach Gelfand triple, if $X \subseteq \mathscr{H} \subseteq X'$, where X is dense in $\mathscr{H}$, and $\mathscr{H}$ is w^*-dense in X'. The prototype of such a triple is $(\ell^1, \ell^2, \ell^\infty)$.

A frame for $\mathscr{H}$ is called a *Gelfand frame* [23] for this triple if there exists a Gelfand triple of sequence spaces (X_d, ℓ^2, X_d'), such that the synthesis operator $D_\Psi : X_d \to X$ and the analysis operator $C_{\tilde{\Psi}} : X \to X_d$ are bounded.

Now for a Gelfand frame Ψ for the Gelfand triple $(X, \mathscr{H}, X')$ with the sequence spaces $(\ell^1, \ell^2, \ell^\infty)$, we define the coorbit space $\mathscr{C}\boldsymbol{o}\,(\ell^p, \Psi) = \{f \in X' : C_\Psi f \in \ell^p\}$. Similarly, one could define the orbit spaces $\mathscr{O}\,(\ell^p, \Psi) = \{D_\Psi c \text{ for } c \in \ell^p\}$. We refer to [30] for an early example and the terminology of coorbit spaces.

3 Localization of Frames

In this section we introduce the concept of localized frames and define the corresponding family of coorbit spaces. In Subsection 3.1 we treat the maximal space $\mathscr{H}_w^\infty$ in detail. In Subsection 3.2 we show the duality relations of these spaces. In Subsection 3.3 we study the frame-related operators.

We call a Banach *-algebra $\mathscr{A}$ of infinite matrices (over the index set K) a *solid spectral matrix algebra*, if

(i) $\mathscr{A} \subseteq \mathfrak{B}(\ell^2)$.
(ii) $\mathscr{A}$ is inverse-closed in $\mathfrak{B}(\ell^2)$, i.e., $A \in \mathscr{A}$ and A is invertible on ℓ^2, then $A^{-1} \in \mathscr{A}$.
(iii) $\mathscr{A}$ is *solid*, i.e., $A \in \mathscr{A}$ and $|b_{k,l}| \leq |a_{k,l}|$, then $B = (b_{k,l}) \in \mathscr{A}$ and $\|B\|_{\mathscr{A}} \leq \|A\|_{\mathscr{A}}$.

Several examples, e.g., the Jaffard class or a Schur-type class, can be found in [35]. In these examples localization is defined by some off-diagonal decay of the Gram matrix. For the systematic construction of spectral matrix algebras, we refer to [49–51, 69], a survey on spectral invariance is contained in [48].

Definition 1. Let $\mathscr{A}$ be a solid spectral matrix algebra. Two frames Ψ and Φ are called $\mathscr{A}$-localized with respect to each other, if their cross-Gram matrix $G_{\Psi,\Phi}$ belongs to $\mathscr{A}$. If $G_{\Psi,\Phi} \in \mathscr{A}$, we write $\Psi \sim_{\mathscr{A}} \Phi$.

A single frame $\Psi = (\psi_k)$ is called (intrinsically) $\mathscr{A}$-localized, if $\Psi \sim_{\mathscr{A}} \Psi$.

Alternative definitions of localized frames can be found in [36] (continuous frames), [43, 45] (localization with respect to a Riesz basis), [1] (ℓ^p-self-localization), or [33, 39, 64] (localization in terms of the intrinsic metric on a manifold). Although all these concepts have their merits, we will focus on the intrinsic localization of Definition 1.

The following connection holds for any chosen dual frame Φ^d [35]:

$$\Psi \sim_{\mathscr{A}} \Phi, \Phi^d \sim_{\mathscr{A}} \Xi \Longrightarrow \Psi \sim_{\mathscr{A}} \Xi. \tag{5}$$

A weight w is called $\mathscr{A}$-admissible, if every $A \in \mathscr{A}$ can be extended to a bounded operator on ℓ^p_w for all $1 \le p \le \infty$, i.e., $\mathscr{A} \subseteq \bigcap_{1\le p\le\infty} \mathfrak{B}\left(\ell^p_w\right)$.

In the following, $\mathscr{A}$ is always a solid spectral Banach algebra of matrices on K. Since $\mathscr{A}$ is a Banach *-algebra, if w is $\mathscr{A}$-admissible, then $1/w$ is admissible, too. This is because for $A : \ell^p_w \to \ell^p_w$, we have $A^* : \ell^q_{1/w} \to \ell^q_{1/w}$ for $q > 1$. For $q = 1$, this argument is valid using the pre-dual.

Definition 2. Let $\mathscr{H}^{00} = \left\{ f = \sum_k c_k \psi_k : c \in \ell^{00} \right\}$ be the subspace of all finite linear combinations over Ψ.

For $1 \le p < \infty$ define $\mathscr{H}^p_w(\Psi, \tilde{\Psi})$ as the completion of $\mathscr{H}^{00}$ with respect to the norm

$$\|f\|_{\mathscr{H}^p_w} = \left\| C_{\tilde{\Psi}}(f) \right\|_{\ell^p_w}.$$

Let $\mathscr{H}^0_w$ be the completion of $\mathscr{H}^{00}$ with respect to the norm

$$\|f\|_{\mathscr{H}^0_w} = \|f\|_{\mathscr{H}^\infty_w} = \left\| C_{\tilde{\Psi}}(f) \right\|_{\ell^\infty_w}.$$

In Section 3.1 we will define the space $\mathscr{H}^\infty_w$ as a weak* completion with respect to the metric $\left\| C_{\tilde{\Psi}}(f) \right\|_{\ell^\infty_w}$. Alternatively, we may define it as the bidual $\mathscr{H}^\infty_w = (\mathscr{H}^0_w)^{**}$.

We note right away that $\mathscr{H}^p_w \subseteq \mathscr{H}^q_w \subseteq \mathscr{H}^0_w$ for $1 \le p \le q$.

As a consequence of this definition the analysis operator can be extended to a bounded operator from $\mathscr{H}^p_w$ into ℓ^p_w.

The main results in [35] are summarized below. The first one describes the independence of $\mathscr{H}^p_w(\Psi, \tilde{\Psi})$ of the defining frame Ψ.

Proposition 1 ([35]). *Let Φ and Ψ be frames for $\mathscr{H}$ and Φ^d and Ψ^d dual frames. If $\Psi^d \sim_{\mathscr{A}} \Psi$, $\Phi^d \sim_{\mathscr{A}} \Psi$ and $\Psi^d \sim_{\mathscr{A}} \Phi$, then $\mathscr{H}^p_w(\Psi, \Psi^d) = \mathscr{H}^p_w(\Phi, \Phi^d)$ with equivalent norms for all $1 \le p \le \infty$.*

The proof of this result relies on the algebra properties of $\mathscr{A}$ and identities for Gram matrices. By this result we may therefore write unambiguously $\mathscr{H}_w^p := \mathscr{H}_w^p(\Psi, \tilde{\Psi}) = \mathscr{H}_w^p(\tilde{\Psi}, \Psi)$.

In particular, let $\Phi = \Psi$. For a frame Ψ and its dual Ψ^d, which are $\mathscr{A}$-localized with respect to each other, it can be shown that they are automatically Banach frames for all involved associated Banach spaces:

Theorem 1 ([35]). *Assume that $\Psi \sim_{\mathscr{A}} \Psi^d$. Then Ψ is a Banach frame for $\mathscr{H}_w^p(\Psi, \Psi^d)$ for $1 \leq p < \infty$ or $p = 0$. The reconstructions $f = \sum_{n \in N} \langle f, \psi_n \rangle \psi_n^d$ and $f = \sum_{n \in N} \langle f, \psi_n^d \rangle \psi_n$ converge unconditionally for $1 \leq p < \infty$.*

The assumptions of Proposition 1 can be weakened for the canonical dual frame, because it can be shown that an intrinsically localized frame is automatically localized with respect to its canonical dual. As a consequence an intrinsically localized frame is automatically a Banach frame for all associated Banach spaces. This is the main theorem about localized frames:

Theorem 2 ([35]). *Let Ψ be an intrinsically $\mathscr{A}$-localized frame, $\Psi \sim_{\mathscr{A}} \Psi$. Then*

$$\tilde{\Psi} \sim_{\mathscr{A}} \tilde{\Psi} \text{ and } \Psi \sim_{\mathscr{A}} \tilde{\Psi}.$$

As a consequence, $\mathscr{H}_w^p(\Psi, \tilde{\Psi}) = \mathscr{H}_w^p(\tilde{\Psi}, \Psi)$ and Ψ is a p-Banach frame for $\mathscr{H}_w^p(\Psi, \tilde{\Psi})$ for $1 \leq p < \infty$ or $p = 0$. The reconstructions

$$f = \sum_{n \in N} \langle f, \psi_n \rangle \tilde{\psi}_n \text{ and } f = \sum_{n \in N} \langle f, \tilde{\psi}_n \rangle \psi_n \tag{6}$$

converge unconditionally in $\mathscr{H}_w^p$ for $1 \leq p < \infty$.

Therefore the norms $\|C_\Psi f\|_{\ell_w^p}$ and $\left\|C_{\tilde{\Psi}} f\right\|_{\ell_w^p}$ are equivalent, and the inequalities

$$\frac{1}{\left\|G_{\tilde{\Psi}}\right\|_{\ell_w^p \to \ell_w^p}} \|f\|_{\mathscr{H}_w^p} \leq \|C_\Psi f\|_{\ell_w^p} \leq \|G_\Psi\|_{\ell_w^p \to \ell_w^p} \|f\|_{\mathscr{H}_w^p} . \tag{7}$$

are valid for $1 \leq p < \infty$ and $p = 0$

The unconditional convergence of the reconstruction formula (6) implies that both synthesis operators D_Ψ and $D_{\tilde{\Psi}}$ map ℓ_w^p onto $\mathscr{H}_w^p$ for $1 \leq p < \infty$ and $p = 0$. Consequently, an equivalent norm on $\mathscr{H}_w^p$ is given by

$$\inf \left\{ \|c\|_{\ell_w^p} : f = D_\psi c \right\} \qquad \text{for } f \in \mathscr{H}_w^p .$$

In particular this means that the orbit and co-orbit definitions of $\mathscr{H}_w^p$ coincide.

The best studied examples of intrinsically localized frames are the following.

(i) Frames of translates [9, 17],
(ii) Frames of reproducing kernels in a shift-invariant space [45, 70]

(iii) Gabor frames [32, 35, 45],
(iv) Frames of reproducing kernels in (general) Bargmann-Fock spaces [58],
(v) Wavelet frames that are orthogonal across different scales [17, 34].

However, not all useful frames are localized in the sense of Definition 1, among them are general wavelet frames, frames of curvelets, shearlets, frames on manifolds, etc. Although these frames do possess some form of localization, they are not part of our theory of localized frames. While many of the constructions discussed in this chapter, such as the definition and characterization of coorbit spaces, can be carried out by hand or with different techniques, the main results for localized frames are not available for wavelets or curvelets. For instance, the decisive Theorem 2 and most of its consequences are false for wavelet frames and their many generalizations.

3.1 $\mathscr{H}_w^\infty$ as a Normed Space

In the following we will focus on the theory of the coorbit spaces $\mathscr{H}_w^p(\Psi, \tilde{\Psi})$ that are associated to a localized frame. We start with the "distribution space" $\mathscr{H}_w^\infty$ and offer a thorough treatment. In [35] "the rigorous discussion was omitted to avoid tedious technicalities."

Let w be an $\mathscr{A}$-admissible weight. We define $\mathscr{H}_w^\infty$ as a certain weak* completion of $\mathscr{H}$. We say that two sequences (f_n) and (g_n) in $\mathscr{H}$ are equivalent, denoted by $f_n \sim g_n$, if $\langle f_n - g_n, \tilde{\psi}_k\rangle \to 0$ as $n \to \infty$. Alternatively, $f_n - g_n \to 0$ in the $\sigma(\mathscr{H}, \mathscr{H}^{00})$-topology.

Definition 3. We define $\mathscr{H}_w^\infty$ as the set of equivalence classes of sequences $f = [f_n]$, such that

(i) $f_n \in \mathscr{H}$ for all $n \in \mathbb{N}$,
(ii) $\lim_{n\to\infty} \langle f_n, \tilde{\psi}_k\rangle = \alpha_k$ exists for all $k \in K$,
(iii) $\sup_n \left\| C_{\tilde{\Psi}} f_n \right\|_{\ell_w^\infty} < \infty$.

In this way $\mathscr{H}_w^\infty$ is well-defined. The definition of f is independent of its representative. Indeed, if $f = [f_n]$ and $f_n \sim g_n$, then $\alpha_k = \lim_{n\to\infty}\langle f_n, \tilde{\psi}_k\rangle = \lim_{n\to\infty}\langle g_n, \tilde{\psi}_k\rangle$.

Furthermore, condition (iii) implies that $|\langle f_n, \tilde{\psi}_k\rangle| w_k \le C$ for all $n \in \mathbf{N}$ and $k \in K$, consequently, $|\alpha_k| w_k = \lim_{n\to\infty} |\langle f_n, \tilde{\psi}_k\rangle| w_k \le C$ and thus $\alpha \in \ell_w^\infty$. Now, write $\langle f, \tilde{\psi}_k\rangle = \alpha_k$. and set

$$\|f\|_{\mathscr{H}_w^\infty} = \|\alpha\|_{\ell_w^\infty}\,. \tag{8}$$

Therefore $C_{\tilde{\Psi}} : \mathscr{H}_w^\infty \to \ell_w^\infty$ is a bounded operator.

Clearly, (8) defines a seminorm, because limits are linear and $\|\cdot\|_{\ell_w^\infty}$ is a norm. Now assume that $\|f\|_{\mathscr{H}_w^\infty} = 0$. This means that for every representative $[f_n]$ of f we have $\lim_n\langle f_n, \tilde{\psi}_k\rangle = 0$, or equivalently $f_n \sim 0$. Thus $f = 0$ in $\mathscr{H}$ and $\|\cdot\|_{\mathscr{H}_w^\infty}$ is indeed a norm.

Lemma 1. *(i) The map* $f = [f_n] \mapsto \alpha = \left(\lim_n \langle f_n, \tilde{\psi}_k \rangle\right)_{k \in K}$ *is an isometric isomorphism from* $\mathscr{H}_w^\infty$ *onto the subspace* $V_\Psi = \{\alpha \in \ell_w^\infty : \alpha = G_{\tilde{\Psi},\Psi}\alpha\}$.

(ii) The subspace V_Ψ *is closed in* ℓ_w^∞ *and thus* $\mathscr{H}_w^\infty$ *is complete.*

Proof. (i) For $f \in \mathscr{H}$ we interpret the reconstruction formula $f = \sum_l \langle f, \tilde{\psi}_l \rangle \psi_l$ weakly as $\sum_l \langle f, \tilde{\psi}_l \rangle \langle \psi_l, \tilde{\psi}_k \rangle$, or in operator notation as

$$C_{\tilde{\Psi}} f = G_{\tilde{\Psi},\Psi} C_{\tilde{\Psi}} f \,. \tag{9}$$

Now let $f = [f_n] \in \mathscr{H}_w^\infty$ as in Definition 3. This is a sequence of vectors $f_n \in \mathscr{H}$ such that $\lim_{n\to\infty} \left\langle f_n, \tilde{\psi}_k \right\rangle = \alpha_k$ and $\|C_{\tilde{\Psi}} f_n\|_{\ell_w^\infty} \le C$ for all n. This means that the sequence $C_{\tilde{\Psi}} f_n$ converges pointwise to α and is dominated by the sequence $C(w_l^{-1})_l \in \ell_w^\infty$. By dominated convergence it now follows that (again with pointwise convergence)

$$\alpha = \lim_{n\to\infty} C_{\tilde{\Psi}} f_n = \lim_{n\to\infty} G_{\tilde{\Psi},\Psi} C_{\tilde{\Psi}} f_n = G_{\tilde{\Psi},\Psi} \lim_{n\to\infty} C_{\tilde{\Psi}} f_n = G_{\tilde{\Psi},\Psi} \alpha \,.$$

Consequently, the limiting sequence $\alpha \in \ell_w^\infty$ satisfies $\alpha = G_{\tilde{\Psi},\Psi}\alpha$ and $\alpha \in V_\Psi$.

Conversely, let $\alpha \in V_\Psi$. Choose a sequence F_n of finite subsets of K, such that $F_n \subset F_{n+1}$ and $\bigcup_{n=1}^\infty F_n = K$ and define

$$f_n = \sum_{l \in F_n} \alpha_l \psi_l \in \mathscr{H} \,.$$

Then clearly

$$\lim_{n\to\infty} \langle f_n, \tilde{\psi}_k \rangle = \lim_{n\to\infty} \sum_{l \in F_n} \alpha_l \left\langle \psi_l, \tilde{\psi}_k \right\rangle = G_{\tilde{\Psi},\Psi}\alpha = \alpha \,,$$

and $\sup_k |\langle f_n, \tilde{\psi}_k \rangle| w_k \le C \|\alpha\|_{\ell_w^\infty}$. This means that $f = [f_n] \in \mathscr{H}_w^\infty$, and as a consequence the map $[f_n] \in \mathscr{H}_w^\infty \mapsto \alpha \in V_\Psi$ is an isometric isomorphism.

(ii) Assume that $\alpha_n \in V_\Psi$ and $\alpha \in \ell_w^\infty$ such that $\|\alpha_n - \alpha\|_{\ell_w^\infty} \to 0$. Since $G_{\tilde{\Psi},\Psi}$ is bounded on ℓ_w^∞, we obtain that $\alpha = \lim_{n\to\infty} \alpha_n = \lim_n G_{\tilde{\Psi},\Psi}\alpha_n = G_{\tilde{\Psi},\Psi}\alpha$, whence $\alpha \in V_\Psi$ and V_Ψ is a (norm)-closed subspace of ℓ_w^∞. By the identification proved in (i), $\mathscr{H}_w^\infty$ is therefore complete. □

Switching the roles of Ψ and $\tilde{\Psi}$ in Definition 3, we see that C_Ψ is an isometry between $\mathscr{H}_w^p(\tilde{\Psi}, \Psi)$ and a closed subspace of ℓ_w^∞. By Equation (7) the corresponding norms are equivalent and so C_Ψ is bounded from $\mathscr{H}_w^p$ into ℓ_w^∞.

We next verify the unconditional weak *-convergence of the sum $D_\Psi c = \sum_{k \in K} c_k \psi_k$ for $(c_k) \in \ell_w^\infty$. Let $\varepsilon > 0$ and $g \in \mathscr{H}^{00}$. Choose a finite set H_0, such that $\sum_{k\notin H_0} |\langle g, \psi_k\rangle| w_k^{-1} < \varepsilon / \|c\|_{\ell_w^\infty}$. Now let $H, J \subseteq K$ be two finite sets such that $H \supseteq H_0$ and $J \supseteq H_0$. Then $H \setminus J \cup J \setminus H \subseteq K \setminus H_0$, and therefore

$$|\langle \sum_{k\in J} c_k \psi_k - \sum_{k\in H} c_k \psi_k, g\rangle| = |\langle \sum_{k\in H\setminus J \cup J\setminus H} c_k \psi_k, g\rangle|$$
$$\leq \|c\|_{\ell_w^\infty} \sum_{k\notin H_0} |\langle \psi_k, g\rangle| w_k^{-1} < \varepsilon\,.$$

Thus the series for $D_\Psi c$ converges weak-* unconditionally. Furthermore, since $G_{\tilde{\Psi},\Psi}$ is bounded on ℓ_w^∞ by the basic assumption on Ψ and Theorem 2, we also deduce the boundedness of D_Ψ as follows:

$$\|D_\Psi c\|_{\mathscr{H}_w^\infty} = \left\|C_{\tilde{\Psi}} D_\Psi c\right\|_{\ell_w^\infty} = \left\|G_{\tilde{\Psi},\Psi} c\right\|_{\ell_w^\infty} \leq \|G_{\tilde{\Psi},\Psi}\|_{\ell_w^\infty \to \ell_w^\infty} \|c\|_{\ell_w^\infty} < \infty.$$

The following lemma summarizes the properties of $\mathscr{H}_w^\infty$.

Lemma 2. *Let Ψ be an $\mathscr{A}$-localized frame and w an $\mathscr{A}$-admissible weight. Then $\left(\mathscr{H}_w^\infty, \|\cdot\|_{\mathscr{H}_w^\infty}\right)$ is a Banach space, and*

(i) $C_\Psi : \left(\mathscr{H}_w^\infty, \|\cdot\|_{\mathscr{H}_w^\infty}\right) \to \ell_w^\infty$ *is continuous.*

(ii) $D_\Psi : \ell_w^\infty \to \left(\mathscr{H}_w^\infty, \|\cdot\|_{\mathscr{H}_w^\infty}\right)$ *is continuous with* $\|D_\Psi\|_{\ell_w^\infty \to \mathscr{H}_w^\infty} \leq \left\|G_{\tilde{\Psi},\Psi}\right\|_{\ell_w^\infty \to \ell_w^\infty}$. *The series* $Dc = \sum_{k\in K} c_k \psi_k$ *is weak-* unconditionally convergent.*

3.2 Duality

The associated Banach spaces $\mathscr{H}_w^p$ are a generalization of the coorbit spaces in [30] and the modulation spaces [29]. We first formulate their duality.

Proposition 2. *Let Ψ be a $\mathscr{A}$-localized frame and w an admissible weight. Let $1 \leq p < \infty$ and q such that $\frac{1}{p} + \frac{1}{q} = 1$ or $(p, q) = (0, 1)$. Then*

$$\left(\mathscr{H}_w^p\right)' \cong \mathscr{H}_{1/w}^q,$$

where the duality for $f \in \mathscr{H}_w^p$ and $h \in \mathscr{H}_{1/w}^q$ is given by

$$\langle f, h\rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} := \left\langle C_{\tilde{\Psi}} f, C_\Psi h\right\rangle_{\ell_w^p, \ell_{1/w}^q}.$$

Proof. Fix $h \in \mathscr{H}^q_{1/w}$. Then, using the duality of ℓ^p_w and $\ell^q_{1/w}$, we define a sesquilinear form by

$$\langle f, h\rangle_{\mathscr{H}^p_w, \mathscr{H}^q_{1/w}} = \left\langle C_{\tilde{\Psi}} f, C_\Psi h\right\rangle_{\ell^p_w, \ell^q_{1/w}},$$

for $f \in \mathscr{H}^p_w$.

Now set $\mathscr{W}(h)(f) = \langle f, h\rangle_{\mathscr{H}^p_w, \mathscr{H}^q_{1/w}}$. Then $|\mathscr{W}(h)(f)| \leq \left\|C_{\tilde{\Psi}} f\right\|_{\ell^p_w} \|C_\Psi h\|_{\ell^q_{1/w}} = \|C_\Psi h\|_{\ell^q_{1/w}} \cdot \|f\|_{\mathscr{H}^p_w}$, since $C_\Psi h \in \ell^q_{1/w}$ by the norm equivalence (7). If $p = 0$, we use the estimate $|\mathscr{W}(h)(f)| \leq \left\|C_{\tilde{\Psi}} f\right\|_{\ell^\infty_w} \|C_\Psi h\|_{\ell^1_{1/w}} = \|C_\Psi h\|_{\ell^1_{1/w}} \cdot \|f\|_{\mathscr{H}^0_w}$, Therefore $\mathscr{W}(h) \in \left(\mathscr{H}^p_w\right)'$. As a consequence $\mathscr{W} : \mathscr{H}^q_{1/w} \to \left(\mathscr{H}^p_w\right)'$ is bounded, with $\|\mathscr{W}(h)\|_{\left(\mathscr{H}^p_w\right)'} \leq \|C_\Psi h\|_{\ell^q_{1/w}}$.

Conversely, let $H \in \left(\mathscr{H}^p_w\right)'$ and $c \in \ell^p_w$ be arbitrary with $1 \leq p \leq \infty$ or $p = 0$. Then $D_\Psi c$ is in $\mathscr{H}^p_w$ and so $H(D_\Psi c) = \sum_k c_k H(\psi_k)$ with unconditional convergence. Therefore the sequence $(H(\psi_k))_k$ is in $\ell^q_{1/w}$ [16]. Now define the operator $\mathscr{V} : \left(\mathscr{H}^p_w\right)' \to \mathscr{H}^q_{1/w}$ by $\mathscr{V}(H) = \sum_k \overline{H(\psi_k)} \tilde{\psi}_k$.

For $f \in \mathscr{H}^p_w$ we have

$$\mathscr{W}(\mathscr{V}(H))(f) = \left\langle C_{\tilde{\Psi}} f, C_\Psi \mathscr{V}(H)\right\rangle_{\ell^p_w, \ell^q_{1/w}} = \sum_l \langle f, \tilde{\psi}_l\rangle \overline{\left\langle \sum_k \overline{H(\psi_k)} \tilde{\psi}_k, \psi_l \right\rangle} =$$

$$= \sum_k H(\psi_k) \left\langle \sum_l \langle f, \tilde{\psi}_l\rangle \psi_l, \tilde{\psi}_k \right\rangle = \sum_k H(\psi_k) \langle f, \tilde{\psi}_k\rangle = H\left(\sum_k \langle f, \tilde{\psi}_k\rangle \psi_k\right) = H(f).$$

The formal manipulations are justified by the unconditional convergence of the series involved, by the continuity of H, and by density arguments. Thus $\mathscr{W} : \mathscr{H}^q_{1/w} \to \left(\mathscr{H}^p_w\right)'$ is onto.

On the other hand,

$$\mathscr{V}(\mathscr{W}(h)) = \sum_l \overline{\mathscr{W}(h)(\psi_l)} \tilde{\psi}_l = \sum_l \overline{\sum_k \langle \psi_l, \tilde{\psi}_k\rangle \overline{\langle h, \psi_k\rangle}} \tilde{\psi}_l =$$

$$= \sum_k \langle h, \psi_k\rangle \left(\sum_l \langle \tilde{\psi}_k, \psi_l\rangle \tilde{\psi}_l\right) = \sum_k \langle h, \psi_k\rangle \tilde{\psi}_k = h.$$

Therefore $\mathscr{W}$ is invertible. □

Similar results appeared in [30] and [23].

Remark 1. Note that the duality is consistent with the inner product $\langle ., .\rangle$ on $\mathscr{H}$, see Lemma 4.

Also, note that the isomorphism $\left(\mathscr{H}_w^p\right)' \cong \mathscr{H}_{1/w}^q$ is not an isometric isomorphism.

By the above result we now have $\left(\mathscr{H}_{1/w}^1\right)' \cong \left(\mathscr{H}_w^\infty, \|\cdot\|_{\mathscr{H}_w^\infty}\right)$. This yields another proof for the completeness of $\left(\mathscr{H}_w^\infty, \|\cdot\|_{\mathscr{H}_w^\infty}\right)$.

3.2.1 Duality for $\mathscr{H}_w^\infty$

For $p = \infty$ we can now prove a reconstruction result, as an extension to Theorem 1.

Lemma 3. *Let Ψ be a $\mathscr{A}$-localized frame and w an $\mathscr{A}$-admissible weight. If $f \in \mathscr{H}_w^\infty$, then $f = \sum \langle f, \psi_k\rangle \tilde{\psi}_k$ and $f = \sum \langle f, \tilde{\psi}_k\rangle \psi_k$ with weak-* unconditional convergence in $\sigma\left(\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1\right)$.*

Therefore $f = D_{\tilde{\Psi}} C_\Psi f = D_\Psi C_{\tilde{\Psi}} f$, and, in particular $D_{\tilde{\Psi}}$ and D_Ψ are onto $\mathscr{H}_w^\infty$.

The norm equivalence (7) is valid for all $1 \le p \le \infty$ and $p = 0$:

$$\frac{1}{\left\|G_{\tilde{\Psi}}\right\|_{\ell_w^p \to \ell_w^p}} \|f\|_{\mathscr{H}_w^p} \le \|C_\Psi f\|_{\ell_w^p} \le \|G_\Psi\|_{\ell_w^p \to \ell_w^p} \|f\|_{\mathscr{H}_w^p} . \tag{7'}$$

Proof. By above, $\mathscr{H}_{1/w}^1$ is the predual of $\mathscr{H}_w^\infty$. Now, let $f \in \mathscr{H}_w^\infty$ and $g \in \mathscr{H}_{1/w}^1$, then we have

$$\left|\sum_{k\in K} \langle f, \psi_k\rangle \left\langle \tilde{\psi}_k, g\right\rangle\right| \le \sum_{k\in K} \left|\langle f, \psi_k\rangle\right| \left|\left\langle \tilde{\psi}_k, g\right\rangle\right| \le \|C_\Psi f\|_{\ell_w^\infty} \left\|C_{\tilde{\Psi}} g\right\|_{\ell_{1/w}^1} ,$$

and the sum of the left-hand side converges absolutely.

By Lemma 2 $D_{\tilde{\Psi}} C_\Psi$ is well-defined on all of $\mathscr{H}_w^\infty$. Let $g \in \mathscr{H}^{00}$, then

$$\left\langle D_{\tilde{\Psi}} C_\Psi f, g\right\rangle = \lim_{\substack{H \to K \\ H \text{ finite}}} \left\langle \sum_{k\in H} \langle f, \psi_k\rangle \cdot \tilde{\psi}_k, g \right\rangle = \lim_{\substack{H \to K \\ H \text{ finite}}} \sum_{k\in H} \langle f, \psi_k\rangle \cdot \left\langle \tilde{\psi}_k, g\right\rangle = \langle f, g\rangle .$$

And so $f = D_{\tilde{\Psi}} C_\Psi f$.

The second reconstruction formula follows by an analogous argument. The norm equivalence (7') follows immediately from the reconstruction formula. □

We can formulate the compatibility of the duality relations in the following way.

Lemma 4. *Let Ψ be a $\mathscr{A}$-localized frame and w an admissible weight. For $f \in \mathscr{H}_w^p$ and $h \in \mathscr{H}_{1/w}^1$ we have*

$$\langle f, h\rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} = \langle f, h\rangle_{\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1} .$$

Proof. The identity follows from the definition of the duality in Proposition 2, because

$$\langle f, h\rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} = \left\langle C_{\tilde{\Psi}} f, C_\Psi h\right\rangle_{\ell_w^p, \ell_{1/w}^q} = \left\langle C_{\tilde{\Psi}} f, C_\Psi h\right\rangle_{\ell_w^\infty, \ell_{1/w}^1} = \langle f, h\rangle_{\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1}.$$

□

After clarifying the meaning of the duality brackets, we can now give the traditional definition of the coorbit space $\mathscr{H}_w^p$ as a subspace of "distributions."

Proposition 3. *Let Ψ be an $\mathscr{A}$-localized frame and w an admissible weight. For $1 \leq p < \infty$ we have*

$$\mathscr{H}_w^p = \left\{ f \in \mathscr{H}_w^\infty : f = \sum_{k \in K} \langle f, \tilde{\psi}_k\rangle_{\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1} \psi_k \text{ with } \langle f, \tilde{\psi}_k\rangle_{\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1} \in \ell_w^p \right\},$$

with unconditional convergence.

Proof. We combine the reconstruction formula in Theorem 2 with the identities $\left(\langle f, \psi_k\rangle_{\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1} \right)_{k \in K} = \left(\langle f, \psi_k\rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} \right)_{k \in K} \in \ell_w^p$ from Lemma 4 and use the unconditional convergence in $\mathscr{H}_w^p$. We obtain

$$f = \sum_{k \in K} \langle f, \psi_k\rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} \tilde{\psi}_k = \sum_{k \in K} \langle f, \psi_k\rangle_{\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1} \tilde{\psi}_k.$$

□

For $p = \infty$ we can state the following characterization (using Proposition 2 and Lemma 3):

Corollary 1. *Let Ψ be a $\mathscr{A}$-localized frame. Let W and w admissible weights satisfying $W \leq w$ so that $\mathscr{H}_w^\infty \subseteq \mathscr{H}_W^\infty$. Then for $f \in \mathscr{H}_W^\infty$ the following properties are equivalent:*

(i) $f \in \mathscr{H}_w^\infty$.
(ii) $\left\| C_{\tilde{\Psi}} f \right\|_{\ell_w^\infty} < \infty$.
(iii) *There is a $c \in \ell_w^\infty$, such that $f = \sum_{k \in K} c_k \psi_k$ with $C_{\tilde{\Psi}} f = G_{\Psi, \tilde{\Psi}} c$ and*

$$\left\| C_{\tilde{\Psi}} f \right\|_{\ell_w^\infty} = \left\| G_{\Psi, \tilde{\Psi}} c \right\|_{\ell_w^\infty}.$$

(iv) $f \in \left(\mathscr{H}_w^1\right)'$.

3.2.2 The Chain of Banach Spaces $\mathscr{H}_w^p$

Formulated for Gelfand triples we obtain the following consequence.

Corollary 2. *Let Ψ be a $\mathscr{A}$-localized frame and w an admissible weight with $\inf_{k\in K} w_k > 0$. Let $1 \le p < 2$ and $1/p + 1/q = 1$, or $(p,q) = (0,1)$. Then Ψ is a Gelfand frame for the Gelfand triples*

$$\mathscr{H}_w^p \subseteq \mathscr{H} \subseteq \mathscr{H}_{1/w}^q,$$

with respect to the duality $\left\langle C_{\tilde{\Psi}} f, C_\Psi h\right\rangle_{\ell_w^p,\ell_{1/w}^q}$ and the sequence spaces $\ell_w^p \subseteq \ell^2 \subseteq \ell_{1/w}^q$.

Proof. By Proposition 2 $\left(\mathscr{H}_w^p\right)' \cong \mathscr{H}_{1/w}^q$. Since $w_k \ge C$ we have, for $1 \le p \le 2$ and $2 \le q \le \infty$, the following inclusions

$$\mathscr{H}_w^1 \subseteq \mathscr{H}_w^p \subseteq \mathscr{H}_w^2 \subseteq \mathscr{H} \subseteq \mathscr{H}_{1/w}^2 \subseteq \mathscr{H}_{1/w}^q \subseteq \mathscr{H}_{1/w}^\infty,$$

For $q < \infty$, these inclusions are norm-dense, continuous embeddings (by the corresponding inclusions (1) for sequence spaces), for $q = \infty$, $\mathscr{H}$ is w*-dense in $\mathscr{H}_{1/w}^\infty$. Theorem 1 asserts that Ψ is a Banach frame for $\mathscr{H}_w^p$ and $\mathscr{H}_{1/w}^q$. □

To summarize the "size" of the coorbit spaces for $1 \le p \le 2$ and $2 \le q \le \infty$ by Equation (1) we obtain the following inclusions:

$$\begin{array}{ccccccccccccc}
\mathscr{H}^{00} & \subseteq & \mathscr{H}_w^1 & \subseteq & \mathscr{H}_w^p & \subseteq & \mathscr{H} & \subseteq & \mathscr{H}_{1/w}^q & \subseteq & \mathscr{H}_{1/w}^0 & \subseteq & \mathscr{H}_{1/w}^\infty \\
 & & & & D_\Psi \uparrow & & & & \downarrow C_\Psi & & & & \\
\ell^{00} & \subseteq & \ell_w^1 & \subseteq & \ell_w^p & \subseteq & \ell^2 & \subseteq & \ell_{1/w}^q & \subseteq & \ell_{1/w}^0 & \subseteq & \ell_{1/w}^\infty
\end{array}$$

All the inclusions but the last one are in fact norm-dense embeddings; $\mathscr{H}^{00}$ is norm dense in $\mathscr{H}_w^p$ for $1 \le p < \infty$ and weak*-dense in $\mathscr{H}_w^\infty$.

Finally we mention that the assumption on the weight serves only to obtain a "small space" with $p = 1$ on the left side of the diagram. By contrast, if $1/w \subseteq \ell^2$, then $\ell_w^\infty \subset \ell^2$, and one obtains the Gelfand pair $\mathscr{H}_w^0 \subseteq \mathscr{H} \subseteq \mathscr{H}_{1/w}^1$, which looks a bit unusual.

3.2.3 Equivalence Result on Inclusion of Sequence Spaces and Associated Banach Spaces

Whereas the inclusions of the coorbit spaces $\mathscr{H}_w^p$ follow from the inclusions of the weighted ℓ^p-spaces, the converse is less obvious and requires more tools.

Theorem 3. *Let Ψ be an $\mathscr{A}$-localized norm-bounded frame, i.e., $\inf_k \|\psi_k\|_{\mathscr{H}} > 0$. Let $1 \leq p_1, p_2 \leq \infty$ and let w_1, w_2 be admissible weights. Then*

$$\mathscr{H}_{w_1}^{p_1} \subseteq \mathscr{H}_{w_2}^{p_2} \Longleftrightarrow \ell_{w_1}^{p_1} \subseteq \ell_{w_2}^{p_2}.$$

Proof. The implication $\Longleftarrow$ is clear.

Conversely, assume that $\mathscr{H}_{w_1}^{p_1} \subseteq \mathscr{H}_{w_2}^{p_2}$ and that $\ell_{w_1}^{p_1} \not\subseteq \ell_{w_2}^{p_2}$.

Since Ψ is a frame, by the Feichtinger conjecture [8] proved in [61], Ψ is a finite union of Riesz sequences. In particular, $\mathscr{H}$ contains an infinite-dimensional subspace with a Riesz basis $\Psi_0 = \{\psi_k \,|k \in K_0\} \subseteq \Psi$.

The Gram matrix G_{Ψ_0} is invertible on $\ell^2(K_0)$ [13]. We extend G_{Ψ_0} to a matrix on $\ell^2(K)$ by defining $Gc = 0$ for $c \in \ell^2(K \backslash K_0) = \ell^2(K_0)^{\perp}$. Note that G is obtained from the Gram matrix G_Ψ by setting $(G_\Psi)_{j,k} = 0$ for $j, k \notin K_0$. Since $\mathscr{A}$ is solid, we conclude that G is in $\mathscr{A}$, and since $\mathscr{A}$ is closed with respect to the pseudo-inversion [35], we also find that $G^\dagger \in \mathscr{A}$. The matrix G possesses the pseudo-inverse $G^\dagger$ with $G^\dagger = G_{\Psi_0}^{-1} c$ for $c \in \ell^2(K_0)$ and $G^\dagger c = 0$ for $c \in \ell^2(K_0)^{\perp}$.

Extend $c \in \ell_{w_1}^{p_1}(K_0) \big\backslash \ell_{w_2}^{p_2}(K_0)$ to a sequence $\tilde{c} \in \ell_{w_1}^{p_1}(K) \big\backslash \ell_{w_2}^{p_2}(K)$ (by setting $\tilde{c}_k = 0$ for $k \in K \backslash K_0$) and set $d = G^\dagger \tilde{c}$. In particular, $d_k = 0$ for $k \notin K_0$. Since $G^\dagger \in \mathscr{A}$ is bounded on $\ell_{w_1}^{p_1}$, it follows that $d \in \ell_{w_1}^{p_1}(K) \backslash \ell_{w_2}^{p_2}(K)$. Therefore $f = \sum\limits_{k \in K} d_k \psi_k \in \mathscr{H}_{w_1}^{p_1}$. Furthermore

$$(C_{\Psi_0} f)_k = \langle f, \psi_k \rangle = \sum_{l \in K_0} d_l \langle \psi_l, \psi_k \rangle = (Gd)_k = c_k \text{ for } k \in K_0.$$

If $f \in \mathscr{H}_{w_1}^{p_1}$, then $C_\Psi f \in \ell_{w_1}^{p_1}$. But $C_\Psi f\big|_{K_0} = c \notin \ell_{w_2}^{p_2}(K_0)$, and so $C_\Psi f \notin \ell_{w_2}^{p_2}$. Therefore $f \notin \mathscr{H}_{w_2}^{p_2}$, which is a contradiction. □

Most likely, the statement could be proved without the full strength of the theorem of Marcus, Spielman, and Srivastava [61].

3.3 Properties of the Frame-Related Operators

We next summarize the properties of the canonical operators associated to every frame. We include the statements for $p = \infty$ and discuss the convergence of series expansions. As a novelty, we discuss all operators with respect to a frame Φ in the same localization class, i.e. $\Phi \sim_{\mathscr{A}} \Psi$. Being pedantic, we always consider the operator as a mapping with a domain. For instance, whereas the synthesis operator is the formula $D_\Psi c = \sum\limits_{k \in K} c_k \psi_k$ we will use the notation $D_\Psi^{p,w}$ to denote the synthesis operator on $\mathscr{H}_w^p$.

Theorem 4. *Let Ψ be an $\mathscr{A}$-localized frame and let w be an $\mathscr{A}$-admissible weight. Let $1 \leq p \leq \infty$ and let $1/p + 1/q = 1$ or $(p, q) = (0, 1)$.*

(i) *The analysis operator* $C_{\Psi}^{p,w} : \mathscr{H}_w^p \to \ell_w^p$ *is given by*

$$C_{\Psi}^{p,w} f = \left(\langle f, \psi_k \rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q}, k \in K \right).$$

Then $C_{\Psi}^{p,w}$ *is bounded, one-to-one and has closed range in* ℓ_w^p. $C_{\Psi}^{p,w}$ *is the restriction of* $C_{\Psi}^{\infty,w}$ *to* $\mathscr{H}_w^p$, *i.e.* $C_{\Psi}^{p,w} = C_{\Psi}^{\infty,w}\big|_{\mathscr{H}_w^p}$. *Furthermore* $\mathrm{ran}\left(C_{\Psi}^{p,w}\right) = \mathrm{ran}\left(C_{\tilde{\Psi}}^{p,w}\right)$ *and this is a complemented subspace:*

$$\ell_w^p = \mathrm{ran}\left(C_{\Psi}^{p,w}\right) \oplus \ker\left(D_{\Psi}^{p,w}\right), \tag{10}$$

Ψ *is an* ℓ_w^p*-Banach frame for all* $\mathscr{H}_w^p$ *with bounds*

$$A = \left\| G_{\tilde{\Psi}}^{p,w} \right\|_{\ell_w^p \to \ell_w^p}^{-1} \text{ and } B = \left\| G_{\Psi}^{p,w} \right\|_{\ell_w^p \to \ell_w^p}. \tag{11}$$

(ii) *The synthesis (or reconstruction) operator* $D_{\Psi}^{p,w} : \ell_w^p \to \mathscr{H}_w^p$ *is given by*

$$D_{\Psi}^{p,w} c = \sum_k c_k \psi_k$$

with unconditional convergence in $\mathscr{H}_w^p$ *for* $1 \leq p < \infty$ *and* $p = 0$, *and weak*-convergence for* $p = \infty$. *Then* $D_{\Psi}^{p,w}$ *is bounded with operator norm* 1 *and it maps onto* $\mathscr{H}_w^p$. *Furthermore* $D_{\Psi}^{p,w}$ *is the restriction of* $D_{\Psi}^{\infty,w}$ *to* $\mathscr{H}_w^p$, *i.e.* $D_{\Psi}^{p,w} = D_{\Psi}^{\infty,w}\big|_{\mathscr{H}_w^p}$. *For* $p < \infty$

$$\left(D_{\Psi}^{p,w}\right)^* = C_{\Psi}^{q,1/w}, \text{ and}$$
$$\left(C_{\Psi}^{p,w}\right)^* = D_{\Psi}^{q,1/w}.$$

(iii) *The frame operator* $S_{\Psi}^{p,w} : \mathscr{H}_w^p \to \mathscr{H}_w^p$ *is defined by*

$$S_{\Psi}^{p,w} f = \sum_k \langle f, \psi_k \rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} \cdot \psi_k = \sum_k \langle f, \psi_k \rangle_{\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1} \cdot \psi_k$$

with unconditional convergence in $\mathscr{H}_w^p$ *for* $1 \leq p < \infty$ *and* $p = 0$. *It is unconditionally weak*-convergent for* $p = \infty$. *Furthermore* $S_{\Psi}^{p,w} = D_{\Psi}^{p,w} C_{\Psi}^{p,w}$ *and* $\left(S_{\Psi}^{p,w}\right)^* = S_{\Psi}^{q,1/w}$, *and* $S_{\Psi}^{p,w} = S_{\Psi}^{\infty,w}\big|_{\mathscr{H}_w^p}$. *The operator* $S_{\Psi}^{p,w}$ *is bounded with bound* $\left\| G_{\tilde{\Psi},\Psi}^{p,w} \right\|_{\ell_w^p \to \ell_w^p} \left\| G_{\Psi}^{p,w} \right\|_{\ell_w^p \to \ell_w^p}$. *It is invertible with inverse* $\left(S_{\Psi}^{p,w}\right)^{-1} = S_{\tilde{\Psi}}^{p,w} = S_{\tilde{\Psi}}^{\infty,w}\big|_{\mathscr{H}_w^p} = \left(S_{\Psi}^{\infty,w}\big|_{\mathscr{H}_w^p}\right)^{-1}$, *and is therefore simultaneously invertible on all* $\mathscr{H}_w^p$.

(iv) For the Gram matrix $(G_\Psi)_{k,l} = \langle \psi_l, \psi_k \rangle_{\mathscr{H}}$ *(which by the admissibility induces a bounded operator* $G_\Psi^{p,w} : \ell_w^q \to \ell_w^p$*) we have* $G_\Psi^{p,w} = C_\Psi^{p,w} D_\Psi^{p,w}$ *and again* $G_\Psi^{p,w} = G_\Psi^{\infty,w}\big|_{\mathscr{H}_w^p}$ *The operator* $G_{\Psi,\tilde{\Psi}}$ *is the projection from* ℓ_w^p *on* $\operatorname{ran}\left(C_\Psi^{p,w}\right)$.

As a consequence of Theorem 4, one may now drop the indices and write simply and unambiguously C_Ψ, D_Ψ, S_Ψ, and G_Ψ.

We split the proof of Theorem 4 into shorter lemmata. Note that we prove them for an arbitrary frame Φ that is localized with respect to the intrinsically localized frame Ψ. So we need the following preparatory result.

Lemma 5. *Let* Φ *and* Ψ *frames with* $\Phi \sim_{\mathscr{A}} \Psi$ *and* $\Psi \sim_{\mathscr{A}} \Psi$ *and let* w *be an* $\mathscr{A}$*-admissible weight. Then* Φ *is intrinsically localized, and* $\tilde{\Phi} \sim_{\mathscr{A}} \tilde{\Psi}$ *and* $\Phi \sim_{\mathscr{A}} \tilde{\Psi}$.

In particular $\phi_k \in \mathscr{H}_{1/w}^1$ *and* $\langle f, \phi_k \rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} = \langle f, \phi_k \rangle_{\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1}$ *for* $f \in \mathscr{H}_w^p$.

Proof. Since $\tilde{\Psi} \sim_{\mathscr{A}} \tilde{\Psi}$ by Theorem 2, we may apply (5) as follows:

$$\Phi \sim_{\mathscr{A}} \Psi, \tilde{\Psi} \sim_{\mathscr{A}} \tilde{\Psi} \Rightarrow \Phi \sim_{\mathscr{A}} \tilde{\Psi}$$

$$\Phi \sim_{\mathscr{A}} \Psi, \tilde{\Psi} \sim_{\mathscr{A}} \Phi \Rightarrow \Phi \sim_{\mathscr{A}} \Phi \, .$$

As a consequence, the frame Φ is $\mathscr{A}$-localized and all results about $\mathscr{A}$-localized frames apply to Φ. In particular, Proposition 1 implies that $\mathscr{H}_w^p(\Psi, \tilde{\Psi}) = \mathscr{H}_w^p(\Phi, \tilde{\Phi})$ with equivalent norms, and we may write unambiguously $\mathscr{H}_w^p$.

Furthermore, since the Gram matrix $G_{\tilde{\Psi},\Phi} \in \mathscr{A}$ is bounded on ℓ_w^1, it follows that every row and column of $G_{\tilde{\Psi},\Phi}$ belongs to ℓ_w^1 and likewise to $\ell_{1/w}^1$. Consequently, $\phi_k = \sum_{l \in K} \langle \phi_k, \tilde{\psi}_l \rangle \psi_l$ is in $\mathscr{H}_w^1 \cap \mathscr{H}_{1/w}^1$. Thus the brackets

$$\langle f, \phi_k \rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} = \langle f, \phi_k \rangle_{\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1}$$

are well-defined by Lemma 4. □

In particular all results shown above are applicable also for Φ, however, with equivalent norms.

Lemma 6. *Let* Φ *and* Ψ *be* $\mathscr{A}$*-localized frames such that* $\Phi \sim_{\mathscr{A}} \Psi$*. Let* w *be an* $\mathscr{A}$*-admissible weight and* $1 \le p, q \le \infty$ *with* $1/p + 1/q = 1$ *or* $(p,q) = (0,1)$*. Then the analysis operator* $C_\Phi^{p,w} : \mathscr{H}_w^p \to \ell_w^p$ *given by*

$$C_\Phi^{p,w} f = \left(\langle f, \phi_k \rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q}, k \in K \right)$$

is bounded, one-to-one and has closed range. Furthermore

$$\frac{1}{\left\| G_{\tilde{\Psi},\tilde{\Phi}}^{p,w} \right\|_{\ell_w^p \to \ell_w^p}} \|f\|_{\mathscr{H}_w^p} \le \left\| C_\Phi^{p,w} f \right\|_{\ell_w^p} \le \left\| G_{\Phi,\Psi}^{p,w} \right\|_{\ell_w^p \to \ell_w^p} \|f\|_{\mathscr{H}_w^p} \, , \tag{12}$$

where both sides of the inequality are bounded. The operator $C_\Phi^{p,w}$ *is the restriction of* $C_\Phi^{\infty,w}$ *to* $\mathscr{H}_w^p$*, i.e.* $C_\Phi^{p,w} = C_\Phi^{\infty,w}\big|_{\mathscr{H}_w^p}$.

Proof. The associated Banach spaces $\mathscr{H}_w^p$ coincide for the frames Ψ and Φ. By Proposition 1 and Lemma 4 $\left(C_\Phi^{p,w} f\right)_k = \langle f, \phi_k \rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} = \langle f, \phi_k \rangle_{\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1}$.

$$\left\| C_{\tilde{\Psi}}^{p,w} f \right\|_{\ell_w^p} = \left\| C_{\tilde{\Psi}}^{p,w} D_{\tilde{\Phi}}^{p,w} C_\Phi^{p,w} f \right\|_{\ell_w^p} \leq \left\| G_{\tilde{\Psi},\tilde{\Phi}}^{p,w} \right\|_{\ell_w^p \to \ell_w^p} \left\| C_\Phi^{p,w} f \right\|_{\ell_w^p}, \text{ and}$$
$$\left\| C_\Phi^{p,w} f \right\|_{\ell_w^p} = \left\| C_\Phi^{p,w} D_\Psi^{p,w} C_{\tilde{\Psi}}^{p,w} f \right\|_{\ell_w^p} \leq \left\| G_{\Phi,\Psi}^{p,w} \right\|_{\ell_w^p \to \ell_w^p} \left\| C_{\tilde{\Psi}}^{p,w} f \right\|_{\ell_w^p}.$$

By Lemma 5 the Gram matrices $G_{\tilde{\Psi},\tilde{\Phi}}^{p,w}$ and $G_{\Phi,\Psi}^{p,w}$ are in $\mathscr{A}$ and are therefore bounded on ℓ_w^p for all p, $1 \leq p \leq \infty$. By Lemma 4 $C_\Psi^{p,w} = C_\Psi^{\infty,w}|_{\mathscr{H}_w^p}$, since $\mathscr{H}_w^p \subseteq \mathscr{H}_w^\infty$. □

As $\tilde{\Psi} \sim_{\mathscr{A}} \Psi$ the analysis operator $C_{\tilde{\Psi}}^{p,w} : \mathscr{H}_w^p \to \ell_w^p$ is given by $C_{\tilde{\Psi}}^{p,w} f = \langle f, \tilde{\psi}_k \rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q}$. By definition, this particular analysis operator is an isometry.

Lemma 7. *Under the assumptions of Lemma 6 the synthesis operator $D_\Phi^{p,w} : \ell_w^p \to \mathscr{H}_w^p$ is bounded and onto with operator norm*

$$\left\| D_\Phi^{p,w} \right\|_{\ell_w^p \to \mathscr{H}_w^p} = \left\| G_{\tilde{\Psi},\Phi}^{p,w} \right\|_{\ell_w^p \to \ell_w^p}.$$

It is given by

$$D_\Phi^{p,w} c = \sum_k c_k \phi_k$$

with unconditional convergence in $\mathscr{H}_w^p$ for $1 \leq p < \infty$ and $p = 0$ and weak-convergence for $p = \infty$. Furthermore, $D_\Phi^{p,w}$ is the restriction of $D_\Phi^{\infty,w}$ to $\mathscr{H}_w^p$, i.e. $D_\Phi^{p,w} = D_\Phi^{\infty,w}|_{\mathscr{H}_w^p}$. For $p < \infty$ we have*

$$\left(D_\Phi^{p,w}\right)^* = C_\Phi^{q,1/w} \quad \textit{and} \quad \left(C_\Phi^{p,w}\right)^* = D_\Phi^{q,1/w}. \tag{13}$$

Proof. By Lemma 5 and Lemma 3 $D_\Phi^{p,w}$ is bounded and onto $\mathscr{H}_w^p(\Phi, \tilde{\Phi}) = \mathscr{H}_w^p(\Psi, \tilde{\Psi})$ (see also Proposition 1). Since $D_\Phi^{p,w} = G_{\Phi,\tilde{\Psi}} D_\Psi^{p,w}$, $D_\Phi^{p,w}$ is bounded on ℓ_w^p. The unconditional convergence of $D_\Phi^{p,w} c = \sum_k c_k \phi_k$ is shown as in Lemma 2. Since $\ell_w^p \subset \ell_w^\infty$, it is clear that $D_\Phi^{p,w} = D_\Phi^{\infty,w}|_{\mathscr{H}_w^p}$.

For the adjoint operator let $c \in \ell_w^p$, and $f \in \mathscr{H}_{1/w}^q \simeq (\mathscr{H}_w^p)'$. Then

$$\left\langle D_\Phi^{p,w} c, f \right\rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} = \left\langle \sum_k c_k \phi_k, f \right\rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} =$$

$$= \sum_k c_k \left\langle \phi_k, f \right\rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} = \left\langle c, C_\Phi^{q,1/w} f \right\rangle_{\ell_w^p, \ell_{1/w}^q},$$

where the change of order is justified because $c \in \ell_w^p, f \in \mathscr{H}_{1/w}^q$ and by Lemma 6.

The operator norm is

$$\left\| D_\Phi^{p,w} \right\|_{\ell^p_w \to \mathcal{H}^p_w} = \sup_{\|c\|_{\ell^p_w}=1} \left\| D_\Phi^{p,w} c \right\|_{\mathcal{H}^p_w} = \sup_{\|c\|_{\ell^p_w}=1} \left\| G^{p,w}_{\tilde{\Psi},\Phi} c \right\|_{\ell^p_w} = \left\| G^{p,w}_{\tilde{\Psi},\Phi} \right\|_{\ell^p_w \to \ell^p_w}.$$

□

Clearly, by above, $\left\| D_\Phi^{p,w} \right\|_{\ell^p_w \to \mathcal{H}^p_w} = \left\| C_\Phi^{q,1/w} \right\|_{(\mathcal{H}^p_w)' \to \ell^q_{1/w}}$. In general we have $\left\| C_\Phi^{q,1/w} \right\|_{(\mathcal{H}^p_w)' \to \ell^q_{1/w}} \neq \left\| C_\Phi^{q,1/w} \right\|_{\mathcal{H}^q_{1/w} \to \ell^q_{1/w}}$, because the isomorphism between $(\mathcal{H}^p_w)'$ and $\mathcal{H}^q_{1/w}$ of Proposition 2 is not an isometry.

Lemma 8. *Let Ψ, Φ, and Ξ be frames with $\Psi \sim_{\mathcal{A}} \Psi$, $\Phi \sim_{\mathcal{A}} \Psi$ and $\Xi \sim_{\mathcal{A}} \Psi$, and let w be an $\mathcal{A}$-admissible weight. Let $1 \leq p \leq \infty$ and let $1/p + 1/q = 1$ or $(p,q) = (0,1)$. The cross-Gram matrix $G_{\Phi,\Xi}$ with entries $(G_{\Phi,\Xi})_{k,l} = \langle \xi_l, \phi_k \rangle_{\mathcal{H}}$ induces a bounded operator $G^{p,w}_{\Phi,\Xi} : \ell^p_w \to \ell^p_w$ and factors as*

$$G^{p,w}_{\Phi,\Xi} = C^{p,w}_\Phi D^{p,w}_\Xi,$$

and so $\left(G^{p,w}_{\Phi,\Xi}\right)^* = G^{q,1/w}_{\Xi,\Phi}$. The Gram matrix $G^{p,w}_{\Phi,\Xi}$ is the restriction $G^{\infty,w}_{\Phi,\Xi}\Big|_{\mathcal{H}^p_w}$.

Furthermore $\mathrm{ran}\left(G^{p,w}_{\Phi,\Xi}\right) = \mathrm{ran}\left(C^{p,w}_\Phi\right)$ *and* $\ker\left(G^{p,w}_{\Phi,\Xi}\right) = \ker\left(D^{p,w}_\Xi\right)$. *The Gram matrix $G^{p,w}_{\Phi,\Xi}$ is a bijective mapping from* $\mathrm{ran}\left(C^{p,w}_{\tilde{\Xi}}\right)$ *onto* $\mathrm{ran}\left(C^{p,w}_\Phi\right)$.

For $\Xi = \tilde{\Phi}$ the Gram matrix satisfies

$$G_{\Phi,\tilde{\Phi}} = G_{\tilde{\Phi},\Phi} = G^*_{\Phi,\tilde{\Phi}}, \tag{14}$$

and $G_{\Phi,\tilde{\Phi}}$ is a bounded projection from ℓ^p_w on the range of $C^{p,w}_\Phi$ with kernel $\ker\left(D^{p,w}_\Phi\right)$. *In addition,* $\mathrm{ran}\left(C^{p,w}_\Phi\right) = \mathrm{ran}\left(C^{p,w}_{\tilde{\Phi}}\right)$ *and*

$$\ell^p_w = \mathrm{ran}\left(C^{p,w}_\Phi\right) \oplus \ker\left(D^{p,w}_\Phi\right),$$

Therefore, we have

$$\left\| D^{p,w}_\Phi \right\|_{\ell^p_w \to \mathcal{H}^p_w} = \left\| G_{\tilde{\Phi},\Phi} \right\|_{\ell^p_w \to \ell^p_w} \geq 1. \tag{15}$$

Proof. For $c \in \ell^{00}$ we have

$$\left(G^{p,w}_{\Phi,\Xi} \cdot c\right)_l = \sum_k (G_{\Phi,\Xi})_{l,k}\, c_k = \sum_k \langle \xi_k, \phi_l \rangle\, c_k = (C_\Phi D_\Xi c)_l\,.$$

Therefore $G_{\Phi,\Xi}^{p,w} = C_{\Phi}^{p,w} D_{\Xi}^{p,w}$ on ℓ^{00}. Since both sides are bounded operators on ℓ_w^p ($G_{\Phi,\Xi}^{p,w}$ because $\Phi \sim_{\mathscr{A}} \Xi$), the factorization can be extended from the dense subspace ℓ^{00} to ℓ_w^p for $p < \infty$.

By Lemma 6 $C_{\Phi}^{p,w}$ is one-to-one, and therefore $\mathsf{ker}\left(G_{\Phi,\Xi}^{p,w}\right) = \mathsf{ker}\left(D_{\Xi}^{p,w}\right)$. Likewise, by Lemma 7 $D_{\Xi}^{p,w}$ is onto $\mathscr{H}_w^p$, and therefore $\mathsf{ran}\left(G_{\Phi,\Xi}^{p,w}\right) = \mathsf{ran}\left(C_{\Phi}^{p,w}\right)$. Since $G_{\Phi,\Xi}^{p,w} C_{\tilde{\Xi}}^{p,w} f = C_{\Phi}^{p,w} f$, the Gram matrix $G_{\Phi,\Xi}^{p,w}$ induces a bijective mapping from $\mathsf{ran}\left(C_{\Xi}^{p,w}\right)$ onto $\mathsf{ran}\left(C_{\Phi}^{p,w}\right)$. (Compare to the "frame transformation" in [2].)

If $\Xi = \tilde{\Phi}$, then

$$(G_{\Phi,\tilde{\Phi}})_{k,l} = \left\langle \phi_l, \tilde{\phi}_k \right\rangle_{\mathscr{H}} = \left\langle \phi_l, S^{-1}\phi_k \right\rangle_{\mathscr{H}} = \left\langle S^{-1}\phi_l, \phi_k \right\rangle_{\mathscr{H}} = (G_{\tilde{\Phi},\Phi})_{k,l},$$

and for the entries of the adjoint matrix

$$(G^*_{\Phi,\tilde{\Phi}})_{k,l} = \overline{(G_{\Phi,\tilde{\Phi}})_{l,k}} = \overline{\langle \phi_k, S^{-1}\phi_l \rangle_{\mathscr{H}}} = \left\langle S^{-1}\phi_l, \phi_k \right\rangle_{\mathscr{H}} = (G_{\Phi,\tilde{\Phi}})_{k,l},$$

and (14) is verified.

Since $D_{\tilde{\Phi}}^{p,w} C_{\Phi}^{p,w} = \mathrm{Id}_{\mathscr{H}_w^p}$ by (6), we obtain

$$(G_{\Phi,\tilde{\Phi}}^{p,w})^2 = C_{\Phi}^{p,w} D_{\tilde{\Phi}}^{p,w} C_{\Phi}^{p,w} D_{\tilde{\Phi}}^{p,w} = C_{\Phi}^{p,w} D_{\tilde{\Phi}}^{p,w} = G_{\Phi,\tilde{\Phi}}^{p,w}.$$

Thus $G_{\Phi,\tilde{\Phi}}^{p,w}$ is a projection operator on ℓ_w^p with range $\mathsf{ran}\left(C_{\Phi}^{p,w}\right)$ and kernel $\mathsf{ker}\left(D_{\tilde{\Phi}}^{p,w}\right) = \mathsf{ker}\left(D_{\Phi}^{p,w}\right)$. In particular, $\|G_{\Phi,\tilde{\Phi}}^{p,w}\| \geq 1$ and (15) follows.

Since $G_{\Phi,\tilde{\Phi}}^{p,w} = G_{\tilde{\Phi},\Phi}^{p,w}$ we get $\mathsf{ran}\left(G_{\Phi,\tilde{\Phi}}^{p,w}\right) = \mathsf{ran}\left(G_{\tilde{\Phi},\Phi}^{p,w}\right)$ and by above we also have $\mathsf{ran}\left(C_{\Psi}^{p,w}\right) = \mathsf{ran}\left(C_{\tilde{\Psi}}^{p,w}\right)$. By the projection property, using [16, Theorem III.13.2], $\mathsf{ran}\left(C_{\Phi}^{p,w}\right)$ and $\mathsf{ker}\left(D_{\Phi}^{p,w}\right)$ are therefore complementary subspaces. □

By combining all properties of $C_{\Psi}^{p,w}$ and $D_{\Psi}^{p,w}$, we finally obtain the following list of properties for the frame operator $S_{\Phi}^{p,w} = D_{\Phi}^{p,w} C_{\Phi}^{p,w}$.

Lemma 9. *The frame operator $S_{\Phi}^{p,w} : \mathscr{H}_w^p \to \mathscr{H}_w^p$ is defined by*

$$S_{\Phi}^{p,w} f = \sum_k \langle f, \phi_k \rangle_{\mathscr{H}_w^p, \mathscr{H}_{1/w}^q} \cdot \phi_k = \sum_k \langle f, \phi_k \rangle_{\mathscr{H}_w^\infty, \mathscr{H}_{1/w}^1} \cdot \phi_k$$

with unconditional convergence in $\mathscr{H}_w^p$ for $p < \infty$ and weak-unconditional convergence for $p = \infty$. The frame operator satisfies the identities $S_{\Phi}^{p,w} = D_{\Phi}^{p,w} C_{\Phi}^{p,w}$, $\left(S_{\Phi}^{p,w}\right)^* = S_{\Phi}^{q,1/w}$, and $S_{\Phi}^{p,w} = S_{\Phi}^{\infty,w}\big|_{\mathscr{H}_w^p}$, and is bounded on all $\mathscr{H}^{p,w}$ with operator norm*

$$\left\| S_{\Phi}^{p,w} \right\|_{\mathscr{H}_w^p \to \mathscr{H}_w^p} \leq \left\| G_{\tilde{\Psi},\Phi}^{p,w} \right\|_{\ell_w^p \to \ell_w^p} \cdot \left\| G_{\Phi,\Psi}^{p,w} \right\|_{\ell_w^p \to \ell_w^p}.$$

Furthermore, $S_{\Phi}^{p,w}$ is simultaneously invertible on all $\mathscr{H}_w^p$ with inverse $\left(S_{\Phi}^{p,w}\right)^{-1} = S_{\tilde{\Phi}}^{p,w}$.

4 Galerkin Matrix Representation of Operators with Localized Frames

For a numerical treatment of operator equations one often uses redundant frame representations for the Galerkin discretization. Such discretizations have been formulated for wavelet frames in [68] and for Gabor frames in [46]. The formalism for general (Hilbert space) frames has been introduced in [3].

For localized frames we formally define the relation between operators and matrices as follows.

Definition 4. Let Ψ, Φ, and Ξ be frames with $\Psi \sim_{\mathscr{A}} \Psi$, $\Phi \sim_{\mathscr{A}} \Psi$ and $\Xi \sim_{\mathscr{A}} \Psi$. Let w_1, w_2 be $\mathscr{A}$-admissible weights and $1 \le p_1, p_2 \le \infty$ or $p_1, p_2 = 0$. Let q_1, q_2 be the dual indices defined as usual.

(i) For the bounded linear operator $O : \mathscr{H}^{p_1}_{w_1} \to \mathscr{H}^{p_2}_{w_2}$ define the matrix $\mathscr{M}_{(\Phi,\Xi)}$ by

$$\left(\mathscr{M}_{(\Phi,\Xi)}(O)\right)_{k,l} = \langle O\xi_l, \phi_k \rangle_{\mathscr{H}^{p_2}_{w_2}, \mathscr{H}^{q_2}_{1/w_2}} = \langle O\xi_l, \phi_k \rangle_{\mathscr{H}^{\infty}_{w_2}, \mathscr{H}^{1}_{1/w_2}} .$$

We call $\mathscr{M}_{(\Phi,\Xi)}(O)$ the *(Galerkin) matrix of O with respect to Φ and Ξ.*

(ii) For the matrix M that induces a bounded operator in $\mathfrak{B}\left(\ell^{p_1}_{w_1}, \ell^{p_2}_{w_2}\right)$ define $\mathscr{O}_{(\Phi,\Xi)} : \mathfrak{B}\left(\ell^{p_1}_{w_1}, \ell^{p_2}_{w_2}\right) \to \mathfrak{B}\left(\mathscr{H}^{p_1}_{w_1}, \mathscr{H}^{p_2}_{w_2}\right)$ by

$$\left(\mathscr{O}_{(\Phi,\Xi)}(M)\right) h = \sum_k \Big(\sum_j M_{k,j} \left\langle h, \xi_j \right\rangle \Big) \phi_k, \tag{16}$$

for $h \in \mathscr{H}^{p_1}_{w_1}$. We call $\mathscr{O}_{(\Phi,\Xi)}(M)$ the *operator of M with respect to Φ and Ξ.*

Theorem 5. *Assume that Φ, Ψ, and Ξ are $\mathscr{A}$-localized frames in $\mathscr{H}$ satisfying $\Phi \sim_{\mathscr{A}} \Psi$ and $\Xi \sim_{\mathscr{A}} \Psi$. Let w_1, w_2 be $\mathscr{A}$-admissible weights, let $1 \le p_1, p_2 \le \infty$ or $p_1, p_2 = 0$ with dual indices q_1, q_2.*

(i) If $O \in \mathfrak{B}\left(\mathscr{H}^{p_1}_{w_1}, \mathscr{H}^{p_2}_{w_2}\right)$, then $\mathscr{M}_{(\Phi,\Xi)}(O) \in \mathfrak{B}\left(\ell^{p_1}_{w_1}, \ell^{p_2}_{w_2}\right)$, and we have

$$\left\| \mathscr{M}_{(\Phi,\Xi)}(O) \right\|_{\ell^{p_1}_{w_1} \to \ell^{p_2}_{w_2}} \le \left\| G_{\Phi,\Psi} \right\|_{\ell^{p_2}_{w_2} \to \ell^{p_2}_{w_2}} \left\| G_{\tilde{\Psi},\Xi} \right\|_{\ell^{p_1}_{w_1} \to \ell^{p_1}_{w_1}} \left\| O \right\|_{\mathscr{H}^{p_1}_{w_1} \to \mathscr{H}^{p_2}_{w_2}} \tag{17}$$

Furthermore,

$$\mathscr{M}_{(\Phi,\Xi)}(O) = C_\Phi \circ O \circ D_\Xi \, .$$

(ii) If $M \in \mathfrak{B}\left(\ell^{p_1}_{w_1}, \ell^{p_2}_{w_2}\right)$, then $\mathscr{O}_{(\Phi,\Xi)}(M) \in \mathfrak{B}\left(\mathscr{H}^{p_1}_{w_1}, \mathscr{H}^{p_2}_{w_2}\right)$, and

$$\mathscr{O}_{(\Phi,\Xi)}(M) = D_\Phi \circ M \circ C_\Xi ,$$

and

$$\left\| \mathscr{O}_{(\Phi,\Xi)}(M) \right\|_{\mathscr{H}^{p_2}_{w_2} \to \mathscr{H}^{p_1}_{w_1}} \le \left\| G_{\tilde{\Psi},\Phi} \right\|_{\ell^{p_1}_{w_1} \to \ell^{p_1}_{w_1}} \left\| G_{\Xi,\Psi} \right\|_{\ell^{p_2}_{w_2} \to \ell^{p_2}_{w_2}} \left\| M \right\|_{\ell^{p_2}_{w_2} \to \ell^{p_1}_{w_1}} . \tag{18}$$

Proof. This result follows directly from the results in Section 3. For example, let $c = (c_k) \in \ell^{p_1}_{w_1}$, then

$$\left(\mathscr{M}_{(\Phi,\Xi)}(O)\,c\right)_l = (C_\Phi \circ O \circ D_\Xi c)_l =$$

$$= \left(C_\Phi\left(\sum_{k\in K} c_k O\xi_k\right)\right)_l = \sum_{k\in K} c_k \left\langle O\xi_k, \phi_l\right\rangle_{\mathscr{H}^{p_2}_{w_2},\mathscr{H}^{q_2}_{1/w_2}}.$$

□

Using tensor products and the results in Section 3, it is easy to extend the results in [3]. For $g \in X_1', f \in X_2$ the *tensor product* $f \otimes g$ is defined as the rank-one operator from X_1 to X_2 by $(f \otimes g)(h) = \langle h, g\rangle_{X_1,X_1'} f$. We will use $(f \otimes g)(h) = \langle h, g\rangle_{\mathscr{H}^p_w,\mathscr{H}^q_{1/w}} f$ for $h \in \mathscr{H}^p_w$.

Proposition 4. *Let Ψ and Φ be $\mathscr{A}$-localized frames in $\mathscr{H}$ satisfying $\Phi \sim_{\mathscr{A}} \Psi$ and w_1, w_2 be $\mathscr{A}$-admissible weights and $1 \le p_1, p_2 \le \infty$ or $p_1, p_2 = 0$. Then the factorization*

$$\left(\mathscr{O}_{(\Phi,\Psi)} \circ \mathscr{M}_{(\tilde\Phi,\tilde\Psi)}\right) = \mathsf{id} = \left(\mathscr{O}_{(\tilde\Phi,\tilde\Psi)} \circ \mathscr{M}_{(\Phi,\Psi)}\right),$$

holds for every space of bounded operators $\mathfrak{B}\left(\mathscr{H}^{p_1}_{w_1}, \mathscr{H}^{p_2}_{w_2}\right)$.

Therefore every $O \in \mathfrak{B}\left(\mathscr{H}^{p_1}_{w_1}, \mathscr{H}^{p_2}_{w_2}\right)$ possesses the representation

$$O = \sum_{k,j} \left\langle O\tilde\psi_j, \tilde\phi_k\right\rangle \phi_k \otimes \psi_j = \sum_{k,j} \left\langle O\psi_j, \phi_k\right\rangle \tilde\phi_k \otimes \tilde\psi_j, \tag{19}$$

and both expansions converge unconditionally in the strong operator topology (respectively weak- unconditionally if either $p_1 = \infty$ or $p_2 = \infty$).*

Proof. By Theorem 5 for an O in any $\mathfrak{B}\left(\mathscr{H}^{p_1}_{w_1}, \mathscr{H}^{p_2}_{w_2}\right)$ we have

$$\left(\mathscr{O}_{(\Phi,\Psi)} \circ \mathscr{M}_{(\tilde\Phi,\tilde\Psi)}\right) O = D_\Phi\left(C_{\tilde\Phi}\, O D_{\tilde\Psi}\right) C_\Psi = O, \tag{20}$$

using the reconstruction formulas in Theorem 2 and Lemma 3.

The representation in (19) converges in the strong operator topology by Theorem 5. □

As in the Hilbert space setting [3] we get the following decomposition.

Proposition 5. *Let $\Psi, \Phi,$ and Ξ be $\mathscr{A}$-localized frames in $\mathscr{H}$ satisfying $\Phi \sim_{\mathscr{A}} \Psi, \Xi \sim_{\mathscr{A}} \Psi$. Let w_1, w_2, w_3 be $\mathscr{A}$-admissible weights and $1 \le p_1, p_2 \le \infty$ or $p_1, p_2 = 0$. Then for $O_1 : \mathscr{H}^{p_1}_{w_1} \to \mathscr{H}^{p_2}_{w_2}$ and $O_2 : \mathscr{H}^{p_3}_{w_3} \to \mathscr{H}^{p_1}_{w_1}$, we have*

$$\mathscr{M}_{(\Phi,\Psi)}(O_1 \circ O_2) = \mathscr{M}_{(\Phi,\Xi)}(O_1) \circ \mathscr{M}_{(\tilde\Xi,\Psi)}(O_2).$$

Proof. The statement follows from the factorization

$$\mathscr{M}_{(\Phi,\Psi)}(O_1 \circ O_2) = C_\Phi \, O_1 \, O_2 \, D_\Psi =$$

$$= C_\Phi \, O_1 \, D_\Xi \, C_{\tilde{\Xi}} \, O_2 \, D_\Psi = \mathscr{M}_{(\Phi,\Xi)}(O_1) \circ \mathscr{M}_{(\tilde{\Xi},\Psi)}(O_2) \,.$$

□

Then we get an extension of results in [4, 5] to coorbit spaces.

Lemma 10. *Let Ψ and Φ be $\mathscr{A}$-localized sequence in $\mathscr{H}$ satisfying $\Phi \sim_{\mathscr{A}} \Psi$, w_1, w_2 be $\mathscr{A}$-admissible weights and $1 \leq p_1, p_2 \leq \infty$ or $p_1, p_2 = 0$. Let O be a linear operator from $\mathscr{H}^{00}$ into $\mathscr{H}^{\infty}_{w_2}$. Then*

$$O \in \mathfrak{B}\big(\mathscr{H}^{p_1}_{w_1}, \mathscr{H}^{p_2}_{w_2}\big) \iff \mathscr{M}_{(\Psi,\Phi)}(O) \in \mathfrak{B}\big(\ell^{p_1}_{w_1}, \ell^{p_2}_{w_2}\big),$$

Proof. The implication $\Rightarrow$ is stated in Theorem 5(i).

For the converse, let O be a linear operator from $\mathscr{H}^{00}$ to $\mathscr{H}^{p_2}_{w_2}$ such that $\mathscr{M}_{(\Psi,\Phi)}(O) = C^{p_2,w_2}_{\Phi} \circ O \circ D^{p_1,w_1}_{\Psi}$ is bounded. Then $\mathsf{ran}\,(D_\Psi) \subseteq \mathsf{dom}\,(O)$ and therefore O is defined everywhere. Since

$$O = D^{p_2,w_2}_{\tilde{\Phi}} \circ C^{p_2,w_2}_{\Phi} \circ O \circ D^{p_1,w_1}_{\Psi} \circ C^{p_1,w_1}_{\tilde{\Psi}},$$

the operator O is also bounded. □

4.1 Characterization of Operator Classes

Combining the matrix representation of an operator with the well-known characterizations of boundedness of operators between ℓ^p-spaces [60] we obtain criteria for the boundedness of operators between certain coorbit spaces.

For the description we recall the following norms on infinite matrices. Considering an index set $K = L \times N$, we can define discrete mixed norm spaces [42], i.e.,

$$\ell^{p_1,p_2}_w = \left\{ M = (M_{l,n})_{l\in L, n\in N} \;\middle|\; \|M\|_{\ell^{p_1,p_2}_w} := \left(\sum_l \left(w_{l,n}\,|M_{l,n}|^{p_2}\right)^{p_1/p_2} \right)^{1/p_2} < \infty \right\}.$$

In particular we consider weights $w = w^{(1)} \otimes w^{(2)}$ with $w_{k,l} = w^{(1)}_k \cdot w^{(2)}_l$.

Proposition 6. *Let Ψ and Φ be $\mathscr{A}$-localized sequence in $\mathscr{H}$ satisfying $\Phi \sim_{\mathscr{A}} \Psi$, $w^{(i)}$ be $\mathscr{A}$-admissible weights and $1 \leq p_i < \infty$ for $i = 1, 2$. Let O be a linear operator from $\mathscr{H}^{00}$ to $\mathscr{H}^{\infty}_{w_2}$, and $M = \mathscr{M}_{(\Phi,\Xi)}(O)$. Then*

$$O \in \mathfrak{B}\left(\mathscr{H}^{\infty}_{w^{(1)}}, \mathscr{H}^{\infty}_{w^{(2)}}\right) \iff O \in \mathfrak{B}\left(\mathscr{H}^{0}_{w^{(1)}}, \mathscr{H}^{\infty}_{w^{(2)}}\right)$$
$$\iff \sup_k \sum_l \left| w_k^{(2)} \cdot \frac{1}{w_l^{(1)}} \langle O\psi_l, \phi_k \rangle \right| < \infty, \tag{21}$$
$$\iff M \in \ell^{\infty,1}_{1/w^{(2)} \otimes w^{(1)}}.$$

$$O \in \mathfrak{B}\left(\mathscr{H}^{\infty}_{w^{(1)}}, \mathscr{H}^{0}_{w^{(2)}}\right) \iff \lim_k \sum_l \left| w_k^{(2)} \cdot \frac{1}{w_l^{(1)}} \langle O\psi_l, \phi_k \rangle \right| = 0. \tag{22}$$

$$O \in \mathfrak{B}\left(\mathscr{H}^{1}_{w^{(1)}}, \mathscr{H}^{\infty}_{w^{(2)}}\right) \iff \sup_{k,l} \left| w_k^{(2)} \cdot \langle O\psi_l, \phi_k \rangle \cdot \frac{1}{w_l^{(1)}} \right| < \infty,$$
$$\iff M \in \ell^{\infty,\infty}_{1/w^{(2)} \otimes w^{(1)}}. \tag{23}$$

$$O \in \mathfrak{B}\left(\mathscr{H}^{1}_{w^{(1)}}, \mathscr{H}^{p}_{w^{(2)}}\right) \iff \sup_l \sum_k \left| w_k^{(2)} \cdot \frac{1}{w_l^{(1)}} \langle O\psi_l, \phi_k \rangle \right|^p < \infty,$$
$$\iff M^* \in \ell^{p,\infty}_{1/w^{(2)} \otimes w^{(1)}} \tag{24}$$

$$O \in \mathfrak{B}\left(\mathscr{H}^{\infty}_{w^{(1)}}, \mathscr{H}^{1}_{w^{(2)}}\right) \iff \sup_{E \text{ finite}} \sum_l \left| \sum_{k \in E} w_k^{(2)} \cdot \frac{1}{w_l^{(1)}} \langle O\psi_l, \phi_k \rangle \right| < \infty. \tag{25}$$

$$O \in \mathfrak{B}\left(\mathscr{H}^{2}_{w^{(1)}}, \mathscr{H}^{2}_{w^{(2)}}\right) \iff \begin{cases} \text{For } \breve{M}_{k,l} = w_k^{(2)} \cdot \frac{1}{w_l^{(1)}} \langle O\psi_l, \phi_k \rangle \text{ we have:} \\ \qquad \sum_l \left| \sum_{k \in E} \breve{M}_{k,l} \right|^2 < \infty, \\ \left(\breve{M}^*\breve{M}\right)^n \text{ is defined for all } n = 1, 2, \dots \\ \sup_n \sup_i \left[\left(\left(\breve{M}^*\breve{M}\right)^n \right)_{i,i} \right]^{1/n} = K < \infty. \end{cases} \tag{26}$$

Proof. The conditions on the matrix M are variations of the well-known Schur test. For instance, M is bounded from ℓ^∞ to ℓ^∞, if and only if the row sums are uniformly bounded, i.e., $\sup_k \sum_l |M_{k,l}| < \infty$. A convenient reference for Schur's test is [60].

To also include weights, we proceed as follows. Let $D_j c = (w_k^{(j)} c_k)_{k \in K}$ be the multiplication operator with weight $w^{(j)}, j = 1, 2$. Then D_j is an isometric isomorphism from $\ell^p_{w^{(j)}}$ onto ℓ^p.

Therefore a matrix M is bounded from $\ell^{p_1}_{w^{(1)}}$ into $\ell^{p_2}_{w^{(2)}}$, if and only if $\breve{M} = D_2 M D_1^{-1}$ is bounded from ℓ^{p_1} into ℓ^{p_2}. We now apply the boundedness characterizations in [60] to $D_2 \mathscr{M}_{(\Psi,\Phi)}(O) D_1^{-1}$. For example, Lemma 10 says that $O \in \mathfrak{B}\left(\mathscr{H}^{\infty}_{w_1}, \mathscr{H}^{\infty}_{w_2}\right) \iff \mathscr{M}_{(\Psi,\Phi)}(O) \in \mathfrak{B}\left(\ell^{\infty}_{w_1}, \ell^{\infty}_{w_2}\right)$, which in turn is equivalent to saying that $\breve{M} = D_2 \mathscr{M}_{(\Psi,\Phi)}(O) D_1^{-1} \in \mathfrak{B}\left(\ell^\infty, \ell^\infty\right)$. Since $\breve{M}_{k,l} = w_k^{(2)} \cdot \frac{1}{w_l^{(1)}} \langle O\psi_l, \phi_k \rangle$, condition (21) follows from [60, Theorem 2.6]. The other characterizations follow in the same way from [60, Theorem 2.12,2.13(a),2.13(b),2.14] and [22], respectively. □

In concrete applications one uses only the sufficient conditions for boundedness and checks that the matrix $\mathcal{M}_{(\Psi,\Phi)}(O)$ satisfies the conditions of Schur's test.

Corollary 3. *Let Ψ and Φ be $\mathcal{A}$-localized frame in $\mathcal{H}$ satisfying $\Phi \sim_{\mathcal{A}} \Psi$, and $1 \le p \le \infty$. Let O be a linear operator from $\mathcal{H}^{00}$ to $\mathcal{H}_w^{\infty}$. If $\sup_k \sum_l |\langle O\psi_l, \phi_k\rangle| < \infty$, and $\sup_l \sum_k |\langle O\psi_l, \phi_k\rangle| < \infty$, then $O \in \mathfrak{B}\left(\mathcal{H}^p, \mathcal{H}^p\right)$.*

For an example of how these abstract results are applied in analysis, we refer to the investigation of the boundedness of pseudodifferential operators with the help of Gabor frames in [46, 51].

For further reference, we remark that the results in Section 4 do not use the full power of intrinsic localization, but remain true under weaker assumptions. In fact, we have only used the norm equivalences for the analysis operators of two frames Φ and Ψ and their duals $\tilde{\Phi}$ and $\tilde{\Psi}$:

$$\|C_\Psi f\|_{\ell^p_w} \asymp \|C_{\tilde{\Psi}} f\|_{\ell^p_w} \asymp \|C_\Phi f\|_{\ell^p_w} \asymp \|C_{\tilde{\Phi}} f\|_{\ell^p_w} \tag{27}$$

for $f \in \mathcal{H}^{00}$. This is all that is needed to define unambiguously a coorbit space $\mathcal{H}_w^p$.

If (27) holds, then all statements of this section, specifically Theorem 5 and Propositions 4–6 remain true.

One of our main points is that these norm equivalences (27) always hold for $\mathcal{A}$-localized frames, as we have seen in Section 3. In addition, (27) also hold for wavelet frames (with sufficiently many vanishing moments and sufficient decay) with the Besov spaces and Sobolev spaces as the corresponding coorbit spaces. In fact, one of the main motivations for wavelets was the investigation of singular integral operators, see [37, 62]. However, for wavelet frames the norm equivalences (27) require different arguments that are not covered by our theory of localized frames.

4.2 Invertibility

For the invertibility we can show, as in the Hilbert space setting [5]:

Lemma 11. *Let Φ and Ψ be $\mathcal{A}$-localized frames for $\mathcal{H}$ satisfying $\Phi \sim_{\mathcal{A}} \Psi$ and $1 \le p \le \infty$. Let $O : \mathcal{H}_{w_1}^p \to \mathcal{H}_{w_2}^p$ be a bounded, linear operator.*

Then O is bijective, if and only if $\mathcal{M}_{(\Phi,\Psi)}(O)$ is bijective as operator from $\operatorname{ran}\left(C_\Psi^{p,w_1}\right)$ to $\operatorname{ran}\left(C_\Psi^{p,w_2}\right)$.

In this case the matrix associated to the inverse is given by

$$\left(\mathcal{M}_{(\Phi,\Psi)}(O)\right)^\dagger := \left(\mathcal{M}_{(\Phi,\Psi)}(O)_{|\operatorname{ran}(C_\Psi)}\right)^{-1} = \mathcal{M}_{(\tilde{\Psi},\tilde{\Phi})}\left(O^{-1}\right).$$

Proof. By Theorem 4, $C_\Psi^{p,w}$ is a bijection from $\mathcal{H}_w^p$ onto $\operatorname{ran}\left(C_\Psi^{p,w}\right)$, and $D_\Psi^{p,w}$ a bijection from $\operatorname{ran}\left(C_\Psi^{p,w}\right)$ onto $\mathcal{H}_w^p$, where $w = w_1$ or w_2.

Therefore O is bijective if and only if $\mathcal{M}_{(\Phi,\Psi)}(O)$ is bijective from $\operatorname{ran}\left(C_{\Psi}^{p_1,w_1}\right)$ to $\operatorname{ran}\left(C_{\Psi}^{p_2,w_2}\right)$.

Furthermore

$$\mathcal{M}_{(\tilde{\Psi},\tilde{\Phi})}\left(O^{-1}\right)\mathcal{M}_{(\Phi,\Psi)}(O) = C_{\tilde{\Psi}}^{p_1,w_1} \circ O^{-1} \circ D_{\tilde{\Phi}}^{p_2,w_2} C_{\Phi}^{p_2,w_2} \circ O \circ D_{\Psi}^{p_1,w_1} =$$

$$= C_{\tilde{\Psi}}^{p_1,w_1} \circ D_{\Psi}^{p_1,w_1} = G_{\tilde{\Psi},\Psi},$$

and therefore the projection on $\operatorname{ran}\left(C_{\Psi}^{p_1,w_1}\right)$. □

Remark 2. Note that for $p_1 \neq p_2$, there does not exist a bijective operator $O : \mathcal{H}_{w_1}^{p_1} \to \mathcal{H}_{w_2}^{p_2}$ by Theorem 3.

The condition number of a matrix (or an operator) plays an important role in numerical analysis [59] and is defined by $\kappa(M) = \|M\| \cdot \left\|M^{-1}\right\|$. For matrices with non-zero kernel we can define the generalized condition number [11] by $\kappa^{\dagger}(M) = \|M\| \cdot \left\|M^{\dagger}\right\|$. By using Lemma 11 and Theorem 5 it is straightforward to show

$$\kappa^{\dagger}\left(\mathcal{M}_{(\Phi,\Psi)}(O)\right) = \kappa^{\dagger}\left(G_{\Phi,\Psi}\right) \cdot \kappa^{\dagger}\left(G_{\tilde{\Psi},\Psi}\right) \cdot \kappa^{\dagger}(O).$$

Theorem 6. *Let Ψ be an $\mathcal{A}$-localized frame for $\mathcal{H}$ and w an $\mathcal{A}$-admissible weight. Assume that $O : \mathcal{H} \to \mathcal{H}$ is invertible and that $\mathcal{M}_{(\Psi,\tilde{\Psi})}(O) \in \mathcal{A}$. Then O is invertible simultaneously on all coorbit spaces $\mathcal{H}_w^p$, $1 \leq p \leq \infty$.*

Proof. By Lemma 11 the matrix of O^{-1} is given by $\mathcal{M}_{(\tilde{\Psi},\Psi)}(O^{-1}) = \left(\mathcal{M}_{(\Psi,\tilde{\Psi})}(O)\right)^{\dagger}$. Since $\mathcal{A}$ is closed with respect to taking a pseudo-inverse and $\mathcal{M}_{(\Psi,\tilde{\Psi})}(O) \in \mathcal{A}$, it follows that also $\mathcal{M}_{(\tilde{\Psi},\Psi)}(O^{-1}) \in \mathcal{A} \subseteq \mathfrak{B}\left(\ell_w^p, \ell_w^p\right)$. By Lemma 10 O^{-1} is therefore bounded on $\mathcal{H}_w^p$ for $1 \leq p \leq \infty$. □

This result is special for intrinsically localized frames and fails for wavelet frames (Fig. 1).

5 Outlook

This manuscript was motivated by many discussions of the first author with applied scientists who work on the numerical solution of integral equations. In applications in acoustics, the solutions of the Helmholtz equation are of particular importance, see, e.g., [55], they are used for example, for the numerical estimation of head-related transfer functions [57, 72].

In general the problem of solving an integral equation can be seen as solving a linear equation

$$O \cdot f = g, \tag{28}$$

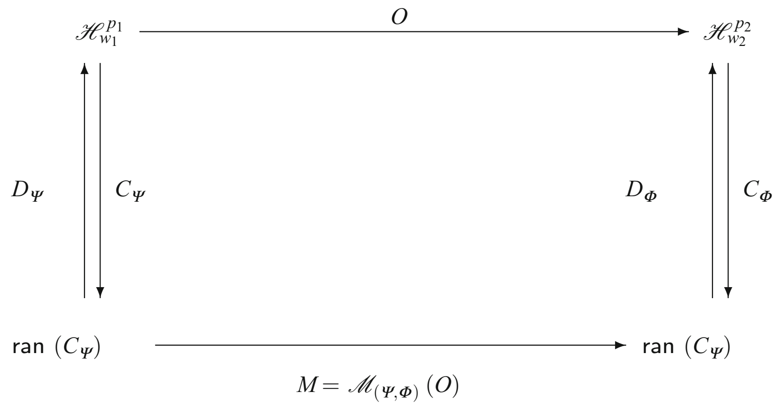

Fig. 1 All operators in the diagram are bijective, if M or equivalently O, is bijective.

where the operator O models the physical system and the right-hand side is given and the solution f is to be determined. For the important example of sound propagation, the right-hand side g is often called the load vector. It is usually assumed that f, g, are in some appropriate function spaces.

For the numerical treatment of such operator equations one needs a reduction to a discrete formulation. This is often done with a so-called Galerkin scheme [66]. As a first step, either the boundary of the considered space or the whole space itself are separated in finite patches or finite volume elements. This leads to the Boundary Element Method [38] or the Finite Element Method [7]. In the Galerkin scheme one first finds the matrix $M = \mathscr{M}_{(\Psi,\Psi)}(0)$ corresponding to the operator O with respect to a given basis or frame Ψ. Instead of solving the operator equation $Of = g$, one converts (28) into a matrix equation as follows:

$$Of = g \Longleftrightarrow \sum_l \langle f, \tilde{\phi}_l\rangle O\phi_l = g \Longleftrightarrow \sum_l \langle f, \tilde{\phi}_l\rangle \langle O\phi_l, \phi_k\rangle = \langle g, \phi_k\rangle \Longleftrightarrow$$
$$\mathscr{M}_{(\Phi,\Phi)}(O) \cdot C_{\tilde{\Phi}} f = C_\Phi g. \tag{29}$$

Here $M = \mathscr{M}^{(\Phi,\tilde{\Phi})}(O)$ is called the *system matrix* or *stiffness matrix*.

In the setting of localized frames, the natural function spaces are the Banach spaces $\mathscr{H}_w^p$. By the results above this is equivalent to the vectors in the matrix equation (29) to be contained in some ℓ_w^p-space.

In finite and boundary element approach the system Φ is usually a spline-like basis [38]. Recently wavelet bases [26], but also frames have been applied, e.g. in [54, 68]. Currently the potential of other frames is investigated for solving operator equations in acoustics, such as α-modulation frames [67]. Note, however, that neither wavelet frames nor α-modulation frames are localized in the sense of Definition 1. As mentioned above, as long as (27) is fulfilled most results of Section 4 can still be applied.

To use a numerical solver, it is necessary to perform a further reduction to a finite-dimensional matrix equation. This means that we have to find a good finite dimensional approximation of $\mathscr{M}_{(\Phi,\Phi)}(O)$. This is done by restricting $\mathscr{M}_{(\Phi,\Phi)}(O)$ to a suitable finite-dimensional subspace. Specifically, let $\{P_n\}$ be a bounded sequence of finite-rank orthogonal projections in $\mathfrak{B}(\mathscr{H})$ with the property that $P_n x \to x$ for all $x \in \mathscr{H}$ and $n \to \infty$. Assume that $A \in \mathfrak{B}(\mathscr{H})$ is invertible. Consider

$$P_n A P_n x = P_n y \tag{30}$$

and solve for $x_n = (P_n A P_n)^{-1} P_n y$. This is the classical *projection method* [40]. The projection method for A is said to *converge*, if for all $y \in \mathscr{H}$ there exists a unique solution x_n to (30) with $x_n \to A^{-1}y$. This is the case [40] if and only if the matrices $A_n = P_n A P_n$ have uniformly bounded inverses, i.e. $\sup\limits_{n \geq N} \left\| A_n^{-1} \right\| < \infty$. In particular, if $\|I - A\| < 1$, then the method converges. The special case when P_n is the orthogonal projection on the first n coordinates in ℓ^2 is called the *finite section* method. In numerical analysis, this approximation scheme is often called the Galerkin scheme.

If $|\langle Ax, x\rangle| \geq c \cdot \|x\|^2$, then A is invertible, $\left\|A^{-1}\right\| \leq c^{-1}$ and it is easy to see that the projection method converges.

A convergence analysis of the finite section method in weighted ℓ^p-spaces is carried out in [52]. The methods are closely related to the methods used for the analysis of localized frames.

The projection method can be combined with frames in several ways:

- The "naive approach": assume that $K = \mathbb{Z}$ and choose $M_{N k,l} = \mathscr{M}_{(\Phi,\Phi)}(O)_{k,l}$ for $|k|, |l| \leq N$. This corresponds to a finite section method.
- Subspace selection: Choose a sequence K_N of finite subsets of K with the following properties:

  (i) $K_i \subseteq K_j$ for $i \leq j$ and $\bigcup_{i=1}^{\infty} K_i = K$.
  (ii) The space $V_N := \operatorname{span}\{\psi_k | k \in K_N\}$ has dimension N.

  For a frame Ψ it may happen that $\dim V_N < \operatorname{card}(K_N)$, but the set $\Psi^{(N)} := \{\psi_k | k \in K_N\}$ is always a frame for V_N. (This need not be the case when $\dim V_N = \infty$). For the numerical treatment the condition numbers of the transforms, i.e. the quotients of the frame bounds, has to be controlled. Therefore we consider $\Psi^{(N)} = \{\psi_k \,| k \in K_N\}$ being a frame for V_N with bounds C,D independent of N. This is called a subframe in [54]. We denote the canonical dual on V_N by $\tilde{\Psi}^{(N)} := \left\{\tilde{\psi}_k^{(N)}\right\}$. Then use the projection $P_N f = \sum\limits_{k \in K_N} \langle f, \psi_k\rangle\, \tilde{\psi}_k^{(N)} = \sum\limits_{k \in K_N} \left\langle f, \tilde{\psi}_k^{(N)} \right\rangle \psi_k$ and solve (30). Since $\mathsf{ran}\left(\mathscr{M}^{(\Phi,\Phi)}(O)\right) \subseteq \mathsf{ran}\,(C_\Phi)$, this set-up leads exactly to the formulation in (29).

  In concrete applications it is a non-trivial problem to find index sets such that the approximation method converges and at the same time is numerically efficient. For wavelet frames this problem can be tackled with a multi-resolution approach with a basis property on each scale, see, e.g., [54].

We note that the matrix M_N cannot have full rank whenever the frame $\{\psi_k | k \in K_N\}$ is redundant for V_N. By Lemma 11 the equation (29) still has a unique solution, although the matrix is not invertible. For the efficient solution of (29), even for frames, one can apply Krylov subspace methods, such as the conjugate gradient method [54]. Other possible methods include versions of Richardson iterations [23] or steepest descent methods [24].

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Computing the Distance Between Frames and Between Subspaces of a Hilbert Space

Travis Bemrose, Peter G. Casazza, Desai Cheng, John Haas, and Hanh Van Nguyen

1 Introduction

A fundamental notion in Hilbert space frame theory is to compute the distance between frames and the distance between subspaces of a Hilbert space. One space these problems arise is with a number of algorithms which serve to approximate frame operators or inverse operators [5]. There are six standard distance functions used in frame theory. In this paper we will establish the exact relationship between all of these distance functions.

A standard result in the field [2] is that the closest Parseval frame (see section 2 for the definitions) to a frame $\Phi = \{\phi_i\}_{i=1}^N$ for H_M is $\{S^{-1/2}\phi_i\}_{i=1}^N$, where S is the frame operator of Φ. Also well known is the closest equal norm frame to a given frame. But the algorithms which carry out these constructions cancel each other out. As a result, a longstanding problem in the field involving distances is the Paulsen Problem.

Paulsen Problem Given a frame Φ, find the closest equal norm tight frame to Φ.

This is not the usual statement of the Paulsen Problem which originally asked: Given a nearly equal norm, nearly Parseval frame, find the closest equal norm Parseval frame. However, the above version is the general result we would like to have. The Paulsen Problem has proved exceptionally difficult because it mixes a

T. Bemrose (✉) • P.G. Casazza • D. Cheng • J. Haas
Mathematics Department, University of Missouri, Columbia, MO 65211, USA
e-mail: tbemrose@gmail.com; casazzap@missouri.edu; chengdesai@yahoo.com; haasji@missouri.edu

H. Van Nguyen
Mathematics Department, University of Alabama, Tuscaloosa, AL 35487, USA
e-mail: hvnguyen@ua.edu

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geometric condition with an algebraic condition, which is often a difficult mixture. There are partial results on the Paulsen Problem [1, 4], but we believe the distances given to date are far from the minimal distance.

In this paper, we will make a detailed study of six major distance measurements for frames and subspaces (see Section 2.2 for the definitions): The **frame distance** d_F, the **Gramian distance** d_G, the **isomorphy distance** d_I, the **subspace distance** d_S, the **chordal distance** d_C, and the **Parseval equivalence class distance** d_P. We will establish the best relationships between these distance measurements and provide numerous equivalent formulations for computing the distances. One very useful result is that the chordal distance between two subspaces is given by two orthonormal bases – one from each subspace – so that the orthonormal bases are biorthogonal.

The main goal in this paper is to establish the relationship between six of the standard distance functions in frame theory so that when one is faced with a distance problem, they may find that an equivalent form is much easier to compute. Each of these distances has certain advantages for given situations. The standard distance is immediate when given two frames Φ and Ψ. However, if we want to compute the minimal distance between all frames unitarily isomorphic to Φ and Ψ, this is an extremely difficult computation. In this case, the Gramian distance has the advantage that unitarily isomorphic frames have the same gram matrix so the minimal distance is now immediate. Also, chordal distance between subspaces has the advantage that there is a simple algorithm for computing chordal distance. Also, chordal distance has a simple representation in terms of projecting an orthonormal basis in the space onto the subspaces which in turn can be computed quickly from the trace of the project of the projections.

The paper is organized as follows. Section 2.1 is an introduction to the basic notions from frame theory and Section 2.2 gives the distance functions we will be working with. In Section 3, we show how the distances between frames relate to the subspace distances in a higher dimensional Hilbert space, and in Section 4, we show how to compute the closest pairs of frames corresponding to fixed pairs of subspaces. Section 5 is a detailed study of the subspace distance and Section 6 is a detailed study of the chordal distance. Finally, in Section 7, we relate all of the distances discussed in this paper.

2 Preliminaries

In this section, we introduce the basics of frame theory and define several notions of distance that we consider in this paper.

2.1 Finite Frames

We write H_M for any real M-dimensional Hilbert space. A family of vectors $\Phi = \{\phi_i\}_{i=1}^N$ in H_M is a **frame** with **lower and upper frame bounds A and B** for real numbers $0 < A \leq B$ if

$$A \|x\|^2 \leq \sum_{i=1}^{N} |\langle x, \phi_i \rangle|^2 \leq B \|x\|^2, \text{ for all } x \in H_M.$$

The frame is **A-tight** if $A = B$ and it is **Parseval** if $A = B = 1$. It is **equal norm** if $\|\phi_i\| = \|\phi_j\|$ for all $1 \leq i, j \leq N$ and it is **unit norm** if every frame vector is a unit vector.

We reserve the letter M to refer to the dimension of the Hilbert space that a frame spans, and we use the letter N to refer to a frame's cardinality. An (N, M) **frame** is thus a frame for H_M consisting of N vectors.

With respect to a fixed orthonormal basis for H_M, the **synthesis matrix** of an (N, M) frame is

$$\Phi = \left[\phi_1 \; \phi_2 \cdots \phi_N\right],$$

the $M \times N$ matrix with columns given by the coordinate representations of the frame vectors. Because it is convenient to identify a frame with its synthesis matrix, we proceed with the tacit understanding that Φ is both a matrix and a set of vectors.

The **analysis matrix** of an (N, M) frame Φ is the $N \times M$ matrix Φ^T and is determined by the mapping

$$\Phi^T x = \sum_{i=1}^{N} \langle x, \phi_i \rangle \, e_i \in H_N,$$

where $\{e_i\}_{i=1}^N$ is the natural basis for $H_N = \ell_2(N)$. The **frame operator matrix** is the $M \times M$ matrix $S = S_\Phi = \Phi\Phi^T$ and is given by

$$Sx = \sum_{i=1}^{N} \langle x, \phi_i \rangle \, \phi_i,$$

for $x \in H_M$. Now,

$$\langle Sx, \phi \rangle = \sum_{i=1}^{N} |\langle x, \phi_i \rangle|^2,$$

so it follows that S is a positive, self-adjoint, invertible matrix satisfying $A \cdot \mathrm{I} \leq S \leq B \cdot \mathrm{I}$. A frame is A-tight if and only if $S = A \cdot \mathrm{I}$, so Φ is a Parseval frame if and only if $S = \mathrm{I}$, in which case

$$\|x\|^2 = \sum_{i=1}^{N} |\langle x, \phi_i \rangle|^2$$

for $x \in H_M$.

The space of $M \times M$ orthogonal matrices is called the **orthogonal group**, denoted $\mathscr{O}_M$, which is a group under normal matrix multiplication. Two (N, M) frames Φ and Ψ are **(unitarily) isomorphic** if there exists $U \in \mathscr{O}_M$ such that $\Psi = U\Phi$, in which case we write $\Phi \simeq \Psi$.

The **Gram matrix** of an (N, M) frame $\Phi = \left[\phi_1\ \phi_2\ \dots\ \phi_N\right]$ is the $N \times N$ matrix

$$G = G_\Phi = \Phi^T \Phi = \left[\langle \phi_j, \phi_i \rangle\right]_{i,j=1}^{N}.$$

If Φ and Ψ are (N, M) frames, then a straightforward application of the singular value decomposition shows that $G_\Phi = \Phi^T \Phi = \Psi^T \Psi = G_\Psi$ if and only if $\Phi \simeq \Psi$ [2], so we sometimes identify unitarily isomorphic frames by their Gram matrix.

For further details on the fundamentals of frame theory, see [2, 3, 5**?**].

2.2 *Distances*

Given real $M \times N$ matrices X and Y, recall that the Hilbert Schmidt trace inner product,

$$\langle X, Y \rangle_{H.S.} = \operatorname{tr}(XY^T),$$

induces the norm $\|\cdot\|_{H.S.}$ on $\mathbb{R}^{M \times N}$, which is in fact the Frobenius norm. This leads to the following well-known fact.

Proposition 1. *If X is an $M \times N$ matrix with column vectors $\{\phi_i\}_{i=1}^N$ and row vectors $\{x_i\}_{i=1}^M$, then*

$$\|X\|_{H.S.}^2 = \sum_{i=1}^{M} \|x_i\|^2 = \sum_{i=1}^{N} \|\phi_i\|^2.$$

With this in mind, we define our notions of distance in terms of the metric induced by the Hilbert Schmidt norm. The fundamental metric on the space of frames is the *frame distance*.

Definition 1. If $\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_i\}_{i=1}^N$ are (N, M) frames, the **frame distance** between them is

$$d_F(\Phi, \Psi) = \|\Phi - \Psi\|_{H.S.}.$$

Next, we define two notions of distance which are pseudometrics on the space of (N, M) frames. Because the Gram matrix encodes much of the geometric information of a frame, we define a distance in terms of the Gram matrices of frames.

Definition 2. If $\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_i\}_{i=1}^N$ are (N, M) frames, the **Gramian distance** between them is

$$d_G(\Phi, \Psi) = \|G_\Phi - G_\Psi\|_{H.S.},$$

where $G_\Phi = \Phi^T \Phi$ and $G_\Psi = \Psi^T \Psi$ are the Gram matrices of Φ and Ψ.

This is a pseudometric because if $\Phi \simeq \Psi$, then $G_\Phi = G_\Psi$, which means $d_G(\Phi, \Psi) = 0$. We also define a distance in terms of unitary isomorphy of frames.

Definition 3. If $\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_i\}_{i=1}^N$ are (N, M) frames, the **isomorphy distance** between them is

$$d_I(\Phi, \Psi) = \inf_{\substack{\Phi' \simeq \Phi \\ \Psi' \simeq \Psi}} \left\|\Phi' - \Psi'\right\|_{H.S.}.$$

As with the Gramian distance, this is a pseudometric because $d_I(\Phi, \Psi) = 0$ whenever $\Phi \simeq \Psi$.

Next, we consider notions of distance between M-dimensional subspaces of H_N. The **Grassmannian space** $\mathscr{G}_{N,M}$ is the space of M-dimensional subspaces of H_N. If W is a M-dimensional subspace of H_N, we let $\mathscr{B}(W)$ denote the family of orthonormal bases of W. Similarly to how it is convenient to identify a frame by its synthesis matrix, it is also convenient to identify an orthonormal basis $X = \{x_i\}_{i=1}^M \in \mathscr{B}(W)$ with the $M \times N$ matrix

$$X = \begin{bmatrix} x_1^T \\ \vdots \\ x_M^T \end{bmatrix},$$

where the rows are the coordinate representations of the transposed elements of X with respect to some fixed basis for H_N. Thus, we think of an element $X \in \mathscr{B}(W)$ both as an $M \times N$ matrix and as a set of row vectors.

Definition 4. If $W_1, W_2 \in \mathscr{G}_{N,M}$, then the **subspace distance** between W_1 and W_2 is

$$d_S(W_1, W_2) = \inf_{\substack{X \in \mathscr{B}(W_1) \\ Y \in \mathscr{B}(W_2)}} \|X - Y\|_{H.S.}.$$

We sometimes identify an M-dimensional subspace W of H_N by its uniquely corresponding **orthogonal projection**, the symmetric, idempotent rank M, $N \times N$ matrix P with the property that

$$Px = x \text{ for } x \in W \quad \text{and} \quad Py = 0 \text{ for } y \in W^\perp.$$

By the Parseval identity, $X = \{x_i\}_{i=1}^{M} \in \mathscr{B}(W)$ if and only if the orthogonal projection P of H_N onto W is given by

$$P = \sum_{i=1}^{M} x_i x_i^T = X^T X.$$

Finally, we recall the *chordal distance*.

Definition 5. Let $W_1, W_2 \in \mathscr{G}(N, M)$. A pair (X, Y), where $X = \{x_i\}_{i=1}^{M} \in \mathscr{B}(W_1)$ and $Y = \{y_i\}_{i=1}^{M} \in \mathscr{B}(W_2)$, is said to be a pair of **principal bases** between W_1 and W_2 if

$$\max_{\substack{x \in W_1 \\ \|x\|=1}} \max_{\substack{y \in W_2 \\ \|y\|=1}} \langle x, y \rangle = \langle x_1, y_1 \rangle$$

and, for $i > 1$, we have

$$\max_{\substack{x \in W_1^i \\ \|x\|=1}} \max_{\substack{y \in W_2^i \\ \|y\|=1}} \langle x, y \rangle = \langle x_i, y_i \rangle,$$

where $W_1^i = W_1 \cap \text{span}(x_1, \ldots, x_{i-1})^{\perp}$ and $W_2^i = W_2 \cap \text{span}(y_1, \ldots, y_{i-1})^{\perp}$. The **principal angles** $\{\theta_i\}_{i=1}^{M} \subset \left[0, \frac{\pi}{2}\right]$ are then defined by

$$\cos\theta_i = \langle x_i, y_i \rangle,$$

for each $i \in \{1, \ldots, M\}$. The **chordal distance** between W_1 and W_2 is then

$$d_C(W_1, W_2) = \sqrt{\sum_{i=1}^{M} \sin^2(\theta_i)}.$$

3 From Frames to Subspaces

In this section, we see that there is a natural relationship between the distance between frames and the distance between the ranges of their analysis matrices. We emphasize this ideas for the case of Parseval frames.

If Φ is an (N, M) frame with transposed row vectors $\{x_i\}_{i=1}^{M}$, then the spanning property of Φ means that $\{x_i\}_{i=1}^{M}$ forms a linearly independent subset of H_N. We call the subspace $W = \text{span}\{x_i\}_{i=1}^{M}$ the **analysis subspace** of Φ, and to facilitate notation, we write $\Gamma(\Phi) = W$ to denote the analysis subspace generated by Φ. To justify this name, we recall a standard result [2, 3, 5].

Theorem 1. *If Φ is an (N, M) frame with analysis subspace $W = \Gamma(\Phi)$, then W is the range of Φ^T.*

By another standard result [2, 3, 5], we can identify unitarily isomorphic frames by their analysis subspaces.

Theorem 2. *If Φ and Ψ are (N, M) frames and $\Phi \simeq \Psi$, then $\Gamma(\Phi) = \Gamma(\Psi)$. Moreover, if Φ and Ψ are Parseval, then $\Phi \simeq \Psi$ if and only if $\Gamma(\Phi) = \Gamma(\Psi)$.*

Let $\mathscr{P}_{N,M}$ denote the class of Parseval (N, M) frames and let $\mathscr{P}_{N,M}/\simeq$ be the partition of $\mathscr{P}_{N,M}$ into the equivalence classes induced by unitary isomorphy. Theorem 2 shows that the elements of $\mathscr{P}_{N,M}/\simeq$ are in one-to-one correspondence with the elements of $\mathscr{G}_{N,M}$. Moreover, if Φ is a Parseval (N, M) frame and $[\Phi] \in \mathscr{P}_{N,M}/\simeq$ is the corresponding equivalence class, then the elements of $[\Phi]$ are in one-to-one correspondence with the elements of $\mathscr{B}(\Gamma(\Phi))$. In order to see this, we recall that the orthonormal bases for a given M-dimensional subspaces are unitarily equivalent.

Lemma 1. *If W is an M-dimensional subspace of H_N and $X = \{x_j\}_{j=1}^M \in \mathscr{B}(W)$ and $Y = \{y_j\}_{j=1}^M \subset H_N$ is any set of orthonormal vectors , then $Y \in \mathscr{B}(W)$ if and only if there exists $Y = UX$ for some $U \in \mathscr{O}_M$.*

Proof. If $Y = UX$ for some orthogonal matrix U, then Y generates the same orthogonal projection as X because

$$Y^T Y = X^T U^T U X = X^T X,$$

which means that $Y \in \mathscr{B}(W)$. Conversely, if $Y \in \mathscr{B}(W)$, then $U = (\langle x_b, y_a\rangle)_{a,b=1}^M$ is an orthogonal matrix because

$$(U^T U)_{ab} = \sum_{j=1}^M \langle y_a, x_j\rangle \langle x_j, y_b\rangle = \left\langle \sum_{j=1}^M \langle y_a, x_j\rangle x_j, y_b \right\rangle = \langle y_a, y_b\rangle = \delta_{ab},$$

where δ_{ab} is the dirac delta function. In this case, the transpose of the j^{th} row of UX is

$$\sum_{l=1}^M \langle y_j, x_l\rangle x_l = \sum_{l=1}^M x_l x_l^T y_j = y_j,$$

where the last equality follows because $\sum_{l=1}^M x_l x_l^T$ is the orthogonal projection of H_N onto W. This means that $Y = UX$, completing the proof. □

If $\Phi_1, \Phi_2 \in [\Phi]$ for some $[\Phi] \in \mathscr{P}_{N,M}/\simeq$, then Theorem 2 shows that Φ_1 and Φ_2 induce the same analysis subspace. Conversely, if $W \in \mathscr{G}_{N,M}$ and $X_1, X_2 \in \mathscr{B}(W)$, then setting $\Phi_1 = X_1$ and $\Phi_2 = X_2$ gives $\Phi_1\Phi_1^T = \Phi_2\Phi_2^T = \mathrm{I}$, so Φ_1 and Φ_2 are Parseval (N, M) frames, and they are unitarily isomorphic by Lemma 1.

Finally, we show that the space of equivalence classes of Parseval (N, M) frames and the space of M-dimensional subspaces of H_N are isometric. In order to do this, we consider a version of the isomorphy distance d_I restricted to $\mathscr{P}_{N,M}$ and lifted to $\mathscr{P}_{N,M}/\simeq$, which is well-defined by Theorem 2.

Definition 6. For $[\Phi], [\Psi] \in \mathscr{P}_{N,M}/\simeq$, the distance between $[\Phi]$ and $[\Psi]$ is

$$d_P([\Phi], [\Psi]) = d_I(\Phi, \Psi).$$

Theorem 3. *The metric spaces $(\mathscr{G}_{N,M}, d_S)$ and $(\mathscr{P}_{N,M}, d_P)$ are isometric. In particular, given Parseval (N, M) frames Φ and Ψ, we have*

$$d_P([\Phi], [\Psi]) = d_S(W_1, W_2),$$

where $W_1 = \Gamma(\Phi)$ and $W_2 = \Gamma(\Psi)$.

Proof. We have already seen that these spaces are in one-to-one correspondence. The claim follows by identifying the elements of $[\Phi]$ and $[\Psi]$ with the elements of $\mathscr{B}(W_1)$ and $\mathscr{B}(W_2)$, respectively. □

4 From Subspaces to Frames

In this section, we compute the isomorphy distance between well-conditioned frames Φ and Ψ. In light of Theorem 2, if $W_1 = \Gamma(\Phi)$ and $W_2 = \Gamma(\Psi)$, then this is equivalent to minimizing the frame distance over all possible pairs of unitarily equivalent frames corresponding to W_1 and W_2.

By definition, the value $d_I(\Phi, \Psi)$ is obtained by minimizing over $\mathscr{O}_M \times \mathscr{O}_M$. In the following lemma, we show that this can be simplified to a maximization over $\mathscr{O}_M$.

Lemma 2. *If Φ and Ψ are (N, M) frames, then*

$$d_I^2(\Phi, \Psi) = \|\Phi\|_{H.S.}^2 + \|\Psi\|_{H.S.}^2 - 2 \sup_{W \in \mathscr{O}_M} \operatorname{tr}(W \Phi \Psi^T).$$

Equivalently, we have

$$d_I(\Phi, \Psi) = \inf_{\Phi' \simeq \Phi} d_F(\Phi', \Psi) = d_F \inf_{\Psi' \simeq \Psi} (\Phi, \Psi').$$

Proof. Because $\mathscr{O}_M$ is a group, if $U, V \in \mathscr{O}_M$, then $W = V^T U \in \mathscr{O}_M$, so using that the Hilbert Schmidt norm is rotationally invariant, we have

$$d_I^2(\Phi, \Psi) = \inf_{U,V \in \mathscr{O}_M} \|U\Phi - V\Psi\|_{H.S.}^2 = \inf_{W \in \mathscr{O}_M} \|W\Phi - \Psi\|_{H.S.}^2 .$$

Expanding the right-hand expression and using the cyclicity of the trace then gives the result. □

In general, the expression $\sup_{W \in \mathscr{O}_M} \operatorname{tr}(W\Phi\Psi^T)$ from Lemma 2 does not have a unique maximizer, for instance, if the analysis subspaces of Φ and Ψ are orthogonal.

Example 1. Let

$$\Phi = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

which are $(4, 2)$ frames. Since $\Phi\Psi^T = 0_2$, where 0_2 is the zero 2×2 matrix, it follows that $\operatorname{tr}(W\Phi\Psi^T) = 0$ for every $W \in \mathscr{O}_2$.

However, if Ψ and Φ are (N, M) frames such that the $M \times M$ matrix $\Phi\Psi^T$ is invertible (i.e., Φ, Ψ are dual frames), then the expression $\sup_{W \in \mathscr{O}_M} \operatorname{tr}(W\Phi\Psi^T)$ is achieved by a unique maximizer $W \in \mathscr{O}_M$, which we compute in the following theorem.

Theorem 4. *If Ψ and Φ are (N, M) frames such that the $M \times M$ matrix $\Phi\Psi^T$ is invertible, then $U = (\Psi\Phi^T\Phi\Psi^T)^{-1/2}\Psi\Phi^T$ is the unique element of $\mathscr{O}_M$ such that*

$$\operatorname{tr}(U\Phi\Psi^T) = \sup_{W \in \mathscr{O}_M} \operatorname{tr}(W\Phi\Psi^T).$$

In particular,

$$\sup_{W \in \mathscr{O}_M} \operatorname{tr}(W\Phi\Psi^T) = \operatorname{tr}\left((\Psi\Phi^T\Phi\Psi^T)^{1/2}\right).$$

Proof. Consider the polar decomposition, $\Phi\Psi^T = VP$, where V is a unitary matrix and $P = (\Psi\Phi^T\Phi\Psi^T)^{1/2}$ is positive semidefinite. In fact, P is positive definite because of the invertibility assumption on $\Phi\Psi^T$, and thus

$$V = \Phi\Psi^T(\Psi\Phi^T\Phi\Psi^T)^{-1/2} \in \mathscr{O}_M.$$

Given any $U \in \mathscr{O}_M$, we have

$$\operatorname{tr}(U\Phi\Psi^T) = \operatorname{tr}(UVP) = \langle U, V^T\rangle_P,$$

where $\langle\cdot,\cdot\rangle_P$ is the inner product induced by P. The Cauchy Schwarz inequality then implies

$$\left\langle U, V^T\right\rangle_P \leq \left|\left\langle U, V^T\right\rangle_P\right| \leq \|U\|_P \left\|V^T\right\|_P,$$

where the right inequality is equality only when U and V^T are collinear, and since U is an orthogonal matrix, the left and right inequalities are both equalities if and only if $U = V^T$, so the claim follows. □

Combining Lemma 2 and Theorem 4, we have the following computation for the isomorphy distance.

Corollary 1. *If Ψ and Φ are (N, M) frames such that the $M \times M$ matrix $\Phi\Psi^T$ is invertible, then*

$$d_I^2(\Phi, \Psi) = \|\Phi\|_{H.S.}^2 + \|\Psi\|_{H.S.}^2 - 2\,\mathrm{tr}\left((\Psi\Phi^T\Phi\Psi^T)^{1/2}\right).$$

Given a frame Φ and an arbitrary subspace $W \in \mathscr{G}_{N,M}$, it may be desirable to find the closest Parseval frame Ψ with analysis subspace W. When the invertibility condition is satisfied, we can identify this frame precisely.

Corollary 2. *Let Φ be an (N, M) frame and let $W \in \mathscr{G}_{N,M}$. If Ψ is any Parseval (N, M) frame such that $\Gamma(\Psi) = W$, the product $\Phi\Psi^T$ is invertible, and we let $\tilde{\Psi} := \Phi\Psi^T(\Psi\Phi^T\Phi\Psi^T)^{-1/2}\Psi$, then*

$$\inf_{\substack{\Psi' \in \mathscr{P}_{N,M} \\ \Gamma(\Psi') = W}} d_F^2(\Phi, \Psi') = \|\Phi\|_{H.S.}^2 + M - 2\,\mathrm{tr}\left((\Phi\Psi^T\Psi\Phi^T)^{1/2}\right) = d_F^2(\Phi, \tilde{\Psi}),$$

and $\tilde{\Psi}$ is the unique Parseval (N, M) frame with analysis subspace W that achieves this minimum. In addition, if Φ is Parseval, then

$$d_F^2(\Phi, \tilde{\Psi}) = 2\left[M - \mathrm{tr}\left((\Phi\Psi^T\Psi\Phi^T)^{1/2}\right)\right].$$

Proof. The distance of interest is the smallest frame distance between Φ and the Parseval frames with analysis subspace W, which, by Theorem 2, is achieved by the frame in $[\Psi]$ that is closest to Φ. By Lemma 2, this distance is the isometry distance between Φ and Ψ. By Theorem 4, $\tilde{\Psi}$ is the unique frame in $[\Psi]$ whose frame distance to Φ gives this isometry distance. □

5 The Subspace Distance

In this section, we study the basic properties of the subspace distance. In order to do this, we show that the subspace distance $d_S(W_1, W_2)$ is characterized by a pair of biorthogonal orthonormal bases from W_1 and W_2.

Definition 7. If $X = \{x_i\}_{i=1}^M \in \mathscr{B}(W_1)$ and $Y = \{y_i\}_{i=1}^M \in \mathscr{B}(W_2)$, then (X, Y) is said to be a pair of **companion bases with respect to** W_1 **and** W_2, or just **companion bases**, if the following conditions hold:

1. $\langle x_i, y_i \rangle \geq 0, \quad 1 \leq i \leq M;$
2. $\langle x_i, y_j \rangle = 0, \quad 1 \leq i \neq j \leq M.$

To begin, we show the existence of companion bases and how to compute them.

Lemma 3. *If W_1 and W_2 are M-dimensional subspaces of H_N, then there exists $X = \{x_i\}_{i=1}^M \in \mathscr{B}(W_1)$ and $Y = \{y_i\}_{i=1}^M \in \mathscr{B}(W_2)$ such that (X, Y) is a pair of companion bases for W_1 and W_2.*

Proof. Let $\tilde{X} = \{\tilde{x}_i\}_{i=1}^M \in \mathscr{B}(W_1)$ and $\tilde{Y} = \{\tilde{y}_i\}_{i=1}^M \in \mathscr{B}(W_2)$. By the singular value decomposition, there exist orthogonal $M \times M$ matrices U and V and a diagonal matrix D with nonnegative diagonal entries such that $\tilde{X}\tilde{Y}^T = UDV^T$. If we set $X = U^T\tilde{X}$ and $Y = V^T\tilde{Y}$, then $X \in \mathscr{B}(W_1)$ and $Y \in \mathscr{B}(W_2)$ by Lemma 1 and

$$\left(\langle x_i, y_j\rangle\right)_{i,j=1}^M = XY^T = U^T\tilde{X}\tilde{Y}^T V = U^T UDV^T V = D,$$

which means that (X, Y) is a pair of companion bases.

Next, we show that when given a pair of orthonormal bases for subspaces, then the Hilbert Schmidt distance between them is minimal when they are companion bases.

Lemma 4. *If $X = \{x_i\}_{i=1}^M \in \mathscr{B}(W_1)$ and $Y = \{y_i\}_{i=1}^M \in \mathscr{B}(W_2)$ are a pair of companion bases, then*

$$\|X - Y\|_{H.S}^2 = \inf_{U \in \mathscr{O}_M} \|UX - Y\|_{H.S}^2$$

and this infimum is achieved if and only if $U = I$.

Proof. If $\tilde{X} = \{\tilde{x}_i\}_{i=1}^M \in \mathscr{B}(W_1)$, then by Lemma 1, there exists $U = (u_{ij})_{i,j=1}^M \in \mathscr{O}_M$ such that $\tilde{X} = UX$, or equivalently $\tilde{x}_i = \sum_{j=1}^M u_{ij}x_j$ for $i \in \{1, \ldots, M\}$. This means that minimizing over $\mathscr{B}(W_1)$ is equivalent to minimizing over $\mathscr{O}_M$; that is,

$$\inf_{\{\tilde{x}_i\}_{i=1}^M \in \mathscr{B}(W_1)} \sum_{i=1}^M \|\tilde{x}_i - y_i\|^2 = \inf_{(u_{ij})_{i,j=1}^M \in \mathscr{O}_M} \sum_{i=1}^M \left\| \sum_{j=1}^M u_{ij}x_j - y_i \right\|^2 .$$

With this in mind and using that $\{x_i\}_{i=1}^M$ and $\{y_i\}_{i=1}^M$ are companion bases for W_1 and W_2, we calculate

$$\begin{aligned} \sum_{i=1}^M \left\| \sum_{j=1}^M u_{ij}x_j - y_i \right\|^2 &= \sum_{i=1}^M \left(2 - 2\left\langle \sum_{j=1}^M u_{ij}x_j, y_i \right\rangle\right) \\ &= \sum_{i=1}^M (2 - 2u_{ii} \langle x_i, y_i\rangle) \\ &\geq \sum_{i=1}^M (2 - 2 \langle x_i, y_i\rangle) = \sum_{i=1}^M \|x_i - y_i\|^2 . \end{aligned}$$

The inequality saturates if and only if $u_{ii} = 1$ for all $1 \leq i \leq M$ if and only if U is the identity matrix. The statement follows by recognizing that

$$\sum_{i=1}^{M} \left\| \sum_{j=1}^{M} u_{ij} x_j - y_i \right\|^2 = \|UX - Y\|_{H.S.}^2 .$$

□

Next, we show that the computation of the subspace distance is equivalent to a minimization over $\mathscr{O}_M$.

Lemma 5. *If W_1 and W_2 are M-dimensional subspaces of H_N, $X \in \mathscr{B}(W_1)$ and $Y \in \mathscr{B}(W_2)$, then*

$$d_S(W_1, W_2) = \inf_{W \in \mathscr{O}_M} \|WX - Y\|_{H.S.} .$$

Proof. Lemma 1 shows that d_S can be rephrased in terms of minimization over $\mathscr{O}_M \times \mathscr{O}_M$,

$$\inf_{\substack{\tilde{X} \in \mathscr{B}(W_1) \\ \tilde{Y} \in \mathscr{B}(W_2)}} \left\| \tilde{X} - \tilde{Y} \right\|_{H.S.} = \inf_{U,V \in \mathscr{O}_M} \|UX - VY\|_{H.S.} .$$

Because $\mathscr{O}_M$ is a group, if given $U, V \in \mathscr{O}_M$, then there exist $W = V^T U \in \mathscr{O}_M$, so using that the Hilbert Schmidt norm is rotationally invariant, d_S can further be simplified to minimizing over just $\mathscr{O}_M$,

$$\inf_{U,V \in \mathscr{O}_M} \|UX - VY\|_{H.S.} = \inf_{W \in \mathscr{O}_M} \|WX - Y\|_{H.S.} .$$

□

Now we are ready to prove the main theorem of this section.

Theorem 5. *Let W_1 and W_2 be M-dimensional subspaces of H_N. If $X = \{x_i\}_{i=1}^M \in \mathscr{B}(W_1)$ and $Y = \{y_i\}_{i=1}^M \in \mathscr{B}(W_2)$ are companion bases for W_1 and W_2, then*

$$d_S^2(W_1, W_2) = \|X - Y\|_{H.S.}^2 .$$

Furthermore, if $\tilde{X} \in \mathscr{B}(W_1)$ and $\tilde{Y} \in \mathscr{B}(W_2)$, then

$$d_S^2(W_1, W_2) = \left\| \tilde{X} - \tilde{Y} \right\|_{H.S.}^2$$

if and only if there exists $U \in \mathscr{O}_M$ such that

$$\tilde{X} = UX \quad \textit{and} \quad \tilde{Y} = UY.$$

Proof. Lemma 5 and Lemma 4 show that $d_S(W_1, W_2) = \|X - Y\|_{H.S.}^2$.

If $\tilde{X} = UX$ and $\tilde{Y} = UY$ for some $U \in \mathscr{O}_M$, then the rotational invariance of the Hilbert Schmidt norm shows that

$$d_S^2(W_1, W_2) = \|X - Y\|_{H.S.}^2 = \|U(X - Y)\|_{H.S.}^2 = \left\|\tilde{X} - \tilde{Y}\right\|_{H.S.}^2.$$

Conversely, suppose that $\tilde{X} \in \mathscr{B}(W_1)$, $\tilde{Y} \in \mathscr{B}(W_2)$, and $d_S(W_1, W_2) = \left\|\tilde{X} - \tilde{Y}\right\|_{H.S.}$. By Lemma 1, there exist $U, V \in \mathscr{O}_M$ such that $\tilde{X} = UX$ and $\tilde{Y} = VY$, so rotational invariance of the Hilbert Schmidt norm gives

$$\left\|\tilde{X} - \tilde{Y}\right\|_{H.S}^2 = \left\|V^T UX - Y\right\|_{H.S.}^2.$$

Thus, the fact that $\left\|\tilde{X} - \tilde{Y}\right\|_{H.S.} = \|X - Y\|_{H.S.}$ together with Lemma 5 and Lemma 1 shows that $V^T U = \mathrm{I}$, which means that $U = V$, completing the proof. □

We end this section with some final results concerning the distance given by companion bases.

Corollary 3. *Given M-dimensional subspaces W_1 and W_2 of H_N with companion bases $X = \{x_i\}_{i=1}^M \in \mathscr{B}(W_1)$ and $Y = \{y_i\}_{i=1}^M \in \mathscr{B}(W_2)$, we have*

$$d_S^2(W_1, W_2) = 2\left(M - \mathrm{tr}(XY^T)\right) = 2\left(M - \sum_{i=1}^{M} \langle x_i, y_i \rangle\right).$$

Proof. This is a straightforward application of Theorem 5. □

Combining these results, we see that the subspace distance can be characterized in terms of the singular value decomposition.

Corollary 4. *If W_1 and W_2 are M-dimensional subspaces of H_M, $\tilde{X} \in \mathscr{B}(W_1)$ and $\tilde{Y} \in \mathscr{B}(W_2)$, then*

$$d_S^2(W_1, W_2) = 2\left(M - \mathrm{tr}(D)\right),$$

where $U, V \in \mathscr{O}_M$ and D is the diagonal matrix with nonnegative entries such that

$$\tilde{X}\tilde{Y}^T = UDV^T$$

is the singular value decomposition of $\tilde{X}\tilde{Y}^T$.

Proof. As in the proof of Lemma 3, the pair $X = U^T\tilde{X}$ and $Y = V^T\tilde{Y}$ forms a pair of companion bases for W_1 and W_2, so the claim follows by applying Corollary 3. □

6 The Chordal Distance

In this section, we recall the basic properties of the chordal distance. To begin, we show that every pair of principal bases is also a pair of companion bases.

Proposition 2. *Let $W_1, W_2 \in \mathscr{G}_{N,M}$. If $X = \{x_i\}_{i=1}^M \in \mathscr{B}(W_1)$ and $Y = \{y_i\}_{i=1}^M \in \mathscr{B}(W_2)$ are a pair of principal bases between W_1 and W_2, then (X, Y) are also a pair of companion bases.*

Proof. If $i, j \in \{1, \ldots, M\}$ with $i < j$, then since every linear combination $\alpha x_i + \beta x_j \in W_1^i = W_1 \cap \text{span}\{x_1, \ldots, x_{i-1}\}^\perp$, it follows from the definition of principal vectors that the function

$$F(t) := \left\|(\cos t)x_i + (\sin t)x_j - y_i\right\|^2$$

obtains its absolute minimum at $t = 0$. By expanding the squared norm in the definition of F and using that X and Y are orthonormal sets, we have

$$F(t) = 2 - 2(\cos t)\langle x_i, y_i\rangle - 2(\sin t)\left\langle x_j, y_i\right\rangle,$$

and since $F'(0) = 0$ and $F''(0) \geq 0$, a direct computation gives $\left\langle x_j, y_i\right\rangle = 0$ and $\langle x_i, y_i\rangle \geq 0$. The argument is similar for the case $i > j$, so the claim follows. □

Fundamental to chordal distance is its relationship to the trace of the product of projections onto the subspaces.

Lemma 6. *Let $W_1, W_2 \in \mathscr{G}_{N,M}$. If P is the orthogonal projection of H_N onto W_1 and Q is the orthogonal projection of H_N onto W_2, then*

$$\text{tr}(PQ) = \sum_{i=1}^{M} cos^2(\theta_i),$$

where $\{\theta_i\}_{i=1}^M$ are the principal angles between W_1 and W_2.

Proof. Let $X \in \mathscr{B}(W_1)$ and $Y \in \mathscr{B}(W_2)$ be a pair of principal bases between W_1 and W_2, so $P = X^T X$ and $Q = Y^T Y$. By Proposition 2, X and Y are also companion bases between W_1 and W_2, so the definitions of principal bases and companion bases together show that XY^T is an $M \times M$ diagonal matrix,

$$XY^T = \text{diag}(\cos\theta_i)_{i=1}^M,$$

which gives

$$\text{tr}(PQ) = \text{tr}(X^T XY^T Y) = \text{tr}(XY^T YX^T) = \text{tr}(XY^T (XY^T)^T) = \sum_{i=1}^{M} cos^2(\theta_i).$$

□

Because of this, the chordal distance is characterized by the trace of orthogonal projections. The following result is in [6].

Proposition 3. *Let $W_1, W_2 \in \mathscr{G}_{N,M}$. If P is the orthogonal projection of H_N onto W_1 and Q is the orthogonal projection of H_N onto W_2, then*

$$d_C^2(W_1, W_2) = M - \mathrm{tr}(PQ).$$

Proof. If $\{\theta_i\}_{i=1}^M$ are the principal angles between W_1 and W_2, then applying Lemma 6 gives

$$d_C^2(W_1, W_2) = \sum_{i=1}^M \sin^2(\theta_i) = M - \sum_{i=1}^M \cos^2(\theta_i) = M - \mathrm{tr}(PQ).$$

□

Another standard result from [6] relates the chordal distance to projections of the orthonormal bases for H_N onto subspaces.

Theorem 6. *Let $M_1, M_2 \in \{1, \ldots, N\}$, let $W_1 \in \mathscr{G}_{N,M_1}$, and let $W_2 \in \mathscr{G}_{N,M_2}$. If P is the orthogonal projection onto W_1, Q is the orthogonal projection onto W_2, and $X = \{x_i\}_{i=1}^N \in \mathscr{B}(H_N)$ is any orthonormal basis for H_N, then*

$$\sum_{i=1}^N \|Px_i - Qx_i\|^2 = (M_1 + M_2) - 2\,\mathrm{tr}(PQ).$$

In particular, if $\dim W_1 = \dim W_2 = M$, *then*

$$d_C^2(W_1, W_2) = \frac{1}{2} \sum_{i=1}^N \|Px_i - Qx_i\|^2 .$$

Proof. Recall that because P and Q are orthogonal projections, we have $P=P^T=P^2$, $Q = Q^T = Q^2$, $\mathrm{tr}(P) = M_1$ and $\mathrm{tr}(Q) = M_2$. Using this, we compute that

$$\begin{aligned}\sum_{i=1}^N \|Px_i - Qx_i\|^2 &= \sum_{i=1}^N \langle Px_i, x_i\rangle + \sum_{i=1}^N \langle Qx_i, x_i\rangle - 2\sum_{i=1}^N \langle PQx_i, x_i\rangle \\ &= \mathrm{tr}(P) + \mathrm{tr}(Q) - 2\,\mathrm{tr}(PQ) \\ &= M_1 + M_2 - 2\,\mathrm{tr}(PQ).\end{aligned}$$

If $\dim W_1 = \dim W_2 = M$, then Proposition 3 shows that

$$\sum_{i=1}^N \|Px_i - Qx_i\|^2 = 2(M - \mathrm{tr}(PQ)) = 2d_C^2(W_1, W_2).$$

□

It follows immediately that the chordal distance between two subspaces is the same as the distance between their orthogonal complements.

Corollary 5. *Let $W_1, W_2 \in \mathcal{G}_{N,M}$. If $W_1^{\perp}$ and $W_2^{\perp}$ are the orthogonal complements of W_1 and W_2, respectively, then*

$$d_C(W_1, W_2) = d_C(W_1^{\perp}, W_2^{\perp}).$$

Proof. If P and Q are the orthogonal projections of H_N onto W_1 and W_2, respectively, then $P' = (\mathrm{I} - P)$ and $Q' = (\mathrm{I} - Q)$ are the orthogonal projections onto $W_1^{\perp}$ and $W_2^{\perp}$, respectively, so given any orthonormal basis $X = \{x_i\}_{i=1}^{M} \in \mathcal{B}(H_N)$ for H_N, it follows from Theorem 6 that

$$\begin{aligned} d_C^2(W_1^{\perp}, W_2^{\perp}) &= \frac{1}{2} \sum_{i=1}^{N} \|(\mathrm{I} - P)x_i - (\mathrm{I} - Q)x_i\|^2 \\ &= \frac{1}{2} \sum_{i=1}^{N} \|Px_i - Qx_i\|^2 \\ &= d_C^2(W_1, W_2). \end{aligned}$$

□

7 Relationships Between Distances

In this section, we relate the subspace and chordal distances. In the special case of Parseval frames, we also relate these to the Gramian and isomorphy distances. The fundamental observation is that companion bases and principal bases are essentially the same objects. We begin with a lemma.

Lemma 7. *Let $W_1, W_2 \in \mathcal{G}_{N,M}$. If $X = \{x_i\}_{i=1}^{M} \in \mathcal{B}(W_1)$ and $Y = \{y_i\}_{i=1}^{M} \in \mathcal{B}(W_2)$ are companion bases, then*

$$\langle x_i, y_i \rangle = \sup_{\substack{y \in W_2 \\ \|y\|=1}} \langle x_i, y \rangle$$

for each $i \in \{1, \ldots, M\}$.

Proof. If $y \in W_2$ with $\|y\| = 1$, then the Parseval identity gives

$$y = \sum_{i=1}^{M} \langle y, y_i \rangle \, y_i \quad \text{and} \quad \sum_{i=1}^{M} |\langle y, y_i \rangle|^2 = 1,$$

so $|\langle y, y_i \rangle| \leq 1$ for every $i \in \{1, \ldots, M\}$. Using this and the fact that $\langle x_i, y_j \rangle = 0$ whenever $i \neq j$, we compute

$$\begin{aligned}\langle x_i, y \rangle &= \left\langle x_i, \sum_{j=1}^{M} \langle y, y_j \rangle y_j \right\rangle \\ &= \langle y, y_i \rangle \langle x_i, y_i \rangle \\ &\leq \langle x_i, y_i \rangle\end{aligned}$$

which completes the proof. □

Thus, up to permutation of indices, we can identify principal bases with companion bases.

Theorem 7. *Let $W_1, W_2 \in \mathcal{G}_{N,M}$. If $X = \{x_i\}_{i=1}^M \in \mathcal{B}(W_1)$ and $Y = \{y_i\}_{i=1}^M \in \mathcal{B}(W_2)$, then (X, Y) is a pair of companion bases for W_1 and W_2 if and only if there exists some permutation σ on $\{1, \ldots, M\}$ such that $(\tilde{X}, \tilde{Y})$ is a pair of principal bases for W_1 and W_2, where $\tilde{X} = \{x_{\sigma(i)}\}_{i=1}^M$ and $\tilde{Y} = \{y_{\sigma(i)}\}_{i=1}^M$.*

Proof. If $\tilde{X} = \{x_{\sigma(i)}\}_{i=1}^M$ and $\tilde{Y} = \{y_{\sigma(i)}\}_{i=1}^M$ is a pair of principal bases for some permutation σ, then it is a pair of companion bases by Proposition 2. To see the converse, suppose that (X, Y) is a pair of companion bases and choose σ so that

$$\langle x_{\sigma(1)}, y_{\sigma(1)} \rangle \leq \langle x_{\sigma(2)}, y_{\sigma(2)} \rangle \leq \cdots \leq \langle x_{\sigma(M)}, y_{\sigma(M)} \rangle .$$

By Lemma 7, it follows that $\tilde{X} = \{x_{\sigma(i)}\}_{i=1}^M$ and $\tilde{Y} = \{y_{\sigma(i)}\}_{i=1}^M$ form a pair of principal bases for W_1 and W_2. □

Because of this, the subspace distance is also characterized by the principal angles. This allows us to relate the two distances.

Theorem 8. *Let $W_1, W_2 \in \mathcal{G}_{N,M}$. If $\{\theta_i\}_{i=1}^M$ are the principal angles between W_1 and W_2, then*

$$d_S^2(W_1, W_2) = 2(M - \sum_{i=1}^{M} \cos\theta_i).$$

In particular,

$$d_S^2(W_1, W_2) \leq 2 d_C^2(W_1, W_2).$$

Proof. If $X \in \mathcal{B}(W_1)$ and $Y \in \mathcal{B}(W_2)$ are a pair of companion bases, then they are also a pair of principal bases by Theorem 7. Thus, by Corollary 3,

$$d_S^2(W_1, W_2) = 2(M - \sum_{i=1}^{M} \cos\theta_i).$$

In particular,

$$d_S^2(W_1, W_2) = 2\sum_{i=1}^{M}(1-\cos\theta_i) \leq 2\sum_{i=1}^{M}(1-\cos^2\theta_i) = 2\sum_{i=1}^{M}\sin^2\theta_i = 2d_C^2(W_1, W_2).$$

□

Next, we consider the special case of Parseval frames. First, we recall a theorem from [1].

Theorem 9. *If $\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_i\}_{i=1}^N$ are Parseval (N, M) frames, then*

$$d_G^2(\Phi, \Psi) \leq 4d_F^2(\Phi, \Psi).$$

Thus, when working with Parseval frames and their analysis subspaces, we are able to relate all of the distances defined in this chapter.

Theorem 10. *Let Φ and Ψ be Parseval (N, M) frames. If $W_1 = \Gamma(\Phi)$ and $W_2 = \Gamma(\Psi)$ are their respective analysis subspaces, then*

$$d_I^2(\Phi, \Psi) = d_S^2(W_1, W_2) \leq 2d_C^2(W_1, W_2) = d_G^2(\Phi, \Psi) \leq 4d_F^2(\Phi, \Psi).$$

Note: From Lemma 2,

$$\inf_{\Phi' \simeq \Phi} d_F^2(\Phi', \Psi) = d_F^2 \inf_{\Psi' \simeq \Psi}(\Phi, \Psi') = d_1^2(\Phi, \Psi)$$

both end of this inequality are bounded by expressions involving d_F^2.

Proof. If $[\Phi], [\Psi] \in \mathscr{P}_{N,M}$, it follows directly from Theorem 3 that $d_I^2(\Phi, \Psi) = d_P^2([\Phi], [\Psi]) = d_S^2(W_1, W_2)$, and we have $d_S^2(W_1, W_2) \leq 2d_C^2(W_1, W_2)$ from Theorem 8.

If $G_\Phi = \Phi^T\Phi$ and $G_\Psi = \Psi^T\Psi$ are the Gram matrices of Φ and Ψ, then they are both symmetric and Parsevality shows that

$$G_\Phi^2 = \Phi^T\Phi\Phi^T\Phi = \Phi^T\Phi = G_\Phi,$$

and, similarly, $G_\Psi^2 = G_\Psi$, so they are both orthogonal projections. If $x \in W_1$, then $x = \Phi^T y$ for some $y \in H_M$, so again Parsevality shows that

$$G_\Phi x = \Phi^T\Phi x = \Phi^T\Phi\Phi^T y = \Phi^T y = x.$$

On the other hand, if $x \in W_1^\perp$, then the fact that $W_1^\perp = \operatorname{ran}(\Phi^T)^\perp = \ker(\Phi)$ shows that

$$G_\Phi x = \Phi^T\Phi x = 0.$$

Thus, G_Φ is the orthogonal projection of H_N onto W_1 and, by the same argument, G_Ψ is the orthogonal projection of H_N onto W_2. Since $\mathrm{tr}(G_\Phi^2) = \mathrm{tr}(G_\Phi) = \sum_{i=1}^M \langle \phi_i, \phi_i \rangle = M$ and similarly $\mathrm{tr}(G_\Psi^2) = M$, it follows from Proposition 3 that

$$\begin{aligned} d_G^2(\Phi, \Psi) &= \|G_\Phi - G_\Psi\|_{H.S.}^2 \\ &= \|G_\Phi\|_{H.S.}^2 + \|G_\Psi\|_{H.S.}^2 - 2\,\mathrm{tr}(G_\Phi G_\Psi) \\ &= 2\,(M - \mathrm{tr}(G_\Phi G_\Psi)) \\ &= 2d_C^2(\Phi, \Psi). \end{aligned}$$

Finally, Theorem 9 shows that $d_G^2(\Phi, \Psi) \leq 4d_F^2(\Phi, \Psi)$. □

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Sigma-Delta Quantization for Fusion Frames and Distributed Sensor Networks

Jiayi Jiang and Alexander M. Powell

1 Introduction

Fusion frames were introduced in [9] as a mathematical tool for problems involving distributed signal processing and data fusion. Fusion frames provide signal decompositions by using projections onto an overlapping and overcomplete collection of subspaces. This provides a framework for applications where one seeks to process or infer global information about a signal in settings where hardware and network constraints only permit access to a collection of coarse local measurements of the signal. For example, fusion frames have been proposed as a natural medium for studying distributed sensor networks [11, 12].

We consider the following sensor network structure as a motivating example throughout the chapter. Suppose that one seeks to measure a signal x over a large environment $\mathscr{E}$ using a collection of remotely dispersed sensor stations $\{s_n\}_{n=1}^N$ that are constrained by limited power, limited computational resources, and limited ability to communicate. The signal of interest might, for example, include:

- electromagnetic signals arising from communications,
- physical measurements of temperature, acoustics, or general pressure waves,
- biological or chemical signatures within a region,
- a combination of multiple signals of differing modalities.

Each sensor station s_n is able to make local measurements in a proximity W_n, but the size of the environment $\mathscr{E}$ and the constraints on s_n prevent individual sensor stations from gaining global insight into the signal x. A key point is that the proximities W_n will have some overlap; utilizing this redundancy or correlation will make it possible

J. Jiang • A.M. Powell (✉)
Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA
e-mail: jiayi.jiang@vanderbilt.edu; alexander.m.powell@vanderbilt.edu

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to accurately measure the signal x in spite of the various constraints on the sensor stations. Further assume that:

- the sensor stations s_n have been dispersed at known pre-determined locations, so that each sensor is aware of the proximities W_n for other nearby sensor stations,
- each sensor station is able to transmit analog signals u_n to a small number of other nearby sensor stations,
- each sensor station is able to transmit a low-bit digital signal q_n to a distantly located central base station $\mathscr{C}$.

The central base station $\mathscr{C}$ is relatively unconstrained in its power and computational resources, and is used to reconstruct a estimate $\widetilde{x}$ of the signal x from the received digital signals q_n.

In this chapter, we examine a mathematical framework for the above sensor network structure when the environment $\mathscr{E} = \mathbb{R}^d$ and the signal $x \in \mathbb{R}^d$. The sensor stations s_n will be mathematically described using fusion frames where the proximity W_n is modeled as a subspace of $\mathbb{R}^d$ and the local measurement $y_n = P_{W_n}(x)$ made by a sensor station is modeled as an orthogonal projection of x onto W_n. The communicated signals u_n and q_n will be implemented by adapting Sigma-Delta ($\Sigma\Delta$) quantization to the setting of fusion frames. The reconstruction procedure at the base station $\mathscr{C}$ will employ an analogue of Sobolev duals for fusion frames.

2 Fusion Frames and Quantization

Let $\{W_n\}_{n=1}^N \subset \mathbb{R}^d$ be a collection of subspaces of $\mathbb{R}^d$ and let $\{v_n\}_{n=1}^N \subset (0, \infty)$ be a collection of positive scalars. The collection $\{(W_n, v_n)\}_{n=1}^N$ is said to be a *fusion frame* with bounds $0 < A \leq B < \infty$ if

$$\forall x \in \mathbb{R}^d, \quad A\|x\|^2 \leq \sum_{n=1}^{N} v_n^2 \|P_{W_n}(x)\|^2 \leq B\|x\|^2, \tag{1}$$

where $P_W(x)$ denotes the orthogonal projection of x onto the subspace W and $\|\cdot\|$ denotes the Euclidean norm. The fusion frame is said to be *tight* if $A = B$. The fusion frame is said to be *unweighted* if each $v_n = 1$. Fusion frames may be viewed as a vector-valued generalization of frame theory and may also be viewed as a generalization of the so-called finite dimensional decompositions to the overcomplete setting, see [10] for further background on fusion frames.

If $\{(W_n, v_n)\}_{n=1}^N \subset \mathbb{R}^d$ is a fusion frame, then the associated *fusion frame operator* $S : \mathbb{R}^d \to \mathbb{R}^d$ is defined by

$$S(x) = \sum_{n=1}^{N} v_n^2 P_{W_n}(x). \tag{2}$$

It is known that S is a positive invertible operator satisfying $AI \leq S \leq BI$, e.g., [9], in the sense that $(S - AI)$ and $(BI - S)$ are positive operators and where I denotes the identity operator. This implies, e.g., [9], that each $x \in \mathbb{R}^d$ can be recovered from the fusion frame measurements $y_n = P_{W_n}(x) \in \mathbb{R}^d$, $1 \leq n \leq N$, by

$$\forall x \in \mathbb{R}^d, \quad x = S^{-1}Sx = \sum_{n=1}^{N} v_n^2 \, S^{-1}(P_{W_n}(x)) = \sum_{n=1}^{N} v_n^2 \, S^{-1}(y_n). \tag{3}$$

In the special case when the fusion frame is tight, it is known that the constant A satisfies

$$A = d^{-1} \sum_{n=1}^{N} v_n^2 \dim(W_n), \tag{4}$$

and the fusion frame operator satisfies $S = AI$ and $S^{-1} = A^{-1}I$, e.g., [10]. In particular, for unweighted tight fusion frames one has the simple and explicit reconstruction formula

$$\forall x \in \mathbb{R}^d, \quad x = \left(\frac{d}{\sum_{j=1}^{N} \dim(W_j)} \right) \sum_{n=1}^{N} P_{W_n}(x) = \left(\frac{d}{\sum_{j=1}^{N} \dim(W_j)} \right) \sum_{n=1}^{N} y_n. \tag{5}$$

The fusion frame measurements $y_n = P_{W_n}(x)$ appearing in (3) and (5) are $\mathbb{R}^d$-valued quantities that can take on a continuum of values and which, in applications, might physically exist as analog quantities. To be amenable to digital processing or storage the measurements must undergo a lossy step that reduces their range to a finite (hence digitally useable) range. For example, in the motivating sensor network structure in Section 1, each sensor station makes a physical measurement y_n that must be converted to a digital quantity q_n to be transmitted to the base station $\mathscr{C}$. The lossy process of reducing the range of measurements from the continuum to the discrete or finite is known as *quantization*.

Memoryless quantization is perhaps the simplest approach to quantizing fusion frame measurements. For each $1 \leq n \leq N$, let $\mathscr{A}_n \subset W_n$ be a finite set, referred to as a *quantization alphabet*, and let $Q_n : W_n \to \mathscr{A}_n$ be an associated *quantizer map* with the property that

$$\forall w \in W_n, \quad \|w - Q_n(w)\| = \min_{a \in \mathscr{A}_n} \|w - a\|. \tag{6}$$

Memoryless quantization of the fusion frame measurements $\{y_n\}_{n=1}^N$ produces quantized measurements $\{q_n\}_{n=1}^N$, with $q_n \in \mathscr{A}_n$, by

$$q_n = Q_n(y_n). \tag{7}$$

The fineness or density of the alphabets $\mathscr{A}_n$ plays an important role in determining how lossy quantization is. Let W be a subspace of $\mathbb{R}^d$ and let $\mathbb{B}(W,\rho) = \{w \in W : \|w\| \leq \rho\}$ denote the closed ball of radius ρ in the subspace W. We shall say that a finite set $\mathscr{N} \subset \mathbb{B}(W,\rho)$ is an *ε-net* for $\mathbb{B}(W,\rho)$ if

$$\forall b \in \mathbb{B}(W,\rho),\ \exists z \in \mathscr{N}, \qquad \text{such that} \qquad \|b - z\| \leq \varepsilon. \tag{8}$$

To illustrate the performance of memoryless quantization, let $\{(W_n, 1)\}_{n=1}^N$ be an unweighted tight fusion frame for $\mathbb{R}^d$, so that (5) holds. Let $x \in \mathbb{R}^d$ satisfy $\|x\| \leq 1$ and note that the fusion frame measurements $y_n = P_{W_n}(x)$ satisfy $\|y_n\| \leq \|x\| \leq 1$, i.e., $y_n \in \mathbb{B}(W_n, 1)$. For each $1 \leq n \leq N$, let $\mathscr{A}_n$ be an ε-net for $\mathbb{B}(W_n, 1)$ and suppose that memoryless quantization (7) is implemented using the alphabets $\{\mathscr{A}_n\}_{n=1}^N$. Since $y_n \in \mathbb{B}(W_n, 1)$, the ε-net property ensures that $\|y_n - Q_n(y_n)\| \leq \varepsilon$. So, motivated by (5), if one reconstructs $\widetilde{x} \in \mathbb{R}^d$ from the quantized measurements $\{q_n\}_{n=1}^N$ by

$$\widetilde{x} = \left(\frac{d}{\sum_{j=1}^N \dim(W_j)}\right) \sum_{n=1}^N q_n,$$

then a coarse upper bound on the overall quantization error gives

$$\|x - \widetilde{x}\| = \left(\frac{d}{\sum_{j=1}^N \dim(W_j)}\right) \left\| \sum_{n=1}^N (y_n - Q(y_n)) \right\| \leq \left(\frac{dN}{\sum_{j=1}^N \dim(W_j)}\right) \varepsilon. \tag{9}$$

Note that the upper bound in (9) does not decrease to zero as N increases. So, a small choice of $\varepsilon > 0$ is needed to ensure that memoryless quantization achieves small overall quantization error. Unfortunately, a small choice of $\varepsilon > 0$ can require extremely large quantization alphabets $\mathscr{A}_n$ which can only be represented using a large number of bits. The cost and complexity of high-bit alphabets might not be suitable for every signal processing problem. For example, this is the case in the sensor network example from Section 1, where the power-constrained sensor stations are only able to transmit low-bit signals to the central base station.

In summary, memoryless quantization does not efficiently utilize overlap or correlation among the subspaces W_n, but instead requires fine alphabets to achieve accurate quantization. The next section describes a different approach to quantization that employs coarse quantization alphabets but compensates for this by efficiently utilizing correlations among the subspaces W_n.

3 Sigma-Delta Quantization for Fusion Frames

In this section we introduce Sigma-Delta ($\Sigma\Delta$) algorithms for fusion frames.

It will be useful to briefly recall the classical $\Sigma\Delta$ algorithm for quantizing a scalar-valued input sequence $\{c_n\} \subset \mathbb{R}$. In the setting of sampling expansions, $c_n = f(n/\lambda) \in \mathbb{R}$ are samples of a bandlimited function f, see [13], whereas in the setting of finite frames, the coefficients $c_n = \langle x, e_n \rangle \in \mathbb{R}$ are frame coefficients of a signal $x \in \mathbb{R}^d$, see [4]. For comparison, the fusion frame setting of this chapter will require an extension to vector-valued measurements $y_n = P_{W_n}(x) \in \mathbb{R}^d$.

To describe classical $\Sigma\Delta$ quantization, let $\mathscr{A} \subset \mathbb{R}$ be a finite scalar-valued quantization alphabet, and let $Q : \mathbb{R} \to \mathscr{A}$ be an associated scalar quantizer that satisfies

$$\forall t \in \mathbb{R}, \quad |t - Q(t)| = \min_{a \in \mathscr{A}} |t - a|.$$

The classical first order $\Sigma\Delta$ quantizer takes coefficients $\{c_n\}_{n=1}^N \subset \mathbb{R}$ as input and produces quantized output coefficients $\{q_n\}_{n=1}^N \subset \mathscr{A}$, by initializing $u_0 = 0$ and iterating the following for $n = 1, \cdots, N$

$$q_n = Q(u_{n-1} + c_n), \tag{10}$$

$$u_n = u_{n-1} + c_n - q_n. \tag{11}$$

See [26] for an overview on $\Sigma\Delta$ quantization in the setting of finite frames. We are now ready to discuss adaptations of this algorithm to vector-valued inputs $y_n = P_{W_n}(x) \in \mathbb{R}^d$ for the setting of fusion frames.

3.1 A First Order $\Sigma\Delta$ Algorithm for Fusion Frames

Let $\{W_n\}_{n=1}^N$ be a collection of subspaces of $\mathbb{R}^d$ with $d_n = \dim(W_n)$. For each $1 \leq n \leq N$, let $\mathscr{A}_n \subset W_n$ be a quantization alphabet and let $Q_n : W_n \to \mathscr{A}_n$ be a quantizer map as in (6).

Suppose that $x \in \mathbb{R}^d$ and denote $y_n = P_{W_n}(x) \in W_n$. The *first order fusion frame $\Sigma\Delta$ quantizer* takes the fusion frame measurements $y_n \in W_n, 1 \leq n \leq N$, as its input, and produces the quantized outputs $q_n \in \mathscr{A}_n, 1 \leq n \leq N$, by initializing $u_0 = 0 \in \mathbb{R}^d$ and iterating the following for $n = 1, \cdots, N$

$$q_n = Q_n(P_{W_n}(u_{n-1}) + y_n), \tag{12}$$

$$u_n = P_{W_n}(u_{n-1}) + y_n - q_n. \tag{13}$$

This algorithm may be viewed as a generalization of $\Sigma\Delta$ quantization or error diffusion from the setting of finite frames to the vector-valued setting of fusion frames, [4, 8, 26]. For related work on vector-valued $\Sigma\Delta$ quantization in the engineering literature, see [19, 22–25]. The work in [1–3] considers delicate stability issues for vector-valued error diffusion algorithms (with connections to digital halftoning). The work in [5] considers $\Sigma\Delta$ quantization for finite frames in $\mathbb{C}^d$ using $\mathbb{C}$-valued alphabets with a lattice structure.

Returning to the sensor network example from Section 1, $y_n = P_{W_n}(x)$ is the signal measurement made by the sensor station s_n, the quantized coefficient q_n is digitally encoded and sent from s_n to the central base station $\mathscr{C}$, and the state variable u_n is the analog signal that is transmitted from s_n to the next station s_{n+1}. Note that, in contrast to classical $\Sigma\Delta$ quantization, the algorithm requires knowledge of the fusion frame $\{W_n\}_{n=1}^N$.

It will be useful to note a matrix formulation of (13). Given sets $S_n \subset \mathbb{R}^d$, $1 \le n \le N$, let $\bigoplus_{n=1}^N S_n$ denote the collection of all N-tuples $f = \{f_n\}_{n=1}^N$ with $f_n \in S_n$. Given subspaces $W_n \subset \mathbb{R}^d$, $1 \le n \le N$, let I denote the identity operator on $\mathbb{R}^d$ and let

$$D : \bigoplus_{n=1}^N W_n \longrightarrow \bigoplus_{n=1}^N W_n$$

denote the following $N \times N$ block operator

$$D = \begin{pmatrix} I & 0 & \cdots & & & 0 \\ -P_{W_2} & I & 0 & \cdots & & 0 \\ 0 & -P_{W_3} & I & 0 & \cdots & 0 \\ & & \ddots & & & \vdots \\ 0 & \cdots & & -P_{W_{N-1}} & I & 0 \\ 0 & \cdots & & 0 & -P_{W_N} & I \end{pmatrix}. \tag{14}$$

Note that D is a bijection and has inverse

$$D^{-1} = \begin{pmatrix} I & 0 & 0 & \cdots & 0 & 0 \\ P_{W_2} & I & 0 & & 0 & 0 \\ (P_{W_3}P_{W_2}) & P_{W_3} & I & \ddots & 0 & 0 \\ (P_{W_4}P_{W_3}P_{W_2}) & (P_{W_4}P_{W_3}) & P_{W_4} & \ddots & & \vdots \\ \vdots & \vdots & \vdots & & I & 0 \\ (P_{W_N}P_{W_{N-1}}\cdots P_{W_2}) & (P_{W_N}P_{W_{N-1}}\cdots P_{W_3}) & (P_{W_N}P_{W_{N-1}}\cdots P_{W_4}) & \cdots & P_{W_N} & I \end{pmatrix}.$$

If one defines $u \in \bigoplus_{n=1}^N W_n$, $y \in \bigoplus_{n=1}^N W_n$, $q \in \bigoplus_{n=1}^N \mathscr{A}_n$ by

$$u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}, \quad q = \begin{pmatrix} q_1 \\ q_2 \\ \vdots \\ q_N \end{pmatrix}, \tag{15}$$

then (13) can be expressed as

$$Du = y - q. \tag{16}$$

3.2 Stability

It is important in practice for one to have control on the size of the state variables u_n in $\Sigma\Delta$ algorithms. A fusion frame $\Sigma\Delta$ quantizer is said to be *stable* if there exist constants $C_1, C_2 > 0$ such that for any $\{y_n\}_{n=1}^{\infty} \in \ell^{\infty}(\mathbb{N})$ and any $N \in \mathbb{N}$ one has

$$\max_{1\leq n\leq N} \|y_n\| \leq C_1 \qquad \Longrightarrow \qquad \max_{1\leq n\leq N} \|u_n\| \leq C_2. \tag{17}$$

Stability plays an important role in the mathematical error analysis of $\Sigma\Delta$ algorithms and it also ensures that the algorithms can be implemented in circuitry without suffering from blow-up of voltages and other physical quantities. In this section, we examine stability of the first order fusion frame $\Sigma\Delta$ quantizer for different choices of the quantization alphabets $\{\mathscr{A}_n\}_{n=1}^{N}$.

Proposition 1. *Fix $\varepsilon, \delta > 0$ and let $\mathscr{A}_n$ be an ε-net for $\mathbb{B}(W_n, \varepsilon + \delta)$. Suppose that the first order fusion frame $\Sigma\Delta$ algorithm is implemented using the alphabets $\{\mathscr{A}_n\}_{n=1}^{N}$. If $\|x\| \leq \delta$, then $\|u_n\| \leq \varepsilon$ holds for all $0 \leq n \leq N$.*

Proof. The proof proceeds by induction and is almost identical to standard stability results for $\Sigma\Delta$ quantization, e.g., see Lemma 2.1 in [13] or Proposition III.2 in [4].

The base case of the induction, $n = 0$, holds since $u_0 = 0$. For the inductive step, suppose that $\|u_{n-1}\| \leq \varepsilon$. Note that

$$\|P_{W_n}(u_{n-1}) + y_n\| = \|P_{W_n}(u_{n-1} + x)\| \leq \|u_{n-1}\| + \|x\| \leq \varepsilon + \delta. \tag{18}$$

Namely, $(P_{W_n}(u_{n-1}) + y_n) \in \mathbb{B}(W_n, \varepsilon + \delta)$. Since $\mathscr{A}_n$ is an ε-net for $\mathbb{B}(W_n, \varepsilon + \delta)$, it follows that

$$\|u_n\| = \|(P_{W_n}(u_{n-1}) + y_n) - Q_n(P_{W_n}(u_{n-1}) + y_n)\| \leq \varepsilon.$$

This completes the proof.

A small choice of ε in Proposition 1 ensures the state variables u_n have small norms, but this requires fine quantization alphabets $\mathscr{A}_n$ that have large cardinality. Since physical implementations of very fine quantizers can be costly, there are instances where it is preferable to use coarse quantizers and low-bit quantization alphabets. To put this in perspective, a desirable property of $\Sigma\Delta$ quantization for scalar-valued coefficients (e.g., as for sampling expansions and finite frames) is that it is possible to achieve highly accurate quantization using only a 1-bit quantization alphabet such as $\mathscr{A} = \{-1, 1\}$ to quantize each scalar-valued coefficient, see [4, 13, 14].

In view of this, we shall consider the question of whether the first order fusion frame $\Sigma\Delta$ quantizer can be stably implemented using low-bit alphabets $\mathscr{A}_n$. We begin by considering the case of equal-norm quantization alphabets.

Let W be a subspace of $\mathbb{R}^d$ and let $\mathbb{S}(W,\rho) = \{w \in W : \|w\| = \rho\}$ denote the sphere of radius ρ in W. We shall say that a finite set $\mathscr{N} \subset \mathbb{S}(W,\rho)$ is an *angular θ-net* for $\mathbb{S}(W,\rho)$ if

$$\forall w \in W,\ \exists z \in \mathscr{N}, \qquad \text{such that} \qquad \cos^{-1}\left(\frac{\langle w, z\rangle}{\|w\|\,\|z\|}\right) \leq \theta.$$

The next result shows that the first order fusion frame $\Sigma\Delta$ algorithm can be stably implemented using equal-norm quantization alphabets.

Proposition 2. *Fix $\{\theta_n\}_{n=1}^N \subset (0,\pi)$ and let $\theta = \max_{1\leq n\leq N}\theta_n$. Let $\mathscr{A}_n$ be an angular θ_n-net for $\mathbb{S}(W_n,\rho)$. Suppose that the first order fusion frame $\Sigma\Delta$ algorithm is implemented using the alphabets $\{\mathscr{A}_n\}_{n=1}^N$.*

If $0 < \delta < \rho\cos\theta$ and $\|x\| \leq \delta$, then

$$\forall\, 0 \leq n \leq N, \qquad \|u_n\| \leq C = \max\left\{\rho, \frac{\delta^2+\rho^2-2\delta\rho\cos(\theta)}{2(\rho\cos\theta-\delta)}\right\}.$$

Proof. The proof proceeds by induction. The base case $n = 0$ holds since $u_0 = 0$.

For the inductive step suppose that $\|u_{n-1}\| \leq C$. To simplify notation let $a_n = \|P_{W_n}(u_{n-1}) + y_n\|$ and $b_n = \|q_n\| = \rho$ and $c_n = \|u_n\|$ and

$$\gamma_n = \cos^{-1}\left(\frac{\langle P_{W_n}(u_{n-1}) + y_n, q_n\rangle}{a_n b_n}\right).$$

Since $\mathscr{A}_n$ is an angular θ_n-net for $\mathbb{S}_n(\rho)$, we have $0 \leq \gamma_n \leq \theta_n \leq \theta$. Thus

$$\begin{aligned}\|u_n\|^2 &= a_n^2 + b_n^2 - 2a_nb_n\cos(\gamma_n)\\ &\leq a_n^2 + \rho^2 - 2a_n\rho\cos(\theta). \qquad (19)\end{aligned}$$

Similar to (18), note that $0 \leq a_n \leq C+\delta$. Let $f(t) = t^2 - 2t\rho\cos(\theta) + \rho^2$. In view of (19), to show that $\|u_n\| \leq C$, it suffices to show that the image of the interval $[0, C+\delta]$ under the map f satisfies

$$f\big([0,C+\delta]\big) \subset \big[0, C^2\big].$$

First, note that the global minimum of f occurs at $t = \rho\cos\theta$, and hence $f(t) \geq f(\rho\cos\theta) = \rho^2\sin^2\theta \geq 0$ holds for all $t \in \mathbb{R}$.

Next, note that the maximal value of f on the interval $[0, C+\delta]$ is achieved at either $t = 0$ or $t = (C+\delta)$. So, it remains to verify that $f(0) \leq C^2$ and $f(C+\delta) \leq C^2$. By the definition of C, we have $f(0) = \rho^2 \leq C^2$. One can also compute that

$$f(C+\delta) = C^2 - 2C(\rho\cos\theta - \delta) + \big(\delta^2 + \rho^2 - 2\delta\rho\cos\theta\big),$$

so that by the definition of C and $0 < \delta < \rho \cos\theta$ it follows that $f(C+\delta) \leq C^2$. This completes the proof.

Next, we specialize Proposition 2 to the case of low-bit alphabets $\mathscr{A}_n$ given by the vertices of the centered regular m-simplex. We recall the following properties of the regular simplex.

Lemma 1. *Let W be an m-dimensional subspace of $\mathbb{R}^d$. There exists a collection $\mathscr{V}(W) = \{v_k\}_{k=1}^{m+1}$ in W such that $\sum_{j=1}^{m+1} v_j = 0$, and each v_j is unit-norm $\|v_j\| = 1$, and*

$$\langle v_j, v_k \rangle = -\frac{1}{m}, \quad \text{for } j \neq k. \tag{20}$$

Moreover, if $\theta = \cos^{-1}(1/m)$, then $\{v_k\}_{k=1}^{m+1}$ is an angular θ-net for $\mathbb{S}(W,1)$.

The θ-net property in Lemma 1 can be found in Theorem 6.5.1 in [7]. The property (20) can be found in the literature on equiangular tight frames, e.g., see (2) in Section 2.2 of [27].

Proposition 3. *Let $\{W_n\}_{n=1}^N$ be subspaces of $\mathbb{R}^d$ with $d^\star = \max_{1\leq n\leq N} \dim(W_n)$. Let $\mathscr{A}_n = \mathscr{V}(W_n)$ be as in Lemma 1. Suppose that the first order fusion frame $\Sigma\Delta$ algorithm is implemented using the alphabets $\{\mathscr{A}_n\}_{n=1}^N$.*

If $0 < \delta < 1/d^\star$ and $\|x\| \leq \delta$, then

$$\forall\, 0 \leq n \leq N, \qquad \|u_n\| \leq C = \max\left\{1, \frac{\delta^2 + 1 - \frac{2\delta}{d^\star}}{2(\frac{1}{d^\star} - \delta)}\right\}. \tag{21}$$

Proof. Since $\mathscr{A}_n$ is an angular θ_n-net with $\theta_n = \cos^{-1}(1/d_n)$ and $d_n = \dim(W_n)$, we have $\theta = \max_{1\leq n\leq N} \theta_n = \cos^{-1}(1/d^\star)$. Since $\cos(\theta) = 1/d^\star$, the result now follows from Proposition 2.

Proposition 3 shows that the first order fusion frame $\Sigma\Delta$ quantizer can be stably implemented with a quantization alphabet of size $(d_n + 1)$ for the d_n-dimensional subspace W_n. If $\mathscr{B}_n$ denotes the number of bits per dimension, then

$$\mathscr{B}_n = \frac{\text{\# bits to encode the alphabet } \mathscr{A}_n}{\text{dimension of subspace } W_n} = \frac{\log_2(d_n+1)}{d_n}.$$

Thus, for subspaces of dimension $d_n > 1$ the first order fusion frame $\Sigma\Delta$ quantizer provides stable “sub-one-bit” quantization.

This means that if the first order fusion frame $\Sigma\Delta$ algorithm is used to implement the sensor network example from Section 1, then the sensor station s_n associated to the proximity W_n only needs to transmit a $\log_2(d_n+1)$-bit signal to the central base station.

3.3 Higher Order Algorithms

In this section we define a higher order version of $\Sigma\Delta$ quantization for fusion frames. Higher order algorithms incorporate additional memory so that the quantization step can utilize several of the previous state variables, instead of just a single preceding state variable as in (13). In the context of Section 1, this corresponds to a sensor network with increased communication capability where each sensor station can share information with more than just one immediately subsequent sensor station.

3.3.1 General *r*th Order Algorithms

To describe rth order $\Sigma\Delta$ quantization for fusion frames, let $\mathscr{G}$ denote the collection of all subspaces of $\mathbb{R}^d$ and let

$$F : \mathbb{R}^d \times \left(\bigoplus_{n=1}^{r} \mathbb{R}^d\right) \times \left(\bigoplus_{n=1}^{r} \mathscr{G}\right) \to \mathbb{R}^d$$

be a fixed function that will be referred to as a *quantization rule*. An *rth order fusion frame $\Sigma\Delta$ quantizer* takes the fusion frame measurements $y_n = P_{W_n}(x) \in W_n, 1 \leq n \leq N$, as its inputs, and produces the quantized outputs $q_n \in \mathscr{A}_n, 1 \leq n \leq N$, by initializing $u_0 = u_{-1} = \cdots = u_{1-r} = 0 \in \mathbb{R}^d$ and iterating the following for $n = 1, \cdots, N$

$$q_n = Q_n\left(F(y_n, \{u_j\}_{j=n-r}^{n-1}, \{W_j\}_{j=n-r+1}^{n})\right) \tag{22}$$

$$u_n = y_n - q_n + \sum_{i=1}^{r} (-1)^{i-1} \binom{r}{i} P_{W_n} P_{W_{n-1}} \cdots P_{W_{n-i+1}}(u_{n-i}). \tag{23}$$

We adopt the convention that $W_j = \emptyset$ when $j \leq 0$. The state variable update (23) is motivated as an analogy with (16) using higher powers of the operator D. In particular, it can be shown that if u, y, q are as in (15), then an equivalent formulation of (23) is given by

$$D^r u = y - q. \tag{24}$$

It is important that the quantization rule F in (22) is chosen so that the rth order fusion frame $\Sigma\Delta$ algorithm is stable in the sense of (17). Analyzing the stability of higher order $\Sigma\Delta$ algorithms can present a number of technically challenging issues, e.g., see [13, 15, 28] for results in the scalar-valued setting. For this reason, we shall focus on a simple and well-understood choice of F in the next section.

3.3.2 The Greedy rth Order Algorithm

The *greedy rth order fusion frame $\Sigma\Delta$ quantizer* takes the fusion frame measurements $y_n = P_{W_n}(x) \in W_n, 1 \le n \le N$, as its inputs, and produces the quantized outputs $q_n \in \mathscr{A}_n, 1 \le n \le N$, by initializing $u_0 = u_{-1} = \cdots = u_{1-r} = 0 \in \mathbb{R}^d$ and iterating the following for $n = 1, \cdots, N$

$$q_n = Q_n\left(y_n + \sum_{i=1}^{r}(-1)^{i-1}\binom{r}{i}P_{W_n}P_{W_{n-1}}\cdots P_{W_{n-i+1}}(u_{n-i})\right), \tag{25}$$

$$u_n = y_n - q_n + \sum_{i=1}^{r}(-1)^{i-1}\binom{r}{i}P_{W_n}P_{W_{n-1}}\cdots P_{W_{n-i+1}}(u_{n-i}). \tag{26}$$

Note that if $r = 1$, then this reduces to the first order fusion frame $\Sigma\Delta$ quantizer in (12) and (13).

Proposition 4. *Fix $\varepsilon, \delta > 0$ and let $\mathscr{A}_n$ be an ε-net for $\mathbb{B}(W_n, 2^r\varepsilon + \delta)$. Suppose that the greedy rth order fusion frame $\Sigma\Delta$ algorithm is implemented using the alphabets $\{\mathscr{A}_n\}_{n=1}^N$. If $\|x\| \le \delta$, then $\|u_n\| \le \varepsilon$ holds for all $(1-r) \le n \le N$.*

Proof. The proof proceeds by induction and is almost identical to standard stability results for the rth order greedy $\Sigma\Delta$ quantizer, e.g., see (18) and (19) in [16].

The base cases $(1-r) \le n \le 0$ of the induction hold since we defined $u_j = 0$ for $(1-r) \le j \le 0$. For the inductive step, suppose that $\|u_k\| \le \varepsilon$ for all $(1-r) \le k \le (n-1)$. Since $\|y_n\| = \|P_{W_n}(x)\| \le \delta$ we have

$$\|y_n + \sum_{i=1}^{r}(-1)^{i-1}\binom{r}{i}P_{W_n}P_{W_{n-1}}\cdots P_{W_{n-i+1}}(u_{n-i})\| \le \delta + \varepsilon\sum_{i=1}^{r}\binom{r}{i} = \delta + (2^r - 1)\varepsilon.$$

In other words,

$$y_n + \sum_{i=1}^{r}(-1)^{i-1}\binom{r}{i}P_{W_n}P_{W_{n-1}}\cdots P_{W_{n-i+1}}(u_{n-i}) \in \mathbb{B}(W_n, \delta + (2^r - 1)\varepsilon).$$

Since $\mathscr{A}_n$ is an ε-net for $\mathbb{B}(W_n, \delta + (2^r - 1)\varepsilon)$, it follows from (25) and (26) that $\|u_n\| \le \varepsilon$. This completes the proof.

It would be interesting to know if there exist choices of F in (22) which provide stable rth order $\Sigma\Delta$ algorithms for fusion frames when a low-bit alphabet $\mathscr{A}_n$ is used (as in Proposition 3 for the first order algorithm).

Question: Do there exist rth order fusion frame $\Sigma\Delta$ algorithms which are stable when measurements using fusion frame subspaces W_n are quantized with suitable alphabets $\mathscr{A}_n$ of cardinality $\#(\mathscr{A}_n) = 1 + \dim(W_n)$?

For perspective, it is known that there exist stable 1-bit $\Sigma\Delta$ algorithms of arbitrary order in the settings of sampling expansions and finite frames, but the constructions can be challenging and technical, e.g., [13].

4 Signal Reconstruction

In this section we shall discuss methods for reconstructing a signal $\widetilde{x}$ from the quantized coefficients $\{q_n\}_{n=1}^N$ produced by fusion frame $\Sigma\Delta$ algorithms. In the sensor network from Section 1, signal reconstruction takes place at the central base station $\mathscr{C}$ and is used to accurately extract content from the low-bit digital signals q_n that were received from the sensor stations s_n.

4.1 Notation

Recall that $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. The spectral norm of a linear operator $P : \mathbb{R}^d \to \mathbb{R}^d$ will be denoted by

$$\|P\|_{\text{spec}} = \sup\{\|Px\| : x \in \mathbb{R}^d \text{ and } \|x\| = 1\}.$$

Let $\{W_n\}_{n=1}^N$ be subspaces of $\mathbb{R}^d$. Let $f = \{f_n\}_{n=1}^N \in \bigoplus_{n=1}^N W_n$. For $1 \leq p < \infty$ define the norms

$$\|f\|_p = \left(\sum_{n=1} \|f_n\|^p\right)^{1/p} \quad \text{and} \quad \|f\|_\infty = \max_{1\leq n\leq N} \|f_n\|.$$

When $p = 2$, the norm $\|f\|_2$ is induced by the inner product

$$\langle f, g\rangle_{\bigoplus W_n} = \sum_{n=1}^N \langle f_n, g_n\rangle.$$

Given a linear operator $R : \bigoplus_{n=1}^N W_n \to \mathbb{R}^d$ and $1 \leq p \leq \infty$, define the following operator norm

$$\|R\|_{p,\bigoplus W_n} = \sup\left\{\|Rf\| : f \in \bigoplus_{n=1}^N W_n \text{ and } \|f\|_p = 1\right\}.$$

To simplify notation for the most convenient of these operator norms (i.e., when $p = 2$), we shall denote

$$\|R\|_{\text{op}} = \|R\|_{2,\bigoplus W_n}.$$

4.2 Left Inverses of the Analysis Operator

Our discussion of signal reconstruction for fusion frame $\Sigma\Delta$ algorithms will be based on choosing a suitable left inverse to the analysis operator of a fusion frame. In this section, we recall some necessary background on left inverses of the analysis operator.

Let $\{(W_n, v_n)\}_{n=1}^N$ be a fusion frame for $\mathbb{R}^d$. The associated *analysis operator* T is defined by

$$T : \mathbb{R}^d \longrightarrow \bigoplus_{n=1}^N W_n,$$
$$x \longmapsto \{v_n P_{W_n}(x)\}_{n=1}^N.$$

The adjoint of the analysis operator is called the *synthesis operator* and is defined by

$$T^* \ : \bigoplus_{n=1}^N W_n \longrightarrow \mathbb{R}^d,$$
$$\{f_n\}_{n=1}^N \longmapsto \sum_{n=1}^N v_n f_n.$$

Since the fusion frame operator (2) satisfies $S = T^*T$ and since S is invertible, one has that $L = (T^*T)^{-1}T^*$ is a left inverse to T, namely, $LT = I$. This canonical choice of left inverse is generally not a unique left inverse since the measurements $y_n = P_{W_n}(x) \in W_n, 1 \leq n \leq N$, provide an overcomplete or redundant representation of $\mathbb{R}^d$ when $\sum_{n=1}^N \dim(W_n) > d$. In fact, we shall later see that a non-canonical choice of left inverse can be preferable in the reconstruction step for fusion frame $\Sigma\Delta$ quantization.

It will be useful to recall the following two lemmas on left inverse operators; the proofs are standard, e.g., [21], but we include them for the sake of completeness. We also refer the reader to [17, 18] for further background on duals of fusion frames.

Lemma 2. *Let $F : \mathbb{R}^d \rightarrow \bigoplus_{n=1}^N W_n$ be a linear operator for which $(F^*F) : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a bijection. A linear operator $L : \bigoplus_{n=1}^N W_n \rightarrow \mathbb{R}^d$ is a left inverse of F if and only if*

$$L = (F^*F)^{-1}F^* + G(I - F(F^*F)^{-1}F^*), \tag{27}$$

for some linear operator $G : \bigoplus_{n=1}^N W_n \longrightarrow \mathbb{R}^d$.

Proof. A direct computation shows that $LF = I$ for any L of the form (27). Next, to see that every left inverse is of the form (27), suppose that $L_0 F = I$, and note that setting $L = L_0$ and $G = L_0$ yields equality in (27). This completes the proof.

Lemma 3. *Let $F : \mathbb{R}^d \to \bigoplus_{n=1}^N W_n$ be a linear operator for which $(F^*F) : \mathbb{R}^d \to \mathbb{R}^d$ is a bijection. Let $\widetilde{F} = (F^*F)^{-1}F^*$. If the linear operator $L : \bigoplus_{n=1}^N W_n \to \mathbb{R}^d$ is a left inverse to F, then $\|\widetilde{F}\|_{\mathrm{op}} \leq \|L\|_{\mathrm{op}}$.*

Proof. Since L is of the form (27), we have that $L = \widetilde{F} + Z$, where Z satisfies $ZF = 0$. Hence $\widetilde{F}Z^* = 0$ and $Z(\widetilde{F})^* = 0$. It follows that

$$\begin{aligned}\|L\|_{\mathrm{op}}^2 &= \|LL^*\|_{\mathrm{spec}} = \|(\widetilde{F}+Z)(\widetilde{F}+Z)^*\|_{\mathrm{spec}} \\ &= \|\widetilde{F}(\widetilde{F})^* + Z(\widetilde{F})^* + \widetilde{F}Z^* + ZZ^*\|_{\mathrm{spec}} = \|\widetilde{F}(\widetilde{F})^* + ZZ^*\|_{\mathrm{spec}}. \end{aligned} \tag{28}$$

Since $\widetilde{F}(\widetilde{F})^*$ is a positive operator, one has

$$\|\widetilde{F}(\widetilde{F})^*\|_{\mathrm{spec}} = \sup_{\|x\|=1} |\langle \widetilde{F}(\widetilde{F})^* x, x\rangle|.$$

and one may select $x_0 \in \mathbb{R}^d$ such that $\|x_0\| = 1$ and $\|\widetilde{F}(\widetilde{F})^*\|_{\mathrm{spec}} = |\langle \widetilde{F}(\widetilde{F})^* x_0, x_0\rangle|$. Since ZZ^* and $(\widetilde{F}(\widetilde{F})^* + ZZ^*)$ are positive operators, one has

$$\begin{aligned}\|\widetilde{F}\|_{\mathrm{op}}^2 &= \|\widetilde{F}(\widetilde{F})^*\|_{\mathrm{spec}} \\ &= |\langle \widetilde{F}(\widetilde{F})^* x_0, x_0\rangle| \\ &\leq |\langle \widetilde{F}(\widetilde{F})^* x_0, x_0\rangle + \langle ZZ^* x_0, x_0\rangle| \\ &= |\langle (\widetilde{F}(\widetilde{F})^* + ZZ^*) x_0, x_0\rangle| \\ &\leq \sup_{\|x\|=1} |\langle (\widetilde{F}(\widetilde{F})^* + ZZ^*) x, x\rangle| \\ &= \|\widetilde{F}(\widetilde{F})^* + ZZ^*\|_{\mathrm{spec}}. \end{aligned} \tag{29}$$

Combining (28) and (29) gives $\|\widetilde{F}\|_{\mathrm{op}} \leq \|L\|_{\mathrm{op}}$, and this completes the proof.

4.3 Error Bounds and Sobolev Left Inverses

In this section, we combine observations from earlier sections to give error bounds for fusion frame $\Sigma\Delta$ algorithms, and we discuss the important issue of selecting suitable left inverses L for reconstruction. Since fusion frame $\Sigma\Delta$ algorithms only act on the measurements $y_n = P_{W_n}(x)$, we will restrict our attention, with little loss of generality, to unweighted fusion frames.

Let $\{(W_n, 1)\}_{n=1}^N$ be an unweighted fusion frame for $\mathbb{R}^d$ with analysis operator T, and let L be a left inverse of T. Consider a stable rth order fusion frame $\Sigma\Delta$ algorithm that satisfies the stability condition (17) with constants $C_1, C_2 > 0$. Given $x \in \mathbb{R}^d$, use the measurements $y_n = P_{W_n}(x)$ as inputs to the rth order fusion frame

$\Sigma\Delta$ algorithm, and denote the quantized output and state variables by $\{q_n\}_{n=1}^N$ and $\{u_n\}_{n=1}^N$, respectively. Let u, y, q be as in (15). Use the left inverse L to reconstruct

$$\widetilde{x} = Lq$$

from the quantized measurements.

Proposition 5. *Assume the set-up of the previous paragraph. If* $\|x\| \leq C_1$*, then the reconstruction error for the rth order fusion frame* $\Sigma\Delta$ *algorithm satisfies*

$$\|x - \widetilde{x}\| \leq C_2 \, \|LD^r\|_{\infty, \bigoplus W_n}. \tag{30}$$

Proof. Since L is a left inverse of T, one has $x = LTx = Ly$. So, (24) gives

$$x - \widetilde{x} = LTx - Lq = L(y - q) = LD^r u. \tag{31}$$

Since $\|x\| \leq C_1$, it follows that $\|y_n\| = \|P_{W_n}(x)\| \leq \|x\| \leq C_1$ holds for all $1 \leq n \leq N$. So, (17) implies

$$\|u\|_\infty = \max_{1 \leq n \leq N} \|u_n\| \leq C_2. \tag{32}$$

Recall that $u \in \bigoplus_{n=1}^N W_n$. So, if $\|x\| \leq C_1$ then by (31) and (32)

$$\|x - \widetilde{x}\| \leq \|LD^r u\| \leq \|LD^r\|_{\infty, \bigoplus W_n} \|u\|_\infty \leq C_2 \, \|LD^r\|_{\infty, \bigoplus W_n}.$$

This completes the proof.

It is natural to ask which choice of left inverse L minimizes the upper bound (30). Unfortunately, the norm $\|\cdot\|_{\infty, \bigoplus W_n}$ is not very convenient for this question. However, it can be shown that

$$\|LD^r\|_{\infty, \bigoplus W_n} \leq \sqrt{N} \|LD^r\|_{2, \bigoplus W_n} = \sqrt{N} \|LD^r\|_{\text{op}}.$$

Combining this with (30) gives an error bound in terms of the norm $\|\cdot\|_{\text{op}}$

$$\|x - \widetilde{x}\| \leq C_2 \sqrt{N} \, \|LD^r\|_{\text{op}}. \tag{33}$$

We now address which choice of left inverse minimizes the upper bound $\|LD^r\|_{\text{op}}$ in (33).

Proposition 6. *Let* $\{(W_n, v_n)\}_{n=1}^N$ *be a fusion frame for* $\mathbb{R}^d$ *with analysis operator* T*. Suppose that* $R : \bigoplus_{n=1}^N W_n \to \bigoplus_{n=1}^N W_n$ *is a bijective linear operator. Let*

$$L_R = ((R^{-1}T)^* R^{-1} T)^{-1} (R^{-1}T)^* R^{-1}.$$

Then L_R is a minimizer of the problem

$$\min_L \|LR\|_{\mathrm{op}} \quad \text{such that} \quad LT = I.$$

Proof. Let $F = R^{-1}T$, and note that $F : \mathbb{R}^d \to \bigoplus_{n=1}^N W_n$. Next, observe that $(F^*F) : \mathbb{R}^d \to \mathbb{R}^d$ is bijective, so that F satisfies the hypotheses of Lemma 3. To see this, note that if $F^*Fz = 0$, then

$$0 = \langle F^*Fz, z\rangle = \langle Fz, Fz\rangle_{\bigoplus W_n} = \|Fz\|_2^2.$$

This implies that $R^{-1}Tz = Fz = 0$. So, $T^*Tz = T^*RR^{-1}T(z) = T^*R(0) = 0$. Since T^*T is injective this implies $z = 0$. Hence $(F^*F) : \mathbb{R}^d \to \mathbb{R}^d$ is injective. It follows from the rank-nullity theorem that $(F^*F) : \mathbb{R}^d \to \mathbb{R}^d$ is bijective.

Now suppose that $LT = I$, i.e., L is a left inverse to T. Since R is a bijection, $LRR^{-1}T = I$, i.e., LR is a left inverse of $F = R^{-1}T$. So, by Lemma 3, we have that $\|LR\|_{\mathrm{op}}$ is minimized when

$$LR = ((R^{-1}T)^*R^{-1}T)^{-1}(R^{-1}T)^*.$$

This completes the proof.

We now return to the error bound (33) for stable rth order fusion frame $\Sigma\Delta$ quantization. Applying Proposition 6 with $R = D^r$ gives the following corollary.

Corollary 1. *Let $\{(W_n, 1)\}_{n=1}^N$ be an unweighted fusion frame for $\mathbb{R}^d$ with analysis operator T. Define the rth order* Sobolev left inverse *by*

$$L_{r,\mathrm{Sob}} = ((D^{-r}T)^*D^{-r}T)^{-1}(D^{-r}T)^*D^{-r}. \tag{34}$$

Then $L_{r,\mathrm{Sob}}$ is a minimizer of the problem

$$\min_L \|LD^r\|_{\mathrm{op}} \quad \text{such that} \quad LT = I.$$

Corollary 1 shows that $L_{r,\mathrm{Sob}}$ is a left inverse to T that minimizes $\|LD^r\|_{\mathrm{op}}$ in (30). In view of this, $L_{r,\mathrm{Sob}}$ is a natural candidate for performing reconstruction from the rth order $\Sigma\Delta$ quantized fusion frame measurements. In the setting of finite frames, the work in [6, 16, 20] shows that $\|LD^r\|_{\mathrm{op}}$ decays significantly faster than $1/\sqrt{N}$ as a function of frame size N; this ensures that $\Sigma\Delta$ efficiently utilizes correlations among frame coefficients. It is reasonable to expect that such theoretical results have extensions to the setting of fusion frames, but this would take us beyond the scope of this chapter.

5 Examples and Experiments

In this section, we provide examples and experiments to illustrate the performance of fusion frame $\Sigma\Delta$ algorithms.

5.1 The First Order Algorithm

The following result gives control on $\|LD\|_{\infty,\bigoplus W_n}$ in (30) for the case of first order fusion frame $\Sigma\Delta$ quantization using unweighted tight fusion frames.

Proposition 7. *Suppose that $\{(W_n, 1)\}_{n=1}^N$ is an unweighted tight fusion frame for $\mathbb{R}^d$ with analysis operator T. Since $\{(W_n, 1)\}_{n=1}^N$ is tight, recall that $L = A^{-1}T^*$ is a left inverse to T, with A defined by (4). The error term $\|LD\|_{\infty,\bigoplus W_n}$ appearing in the fusion frame $\Sigma\Delta$ error bound (30) for $r = 1$ can be bounded by*

$$\|LD\|_{\infty,\bigoplus W_n} \leq \left(\frac{d}{\sum_{n=1}^N \dim(W_n)}\right)\left(1 + \sum_{n=1}^{N-1} \|P_{W_{n+1}} - P_{W_n}\|_{\text{spec}}\right). \tag{35}$$

Proof. If $u = \{u_n\}_{n=1}^N \in \bigoplus_{n=1}^N W_n$, then

$$Du = \begin{pmatrix} u_1 \\ u_2 - P_{W_2}(u_1) \\ u_3 - P_{W_2}(u_2) \\ \vdots \\ u_N - P_{W_N}(u_{N-1}) \end{pmatrix}.$$

By the definition of L and since $u_n \in W_n$

$$\begin{aligned} LDu &= A^{-1}\left(u_1 + \sum_{n=2}^{N}(u_n - P_{W_n}(u_{n-1}))\right) \\ &= A^{-1}\left(P_{W_1}(u_1) + \sum_{n=2}^{N}(P_{W_n}(u_n) - P_{W_n}(u_{n-1}))\right) \\ &= A^{-1}\left(P_{W_N}(u_N) + \sum_{n=1}^{N-1}(P_{W_n}(u_n) - P_{W_{n+1}}(u_n))\right). \end{aligned}$$

It follows that

$$\|LDu\| \leq A^{-1}\left(\|u_N\| + \sum_{n=1}^{N-1} \|P_{W_n} - P_{W_{n+1}}\|_{\text{spec}}\, \|u_n\|\right)$$
$$\leq A^{-1}\|u\|_\infty \left(1 + \sum_{n=1}^{N-1} \|P_{W_n} - P_{W_{n+1}}\|_{\text{spec}}\right).$$

This completes the proof.

The error term $\sum_{n=1}^{N-1} \|P_{W_{n+1}} - P_{W_n}\|_{\text{spec}}$ in (35) is analogous to the frame variation for $\Sigma\Delta$ quantization of setting of finite frames, see [4]. Since $\sum_{n=1}^{N-1} \|P_{W_{n+1}} - P_{W_n}\|_{\text{spec}}$ remains well-controlled when the consecutive subspaces W_n, W_{n+1} are close to each other, (30) and (35) show that the error for the first order fusion frame $\Sigma\Delta$ algorithm will decrease as a function of N, i.e., the algorithm efficiently utilizes correlations among the measurements. The next example illustrates this for a particular family of fusion frames.

Example 1. For each $N \geq 3$, define the system $\Phi_N = \{\varphi_n^N\}_{n=1}^N \subset \mathbb{R}^3$ by

$$\varphi_n^N = \sqrt{\frac{2}{3}}\left(\frac{1}{\sqrt{2}}, \cos\left(\frac{2\pi n}{N}\right), \sin\left(\frac{2\pi n}{N}\right)\right) \in \mathbb{R}^3.$$

The collection Φ_N is a unit-norm tight frame for $\mathbb{R}^3$, i.e.,

$$\forall x \in \mathbb{R}^d, \quad x = \frac{3}{N}\sum_{n=1}^{N} \langle x, \varphi_n^N\rangle \varphi_n^N,$$

e.g., see [29]. For each $N \geq 3$, define the collection of 2-dimensional subspaces $\{W_n^N\}_{n=1}^N \subset \mathbb{R}^3$ by

$$W_n^N = \{x \in \mathbb{R}^3 : \langle x, \varphi_n^N\rangle = 0\}. \tag{36}$$

It is easily shown that for each fixed $N \geq 3$, $\mathscr{W}_N = \{(W_n^N, 1)\}_{n=1}^N$ is an unweighted tight fusion frame for $\mathbb{R}^3$ with constant $A_N = 2N/3$, cf., Lemma 3 in [10].

For each $N \geq 3$, and $1 \leq n \leq N$, let

$$e_{1,n}^N = \left(0, \sin\left(\frac{2\pi n}{N}\right), -\cos\left(\frac{2\pi n}{N}\right)\right), \tag{37}$$

$$e_{2,n}^N = \sqrt{\frac{1}{3}}\left(-\sqrt{2}, \cos\left(\frac{2\pi n}{N}\right), \sin\left(\frac{2\pi n}{N}\right)\right). \tag{38}$$

Note that $\{e_{1,n}^N, e_{2,n}^N\}$ forms an orthonormal basis for W_n^N. Define the quantization alphabet $\mathscr{A}_n^N \subset W_n^N$ by

$$\mathscr{A}_n^N = \left\{ e_{1,n}^N, \left(\frac{-1}{2}e_{1,n}^N + \frac{\sqrt{3}}{2}e_{2,n}^N\right), \left(\frac{-1}{2}e_{1,n}^N - \frac{\sqrt{3}}{2}e_{2,n}^N\right)\right\}.$$

Note that $\mathscr{A}_n^N$ satisfies the conditions of the alphabet in Lemma 1 in the case of 2-dimensional subspaces.

Since $\{e_{1,n}^N, e_{2,n}^N\}$ is an orthonormal basis for W_n^N, we have that for each $x \in \mathbb{R}^3$

$$\begin{aligned} P_{W_n^N}(x) - P_{W_{n+1}^N}(x) &= \sum_{j=1}^{2} \langle x, e_{j,n}^N \rangle e_{j,n}^N - \sum_{j=1}^{2} \langle x, e_{j,n+1}^N \rangle e_{j,n+1}^N \\ &= \sum_{j=1}^{2} \left(\langle x, e_{j,n}^N \rangle - \langle x, e_{j,n+1}^N \rangle \right) e_{j,n}^N + \sum_{j=1}^{2} \langle x, e_{j,n+1}^N \rangle \left(e_{j,n}^N - e_{j,n+1}^N \right). \end{aligned}$$

So,

$$\begin{aligned} \|P_{W_n^N}(x) - P_{W_{n+1}^N}(x)\| &\leq \left(\sum_{j=1}^{2} \left| \langle x, e_{j,n}^N \rangle - \langle x, e_{j,n+1}^N \rangle \right|^2 \right)^{1/2} \\ &\quad + \sum_{j=1}^{2} |\langle x, e_{j,n+1}^N \rangle| \, \|e_{n,j,}^N - e_{n+1,j}^N\| \\ &\leq \|x\| \left(\sum_{j=1}^{2} \|e_{j,n}^N - e_{j,n+1}^N\|^2 \right)^{1/2} + \|x\| \sum_{j=1}^{2} \|e_{j,n,}^N - e_{j,n+1}^N\| \\ &\leq \|x\|(1+\sqrt{2}) \left(\sum_{j=1}^{2} \|e_{j,n}^N - e_{j,n+1}^N\|^2 \right)^{1/2}. \end{aligned}$$

Using that $1 - \cos(\theta) \leq \frac{\theta^2}{2}$ for $0 \leq \theta \leq \pi/2$, it can be shown that

$$\begin{aligned} \|e_{1,n}^N - e_{1,n+1}^N\| &\leq \frac{2\pi}{N}, \\ \|e_{2,n}^N - e_{2,n+1}^N\| &\leq \frac{2\pi}{N\sqrt{3}} \leq \frac{2\pi}{N}. \end{aligned}$$

Thus,

$$\|P_{W_n^N}(x) - P_{W_{n+1}^N}(x)\| \leq \|x\| \frac{2\pi\sqrt{2}(1+\sqrt{2})}{N} \leq \|x\| \left(\frac{8\pi}{N} \right).$$

So, the spectral norm satisfies

$$\|P_{W_n^N} - P_{W_{n+1}^N}\|_{\text{spec}} \leq \frac{8\pi}{N}. \tag{39}$$

Let T_N denote the analysis operator for the tight fusion frame $\mathscr{W}_N$. We have that $L_N = A_N^{-1}T_N^* = \left(\frac{3}{2N}\right)T_N^*$ is a left inverse to T_N. By Proposition 7 and (39) we have

$$\|LD\|_{\infty,\bigoplus W_n} \leq \left(\frac{3}{2N}\right)\left(1+\sum_{n=1}^{N-1}\frac{8\pi}{N}\right) \leq \frac{3(8\pi+1)}{2N} \leq \frac{18\pi}{N}. \tag{40}$$

Given $x \in \mathbb{R}^3$, let $y_n^N = P_{W_n^N}(x), 1 \leq n \leq N$, denote the fusion frame measurements of x. Use the first order fusion frame $\Sigma\Delta$ algorithm with alphabets $\{\mathscr{A}_n^N\}_{n=1}^N$, to quantize the measurements $\{y_n^N\}_{n=1}^N$, and let $\{q_n^N\}_{n=1}^N$ be the algorithm's quantized outputs and let $\{u_n^N\}_{n=1}^N$ be the associated state variables.

If $0 < \delta < 1/2$ and $\|x\| < \delta$, then Proposition 3 implies that $\|u_n^N\| \leq C$ for all $1 \leq n \leq N$, where C is given by (21). Reconstruct the signal $\widetilde{x}_N$ from $q^N = \{q_n^N\}_{n=1}^N$ by

$$\widetilde{x}_N = L_N(q^N).$$

Combining (30) and (40) shows that

$$\|x - \widetilde{x}_N\| \leq C\|LD\|_{\infty,\bigoplus W_n} \leq \frac{18C\pi}{N}. \tag{41}$$

In particular, in this example, the quantization error decays like $1/N$ as the number N of subspaces increases.

Example 2. This example shows a numerical experiment to illustrate the error bound (41) from Example 1.

Assume the set-up of Example 1 with $x = (\frac{1}{10}, \frac{\pi}{25}, -\frac{1}{\sqrt{57}})$ and $\delta = 1/4$. The constant C from (21) satisfies $C = \frac{13}{8} < 2$. Noting that $\|x\| \approx 0.2082 < \delta$, the error bound (41) gives

$$\|x - \widetilde{x}_N\| \leq \frac{36\pi}{N}.$$

Figure 1 shows a log-log plot of the error $\|x - \widetilde{x}_N\|$ against N. For comparison, the figure also shows a log-log plot of upper bound $36\pi/N$ against N.

5.2 Second Order Algorithm

In this section, we examine the performance of the second order fusion frame $\Sigma\Delta$ algorithm.

Example 3. For each $N \geq 3$, let the 2-dimensional subspaces $\{W_n^N\}_{n=1}^N$ be defined as in (36) and let $\{e_{1,n}^N, e_{2,n}^N\}$ be the orthonormal basis for W_n^N defined by (37) and (38). Recall that for each $N \geq 3$, $\{(W_n^N, 1)\}_{n=1}^N$ is an unweighted tight fusion frame for $\mathbb{R}^3$. Let T_N be the associated analysis operator.

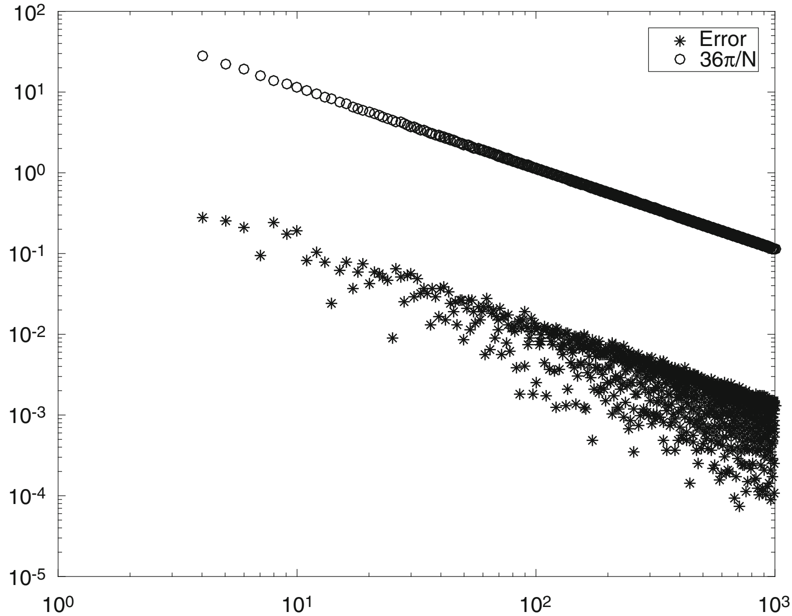

Fig. 1 Log-log plot of the error $\|x - \widetilde{x}_N\|$ versus N, and log-log plot of $36\pi/N$ versus N, see Example 2. The first order fusion frame $\Sigma\Delta$ algorithm was used.

The greedy second order fusion frame $\Sigma\Delta$ algorithm takes the following form

$$q_n = Q_n\left(y_n + 2P_{W_n}(u_{n-1}) - P_{W_n}P_{W_{n-1}}(u_{n-2})\right),$$
$$u_n = y_n + 2P_{W_n}(u_{n-1}) - P_{W_n}P_{W_{n-1}}(u_{n-2}) - q_n.$$

Define the quantization alphabet $\mathscr{A}_n^N \subset W_n^N$ by

$$\mathscr{A}_n^N = \left\{\frac{k}{4}\cos\left(\frac{\pi j}{4}\right)e_{1,n}^N + \frac{k}{4}\sin\left(\frac{\pi j}{4}\right)e_{2,n}^N : 1 \leq j \leq 8 \text{ and } 1 \leq k \leq 6\right\}.$$

Let $x = \left(\frac{\sqrt{2}}{2}, \frac{\pi-3}{2}, \frac{-1}{200}\right)$. For $N \geq 4$, let $y_n^N = P_{W_n^N}(x)$, $1 \leq n \leq N$, denote the fusion frame measurements of x. Use the greedy second order fusion frame $\Sigma\Delta$ algorithm with alphabets $\{\mathscr{A}_n^N\}_{n=1}^N$, to quantize the measurements $\{y_n^N\}_{n=1}^N$, and let $q^N = \{q_n^N\}_{n=1}^N$ be the algorithm's quantized outputs.

Let $L_N = \frac{3}{2N}T_N^*$ be as in Example 1, and recall that L_N is a left inverse of T_N. Let $L_{\text{Sob},N}$ be the Sobolev left inverse of T_N, as defined by (34) with $r = 2$. Reconstruct signals from the quantized measurements using the canonical reconstruction and the Sobolev inverse:

$$\widetilde{x}_{N,\text{can}} = L_N(q^N) \quad \text{and} \quad \widetilde{x}_{N,\text{Sob}} = L_{\text{Sob},N}(q^N).$$

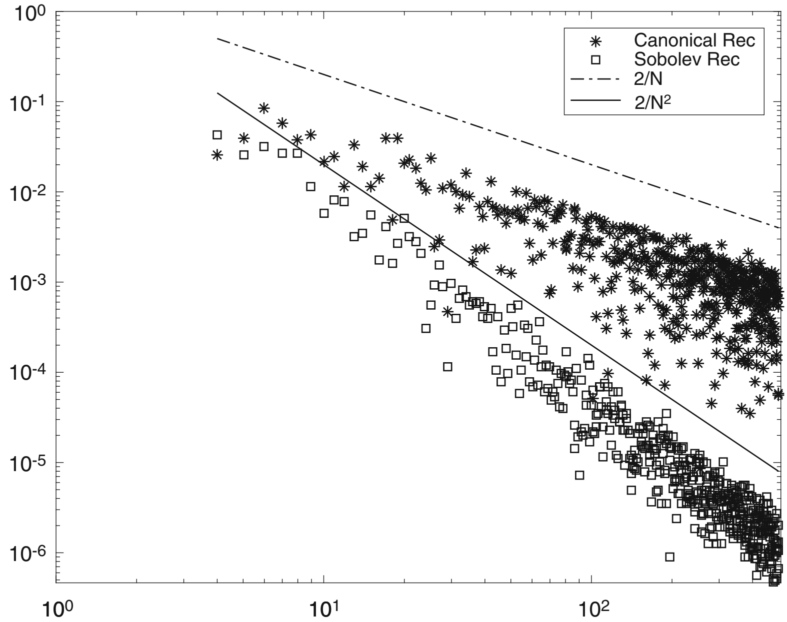

Fig. 2 Log-log plot of the error versus N when the signal is reconstructed using either the canonical left inverse or the Sobolev left inverse, see Example 3. The greedy second order fusion frame $\Sigma\Delta$ algorithm was used. For comparison, log-log plots of $2/N$ and $2/N^2$ versus N are given.

Figure 2 shows log-log plots of $\|x - \widetilde{x}_{N,\text{can}}\|$ and $\|x - \widetilde{x}_{N,\text{Sob}}\|$ against N. For comparison, log-log plots of $2/N$ and $2/N^2$ against N are also given. The figure shows that reconstruction with the Sobolev left inverse leads to significantly smaller error.

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Part III
Space-Frequency Analysis in Function Spaces on $\mathbf{R}^n$

Recent Progress in Shearlet Theory: Systematic Construction of Shearlet Dilation Groups, Characterization of Wavefront Sets, and New Embeddings

Giovanni S. Alberti, Stephan Dahlke, Filippo De Mari, Ernesto De Vito, and Hartmut Führ

1 Introduction

This chapter is concerned with several important aspects of modern signal analysis. Usually, signals are modeled as elements of function spaces such as L^2 or Sobolev spaces. To analyze such a signal and to extract the information of interest from it, the first step is always to decompose the signal into suitable building blocks. This is performed by transformation, i.e., the signal is mapped into function spaces on an underlying parameter set, and then the signal is processed and analyzed by studying and modifying the resulting coefficients. By now, a whole zoo of suitable transforms have been introduced and analyzed such as the Fourier transform, the Gabor transform, or the wavelet transform, just to name a few. Some of them have already been very successful, e.g., the Fourier transform works excellently for signals that are well localized in the frequency domain, whereas wavelets are often the method of choice for the analysis of piecewise smooth signals with well-localized singularities such as edges in an image. Which transform to choose obviously

G.S. Alberti (✉)
Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, Genova, Italy
e-mail: alberti@dima.unige.it

S. Dahlke
FB12 Mathematik und Informatik, Philipps-Universität Marburg, Hans-Meerwein-Straße, Lahnberge, 35032 Marburg, Germany
e-mail: dahlke@mathematik.uni-marburg.de

F. De Mari • E. De Vito
Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, Genova, Italy
e-mail: demari@dima.unige.it; devito@dima.unige.it

H. Führ
Lehrstuhl A für Mathematik, RWTH Aachen University, 52056 Aachen, Germany
e-mail: fuehr@matha.rwth-aachen.de

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depends on the application, i.e., on the type of information one wants to detect from the signal. However, in recent years, it has turned out that a serious bottleneck still has to be removed. Most of the classical transforms such as the wavelet transform perform suboptimally when it comes to the detection of directional information. The reason is very simple: most of these transforms are essentially isotropic, whereas directional information is of anisotropic nature. This observation triggered many innovative studies how to design new building blocks that are particularly tuned to this problem, such as curvelets [4], contourlets [16], ridgelets [3], and many others. In this chapter, we are in particular interested in one specific contribution to this problem, i.e., the shearlet approach. Shearlets are new affine representation systems that are based on translations, shearings, and anisotropic dilations. We refer to the monograph [35] for an overview. Among all the new approaches, the shearlet transform stands out for the following reason: the continuous shearlet transform can be derived from a square-integrable representation of a specific group, the full shearlet group [7–9]. This property is not only of academic interest but also has the important consequence that the whole powerful machinery derived in the realm of square-integrable group representations such as reproducing kernels, inversion formulas, etc. can directly be employed. This feature of the shearlet transform clearly has strengthened the interest in the shearlet theory, and many important results concerning the group-theoretical background have been derived so far. It is the aim of this chapter to push forward, to clarify, and to unify this theory with respect to several important aspects. Our main objectives can be described as follows.

After the full shearlet group has been discovered, the question arose if other suitable concepts of shearlet groups could be constructed. A first example was the shearlet Toeplitz group [6], where the shearing part of the group has a Toeplitz structure. As we will see later in Subsection 3.3 of this chapter, the full shearlet group and the shearlet Toeplitz group are in a certain sense the "extreme" cases of a general construction principle. In this view, the full shearlet group corresponds to the nilpotency class $n = 2$, whereas the Toeplitz case corresponds to the nilpotency class $n = d$, where d denotes the space dimension. Therefore, one would conjecture that there should be a lot of examples "in between". Indeed, in [26], a positive answer has been given, and a first classification of low-dimensional shearlet groups has been derived. In this chapter, we further extend these results and present an approach to the systematic construction of suitable shearlet groups. The starting point is a general class of shearlet groups introduced in [26]. We say that a dilation group H is a shearlet group if every $h \in H$ can be written as $h = \pm ds, d \in D, s \in S$ where D is a diagonal *scaling subgroup* and S denotes a connected, closed abelian matrix group, the *shearing subgroup*. The key to understanding and constructing shearing subgroups lies in the realization that their associated Lie algebras carry a very useful associative structure. This associative structure also greatly facilitates the task of identifying the diagonal scaling groups compatible with a given shearing subgroup. Through the notion of Jordan-Hölder bases the problem of characterizing all suitable scaling group generators translates to a rather simple linear system of equations. It turns out that all examples known so far are special cases of this general construction.

In recent studies, it has also been observed that shearlets provide a very powerful tool in microlocal analysis [28], e.g., to determine the local regularity of a function. In the one-dimensional case, pointwise smoothness can very efficiently be detected by examining the decay of the continuous wavelet transform as the scale parameter a tends to zero [32]. In the multivariate setting, pointwise smoothness does not cover all the geometric information one might be interested in. E.g., if the function under consideration exhibits singularities, one usually wants to know in which direction the function is singular. This can be described by the so-called *wavefront set* of a distribution. It has turned out that the continuous shearlet transform can be employed to detect this wavefront set, once again by studying its decay as the scaling parameter tends to zero. This property has been first observed in [34]; we also refer to [28] for an overview, and to [29] for a sample application of these techniques in image processing. In [21], these concepts have been generalized to much more general classes of dilation groups. It has been shown that under natural assumptions, a wavefront set can again be detected by the decay of the voice transform. Essentially, two fundamental conditions are needed, that are related with the *dual action* of the dilation group H: the dual action must be microlocally admissible in direction ξ and it must satisfy the V-cone approximation property at ξ, see Section 4.1 for the precise definitions. If these properties hold for one point ξ_0 in the open dual orbit, a characterization of wavefront sets is possible. In this chapter, we show that both properties are satisfied for our general construction of shearlet dilation groups, provided that the infinitesimal generator Y of the scaling subgroup satisfies $Y = \mathrm{diag}(1, \lambda_2, \ldots, \lambda_d), 0 < \lambda_i < 1, 2 \leq i \leq d$. Consequently, characterizations of wavefront sets are possible for a huge subclass of our general construction. It is worth mentioning that anisotropic dilations are necessary for the detection of wavefront sets, in particular the classical (isotropic) continuous wavelet transform would not do the job.

A third important issue we will be concerned with in this chapter is the relations of our general shearlet groups to other classical groups, in particular to the symplectic groups $Sp(d, \mathbb{R})$. The symplectic groups are among the most important classical groups, because they play a prominent role in classical mechanics. We therefore investigate to which extent our shearlet dilation groups can be embedded into symplectic groups, in a way that intertwines the quasi-regular representation with the metaplectic representation. For the full shearlet groups and the shearlet Toeplitz groups, these issues have been studied in [14], see also [33]. Their connected components can indeed be embedded into the symplectic groups, which yields group isomorphisms of the positive parts of shearlet groups with the so-called TDS(d) subgroups that have already been studied in [15]. In this chapter, we generalize this result to dilation groups of the form $G = \mathbb{R}^d \rtimes H$, where H is a subgroup of $T(d, \mathbb{R})_+ = \{h \in \mathrm{GL}(d, \mathbb{R}) : h_{1,1} > 0 \text{ and } h_{i,j} = 0 \text{ for every } i > j\}$. We show that for any such group there exists a group embedding $\phi : G \to Sp(d, \mathbb{R})$, and that its quasi-regular representation is unitarily equivalent to $\mu \circ \phi$, where μ denotes the metaplectic representation of $Sp(d, \mathbb{R})$. Since the positive part of any shearlet group falls into this general category, the desired embeddings for shearlet groups follow from this result. Let us also mention the following very interesting fact: for the full shearlet dilation groups, such embeddings are never possible. Indeed, in [14] it has been shown that already for the two-dimensional full shearlet group there does not exist an injective continuous homomorphism into $Sp(2, \mathbb{R})$ or into any of its coverings.

Let us also mention a nice by-product of our construction. In recent studies [8, 9, 11, 12], an interesting relation of the shearlet approach to the coorbit theory derived by Feichtinger and Gröchenig [17–20, 27] has been established. Based on a square-integrable group representation, coorbit space theory gives rise to canonical associated smoothness spaces, where smoothness is measured by the decay of the underlying voice transform. In [8–13], it has been shown that all the conditions needed in the coorbit setting can be satisfied for the full shearlet and the shearlet Toeplitz groups. In [24], the coorbit approach has been extended to much more general classes of dilation groups, and it turns out that the analysis from [24] also carries over to the construction presented in this chapter, so that we obtain many new examples of coorbit spaces. In particular, we refer to [25, 26] for explicit criteria for compactly supported functions that can serve as atoms in the coorbit scheme.

This chapter is organized as follows. In Sections 2 and 3, we present our construction of generalized shearlet dilation groups. After discussing the basic notations and definitions in the Subsections 2.1 and 2.2, in Subsection 2.3 we start with the systematic investigation of the Lie algebras of shearing subgroups. One of the main results is Lemma 6 which provides a complete description of a shearing subgroup in terms of the canonical basis of its Lie algebra. This fact can be used to derive linear systems whose nonzero solutions determine the anisotropic scaling subgroups that are compatible with S (Lemma 9). These relationships are then used in Section 3 to derive a systematic construction principle. The canonical basis can be directly computed from the structure constants of a Jordan-Hölder basis (Lemma 13). The power of this approach is demonstrated by several examples. In Section 4, we study the suitability of shearlet dilation groups for the characterization of wavefront sets. Here the main result is Theorem 28 which shows that shearlet groups with anisotropic dilations and suitable infinitesimal generators for the scaling subgroups do the job. The proof is performed by verifying the basic conditions from [21]. The last section is concerned with the embeddings of shearing dilation groups into symplectic groups. The main result of this section is Theorem 33 which shows that the huge class of semidirect products of the form $G = \mathbb{R}^d \rtimes H$, where H is a subgroup of $T(d, \mathbb{R})_+$ can be embedded into $Sp(d, \mathbb{R})$.

2 Generalities on Shearlet Dilation Groups

2.1 Basic Notations and Definitions

As explained in the introduction to this chapter, we are concerned with the construction and analysis of large classes of generalized shearlet transforms. These transforms are constructed by fixing a suitable matrix group, the so-called *shearlet* *dilation group*. By construction, these groups have a naturally associated isometric continuous wavelet transform, which will be the generalized shearlet transform. In this subsection, we summarize the necessary notation related to general continuous wavelet transforms in higher dimensions. We let $\mathrm{GL}(d, \mathbb{R})$ denote the group of invertible $d \times d$-matrices. We use I_d to denote the $d \times d$ identity matrix. The Lie algebra of $\mathrm{GL}(d, \mathbb{R})$ is denoted by $\mathfrak{gl}(d, \mathbb{R})$, which is the space of all $d \times d$ matrices, endowed with the Lie bracket $[X, Y] = XY - YX$. Given $h \in \mathfrak{gl}(d, \mathbb{R})$ its (operator) norm is denoted by

$$\|h\| = \sup_{|x| \leq 1} |hx|.$$

We let $\exp : \mathfrak{gl}(d, \mathbb{R}) \to \mathrm{GL}(d, \mathbb{R})$ denote the exponential map, defined by

$$\exp(X) = \sum_{k=0}^{\infty} \frac{X^k}{k!},$$

known to converge absolutely for every matrix X. Given a closed subgroup $H < \mathrm{GL}(d, \mathbb{R})$, the associated Lie algebra of H is denoted by $\mathfrak{h}$, and it is defined as tangent space of H at I_d, or, equivalently, as the set of all matrices X with $\exp(\mathbb{R}X) \subset H$. It is a Lie subalgebra of $\mathfrak{gl}(d, \mathbb{R})$, i.e., it is closed under taking Lie brackets.

A matrix group of particular importance for the following is the group $T(d, \mathbb{R})$ of upper triangular matrices with ones on the diagonal. Elements of $T(d, \mathbb{R})$ are called *unipotent*. Its Lie algebra is the subspace $\mathfrak{t}(d, \mathbb{R}) \subset \mathfrak{gl}(d, \mathbb{R})$ of all strictly upper triangular matrices. It is well known that $\exp : \mathfrak{t}(d, \mathbb{R}) \to T(d, \mathbb{R})$ is a homeomorphism [31]. In particular, whenever $\mathfrak{s} \subset \mathfrak{t}(d, \mathbb{R})$ is a Lie subalgebra, the exponential image $\exp(\mathfrak{s})$ is a closed, simply connected and connected matrix group with Lie algebra given by $\mathfrak{s}$. Conversely, any connected Lie subgroup S of $T(d, \mathbb{R})$ is closed, simply connected and $S = \exp(\mathfrak{s})$ where $\mathfrak{s} \subset \mathfrak{t}(d, \mathbb{R})$ is the corresponding Lie algebra, see Theorem 3.6.2 of [36].

For the definition of generalized wavelet transforms, we fix a closed matrix group $H < \mathrm{GL}(d, \mathbb{R})$, the so-called *dilation group*, and let $G = \mathbb{R}^d \rtimes H$. This is the group of affine mappings generated by H and all translations. Elements of G are denoted by pairs $(x, h) \in \mathbb{R}^d \times H$, and the product of two group elements is given by $(x, h)(y, g) = (x + hy, hg)$. The left Haar measure of G is given by $d\mu_G(x, h) = |\det(h)|^{-1} dx dh$, where dx and dh are the Lebesgue measure and the (left) Haar measure of $\mathbb{R}^d$ and H, respectively.

The group G acts unitarily on $\mathrm{L}^2(\mathbb{R}^d)$ by the *quasi-regular representation* defined by

$$[\pi(x, h)f](y) = |\det(h)|^{-1/2} f\left(h^{-1}(y - x)\right) . \quad (1)$$

We assume that H is chosen *irreducibly admissible*, i.e. such that π is an *(irreducible) square-integrable representation*. Recall that a representation is irreducible if the only invariant closed subspaces of the representation space are the trivial ones. Square-integrability of the representation means that there exists at least one nonzero *admissible vector* $\psi \in \mathrm{L}^2(\mathbb{R}^d)$ such that the matrix coefficient

$$(x,h) \mapsto \langle \psi, \pi(x,h)\psi \rangle$$

is in $\mathrm{L}^2(G)$, which is the L^2-space associated to the left Haar measure $d\mu_G$. In this case the associated wavelet transform

$$W_\psi : \mathrm{L}^2(\mathbb{R}^d) \ni f \mapsto ((x,h) \mapsto \langle f, \pi(x,h)\psi \rangle) \in L^2(G) \tag{2}$$

is a scalar multiple of an isometry, which gives rise to the *wavelet inversion formula*

$$f = \frac{1}{c_\psi} \int_G W_\psi f(x,h)\pi(x,h)\psi \; d\mu_G(x,h) \;, \tag{3}$$

where the integral is in the weak sense, and $c_\psi > 0$ is a normalization constant depending on the wavelet.

We note that the definition of $W_\psi f$ also makes sense for tempered distributions f, as soon as the wavelet ψ is chosen as a Schwartz function and the L^2-scalar product is properly extended to a sesquilinear map $\mathscr{S}' \times \mathscr{S} \to \mathbb{C}$. Analogs of the wavelet inversion formula are not readily available in this general setting, but it will be seen below that the transform has its uses, for example in the characterization of wavefront sets.

Most relevant properties of the wavelet transform are in some way or another connected to the *dual action*, i.e., the (right) linear action $\mathbb{R}^d \times H \ni (\xi, h) \mapsto h^T\xi$. For example, H is irreducibly admissible if and only if the dual action has a single open orbit $\mathscr{O} = \{h^T\xi_0 : h \in H\} \subset \mathbb{R}^d$ of full measure (for some $\xi_0 \in \mathscr{O}$), such that in addition the stabilizer group $H_{\xi_0} = \{h \in H : h^T\xi_0 = \xi_0\}$ is compact [23]. This condition does of course not depend on the precise choice of $\xi_0 \in \mathscr{O}$. The dual action will also be of central importance to this chapter.

2.2 Shearlet Dilation Groups

The original shearlet dilation group was introduced in [7, 8], as

$$H = \left\{ \pm \begin{pmatrix} a & b \\ 0 & a^{1/2} \end{pmatrix} : a > 0, b \in \mathbb{R} \right\} \;.$$

The rationale behind this choice was that the anisotropic scaling, as prescribed by the exponents $1, 1/2$ on the diagonal, combines with the shearing (controlled by the parameter b) to provide a system of generalized wavelets that are able to swiftly adapt to edges of all orientations (except one). A mathematically rigorous formulation of this property is the result, due to Kutyniok and Labate, that the continuous shearlet transform characterizes the wavefront set [34]. Approximation-theoretic properties of a different, more global kind were the subject of the chapter [8] describing the so-called *coorbit spaces* defined in terms of weighted integrability conditions on the wavelet coefficients.

The original shearlet dilation group has since been generalized to higher dimensions. Here, the initial construction was introduced in [9], and further studied, e.g., in [5, 11]. It is a matrix group in dimension $d \geq 3$ defined by

$$S = \left\{ \pm \begin{pmatrix} a & s_1 & \dots & s_{d-1} \\ & a^{\lambda_2} & & \\ & & \ddots & \\ & & & a^{\lambda_d} \end{pmatrix} : a > 0,\ s_1, \dots, s_{d-1} \in \mathbb{R} \right\}. \tag{4}$$

Here $\lambda_2, \dots, \lambda_d$ are positive exponents, often chosen as $\lambda_2 = \dots = \lambda_d = 1/2$. It should, however, be noted that they can be chosen essentially arbitrarily (even negative), without affecting the wavelet inversion formula. Coorbit space theory is applicable to all these groups as well [11, 24]. Furthermore, it was recently shown that, for suitable choices of the analyzing wavelet, the associated shearlet transform also characterizes the wavefront set [21], as long as the exponents $\lambda_2, \dots, \lambda_d$ are strictly between zero and one.

A second, fundamentally different class of shearlet groups are the *Toeplitz shearlet groups* introduced in [6] and further studied in [12]. These groups are given by

$$H = \left\{ \pm \begin{pmatrix} a & s_1 & s_2 & \dots & \dots & \dots & s_{d-1} \\ & a & s_1 & s_2 & \dots & \dots & s_{d-2} \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & \ddots & \ddots & \ddots & \vdots \\ & & & & \ddots & \ddots & s_2 \\ & & & & & \ddots & s_1 \\ & & & & & & a \end{pmatrix} : a > 0,\ s_1, \dots, s_{d-1} \in \mathbb{R} \right\}. \tag{5}$$

Coorbit space theory can be applied to these groups as well [12, 24]. By [21, Lemma 4.10], the fact that H contains nontrivial multiples of the identity implies that H does not characterize the wavefront set. However, it will be shown below that by properly adjusting the diagonal entries, it is possible to construct a closely related group H' that does lend itself to the characterization of the wavefront set.

A closer inspection of the two higher-dimensional families of shearlet groups reveals several common traits: fix one of the above-listed groups H. Then each $h \in H$ factors as

$$h = \pm \mathrm{diag}(a, a^{\lambda_2}, \dots, a^{\lambda_d}) \cdot u$$

where the first factor denotes the diagonal matrix with the same diagonal entries as h, and the second factor u is unipotent. In fact, this factorization is necessarily unique. Furthermore, denoting by D the set of all diagonal matrices occurring in such factorizations, and by S the set of all unipotent ones that arise, it is easy to see that D (and consequently S) are closed subgroups of H. Finally, one readily verifies that the groups S that occur in the examples are in fact commutative. We will now use these properties to define a general class of shearlet dilation groups, that we will study in this chapter:

Definition 1. Let $H < \mathrm{GL}(d, \mathbb{R})$ denote an irreducibly admissible dilation group. H is called *generalized shearlet dilation group*, if there exist two closed subgroups $S, D < H$ with the following properties:

(i) S is a connected abelian Lie subgroup of $T(d, \mathbb{R})$;
(ii) $D = \{\exp(rY) : r \in \mathbb{R}\}$ is a one-parameter group, where Y is a diagonal matrix;
(iii) Every $h \in H$ can be written uniquely as $h = \pm ds$, with $d \in D$ and $s \in S$.

S is called the *shearing subgroup* of H, and D is called the *diagonal complement* or *scaling subgroup* of H.

Remark 2. As noted in Subsection 2.1, S is closed and simply connected and the exponential map is a diffeomorphism from its Lie algebra $\mathfrak{s}$ onto S.

Remark 3. The class of shearlet dilation groups, as considered in this chapter, was initially defined in [26], and for some of the following results and observations, more detailed proofs can be found in that paper. In particular, it was shown there that coorbit space theory applies to all generalized shearlet dilation groups. In fact, it is possible to construct wavelet frames with compactly supported atoms, with frame expansions that, depending on the provenance of the signal, converge in a variety of coorbit space norms simultaneously.

The notion of a *shear group* occurred prior to [26] in [30], where it refers to a discrete group of matrices generated by commuting matrices b with $(b - I_d)^2 = 0$. Up to a suitable change of coordinates, these groups are contained in $T(d, \mathbb{R})$, just as our class of shearing subgroups. Apart from this rather general observation, the precise relationship between shear groups according to [30] and shearlet dilation subgroups according to Definition 1 is currently unclear.

As will be seen below, shearlet dilation groups can be constructed systematically. The natural order in finding the constituent subgroups S, D is to first pick a candidate for S, and then determine the infinitesimal generators of the one-parameter group D that are compatible with S. The details of this programme are given in the next subsections.

2.3 Shearlet Dilation Groups and Their Lie Algebras

It is the aim of this subsection to give an overview of the most important structural properties of shearlet dilation groups. The following proposition gives a first characterization of these groups, see [26, Proposition 4.3]. In the following, we will make repeated use of associative commutative algebras. If $\mathscr{A}$ denotes such an algebra containing an identity element $1_{\mathscr{A}}$, we let $\mathscr{A}^{\times}$ denote the group of invertible elements in $\mathscr{A}$, i.e., the set of all elements $a \in \mathscr{A}$ for which $b \in \mathscr{A}$ exists with $ab = 1_{\mathscr{A}}$.

Proposition 4. *Let S denote a connected abelian subgroup of $T(d, \mathbb{R})$. Then the following are equivalent:*

(i) S is the shearing subgroup of a generalized shearlet dilation group;
(ii) There is $\xi \in \mathbb{R}^d$ such that S acts freely on $S^T\xi$ via the dual action, and in addition, $\dim(S) = d - 1$*;*
(iii) The matrix group $A = \{rs : s \in S, r \in \mathbb{R}^{\times}\}$ is an abelian irreducibly admissible dilation group. It is also a shearlet dilation group.

The fundamental observation made in [22, Remark 9] is that if A is abelian and admissible, as in part (iii) of the above proposition, then its Lie algebra $\mathfrak{a}$ is in fact an *associative* subalgebra containing the identity element, hence it is closed under matrix multiplication. This associative structure is in many ways decisive. To begin with, one has the relations

$$\mathfrak{a} = \mathrm{span}(A), \qquad A = \mathfrak{a}^{\times}.$$

We will see in Subsection 3.1 below that this connection to associative algebras can be used for the systematic –even exhaustive– construction of shearing subgroups.

There is however a second ingredient, that is more directly related to the properties of the dual action. It is described in the following lemma, see [26, Corollary 4.7]. We use $e_1, \ldots, e_d$ for the canonical basis of $\mathbb{R}^d$.

Lemma 5. *Let S denote a connected abelian subgroup of $T(d, \mathbb{R})$ of dimension $d - 1$, with Lie algebra $\mathfrak{s}$. Then the following are equivalent:*

(i) S is a shearing subgroup;
(ii) There exists a unique basis $X_2, \ldots, X_d$ of $\mathfrak{s}$ with $X_i^T e_1 = e_i$, for all $i = 2, \ldots, d$.

We call the basis from part (ii) the canonical basis *of $\mathfrak{s}$.*

The canonical basis plays a special role for the description of shearing subgroups. As a first indication of its usefulness, we note that all off-diagonal entries of the elements of shearing groups depend linearly on the entries in the first row.

Lemma 6. *Let S denote a shearing subgroup with Lie algebra $\mathfrak{s}$, and canonical basis $X_2, \ldots, X_d$ of $\mathfrak{s}$. Then the following holds:*

(a) $S = \{I_d + X : X \in \mathfrak{s}\}$.

(b) Let $h \in S$ be written as

$$h = \begin{pmatrix} 1 & h_{1,2} & \dots & \dots & \dots & h_{1,d} \\ 0 & 1 & h_{2,3} & \dots & \dots & h_{2,d} \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 & h_{d-1,d} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$$h = I_d + \sum_{i=2}^{d} h_{1,i} X_i \ .$$

Proof. For part (a), denote the right-hand side by S_1. Since $\mathfrak{s}$ is an associative subalgebra consisting of nilpotent matrices, S_1 consists of invertible matrices, and it is closed under multiplication. Furthermore, the inverse of any element of S_1 can be computed by a Neumann series that breaks off after at most d terms:

$$(I_d + X)^{-1} = I_d + \sum_{k=2}^{d-1} (-1)^{k-1} X^k \ ,$$

and the result is again in S_1. Hence S_1 is a matrix group. It is obviously closed and connected, with tangent space of S_1 at the identity matrix given by $\mathfrak{s}$. It follows that S_1 is a Lie subgroup of $T(d, \mathbb{R})$ and, hence, it is simply connected. Thus S and S_1 are closed, connected, and simply connected subgroups sharing the same Lie algebra, hence they are equal. Now part (b) directly follows from (a) and the properties of the canonical basis.

We now turn to the question of characterizing the scaling subgroups D that are compatible with a given shearing subgroup S. It is convenient to describe D in terms of its Lie algebra as well. Since D is one-dimensional, we have $D = \exp(\mathbb{R}Y)$, with a diagonal matrix $Y = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_d)$. We then have the following criterion [26, Proposition 4.5]:

Proposition 7. *Let $S < \mathrm{GL}(d, \mathbb{R})$ denote a shearing subgroup. Let Y denote a nonzero diagonal matrix, and let $D := \exp(\mathbb{R}Y)$ the associated one-parameter group with infinitesimal generator Y. Then the following are equivalent:*

(i) $H = DS \cup (-DS)$ is a shearlet dilation group;
(ii) For all $X \in \mathfrak{s}$ we have $[X, Y] = XY - YX \in \mathfrak{s}$, and in addition the first diagonal entry of Y is nonzero.

Remark 8. The above proposition states that $H = S \rtimes \mathbb{R}^\times$, so that H is solvable group with two connected components, and each of them is simply connected.

Since Y and rY, for nonzero $r \in \mathbb{R}$, determine the same one-parameter subgroup, part (ii) of the proposition allows to fix $\lambda_1 = 1$. Note that part (ii) is trivially fulfilled by *isotropic scaling*, which corresponds to taking $1 = \lambda_1 = \lambda_2 = \ldots = \lambda_d$. In what follows, we will be particularly interested in anisotropic solutions; our interest in these groups is mainly prompted by the crucial role of anisotropic scaling for wavefront set characterization.

It turns out that the relation $[Y, \mathfrak{s}] \subset \mathfrak{s}$ translates to a fairly transparent system of linear equations. Once again, the canonical basis $X_2, \ldots, X_d$ of $\mathfrak{s}$ proves to be particularly useful: As the following lemma shows, the adjoint action $\mathfrak{s} \ni X \mapsto [Y, X]$ maps $\mathfrak{s}$ into itself if and only if the X_i are eigenvectors of that map. The lemma uses the notation $E_{i,j}$ for the matrix having entry one at row i and column j, and zeros everywhere else.

Lemma 9. *Let $\mathfrak{s}$ denote the Lie algebra of a shearing subgroup, and let $X_2, \ldots, X_d$ denote the canonical basis of $\mathfrak{s}$, given by*

$$X_i = E_{1,i} + \sum_{j=2}^{d} \sum_{k=j+1}^{d} d_{i,j,k} E_{j,k} \tag{6}$$

with suitable coefficients $d_{i,j,k}$. Let $Y = \mathrm{diag}(1, \lambda_2, \ldots, \lambda_d)$ be given. Then $[Y, \mathfrak{s}] \subset \mathfrak{s}$ if and only if

$$\text{for all } i = 2, \ldots, d \ : \ \lambda_i = 1 + \mu_i \ ,$$

and the vector $(\mu_2, \ldots, \mu_d)$ is a solution of the system of linear equations given by

$$\text{for all } (i, j, k) \in \{2, \ldots, d\}^3 \text{ with } d_{i,j,k} \neq 0 \ : \ \mu_i + \mu_j = \mu_k \ . \tag{7}$$

In particular, $(\mu_2, \ldots, \mu_d) \mapsto (1, 1 + \mu_2, \ldots, 1 + \mu_d)$ sets up a bijection between the nonzero solutions of (7) on the one hand and the anisotropic scaling subgroups D compatible with S on the other.

Remark 10. Note that (6) shows that $d_{i,j,k} = (X_i)_{jk}$.

Proof. We first note that the $E_{j,k}$ are eigenvectors under the adjoint action of any diagonal matrix:

$$[Y, E_{j,k}] = (\lambda_j - \lambda_k) E_{j,k} \ . \tag{8}$$

As a consequence, given any matrix X, the support of the matrix $[Y, X]$ (i.e., the set of indices of its nonzero entries) is contained in the support of X.

Note that Y normalizes $\mathfrak{s}$ if and only if $[Y, X_i] \in \mathfrak{s}$ for $i = 2, \ldots, d$. Now the calculation

$$[Y, X_i] = [Y, E_{1,i}] + \sum_{(j,k)} d_{i,j,k}[Y, E_{j,k}] = (1 - \lambda_i)E_{1,i} + \sum_{(j,k)} d_{i,j,k}(\lambda_j - \lambda_k)E_{j,k} \tag{9}$$

shows that the only (potentially) nonzero entry in the first row of $[Y, X_i]$ occurs at the ith column, hence $[Y, X_i]$ is in $\mathfrak{s}$ if and only if it is a scalar multiple of X_i. In view of (9) and the linear independence of the $E_{j,k}$, this holds precisely when

$$\text{for all } (i,j,k) \in \{2, \ldots, d\}^3 \text{ with } d_{i,j,k} \neq 0 \ : \ 1 - \lambda_i = \lambda_j - \lambda_k \, . \tag{10}$$

Rewriting this system for $\mu_i = \lambda_i - 1$, for $i = 2, \ldots, d$, yields (7).

Finally, let us return to properties of the associated shearlet transforms. In view of the central role of the dual action, it is important to compute the associated open dual orbit. Here we have the following, see [26, Proposition 4.5]:

Proposition 11. *Let S be a shearing subgroup, and D any diagonal complement of S. Then $H = DS \cup -DS$ acts freely on the unique open dual orbit given by $\mathcal{O} = \mathbb{R}^\times \times \mathbb{R}^{d-1}$.*

Note that the dual orbit is the same for all shearing groups. Somewhat surprisingly, the same can be said of the admissibility condition [26, Theorem 4.12]:

Theorem 12. *Let $H < \mathrm{GL}(\mathbb{R}^d)$ denote a generalized shearlet dilation group. Then $0 \neq \psi \in \mathrm{L}^2(\mathbb{R}^d)$ is admissible iff*

$$\int_{\mathbb{R}^d} \frac{|\widehat{\psi}(\xi)|^2}{|\xi_1|^d} d\xi < \infty \, .$$

3 A Construction Method for Shearlet Dilation Groups

3.1 Constructing Shearing Subgroups

In this subsection we want to describe a general method for the systematic construction of shearing subgroups. Recall that given a shearing subgroup S with Lie algebra $\mathfrak{s}$, taking the Lie algebra $\mathfrak{a} = \mathbb{R}I_d \oplus \mathfrak{s}$ and its associated closed matrix group A results in an abelian irreducibly admissible matrix group. Following [22], this entails that $\mathfrak{a}$ is an associative matrix algebra. Furthermore, note that $\mathfrak{s}$ consists of strictly upper triangular matrices, which entails that any product of d elements of $\mathfrak{s}$ vanishes.

These features of $\mathfrak{a}$ can be described in general algebraic terms. Given a finite-dimensional, associative commutative algebra $\mathscr{A}$, we call an element $a \in \mathscr{A}$ *nilpotent* if there exists $n \in \mathbb{N}$ such that $a^n = 0$. The set of all nilpotent elements in $\mathscr{A}$ is called the *nilradical* of $\mathscr{A}$, denoted by $\mathscr{N}$. We call $\mathscr{A}$ nilpotent if every element of $\mathscr{A}$ is nilpotent. $\mathscr{N}$ is an *ideal* in $\mathscr{A}$, i.e., given $a \in \mathscr{N}$ and an arbitrary $b \in \mathscr{A}$, one has $(ab)^n = a^n b^n = 0$ for sufficiently large n, i.e. ab is again in the nilradical. We call the algebra $\mathscr{A}$ *irreducible (over* $\mathbb{R}$*)* if it has a unit element $1_{\mathscr{A}}$ satisfying $1_{\mathscr{A}} b = b$ for all $b \in \mathscr{A}$, and such that $\mathscr{A} = \mathbb{R} \cdot 1_{\mathscr{A}} \oplus \mathscr{N}$ holds. Note that $\mathscr{N}$ determines $\mathscr{A}$ in this case, and we will freely switch between $\mathscr{A}$ and $\mathscr{N}$ in the following.

Now the above considerations show that $\mathfrak{a}$ is an irreducible associative commutative algebra. In the remainder of this subsection, we will be concerned with a converse to this statement, i.e., with the construction of shearing subgroups from an abstractly given irreducible associative algebra. Assume that $\mathscr{A}$ is an irreducible commutative associative algebra of dimension d, and denote its nilradical by $\mathscr{N}$. We let

$$n(\mathscr{A}) = \min\{k \in \mathbb{N} : a^k = 0, \ \forall a \in \mathscr{N}\} ,$$

which is called the *nilpotency class* of $\mathscr{A}$. Letting

$$\mathscr{N}^k = \{a_1 \dots a_k : a_i \in \mathscr{N}\},$$

for $k \geq 1$, and $\mathscr{N}^0 = \mathscr{N}$, one can prove that

$$n(\mathscr{A}) = \min\{k \in \mathbb{N} : \mathscr{N}^k = \{0\}\} \leq d .$$

By definition of the nilpotency class, we obtain that $\mathscr{N}^{n(\mathscr{A})-1} \neq \{0\}$, and for all $a \in \mathscr{N}^{n(\mathscr{A})-1}$ and $b \in \mathscr{N}$, it follows that $ab = 0$.

Hence, choosing a nonzero $a_d \in \mathscr{N}^{n(\mathscr{A})-1}$, we find that $\mathscr{I}_d := \mathbb{R} \cdot a_d$ is an ideal in $\mathscr{N}$; in fact, we get $\mathscr{N}\mathscr{I}_d = \{0\}$. Applying the same reasoning to the algebra $\mathscr{N}/\mathscr{I}_d$ (and choosing any representative modulo $\mathscr{I}_d$) produces a second element a_{d-1} with the property that $\mathscr{I}_{d-1} = \mathrm{span}(a_{d-1}, a_d)$ fulfills $\mathscr{N}\mathscr{I}_{d-1} \subset \mathscr{I}_d$. Further repetitions of this argument finally yield a basis $a_2, \dots, a_d$ of $\mathscr{N}$, that we supplement by $a_1 = 1_{\mathscr{A}}$ to obtain a basis of $\mathscr{A}$ with the property

$$\mathscr{N}\mathscr{I}_k \subset \mathscr{I}_{k+1} \text{ for } 1 \leq k < d , \tag{11}$$

and $\mathscr{I}_2 = \mathscr{N}$. We call a basis $a_2, \dots, a_d$ of $\mathscr{N}$ satisfying condition (11) a *Jordan-Hölder basis* of $\mathscr{N}$.

The existence of a Jordan-Hölder basis can be also proved by referring to a general result about nilpotent representations of nilpotent algebras, see Theorem 3.5.3 of [36]. Indeed, regard $\mathscr{N}$ as nilpotent algebra and $\mathscr{A}$ as a vector space. It is easy to check that the (regular) representation ρ of the Lie algebra $\mathscr{N}$ acting on $\mathscr{A}$ as $\rho(a)b = ab$ is nilpotent, so that there exists a basis $\{a_1, \dots, a_d\}$ of $\mathscr{A}$ such that for each $a \in \mathscr{N}$, the endomorphism $\rho(a)$ is represented by a strictly upper triangular matrix $\Psi(a) \in \mathfrak{gl}(d, \mathbb{R})$ according to the canonical isomorphism

$$\rho(a)a_j = \sum_{k=1}^{d} \Psi(a)_{j,k}\, a_k \qquad j = 1, \dots, d .$$

Since $\rho(a)1_{\mathscr{A}} = a$, it is always possible to choose $a_1 = 1_{\mathscr{A}}$ and, by construction, for all $a \in \mathscr{N}$ and for $i = 1, \ldots, d-1$

$$\rho(a)\mathrm{span}\{a_i, \ldots, a_d\} \subset \mathrm{span}\{a_{i+1}, \ldots, a_d\} \qquad \rho(a)a_d = 0 .$$

These bases provide access to an explicit construction of an associated shearing subgroup, explained in detail in the next lemma. Recall the notation $E_{i,j}$ for the matrix possessing entry one in row i, column j, and zeros elsewhere. Note that the map $\Psi : \mathscr{A} \to \mathfrak{gl}(d, \mathbb{R})$ in the following lemma coincides with the identically denoted map that we just introduced.

Lemma 13. *Let $\mathscr{A}$ denote an irreducible commutative associative algebra of dimension d with nilradical $\mathscr{N}$ possessing the Jordan-Hölder basis*

$$a_2, \ldots, a_d \in \mathscr{A} .$$

Let $a_1 = 1_{\mathscr{A}}$, and let $\psi : \mathbb{R}^d \to \mathscr{A}$ denote the induced linear isomorphism

$$\psi((x_1, \ldots, x_d)^T) = \sum_{i=1}^{d} x_i a_i .$$

Let $\Psi : \mathscr{A} \to \mathfrak{gl}(d, \mathbb{R})$ denote the associated linear map satisfying for all $\tilde{a} \in \mathscr{A}$ and for all $x \in \mathbb{R}^d$:

$$\psi^{-1}(\tilde{a} \cdot \psi(x)) = \Psi(\tilde{a}) \cdot x .$$

(a) The set

$$S = \{I_d + \Psi(a)^T : a \in \mathscr{N}\}$$

is a shearing subgroup, with associated Lie algebra given by

$$\mathfrak{s} = \{\Psi(a)^T : a \in \mathscr{N}\} .$$

(b) Defining $X_i = \Psi(a_i)^T$, for $i = 1, \ldots, d$, we get that X_1 is the identity matrix, and $X_2, \ldots, X_d$ is the canonical basis of $\mathfrak{s}$ in the sense of Lemma 5.

(c) Let $(d_{i,j,k})_{1 \le i,j,k \le d}$ denote the structure constants associated to the basis, defined by the equations

$$\text{for all } 1 \le i,j \le d \ : \ a_i a_j = \sum_{k=1}^{d} d_{i,j,k} a_k . \qquad (12)$$

Then

$$X_i = \begin{pmatrix} d_{i,1,1} & d_{i,1,2} & \dots & d_{i,1,d} \\ \vdots & \vdots & \vdots & \vdots \\ d_{i,d,1} & d_{i,d,2} & \dots & d_{i,d,d} \end{pmatrix} . \tag{13}$$

(d) *We note the following nontrivial properties of the $d_{i,j,k}$, valid for all $1 \leq i, j, k \leq d$:*

$$d_{i,j,k} = d_{j,i,k}, \quad d_{1,j,k} = \delta_{j,k}, \quad d_{i,j,k} = 0 \quad \text{whenever } k \leq \max(i,j).$$

In particular, we get for $2 \leq j \leq d$

$$X_i = E_{1,i} + \sum_{j=2}^{d} \sum_{k=j+1}^{d} d_{i,j,k} E_{j,k} . \tag{14}$$

Proof. We start with part (c). Since multiplication with $a_1 = 1_{\mathscr{A}}$ is the identity operator, the statement about X_1 is clear. Let $1 \leq i,j \leq d$. By definition of ψ, we have $\psi(e_j) = a_j$, and hence by definition of Ψ

$$\begin{aligned} \Psi(a_i)e_j &= \psi^{-1}(a_i \cdot \psi(e_j)) = \psi^{-1}(a_i a_j) \\ &= \psi^{-1}\left(\sum_{k=1}^{d} d_{i,j,k} a_k\right) = \sum_{k=1}^{d} d_{i,j,k} e_k . \end{aligned}$$

Hence the jth column of $\Psi(a_i)$ is the vector $(d_{i,j,1}, \dots, d_{i,j,d})^T$, and its transpose is the jth row of $\Psi(a_i)^T$. This shows (13).

Now, with (c) established, the equation

$$a_1 a_j = a_j$$

for $i = 2, \dots, d$, yields that $d_{1,j,k} = \delta_{j,k}$, which also takes care of part (b). Furthermore, the fact that $a_i a_j = a_j a_i$ ensures that $d_{i,j,k} = d_{j,i,k}$. Finally, recall that $\mathscr{A}\mathscr{I}_i \subset \mathscr{I}_{i+1}$ by (11), which entails $a_i a_j \in \mathscr{I}_{i+1}$, and thus $d_{i,j,k} = 0$ whenever $k \leq i$. Since $d_{i,j,k} = d_{j,i,k}$, we then obtain more generally that $k \leq \max(i,j)$ entails $d_{i,j,k} = 0$. Now equation (14) is clear, and (d) is shown.

In order to prove (a), we first note that Ψ is a homomorphism of associative algebras, hence $\mathfrak{s} = \Psi(\mathscr{N})^T$ is a commutative associative matrix algebra. In particular, it is also an abelian Lie-subalgebra. Furthermore, the relation $d_{i,j,k} = 0$ whenever $k \leq \max(i,j)$ ensures that the basis $X_2, \dots, X_d$ consists of strictly upper triangular matrices. In addition, $d_{1,j,k} = \delta_{j,k}$ entails that $X_2, \dots, X_d$ is indeed a canonical basis, and thus Lemma 5 gives that the associated Lie group is a shearing subgroup. Now part (a) of Lemma 6 yields (a) of the current lemma, and (b) is also shown.

Remark 14. It is natural to ask whether the construction of shearing subgroups S from irreducible commutative associative algebras $\mathscr{A}$, as described in Lemma 13, is exhaustive. The answer is yes. To see this, consider the Lie algebra $\mathfrak{s}$ of a shearing subgroup S. Let $X_2, \ldots, X_d$ be the canonical basis of $\mathfrak{s}$. Since X_j is strictly upper triangular, and the first row of X_i equals e_i^T, it follows that the first i entries of the first row of $X_i X_j$ vanish. This product is again in the span of the X_k, hence

$$X_i X_j = \sum_{k>i} d_{i,j,k} X_k \ ,$$

with suitable coefficients $d_{i,j,k}$. But the fact that the sum on the right-hand side starts with $k = i + 1$ shows that the basis $X_2, \ldots, X_d$ is a Jordan-Hölder basis of the nilpotent associative matrix algebra $\mathfrak{s}$. If one now applies the procedure from Lemma 13 (with $a_i = X_i$), direct calculation allows to verify that $\Psi(X)^T = X$ for all $X \in \mathfrak{s}$. Hence every shearing subgroup arises from the construction in Lemma 13.

In particular, the observations concerning the structure constants $d_{i,j,k}$ made in part (d) of Lemma 13 also apply to the $d_{i,j,k}$ in Lemma 9.

Remark 15. A further benefit of the above construction of shearing groups via associative algebras is that it settles the question of conjugacy as a by-product. By Theorem 13 in [22] and the remarks prior to that result, one sees that two shearing subgroups S_1 and S_2 are conjugate iff their Lie algebras are isomorphic as associative algebras.

In particular, following the observation made in [22, Theorem 15], in dimension $d \geq 7$ there exist uncountably many nonconjugate shearing subgroups.

3.2 An Inductive Approach to Shearlet Dilation Groups

For possible use in inductive proof strategies, we note a further consequence of the block structure:

Proposition 16. *Let $H = \pm DS < Gl(d, \mathbb{R})$ denote a shearlet dilation group, with $d \geq 3$, and let*

$$H_1 = \left\{ h' \in \mathrm{GL}(d-1, \mathbb{R}) : \exists\, h \in H, z \in \mathbb{R}^{d-1}, s \in \mathbb{R} \setminus \{0\} \ \textit{with}\ h = \begin{pmatrix} h' & z \\ 0 & s \end{pmatrix} \right\} \ .$$

Then H_1 is a shearlet dilation group as well.

Conversely, the elements of H can be described in terms of H_1 as follows: There exists a map $y : H_1 \to \mathbb{R}^{d-1}$ such that we can write each $h \in H$ uniquely as

$$h(h_1, r) = \begin{pmatrix} & & & r \\ & & & y_1(h_1) \\ & & h_1 & y_2(h_1) \\ & & & \vdots \\ & & & y_{d-2}(h_1) \\ 0 & \ldots & \ldots & 0 \; y_{d-1}(h_1) \end{pmatrix},$$

with $h_1 \in H_1, r \in \mathbb{R}$.

3.3 Examples

As a result of the previous subsections, we obtain the following general procedure for the systematic construction of shearlet dilation groups:

1. Fix a nilpotent associative algebra $\mathscr{N}$.
2. Pick a Jordan-Hölder basis $a_2, \ldots, a_d$ of $\mathscr{N}$, and compute the canonical basis $X_2, \ldots, X_d$ of the Lie algebra $\mathfrak{s}$ of the associated shearing subgroup. Note that this amounts to determining the structure constants $(d_{i,j,k})_{1 \le i,j,k \le d}$. The shearing subgroup is then determined as $S = I_d + \mathfrak{s}$.
3. In order to determine the diagonal scaling groups that are compatible with S, set up and solve the linear system (7) induced by the nonvanishing $d_{i,j,k}$.

We will now go through this procedure for several examples or classes of examples.

Example 17. We start out with the simplest case of a nilpotent algebra $\mathscr{N}$ of dimension $d-1$, namely that of nilpotency class 2. Here one has $ab = 0$ for any $a, b \in \mathscr{N}$, and it is clear that for two such algebras, any linear isomorphism is an algebra isomorphism as well. Picking any basis $a_2, \ldots, a_n$ of $\mathscr{N}$, we obtain $X_i = E_{1,i}$. In particular, the linear system (7) is trivial. Hence any one-parameter diagonal group can be used as scaling subgroup. We thus recover the groups described in (4).

Example 18. Another extreme class of nilpotent algebras of dimension d is that of nilpotency class d. Here there exists $b \in \mathscr{N}$ with $b^{d-1} \neq 0$. This implies that $b, \ldots, b^{d-1}$ are linearly independent, and then it is easily seen that $a_i = b^{i-1}$, for $i = 2, \ldots, d$, defines a Jordan-Hölder basis of $\mathscr{N}$. In this example, the defining relations read

$$a_i a_j = a_{i+j-1}, \qquad 2 \le i, j, i+j-1 \le d, \tag{15}$$

and the resulting canonical Lie algebra basis is then determined as

$$X_2 = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & \ddots & \ddots & & \\ & & & \ddots & \ddots & \\ & & & & 0 & 1 \\ & & & & & 0 \end{pmatrix}, \; X_3 = \begin{pmatrix} 0 & 0 & 1 & & & \\ & 0 & 0 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 0 & 0 & 1 \\ & & & & 0 & 0 \\ & & & & & 0 \end{pmatrix}, \ldots,$$

$$X_d = \begin{pmatrix} 0 & \cdots\cdots & 0 & 1 \\ & & & 0 \\ & & & \vdots \\ & & \mathbf{0} & \vdots \\ & & & \vdots \\ & & & 0 \end{pmatrix}.$$

Thus we see that the resulting shearing subgroup is that of the Toeplitz shearlet group from (5). The linear system (7) becomes

$$\mu_i + \mu_j = \mu_{i+j-1}, \quad \text{for } 2 \leq i,j,\ i+j-1 \leq d.$$

It is easy to see that all solutions of this system are given by

$$\mu_j = (j-1)\delta, \quad j = 2, \ldots, d$$

with δ an arbitrary real parameter. Thus the scaling subgroups compatible with the Toeplitz dilation group are precisely given by

$$\exp(\mathbb{R}\mathrm{diag}(1, 1+\delta, \ldots, 1+(d-1)\delta)),$$

with $\delta \in \mathbb{R}$ arbitrary.

Remark 19. For $d = 3$, the two above listed cases are all possible examples of shearing subgroups, and not even just up to conjugacy. In particular, we find that all shearing subgroups in dimension 3 are compatible with anisotropic dilations.

We now turn to the shearing subgroups in dimension 4, with focus on the groups not covered by (4) and (5).

Example 20. Since the nilpotency classes $n = 2, 4$ are already covered by the previous examples, the remaining 4-dimensional cases of irreducible algebras $\mathscr{A}$ all have nilpotency class 3. It is shown in [22] that

$$\mathscr{A} \cong \mathbb{R}[Y_1, Y_2]/(Y_1^3, Y_2^2 - \alpha Y_1^2, Y_1 Y_2),$$

with $\alpha \in \{-1, 0, 1\}$. Here, $\mathbb{R}[Y_1, Y_2]$ denotes the algebra of polynomials with real coefficients and indeterminates Y_1, Y_2, and $\mathscr{J} = (Y_1^3, Y_2^2 - \alpha Y_1^2, Y_1 Y_2)$ denotes the ideal generated by the three polynomials. Then the nilradical $\mathscr{N}$ is generated by $Y_1 + \mathscr{J}, Y_2 + \mathscr{J}$. We choose the basis $a_2 = Y_1 + \mathscr{J}$, $a_3 = Y_2 + \mathscr{J}$, $a_4 = Y_1^2 + \mathscr{J}$, and obtain as the only nonzero relations

$$a_2^2 = a_4, \qquad a_3^2 = \alpha a_4.$$

This allows to conclude that a_2, a_3, a_4 is indeed a Jordan-Hölder basis. Following Lemma 13 (c), we can read off the canonical basis of the associated shearing subgroup as

$$X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We next determine the compatible scaling subgroups. In the case $\alpha \neq 0$, we obtain the system of equations

$$2\mu_2 = \mu_4, \qquad 2\mu_3 = \mu_4.$$

Thus the infinitesimal generators of scaling subgroups are of the form $Y = \mathrm{diag}(1, 1+\delta, 1+\delta, 1+2\delta)$, with $\delta \in \mathbb{R}$ arbitrary.

In the case $\alpha = 0$, we only get one equation, namely

$$2\mu_2 = \mu_4,$$

showing that here the compatible infinitesimal generators are of the form $Y = \mathrm{diag}(1, 1+\delta_1, 1+\delta_2, 1+2\delta_1)$, with $\delta_1, \delta_2 \in \mathbb{R}$ arbitrary.

Finally, we give an example of a shearing subgroup which is only compatible with isotropic scaling. It is based on the same algebra as Example 18 (with $d = 4$), and as a result the associated shearing subgroups are conjugate. Recall that the groups in Example 18 are compatible with anisotropic scaling. This illustrates an important, somewhat subtle point: While the precise choice of Jordan-Hölder basis in the procedure described in Lemma 13 is immaterial if one is just interested in guaranteeing the shearing subgroup property, it may have a crucial influence on the availability of compatible *anisotropic* scaling subgroups.

Example 21. Let $\mathscr{A} = \mathbb{R}[X]/(X^4)$. We use the Jordan-Hölder algebra $a_2 = X + X^2 + (X^4)$, $a_3 = X^2 + (X^4)$, $a_4 = X^3 + (X^4)$. This leads to the following nonzero relations

$$a_2^2 = a_3 + 2a_4, \qquad a_2 a_3 = a_4,$$

which gives rise to the basis

$$X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad X_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now the nonzero entries in the matrix X_2 imply that the linear system (7) contains the equations

$$2\mu_2 = \mu_3, \qquad 2\mu_2 = \mu_4, \qquad \mu_2 + \mu_3 = \mu_4.$$

The first two equations imply $\mu_3 = 2\mu_2 = \mu_4$, and then the third equation yields $\mu_2 = 0$. Hence this shearing subgroup is only compatible with isotropic scaling.

4 Anisotropic Scaling and Wavefront Set Characterizations

In this section we investigate the suitability of the various groups for microlocal analysis. The idea is to verify the criteria derived in [21] that allow to establish the suitability of a dilation group for the characterization of the wavefront set via wavelet coefficient decay. As it will be seen, this property only depends on the scaling subgroup.

4.1 Criteria for Wavefront Set Characterization

Throughout this subsection H is an irreducibly admissible matrix group, i.e. its dual action has a single open orbit $\mathcal{O} \subset \mathbb{R}^d$, with associated compact fixed groups. We use $V \Subset \mathcal{O}$ to denote that the closure of V inside $\mathcal{O}$ is compact.

Given $R > 0$ and $x \in \mathbb{R}^d$, we let $B_R(x)$ and $\overline{B_R}(x)$ denote the open/closed ball with radius R and center x, respectively. We let $S^{d-1} \subset \mathbb{R}^d$ denote the unit sphere. By a neighborhood of $\xi \in S^{d-1}$, we will always mean a *relatively open* set $W \subset S^{d-1}$ with $\xi \in W$. Given $R > 0$ and an open set $W \subset S^{d-1}$, we let

$$C(W) := \{r\xi' : \xi' \in W, r > 0\} = \left\{\xi \in \mathbb{R}^d \setminus \{0\} : \frac{\xi}{|\xi|} \in W\right\},$$
$$C(W,R) := C(W) \setminus \overline{B_R}(0).$$

Both sets are clearly open subsets of $\mathbb{R}^d \setminus \{0\}$ and thus of $\mathbb{R}^d$.

Given a tempered distribution u, we call $(x, \xi) \in \mathbb{R}^d \times S^{d-1}$ a *regular directed point of u* if there exists $\varphi \in C_c^\infty(\mathbb{R}^d)$, identically one in a neighborhood of x, as well as a ξ-neighborhood $W \subset S^{d-1}$ such that for all $N \in \mathbb{N}$ there exists a constant $C_N > 0$ with

$$\text{for all } \xi' \in C(W) \ : \ \left|\widehat{\varphi u}(\xi')\right| \le C_N(1 + |\xi'|)^{-N}. \tag{16}$$

We next formally define the sets K_i and K_o which will allow to associate group elements to directions.

Definition 22. Let $\emptyset \neq W \subset S^{d-1}$ be open with $W \subset \mathcal{O}$ (which implies $C(W) \subset \mathcal{O}$). Furthermore, let $\emptyset \neq V \Subset \mathcal{O}$ and $R > 0$. We define

$$K_i(W, V, R) := \left\{h \in H : h^{-T}V \subset C(W, R)\right\}$$

as well as

$$K_o(W, V, R) := \left\{h \in H : h^{-T}V \cap C(W, R) \neq \emptyset\right\}.$$

If the parameters are provided by the context, we will simply write K_i and K_o. Here, the subscripts i/o stand for "inner/outer".

We now define what we mean by dilation groups characterizing the wavefront set. We first extend the continuous wavelet transform to the space of tempered distributions, i.e., we use $W_\psi u$, for a Schwartz wavelet ψ and a tempered distribution u.

Definition 23. The dilation group H *characterizes the wavefront set* if there exists a nonempty open subset $V \Subset \mathcal{O}$ with the following property: For all $0 \neq \psi \in \mathcal{S}(\mathbb{R}^d)$ with $\text{supp}(\widehat{\psi}) \subset V$, for every $u \in \mathcal{S}'(\mathbb{R}^d)$ and all $(x, \xi) \in \mathbb{R}^d \times (\mathcal{O} \cap S^{d-1})$, the following statements are equivalent:

(a) (x, ξ) is a regular directed point of u.
(b) There exists a neighborhood U of x, some $R > 0$ and a ξ-neighborhood $W \subset S^{d-1}$ such that for all $N \in \mathbb{N}$ there exists a constant $C_N > 0$ such that for all $y \in U$, and for all $h \in K_o(W, V, R)$ the following estimate holds:

$$|W_\psi u(y, h)| \le C_N \|h\|^N.$$

Note that the definition excludes a set of directions ξ from the analysis of the wavefront set, namely the directions not contained in $\mathcal{O} \cap S^{d-1}$. These directions always constitute a set of measure zero. Recall from Proposition 11 that in the case of shearlet dilation groups, this exceptional set is given by $(\{0\} \times \mathbb{R}^{d-1}) \cap S^{d-1}$.

We next recall the sufficient conditions for dilation groups that characterize the wavefront set, as established in [21]. The first one is related to the problem that one would like to interpret the norm as a scale parameter.

Definition 24. Let $\xi \in \mathcal{O} \cap S^{d-1}$ and $\emptyset \neq V \Subset \mathcal{O}$. The dual action is called *V-microlocally admissible in direction ξ* if there exists a ξ-neighborhood $W_0 \subset S^{d-1} \cap \mathcal{O}$ and some $R_0 > 0$ such that the following hold:

1. There exist $\alpha_1 > 0$ and $C > 0$ such that

$$\|h^{-1}\| \leq C \cdot \|h\|^{-\alpha_1}$$

holds for all $h \in K_o(W_0, V, R_0)$.
2. There exists $\alpha_2 > 0$ such that

$$\int_{K_o(W_0,V,R_0)} \|h\|^{\alpha_2} \, \mathrm{d}h < \infty.$$

The dual action is called *microlocally admissible in direction* ξ if it is V-microlocally admissible in direction ξ for some $\emptyset \neq V \Subset \mathcal{O}$.

The second important condition is contained in the following definition. It can be understood as formalizing the ability of the associated wavelet systems o make increasingly fine distinctions between different directions, as the scales go to zero.

Definition 25. Let $\xi \in \mathcal{O} \cap S^{d-1}$ and $\emptyset \neq V \Subset \mathcal{O}$. The dual action has the *V-cone approximation property at* ξ if for all ξ-neighborhoods $W \subset S^{d-1}$ and all $R > 0$ there are $R' > 0$ and a ξ-neighborhood $W' \subset S^{d-1}$ such that

$$K_o(W', V, R') \subset K_i(W, V, R).$$

We now have the following [21, Corollary 4.9]:

Theorem 26. *Assume that the dual action is V-microlocally admissible at some $\xi_0 \in \mathcal{O}$ and has the V-cone approximation property at ξ_0, for some nonempty open subset $V \subset \mathcal{O}$. Then H characterizes the wavefront set.*

Remark 27. The property of characterizing the wavefront set is linked to anisotropic scaling, in the following sense: If H characterizes the wavefront set, then

$$H \cap \mathbb{R}^+ \cdot I_d = \{I_d\} \,,$$

by [21, Lemma 4.10]. Hence if H is a shearlet dilation group characterizing the wavefront set, its scaling subgroup must be anisotropic. This excludes the shearing group constructed in Example 21.

Theorem 26 therefore implies that every group failing the anisotropy criterion $H \cap \mathbb{R}^+ \cdot I_d = \{I_d\}$ must necessarily fail either the microlocal admissibility or the cone approximation property. It is in fact the latter that breaks down, as noted in [21, Lemma 4.4].

These considerations highlight the importance of understanding when a given shearing group admits anisotropic scaling.

4.2 Characterization of the Wavefront Set for Shearlet Dilation Groups

We can now state a very general theorem concerning the ability of shearlet groups to characterize the wavefront set. Note that there are no conditions on the shearing subgroups.

Theorem 28. *Assume that H is a shearlet dilation group. Let $Y = \mathrm{diag}(1, \lambda_2, \ldots, \lambda_d)$ denote the infinitesimal generator of the scaling subgroup. If $0 < \lambda_i < 1$ holds, for all $2 \le i \le d$, then H characterizes the wavefront set.*

Remark 29. We can now quickly go through the examples of shearing subgroups in Subsection 3.3 and show that for most cases, there exists a compatible anisotropic scaling subgroup that allows to characterize the wavefront set. Writing $\lambda_i = 1 + \mu_i$ as in Lemma 5, the condition from Theorem 28 translates to $-1 < \mu_i < 0$, for $2 \le i \le d$. Apart from the group in Example 21, which was specifically constructed to not allow any anisotropic scaling, all other shearing groups can be combined with a compatible scaling group in such a way that the resulting shearlet transform fulfills the conditions of Theorem 28, and therefore characterizes the wavefront set. Note that this was previously known only for the original shearlet group [21, 34]. In particular, we may combine the Toeplitz shearing subgroup with the scaling subgroup with exponents $(1, 1-\delta, \ldots, 1-(d-1)\delta)$, and choosing $\delta \in (0, 1/(d-1))$ guarantees that the Toeplitz shearlet transform characterizes the wavefront set.

The proof of the Theorem amounts to verifying the cone approximation property and microlocal admissibility of the dual action, and this will be carried out in the following two propositions. For the remainder of this section, we fix a shearlet dilation group H with infinitesimal generator $\mathrm{diag}(1, \lambda_2, \ldots, \lambda_d)$ of the scaling subgroup. We let $\lambda_{\max} = \max_{i \ge 2} \lambda_i$, and $\lambda_{\min} = \min_{i \ge 2} \lambda_i$.

Proposition 30. *If $\lambda_{\max} < 1$, there exists an open subset $\emptyset \ne V \Subset \mathscr{O}$ such that the dual action of H on the orbit $\mathscr{O}$ has the V-cone approximation property at $(1, 0, \ldots, 0)^T \in S^{d-1} \cap \mathscr{O}$.*

Proof. We will employ the structural properties of shearing subgroups derived in Section 2.2. We let S and D denote the shearing and scaling subgroups of H, respectively. The infinitesimal generator of D is a diagonal matrix with the entries $1, \lambda_2, \ldots, \lambda_d$. We let $X_2, \ldots, X_d$ denote the canonical basis of $\mathfrak{s}$, consisting of strictly upper triangular matrices X_i. By Lemma 6, each $h \in S$ is uniquely described by

$$h = h(t, 1) = I_d + \sum_{i=2}^{d} t_i X_i \,,$$

where $t = (t_2, \ldots, t_d)^T$ denotes the vector of first row entries of $h(t, 1)$. For H, we thus obtain the global chart

$$h(t, a) = \left(I_d + \sum_{i=2}^{d} t_i X_i \right) \mathrm{sgn}(a) \mathrm{diag}(|a|, |a|^{\lambda_2}, \ldots, |a|^{\lambda_d}) \in GL(d, \mathbb{R}), \ (t, a) \in \mathbb{R}^{d-1} \times \mathbb{R}^{\times} \,.$$

For the purpose of the following computations, it is possible and beneficial to slightly modify this construction and replace $h(t, 1)$ by its inverse. Thus, every $h \in H$ can be written (uniquely) as $h = \pm h(t, 1)^{-1} h(0, a)$ with $t \in \mathbb{R}^{d-1}$ and $a \in (0, +\infty)$. The dual action is then given by

$$(h^{-1})^T = \pm(h(0,a)^{-1}h(t,1))^T = \pm\left(I_d + \sum_{i=2}^{d} t_i X_i^T\right) h(0, a^{-1}), \tag{17}$$

where by construction

$$I_d + \sum_{i=2}^{d} t_i X_i^T = \begin{pmatrix} 1 & 0^T \\ t & I_{d-1} + A(t)^T \end{pmatrix}, \tag{18}$$

with $A(t)$ being a $(d-1) \times (d-1)$ strictly lower-triangular matrix satisfying

$$\|A(t)\| \leq C|t| \tag{19}$$

with a constant C depending only on H.

We now parametrise the open orbit $\mathscr{O}$ by the global chart provided by affine coordinates

$$\Omega : \mathbb{R}^\times \times \mathbb{R}^{d-1} \to \mathscr{O} \qquad \Omega(\tau, v) = \tau(1, v^T)^T,$$

and $S^{d-1} \cap \mathscr{O}$ by the corresponding diffeomorphism to its image

$$\omega : \mathbb{R}^{d-1} \to S^{d-1} \cap \mathscr{O}, \qquad \omega(v) = \frac{(1, v^T)^T}{\sqrt{1 + |v|^2}}.$$

Given $\varepsilon > 0$, we set

$$W_\varepsilon = \{v \in \mathbb{R}^{d-1} : |v| < \varepsilon\} = B_\varepsilon(0),$$

since $\{W_\varepsilon : \varepsilon > 0\}$ is a neighborhood basis of the origin in $\mathbb{R}^{d-1}$ and $\{\omega(W_\varepsilon) : \varepsilon > 0\}$ is a neighborhood basis of $\xi_0 = (1, 0, \ldots, 0) \in S^{d-1} \cap \mathscr{O}$.

Furthermore, for fixed $0 < \tau_1 < \tau_2$ and $\varepsilon_0 > 0$ the set

$$V = \Omega(\,(\tau_1, \tau_2) \times W_{\varepsilon_0}\,)$$

is an open subset with $V \Subset \mathscr{O}$.

Given $h \in H$, as in (17), and $\xi \in V$, then $\xi = \Omega(\tau, v)$ with $\tau_1 < \tau < \tau_2$ and $v \in W_{\varepsilon_0}$, and we get

$$(h^{-1})^T \xi = \pm\tau \begin{pmatrix} 1 & 0^T \\ t & I_{d-1} + A(t)^T \end{pmatrix} \begin{pmatrix} a^{-1} \\ v' \end{pmatrix} = \pm a^{-1}\tau \begin{pmatrix} 1 \\ t + (I_{d-1} + A(t)^T)v'' \end{pmatrix}$$

where $v', v'' \in \mathbb{R}^{d-1}$ have components given by $v'_i = a^{-\lambda_i} v_i$ and $v''_i = a^{1-\lambda_i} v_i$ for all $i = 2, \ldots, d$. Hence

$$(h^{-1})^T(V) = \Omega\left((\pm a^{-1}\tau_1, \pm a^{-1}\tau_2) \times (t + (I_{d-1} + A(t)^T) W^a_{\varepsilon_0})\right)$$

where $W^a_{\varepsilon_0} = \{v'' \in \mathbb{R}^{d-1} : v''_i = a^{1-\lambda_i} v_i,\ |v| < \varepsilon_0\}$.

Fix now $R > 0$ and a neighborhood $W \subset S^{d-1} \cap \mathscr{O}$ of ξ_0. Without loss of generality we can assume that $W = \omega(W_\varepsilon)$ for some $\varepsilon > 0$. Furthermore, since

$$(R, +\infty) \times W_\varepsilon \subset \Omega^{-1}(C(\omega(W_\varepsilon), R)) \subset (\frac{R}{\sqrt{1+\varepsilon^2}}, +\infty) \times W_\varepsilon \subset (\frac{R}{2}, +\infty) \times W_\varepsilon,$$

where the last inclusion on the right holds if $\varepsilon \leq 1$, then the V-cone approximation property holds true if there exist $R' > 0$ and $0 < \varepsilon' \leq 1$ such that for all $h \in H$ satisfying

$$(h^{-1})^T(V) \cap \Omega\left((\frac{R'}{2}, +\infty) \times W_{\varepsilon'}\right) \neq \emptyset, \tag{20a}$$

it holds that

$$(h^{-1})^T(V) \subset \Omega\left((R, +\infty) \times W_\varepsilon\right). \tag{20b}$$

Take $R' > 0$ and $0 < \varepsilon' < \sqrt{3}$, which we will fix later on as functions of R and ε, and $h \in H$ as in (17). If $h = -(h(0,a)^{-1} h(t,1))^T$, then

$$\left((\frac{R'}{2}, +\infty) \times W_{\varepsilon'}\right) \cap \left((-a^{-1}\tau_2, -a^{-1}\tau_1) \times (t + (I_{d-1} + A(t)^T) W^a_{\varepsilon_0})\right) = \emptyset,$$

so that (20a) implies that $h = +(h(0,a)^{-1} h(t,1))^T$ and

$$\left((\frac{R'}{2}, +\infty) \times W_{\varepsilon'}\right) \cap \left((a^{-1}\tau_1, a^{-1}\tau_2) \times (t + (I_{d-1} + A(t)^T) W^a_{\varepsilon_0})\right) \neq \emptyset.$$

Hence

$$R' < 2a^{-1}\tau_2, \qquad W_{\varepsilon'} \cap \left(t + (\mathrm{Id}_{d-1} + A(t)^T) W^a_{\varepsilon_0}\right) \neq \emptyset.$$

If we choose $R' > 2\tau_2$, the first inequality gives

$$a < \frac{2\tau_2}{R'} < 1, \tag{21}$$

and, since $a < 1$, setting $\lambda_{\max} = \max\{\lambda_2, \ldots, \lambda_d\}$, clearly

$$W^a_{\varepsilon_0} \subset W_{\varepsilon_0 a^{1-\lambda_{\max}}}. \tag{22}$$

By the above inclusion, since $W_{\varepsilon'} \cap \big(t + (I_{d-1} + A(t)^T)W^a_{\varepsilon_0}\big) \neq \emptyset$, then there exists $\xi \in W_{\varepsilon_0 a^{1-\lambda_{\max}}}$ such that $|t + \xi + A(t)^T\xi| < \varepsilon'$. Hence, triangle inequality, (19) and (21) give

$$\begin{aligned}|t| &< \varepsilon' + (1 + \|A(t)^T\|)|\xi| \leq \varepsilon' + (1 + C|t|)a^{1-\lambda_{\max}}\varepsilon_0 \\ &\leq \varepsilon' + (\frac{2\tau_2}{R'})^{1-\lambda_{\max}}(1 + C|t|)\varepsilon_0 \leq 2\varepsilon' + \frac{1}{2}|t|,\end{aligned}$$

where the last inequality holds true provided that

$$R' > 2\tau_2 \max\{1, (\frac{\varepsilon_0}{\varepsilon'})^{\frac{1}{1-\lambda_{\max}}}, (2C\varepsilon_0)^{\frac{1}{1-\lambda_{\max}}}\}. \tag{23}$$

Hence, if (20a) holds true with R' satisfying (23), then

$$a < \frac{2\tau_2}{R'} < 1 \tag{24a}$$

$$|t| < 4\varepsilon' \tag{24b}$$

$$(\frac{2\tau_2}{R'})^{1-\lambda_{\max}}\varepsilon_0 < \min\{\varepsilon', \frac{1}{2C}\}. \tag{24c}$$

The condition (20b) is equivalent to

$$(a^{-1}\tau_1, a^{-1}\tau_2) \times (t + I_{d-1} + A(t)^T)W^a_{\varepsilon_0}) \subset (R, +\infty) \times W_\varepsilon,$$

which is ensured by $a^{-1}\tau_1 > R$ and, recalling (22), by $t + (I_{d-1} + A(t)^T)W_{\varepsilon_0 a^{1-\lambda_{\max}}} \subset W_\varepsilon$.

By (24a) the first condition is satisfied if $\tau_1/R > \frac{2\tau_2}{R'}$. Taking into account (23), it is sufficient to assume that

$$R' > 2\tau_2 \max\{1, (\frac{\varepsilon_0}{\varepsilon'})^{\frac{1}{1-\lambda_{\max}}}, (2C\varepsilon_0)^{\frac{1}{1-\lambda_{\max}}}, \frac{R}{\tau_1}\}. \tag{25}$$

To ensure that $t + (I_{d-1} + A(t)^T)W_{\varepsilon_0 a^{1-\lambda_{\max}}} \subset W_\varepsilon$, note that, for all $\xi \in W_{\varepsilon_0 a^{1-\lambda_{\max}}}$, conditions (19), (24a), and (24b) give

$$\begin{aligned}|t + (I_{d-1} + A(t)^T)\xi| &\leq |t| + (1 + C|t|)|\xi| \leq |t| + (1 + C|t|)a^{1-\lambda_{\max}}\varepsilon_0 \\ &< 4\varepsilon' + (1 + C4\varepsilon')(\frac{2\tau_2}{R'})^{1-\lambda_{\max}}\varepsilon_0 \\ &\leq 4\varepsilon' + \varepsilon' + 2\varepsilon' = 7\varepsilon',\end{aligned}$$

where the last inequality follows from (24c). Hence, with the choice $\varepsilon' = \min\{1, \varepsilon/7\}$ and R' satisfying (25) for all $\xi \in W_{\varepsilon_0 a^{1-\lambda_{\max}}}$,

$$|t + (I_{d-1} + A(t)^T)\xi| < \varepsilon,$$

so that (20b) holds true for all $h \in H$ satisfying (20a).

Remark 31. The proof does not make use of the fact that the shearlet group $\mathscr{S}$ is abelian. The proof is based only on the following two properties of S

a) a global smooth chart $t \mapsto s(t)$ from $\mathbb{R}^{d-1}$ onto S;
b) for all $t \in \mathbb{R}^d$ the dual action of $s(t)$ is of the form

$$(s(t)^{-1})^T = \begin{pmatrix} 1 & 0^T \\ t & B(t) \end{pmatrix}$$

where $\|B(t)\| \leq C_1 + C_2|t|$ for a suitable choice of C_1 and C_2.

With the cone approximation property already established, the remaining condition is quite easy to check.

Proposition 32. *If* $0 < \lambda_{\min} \leq \lambda_{\max} < 1$*, there exists an open subset* $\emptyset \neq V \Subset \mathscr{O}$ *such that the dual action of* H *on the orbit* $\mathscr{O}$ *is* V*-microlocally admissible in direction* $(1, 0, \ldots, 0) \in S^{d-1} \cap \mathscr{O}$.

Proof. We retain the notations from the previous proof, as well as the open set

$$V = \Omega(\,(\tau_1, \tau_2) \times W_{\varepsilon_0}\,)\,,$$

with $\tau_1 < 1 < \tau_2$. Since we assume $\lambda_{\max} < 1$, the cone approximation property holds, and then condition (2) of Definition 24 follows from condition (1) by [21, Lemma 4.7]. In addition, the cone approximation property allows to replace K_o in that condition by the smaller set K_i. In short, it remains to prove the existence of $\alpha > 0$ and $C'' > 0$ such that

$$\|h^{-1}\| \leq C''\|h\|^{-\alpha}$$

holds for all $h \in K_i(\omega(W_\varepsilon), V, R)$, for suitable $\varepsilon, R > 0$. In the following computations, we let $\varepsilon = 1$ and $R > 2$. Now assume that $h = \pm h(t,1)^{-1}h(0,a) \in K_i(\omega(W_\varepsilon), V, R)$, which means that $h^{-T}V \subset C(\omega(W_\varepsilon), R)$. This implies in particular that

$$h^{-T}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \pm h(t,1)^T h(0,a)^{-1}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \pm h(t,1)^T\begin{pmatrix} a^{-1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$
$$= \pm a^{-1}\begin{pmatrix} 1 \\ t \end{pmatrix} \in C(\omega(W_\varepsilon), R)\,.$$

This implies that the sign is in fact positive. Furthermore, we have $|t| \leq \varepsilon = 1$, and then

$$2a^{-1} \geq \left| h^{-T}\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right| > R > 2\,,$$

which implies $a < 1$. By using the fact $\|h(t,1)\| \le \sqrt{2}(1+(1+C)|t|) \le \sqrt{2}(2+C)$, where C was the constant from (19), we can now estimate

$$\|h^{-1}\| = \|h(0,a)^{-1}h(t,1)\| \le \|h(0,a)^{-1}\|\|h(t,1)\| \le \sqrt{2}(2+C)a^{-1},$$

where we used $a < 1$ and $\lambda_{\max} \le 1$ to estimate the norm of $h(0,a)^{-1}$. In addition,

$$\|h\| = \|h(t,1)^{-1}h(0,a)\| \le \|h(t,1)^{-1}\|\|h(0,a)\| \le C' a^{\lambda_{\min}}.$$

Here we used that the set $\{h(t,1) : |t| \le 1\} \subset H$ is compact to uniformly estimate the norm of the inverses by a suitable C', and $a < 1$ to estimate the norm of $h(0,a)$. But these estimates combined yield

$$\|h^{-1}\| \le \sqrt{2}(2+C)a^{-1} \le \sqrt{2}(2+C)(C')^{1/\lambda_{\min}}\|h\|^{-1/\lambda_{\min}}.$$

Since we assume that $\lambda_{\min} > 0$, the proof is finished.

5 Embeddings into the Symplectic Group

From the analytical point of view, we saw that shearlet dilation groups are a useful tool for the characterization of the wavefront set of distributions. On the other hand, from the algebraic and geometrical points of view, these groups and the associated generalized wavelet representation exhibit an interesting link with the symplectic group and the metaplectic representation. More precisely, in this section we show that the positive part DS of any shearlet dilation group $DS \cup (-DS)$ may be embedded into the symplectic group. Note that the full group $DS \cup (-DS)$ cannot be expected to be imbedded into $Sp(d, \mathbb{R})$ [14, Theorem 3.5]. Moreover, we prove that the wavelet representation is unitarily equivalent to the metaplectic representation, provided that they are restricted to a suitable subspace of $L^2(\mathbb{R}^d)$. In fact, a much more general class of groups is allowed, see Theorem 33.

The relevance of the symplectic group and of the metaplectic representation in this context has already been shown in several works [1, 2, 14, 15]. In particular, the argument given here generalizes [14].

Let $T(d,\mathbb{R})_+$ denote the subgroup of $\mathrm{GL}(d,\mathbb{R})$ consisting of the upper triangular matrices with positive entry in position $(1,1)$, namely

$$T(d,\mathbb{R})_+ = \{h \in \mathrm{GL}(d,\mathbb{R}) : h_{1,1} > 0 \text{ and } h_{i,j} = 0 \text{ for every } i > j\}.$$

We consider the following subspace of $L^2(\mathbb{R}^d)$:

$$\mathscr{H} = \{f \in L^2(\mathbb{R}^d) : \mathrm{supp}\hat{f} \subseteq \Theta_L\}, \quad \text{where} \quad \Theta_L = \{\xi \in \mathbb{R}^d : \xi_1 \le 0\}.$$

The main result of this section reads as follows.

Theorem 33. *Take $H < T(d,\mathbb{R})_+$. The group $G = \mathbb{R}^d \rtimes H$ may be embedded into the symplectic group, namely there exists a group embedding $\phi\colon G \to Sp(d,\mathbb{R})$. Moreover, the restriction to $\mathscr{H}$ of the quasi-regular representation π defined in (1) is unitarily equivalent to $\mu \circ \phi$ restricted to $\mathscr{H}$, where μ is the metaplectic representation of $Sp(d,\mathbb{R})$.*

The rest of this section is devoted to the proof of this theorem. The embedding ϕ, the subgroup $\phi(G)$, as well as the intertwining operator between the quasi-regular representation and the metaplectic representation will be explicitly constructed.

First, we construct the subgroup $\phi(G) < Sp(d,\mathbb{R})$ and the map ϕ. The vectorial part of $G = \mathbb{R}^d \rtimes H$ will correspond to the subspace of the d-dimensional symmetric matrices given by

$$\Sigma := \{\sigma_b := \begin{pmatrix} b_1 & b_2/2 \cdots b_d/2 \\ b_2/2 & \\ \vdots & \mathbf{0} \\ b_d/2 & \end{pmatrix} :\ b \in \mathbb{R}^d\}.$$

We shall need the following preliminary result concerning the map

$$\rho\colon T(d,\mathbb{R})_+ \to GL(d,\mathbb{R}), \qquad h \mapsto \sqrt{h_{1,1}}\, h^{-T}. \tag{26}$$

Lemma 34. *The map ρ is a group homomorphism and for all $b \in \mathbb{R}^d$ and $h \in T(d,\mathbb{R})_+$ there holds*

$$\rho(h)^{-T}\sigma_b\rho(h)^{-1} = \sigma_{hb}. \tag{27}$$

Proof. The first part is trivial, since the matrices in H are upper triangular with $h_{1,1} > 0$. The second part can be proven as follows. Fix $b \in \mathbb{R}^d$ and $h \in T(d,\mathbb{R})_+$. The assertion is equivalent to

$$h\sigma_b\, h^T = h_{1,1}\sigma_{hb}.$$

Write for $i = 2,\dots,d$

$$h = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_d \end{pmatrix}, \quad h_1 = [h_{1,1}\ h_1'], \quad h_i = [0\ h_i'], \quad b' = \begin{pmatrix} b_2 \\ \vdots \\ b_d \end{pmatrix}.$$

We have

$$h\sigma_b = \begin{pmatrix} h_{1,1} & h_1' \\ 0 & h_2' \\ \vdots & \vdots \\ 0 & h_d' \end{pmatrix} \begin{pmatrix} b_1 & b'^T/2 \\ b'/2 & \mathbf{0} \end{pmatrix} = \begin{pmatrix} h_{1,1}b_1 + h_1'b'/2 & h_{1,1}b'^T/2 \\ h_2'b'/2 & \mathbf{0} \\ \vdots & \vdots \\ h_d'b'/2 & \mathbf{0} \end{pmatrix},$$

whence

$$h\sigma_b\, h^T = \begin{pmatrix} h_{1,1}b_1 + h_1' b'/2 & h_{1,1}b'^T/2 \\ h_2' b'/2 & \mathbf{0} \\ \vdots & \vdots \\ h_d' b'/2 & \mathbf{0} \end{pmatrix} \begin{pmatrix} h_{1,1} & 0 & \cdots & 0 \\ h_1'^T & h_2'^T & \cdots & h_d'^T \end{pmatrix}$$

$$= \begin{pmatrix} h_{1,1}(h_{1,1}b_1 + h_1' b'/2) + h_{1,1}b'^T h_1'^T/2 & h_{1,1}b'^T h_2'^T/2 \cdots & h_{1,1}b'^T h_d'^T/2 \\ h_{1,1}h_2' b'/2 & & \\ \vdots & & \mathbf{0} \\ h_{1,1}h_d' b'/2 & & \end{pmatrix}.$$

Therefore, since $b'^T h_i'^T = h_i' b'$ for every i and $h_i' b' = h_i b$ for $i \geq 2$, we obtain

$$h\sigma_b\, h^T = h_{1,1} \begin{pmatrix} h_{1,1}b_1 + h_1' b' & h_2' b'/2 \cdots & h_d' b'/2 \\ h_2' b'/2 & & \\ \vdots & & \mathbf{0} \\ h_d' b'/2 & & \end{pmatrix} = h_{1,1} \begin{pmatrix} h_1 b_1 & h_2 b/2 \cdots & h_d b/2 \\ h_2 b/2 & & \\ \vdots & & \mathbf{0} \\ h_d b/2 & & \end{pmatrix},$$

whence $h\sigma_b\, h^T = h_{1,1}\sigma_{hb}$, as desired.

We use the notation

$$g(\sigma, h) = \begin{pmatrix} h & \\ \sigma h & h^{-T} \end{pmatrix} \in Sp(d, \mathbb{R}), \qquad \sigma \in Sym(d, \mathbb{R}), h \in GL(d, \mathbb{R}).$$

The product law is

$$g(\sigma_1, h_1)g(\sigma_2, h_2) = g(\sigma_1 + h_1^{-T}\sigma_2 h_1^{-1}, h_1 h_2). \tag{28}$$

In the following result we show that $G = \mathbb{R}^d \rtimes H$ is isomorphic to the subgroup of $Sp(d, \mathbb{R})$ given by $\Sigma \rtimes \rho(H) := g(\Sigma, \rho(H))$. This proves the first part of Theorem 33.

Proposition 35. *Take $H < T(d, \mathbb{R})_+$. Then the map*

$$\phi\colon \mathbb{R}_d \rtimes H \to g(\Sigma, \rho(H)) < Sp(d, \mathbb{R}), \quad (b, h) \mapsto g(\sigma_b, \rho(h))$$

is a group isomorphism.

It is worth mentioning that Lemma 2.3 in [14] immediately follows from this result.

Proof. Recall that the product in $\mathbb{R}^d \rtimes H$ is defined by

$$(b_1, h_1)(b_2, h_2) = (b_1 + h_1 b_2, h_1 h_2), \quad b_i \in \mathbb{R}^d, h_i \in H.$$

By definition of ϕ and using (28) there holds

$$\begin{aligned}\phi(b_1,h_1)\phi(b_2,h_2) &= g(\sigma_{b_1},\rho(h_1))g(\sigma_{b_2},\rho(h_2))\\ &= g(\sigma_{b_1} + \rho(h_1)^{-T}\sigma_{b_2}\rho(h_1)^{-1},\rho(h_1)\rho(h_2)).\end{aligned}$$

Therefore, Lemma 34 gives

$$\begin{aligned}\phi(b_1,h_1)\phi(b_2,h_2) &= g(\sigma_{b_1}+\sigma_{h_1b_2},\rho(h_1h_2))\\ &= g(\sigma_{b_1+h_1b_2},\rho(h_1h_2))\\ &= \phi((b_1,h_1)(b_2,h_2)),\end{aligned}$$

as desired. Note that the fact that $g(\Sigma,\rho(H))$ is a subgroup follows a posteriori.

Intertwining the quasi-regular representation π, given in (1), with the Fourier transform $\mathscr{F}\colon\mathscr{H}\to L^2(\Theta_L)$ we obtain the representation $\hat{\pi}(b,h) := \mathscr{F}\pi(b,h)\mathscr{F}^{-1}$ on $L^2(\Theta_L)$ given by

$$\hat{\pi}(b,h)\hat{f}(\xi) = |\det h|^{1/2}e^{-2\pi i\langle b,\xi\rangle}\hat{f}(h^T\xi), \qquad \hat{f}\in L^2(\Theta_L).$$

The metaplectic representation restricted to $\Sigma\rtimes\rho(H)$ takes the form

$$\mu(\phi(b,h))\hat{f}(\xi) = |\det\rho(h)|^{-1/2}e^{\pi i\langle\sigma_b\xi,\xi\rangle}\hat{f}(\rho(h)^{-1}\xi), \qquad \hat{f}\in L^2(\Theta_L). \tag{29}$$

We now show that $\hat{\pi}$ and μ are unitarily equivalent, which concludes the proof of Theorem 33. The intertwining operator is given by

$$\Psi\colon L^2(\Theta_L)\to L^2(\Theta_L), \quad \Psi\hat{f}(\xi) = |\det J_{Q^{-1}}(\xi)|^{1/2}\hat{f}(Q^{-1}(\xi)),$$

where $Q\colon\Theta_L\to\Theta_L$ is defined by $Q(\xi) = -\frac{1}{2}\xi_1\xi$.

Proposition 36. *Let ϕ be the group isomorphism given by Proposition 35. For every $(b,h)\in\mathbb{R}^d\rtimes H$ there holds*

$$\Psi\mu(\phi(b,h))\Psi^{-1} = \mathscr{F}\pi(b,h)\mathscr{F}^{-1} = \hat{\pi}(b,h).$$

Proof. We start by giving a few identities without proof [14]:

$$|\det J_Q(\xi)| = 2^{1-d}|\xi_1|^d, \tag{30}$$

$$|\det J_{Q^{-1}}(\xi)| = 2^{\frac{d}{2}-1}|\xi_1|^{-\frac{d}{2}}, \tag{31}$$

$$\langle\sigma_b\xi,\xi\rangle = -2\langle b,Q(\xi)\rangle, \tag{32}$$

$$Q^{-1}(\xi) = \sqrt{2}\xi/\sqrt{-\xi_1}. \tag{33}$$

By (32) and (27) there holds

$$-2\langle b, Q(h^T\xi)\rangle = \langle \sigma_b\, h^T\xi, h^T\xi\rangle = \langle h\sigma_b\, h^T\xi, \xi\rangle = h_{1,1}\langle \sigma_{hb}\xi, \xi\rangle.$$

Therefore, using again (32) we obtain

$$-2\langle b, Q(h^T\xi)\rangle = -2h_{1,1}\langle hb, Q(\xi)\rangle = -2\langle b, h_{1,1}\, h^T Q(\xi)\rangle,$$

whence

$$Q(h^T\xi) = h_{1,1}\, h^T Q(\xi). \tag{34}$$

By using the definition of Ψ, (29), (32), (26) and once again the definition of Ψ, we can now compute for $\hat{f} \in L^2(\Theta_L)$, $b \in \mathbb{R}^d$ and $h \in H$

$$\begin{aligned}
&\Psi\mu(\phi(b,h))\Psi^{-1}\hat{f}(\xi) = |\det J_{Q^{-1}}(\xi)|^{1/2}(\mu(\phi(b,h))\Psi^{-1}\hat{f})(Q^{-1}(\xi))\\
&= |\det J_{Q^{-1}}(\xi)|^{1/2}|\det\rho(h)|^{-1/2}e^{\pi i\langle\sigma_b Q^{-1}(\xi), Q^{-1}(\xi)\rangle}\Psi^{-1}\hat{f}(\rho(h)^{-1}Q^{-1}(\xi))\\
&= |\det J_{Q^{-1}}(\xi)|^{1/2}|\det\rho(h)|^{-1/2}e^{-2\pi i\langle b,\xi\rangle}\Psi^{-1}\hat{f}(\rho(h)^{-1}Q^{-1}(\xi)) \qquad (35)\\
&= |\det J_{Q^{-1}}(\xi)|^{1/2}h_{1,1}^{-\frac{d}{4}}|\det h|^{1/2}e^{-2\pi i\langle b,\xi\rangle}\Psi^{-1}\hat{f}(h_{1,1}^{-\frac{1}{2}}\, h^T Q^{-1}(\xi))\\
&= |\det J_{Q^{-1}}(\xi)|^{1/2}h_{1,1}^{-\frac{d}{4}}|\det h|^{1/2}e^{-2\pi i\langle b,\xi\rangle}\\
&\qquad\cdot|\det J_Q(h_{1,1}^{-\frac{1}{2}}\, h^T Q^{-1}(\xi))|^{1/2}\hat{f}(Q(h_{1,1}^{-\frac{1}{2}}\, h^T Q^{-1}(\xi))).
\end{aligned}$$

Now note that by (34) and by the fact that Q is quadratic there holds

$$Q(h_{1,1}^{-\frac{1}{2}}\, h^T Q^{-1}(\xi)) = h_{1,1}\, h^T Q(h_{1,1}^{-\frac{1}{2}} Q^{-1}(\xi)) = h^T Q(Q^{-1}(\xi)) = h^T\xi. \tag{36}$$

Moreover we have

$$\begin{aligned}
|\det J_Q(h_{1,1}^{-\frac{1}{2}}\, h^T Q^{-1}(\xi))| &= 2^{1-d}|(h_{1,1}^{-\frac{1}{2}}\, h^T Q^{-1}(\xi))_1|^d\\
&= 2^{1-d}h_{1,1}^{-\frac{d}{2}}|(\, h^T Q^{-1}(\xi))_1|^d\\
&= 2^{1-d}h_{1,1}^{-\frac{d}{2}}h_{1,1}^d|Q^{-1}(\xi)_1|^d\\
&= 2^{1-d}h_{1,1}^{\frac{d}{2}}2^{\frac{d}{2}}|\xi_1|^{\frac{d}{2}},
\end{aligned}$$

where the first equality follows from (30), the third one from the fact that h^T is lower triangular and the fourth one from (33). Therefore by (31)

$$|\det J_{Q^{-1}}(\xi)|^{1/2}|\det J_Q(h_{1,1}^{-\frac{1}{2}}\, h^T Q^{-1}(\xi))|^{1/2} = 2^{\frac{d}{4}-\frac{1}{2}}|\xi_1|^{-\frac{d}{4}}2^{\frac{1}{2}-\frac{d}{2}}h_{1,1}^{\frac{d}{4}}2^{\frac{d}{4}}|\xi_1|^{\frac{d}{4}} = h_{1,1}^{\frac{d}{4}}. \tag{37}$$

Finally, inserting (36) and (37) into (35) we obtain

$$\Psi\mu(\phi(b,h))\Psi^{-1}\hat{f}(\xi) = |\det h|^{1/2} e^{-2\pi i\langle b,\xi\rangle}\hat{f}(h^T\xi) = \hat{\pi}(b,h)\hat{f}(\xi),$$

as desired.

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Numerical Solution to an Energy Concentration Problem Associated with the Special Affine Fourier Transformation

Amara Ammari, Tahar Moumni, and Ahmed Zayed

1 Introduction

The SAFT, which was introduced in [1], is an integral transformation associated with a general inhomogeneous lossless linear mapping in phase-space that depends on six parameters independent of the phase-space coordinates. It maps the position x and the wave number k into

$$\begin{bmatrix} x' \\ k' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ k \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix} \tag{1}$$

with

$$ad - bc = 1. \tag{2}$$

This transformation, which can model many general optical systems [1], maps any convex body into another convex body and (2) guarantees that the area of the body is preserved by the transformation. Such transformations form the inhomogeneous special linear group $\mathrm{ISL}(2, \mathbf{R})$.

A. Ammari
Department of Mathematics, Faculty of Sciences of Tunis, University of Manar, Tunis, Tunisia
e-mail: ammari.ipeib@gmail.com

T. Moumni
Department of Mathematics, Faculty of Sciences of Bizerte, University of Carthage, Jarzouna, Tunisia
e-mail: moumni.tahar1@gmail.com

A. Zayed (✉)
Department of Mathematical Sciences, DePaul University, Chicago, IL 60614, USA
e-mail: azayed@condor.depaul.edu

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The SAFT offers a unified viewpoint of known signal processing transformations as well as optical operations on light waves.

The integral representation of the wave-function transformation associated with the transformation (1) and (2) is given by

$$F(\omega) = \widehat{f}_{\mathbf{A}}(\omega) = \int_{\mathbf{R}} k(t,\omega) f(t) dt \tag{3}$$
$$= \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbf{R}} \exp\left\{ \frac{i}{2b} \left(at^2 + d\omega^2 - 2t\omega + 2pt + 2(bq - dp)\omega \right) \right\} f(t) dt,$$

where $\mathbf{A}$ stands for the six parameters (a, b, c, d, p, q), and

$$k(t,\omega) = \frac{1}{\sqrt{2\pi|b|}} \exp\left\{ \frac{i}{2b} \left(at^2 + d\omega^2 - 2t\omega + 2pt + 2(bq - dp)\omega \right) \right\}.$$

The inversion formula for the SAFT is easily shown to be

$$f(t) = \frac{1}{\sqrt{2\pi|b|}} \int_{\mathbf{R}} F(\omega) \exp\left\{ \frac{-i}{2b} \left(d\omega^2 + at^2 - 2t\omega + 2\omega(bq - dp) + 2pt \right) \right\} d\omega, \tag{4}$$

which may be considered as the SAFT evaluated using $\mathbf{A}^{-1}$, where[1]

$$\mathbf{A}^{-1} \stackrel{def}{=} \left[\mathbf{A}^{-1} \middle| \underline{\lambda}^{-1} \right] \equiv \left[\begin{array}{cc|c} +d & -b & bq - dp \\ -c & +a & cp - aq \end{array} \right],$$

and to be precise,

$$\mathbf{A}^{-1} = \begin{bmatrix} +d & -b \\ -c & +a \end{bmatrix} \text{ and } \underline{\lambda}^{-1} \stackrel{def}{=} \begin{bmatrix} bq - dp \\ cp - aq \end{bmatrix}.$$

We also have

$$\langle f, g \rangle = \int_{\mathbf{R}} f(t)\overline{g}(t) dt = \int_{\mathbf{R}} F(\omega)\overline{G}(\omega) d\omega = \langle F, G \rangle,$$

from which we obtain $\|f\| = \|F\|$. When $p = 0 = q$, we obtain the homogeneous special group $\mathrm{SL}(2, \mathbf{R})$, which is represented by the unimodular matrix

$$\mathbf{M} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

[1]With a little abuse of notation, we use $\underline{\lambda}^{-1}$ which should be understood as a parameter vector corresponding to the inverse–SAFT.

One of the fundamental problems in communication engineering is the energy concentration problem, i.e. the problem of finding a signal bandlimited to $[-\sigma, \sigma]$ with maximum energy concentration in the interval $[-\tau, \tau]$, $0 < \sigma, \tau$, in the time domain or equivalently, finding a signal that is time limited to the interval $[-\tau, \tau]$ with maximum energy concentration in $[-\sigma, \sigma]$ in the frequency domain. This problem was solved by a group of mathematicians, D. Slepian, H. Landau, and H. Pollak at Bell Labs [7, 8, 10, 11] in the early 1960s. The solution involved the prolate spheroidal wave functions (PSWF) which are eigenfunctions of a differential and an integral equations.

In [13] a solution of the energy concentration problem for a general class of integral transforms, including SAFT, was proposed. In one dimension, the problem for the SAFT was not difficult to solve in closed form; however, the two-dimensional problem was more challenging when the signal is bandlimited to a disc. The latter problem may be phrased as follows: Among all signals bandlimited to a disc E_1 centered at the origin with radius δ in the SAFT domain, find the signal with maximum energy concentration in a disc T_1 centered at the origin with radius R. Because of the uncertainty principle, the energy concentration in T_1 cannot be 100%.

A. Zayed has showed in [13] that the solution of the energy concentration problem is the eigenfunction that corresponds to the largest eigenvalue of the integral equation

$$\int_{E_1} f(\mathbf{p}) K_1(\mathbf{q}, \mathbf{p}) d\mathbf{p} = \lambda f(\mathbf{q}), \tag{5}$$

where $\mathbf{p}, \mathbf{q} \in \mathbf{R}^2$ and K_1 depends on R and the kernel of the SAFT and is given explicitly by

$$K_1(\mathbf{p}, \mathbf{q}) = \frac{RF(p,q)}{2\pi b} \frac{J_1\left(R\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}/b\right)}{\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}},$$

where

$$F(p,q) = \exp\left\{ \frac{i}{2b} \left[d\left(|\mathbf{p}|^2 - |\mathbf{q}|^2\right) + \eta(\mathbf{p} - \mathbf{q}) \right] \right\}, \tag{6}$$

where $\eta = 2(bn - dm)$ and J_1 is the Bessel function of the first kind and order one.

The aim of this chapter is to solve numerically the above energy concentration problem for the SAFT, i.e. to find numerically the largest eigenvalues and their corresponding eigenfunctions. We use the Gaussian quadrature method in two dimensions to solve that problem, as well as a concentration of energy problem in other cases involving the so-called generalized prolate spheroidal wave functions (GPSWFs) and another problem involving an integral equation with kernel of convolution type.

2 Gaussian Quadrature for General Quadrilateral

In this section we discuss the Gaussian quadrature method for evaluating two-dimensional integrals over general quadrilateral domains, for more details, see [12]. We begin with the evaluation of two-dimensional integrals over a standard quadrilateral element. Let R_{st} be the standard quadrilateral element $R_{st} = [-1, 1]^2$ (Fig. 1).

To compute the following integral

$$I = \int\int_{R_{st}} g(\xi,\ \eta)d\xi d\eta = \int_{-1}^{1}\int_{-1}^{1} g(\xi,\ \eta)d\xi d\eta,$$

we use the Gaussian quadrature method in one dimension twice. More precisely, for any fixed η, we integrate numerically with respect to ξ:

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\ \eta)d\xi d\eta \approx \int_{-1}^{1}(\sum_{i=1}^{M} w_i g(\xi_i,\ \eta))d\eta,$$

where ξ_i and w_i are Gaussian quadrature points and weights of order M in the ξ direction. Next, integrating numerically with respect to η, we have

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\ \eta)d\xi d\eta \approx \sum_{i=1}^{M}\sum_{j=1}^{N} w_i \widehat{w}_j g(\xi_i,\ \eta_j),$$

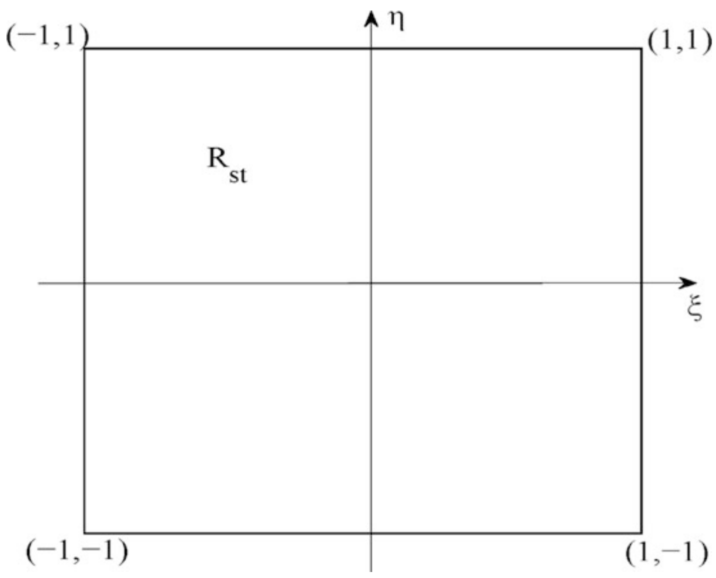

Fig. 1 The standard quadrilateral element $R_{st} = [-1, 1]^2$.

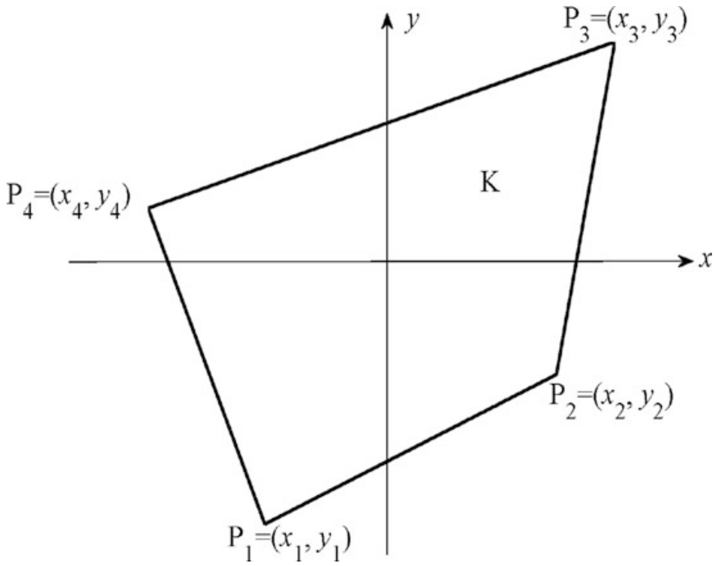

Fig. 2 A quadrilateral element with straight boundary lines.

where η_j and $\widehat{w}_j$ are Gaussian quadrature points and weights of order N in the η direction. Usually $M = N(so\ \eta_i = \xi_i, \widehat{w}_i = w_i)$, and we have Gaussian quadrature of order N for the standard quadrilateral element

$$\int_{-1}^{1}\int_{-1}^{1} g(\xi,\ \eta)d\xi d\eta \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j g(\xi_i,\ \xi_j). \tag{7}$$

Now we use the Gaussian quadrature method to evaluate two-dimensional integrals over a general quadrilateral region. Let K be a general quadrilateral element with straight boundary lines and vertices $(x_i,\ y_i),\ i = 1, 2, 3, 4$ arranged in the counter-clockwise order (Fig. 2): We would like to evaluate

$$I = \int\int_K F(x,\ y)dxdy.$$

Firstly, we transform the quadrilateral element K to the standard quadrilateral element R_{st} and then apply the Gaussian quadrature (7). The idea is to construct a linear mapping to map the quadrilateral element K with straight boundary lines to the standard quadrilateral element R_{st} (Fig. 3):

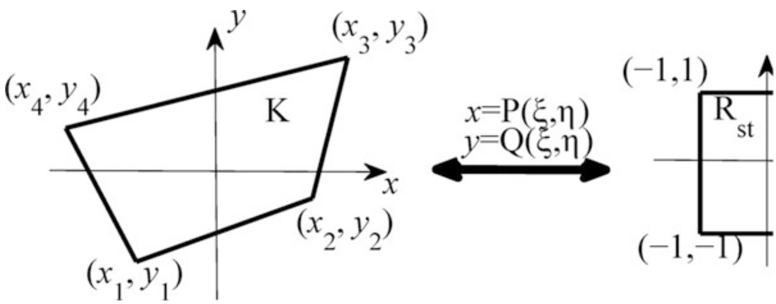

Fig. 3 Linear mapping between K and R_{st}.

The mapping can be achieved conveniently by using the nodal shape functions as follows:

$$x = P(\xi,\ \eta) = \sum_{i=1}^{4} x_i N_i(\xi,\ \eta) = x_1N_1(\xi,\ \eta)+x_2N_2(\xi,\ \eta)+x_3N_3(\xi,\ \eta)+x_4N_4(\xi,\ \eta),$$

$$y = Q(\xi,\ \eta) = \sum_{i=1}^{4} y_i N_i(\xi,\ \eta) = y_1N_1(\xi,\ \eta)+y_2N_2(\xi,\ \eta)+y_3N_3(\xi,\ \eta)+y_4N_4(\xi,\ \eta),$$

where the nodal shape functions for quadrilaterals are given by:

$$N_1(\xi,\ \eta) = \frac{1}{4}(1-\xi)(1-\eta),$$

$$N_2(\xi,\ \eta) = \frac{1}{4}(1+\xi)(1-\eta),$$

$$N_3(\xi,\ \eta) = \frac{1}{4}(1+\xi)(1+\eta),$$

$$N_4(\xi,\ \eta) = \frac{1}{4}(1-\xi)(1+\eta).$$

Then we have

$$\int\int_K F(x,\ y)dxdy = \int\int_{R_{st}} F(P(\xi,\ \eta),\ Q(\xi,\ \eta))|J(\xi,\ \eta)|d\xi d\eta,$$

where $J(\xi,\ \eta)$ is the Jacobian of the transformation defined by

$$J(\xi,\ \eta) = |\frac{\partial(x,y)}{\partial(\xi,\eta)}| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix}.$$

Secondly, by applying the Gaussian quadrature methods for the standards quadrilateral element, we obtain the Gaussian quadrature of order N for general quadrilateral elements,

$$\int\int_K F(x,\ y)dxdy \approx \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j F(P(\xi_i,\ \xi_j),\ Q(\xi_i,\ \xi_j))|J(\xi_i,\ \xi_j)|.$$

3 The Gaussian Quadrature Method and Integral Equations

In this section we use the Gaussian quadrature method to solve two-dimensional integral equations. Suppose that now we have to solve the following integral equation

$$\int_K k(x,y)\varphi(y)dy = \alpha\varphi(x), \quad x \in K, \tag{8}$$

where K is a general quadrilateral element or the unit disc of $\mathbf{R}^2$. In the special case where K is a general quadrilateral element we use the mapping cited above which transforms K to the standard quadrilateral element. In the special case where K is the unit disc of $\mathbf{R}^2$ we use the polar coordinates to transform (8) to the following integral equation:

$$\int_0^1 \int_0^{2\pi} \mathscr{K}(r,r',\theta,\theta')\psi(r',\theta')d\theta' r' dr' = \alpha\psi(r,\theta). \tag{9}$$

To proceed further, we assume that $\mathscr{K}(r,r',\theta,\theta')$ can be written as follows:

$$r'\mathscr{K}(r,r',\theta,\theta') = \sum_{N=-\infty}^{+\infty} k_N(r,r')e^{iN(\theta-\theta')}. \tag{10}$$

As was done in [10], let us search for a solution of (10) of the form,

$$\psi(r,\theta) = \sum_{N=-\infty}^{+\infty} \varphi_N(r)e^{iN\theta} \tag{11}$$

Substituting (10) and (9) into (11), one gets

$$\int_0^1 k_N(r,r')\varphi_N(r')dr' = \beta\varphi_N(r), \quad r \in [0,1], N \in Z, \tag{12}$$

where $\alpha = 2\pi\beta$. By applying the Gaussian quadrature method in one dimension [3] we obtain an approximate solution of the integral equation (12) and therefore we obtain an approximate solution of (11).

3.1 Examples

In this subsection we demonstrate the above techniques by examples.

Example 1: Prolate Spheroidal Wave Function (PSWF)

In this example and without loss of generality, we will consider the standard quadrilateral element $R_s = [-1, 1]^2$. We consider the 2-D PSWFs associated with R_s and defined by:

$$\int_{R_s} e^{ic<x,y>} \Psi(y)dy = \alpha\Psi(x). \tag{13}$$

Here, $< ., . >$ is the classical inner product on $\mathbf{R}^2$. It is well known that the solutions of (13) of product form are

$$\Psi_{mn}(x) = \psi_m(x_1)\psi_n(x_2), \quad m,\ n \in \mathbf{N},$$

associated with the eigenvalue $\alpha_{m,n}(c)$. Here, ψ_m is the classical PSWFs in one dimension given by

$$\int_{-1}^{1} e^{icxy} \psi_m(y)dy = \widetilde{\alpha}\psi_m(x). \tag{14}$$

Although the eigenfunctions can be found in closed form, numerical methods are used to calculate the eigenvalues; see [11].

Example 2: Generalized Prolate Spheroidal Wave Function (GPSWF) The Generalized Prolate Spheroidal Wave Function was introduced in [9]. The N-dimensional GPSWFs are solution of the integral equation

$$\int_T k(x, y)\varphi(y)d\mu(y) = \alpha\varphi(x), \tag{15}$$

where $k(x, y)$ is the reproducing kernel of an appropriate reproducing-kernel Hilbert space of functions and T is a general domain in $\mathbf{R}^N$. For the computation of GPSWF in the special case where T is a general quadrilateral element, we use the Gaussian quadrature for general quadrilateral elements.

Example 3: Circular Prolate Spheroidal Wave Functions:

In this example, we consider the 2-D PSWFs defined by

$$\int_D K(x, y)\Psi(y)dy = \alpha\Psi(x), \tag{16}$$

where

- $D = \{(x_1, x_2) \in \mathbf{R}^2; x_1^2 + x_2^2 \leq 1\}$ is the unit disc of $\mathbf{R}^2$,
- $K(x, y) = e^{ic<x,y>}$, where $< x, y > = x_1y_1 + x_2y_2$ is the usual inner product on $\mathbf{R}^2$, and c is a positive real number.

To solve the integral equation (16), Slepian [10] used polar coordinates and converted (16) into

$$\int_0^1 r'dr' \int_0^{2\pi} \mathscr{K}(r, r', \theta, \theta')\widetilde{\Psi}(r', \theta')d\theta' = \alpha\widetilde{\Psi}(r, \theta). \tag{17}$$

Here, $\mathscr{K}(r, r', \theta, \theta') = e^{icrr'cos(\theta-\theta')}$. Note that we can use the Gaussian quadrature method to solve (17), but since the kernel $\mathscr{K}(r, r', \theta, \theta')$ is of convolution type, let us proceed and search for a solution of (17) in the form

$$\widetilde{\Psi}(r, \theta) = \sum_{N=-\infty}^{+\infty} \varphi_N(r)e^{iN\theta} \tag{18}$$

In view of the relation [2, P. 344],

$$e^{itsin\theta} = \sum_{N=-\infty}^{+\infty} J_N(t)e^{iN\theta} \tag{19}$$

or equivalently,

$$e^{icrrcos(\theta-\theta')} = \sum_{N=-\infty}^{+\infty} i^N J_N(crr')e^{iN(\theta-\theta')}. \tag{20}$$

Substituting (20) and (17) into (18), one gets

$$\widetilde{H}_N(\varphi_N)(x) \stackrel{def}{=} \int_0^1 yJ_N(cxy)\varphi_N(y)dy = \beta\varphi_N(x), \quad x \in [0, 1], \quad N \in \mathbf{Z} \tag{21}$$

where $J_N(.)$ is the Bessel function of the first kind of degree N and $\alpha = 2\pi\beta$. By making the substitution

$$\gamma = \sqrt{c}\beta, \;\; \phi_N(y) = \sqrt{y}\varphi_N(y),$$

in (21), we get the equivalent integral equation

$$\widetilde{\mathscr{H}}_N(\phi_N)(x) \stackrel{def}{=} \int_0^1 \sqrt{cxy}\, J_N(cxy)\phi_N(y)dy = \gamma\phi_N(x), \quad x \in [0, 1] \quad N \in \mathbf{Z}. \tag{22}$$

Circular prolate spheroidal wave functions (CPSWFs) are the solutions of (22). In [5], we have showed that the CPSWFs share similar properties with the PSWFs, notably the double orthogonality over finite and infinite intervals. We have also used the Gaussian quadrature methods, among other methods, to solve the integral equation (22).

4 Solution of the SAFT Energy Concentration Problem by the Gaussian Quadrature Method

The aim of this section is to solve the integral equation (5) or at least to find the largest eigenvalues and the corresponding eigenfunctions for different values of b and fixed δ, and then for fixed b and different values of δ. It is shown that for fixed b, the largest eigenvalues increase as δ increases, while for fixed δ the largest eigenvalues decrease as b increases. Because of the phase factor in (6), these eigenfunctions are generally complex-valued. However, for practical purposes we are interested in the modulus of this function. Therefore, without loss of generality, we will consider the integral equation:

$$\mathrm{K}f(x) = \int_E f(y)K(y,x)dy = \lambda f(x), \tag{23}$$

where

$$K(x,y) = \frac{J_1(\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}/b)}{2\pi b\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}},$$

$b > 0$, and J_1 is the Bessel function of the first kind and E is a disc of radius $\delta \le 1$.

Using polar coordinates, $x = (\rho, \phi)$ and $y = (r, \theta)$, Equation (23) becomes

$$\int_0^\delta \int_0^{2\pi} g(r,\theta)\mathscr{K}(r,\theta,\rho,\phi)rdrd\theta = \lambda g(\rho,\phi), \tag{24}$$

where

$$\mathscr{K}(r,\theta,\rho,\phi) = \frac{J_1(\sqrt{r^2+\rho^2-2r\rho cos(\theta-\phi)}/b)}{2\pi b\sqrt{r^2+\rho^2-2r\rho cos(\theta-\phi)}}.$$

Using the following change of variables, $\widetilde{r} = \frac{r}{\delta}$, $\widetilde{\rho} = \frac{\rho}{\delta}$, $\widetilde{\theta} = \frac{\theta}{2\pi}$, $\widetilde{\phi} = \frac{\phi}{2\pi}$, Equation (24) becomes

$$\int_0^1 \int_0^1 g(\delta\widetilde{r},2\pi\widetilde{\theta})\mathscr{K}(\delta\widetilde{r},2\pi\widetilde{\theta},\delta\widetilde{\rho},2\pi\widetilde{\phi})\widetilde{r}d\widetilde{r}d\widetilde{\theta} = \widetilde{\lambda}g(\delta\widetilde{\rho},2\pi\widetilde{\phi}), \tag{25}$$

where $\widetilde{\lambda} = \frac{\lambda}{2\pi\delta^2}$.

For simplicity of the notation, we denote by $\widetilde{K}(r,\theta,\rho,\phi) = \mathscr{K}(\delta r, 2\pi\theta, \delta\rho, 2\pi\phi)$ and $\widetilde{g}(r,\theta) = g(\delta r, 2\pi\theta)$. Thus, our aim now is to solve the integral equation

$$\mathbf{K}\widetilde{g}(x) = \int_0^1 \int_0^1 \widetilde{g}(r,\theta)\widetilde{K}(r,\theta,\rho,\phi) r dr d\theta = \widetilde{\lambda}\widetilde{g}(\rho,\phi), \tag{26}$$

Let us denote by $\varphi_{n,b}$ the nth eigenfunction of $\mathbf{K}$ and by $\widetilde{\lambda}_n(b)$ its associate eigenvalue. Here the eigenvalues $\widetilde{\lambda}_n(b)$ are arranged as follows $|\widetilde{\lambda}_1(b)| \geq |\widetilde{\lambda}_2(b)| \geq \cdots |\widetilde{\lambda}_n(b)| \geq |\widetilde{\lambda}_{n+1}(b)| \geq \cdots$. When b is known and fixed, we may write φ_n instead of $\varphi_{n,b}$ and $\widetilde{\lambda}_n$ instead of $\widetilde{\lambda}_n(b)$.

For the computation of φ_n and its corresponding eigenvalue, we use shifted Legendre polynomials given by the following Rodriguez formula, see [6]

$$P_n(x) = c_n(-1)^n \frac{d^n[x^n(x-1)^n]}{dx^n}, \tag{27}$$

where c_n is a normalization coefficient to be determined in the sequel. Straightforward computations show that

$$\int_0^1 P_n(x)P_m(x)dx = (-1)^n (c_n)^2 (2n)! B(n-1, n-1)\delta_{mn}, \tag{28}$$

where $B(x,y)$ is the Beta function given by $B(x,y) = \displaystyle\int_0^1 t^{x-1}(1-t)^{y-1}dt$ for $x > 0$ and $x > 0$. Hence, the normalized polynomials are given by

$$P_n(x) = \frac{\sqrt{(2n+1)}}{n!} \frac{d^n[x^n(1-x)^n]}{dx^n}. \tag{29}$$

It is interesting to mention that P_n, $n \geq 0$, has all its zeros in $[0,1]$. These zeros denoted by $x_1, \cdots, x_n$, are simply given as the eigenvalues of the following tridiagonal matrix D

$$D = [d_{i,j}]_{1\leq i,j\leq n}, \quad d_{j,j} = \frac{1}{2}, \quad d_{j,j+1} = d_{j+1,j} = \frac{j}{2\sqrt{4j^2-1}}, \tag{30}$$

and $d_{i,j} = 0$ if $i \neq j-1, j, j+1$. For more details, see [3]. Also, it is well known [3], that if $f \in C^{2n}([0,1];\mathbf{R})$, then we have the following Gaussian quadrature formula to evaluate the integral

$$\int_0^1 f(x)dx = \sum_{k=1}^n w_k f(x_k) + \frac{1}{a_n^2}\frac{f^{(2n)}(\eta)}{(2n)!}, \text{for some } 0 \leq \eta \leq 1, \tag{31}$$

where $a_n = \frac{\sqrt{2n+1}}{n!}$ is the highest coefficient of P_n and $w_k = -\frac{a_{n+1}}{a_n} \frac{1}{P_{n+1}(x_k) P'_n(x_k)}$.

The formula for the Gaussian quadrature method in two dimensions is given by

$$\int_0^1 \int_0^1 f(x, y) dx dy \approx \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j f(x_i, x_j). \tag{32}$$

By combining (32) and (25), one obtains:

$$\sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j x_i \widetilde{K}(x_i, x_j, \rho, \phi) \varphi_n(x_i, x_j) \approx \widetilde{\lambda}_n \varphi_n(\rho, \phi). \tag{33}$$

By making the following substitutions in (33), $\rho = x_l$ and $\phi = x_k$, where $1 \leq \ell, k \leq N$, one obtains the following system of N^2 equations

$$\sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j x_i \widetilde{K}(x_i, x_j, x_\ell, x_k) \varphi_n(x_i, x_j) \approx \widetilde{\lambda}_n \varphi_n(x_\ell, x_k). \tag{34}$$

For the simplicity of notation, we let $a(i, j, \ell, k) = w_i w_j x_i \widetilde{K}(x_i, x_j, x_\ell, x_k)$ and $b_n(\ell, k) = \varphi_n(x_\ell, x_k)$. Note here that the N^2 equations given by (33) are equivalent to

$$A \cdot V_n = \widetilde{\lambda}_n V_n,$$

where A is the $N^2 \times N^2$ matrix given by

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1N} \\ A_{21} & A_{22} & \cdots & A_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ A_{N1} & A_{N2} & \cdots & A_{NN} \end{pmatrix}$$

and

$$V_n^T = (b_n(1, 1), \cdots, b_n(1, N), b_n(2, 1), \cdots, b_n(2, N), \cdots, b_n(N, 1), \cdots, b_n(N, N))$$

Here A_{lk} is the $N \times N$ matrix whose coefficients are nothing else but $(a(k, j, \ell, i)_{1 \leq i, j \leq N})$.

5 Numerical Results

Example 1: The case of the disc (SAFT).
By letting $N = 10$ in the case of the SAFT, we obtain the following numerical results (Fig. 4):

Remark 1. We remark from the numerical results of Tables 1–3 that for fixed n and fixed δ the function $|\lambda_n(b, \delta)|$ decreases as b increases.

Remark 2. We remark from the numerical results of Tables 4–6 that for fixed n and b the function $|\lambda_n(b, \delta)|$ increases as δ increases.

To proceed further, let denote by Φ_n the nth eigenfunction of K given by (23). Note here that Φ_n is defined on the disc of radius δ centered at the origin (Figs. 5, 6, 7, 8, and 9).

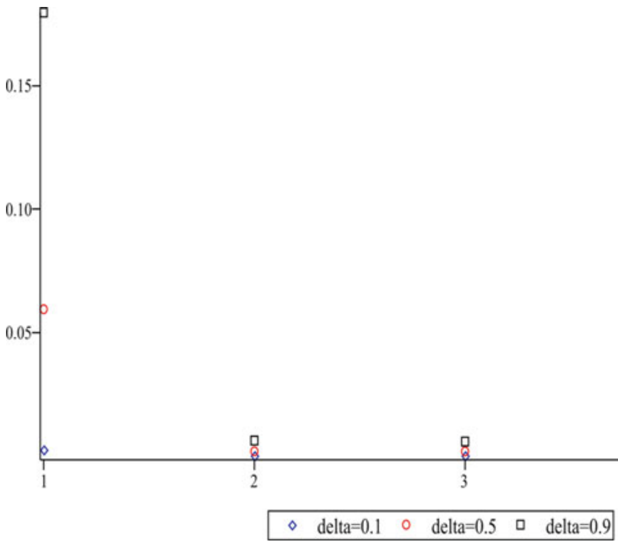

Fig. 4 Graphs of $|\lambda_n(b, \delta)|$ for different values of the parameter δ and for $b = 1$.

Table 1 Values of $|\lambda_n(b, \delta)|$ for different values of b and fixed $\delta = 0.1$.

n	$\|\lambda_n(1, 0.1)\|$	$\|\lambda_n(10, 0.1)\|$	$\|\lambda_n(20, 0.1)\|$
n = 1	0.2454022238e-2	0.2457110612e-4	0.6142835069e-5
n = 2	0.7125526937e-4	0.7128850760e-6	0.1782219658e-6
n = 3	0.7042877369e-4	0.7046377014e-6	0.1761601553e-6
n = 4	0.6791566596e-4	0.6795821515e-6	0.1698964494e-6
n = 5	0.6694493407e-4	0.6708114218e-6	0.1677055989e-6

Table 2 Values of $|\lambda_n(b, \delta)|$ for different values of b and fixed $\delta = 0.5$.

n	$\|\lambda_n(1, 0.5)\|$	$\|\lambda_n(10, 0.5)\|$	$\|\lambda_n(20, 0.5)\|$
n = 1	0.0595157513	0.6140903665e-3	0.1535591695e-3
n = 2	0.0017757813	0.1781995457e-4	0.4455410558e-5
n = 3	0.0017547996	0.1761366356e-4	0.4403858648e-5
n = 4	0.0016916250	0.1698672813e-4	0.4247230158e-5
n = 5	0.0016446913	0.1676164059e-4	0.4192093072e-5

Table 3 Values of $|\lambda_n(b, \delta)|$ for different values of b and fixed $\delta = 0.9$.

n	$\|\lambda_n(1, 0.9)\|$	$\|\lambda_n(10, 0.9)\|$	$\|\lambda_n(20, 0.9)\|$
n = 1	0.1798154920	0.1988237807e-2	0.4974432166e-3
n = 2	0.6158608947e-2	0.5772152659e-4	0.1443450443e-4
n = 3	0.5743168460e-2	0.5705233661e-4	0.1426742584e-4
n = 4	0.5678783994e-2	0.5501767092e-4	0.1375969164e-4
n = 5	0.5620593759e-2	0.5424552928e-4	0.1357829818e-4

Table 4 Values of $|\lambda_n(b, \delta)|$ for different values of δ and fixed $b = 0.01$.

n	$\|\lambda_n(0.01, 0.1)\|$	$\|\lambda_n(0.01, 0.5)\|$	$\|\lambda_n(0.01, 0.9)\|$
n = 1	1.3516387262	3.7148184478	5.0445557883
n = 2	1.3496428804	3.6615475731	4.9869102652
n = 3	1.2944308861	3.3673717133	4.8933207101
n = 4	1.2890502395	3.3359956169	4.8449264773
n = 5	0.9894122515	3.3281710198	4.5778648879

Table 5 Values of $|\lambda_n(b, \delta)|$ for different values of δ and fixed $b = 0.1$.

n	$\|\lambda_n(0.1, 0.1)\|$	$\|\lambda_n(0.1, 0.5)\|$	$\|\lambda_n(0.1, 0.9)\|$
n = 1	0.21686083297	0.9220108659	1.3887067139
n = 2	0.9969863516e-2	0.8296680510	1.2724173455
n = 3	0.9313107509e-2	0.8171040493	1.1452172446
n = 4	0.7086460202e-2	0.5970233994	1.0226225145
n = 5	0.7009256906e-2	0.5933987767	0.8873584713

Table 6 Values of $|\lambda_n(b, \delta)|$ for different values of δ and fixed $b = 1$.

n	$\|\lambda_n(1, 0.01)\|$	$\|\lambda_n(1, 0.5)\|$	$\|\lambda_n(1, 0.9)\|$
n = 1	0.2457110612e-4	0.5951575130e-1	0.1798154920
n = 2	0.7128850760e-6	0.1775781386e-2	0.6158608947e-2
n = 3	0.7046377014e-6	0.1754799680e-2	0.5743168460e-2
n = 4	0.6795821515e-6	0.1691625026e-2	0.5678783994e-2
n = 5	0.6708114218e-6	0.1644691357e-2	0.5620593759e-2

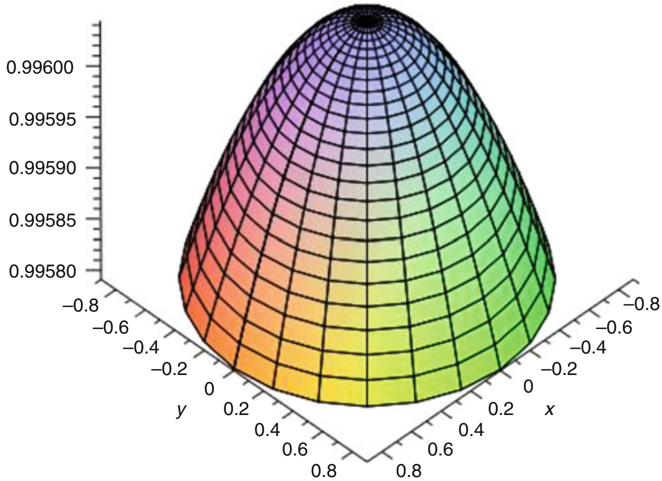

Fig. 5 Graphs of Φ_1 associated with the parameter $b = 20$ and $\delta = 0.9$.

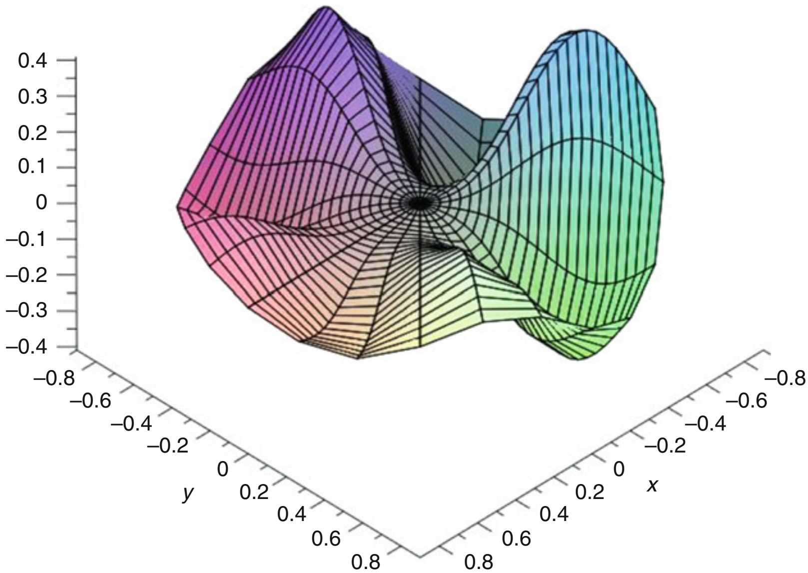

Fig. 6 Graphs of Φ_1 associated with the parameter $b = 0.1$ and $\delta = 0.9$.

Example 2: The 2-D PSWFs associated with the standard quadrilateral.

By letting $N = 10$ in the Gaussian quadrature formula in two dimensions for the case of the 2-D PSWFs associated with the standard quadrilateral, we obtain the following numerical results: In Table 7 we list eigenvalues $|\alpha_{m,n}(c)|$ associated with $\Psi_{m,n}$ for several pairs (m, n)

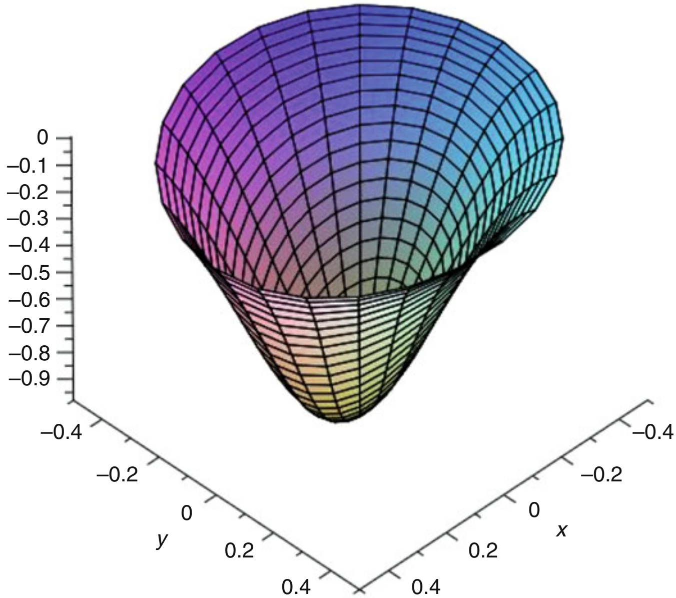

Fig. 7 Graphs of Φ_1 associated with the parameter $b = 0.1$ and $\delta = 0.5$.

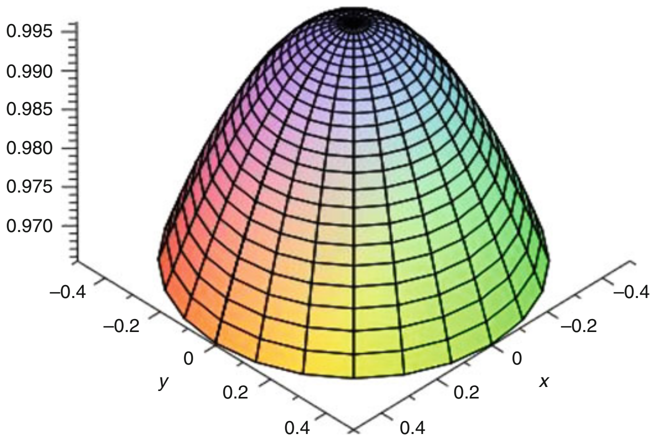

Fig. 8 Graphs of Φ_1 associated with the parameter $b = 1$ and $\delta = 0.5$.

To compare the results of this example to those given in [4], we recall that the mth PSWFs given by (14) is also solution of the following integral equation

$$\int_{-1}^{1} \frac{\sin(c(x-y))}{\pi(x-y)} \psi_m(y)dy = \widetilde{\lambda}_m(c)\psi_m(x). \tag{35}$$

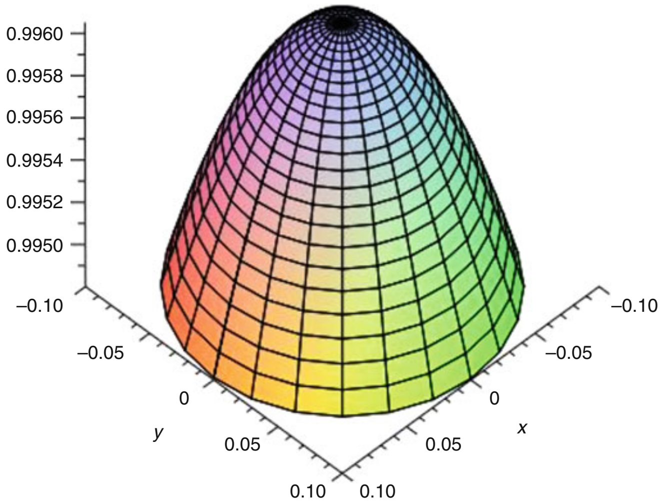

Fig. 9 Graphs of Φ_1 associated with the parameter $b = 1$ and $\delta = 0.1$.

Table 7 Values of $|\alpha_{m,n}(c)|$ for different values of c.

(m, n)	$\lvert\alpha_{m,n}(0.1)\rvert$	$\lvert\alpha_{m,n}(4)\rvert$	$\lvert\alpha_{m,n}(10)\rvert$
$(1, 1)$	3.9955600952	1.5643332702	0.6283185030
$(1, 2)$	0.1331793883	1.4970888465	0.6283175023
$(2, 2)$	0.4439114679e-2	1.4327349913	0.6283165016

Table 8 Values of $|\widetilde{\lambda}_n(c)|$ for $c = 4$ and for $c = 10$.

n	$\lvert\widetilde{\lambda}_n(4)\rvert$	$\lvert\widetilde{\lambda}_n(4)\rvert$ see [4]	$\lvert\widetilde{\lambda}_n(10)\rvert$	$\lvert\widetilde{\lambda}_n(10)\rvert$ see [4]
1	0.**9958854**897	**0.958854**904	0.**999**5963049	0.**999**9999559
2	0.**9121074**308	0.**9121074**240	0.**99**68501088	0.**99**99967707
3	0.**5190548**175	0.**5190548**374	0.**99**36927993	0.**99**98927329
5	0.**381**3008980**e-3**	0.**381**2917217**e-3**	0.**8**976748606	0.**8**251463487

Note that the relation between $\widetilde{\alpha}_n(c)$ and $\widetilde{\lambda}_n(c)$ is given by $|\widetilde{\lambda}_n(c)| = \frac{c}{2\pi}|\widetilde{\alpha}_n(c)|^2 = \frac{c}{2\pi}|\alpha_{nn}(c)|$, see [11]. In Table 8 we list in the first and third columns the eigenvalues $\widetilde{\lambda}_n(c)$ calculated by the quadrature method in two dimensions for $c = 4$ and $c = 10$, respectively, while in the second and the fourth columns we list the eigenvalues $\widetilde{\lambda}_n(c)$ obtained in [4] for the same values of c. Note here that the bold digits indicate the agreement between our results and those of Ref. [4].

In what follows we present the plots of some 2-D PSWFs associated with the standard equilateral $\Psi_{m,n}$ for different values of m, n and c (Figs. 10, 11, 12, 13, 14, and 15).

Example 3: The 2-D PSWFs associated with the disc of radius 1.

We note here that in this example, the kernel given in (17) can be written as in (10). Hence, finding the solution of (16) is reduced to finding the solution of (22).

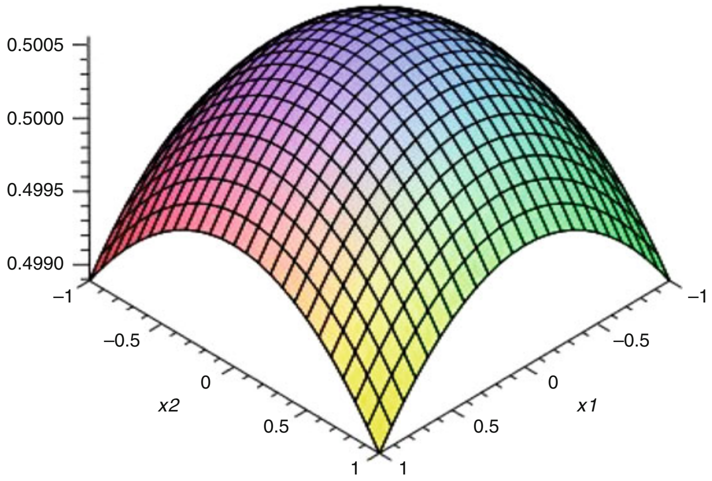

Fig. 10 Graphs of Ψ_{11} associated with the parameter $c = 0.1$.

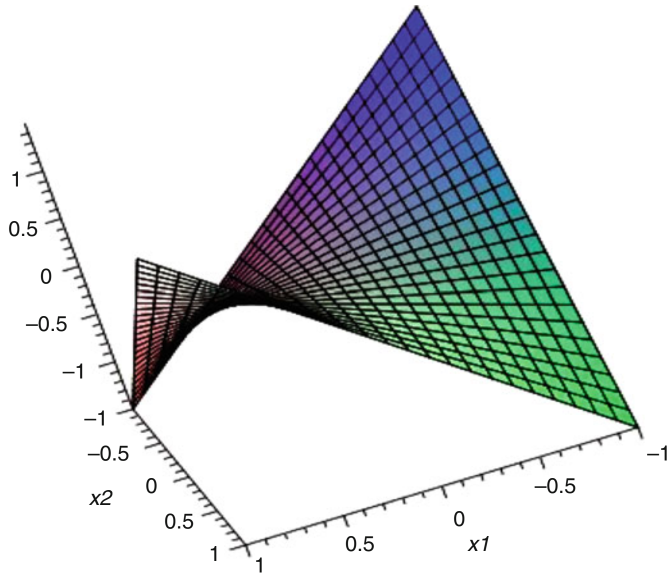

Fig. 11 Graphs of Ψ_{22} associated with the parameter $c = 0.1$.

By using the Gaussian quadrature method in one dimension and letting $N = 40$, we obtain the following numerical results (Table 9):

In what follows we present the plots of some 2-D PSWFs $\Psi_m(x, y)$, given by (16), associated with the unit disc for different values of m, and c (Figs. 16, 17, 18, 19, 20, and 21).

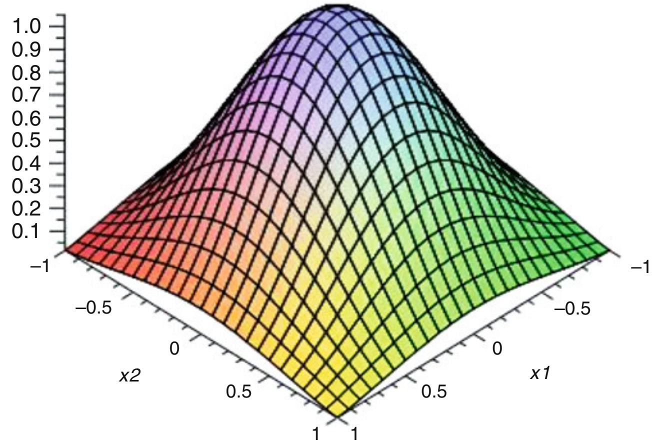

Fig. 12 Graphs of Ψ_{11} associated with the parameter $c = 4$.

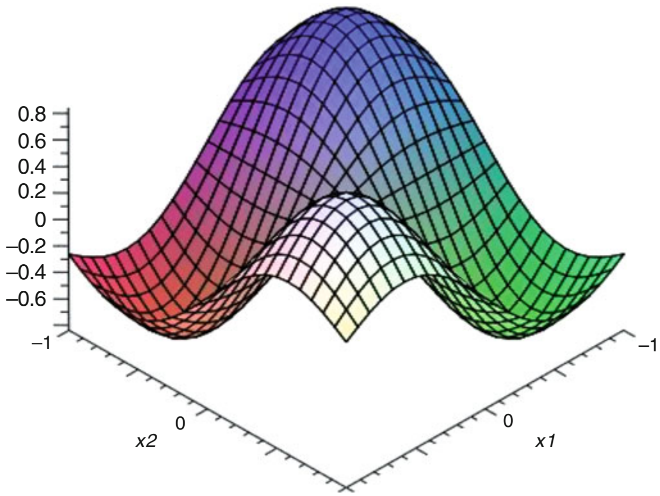

Fig. 13 Graphs of Ψ_{22} associated with the parameter $c = 4$.

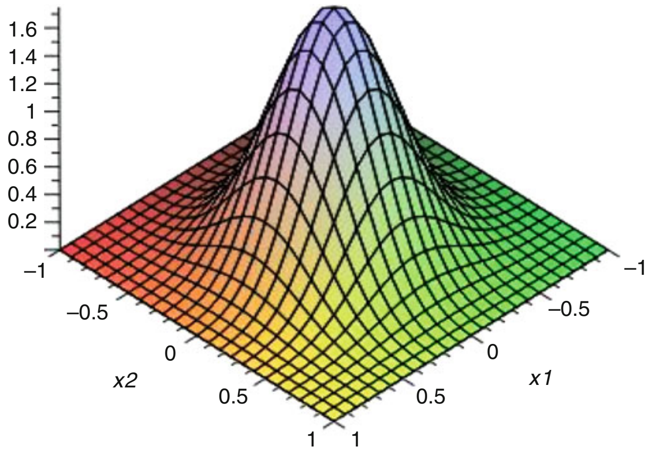

Fig. 14 Graphs of Ψ_{11} associated with the parameter $c = 10$.

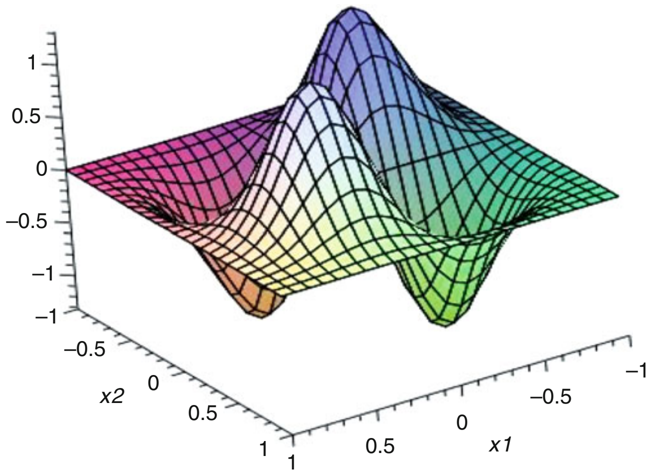

Fig. 15 Graphs of Ψ_{22} associated with the parameter $c = 10$.

Table 9 Values of $|\alpha_n(c)|$ for different values of c.

n	$\lvert\alpha_n(0,1)\rvert$	$\lvert\alpha_n(7)\rvert$	$\lvert\alpha_n(10)\rvert$
1	0.7849619636e-1	0.8962680659	0.25132741228718
2	0.5453911469e-5	0.7588496584	0.25132741228717
3	0.1363521777e-9	0.1896482892	0.25132741228037

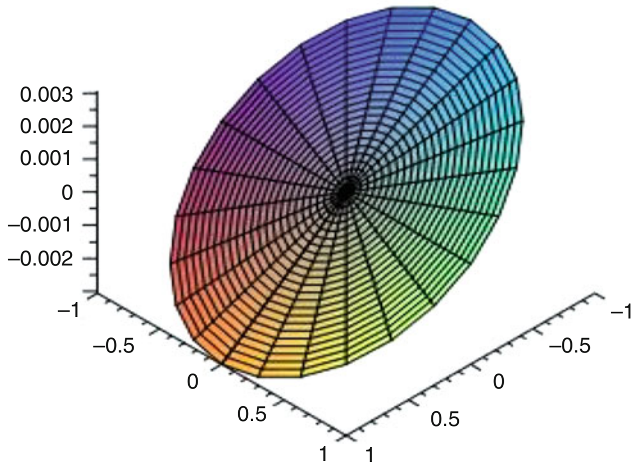

Fig. 16 Graphs of $\Psi_1(x, y)$ associated with the parameter $c = 0.1$.

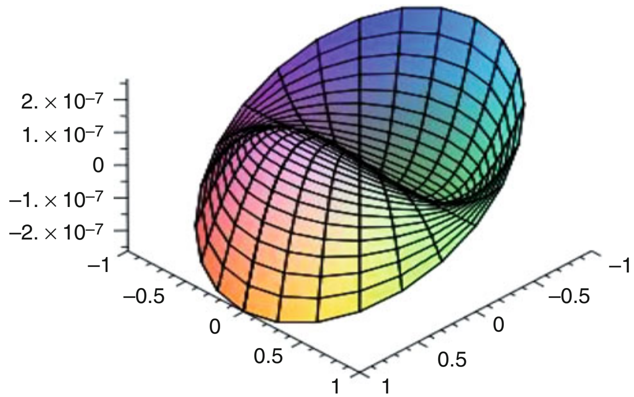

Fig. 17 Graphs of $\Psi_2(x, y)$ associated with the parameter $c = 0.1$.

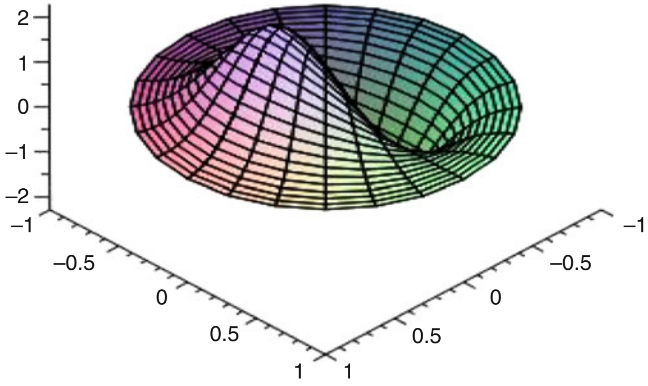

Fig. 18 Graphs of $\Psi_1(x, y)$ associated with the parameter $c = 10$.

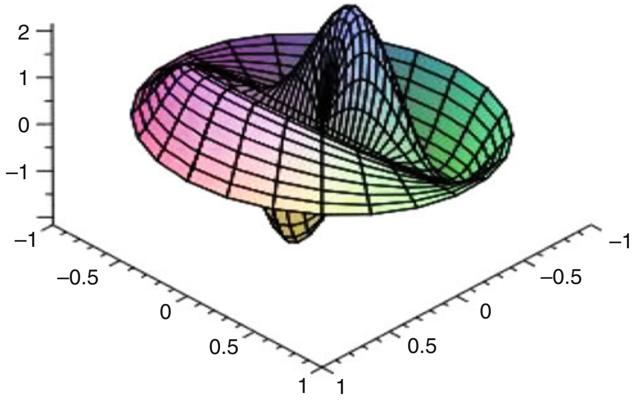

Fig. 19 Graphs of $\Psi_2(x, y)$ associated with the parameter $c = 10$.

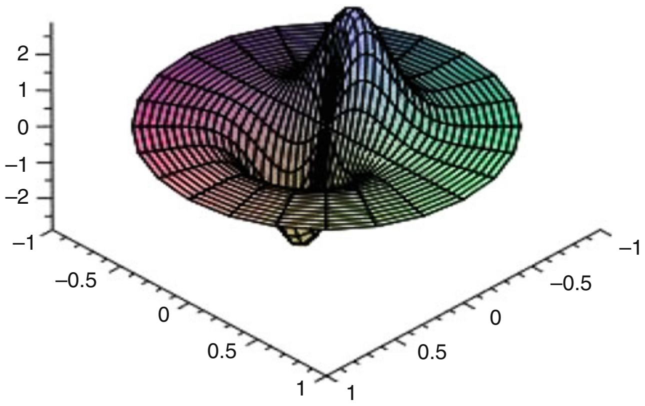

Fig. 20 Graphs of $\Psi_1(x, y)$ associated with the parameter $c = 25$.

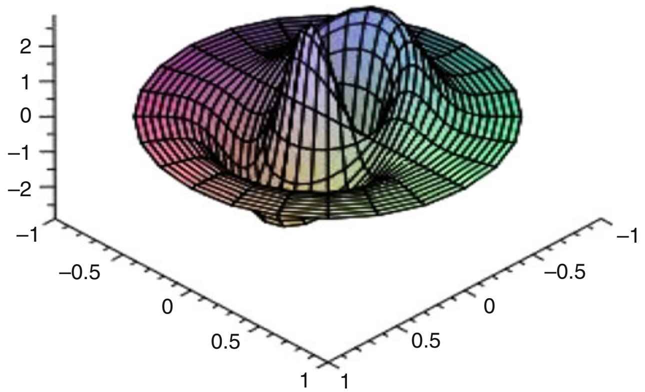

Fig. 21 Graphs of $\Psi_2(x, y)$ associated with the parameter $c = 25$.

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A Frame Reconstruction Algorithm with Applications to Magnetic Resonance Imaging

John J. Benedetto, Alfredo Nava-Tudela, Alexander M. Powell, and Yang Wang

1 Introduction

1.1 Background

We introduce a combined Fourier and finite frame technique to resolve a class of signal reconstruction problems, where efficient noise reduction and stable signal reconstruction are essential. This class includes the special case of obtaining fast spectral data acquisition in magnetic resonance imaging (MRI) [32]. Fast data acquisition is important for a variety of reasons. For example, human subject motion during the MRI process should be analyzed by methods that do not blur essential features, and speed of data acquisition lessens the effect of such motion. We shall use

J.J. Benedetto (✉)
Norbert Wiener Center, Department of Mathematics, University of Maryland, College Park, MD 20742, USA
e-mail: jjb@math.umd.edu

A. Nava-Tudela
Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA
e-mail: ant@umd.edu

A.M. Powell
Department of Mathematics, Vanderbilt University, Nashville, TN 37240, USA
e-mail: alexander.m.powell@vanderbilt.edu

Y. Wang
Department of Mathematics, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong
e-mail: yangwang@ust.hk

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the MRI case as a prototype to explain our idea. Generally, our approach includes the transition from a theoretically conclusive reconstruction method using Fourier frames to a finite frame algorithm designed for effective computation.

To begin, the use of interleaving spirals in the spectral domain, so-called k–space when dealing with MRI, is one viable setting for attaining fast MRI signal reconstruction in the spatial domain; and a major method in this regard is spiral–scan echo planar imaging (SEPI), e.g., see [23]. With this in mind, the Fourier frame component of our technique goes back to results of Beurling [13] and Henry J. Landau [42], as well as a reformulation of the Beurling–Landau theory in the late 1990s that was made with Hui-Chuan Wu, see [9–11]. This reformulation is in terms of quantitative coverings of a spectral domain by translates of the polar set of the target/subject disk space D in the spatial domain. In this context, harmonics for Fourier frames can be constructed by means of the Beurling–Landau theory on interleaving spirals in the spectral domain, allowing for the reconstruction of signals in D.

The finite frame component of our technique was developed in 2002, when the four co-authors worked together, see [12].

There has been major progress with regard to MRI and fMRI, *and* the importance of effective SEPI has *not* been diminished.

With regard to the progress, MRI and fMRI are often essentially effected in real–time [22], and technologies such as wavelet theory [33], compressed sensing [45], and non-uniform FFTs [25, 26, 30, 37] have also been used to advantage.

Although SEPI is faster than conventional rectilinear sampling, the fastest rectilinear echo planar imaging (EPI), which can be faster than SEPI, is prone to artifacts from gradient switching which is often ameliorated in SEPI. Further, SEPI is still of potential great importance with regard to spectroscopic imaging [2] and fMRI, e.g., dynamic imaging of blood flow [50].

Amidst all of this complexity, a distinct advantage of frame oriented techniques, such as ours, is the potential for effective noise reduction and stable signal reconstruction in the MRI process. With regard to frames, noise reduction, and stable reconstruction, we refer to [5, 7], and see [38, 39] for an authoritative more up-to-date review. The point is that noise reduction can be effected by modeling in which information bearing signals can be moved into a coefficient subspace relatively disjoint from coefficients representing noise in the system. This idea has a long history in the engineering community, and the theory of frames provides an excellent model to effect such a transformation. In fact, frames that are not bases allow one to construct Bessel mappings, see Section 3, that are not surjective, giving rise to the aforementioned subspaces; and the overcompleteness inherent in frames guarantees stable signal representation, e.g., see [20] and [4], Chapter 7.

1.2 Outline

Section 2 describes spiral–scan echo planar imaging (SEPI), beginning with the imaging equation for MR in which the NMR (nuclear magnetic resonance) signal $S(t)$ is obtained by integrating the solution of Bloch's differential equation.

The phenomenon of NMR was discovered independently by Felix Bloch and Edward Purcell, see [18], page 13, for historical comments (the word nuclear gives the false impression that nuclear material is used). Section 7 expands on this material by means of a sequence of images with brief explanations.

Section 3 provides the mathematical background for our theory and algorithm. This includes the theory of frames and a fundamental condition for the existence of Fourier frames due to Beurling and Landau. We also have an alternative parallel approach depending on a multidimensional version of Kadec's sufficient condition for Riesz bases in the Fourier frame case. In Section 4, we first describe our algorithm conventionally and, keeping in mind our interest in noise reduction and stable reconstruction, we then formulate it in frame theoretic terms. This allows us to prove a basic theorem on computational stability (Theorem 3) indicating the importance of designing frames that are tight or, at least, almost tight. Naturally, our algorithm, which is discrete, should also have the theoretical property that, in the limit, it will be a constructive way of genuinely approximating analogue images, whose discrete versions are computed by the algorithm. This is the content of Section 5.

Section 6 is devoted to refinements of the formulation in Section 4 in order to effect useful implementation.

Finally, after Section 7 we close with Section 8, that outlines the paradigm we have used to manufacture data in which to evaluate our algorithm when MRI generated data is not available.

2 An MRI Problem

A standard MRI equation is a consequence of Felix Bloch's equation for transverse magnetization M_{tr} in the presence of a linear magnetic field gradient [18] pages 269–270, see Section 7. In fact, an MR signal $S(t)$ is the integration of M_{tr}; and the corresponding imaging equation is

$$S(t) = S(k(t)) = S(k_x(t), k_y(t), k_z(t)) \tag{1}$$

$$= \int\int\int \rho(x, y, z) \exp[-2\pi i \langle (x, y, z),\ (k_x(t), k_y(t), k_z(t))\rangle] e^{-t/T_2} dx\, dy\, dz,$$

e.g., see [17, 18, 33], pages 269–270, [14], Subsection 16.2, page 344. $S(t)$ is also referred to as an *echo* or *FID* (free induction decay), and can be measured for the sake of imaging. Equation 1 is a natural physical Fourier transform associated with magnetization, analogous to the natural physical wavelet transform effected by the behavior of the basilar membrane within certain frequency ranges, e.g., [6].

The parameters, variable, and inputs in Equation 1 are the following:

$$k_x(t) = \gamma \int_0^t G_x(u)du \qquad (2)$$

and $G_x(u)$ is an x-directional time-varying gradient with similar definitions for the y and z variables, T_2 is the transverse relaxation time, the exponential term e^{-t/T_2} representing the T_2 decay appears as a limiting factor in echo planar imaging [1], γ is the gyromagnetic ratio, and $\rho(\mathbf{r}) = \rho(x,y,z) = \rho(\mathbf{r},T_2)$ is the spatial spin density distribution from which the spin density image is reconstructed.

Since $S(t)$ is a measurable quantity in the MR process and since precise knowledge of $\rho(x,y,z)$ is desired, it is natural to compute the inverse Fourier transform of S, properly adapted to the format in Equation 2. Because of significant issues which arise and goals which must be addressed, the inversion process has to be treated carefully. In particular, there is a significant role for the time-varying gradients. First, the gradients are inputs to the process, and must be designed theoretically in order to be realizable and goal oriented. Once the gradients have been constructed, the imaging data $S(t)$ at time t is really of the form $S(k(t))$ as seen in Equations 1 and 2; and it is usual to refer to the *spectral* domain of S as k–space. See Section 7 for more detail for this process.

Example 1. Let

$$G_x(t) = \eta \cos \xi t - \eta \xi t \sin \xi t$$

and

$$G_y(t) = \eta \sin \xi t + \eta \xi t \cos \xi t.$$

By the definition of k_x, k_y, see Equation 2, we compute $k_x(t) = \gamma \eta t \cos \xi t$ and $k_y(t) = \gamma \eta t \sin \xi t$. Combining k_x and k_y we obtain the Archimedean spiral,

$$A_c = \{(c\theta \cos 2\pi\theta, c\theta \sin 2\pi\theta) : \theta \geq 0\} \subseteq \widehat{\mathbb{R}}^2,$$

where γ, η, and $\xi > 0$ are considered as constants, $\theta = \theta(t) = (1/2\pi)\,\xi t$, and $c = 2\pi\gamma\eta/\xi$. Clearly, we have $\theta(t) \to \infty$ as $t \to \infty$.This idea for formulating time domain gradient pulse forms is due to Ljunggren [44]. They clearly generate a spiral scan in the k–domain and are not difficult to realize, see Figure 1.

Remark 1. a. The echo planar imaging (EPI) method, developed by Mansfield (1977) [18], page 306, theoretically and usually provides high speed data acquisition within the time interval of a few hundredths of seconds. The method utilizes multiple echos by fast gradient alternation. As such, realizable gradient design giving rise to large high speed gradient fields is essential. A solution to this design problem has to be coupled with controlling spatial resolution limits imposed by the T_2-decay in Equation 1, e.g., [18] pages 314–315.

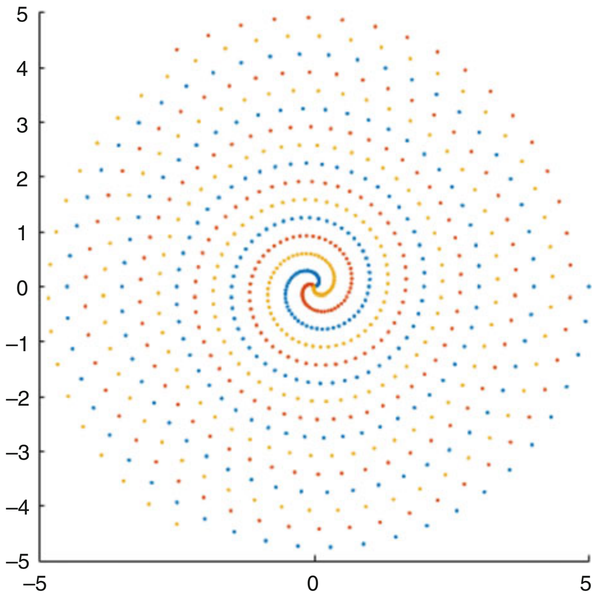

Fig. 1 Archimedean sampling example with three Archimedean spirals in the k domain.

A weakness of this technique as originally formulated is that the alternating gradient to be applied is a series of rectangular pulses which are difficult to generate for high gradient power and frequency, see [1], pages 2–3, for this and a fuller critique.

b. SEPI ensures rapid scanning for fast data acquisition. Spiral scanning also simplifies the scanning of data in radial directions once the span is completed. In this regard the inherent T_2 effect appears as an almost circular blurring which is preferable to the one-dimensional blurs observed in earlier EPI. Further, there is a reduction in transient and steady state distortion, since SEPI eliminates the discontinuities of gradient waveforms which arise in uniform rectilinear scanning that proceeds linearly around corners while transversing k-space, e.g., [1, 18].

Interleaving spiraling from rapid spiral scans proceeds from dc levels to high frequencies. That such multiple pulsing can be implemented in SEPI is due to its locally circular symmetry property in data acquisition, and the resulting interleaving spirals yield high resolution imaging when accompanied by effective k–space sampling and reconstruction methods, see [35]. In fact, interleaving spiral scans not only improve k-space sampling strategies, but they also overcome the gradient requirement and T_2-decay limitations for standard EPI.

c. EPI and SEPI are both fast in terms of image data acquisition, but the off resonance and flow properties of the two methods differ; and, in fact, total scan time spiral imaging requires lower gradient power than EPI, e.g., [43]. Further, SEPI has more significantly reduced artifact intensities than the 2–dimensional FFT since its spiral trajectories collect low spatial frequencies with every view; and it also seems to be superior vis-à-vis motion insensitivity, see [28, 29].

3 Fourier Frames and Beurling's Theorem

3.1 Frames and Beurling's Theorem

Definition 1. *a.* Let H be a separable Hilbert space. A sequence $\{x_n : n \in \mathbb{Z}^d\} \subseteq H$ is a *frame* for H if there exist constants $0 < A \leq B < \infty$ such that

$$\forall y \in H, \quad A\|y\|^2 \leq \sum |\langle y, x_n\rangle|^2 \leq B\|y\|^2.$$

The optimal constants, viz., the supremum over all such A and infimum over all such B, are called the *lower* and *upper frame bounds*, respectively. When we refer to *frame bounds* A and B, we shall mean these optimal constants.

b. A frame X for H is a *tight frame* if $A = B$. If a tight frame has the further property that $A = B = 1$, then the frame is a *Parseval frame* for H.

c. A frame X for H is *equal-norm* if each of the elements of X has the same norm. Further, a frame X for H is a *unit norm tight frame* (UNTF) if each of the elements of X has norm 1. If H is finite dimensional and X is a UNTF for H, then X is a *finite unit norm tight frame* (*FUNTF*).

d. A sequence of elements of H satisfying an upper frame bound, such as $B\,\|x\|^2$ in part *a*, is a *Bessel sequence*.

There is an extensive literature on frames, e.g., see [4, 8, 15, 19, 20, 24, 52].

Let $X = \{x_j\}$ be a frame for H. We define the following operators associated with every frame. They are crucial to frame theory. The *analysis operator* $L : H \to \ell^2(\mathbb{Z}^d)$ is defined by

$$\forall x \in H. \quad Lx = \{\langle x, x_n\rangle\}.$$

The second inequality of Definition 1, part *a*, ensures that the analysis operator L is bounded. If H_1 and H_2 are separable Hilbert spaces and if $T : H_1 \to H_2$ is a linear operator, then the *operator norm* $\|T\|_{op}$ of T is

$$\|T\|_{op} = \sup_{\|x\|_{H_1} \leq 1} \|T(x)\|_{H_2}.$$

Clearly, we have $\|L\|_{op} \leq \sqrt{B}$. The adjoint of the analysis operator is the *synthesis operator* $L^* : \ell^2(\mathbb{Z}^d) \to H$, and it is defined by

$$\forall a \in \ell^2(\mathbb{Z}^d), \quad L^* a = \sum_{n \in \mathbb{Z}^d} a_n x_n.$$

From Hilbert space theory, we know that any bounded linear operator $T : H \to H$ satisfies $\|T\|_{op} = \|T^*\|_{op}$. Therefore, the synthesis operator L^* is bounded and $\|L^*\|_{op} \leq \sqrt{B}$.

The *frame operator* is the mapping $S : H \to H$ defined as $S = L^* L$, i.e.,

$$\forall x \in H, \quad Sx = \sum_{n \in \mathbb{Z}^d} \langle x, x_n \rangle \, x_n.$$

We shall describe S more fully. First, we have that

$$\forall x \in H, \quad \langle Sx, x \rangle = \sum_{n \in \mathbb{Z}^d} |\langle x, x_n \rangle|^2 .$$

Thus, S is a positive and self-adjoint operator, and the inequalities of Definition 1, part a, can be rewritten as

$$\forall x \in H, \quad A \, \|x\|^2 \le \langle Sx, x \rangle \le B \, \|x\|^2 ,$$

or, more compactly, as

$$AI \le S \le BI.$$

It follows that S is invertible ([4, 20]), S is a multiple of the identity precisely when X is a tight frame, and

$$B^{-1} I \le S^{-1} \le A^{-1} I. \tag{3}$$

Hence, S^{-1} is a positive self-adjoint operator and has a square root $S^{-1/2}$ (Theorem 12.33 in [48]). This square root can be written as a power series in S^{-1}; consequently, it commutes with every operator that commutes with S^{-1}, and, in particular, with S. Utilizing these facts we can prove a theorem that tells us that frames share an important property with orthonormal bases, viz., there is a reconstruction formula [8], Theorem 3.2.

Theorem 1 (Frame reconstruction formula). *Let H be a separable Hilbert space, and let $X = \{x_n\}_{n \in \mathbb{Z}^d}$ be a frame for H with frame operator S. Then,*

$$\forall x \in H, \quad x = \sum_{n \in \mathbb{Z}^d} \langle x, x_n \rangle \, S^{-1} x_n = \sum_{n \in \mathbb{Z}^d} \left\langle x, S^{-1} x_n \right\rangle x_n = \sum_{n \in \mathbb{Z}^d} \left\langle x, S^{-1/2} x_n \right\rangle S^{-1/2} x_n.$$

Proof. The proof is given by the following computations. From $I = S^{-1} S$, we have

$$\forall x \in H, \quad x = S^{-1} S x = S^{-1} \sum_{n \in \mathbb{Z}^d} \langle x, x_n \rangle \, x_n = \sum_{n \in \mathbb{Z}^d} \langle x, x_n \rangle \, S^{-1} x_n;$$

from $I = S S^{-1}$, we have

$$\forall x \in H, \quad x = S S^{-1} x = \sum_{n \in \mathbb{Z}^d} \left\langle S^{-1} x, x_n \right\rangle x_n = \sum_{n \in \mathbb{Z}^d} \left\langle x, S^{-1} x_n \right\rangle x_n;$$

and from $I = S^{-1/2}SS^{-1/2}$, it follows that

$$\forall x \in H,\; x = S^{-1/2}SS^{-1/2}x = S^{-1/2}\sum_{n\in\mathbb{Z}^d}\left\langle S^{-1/2}x, x_n\right\rangle x_n = \sum_{n\in\mathbb{Z}^d}\left\langle x, S^{-1/2}x_n\right\rangle S^{-1/2}x_n.$$

□

Let $\mathbb{R}^d$ be d-dimensional Euclidean space, and let $\widehat{\mathbb{R}}^d$ denote $\mathbb{R}^d$ when it is considered as the domain of the Fourier transforms of signals defined on $\mathbb{R}^d$. The Fourier transform of $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ is formally defined as

$$\varphi(\gamma) = \widehat{f}(\gamma) = \int_{\mathbb{R}^d} f(x)\, e^{-2\pi i x\cdot\gamma}\, dx.$$

The *Paley-Wiener* space PW_D is

$$PW_D = \left\{\varphi \in L^2(\widehat{\mathbb{R}}^d) : \operatorname{supp}\varphi^\vee \subseteq D\right\},$$

where $D \subseteq \mathbb{R}^d$ is closed, $L^2(\widehat{\mathbb{R}}^d)$ is the space of finite energy signals φ on $\widehat{\mathbb{R}}^d$, i.e.,

$$\|\varphi\|_{L^2(\widehat{\mathbb{R}}^d)} = \left(\int_{\widehat{\mathbb{R}}^d} |\varphi(\gamma)|^2\, d\gamma\right)^{1/2} < \infty,$$

$\varphi^\vee$ is the inverse Fourier transform of φ defined as

$$\varphi^\vee(x) = \int_{\widehat{\mathbb{R}}^d} \varphi(\gamma) e^{2\pi i x\cdot\gamma} d\gamma,$$

and $\operatorname{supp}\varphi^\vee$ denotes the support of $\varphi^\vee$.

Notationally, let $e_\lambda(x) = e^{2\pi i x\cdot\lambda}$, where $x \in \mathbb{R}^d$ and $\lambda \in \widehat{\mathbb{R}}^d$.

Definition 2. Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be a sequence, and let $D \subseteq \mathbb{R}^d$ be a closed set having finite Lebesgue measure. It is elementary to see that $\mathcal{E}(\Lambda) = \{e_{-\lambda} : \lambda \in \Lambda\}$ is a frame for the Hilbert space $L^2(D)$ if and only if there exist $0 < A \leq B < \infty$ such that

$$\forall \varphi \in PW_D, \quad A\|\varphi\|^2_{L^2(\widehat{\mathbb{R}}^d)} \leq \sum_{\lambda\in\Lambda} |\varphi(\lambda)|^2 \leq B\|\varphi\|^2_{L^2(\widehat{\mathbb{R}}^d)}.$$

In this case, and because of this equivalence, we say that Λ is a *Fourier frame* for PW_D.

It is elementary to verify the following equivalence.

Proposition 1. *$\mathscr{E}(\Lambda) = \{e_{-\lambda} : \lambda \in \Lambda\}$ is a frame for the Hilbert space $L^2(D)$ if and only if the sequence,*

$$\{\widehat{(e_{-\lambda}\,\mathbb{1}_D)} : \lambda \in \Lambda\} \subseteq PW_D,$$

is a frame for PW_D, in which case it is also called a Fourier frame *for PW_D.*

Recall that a set Λ is *uniformly discrete* if there is $r > 0$ such that

$$\forall \lambda, \gamma \in \Lambda, \quad |\lambda - \gamma| \geq r,$$

where $|\lambda - \gamma|$ is the Euclidean distance between λ and γ.

Beurling [13] proved the following theorem for the case that D is the closed ball $B(0, R) \subseteq \mathbb{R}^d$ centered at $0 \in \mathbb{R}^d$ and with radius R.

Theorem 2. *Let $\Lambda \subseteq \widehat{\mathbb{R}}^d$ be uniformly discrete, and define*

$$\rho = \rho(\Lambda) = \sup_{\zeta \in \widehat{\mathbb{R}}^d} \operatorname{dist}(\zeta, \Lambda),$$

where $\operatorname{dist}(\zeta, \Lambda) = \inf\{|\zeta - \lambda| : \lambda \in \Lambda\}$. *If $R\rho < 1/4$, then Λ is a Fourier frame for $PW_{B(0,R)}$.*

By the definition of Fourier frame the assertion of Beurling's theorem is that every finite energy signal f defined on D has the representation,

$$f(x) = \sum_{\lambda \in \Lambda} a_\lambda(f) e^{2\pi i x \cdot \lambda}, \tag{4}$$

in L^2-norm on D, where $\sum_{\lambda \in \Lambda} |a_\lambda(f)|^2 < \infty$. Beurling [13] and [42] used the term *set of sampling* instead of *Fourier frame*. In practice, signal representations such as Equation 4 often undergo an additional quantization step to achieve analog-to-digital conversion of the signal, e.g., [47].

In theory, for the case $D = B(0, R)$, we cannot expect to construct either tight or exact Fourier frames for the spiral in Subsection 3.2, see [27].

It is possible to make a quantitative comparison between Kadec's $1/4$-theorem and Theorem 2. For now we provide the following remark which shows that the construction of Subsection 3.2 can also be achieved by use of Kadec's theorem.

Remark 2. Kadec (1964) [36] proved that *if* $\Lambda = \{\lambda_m : m \in \mathbb{Z}\} \subseteq \widehat{\mathbb{R}}$ *and* $\sup_{m\in\mathbb{Z}} |\lambda_m - \frac{m}{2R}| < 1/4$, *then* Λ *is a Riesz basis for* $PW_{[-R,R]}$. This means that $\{e^{2\pi i \lambda_m / R}\}$ is an exact frame for $PW_{[-R,R]}$, which, in turn, means it is a bounded unconditional basis for $PW_{[-R,R]}$ or, equivalently, that it is a frame which ceases to be a frame if any of its elements is removed, see, e.g., [52].

3.2 Fourier Frames on Interleaving Spirals

We can now address the problem of imaging speed in the data acquisition process of MRI in terms of the imaging equation, Equation 1, translated into a Fourier frame decomposition. In fact, $\lambda \in \Lambda \subseteq \widehat{\mathbb{R}}^2$ in Equation 4 corresponds to $(k_x(t), k_y(t), k_z(t))$ in Equation 1 in the case $k_z(t)$ is identically 0.

We use Theorem 2 to give a constructive non-uniform sampling signal reconstruction method in the setting of spirals and their interleaves. The method is much more general than the geometry of interleaving spirals.

For the case of spirals there are three cases: given an Archimedean spiral A, to show there is $R > 0$, generally small, and a Fourier frame $\Lambda \subseteq A$ for $PW_{B(0,R)}$ (the calculation for this case uses techniques from the following case); given an Archimedean spiral A and $R > 0$, to show there are finitely many interleaves of A and a Fourier frame Λ, contained in their union, for $PW_{B(0,R)}$ (Example 2); given $R > 0$, to show there is an Archimedean spiral A and a Fourier frame $\Lambda \subseteq A$ for $PW_{B(0,R)}$ (Example 3).

Example 2. a. Given *any* $R > 0$ and $c > 0$. Notationally, we write any given $\xi_0 \in \widehat{\mathbb{R}}^2$ as $\xi_0 = r_0 e^{2\pi i \theta_0} \in \mathbb{C}$, where $r_0 \geq 0$ and $\theta_0 \in [0, 1)$. We shall show how to construct a finite interleaving set $B = \cup_{k=1}^{M-1} A_k$ of spirals,

$$A_k = \left\{ c\theta e^{2\pi i(\theta - k/M)} : \theta \geq 0 \right\}, \quad k = 0, 1, \ldots, M-1,$$

and a uniformly discrete set $\Lambda_R \subseteq B$ such that Λ_R is a Fourier frame for $PW_{B(0,R)}$. Thus, all of the elements of $L^2(B(0,R))$ will have a decomposition in terms of the Fourier frame $\{e_\lambda : \lambda \in \Lambda_R\}$, see [9–11] for the original details.

b. We begin by choosing M such that $cR/M < 1/2$. Then, either $0 \leq r_0 < c\theta_0 < c$ or there is $n_0 \in \mathbb{N} \cup \{0\}$ for which

$$c(n_0 + \theta_0) \leq r_0 < c(n_0 + 1 + \theta_0).$$

In this latter case, we can find $k \in \{0, \ldots, M-1\}$ such that

$$c(n_0 + \theta_0 + \frac{k}{M}) \leq r_0 < c(n_0 + \theta_0 + \frac{k+1}{M}).$$

Thus,

$$\operatorname{dist}(\xi_0, B) \leq \frac{c}{2M}.$$

Next, we choose $\delta > 0$ such that $R\rho < 1/4$, where $\rho = (c/2M + \delta)$.

Then, for each $k = 0, 1, \ldots, M-1$, we choose a uniformly discrete set Λ_k of points along the spiral A_k having curve distance between consecutive points less than 2δ, and beginning within 2δ of the origin. The curve distance, and consequently the ordinary distance, from any point on the spiral A_k to Λ_k is less than δ. Finally, we set $\Lambda_R = \cup_{k=0}^{M-1} \Lambda_k$. Thus, by the triangle inequality, we have

$$\forall \xi \in \widehat{\mathbb{R}}^2, \quad \operatorname{dist}(\xi, \Lambda_R) \leq \operatorname{dist}(\xi, B) + \operatorname{dist}(B, \Lambda_R) \\ \leq \frac{c}{2M} + \delta = \rho,$$

where $\operatorname{dist}(B, \Lambda_R) = \inf\{|\zeta - \lambda| : \zeta \in B \text{ and } \lambda \in \Lambda_R\}$. Hence, $R\rho < 1/4$ by our choice of M and δ; and so we invoke Beurling's theorem, Theorem 2, to conclude that Λ_R is a Fourier frame for $PW_{B(0,R)}$.

Example 3. Note that since we are reconstructing signals on a space domain having area about R^2, we require essentially R interleaving spirals. On the other hand, if we are allowed to choose the spiral(s) after we are given $PW_{B(0,R)}$, then we can choose Λ_R contained in a *single* spiral A_c for $c > 0$ small enough.

Remark 3. *a.* There have been refinements and generalizations of Kadec's theorem (Remark 2), that are relevant to our approach, e.g., Sun and Zhou [51]. In fact, given $R > 0$, the Sun and Zhou result gives rise to exact frames for $L^2([-R, R]^d)$ which become frames for $L^2(B(0, R))$. For $d = 2$, the corresponding set $\Lambda \subseteq \widehat{\mathbb{R}}^2$ can be chosen on interleaves of a given spiral $A \subseteq \widehat{\mathbb{R}}^2$. This allows us to replace the application of Beurling's theorem in Examples 2 and 3 by a multidimensional Kadec theorem.

b. It can be proved that it is not possible to cover a separable lattice by finitely many interleaves of an Archimedean spiral, see [46]. In particular, sampling for the spiral MRI problem cannot be accomplished by simply overlaying spirals on top of a lattice, and then appealing to classical sampling theory on lattices. Consequently, it is a theoretical necessity that non-uniform sampling results, such as the Beurling's or Kadec's theorem, are required in the spiral case.

4 The Transition to Finite Frames

4.1 Algorithm

Let $D = [0, 1)^2$ and let $N > 1$. The space $L^2_N(D)$ of *N-digital images* is the closed subspace of $L^2(D)$ consisting of all piecewise constant functions, $f \in L^2(D)$, of the form

$$f(x, y) = f_{k,l} \quad \text{for } (x, y) \in \left[\frac{k}{N}, \frac{k+1}{N}\right) \times \left[\frac{l}{N}, \frac{l+1}{N}\right), \; 0 \leq k, l < N.$$

We use the notation, $\alpha = (\lambda, \mu) \in \widehat{\mathbb{R}}^2$ and $e(s) = e^{-2\pi is}, s \in \mathbb{R}$. For a given $f \in L^2_N(D)$, we compute

$$\widehat{f}(\alpha) = -\frac{1}{4\pi^2\lambda\mu}\sum_{k,l=0}^{N-1} f_{k,l} \cdot e\left(\frac{k\lambda + l\mu}{N}\right)\left[e\left(\frac{\lambda}{N}\right) - 1\right]\left[e\left(\frac{\mu}{N}\right) - 1\right].$$

Setting

$$G_{k,l}(\lambda, \mu) = e\left(\frac{k\lambda + l\mu}{N}\right)\left[e\left(\frac{\lambda}{N}\right) - 1\right]\left[e\left(\frac{\mu}{N}\right) - 1\right],$$

we have

$$\widehat{f}(\alpha) = \widehat{f}(\lambda, \mu) = -\frac{1}{4\pi^2\lambda\mu}\sum_{k,l=0}^{N-1} f_{k,l}G_{k,l}(\lambda, \mu). \tag{5}$$

Since there are N^2 unknowns, $f_{k,l}$, if we have N^2 or more samples of $\widehat{f}(\alpha)$, say $\{\widehat{f}(\alpha_m)\}_{m=0}^{M-1}$ with $M \geq N^2$, where $\Lambda = \{\alpha_m\}_{m=0}^{M-1}$ is properly chosen in the square $\left[-\frac{N}{2}, \frac{N}{2}\right]^2$, then we have a necessary condition for being able to reconstruct $\{f_{k,l}\}$. In fact, we suppose that the following *conditions* are satisfied.

1. $M \geq N^2$, and, in fact, we may want sufficient over-sampling so we may choose $M \gg N^2$, e.g., $M \approx 4N^2$.
2. The periodic extension $\Lambda + K\mathbb{Z}^2$ gives rise to a frame $\mathscr{E}(\Lambda + K\mathbb{Z}^2)$ for $L^2_N(D)$ with frame constants A, B, see Proposition 1. This can be proved for Λ constructed in the square $\left[-\frac{N}{2}, \frac{N}{2}\right)^2$. In the case of SEPI for MRI imagery, this is achieved by taking $\{\alpha_m\}$ on a few tightly wound spirals.

We shall show that the samples $\Lambda = \{\alpha_m\}_{m=0}^{M-1}$ allow us to reconstruct f in a stable manner. We begin by writing

$$H_{k,l}(\lambda, \mu) = -\frac{1}{4\pi^2\lambda\mu}G_{k,l}(\lambda, \mu). \tag{6}$$

Hence, by (5), we have

$$\widehat{f}(\lambda, \mu) = \sum_{k,l=0}^{N-1} f_{k,l}H_{k,l}(\lambda, \mu). \tag{7}$$

Ordering $\{(k, l) : 0 \leq k, l < N\}$ lexicographically as $\{a_n : n = 0, \ldots, N^2 - 1\}$, we obtain

$$\widehat{f}(\lambda, \mu) = \sum_{n=0}^{N^2-1} f_{a_n}H_{a_n}(\lambda, \mu). \tag{8}$$

Therefore, we can write

$$\widehat{f}(\alpha_m) = \sum_{n=0}^{N^2-1} f_{a_n} H_{a_n}(\alpha_m). \tag{9}$$

We define the vectors,

$$\mathbb{F} = \begin{pmatrix} f_{a_0} \\ \vdots \\ f_{a_{N^2-1}} \end{pmatrix} \quad \text{and} \quad \widehat{F} = \begin{pmatrix} \widehat{f}(\alpha_0) \\ \vdots \\ \widehat{f}(\alpha_{M-1}) \end{pmatrix},$$

and the matrix,

$$\mathbb{H} = \left(H_{a_n}(\alpha_m)\right)_{m,n}. \tag{10}$$

It is convenient notationally to set $H_n = H_{a_n}$. and so $\mathbb{H}$ can be written as

$$\mathbb{H} = \begin{pmatrix} H_0(\alpha_0) & \dots & H_{N^2-1}(\alpha_0) \\ H_0(\alpha_1) & \dots & H_{N^2-1}(\alpha_1) \\ \vdots & & \vdots \\ H_0(\alpha_{M-1}) & \dots & H_{N^2-1}(\alpha_{M-1}) \end{pmatrix}. \tag{11}$$

We obtain

$$\widehat{\mathbb{F}} = \mathbb{H}\mathbb{F}. \tag{12}$$

Since (12) is an over-determined system, we find the least-square approximation, yielding

$$\mathbb{F} = (\mathbb{H}^*\mathbb{H})^{-1}\mathbb{H}^*\widehat{F}, \tag{13}$$

where $\mathbb{H}^* = \overline{\mathbb{H}}^{\mathrm{T}}$ and T denotes the transpose operation, i.e., $*$ is the usual Hermitian involution for matrices. Note that $\mathbb{H}$ is an $M \times N^2$ matrix, and so $\mathbb{H}^*$ is an $N^2 \times M$ matrix and $\mathbb{H}^*\mathbb{H}$ is an $N^2 \times N^2$ matrix.

Equation 13 asserts that $\mathbb{F}$ is the Moore-Penrose pseudo-inverse of $\widehat{F}$, and a major goal is to mold this equation into a viable algorithm and computational tool with regard to noise reduction and stable reconstruction, see Section 6. It should be pointed out that Moore-Penrose has played a role in the reconstruction of MR images, going back to Van de Walle et al. (2000) and Knutsson et al. (2002). However, unprocessed application of Moore-Penrose is not feasible for typical MR image sizes, as the work of Samsonov et al. and Fessler illustrates. In fact, our frame theoretic approach is meant to provide a new technique for experts in MRI to develop.

Equation 13 can be written in frame-theoretic terminology. In fact, $\mathbb{H}$ is the analysis operator $L : l^2(\{0,\ldots N^2-1\}) \to l^2(\{0,\ldots M-1\})$, $\mathbb{H}^*$ is its adjoint synthesis operator L^*, and the frame operator $S = L^*L$ is $\mathbb{H}^*\mathbb{H}$. Defining the Gramian $R = LL^*$, we have

$$f = (S^{-1}L^*)Lf,$$

and

$$f = (L^*R^{-1})Lf,$$

where $f \in l^2(\{0,\ldots N^2-1\})$. Clearly, Equation 13 is $f = (S^{-1}L^*)Lf$ in our frame theoretic notation.

Remark 4. Define the space $F_N^2(D) \subseteq L^2(D)$ of *continuous N-digital images* as

$$F_N^2(D) = \left\{ \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f_{m,n}\Delta\left(x-\frac{m}{N}, y-\frac{n}{N}\right) : \{f_{m,n}\}_{m,n=0}^{N-1} \subseteq \mathbb{R} \right\},$$

where $\Delta(x,y) = \Delta(x)\Delta(y) = \Delta^N(x)\Delta^N(y)$, and $\Delta^N(x)$ is the triangle function supported by $[0,1/N]$ whose Fourier transform is the usual Fejér kernel. We introduce $F_N^2(D)$ in order to increase the speed of our algorithm, Equation 13. In fact, in forthcoming work we provide a Fejér kernel reconstruction algorithm with which to refine Equation 13.

4.2 Computational Stability

We must find out to what extent the reconstruction scheme of Section 4.1, in which we evaluate the coefficients $f_{k,\ell}$ in Equation 5, is stable. To this end,we would like to show that the condition number,

$$\kappa(\mathbb{H}^*\mathbb{H}) = \mathrm{cond}(\mathbb{H}^*\mathbb{H}) = \frac{|\lambda_{\max}(\mathbb{H}^*\mathbb{H})|}{|\lambda_{\min}(\mathbb{H}^*\mathbb{H})|}, \tag{14}$$

is not too large, where $\lambda_{\max}$, $\lambda_{\min}$ denote the *maximum* and *minimum* eigenvalues of $\mathbb{H}^*\mathbb{H}$. Thus, the problem is precisely that such a reconstruction scheme is not necessarily stable because the matrix $\mathbb{H}^*\mathbb{H}$ may have a large condition number. Consequently, if the sampled values $\widehat{f}(\alpha)$ are noisy, the reconstruction may not be useful. This is where the theory of frames can be applied to yield a stable reconstruction.

We can check that $\mathbb{H}^*\mathbb{H}$ is positive definite, and so the absolute values on the right side of Equation 14 can be omitted. Further, $\mathbb{H}^*\mathbb{H}$ is a normal operator (matrix).

The following theorem underlies a useful algorithmic approach, but *must* be made more precise in the sense that the *conditions* of Subsection 4.1 be made with more specificity.

Theorem 3. *Given $\mathbb{H}$ as defined in Equation 10, and assume $X = \Lambda + N\mathbb{Z}^2$ is a Fourier frame for $L^2(D)$ with frame bounds A and B. Then,*

$$\mathrm{cond}(\mathbb{H}^*\mathbb{H}) \leq \left(\frac{\pi}{2}\right)^4 \frac{B}{A}.$$

Proof. *Let $\alpha = (\lambda, \mu) \in \Lambda + N\mathbb{Z}^2$. Then, $\alpha = \alpha_m + N\gamma$ for some $0 \leq m < M$ and $\gamma \in \mathbb{Z}^2$. Let $g \in L^2([0,1]^2)$ be an N-digital image, i.e.,*

$$g(x, y) = g_{k,l} \quad \text{for } a = (x, y) \in \left[\frac{k}{N}, \frac{k+1}{N}\right) \times \left[\frac{l}{N}, \frac{l+1}{N}\right).$$

By 5, we compute

$$\begin{aligned}\widehat{g}(\alpha) &= \sum_{k,l=0}^{N-1} \int_{\frac{k}{N}}^{\frac{k+1}{N}} \int_{\frac{l}{N}}^{\frac{l+1}{N}} g_{k,l} e(a \cdot \alpha) da, \\ &= -\frac{1}{4\pi^2 \lambda\mu} \sum_{k,l=0}^{N-1} g_{k,l} G_{k,l}(\alpha),\end{aligned}$$

where $\alpha = (\lambda, \mu)$ and $e(s) = e^{-2\pi i s}$, $s \in \mathbb{R}$. Let $\alpha_m = (\lambda_m, \mu_m) \in \Lambda$. Then,

$$\widehat{g}(\alpha) = -\frac{1}{4\pi^2 \lambda\mu} \sum_{k,l=0}^{N-1} g_{k,l}\, G_{k,l}(\alpha_m), \tag{15}$$

$$= \frac{\lambda_m \mu_m}{\lambda\mu} \widehat{g}(\alpha_m).$$

Therefore, with $\gamma = (\gamma_x, \gamma_y)$, we compute

$$\begin{aligned}\sum_{\alpha \in \Lambda + N\mathbb{Z}^2} |\widehat{g}(\alpha)|^2 &= \sum_{m=0}^{M-1} \sum_{\gamma \in \mathbb{Z}^2} |\widehat{g}(\alpha_m + N\gamma)|^2 \\ &= \sum_{m=0}^{M-1} \sum_{\gamma \in \mathbb{Z}^2} \left(\frac{\lambda_m \mu_m}{(\lambda_m + N\gamma_x)(\mu_m + N\gamma_y)}\right)^2 |\widehat{g}(\alpha_m)|^2.\end{aligned}$$

It is easy to check that, since $(\lambda_m, \mu_m) = \alpha_m \in (-N/2, N/2)^2$, *we have*

$$\begin{aligned}
1 &\leq \sum_{\gamma\in\mathbb{Z}^2} \left(\frac{\lambda_m \mu_m}{(\lambda_m + N\gamma_x)(\mu_m + N\gamma_y)} \right)^2 \\
&= \sum_{\gamma_x\in\mathbb{Z}} \left(\frac{\lambda_m}{\lambda_m + N\gamma_x} \right)^2 \cdot \sum_{\gamma_y\in\mathbb{Z}} \left(\frac{\mu_m}{\mu_m + N\gamma_y} \right)^2 \\
&= \frac{1}{\operatorname{sinc}^2(\frac{\pi\lambda_m}{N})} \cdot \frac{1}{\operatorname{sinc}^2(\frac{\pi\mu_m}{N})},
\end{aligned}$$

where we use the identity,

$$\sum_{n\in\mathbb{Z}} \frac{t^2}{(t+Nn)^2} = \frac{1}{\operatorname{sinc}^2(\frac{\pi t}{N})},$$

with $\operatorname{sinc}(t) = \frac{\sin t}{t}$, *see [34], Equation (10).*

We know that

$$\frac{1}{\operatorname{sinc}^2(\frac{\pi\lambda_m}{N})} \frac{1}{\operatorname{sinc}^2(\frac{\pi\mu_m}{N})} \leq \left(\frac{\pi}{2}\right)^4.$$

Therefore, the fact that

$$\sum_{\alpha\in\Lambda+N\mathbb{Z}^2} |\widehat{g}(\alpha)|^2 = \sum_{m=0}^{M-1} \sum_{\gamma\in\mathbb{Z}^2} \left[\frac{\lambda_m\mu_m}{(\lambda_m + N\gamma_x)(\mu_m + N\gamma_y)} \right]^2 |\widehat{g}(\alpha_m)|^2$$

allows us to make the estimate,

$$\sum_{m=0}^{M-1} |\widehat{g}(\alpha_m)|^2 \leq \sum_{\alpha\in\Lambda+N\mathbb{Z}^2} |\widehat{g}(\alpha)|^2 \leq \left(\frac{\pi}{2}\right)^4 \sum_{m=0}^{M-1} |\widehat{g}(\alpha_m)|^2.$$

Hence, it follows from the inequalities,

$$A\|g\|^2_{L^2} \leq \sum_{\alpha\in\Lambda+N\mathbb{Z}^2} |\widehat{g}(\alpha)|^2 \leq B\|g\|^2_{L^2},$$

that

$$\left(\frac{2}{\pi}\right)^4 A\|g\|^2_{L^2} \leq \sum_{m=0}^{M-1} |\widehat{g}(\alpha_m)|^2 \leq B\|g\|^2_{L^2}. \tag{16}$$

Now, replacing f with g in Equation 12, we obtain

$$\widehat{\mathbb{G}} = \mathbb{H}\,\mathbb{G}.$$

Therefore,

$$\mathbb{G}^*(\mathbb{H}^*\mathbb{H})\mathbb{G} = \widehat{\mathbb{G}}^*\widehat{\mathbb{G}} = \sum_{m=0}^{M-1} |\widehat{g}(\alpha_m)|^2. \tag{17}$$

Observe that

$$\|\mathbb{G}\|^2 = \sum_{k,l=0}^{N-1} |g_{k,l}|^2 = N^2 \|g\|^2_{\mathrm{L}^2}.$$

Combining 16 and 17 leads to

$$\frac{(\frac{2}{\pi})^4 A}{N^2}\|\mathbb{G}\|^2 \le \mathbb{G}^*(\mathbb{H}^*\mathbb{H})\mathbb{G} \le \frac{B}{N^2}\|\mathbb{G}\|^2; \tag{18}$$

and so

$$\lambda_{\max}(\mathbb{H}^*\mathbb{H}) \le \frac{B}{N^2},$$

and

$$\lambda_{\min}(\mathbb{H}^*\mathbb{H}) \ge \left(\frac{2}{\pi}\right)^4 \frac{A}{N^2}.$$

Hence, we conclude that

$$\operatorname{cond}(\mathbb{H}^*\mathbb{H}) \le \left(\frac{\pi}{2}\right)^4 \frac{B}{A}.$$

□

5 Asymptotic Properties of the Algorithm

Given the samples $\{\widehat{f}(\alpha_j)\}_{j=0}^{M-1}$ and $N \in \mathbb{N}$, where $f \in L^2(D)$, $M > N^2$, and $\{\alpha_j\}_{j=0}^{M-1} \subseteq [-k/2, k/2] \times [-k/2, k/2] \subseteq \widehat{\mathbb{R}}^2$, the reconstruction $f_{\text{recon}} \in L^2_N(D)$, should serve as an approximation to f, see Equation 13. We quantify that wish in this subsection. We begin with the following, which is not difficult to verify.

Proposition 2. *Given $f \in L^2(D)$ and $N \in \mathbb{N}$. The function $g \in L^2_N(D)$, that minimizes $\|f - g\|_2$ is*

$$g(x,y) = \sum_{k,l=0}^{N-1} g_{k,l} \mathbb{1}_{\left[\frac{k}{N},\frac{k+1}{N}\right)}(x)\mathbb{1}_{\left[\frac{l}{N},\frac{l+1}{N}\right)}(y),$$

where

$$g_{k,l} = \frac{1}{\left|\left[\frac{k}{N},\frac{k+1}{N}\right)\times\left[\frac{l}{N},\frac{l+1}{N}\right)\right|}\int_{\left[\frac{k}{N},\frac{k+1}{N}\right)\times\left[\frac{l}{N},\frac{l+1}{N}\right)} f(x,y)\,dx\,dy,$$

i.e., $g_{k,l}$ is the average of f over $\left[\frac{k}{N},\frac{k+1}{N}\right)\times\left[\frac{l}{N},\frac{l+1}{N}\right)$.

From the definition of $H_{k,l}$ in Equation 6, we have

$$H_{k,l}(\lambda,\mu) = \widehat{\mathbb{1}}_{\left[\frac{k}{N},\frac{k+1}{N}\right)}(\lambda)\widehat{\mathbb{1}}_{\left[\frac{l}{N},\frac{l+1}{N}\right)}(\mu), \tag{19}$$

and, as in Subsection 4.1, recall that we order $\{(k,l) : 0 \leq k,l < N\}$ lexicographically as $\{a_n\}_{n=0}^{N^2-1}$. Also, let $D^N_{a_n}$ be the square, $\left[\frac{k}{N},\frac{k+1}{N}\right)\times\left[\frac{l}{N},\frac{l+1}{N}\right)$, where a_n is the lexicographic integer corresponding to the word (k,l). For convenience, we write $D_n = D^N_{a_n}$.

The asymptotic behavior of the algorithm is formulated in the following assertion. The mathematical calculation to verify this behavior follows the assertion, see Remark 5.

Asymptotic behavior of the algorithm. *Let $f \in L^2(D)$ and fix N. Assume $K \gg 0$ and assume $\{\alpha_j\}_{j=0}^{M-1}$ is essentially uniformly distributed [40] in the square, $\left[-\frac{K}{2},\frac{K}{2}\right]\times\left[-\frac{K}{2},\frac{K}{2}\right]$, as $M \to \infty$. Then, for $M \gg 0$, we obtain the approximation,*

$$\forall n = 1,\ldots,N^2-1, \quad f_{a_n} \approx \frac{1}{|D_n|}\int_{D_n} f(x,y)dxdy, \tag{20}$$

where $|D_n| = 1/N^2$ is the area of D_n, and where the f_{a_n} are the coefficients of f_{recon} for a given element of $L^2_N(D)$. Thus, comparing Equation 20 with Proposition 2, we see that, as $M \to \infty$, the algorithm reconstruction, f_{recon}, approaches the optimal $L^2_N(D)$ approximation of f.

Note that

$$\mathbb{H}^*\mathbb{H} = \begin{pmatrix} c_{0,0} & \cdots & c_{0,N^2-1} \\ \vdots & & \vdots \\ c_{N^2-1,0} & \cdots & c_{N^2-1,N^2-1} \end{pmatrix},$$

where $c_{k,l} = \sum_{j=0}^{M-1} H_l(\alpha_j)\overline{H_k(\alpha_j)}$. Also, we compute

$$\mathbb{H}^*\widehat{\mathbb{F}} = \begin{pmatrix} \sum_{j=0}^{M-1} \overline{H_0(\alpha_j)}\widehat{f}(\alpha_j) \\ \vdots \\ \sum_{j=0}^{M-1} \overline{H_{N^2-1}(\alpha_j)}\widehat{f}(\alpha_j) \end{pmatrix}.$$

Consequently, we have

$$\begin{aligned} f_{\text{recon}} &= (\mathbb{H}^*\mathbb{H})^{-1}\mathbb{H}^*\widehat{\mathbb{F}} \\ &= \begin{pmatrix} \frac{1}{M}\sum_{j=0}^{M-1} H_0(\alpha_j)\overline{H_0(\alpha_j)} & \dots & \frac{1}{M}\sum_{j=0}^{M-1} H_{N^2-1}(\alpha_j)\overline{H_0(\alpha_j)} \\ \vdots & & \vdots \\ \frac{1}{M}\sum_{j=0}^{M-1} H_0(\alpha_j)\overline{H_{N^2-1}(\alpha_j)} & \dots & \frac{1}{M}\sum_{j=0}^{M-1} H_{N^2-1}(\alpha_j)\overline{H_{N^2-1}(\alpha_j)} \end{pmatrix}^{-1} \\ &\qquad \times \begin{pmatrix} \frac{1}{M}\sum_{j=0}^{M-1} \overline{H_0(\alpha_j)}\widehat{f}(\alpha_j) \\ \vdots \\ \frac{1}{M}\sum_{j=0}^{M-1} \overline{H_{N^2-1}(\alpha_j)}\widehat{f}(\alpha_j) \end{pmatrix}, \end{aligned}$$

which tends to

$$\begin{pmatrix} \int_{[\frac{K}{2},\frac{K}{2}]^2} H_0(\lambda)\overline{H_0(\lambda)}d\lambda & \dots & \int_{[\frac{K}{2},\frac{K}{2}]^2} H_{N^2-1}(\lambda)\overline{H_0(\lambda)}d\lambda \\ \vdots & & \vdots \\ \int_{[\frac{K}{2},\frac{K}{2}]^2} H_0(\lambda)\overline{H_{N^2-1}(\lambda)}d\lambda & \dots & \int_{[\frac{K}{2},\frac{K}{2}]^2} H_{N^2-1}(\lambda)\overline{H_{N^2-1}(\lambda)}d\lambda \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} \int_{[\frac{K}{2},\frac{K}{2}]^2} \overline{H_0(\lambda)}\widehat{f}(\lambda)d\lambda \\ \vdots \\ \int_{[\frac{K}{2},\frac{K}{2}]^2} \overline{H_{N^2-1}(\lambda)}\widehat{f}(\lambda)d\lambda \end{pmatrix},$$

as $M \to \infty$ and for $K >> 0$. This last matrix product is *approximately*

$$\begin{pmatrix} \langle H_0, H_0\rangle & \dots & \langle H_{N^2-1}, H_0\rangle \\ \vdots & & \vdots \\ \langle H_0, H_{N^2-1}\rangle & \dots & \langle H_{N^2-1}, H_{N^2-1}\rangle \end{pmatrix}^{-1} \begin{pmatrix} \langle \widehat{f}, H_0\rangle \\ \vdots \\ \langle \widehat{f}, H_{N^2-1}\rangle \end{pmatrix}$$

$$= \begin{pmatrix} |D_0| & & & 0 \\ & |D_1| & & \\ & & \ddots & \\ 0 & & & |D_{N^2-1}| \end{pmatrix}^{-1} \begin{pmatrix} \langle f, \mathbb{1}_{D_0}\rangle \\ \langle f, \mathbb{1}_{D_1}\rangle \\ \vdots \\ \langle f, \mathbb{1}_{D_{N^2-1}}\rangle \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{|D_0|}\langle f, \mathbb{1}_{D_0}\rangle \\ \vdots \\ \frac{1}{|D_{N^2-1}|}\langle f, \mathbb{1}_{D_{N^2-1}}\rangle \end{pmatrix} = \begin{pmatrix} \frac{1}{|D_0|}\int_{D_0} f(x,y)dxdy \\ \vdots \\ \frac{1}{|D_{N^2-1}|}\int_{D_{N^2-1}} f(x,y)dxdy \end{pmatrix},$$

where we use Equation 19 and Parseval's theorem for the first equality.

Therefore, for a given $f \in L^2_N(D)$, we have

$$f_{\text{recon}} = (\mathbb{H}^*\mathbb{H})^{-1}\mathbb{H}^*\widehat{\mathbb{F}} \approx \left(\frac{1}{|D_0|}\int_{D_0} f(x,y)dxdy, \ldots, \frac{1}{|D_{N^2-1}|}\int_{D_{N^2-1}} f(x,y)dxdy\right)^T,$$

as asserted.

Remark 5. The above approximation of integrals by sums can be justified using results from the theory of uniformly distributed sequences, especially Theorem 5.5 (the Koksma-Hlawka inequality) and Theorem 6.1 and related techniques dealing with the discrepancy of sequences [40], Chapter 2. These methods are important with regard to exact frames, see [3, 49]. Further, continuity properties of matrix inversion enable the interchange of limits with matrix inverses in the calculation.

6 Computational Aspects of the Algorithm

6.1 Computational Feasibility

To solve the basic problem of Section 4, i.e., reconstructing $f \in L^2_N(D)$ through

$$\widehat{\mathbb{F}} = \mathbb{H}\mathbb{F}, \tag{21}$$

and develop the associated algorithm formula, Equation 13, as we did in Subsection 4.1, we begin by addressing the system,

$$(\mathbb{H}^*\mathbb{H})\mathbb{F} = \mathbb{H}^*\widehat{\mathbb{F}}. \tag{22}$$

The dimensions of the vectors and matrices are:

- $\mathbb{F}$ is $N^2 \times 1$
- $\mathbb{H}$ is $M \times N^2$, where $M \geq N^2$
- $A = \mathbb{H}^*\mathbb{H}$ is $N^2 \times N^2$
- $\widehat{\mathbb{F}}$ is $M \times 1$
- $b = \mathbb{H}^*\widehat{\mathbb{F}}$ is $N^2 \times 1$.

Therefore, a direct implementation requires *memory* for

$$N^4 + (M+1)\,N^2 + M \geq 2\,(N^4 + N^2)$$

scalars. With regard to *operation count*, we have the following situation. The computations to solve Equation 22, assuming that $\mathbb{H}^*$ and $\mathbb{H}^*\mathbb{H}$ are given to us, involve computing $\mathbb{H}^*\widehat{\mathbb{F}}$ and $(\mathbb{H}^*\mathbb{H})^{-1}(\mathbb{H}^*\widehat{\mathbb{F}})$. The first term requires $O(MN^2)$ operations and the second term requires $O((N^2)^3)$ operations.

6.2 Transpose Reduction

Set

$$\mathbb{H} = \begin{pmatrix} H_0(\alpha_0) & \dots & H_{N^2-1}(\alpha_0) \\ H_0(\alpha_1) & \dots & H_{N^2-1}(\alpha_1) \\ \vdots & & \vdots \\ H_0(\alpha_{M-1}) & \dots & H_{N^2-1}(\alpha_{M-1}) \end{pmatrix} = \begin{pmatrix} V_0^{\mathrm{T}} \\ V_1^{\mathrm{T}} \\ \vdots \\ V_{M-1}^{\mathrm{T}} \end{pmatrix},$$

where each V_j is $N^2 \times 1$, and $V_j = (H_0(\alpha_j), \dots, H_{N^2-1}(\alpha_j))^{\mathrm{T}}$. We compute

$$\begin{aligned}
\mathbb{H}^*\mathbb{H} &= \begin{pmatrix} \sum_{k=0}^{M-1} \overline{H_0(\alpha_k)} H_0(\alpha_k), \dots, \sum_{k=0}^{M-1} \overline{H_0(\alpha_k)} H_{N^2-1}(\alpha_k) \\ \vdots \\ \sum_{k=0}^{M-1} \overline{H_{N^2-1}(\alpha_k)} H_0(\alpha_k), \dots, \sum_{k=0}^{M-1} \overline{H_{N^2-1}(\alpha_k)} H_{N^2-1}(\alpha_k) \end{pmatrix}, \\
&= \sum_{k=0}^{M-1} \begin{pmatrix} \overline{H_0(\alpha_k)} H_0(\alpha_k), \dots, \overline{H_0(\alpha_k)} H_{N^2-1}(\alpha_k) \\ \vdots \\ \overline{H_{N^2-1}(\alpha_k)} H_0(\alpha_k), \dots, \overline{H_{N^2-1}(\alpha_k)} H_{N^2-1}(\alpha_k) \end{pmatrix}, \\
&= \sum_{k=0}^{M-1} \begin{pmatrix} \overline{H_0(\alpha_k)} \\ \overline{H_1(\alpha_k)} \\ \vdots \\ \overline{H_{N^2-1}(\alpha_k)} \end{pmatrix} (H_0(\alpha_k), H_1(\alpha_k), \dots, H_{N^2-1}(\alpha_k)), \\
&= \sum_{k=0}^{M-1} \overline{V_k} V_k^{\mathrm{T}}.
\end{aligned}$$

Also, we have

$$\mathbb{H}^*\widehat{\mathbb{F}} = \begin{pmatrix} \sum_{k=0}^{M-1} \overline{H_0(\alpha_k)} \widehat{f}_k \\ \vdots \\ \sum_{k=0}^{M-1} \overline{H_{N^2-1}(\alpha_k)} \widehat{f}_k \end{pmatrix} = \sum_{k=0}^{M-1} \widehat{f}_k \overline{V_k}.$$

Consequently, our algorithm for calculating $\mathbb{H}^*\mathbb{H}$ and $\mathbb{H}^*\widehat{f}$ requires the variables $A, V, \widehat{\mathbb{F}}$, and b. The algorithm is constructed as follows. Given $\{\alpha_0, \cdots, \alpha_{M-1}\}$ and $\widehat{\mathbb{F}} = (\widehat{f}_0, \ldots, \widehat{f}_{M-1})^{\mathrm{T}}$.

1. Let $V = [H_0(\alpha_0), \ldots, H_{N^2-1}(\alpha_0)]^{\mathrm{T}}$, where a "for loop" of length N^2 is required to compute V;
2. Define $A = \overline{V}V^{\mathrm{T}}$;
3. Define $b = \widehat{f_0}\overline{V}$;
4. For $j = 1$ to $M-1$,
   - Let $V = \left[H_0(\alpha_j), \ldots, H_{N^2-1}(\alpha_j)\right]^{\mathrm{T}}$;
   - Let $A = A + \overline{V}V^{\mathrm{T}}$;
   - $b = b + \widehat{f_j}\overline{V}$.

   end

Therefore, the algorithm requires memory for $N^2 \times N^2 + 2\,(M \times 1) + N^2 \times 1 + N^2 \times 1$ scalars. This is better than the direct implementation Equation 21 of Subsection 6.1.

The computational cost requires:

- $O(MN^2)$ calculations to compute the V vectors,
- $O(MN^4)$ calculations to compute $A = \mathbb{H}^*\mathbb{H}$, and
- $O(MN^2)$ calculations to compute $b = \mathbb{H}^*\widehat{\mathbb{F}}$.

Remark 6. The direct implementation uses more memory than the transpose reduction algorithm by a factor of roughly $(M/N^2) + 1$.

6.3 An Alternative

As before, we begin with the system,

$$\mathbb{H}^*\mathbb{H}\mathbb{F} = \mathbb{H}^*\widehat{\mathbb{F}},$$

where $\mathbb{H}^*\mathbb{H}$ is of size $N^2 \times N^2$.

A problem arises from the fact that we have to build an $N^2 \times N^2$ matrix, when in fact we only need a set of N^2 coefficients to describe the image that we want to reconstruct from the frequency information contained in $\widehat{\mathbb{F}}$.

Let us review the process:

The unit square D is divided in N^2 smaller elements, in a grid-like fashion; and, as such, we deal with the characteristic functions for each of the $\left[\frac{k}{N}, \frac{k+1}{N}\right) \times \left[\frac{l}{N}, \frac{l+1}{N}\right)$ sub-squares.

Thus, an N-digital image $f \in L^2_N(D)$ is defined as

$$\sum_{k=0,l=0}^{N-1} f_{k,l}\mathbb{1}_{\left[\frac{k}{N}, \frac{k+1}{N}\right) \times \left[\frac{l}{N}, \frac{l+1}{N}\right)}.$$

When we have $M = N^2$ values of $\widehat{f}$, we are dealing with the exact and unique solution of $\mathbb{H}^*\mathbb{H}\mathbb{F} = \mathbb{H}^*\widehat{\mathbb{F}}$. When we have more than N^2 values of $\widehat{f}$, i.e., when $M > N^2$, then we are dealing with a *minimum squares solution*.

It is natural to ask how one can formulate this situation in terms of some *energy*. Consider the function,

$$E(\mathbf{v})(\lambda, \mu) = \sum_{i=0}^{N^2-1} v_i \widehat{\mathbb{1}}_i(\lambda, \mu),$$

where $\mathbf{v} = (v_0, \ldots, v_{N^2-1})^{\mathrm{T}} \in \mathbb{R}^{N^2}$ and

$$\mathbb{1}_i = \mathbb{1}_{\left[\frac{k_i}{N}, \frac{k_i+1}{N}\right) \times \left[\frac{l_i}{N}, \frac{l_i+1}{N}\right)},$$

for $0 \leq k_i, l_i \leq N - 1$.

Also, consider the data set $\{\widehat{f}_j = \widehat{f}(\lambda_j, \mu_j) : (\lambda_j, \mu_j) \in \widehat{\mathbb{R}}^2, \ 0 \leq j \leq M - 1\}$, where $\widehat{f}$ is the Fourier transform of $f : \mathbb{R}^2 \to \mathbb{R}$.

We build the function $\mathscr{F} : \mathbb{R}^{N^2} \longrightarrow \mathbb{R}$ as follows:

$$\mathscr{F}(\mathbf{v}) = \sum_{j=0}^{M-1} \left| \sum_{i=0}^{N^2-1} v_i \widehat{\mathbb{1}}_i(\lambda_j, \mu_j) - \widehat{f}_j \right|^2 = \sum_{j=0}^{M-1} \left| E(\mathbf{v})(\lambda_j, \mu_j) - \widehat{f}_j \right|^2.$$

We want to find $\mathbf{v}_* \in \mathbb{R}^{N^2}$ such that

$$\mathscr{F}(\mathbf{v}_*) = \min_{\mathbf{v} \in \mathbb{R}^{N^2}} \mathscr{F}(\mathbf{v}).$$

We shall take the following course of action. First, the minimization approach will not be pursued because of the calculation of $\mathscr{F}(\mathbf{v})$ is generally too expensive. In fact, we shall take the *conjugate gradient approach* to solving the system,

$$\mathbb{H}^*\mathbb{H}\mathbb{F} = \mathbb{H}^*\widehat{\mathbb{F}}. \tag{23}$$

It makes sense to take this approach for the following reasons.

1. Modulo the problem of storing $\mathbb{H}$, we can solve in a finite number of steps equation 23 perfectly, if perfect arithmetic, as opposed to other iterative methods.
2. Since the storage of $\mathbb{H}$ is prohibitively expensive, we shall have to resort to computing $\mathbb{H}^*\mathbb{H}p_k$ iteratively, where p_k is from the usual conjugate gradient algorithm notation. Note that $\mathbb{H}^*\mathbb{H}$ is implicitly stored that way.
3. The storage requirements are reduced to 4 vectors, in our case, of size $N^2 \times 1$. In reality we need an extra vector that grows as $M \times 1$ to be able to compute $\mathbb{H}^*\mathbb{H}p_k$.

This method makes sense when $\mathbb{H}^*\mathbb{H}$ is positive-definite.

For perspective, the Kaczmarz algorithm is a different approach to signal reconstruction that can operate with low memory requirements by using simple row-action updates, e.g., [16]. The Kaczmarz algorithm has figured prominently in computerized tomography.

7 An MRI Primer

The ideas behind the discovery of *magnetic resonance imaging*, are due to Paul Lauterbur, see [21]. We outline and illustrate them.

A *magnetic dipole* is a spinning charged particle. A magnetic dipole has a *magnetic dipole moment* that is characterized by its *gyromagnetic ratio* γ and its *spin angular momentum* $\mathbf{S}$. We call this magnetic dipole moment μ, and $\mu = \gamma\mathbf{S}$, [31]. See Figure 2.

If we place a magnetic dipole in the presence of a static magnetic field $\mathbf{B}_0$, and its magnetic dipole moment is not aligned with the magnetic field, we observe that the magnetic dipole moment precesses about the magnetic field at a frequency ω_0 called the *Larmor frequency*. The Larmor frequency is proportional to the strength of the magnetic field. The constant of proportionality is the gyromagnetic ratio, i.e., $\omega_0 = \gamma\|\mathbf{B}_0\|_2$, [31, 41]. See Figure 3.

If a macroscopic sample of magnetic dipoles in solid, liquid, or gaseous form (for example, about 10^{23} hydrogen nuclei in water per cm^3) is placed in the presence of a static magnetic field $\mathbf{B}_0$, then the energy in this sample will be minimized when the majority of the magnetic dipole moments align with $\mathbf{B}_0$. This minimum energy state gives rise to a local *magnetization* M of the sample, and $M = \chi\mathbf{B}_0$, where χ is called the *nuclear susceptibility* of the sample, [41]. See Figure 4.

Suppose that we place a circular coil centered on a macroscopic sample of magnetic dipoles that has been magnetized by a static magnetic field $\mathbf{B}_0$, and

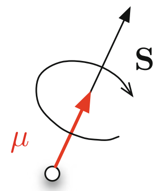

Fig. 2 A *magnetic dipole* is a spinning charged particle.

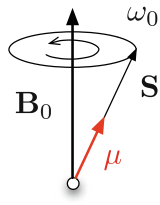

Fig. 3 Magnetic dipole precession in the presence of a magnetic field.

Fig. 4 Magnetization of a macroscopic sample of magnetic dipoles.

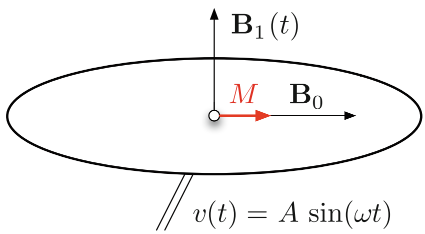

Fig. 5 Magnetization of a sample and coil experiment.

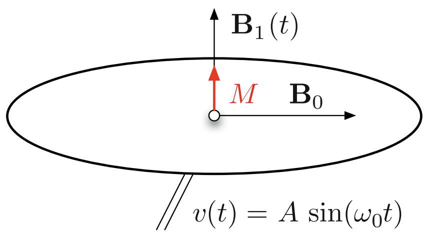

Fig. 6 Nuclear magnetic resonance (NMR) observed at the Larmor frequency.

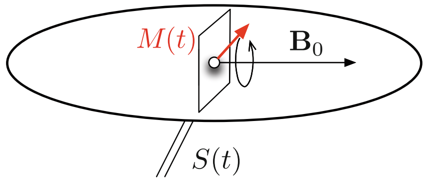

Fig. 7 Relaxation of the magnetization of a sample to its steady state.

suppose that the coil is embedded in a plane containing $\mathbf{B}_0$. See Figure 5. We then apply a time-varying sinusoidal voltage $v(t) = A\sin(\omega t)$ at the coil with amplitude A and frequency ω. We observe a time-varying magnetic field $\mathbf{B}_1(t)$ perpendicular to $\mathbf{B}_0$ that will grow and shrink, coming in and out of the plane containing the coil.

The *nuclear magnetic resonance* or NMR phenomenon can then be observed at the Larmor voltage frequency ω_0 in the following way: the magnetization in the sample is rotated and placed in the transversal plane to $\mathbf{B}_0$, [31, 41]. See Figure 6.

When the voltage pulse that generated the magnetic field $\mathbf{B}_1$ is turned off, we then observe an induced voltage $S(t)$ in the coil as the magnetization of the sample M precesses around $\mathbf{B}_0$ eventually aligning with it. This relaxation process is triggered by thermal noise in the sample, [41]. See Figure 7.

The magnetization M can be decomposed in longitudinal and transversal components, M_{lon} and M_{tr}, respectively. The longitudinal component will be parallel to $\mathbf{B}_0$ and the transversal component will be in the transversal plane perpendicular to $\mathbf{B}_0$, [41]. See Figure 8.

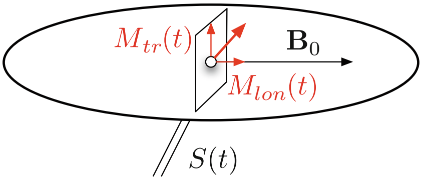

Fig. 8 Transversal and longitudinal magnetizations.

Bloch's equations predict in a variety of cases that the decay to the steady state of the magnetization will be exponential, i.e.,

$$M_{lon} - \chi \mathbf{B}_0 \propto \exp(-t/T_1), \tag{24}$$

$$\|M_{tr}\|_2 \propto \exp(-t/T_2), \tag{25}$$

where $\propto$ denotes *proportional to* and where the characteristic relaxation times T_1 and T_2 are particular to the magnetization sample, [41].

With this physical setup, we arrive at the imaging equation, Equation 1, of Section 2, viz.,

$$\begin{aligned} S(t) &= S(k(t)) = S(k_x(t), k_y(t), k_z(t)) \\ &= \int\int\int M_{tr}(x,y,z) e^{-2\pi i \langle (x,y,z),(k_x,k_y,k_z)\rangle} dx\, dy\, dz \\ &= \int\int\int \rho(x,y,z) e^{-t/T_2} e^{-2\pi i \langle (x,y,z),(k_x,k_y,k_z)\rangle} dx\, dy\, dz, \end{aligned} \tag{26}$$

where we have introduced the transversal component M_{tr}, where ρ is a magnetic dipole density function of space, where $\mathbf{r} = (x, y, z)$, and where $k_s(t)$ for $s = x, y, z$ is proportional to

$$\int_0^t \frac{\partial}{\partial s} \mathbf{B}_0(u) du, \tag{27}$$

cf. Equation 2 of Section 2. Thus, the induced signal S can be seen as the Fourier transform of the transversal magnetization M_{tr} in the k-spectral domain. Hence, to recover the transversal magnetization we take the inverse Fourier transform of the signal S. The transversal magnetization $M_{tr} = M_{tr}(\mathbf{r}, t) = \rho(\mathbf{r}) e^{-t/T_2}$ is also a function of time t, as seen in Equation (26). In practice, the values of k_r in the k-spectral domain are obtained by sampling (27) at regular time intervals. For this strategy to work, the magnetic field $\mathbf{B}_0$ must have a non-zero gradient. Hence, the design of magnetic gradients plays an important role in the sampling strategy of the k-spectral domain from which we recover an image in the spatial domain of M_{tr}, and from which we obtain an image of the density ρ, [41].

8 Synthetic Data Generation

We give the logic for the empirical evaluation of the algorithm in the case when data is not machine provided automatically, e.g., from an actual MRI.

- Given a high resolution image I (1024×1024).
- Downsample I (e.g., by taking averages) to I_N, $N \times N$, where, for example, N could be 128 or 256.
- Therefore, for comparison purposes, I_N is the optimal, *available* image at the $N \times N$ level.
- Calculate $\widehat{I} = \sum I_{a_k} H_{a_k}$, i.e., 10^6 terms for each $\alpha_m \in \widehat{\mathbb{R}}^2$.
- Choose $\widehat{I}(\alpha_m)$, $m = 0, 1, \ldots, M-1 \geq N^2 - 1$, appropriately, where the α_m are on a finite union of sufficiently tightly wound Archimedean spirals, for example, and are restricted to a $[K, K]^2$ square.
- Set $LI = \widehat{I}$, an $M \times 1$ vector.
- Implementation gives

  $$\tilde{I} = S^{-1} L^* \widehat{I},$$

  that has matrix dimension,

  $$(N^2 \times N^2)(N^2 \times M)(M \times 1) = N^2 \times 1.$$

- Quantitatively analyze the difference $I_N - \tilde{I}$, an $N \times N$ matrix.

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Frame Properties of Shifts of Prolate and Bandpass Prolate Functions

Jeffrey A. Hogan and Joseph D. Lakey

1 Introduction: Prolate Functions, Bandpass Prolates, and Frames

We study certain frames for spaces of bandlimited or bandpass-limited functions that are generated by shifts of those functions that are most concentrated in a given time interval. In the baseband case the generators correspond to the most concentrated eigenfunctions of the operator $P_\Omega Q_T$ that first truncates an $L^2(\mathbb{R})$-function to $[-T, T]$ then bandlimits to $[-\Omega/2, \Omega/2]$. These eigenfunctions are dilated *prolate spheroidal wave functions* (prolates for short) and the number of such eigenfunctions with good concentration is approximately $c = 2\Omega T$—the *time-bandwidth product*. In [7] we showed that by shifting the most concentrated prolates one can generate a frame for the Paley–Wiener space $\mathrm{PW}_\Omega = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset [-\Omega/2, \Omega/2]\}$ where $\mathscr{F}(f) = \widehat{f}$ denotes the Fourier transform $\widehat{f}(\xi) = \int f(t)\, \mathrm{e}^{-2\pi i t\xi}\, \mathrm{d}t$, $f \in L^1 \cap L^2(\mathbb{R})$. The frame property holds if there are enough prolate shifts—essentially one prolate shift per unit time per unit bandwidth is required. Denoting by φ_n the nth eigenfunction of $P_\Omega Q_T$ (ordered by decreasing eigenvalue), this means the shifts $\{\varphi_n(\cdot - \alpha k) : n = 0, \ldots, P-1,\ k \in \mathbb{Z}\}$ form a frame for PW_Ω if $P/\alpha \geq 2\Omega T$ (see Theorems 2 and 3).

Concentration of bandpass-limited signals is important in applications. Define $P_{\Omega',\Omega} = P_\Omega - P_{\Omega'}$ where $0 < \Omega' < \Omega$. We refer to eigenfunctions of $P_{\Omega',\Omega} Q_T$

J.A. Hogan (✉)
School of Mathematical and Physical Sciences, University of Newcastle, Callaghan, NSW 2308, Australia
e-mail: jeff.hogan@newcastle.edu.au

J.D. Lakey
Department of Mathematical Sciences, New Mexico State University, Las Cruces, NM 88003-8001, USA
e-mail: jlakey@nmsu.edu

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as *bandpass prolates*. Whether the density property—one bandpass prolate shift per unit bandwidth per unit time—is sufficient for shifts of the most concentrated bandpass prolates to form frames is an open question but we can verify explicit frame bounds in several redundant cases, e.g., Corollary 1, cf., [7] and [9]. In the baseband case the generators of corresponding canonical dual frames can be computed through their coefficients in the prolate shift frames.

An application of bandpass prolate frames in the study of EEG involving *phase locking*, e.g., [11], is outlined here. A second application to *cross-frequency coupling*, specifically, the question of whether certain cognitive functions exhibit coupling of power in a high frequency band with phase of a corresponding low frequency band—is also outlined very briefly.

2 Prolate Spheroidal Wave Functions and Frames

The *Bell Labs Theory* of time and bandlimiting was developed in several papers authored by combinations of Landau, Slepian, and Pollak [16, 17, 21–23] and published in the *Bell System Technical Journal* beginning in the early 1960s. We review basic concepts of time and bandlimiting here.

2.1 Basic Concepts of Time and Bandlimiting and Frames

The time-limiting operator Q_T multiplies $f \in L^2(\mathbb{R})$ by the characteristic function $\mathbb{1}_{[-T,T]}$ of the interval $[-T, T]$ (we write $Q = Q_1$). We let P_Ω denote the bandlimiting operator $P_\Omega = \mathscr{F}^{-1} Q_{\Omega/2} \mathscr{F}$. We have changed the roles of the symbols P and Q relative to some of the standard literature, e.g., [12, 29]. The composition $P_\Omega Q_T$ is compact and self-adjoint on the Hilbert space $\mathrm{PW}_\Omega = P_\Omega(L^2(\mathbb{R}))$ and with c the time-bandwidth product, $P_{c/\pi} Q$ commutes with the differential operator [23]

$$\mathscr{P}_c = \frac{\mathrm{d}}{\mathrm{d}t}(t^2 - 1)\frac{\mathrm{d}}{\mathrm{d}t} + c^2 t^2 \tag{1}$$

whose eigenfunctions φ_n ($n \geq 0$) associated with the eigenvalues χ_n with $\chi_0 < \chi_1 < \chi_2 < \cdots$ are the *prolate spheroidal wave functions* which arise in mathematical physics (e.g., [20], cf., [24]). The normalized eigenfunctions of $P_\Omega Q_T$ are unitary dilates by $1/T$ of those of $P_{T\Omega} Q$. We refer to the former as (T, Ω)-prolates, or just prolates when their dependence on the duration–bandwidth product $2T\Omega$ is clear.

The $(1, \Omega)$-prolates $\varphi_n = \varphi_n^\Omega$ form an orthonormal basis for PW_Ω with L^2-inner product as well as an orthogonal basis for $L^2[-1, 1]$ with $\int_{-1}^1 |\varphi_n|^2 = \lambda_n$ when $\|\varphi_n\|_{L^2(\mathbb{R})} = 1$. Thus, if $f \in \mathrm{PW}_\Omega$, then $f = \sum_{n=0}^\infty \langle f, \varphi_n^\Omega \rangle \varphi_n^\Omega$ and $\int_{-1}^1 |f|^2 = \sum \lambda_n |\langle f, \varphi_n^\Omega \rangle|^2$. The prolate $\varphi_n = \varphi_n^\Omega$ is real-valued, even (odd) if n is even (odd), has n zeros on $[-1,1]$, and is also an eigenfunction of $f \mapsto \sqrt{2/\Omega}\, D_{\Omega/2}\mathscr{F}^{-1}Qf$, where $(D_a f)(t) = \sqrt{a} f(at)$, with eigenvalue $\mu_n = \mathrm{i}^n\sqrt{2\lambda_n(\Omega)/\Omega}$, see, e.g., [6]. That is,

$$\widehat{\varphi}_n\Big(\frac{\Omega}{2}\xi\Big) = (-\mathrm{i})^n\sqrt{\frac{2}{\Omega\lambda_n}}\, Q\varphi_n(\xi)\,. \tag{2}$$

Landau and Widom's $2\Omega T$ theorem [18] states that $P_\Omega Q_T$ has approximately $2\Omega T$ eigenvalues close to one, with a logarithmic plunge from eigenvalues close to one to very small eigenvalues. Denote by $P(\Omega, T, \alpha)$ the number of eigenvalues of $P_\Omega Q_T$ larger than $\alpha \in (0, 1)$. Then

$$P(\Omega, T, \alpha) = 2\Omega T + \log(2\Omega T)\log\Big(\frac{1-\alpha}{\alpha}\Big) + o(\log(\Omega T))\,. \tag{3}$$

A corresponding identity applies to $P_\Sigma Q_T$ when Σ is a finite union of intervals, e.g., [6, Theorem 4.1.1] but the width of the plunge region is proportional to the number of intervals comprising Σ. When $\alpha = 1/2$ one also has $\lfloor 2\Omega T\rfloor + 1 \geq P(\Omega, T, 1/2) \geq \lfloor 2\Omega T\rfloor - 1$, see [6, 15].

For the sake of completeness we include here the definitions of frames and Riesz bases, see, e.g., [5]. Let $\mathscr{H}$ be an infinite dimensional, separable Hilbert space. A family $\{f_1, f_2, \dots\}$ is a frame for $\mathscr{H}$ if there exist constants $0 < A \leq B < \infty$ such that for any $f \in \mathscr{H}$ one has

$$A\|f\|^2_{\mathscr{H}} \leq \sum_n |\langle f, f_n\rangle|^2 \leq B\|f\|^2_{\mathscr{H}}\,.$$

A sequence $\{g_n\}$ is a Riesz basis for its closed span in $\mathscr{H}$ if there exist constants $0 < c \leq C < \infty$ such that for any sequence $\{c_n\}_{n=1}^\infty \in \ell^2(\mathbb{N})$ one has

$$c\sum_{n=1}^\infty |c_n|^2 \leq \|\sum c_n g_n\|_{\mathscr{H}} \leq C\sum_{n=1}^\infty |c_n|^2\,.$$

2.2 Basic Fourier Identities for Time and Frequency Limiting

In this section we prove a general identity for eigenfunctions of operators of the form $P_\Sigma Q_S$ where Q_S denotes multiplication by $\mathbb{1}_S$ and $P_\Sigma = \mathscr{F}^{-1}Q_\Sigma\mathscr{F}$. We assume that S and Σ are compact and that $\Sigma = -\Sigma$. The latter implies that the kernel of P_Σ is real and symmetric, and the compactness then implies that the eigenvalues $\{\lambda_n\}$ of $P_\Sigma Q_S$ satisfy $1 > \lambda_0 \geq \lambda_1 \geq \cdots > 0$.

Lemma 1. *Let* $\{\widehat{\psi}_n\}_{n=0}^{\infty}$ *be an orthonormal basis for* $L^2(\Sigma)$ *where* $\Sigma \subset \mathbb{R}$ *is compact. Then* $\{Q_I\psi_n\}_{n=0}^{\infty}$ *is complete in* $L^2(I)$ *for any bounded interval* I.

Proof. By Plancherel's theorem, if $\sum |\langle Q_I f,\, \psi_n\rangle|^2 = \sum |\langle \widehat{Q_I f},\, \widehat{\psi}_n\rangle|^2 = 0$, then by completeness of $\widehat{\psi}_n$, $\widehat{Q_I f} = 0$ a.e. on Σ. However, $(Q_I f)^\wedge$ is real-analytic so $(Q_I f)^\wedge$ must then vanish identically. Then $Q_I f = 0$ a.e. by Fourier uniqueness. ∎

Lemma 2. *Suppose that* ρ *is a real-valued, symmetric kernel* $\rho(t,s) = \rho(t-s)$ *with the reproducing property* $\int \rho(t-s)\,\rho(t-u)\,\mathrm{d}t = \int \rho(s-t)\,\rho(t-u)\,\mathrm{d}t = \rho(u-s)$ *and let* $Rf(t) = \int_S \rho(t-s) f(s)\,\mathrm{d}s$ *have a discrete spectrum. Then the eigenfunctions* ψ_n *of* R *are orthogonal over* S *as well as over* R.

Proof. Let μ_n be the nth eigenvalue of R. By orthogonality,

$$\delta_{nm} = \int_{-\infty}^{\infty} \psi_n(t)\,\psi_m(t)\,\mathrm{d}t = \frac{1}{\mu_n\,\mu_m}\int_{-\infty}^{\infty}\int_S \rho(t-s)\psi_n(s)\,\mathrm{d}s \int_S \rho(t-u)\psi_n(u)\,\mathrm{d}u\,\mathrm{d}t$$

$$= \frac{1}{\mu_n\,\mu_m}\int_S \psi_n(s)\int_S \psi_n(u)\int_{-\infty}^{\infty} \rho(t-s)\rho(t-u)\,\mathrm{d}t\,\mathrm{d}u\,\mathrm{d}s$$

$$= \frac{1}{\mu_n\,\mu_m}\int_S \psi_n(s)\int_S \psi_m(u)\,\rho(s-u)\mathrm{d}u\,\mathrm{d}s = \frac{1}{\mu_n}\int_S \psi_n(s)\,\psi_m(s)\,\mathrm{d}s\,. \qquad ∎$$

Lemma 3. *If* $\Sigma = -\Sigma$ *is compact, then the eigenfunctions* ψ_n *of* $P_\Sigma Q_S$ *with* $\|\psi_n\|_{L^2(\mathbb{R})} = 1$ *form a complete orthogonal family in* $L^2(S)$ *and* $\sum \lambda_n |\widehat{\psi}_n(\xi)|^2 = |S|$, $\xi \in \Sigma$.

Proof. Completeness follows from Lemma 1. Orthogonality follows from Lemma 2. For each $\xi \in \Sigma$, the orthonormal expansion $\mathrm{e}^{2\pi i t\xi} = \sum_n \langle \mathrm{e}^{2\pi i\cdot\xi}, \psi_n\rangle_{L^2(S)}$ $\psi_n(t)/\lambda_n$ of the exponential $\exp(2\pi i t\xi)$ converges in $L^2(S)$. This gives

$$\sum \lambda_n |\widehat{\psi}_n(\xi)|^2 = \sum_{n=0}^{\infty} \frac{1}{\lambda_n}\int_S \psi_n(t)\,\mathrm{e}^{-2\pi \mathrm{i}t\xi}\,\mathrm{d}t \int_S \overline{\psi}_n(s)\,\mathrm{e}^{2\pi \mathrm{i}s\xi}\,\mathrm{d}s$$

$$= \int_S \Big(\sum_{n=0}^{\infty}\frac{1}{\lambda_n}\Big(\int_S \mathrm{e}^{2\pi \mathrm{i}s\xi}\overline{\psi}_n(s)\,\mathrm{d}s\Big)\psi_n(t)\Big)\mathrm{e}^{-2\pi \mathrm{i}t\xi}\,\mathrm{d}t = \int_S \mathrm{e}^{2\pi \mathrm{i}t\xi}\mathrm{e}^{-2\pi \mathrm{i}t\xi}\,\mathrm{d}t = \int \mathbb{1}_S = |S|\,.$$

That is, $\sum \lambda_n |\widehat{\psi}_n(\xi)|^2 = |S|$ on Σ. ∎

By Dini's theorem, convergence is uniform in ξ. As such, given $A < |S|$ there is a P such that $A \le \sum_{n=0}^{P-1} \lambda_n |\widehat{\psi}_n(\xi)|^2 \le |S|$ for all $\xi \in \Sigma$.

3 Frames Generated by Shifting Prolates

In this section we review some results first presented in [7] and [9] regarding frames of shifts of baseband prolates and their duals in certain cases.

3.1 Normalized Prolate Shifts

We write $\phi_n(t) = \sqrt{\lambda_n}\varphi_n(t)$ to denote a $(1, 2\Omega)$-baseband prolate normalized to have $\int_{-\infty}^{\infty}\phi_n^2(t)\,\mathrm{d}t = \lambda_n$ and $\phi_{n,\alpha}(t) = \sqrt{\lambda_n}\varphi_n(t-\alpha)$.

Theorem 1 ([7, Thm. 2]). *For $\alpha \le 1$ fixed, $\mathscr{F}_\alpha = \{\phi_{n,2\alpha\ell} : \ell \in \mathbb{Z}, n \ge 0\}$ forms a frame for $\mathrm{PW}_{2\Omega}$ with lower frame bound $A \ge \lfloor 1/\alpha \rfloor$ and upper bound $B \le \lceil 1/\alpha \rceil$.*

As a special case, $\mathscr{F}_1$ forms a Parseval frame for $\mathrm{PW}_{2\Omega}$. The generator ϕ_n is not concentrated on $[-1, 1]$ when $n > 4\Omega$. However, the normalization $\|\phi_{n,\alpha}\|^2 = \lambda_n$ implies that the $\phi_{n,\alpha}$ term just adds fine detail for large n. The proof of Theorem 1 uses the completeness of the prolates in $\mathrm{PW}_{2\Omega}$ and thus casts doubt on whether a frame can be generated from shifts of the most concentrated prolates only. However, a different frame bound also applies when $1/\alpha$ is a multiple of the time-bandwidth product and it suggests that $\mathscr{F}_{P,\alpha} = \{\phi_{n,2\alpha\ell} : \ell \in \mathbb{Z}, 0 \le n < P\}$ can form a frame for appropriate P and α.

Theorem 2 ([7, Cor. 7]). *If $1/\alpha = 4m\Omega$ for some $m \in \mathbb{N}$, then $\mathscr{F}_{P,\alpha} = \{\phi_{n,2\alpha\ell} : \ell \in \mathbb{Z}, 0 \le n < P\}$ forms a frame for $\mathrm{PW}_{2\Omega}$ with frame bounds*

$$\inf \sum_{n=0}^{P-1} \frac{1}{\lambda_n}\left|\phi_n\left(\frac{\xi}{\Omega}\right)\right|^2 \le A \le B \le \sup \sum_{n=0}^{P-1} \frac{1}{\lambda_n}\left|\phi_n\left(\frac{\xi}{\Omega}\right)\right|^2 .$$

In the limit $P \to \infty$ the corresponding frame $\mathscr{F}_\alpha$ is tight with $A = B = 4m\Omega$.

The limiting case effectively states that one can replace the upper and lower frame bound estimates $\lfloor 1/\alpha \rfloor \le A \le B \le \lceil 1/\alpha \rceil$ of Theorem 1 with $A = B = 1/\alpha$ when $1/\alpha = 4\Omega\, m$. The proof of Theorem 2 combines the identity

$$\sum_{\ell}\sum_{n=0}^{P-1} |\langle f, \phi_{n,2\alpha\ell}\rangle|^2 = 2\int_{-\Omega}^{\Omega} |\widehat{f}(\xi)|^2 \sum_{n=0}^{P-1} \frac{1}{\lambda_n}\left|\phi_n\left(\frac{\xi}{\Omega}\right)\right|^2 \mathrm{d}\xi$$

with a variant of Lemma 3 that uses the identity (2). When $\mathscr{F}_{P,\alpha}$ forms a frame for $\mathrm{PW}_{2\Omega}$, the modulates $\mathrm{e}^{4\pi \mathrm{i}\Omega m t}\phi_{n,2\alpha\ell},\ \ell, m \in \mathbb{Z}, 0 \le n < P$ form a frame for $L^2(\mathbb{R})$ with the same bounds. This is because the modulates for different m are orthogonal.

When $1/\alpha = 4\Omega$ the frames $\mathscr{F}_{P,\alpha}$ in Theorem 2 have P generators per unit shift, and 4Ω shifts—the time-bandwidth product—per unit time. The lower frame bound is positive for $P = 1$ and increases with P. When $P = 1$ the family $\mathscr{F}_{P,1/(4\Omega)}$

in fact forms a Riesz basis for $\mathrm{PW}_{2\Omega}$, see [28]. However, the lower frame bound A_1 is very small so numerical reconstructions can be unstable. When $P > 1$, $\mathscr{F}_{P,1/(4\Omega)}$ is redundant with a factor P but the frame also becomes more snug, meaning that $(B-A)/(B+A)$ becomes smaller as P gets larger. It turns out that if $P/\alpha = 4\Omega$ then the family $\mathscr{F}_{P,\alpha}$ is also a Riesz basis as the following result in [7] shows. The Riesz bounds in this case are equivalent to the respective infimum and supremum of the spectrum of the positive-definite matrix valued function

$$Q = \sum_{\ell=0}^{P-1} Q^{\ell}; \quad Q^{\ell}_{n,m}(\xi) = \frac{1}{\Omega}\phi_n\Big(\frac{\xi+\ell/\alpha}{\Omega}-1\Big)\phi_m\Big(\frac{\xi+\ell/\alpha}{\Omega}-1\Big),$$
$$0 \le n,m < P, 0 \le \xi < 1/\alpha \qquad (4)$$

where $P = 2\Omega\alpha \in \mathbb{N}$. The values of $\|Q(\xi)\|$ and $\|Q^{-1}(\xi)\|$ are plotted in [7]. We consider a finite dimensional analogue in Sect. 5.

The argument depends crucially on the fact that the prolate functions $\{\varphi_n\}_{n=0}^{P-1}$ form a Chebyshev system on $[-1,1]$. Specifically, if $-1 \le t_0 < t_1 < \ldots t_{P-1} < 1$, then the matrix $\Phi_{nm} = \varphi_n(t_m)$, $0 \le n,m < P$ is nonsingular. It is natural to ask, for fixed bandwidth Ω, which P provides a Riesz basis with best properties, either from the point of view of Riesz bounds or in terms of localization of the basis elements and their duals. This discussion may be summarized as follows.

Theorem 3 ([7, Thm. 12]). *Let φ_n denote the nth prolate bandlimited to $[-\Omega, \Omega]$ and time-concentrated in $[-1,1]$. If $P = 2\Omega\alpha \in \mathbb{N}$, then the functions $\{\varphi_n(\cdot-\alpha k) : 0 \le n \le P-1,\ k \in \mathbb{Z}\}$ form a Riesz basis for $\mathrm{PW}_{2\Omega}$. That is, there exist constants $0 < A \le B < \infty$ such that for any sequence $\{c_{nk}\}_{n=0,\,k\in\mathbb{Z}}^{P-1} \in \ell^2(\mathbb{Z}^N)$ one has*

$$A\sum_{nk}|c_{nk}|^2 \le \|\sum_{nk} c_{nk}\,\varphi_n(\cdot-\alpha k)\|^2 \le B\sum_{nk}|c_{nk}|^2 .$$

Riesz bounds A, B depend on the parameters P, α. A Riesz basis is, in a sense, closer to an orthogonal basis if B/A is close to one. We illustrate in a finite dimensional analogue that Riesz bounds are closer to one when P is close to the time-bandwidth product, see Fig. 6. The prolate Riesz duals can be computed in the Fourier domain essentially by inverting the matrix Q in (4).

3.2 Duals of Redundant Prolate Shift Frames

Any dual frame of a shift-invariant frame is also shift invariant and computing a dual frame is a matter of computing its generators. We can provide a concrete method to compute generators of canonical duals of prolate-shift frames when the redundancy is expressed in the number of generators. Thus, assuming that $\{\phi_n(\cdot - \alpha k) : 0 \le n \le P-1, k \in \mathbb{Z}\}$ generates a frame for $\mathrm{PW}_{2\Omega}$—as before, $\phi_n = \sqrt{\lambda_n}\varphi_n$ where φ_n is the nth baseband prolate—we seek generators $\widetilde{\phi}_n$ of the canonical dual frame.

If $L \ge 2\Omega$ then, as a consequence of the sampling theorem, if $f, g \in \mathrm{PW}_\Omega$ then

$$\langle f, g\rangle = \frac{1}{L}\sum_{\ell\in\mathbb{Z}} f\Big(\frac{\ell}{L}\Big)\,\overline{g}\Big(\frac{\ell}{L}\Big).$$

The frame operator $Tf = \sum_{n=0}^{P-1}\sum_{k=-\infty}^{\infty}\langle f, \phi_{n,\alpha k}\rangle\phi_{n,\alpha k}$ can be written

$$\begin{aligned}
Tf(t) &= \sum_{n=0}^{P-1}\sum_{k=-\infty}^{\infty}\lambda_n\Big(\int_{-\infty}^{\infty} f(x)\varphi_n(x-\alpha k)\,\mathrm{d}x\Big)\,\varphi_n(t-\alpha k)\\
&= \sum_{n=0}^{P-1}\lambda_n\int_{-\infty}^{\infty} f(x)\Big(\sum_{k=-\infty}^{\infty}\varphi_n(x-\alpha k)\,\varphi_n^*(\alpha k - t)\Big)\,\mathrm{d}x\\
&= \sum_{n=0}^{P-1}\lambda_n\frac{1}{\alpha}\Big(\int_{-\infty}^{\infty} f(x)\int_{-\infty}^{\infty}\varphi_n(x-s)\,\varphi_n^*(s-t)\mathrm{d}s\,\mathrm{d}x\\
&= \frac{1}{\alpha}\sum_{n=0}^{P-1}\lambda_n f * \varphi_n * \varphi_n^*(t)
\end{aligned}$$

where $f * g$ denotes the convolution of f and g and $g^*(t) = \overline{g}(-t)$. Taking Fourier transforms and using (2) for bandwidth 2Ω one obtains

$$\widehat{Tf}(\xi) = \widehat{f}(\xi)\,\mu(\xi);\quad \mu(\xi) = \frac{1}{\alpha}\sum_{n=0}^{P-1}\lambda_n|\widehat{\varphi}_n(\xi)|^2 = \frac{1}{\alpha}\sum_{n=0}^{P-1}|\varphi_n\big(\xi/\Omega\big)|^2. \tag{5}$$

In other words, the frame operator is a Fourier multiplier. Since μ is nonvanishing on its support $[-\Omega, \Omega]$, one can invert μ there and thus invert the frame operator, that is,

$$(\widehat{T^{-1}f})(\xi) = \frac{\widehat{f}(\xi)}{\mu(\xi)},\quad |\xi| \le \Omega.$$

Finite dimensional analogues of μ are plotted in Fig. 1, see also Sect. 5. Recall that the canonical dual frame is obtained by applying T^{-1} to each of the frame elements. In our case, the canonical dual is also a shift frame whose generators $\widetilde{\varphi}_n$ are obtained by applying T^{-1} to the generators of the primal frame, namely,

$$\widehat{\widetilde{\varphi}}_n(\xi) = \frac{\widehat{\varphi}_n(\xi)}{\mu(\xi)},\quad |\xi| \le \Omega.$$

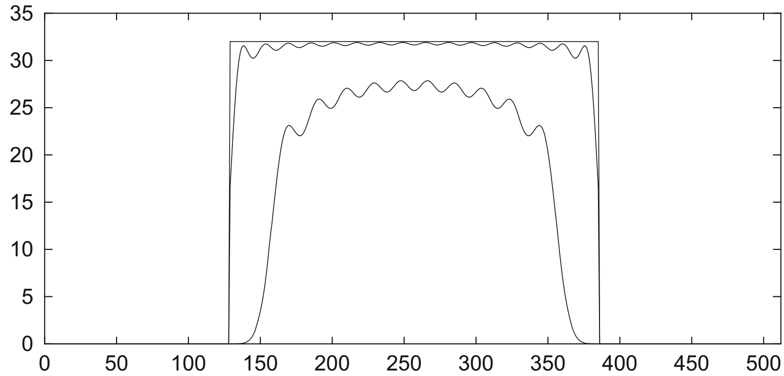

Fig. 1 512-dimensional DFT analogues of the multiplier μ in (5) using $P = 10, 16$, and 20 terms with a normalized time-bandwidth product of 16.

In particular, the nth generator of the dual frame can be expressed solely in terms of shifts of the nth generator of the primal frame, and the coefficients of these shifts are independent of the generator (though they do depend ultimately on the number of prolate generators, reflected in the definition of μ). Specifically, define

$$\frac{1}{\mu} = \sum_{\ell=-\infty}^{\infty} b_\ell \, \mathrm{e}^{2\pi \mathrm{i} \ell \xi/(2\Omega)}; \quad b_\ell = \frac{1}{2\Omega} \int_{-\Omega}^{\Omega} \frac{\mathrm{e}^{-2\pi \mathrm{i} \ell \eta/(2\Omega)}}{\mu(\eta)} \, \mathrm{d}\eta \, .$$

Then

$$\tilde{\varphi}_n(t) = \int_{-\Omega}^{\Omega} \frac{\hat{\varphi}_n(\xi)}{\mu(\xi)} \, \mathrm{e}^{2\pi \mathrm{i} t \xi} \, \mathrm{d}\xi = \sum b_\ell \int_{-\Omega}^{\Omega} \hat{\varphi}_n(\xi) \, \mathrm{e}^{2\pi \mathrm{i} \xi (t + \frac{\ell}{2\Omega})} \, \mathrm{d}\xi = \sum b_\ell \varphi_n\Big(t + \frac{\ell}{2\Omega}\Big) \, .$$

If μ_ℓ denotes the ℓth Fourier coefficient of μ, then $\{b_\ell\}$ is a convolution inverse of μ_ℓ, that is,

$$\sum_{\ell=-\infty}^{\infty} b_\ell \mu_{k-\ell} = \delta_k \, .$$

One can estimate the dual prolates numerically by estimating the coefficients b_ℓ, provided that one has a means to compute the prolates themselves. The following calculation shows that one can compute the b_ℓ from the samples of the prolate generators:

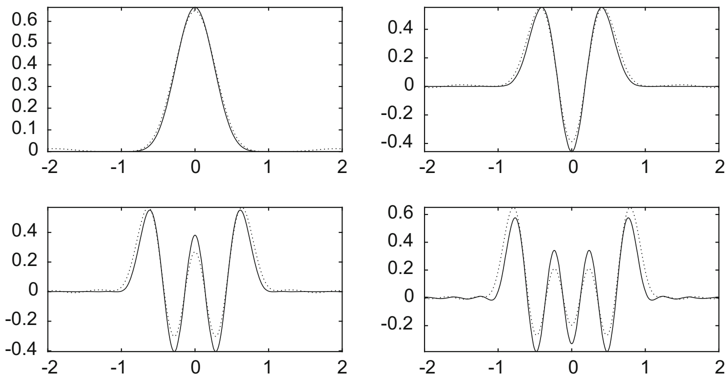

Fig. 2 The most concentrated generators φ_n (solid) and their duals (suitably normalized–dotted) are plotted for $n = 0, 2, 4, 6$ using $P = 10$ and shift parameter $\alpha = 1/4\Omega$ with a time-bandwidth product equal to 10. The generators are computed by computing 8000 Nyquist samples and sinc interpolating.

$$\mu_\ell = \frac{1}{2\Omega}\int_{-\Omega}^{\Omega} \mu(\eta)\mathrm{e}^{-2\pi\mathrm{i}\ell\eta/(2\Omega)}\,\mathrm{d}\eta = \sum_{n=0}^{P-1}\lambda_n \int_{-\Omega}^{\Omega} |\widehat{\varphi}_n(\eta)|^2 \mathrm{e}^{-2\pi\mathrm{i}\ell\eta/(2\Omega)}\,\mathrm{d}\eta$$
$$= \sum_{n=0}^{P-1}\lambda_n\varphi_n * \varphi_n^*\Big(\frac{\ell}{2\Omega}\Big) = \sum_{n=0}^{P-1}\lambda_n \int_{-\infty}^{\infty} \varphi_n\Big(\frac{\ell}{2\Omega} - x\Big)\varphi_n^*(x)\,\mathrm{d}x$$
$$= \frac{1}{2\Omega}\sum_{n=0}^{P-1}\lambda_n \sum_{k=-\infty}^{\infty} \varphi_n\Big(\frac{\ell-k}{2\Omega}\Big)\varphi_n^*\Big(\frac{k}{2\Omega}\Big).$$

To compute the coefficients μ_ℓ it suffices to have the samples $\varphi_n(k/2\Omega)$, $k \in \mathbb{Z}$ and $n = 0, \ldots, P-1$. Plots of the four most concentrated symmetric prolates and their duals for $\Omega = 2.5$ are given in Fig. 2. To estimate μ_ℓ numerically one needs to use finitely many samples. In general the pointwise decay of a $(1, 2\Omega)$-prolate φ_n is like that of a sinc function, e.g., [1] when $n \gtrsim 4\Omega$. On the other hand, the samples can be controlled by the eigenvalue for small n as the following result proved in [8] shows.

Theorem 4 ([8, Thm. 5]). *Let φ_n denote the nth prolate bandlimited to $[-\Omega, \Omega]$ and time concentrated on $[-1, 1]$. Then there are constants C_1 and C_2 such that if $M = (1 + C_1\Omega/\log(4\Omega))$ then $\sum_{|k|>M}\big|\varphi_n(k/(2\Omega))\big|^2 \le C_2(1-\lambda_n)$.*

4 Bandpass Prolate Shift Frames

In this section we discuss bandpass prolate functions and frames generated by their shifts.

4.1 Bandpass Prolate Functions

Set $P_{\Omega',\Omega} = P_\Omega - P_{\Omega'}$ where $0 < \Omega' < \Omega$. By a $(1,\Omega',\Omega)$-bandpass prolate (BPP) we mean an eigenfunction of $P_{\Omega',\Omega}Q$ for fixed Ω', Ω. Unlike baseband prolates, BPPs do not arise as eigenfunctions of a second or fourth order differential equation with polynomial coefficients [19] so computing them numerically requires a different approach. For fixed $\Delta = (\Omega - \Omega')/2$, it can be shown that as $\Omega \to \infty$, the eigenfunctions of $P_{\Omega',\Omega}Q$ become sine or cosine modulations of the baseband prolate eigenfunctions of $P_\Delta Q$. However, for a fixed finite value of the center bandpass frequency $\Omega_0 = (\Omega + \Omega')/2$, the BPPs are expressed as linear combinations of modulations

$$\psi = \sum_{n=0}^{\infty}(a_n e^{\pi i\Omega_0 t} + b_n e^{-\pi i\Omega_0 t})\,\varphi_n \tag{6}$$

of baseband prolate eigenfunctions φ_n of $P_\Delta Q$ whose coefficients a_n, b_n are components of eigenvectors $\boldsymbol{a}$, $\boldsymbol{b}$ solving the matrix eigenvalue problem

$$\lambda\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix} = \begin{pmatrix}\Lambda & \Gamma^*\\ \Gamma & \Lambda\end{pmatrix}\begin{pmatrix}\boldsymbol{a}\\ \boldsymbol{b}\end{pmatrix} \tag{7}$$

in which Λ is the diagonal matrix of eigenvalues of $P_\Delta Q$ and the entries of Γ are the integrals $\Gamma_{nm} = \int_{-1}^{1} e^{2\pi i\Omega_0 t}\,\varphi_n\,\varphi_m\,dt$ [10].

Let $P_n(t)$ denote the nth Legendre polynomial

$$P_n(t) = \frac{1}{2^n\,n!}\frac{d^n}{dx^n}[(x^2-1)^n]$$

and denote by $\bar{P}_n(t) = \sqrt{n+1/2}\,P_n(t)$ the normalization satisfying $\int_{-1}^{1}\bar{P}_n^2(t)\,dt = 1$. The baseband prolate φ_n can be expressed as

$$\varphi_n(t) = \frac{1}{\sqrt{\lambda_n}}\sum_{k=0}^{\infty} b_{kn}\bar{P}_k(t)\,. \tag{8}$$

Denote by $B = B_\Delta$ the unitary matrix with nth column $\{b_{kn}\}_k$ and by $\sqrt{\Lambda}$ the diagonal matrix with diagonal entries $\sqrt{\lambda_n}$. Finally let T denote the matrix with entries

$$T_{mn} = \int_{-1}^{1} t\,\bar{P}_n(t)\,\bar{P}_m(t)\,dt = \frac{n}{\sqrt{4n^2-1}}\,\delta_{m,n-1}, \quad (m \le n)\,. \tag{9}$$

The following was proved in [10].

Proposition 1 ([10, Prop. 3]). *The matrices Γ and T in (7) and (9) are related by*

$$\sqrt{\Lambda} B e^{2\pi i \Omega_0 T} B^* \sqrt{\Lambda} = \Gamma \, .$$

where the entries of B are defined by (8).

Let T_N denote the truncation of T to its principal $N \times N$ minor. The following estimate, also proved in [10], together with superexponential decay of the eigenvalues λ_n of $P_\Delta Q$ implies that the coefficients a_n, b_n of (6) are negligible for $N \gg 2\Delta$:

$$\|\Gamma - \sqrt{\Lambda} B \Big(\sum_{n=0}^{N} \frac{(2\pi i \Omega_0 T_{2N})^n}{n!} \Big)_N B^* \sqrt{\Lambda} \|_{\ell^2 \to \ell^2} \leq 2 \frac{(2\pi \Omega_0)^N}{N!} \, .$$

In particular, a solution of (7) can be well-approximated in ℓ^2 by a solution of the truncated problem with Λ and Γ replaced by truncations to their corresponding principal $N \times N$ minors, when $N \gg 2\Delta$.

Further details of the numerical construction of BPPs can be found in [10]. One important property of BPPs is that they are real-valued and symmetric or antisymmetric. The eigenvalues of $P_{\Omega',\Omega} Q$ can be degenerate, but only when a symmetric and antisymmetric BPP share the same eigenvalue. As such, the symmetric BPPs can be indexed by decreasing eigenvalue as can the antisymmetric BPPs.

4.2 Bandpass Prolate Shift Frames

For notational convenience we denote by ψ_n the nth eigenfunction of $P_{2\Omega',2\Omega} Q$ (i.e., the kth symmetric eigenfunction if $n = 2k$ and the k-th antisymmetric eigenfunction if $n = 2k - 1$). We continue to denote by λ_n the corresponding eigenvalue of $P_{2\Omega',2\Omega} Q$ (Fig. 3). The functions $\{\sqrt{\lambda_n}\psi_n(\cdot - \frac{\ell}{2\Omega})\}$ form a tight frame for $\mathrm{PW}_{2\Omega',2\Omega} = \{f \in L^2(\mathbb{R}) : \operatorname{supp} \widehat{f} \subset \{\Omega' \leq |\xi| \leq \Omega\}\}$, see [9]. Using Plancherel's theorem we have

$$\begin{aligned}
\sum_{\ell} \sum_{n=0}^{P-1} \Big| \Big\langle f, \sqrt{\lambda_n} \psi_n\Big(\cdot - \frac{\ell}{2\Omega}\Big) \Big\rangle \Big|^2 &= \sum_{\ell} \sum_{n=0}^{P-1} \lambda_n \Big| \Big\langle \widehat{f}, e^{\pi i \ell \cdot / \Omega} \widehat{\psi}_n(\cdot) \Big\rangle \Big|^2 \\
&= \sum_{\ell} \sum_{n=0}^{P-1} \lambda_n \Big| \int_{-\Omega}^{\Omega} \widehat{f}(\xi) \, e^{-\pi i \ell \xi / \Omega} \, \overline{\widehat{\psi}_n(\xi)} \, d\xi \Big|^2 \\
&= 2\Omega \sum_{n=0}^{P-1} \lambda_n \int_{-\Omega}^{\Omega} \big| \widehat{f}(\xi) \, \overline{\widehat{\psi}_n(\xi)} \big|^2 \, d\xi
\end{aligned}$$

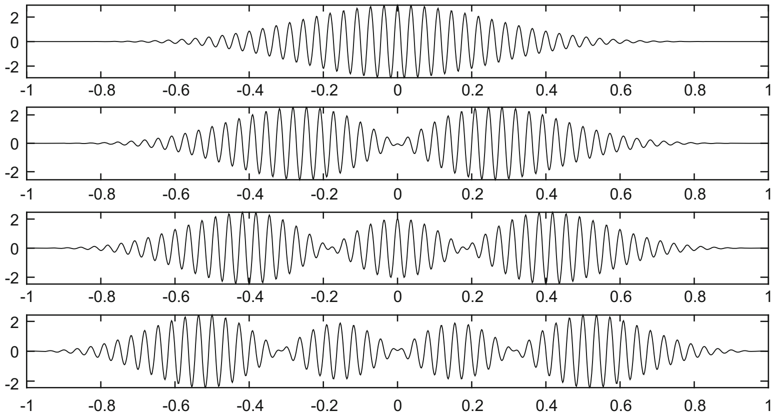

Fig. 3 The four most concentrated symmetric bandpass prolates are plotted for a time-bandwidth product of ten ($\Delta = 2.5$) and $\Omega_0 = 55$. In this case the bandpass prolates are approximately sine ($n = 1, 3$) or cosine ($n = 0, 2$) modulates of the Δ-baseband prolates φ_0, φ_1, φ_2, and φ_3. The plots are generated from samples of numerical solutions of (6) and (7).

$$= 2\Omega \int_{-\Omega}^{\Omega} |\widehat{f}(\xi)|^2 \sum_{n=0}^{P-1} \lambda_n \left|\widehat{\psi}_n(\xi)\right|^2 \mathrm{d}\xi$$

$$= 2\Omega \int_{\Omega' \le |\xi| \le \Omega} |\widehat{f}(\xi)|^2 \sum_{n=0}^{P-1} \lambda_n \left|\widehat{\psi}_n(\xi)\right|^2 \mathrm{d}\xi.$$

Lemma 3 indicates that $\sum_{n=0}^{P-1} \lambda_n \left|\widehat{\psi}_n(\xi)\right|^2$ converges (monotonically) to two on $\Omega' \le |\xi| \le \Omega$ as $P \to \infty$. By Dini's theorem the convergence is uniform. See Fig. 4 for a finite dimensional illustration. One can verify that, in fact, one already has a frame for $P = 1$.

Corollary 1. *For P sufficiently large, the shifts $\{\sqrt{\lambda_n}\psi_n(t - k/(2\Omega)) : 0 \le n \le P - 1,\ k \in \mathbb{Z}\}$ form a (non-tight) frame for* $\mathrm{PW}_{2\Omega',2\Omega}$.

These BPP shift frames are redundant for two reasons. First, the shift rate corresponds to the Nyquist rate for $\mathrm{PW}_{2\Omega}$ whereas the total width of the passband is only $2(\Omega - \Omega')$. Additionally, in principle only one generator should be needed for completeness and we are using P of them.

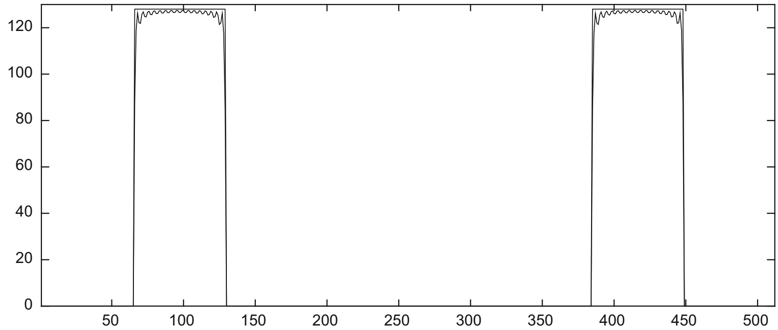

Fig. 4 Sums of square-moduli of DFTs of first P finite bandpass prolates (FBPs) for $P = 32$ (wiggly) and $P = 64$ (straight). The FBPs are discussed in Sect. 5. The terms are 512 dimensional vectors concentrated in the middle 128 dimensions and frequency limited to discrete Fourier coordinates $64 < \omega \leq 128 \bmod 512$.

4.3 Bandpass Shift Frame Duals: Some Unsettled Matters

Duals of bandpass prolate shift frames in the redundant case can be computed, in principle, along the same lines as in Sect. 3.2. That is, by inverting the Fourier series of the partial sum of squares of Fourier transforms of bandpass prolate generators on the passbands. It should be possible also to compute the corresponding multiplier from BPP samples just as in the baseband case. Doing so will require justification of all of the steps in Sect. 3.2 including Thm. 4.

As in the baseband case, shifts of the P most concentrated (symmetric and antisymmetric) bandpass prolates $\sqrt{\lambda_n}\,\psi_n(\cdot - \alpha k)$ should form a Riesz basis for $\mathrm{PW}_{\Omega',\Omega}$ when $1/\alpha = 4\Delta P$, that is, when there is one bandpass prolate shift per unit time per unit bandwidth, as follows.

Conjecture 1. The bandpass prolate functions $\{\sqrt{\lambda_n}\psi_n(t-\alpha k) : 0 \leq n \leq P-1, k \in \mathbb{Z}\}$ form a frame for $\mathrm{PW}_{\Omega',\Omega}$ if $P/\alpha \geq 2(\Omega - \Omega')$, that is, there is at least one bandpass prolate shift per unit bandwidth per unit time. If $P/\alpha = 2(\Omega - \Omega')$, then the bandpass prolate shifts form a Riesz basis.

We have verified a version of the conjecture in which the bandpass prolates are replaced by eigenfunctions of the operator $P^P_{\Omega_0,\Delta}Q$ in which $P^P_{\Omega_0,\Delta}$ denotes projection onto the span of the functions $\{\mathrm{e}^{\pm\pi\mathrm{i}\Omega_0 t}\varphi_n(t)\}$, $n = 0,\ldots,P-1$, and φ_n is the nth eigenfunction of $P_\Delta Q$. Details will be provided elsewhere.

The Chebyshev property of the baseband prolates—that $\{\varphi_n(t_k)\}_{n,k=0}^{N-1}$ is non-singular for $-1 \leq t_0 < t_1 < \cdots < t_{N-1}$—a key element of the baseband proof—is not known to hold in the bandpass case. One expects that the separate symmetric and antisymmetric BPPs form Chebyshev systems as their respective eigenvalues are separately nondegenerate. One can test the conjecture in the finite dimensional setting. Here, analogues of prolates and bandpass prolates can be formed as eigenvectors if the composition of a time-truncation matrix and DFT truncation, and shifts are replaced by cyclic translations. The corresponding Paley–Wiener space is the span of appropriate DFT vectors and one can define the redundancy simply as the number of prolate generators times the number of shifts (the ambient dimension divided by the amount of the basic shift) divided by the dimension of the DFT subspace. We will see in examples that when the redundancy is equal to one, the shifts of multiple bandpass prolates have desirable spectral properties—in particular, they form a basis for their span, which is the set of vectors whose DFTs are nonvanishing on the Fourier supports of the prolates—except in degenerate cases.

5 Finite Dimensional Analogues

The finite dimensional analogue of $\mathrm{PW}_{\Omega',\Omega}$ is the space of (K',K)-bandlimited vectors in $\mathbb{C}^N$ whose discrete Fourier transforms are supported in frequencies between K' and K (mod N). The baseband prolate theory for this case was originally worked out by Chamzas and Xu [30]; see also [6, pp. 32–33], which also discusses other contributions to the finite theory. Fix a finite dimension N and think of vectors $\boldsymbol{x} = [x(0),\ldots,x(N-1)]^T \in \mathbb{C}^N = \ell^2(\mathbb{Z}_N)$ as one period of a periodic sequence. The finite dimensional analogue of the reproducing kernel for a Paley–Wiener space is a Toeplitz matrix $A = A^K$ where K is fixed with $2K+1 \leq N$. Define $A_{k\ell} = a_{k-\ell} = \sin((2K+1)(k-\ell)\pi/N)/(N\sin((k-\ell)\pi/N), k,\ell = 0,\ldots,N-1$. Vectors in the image of A are said to be K-bandlimited, as the DFTs of the columns of A vanish at indices $m \in \mathbb{Z}_N$ such that $m > K \mod N$. Let $A_M = A_M^K$ be the principal $M\times M$ minor of A. The eigenproblem $A_M\boldsymbol{s} = \lambda\boldsymbol{s}$, $s = [s(0),\ldots,s(M-1)]^T$ is a finite dimensional analogue of the problem $QP_\Omega Q\varphi = \lambda Q\varphi$. It is proved in [30] that when $M + 2K < N$ and $M > 2K$, the ordered eigenvalues λ_n of A_M are nondegenerate: $1 > \lambda_0 > \lambda_1 > \cdots > \lambda_{2K}$. The range of A^K has dimension $2K+1$ so $\lambda_{2K+1} = \cdots = \lambda_{M-1} = 0$. When $M + 2K \geq N$, A_M will have $M + 2K + 1 - N$ unit eigenvalues (loc. cit.). One can argue as in [15] (when $M + 2K < N$) that the number of eigenvalues larger than $1/2$ cannot exceed the *normalized time-bandwidth product* $(2K+1)\times M/N$.

Just as $P_\Omega Q$ commutes with a differential operator from which its eigenfunctions can be computed, A_M commutes with a symmetric tridiagonal matrix T_M defined by

$$T_{kk} = -\cos\frac{\pi}{N}(2k+1-M)\cos\frac{\pi}{N}(2K+1),$$

$$T_{k,k+1} = \sin\frac{\pi(k+1)}{N}\,\sin\frac{\pi}{N}(M-1-k), \quad k = 0,\ldots,M-1$$

and $T_{k-1,k} = T_{k,k+1}$ and $T_{k\ell} = 0$ if $|k-\ell| > 1$. The *finite prolate* (FPS) *sequences* $\boldsymbol{s}_n = [s_n(0),\ldots,s_n(M-1)]^T$, $n = 0,\ldots,2K$ can be computed as eigenvectors of T_M and extended to K-bandlimited vectors in $\ell^2(\mathbb{Z}_N)$ by applying the $N\times N$ Toeplitz matrix A to the vectors $[s_n(0), \ldots, s_n(M-1), 0, \ldots, 0] \in \mathbb{C}^N$. These extensions possess the same double orthogonality properties of the baseband prolates, *viz.* Lemma 2 and form a complete family for the space of K-bandlimited vectors.

5.1 Finite Bandpass Prolates

Here we develop a method to compute *finite bandpass prolates* (FBPs). For frames generated by certain shifted FBPs, one can also compute canonical dual frames using the *frame conjugate gradient method*, e.g., [4].

By analogy with the continuous case in which the (Ω', Ω) bandpass prolates are defined as eigenfunctions of $(P_\Omega - P_{\Omega'})Q$, we then *define* the M-concentrated and (K', K)-bandlimited FBPs as eigenvectors of $A_M^K - A_M^{K'}$. In turn, these eigenvectors can be expressed in terms of their coefficients when expanded in eigenvectors of A_M^K. These coefficients solve an eigenvalue problem analogous to (7) as the following result, whose proof can be found in [9], shows.

Lemma 4 ([9, Lem. 5]). *Let* $\boldsymbol{s}_n(M, K)$ *denote the nth eigenvector of* A_M^K, $0 \le n \le 2K$, *and let* $\boldsymbol{\sigma}$ *be an eigenvector of* $A_M^K - A_M^{K'}$ *expressed as* $\boldsymbol{\sigma} = \sum_n \alpha_n \boldsymbol{s}_n(M, K)$. *Then the coefficients* $\boldsymbol{\alpha} = [\alpha_0, \ldots, \alpha_M]^T$ *of* $\boldsymbol{\sigma}$ *form an eigenvector of the matrix* $(I_{2K+1} - T^{K,K'})\Lambda^K$ *where* Λ^K *is the diagonal matrix whose entries are the nonzero eigenvalues of* A_M^K *and* $T_{k,\ell}^{K,K'} = \boldsymbol{s}_k(M, K)^T \boldsymbol{s}_\ell(M, K')$, $0 \le k, \ell \le 2K$.

Figure 5 illustrates the first four symmetric FBPs when $N = 1024$, $M = 512$, $K = 120$ and $K' = 100$; cf., the continuous BPPs in Fig. 3.

5.2 Numerical FBPs and Shift Frames

Conjecture 1 is that a family of shifts of the first P $(1, \Omega', \Omega)$-BPPs forms a Riesz basis for the space $\mathrm{PW}_{\Omega',\Omega}$ if $1/\alpha = 2\Delta P$, that is, there is one BPP shift per unit time per unit bandwidth. Shen and Walter [28] actually proved that the shifts of a modulated prolate can form a Riesz basis for $\mathrm{PW}_{\Omega',\Omega}$, using a special characterization of Riesz bases generated by shifts of a single function. As indicated in Fig. 1 the Riesz bounds will not be snug in this case. Practically speaking, the Riesz bounds should be snug in order that the frame operator is well conditioned.

Since the space of (K', K)-bandlimited vectors has dimension $2(K - K')$, and since there are S (circular) shifts of size N/S, the finite dimensional (FBP) version of the density conjecture for BPP shifts is that taking $S = 2(K - K')/P$ shifts by multiples of $N \times P/(2(K - K'))$ of each FBP provides a basis for the (K', K)-bandlimited vectors. Observe that, unlike the continuous case, this calculation does not take into account the duration of the FBPs, as there is not a natural rescaling of duration versus bandwidth in the finite dimensional case as there is in the

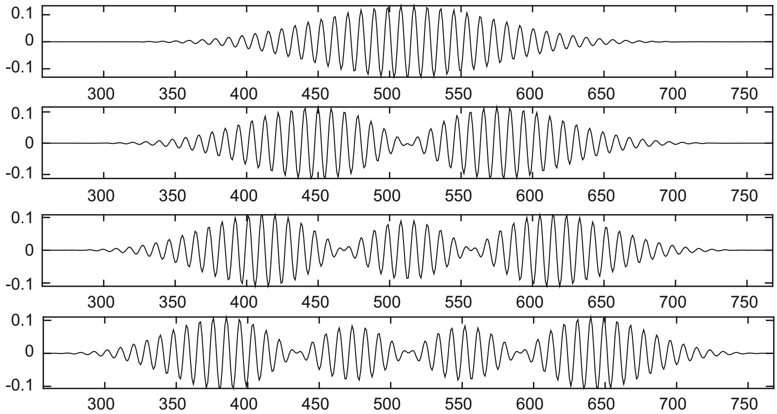

Fig. 5 Middle sections of the four most concentrated symmetric finite bandpass prolates are plotted for $N = 1024$, $M = 512$, and $(K', K) = (100, 120)$. The normalized time-bandwidth product $2(K - K')M/N$ is equal to the TBP in Fig. 3 and other parameters reflect those in the corresponding continuous case. The plots are generated from numerical solutions outlined in Lemma 4. This illustrates that the continuous case can be viewed as a limit of finite dimensional cases with corresponding parameter values.

continuous case. Nevertheless, the number of prolate generators P is a parameter in the finite dimensional case and it is natural to ask, for a fixed duration M, how the Riesz bounds depend on P. Figure 6 plots the singular values of the matrix whose columns consist of shifted FBP Riesz bases for the 32-dimensional space of $(16, 32)$-bandlimited vectors inside $\mathbb{C}^{512}$. The squares of the largest and smallest singular values are the upper and lower Riesz bounds. The duration $M = 128$ is fixed so the normalized time-bandwidth product is $32 \times 128/512 = 8$. The figure shows that the ratio of largest to smallest singular values is smallest in the case $P = 8$ and $S = 4$. Figure 7 plots the most concentrated FBP and its corresponding dual in the particular case $N = 512, M = 128, (K', K) = (16, 32)$ using $P = 2, 4, 8,$ and 16. When $P = 8$ the basis is nearly tight. In other cases the duals are not as well concentrated, indicating that the basis may be less effective in temporally localizing signal features.

6 A Brief Outline of Certain Applications in EEG

The prolate frames outlined here could be of use in any application that requires a quantification of concentration in space or time of a bandlimited or bandpass limited signal, as is illustrated in Fig. 8. We briefly outline one application in the study of electroencephalography and mention one other. Figure 9 shows two one second records of two different channels of EEG data recorded simultaneously from the same subject. The first application was discussed in more detail in [11] and

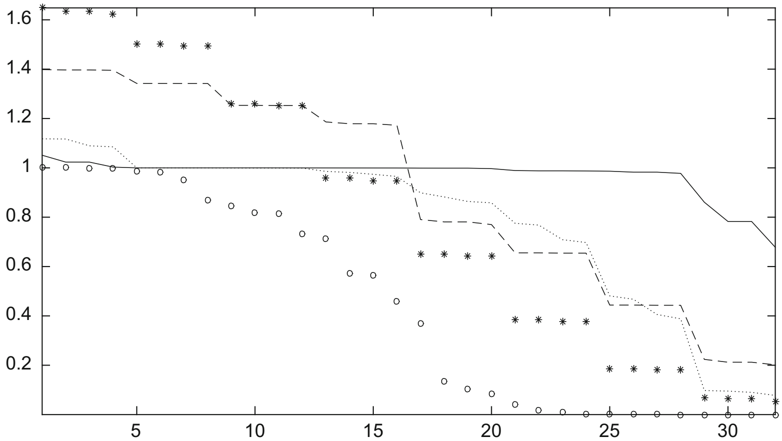

Fig. 6 Singular values of the 512 × 32 matrix whose columns are the FBPs corresponding to $N = 512$, $M = 128$, $K = 32$, and $K' = 16$ for number of prolate generators NP = 2 (∗), NP = 4 (dashed), NP = 8 (solid), NP = 16 (dots), and NP = 32 (∘).

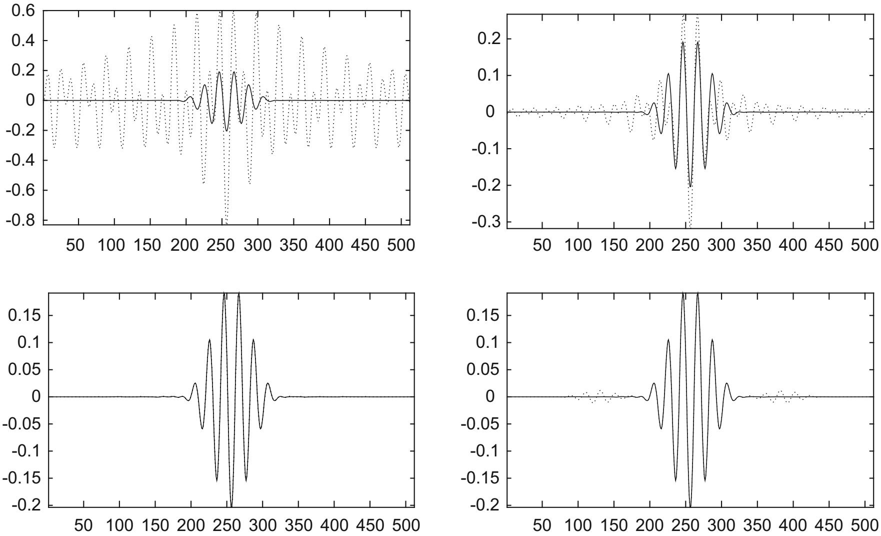

Fig. 7 Plots of most concentrated FBP (solid) and corresponding canonical shift-dual prolate (dots) in the Riesz basis cases, $N = 512$, $M = 128$, $K = 32$, and $K' = 16$ for number of prolate generators and shift size (NP, $S = 512 \times \text{NP}/32$). Top left: (NP, S) = (2, 32); top right: (NP, S) = (4, 64); bottom left: (NP, S) = (8, 128); bottom right: (NP, S) = (16, 256).

it involves quantifying locking of phases of the parts of different signals—in this case measured scalp voltage channels in EEG—in the same frequency range. Such behavior was first measured in clean electrocorticogram (ECoG) signals measured using embedded electrodes.

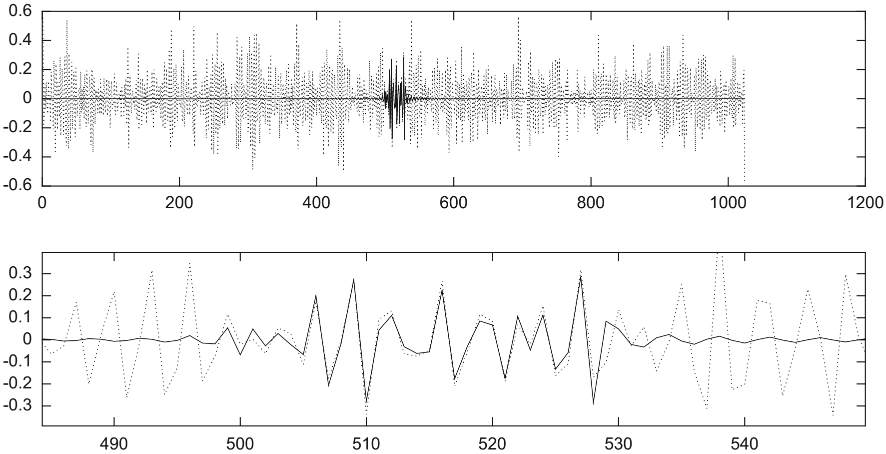

Fig. 8 Projection of a 1024 Hz sampled signal onto the highpass component $256 \le |\omega| < 512$ Hz using $N = 1024$ (dashed), and its local approximation by the terms of an FBP shift frame with eight FBP generators and redundancy four (solid). The second figure zooms in around the time center. The local approximation has the form $\sum_{n=0}^{7} \langle \boldsymbol{x}, \tilde{\boldsymbol{v}}_{n,s} \rangle \boldsymbol{v}_{n,s}$ where $\boldsymbol{v}_{n,s}$ is the nth most concentrated FBP centered at $s = 512$ and $\tilde{\boldsymbol{v}}_{n,s}$ is the corresponding dual frame generator.

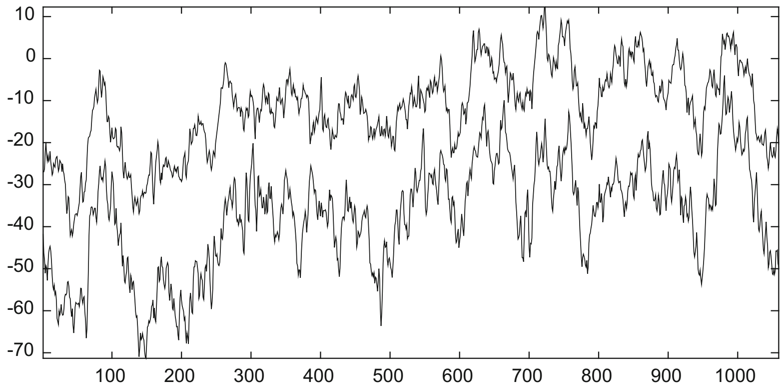

Fig. 9 One second records of EEG data. Lower curve channel is from prefrontal cortex. Upper curve is from left dorsolateral prefrontal cortex. Channels were recorded in the middle of an integration task visualizing a fixed shape and color.

Based on ECoG data, it was observed [13, 14, 25, 26] that phase locking of gamma-band (30–100 Hz) activity in different cortical regions is associated with certain cognitive tasks that involve integrating different features of recalled information such as shape and color. Specifically, phase locking is believed to reflect switching of communication between different cortical regions on and off, and this switching occurs over a few oscillations in the given frequency band.

Given any pair of complex signals $z_1(t)$, $z_2(t)$ that possess defined amplitude–phase decompositions $z_i(t) = A_i(t)\, e^{i\theta_i(t)}$, one can measure phase locking near time t_0 with respect to a weight p, considered as a nonnegative function with integral one, by setting

$$\mathrm{PL}(z_1, z_2, t_0) = \left| \int p(t_0 - t)\, e^{i(\theta_1(t) - \theta_2(t))}\, dt \right|. \tag{10}$$

A typical choice of p is $p(t) = \mathbb{1}_{[-\delta,\delta]}(t)/(2\delta)$. Then $\mathrm{PL}(z_1, z_2, t_0)$ is the modulus of the average of the unit vectors $e^{i(\theta_1(t)-\theta_2(t)}$ on $[t_0 - \delta, t_0 + \delta]$. If the differences $\theta_1(t) - \theta_2(t)$ wander over this interval, then the modulus of the average will be small, whereas if $\theta_1(t) - \theta_2(t)$ varies little over this interval, then the modulus of the average will be close to one.

Because EEG signals are noisy and less well associated with cortical activity compared to ECoG, frequency specific phase relationships are harder to measure in EEG. If phase in the gamma band (30–100 Hz) is to be associated with switching activity over 3–5 cycles, then the duration should be approximately 50–100 ms. An alternative to defining gamma phase by narrowband filtering and Hilbert transforming is to project onto the span of the bandpass prolates most concentrated in the desired time–frequency region. In our setting, assuming a sampling rate of 1024 Hz, it is reasonable to define finite bandpass prolates in the passband to be 64 ± 32 Hz and with a concentration length of 64 samples, say. In this case the normalized time-bandwidth product is eight. A phase of the signal near a given time center can be defined by taking the Hilbert transform of the projection of the signal shifted to that center onto the span of the first eight prolates (using dual generators computed at the highest shift rate to compute the expansion coefficients). A phase locking value for two such signals can then be defined by computing (10) using $p(t) = \mathbb{1}_{[-\delta,\delta]}/(2\delta)$ with $\delta = 1/32$s. The PLVs themselves tend to vary sharply and a graphical representation is usually produced by averaging PLVs over a duration δ (Fig. 10). Strong peaks in the PLV curve are believed to indicate switching on or off of communication between regions of cortex in vicinity of the measured voltages. See [11] for more details and comparison with other methods in the literature. In particular, the method proposed by Lachaux et al. [14] amounts to using phase computed from projection on the span of the single most concentrated prolate.

A second potential application of BPPs, in particular of wavelets generated by families of BPPs, is the study of cross-frequency coupling in EEG data, e.g., [2, 3], which is thought to reflect mechanisms to transfer information from large-scale brain networks operating at behavioral timescales to the fast, local cortical processing. Such coupling was first observed in time–frequency distributions of ECoG data. Most studies address coupling between low frequency phase and high frequency power. Such coupling seems functionally dependent, for example, theta/gamma coupling has been observed in auditory tasks while alpha/high gamma coupling seems more common in visual processing, e.g., [27]. Evidently, ability to localize high-frequency energy in a part of phase is critical and snug BPP-shift

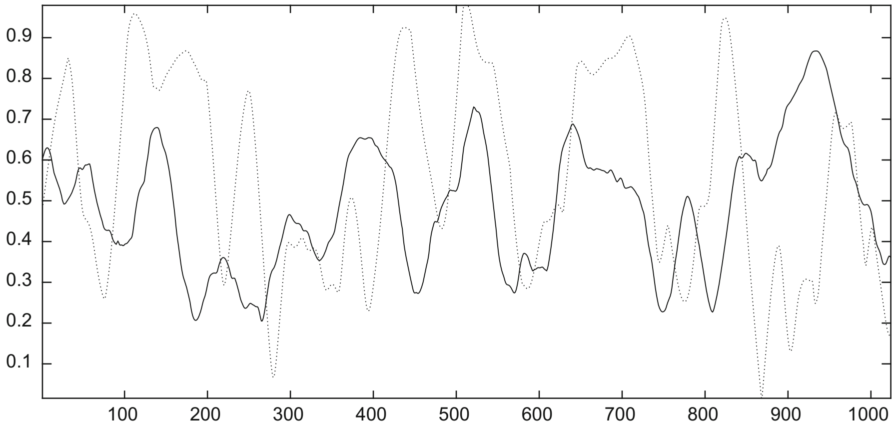

Fig. 10 Smoothed phase locking values produced by projecting channels in Fig. 9 onto 8 most concentrated FBPs bandlimited to $32 \leq |\omega| \leq 96$ Hz and concentrated in $M = 64$ samples (solid) compared to modulated Gaussian method of Lachaux et al. [14] (dots). The modulated Gaussian method corresponds to projection onto shifts of a single prolate versus the shifts of all well-concentrated prolates, which account for temporal variation in signal phase. Rightmost peak should indicate switching of cortical communication in vicinities of measurements and is missed by Gaussian method.

frames can do so because the frame inequality precisely quantifies such energy localization. Experimental results from EEG will be reported elsewhere.

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Fast Fourier Transforms for Spherical Gauss-Laguerre Basis Functions

Jürgen Prestin and Christian Wülker

1 Introduction

Since its popularization by Cooley and Tukey [5], the *Fast Fourier Transform* (FFT) on the unit circle $\mathbb{T}$ and its inverse (iFFT) have been generalized to several other domains and corresponding sets of basis functions. For example, many applications in signal processing and data analysis nowadays benefit from an extension of the univariate FFTs to the d-dimensional Torus $\mathbb{T}^d$ ($d > 1$), where multivariate trigonometric polynomials are used in analogy to the univariate case (see, e.g., [10, Sect. 2]). Another example is the two-dimensional unit sphere $\mathbb{S}^2$. Here, the spherical harmonics are used as an orthonormal basis of the space $L^2(\mathbb{S}^2)$ of functions square-integrable over $\mathbb{S}^2$ (see [8, 14, 17, 20], for instance). This has also initiated the development of fast Fourier transforms on the three-dimensional rotation group SO(3), where the spherical harmonics are replaced by so-called Wigner-D functions (see [16, 21, 23], for example). Recently, certain combinations of spherical harmonics, generalized Laguerre polynomials, and an exponential radial decay factor were used as orthonormal basis functions of the space $L^2(\mathbb{B}^3)$ of square-integrable functions on the three-dimensional unit ball $\mathbb{B}^3$; a fast Fourier transform was developed in this setting as well (see [18] for more information).

In this work, we introduce fast Fourier transforms on the entire three-dimensional real space $\mathbb{R}^3$. On the one hand, this extends the above collection of domains in a

A MATLAB implementation of the algorithms presented in this paper is available at https://github.com/cwuelker/SGLPack.

J. Prestin • C. Wülker (✉)
Institute of Mathematics, University of Lübeck, Ratzeburger Allee 160, Lübeck 23562, Germany
e-mail: wuelker@math.uni-luebeck.de

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natural direction; on the other hand, due to the non-compactness of $\mathbb{R}^3$, we find ourselves in a somewhat new situation.

Of course, the non-compactness of the underlying domain has to be accounted for. While it is conceivable to consider basis functions that exhibit an appropriate decay behavior, in this work, we endow the space $L^2(\mathbb{R}^3)$ with the Gaussian weight function $\exp(-|\cdot|^2)$, where $|\cdot|$ denotes the standard Euclidean norm (such weight function is also referred to as a *multivariate Hermite weight* in literature). In particular, we consider the weighted L^2 space

$$H := \left\{ f : \mathbb{R}^3 \to \mathbb{C} : f \text{ (Lebesgue) measurable and } \int_{\mathbb{R}^3} |f(\boldsymbol{x})|^2 \exp(-|\boldsymbol{x}|^2)\,\mathrm{d}\boldsymbol{x} < \infty \right\},$$

endowed with the inner product

$$\langle f, g \rangle_H := \int_{\mathbb{R}^3} f(\boldsymbol{x})\,\overline{g(\boldsymbol{x})}\,\exp(-|\boldsymbol{x}|^2)\,\mathrm{d}\boldsymbol{x}, \qquad f, g \in H, \tag{1}$$

and induced norm $\|\cdot\|_H := \sqrt{\langle\cdot,\cdot\rangle_H}$.

A crucial feature of the space H is that it allows to work with such structurally simple functions as *polynomials*. Particularly, as recently noted by Maizlish and Prymak [19, Sect. 1], we have the following result, essential for everything to follow:

Theorem 1.1. *The class of (complex-valued) polynomials on $\mathbb{R}^3$ is dense in H, i.e., any function $f \in H$ can be approximated arbitrarily well by polynomials with respect to $\|\cdot\|_H$.*

Having this in mind, it appears natural to employ appropriately normalized orthogonal polynomials as an orthonormal basis of the Hilbert space H. In view of this, however, we note that orthogonal polynomials in H are not unique, as we should expect in the univariate setting. In fact, a review of the relevant literature reveals several different variants of such, arising from different construction approaches (see, e.g., [11, Sect. 5.1.3]).

By a *separation-of-variables* approach, Ritchie and Kemp [25] constructed particular orthogonal polynomials in H from the well-known spherical harmonics (Definition 2.1) and generalized Laguerre polynomials (Theorem 2.3). We call these *spherical Gauss-Laguerre* (SGL) *basis functions* (the term "Gaussian" is to account for the Gaussian weight on H).

Definition 1.2 (SGL basis functions). The SGL basis function of *degree* $n \in \mathbb{N}$ and *orders* $l \in \{0, \ldots, n-1\}$ and $m \in \{-l, \ldots, l\}$ is defined in spherical coordinates (see Section 2) as

$$H_{nlm} : \mathbb{R}^3 \to \mathbb{C}, \qquad H_{nlm}(r, \vartheta, \varphi) := N_{nl}\, R_{nl}(r)\, Y_{lm}(\vartheta, \varphi), \tag{2}$$

where N_{nl} is a normalization constant,

$$N_{nl} := \sqrt{\frac{2(n-l-1)!}{\Gamma(n+1/2)}},$$

Y_{lm} is the spherical harmonic of degree l and order m, while the radial part R_{nl} is defined as

$$R_{nl}(r) := L_{n-l-1}^{(l+1/2)}(r^2)\, r^l,$$

$L_{n-l-1}^{(l+1/2)}$ being a generalized Laguerre polynomial.

By construction of the SGL basis functions, these polynomials are orthonormal in H and span the space of all polynomials on $\mathbb{R}^3$ (see Section 2). The completeness of this orthonormal system thus follows from Theorem 1.1.

Corollary 1.3. *The SGL basis functions H_{nlm} constitute an orthonormal basis (i.e., a complete orthonormal system) in H. In particular, for any $f \in H$, the* Fourier partial sums

$$\sum_{n=1}^{B} \sum_{l=0}^{n-1} \sum_{m=-l}^{l} \langle f, H_{nlm} \rangle_H H_{nlm}, \qquad B \in \mathbb{N}, \tag{3}$$

converge to f in the norm of H as B approaches ∞.

In this paper, we present a general class of algorithms for the efficient numerical computation of the SGL Fourier coefficients $\hat{f}_{nlm} := \langle f, H_{nlm} \rangle_H$ in (3) – that is, we present *fast SGL Fourier transforms*. As is commonly done in generalized FFTs, we develop our algorithms starting out from a concomitant quadrature formula, so that these algorithms are exact (in exact arithmetics) for bandlimited functions (see Section 3). Inspired by the construction of the SGL basis functions, our approach is based on a separation of variables, separating the radius r from the angles ϑ and φ. For the radial part of our fast transforms, we introduce the *discrete R transform* (Section 3.3.2). The spherical part of our transforms is constituted by a fast spherical Fourier transform, i.e., a generalized FFT for the spherical harmonics. Notably, our approach also results in fast inverse transforms with the same asymptotic complexity as the forward transforms: All of our fast algorithms have an asymptotic complexity of $\mathcal{O}(B^4)$, B being the respective bandlimit, while the number of sample points on $\mathbb{R}^3$ scales with B^3. This clearly improves the naive bound of $\mathcal{O}(B^7)$.

Applications of our fast algorithms arise, for example, in the simulation of biomolecular recognition processes, such as *protein-protein* or *protein-ligand docking* (see Section 5).

The rest of this paper is organized as follows: In Section 2, we review the construction of the SGL basis functions. This section is optional to the reader interested solely in our fast algorithms. Subsequently, in Section 3, we develop fast SGL Fourier transforms. The resulting algorithms are tested in a prototypical numerical experiment in Section 4. In Section 5, we discuss the results, draw final conclusions, and give an outlook on future developments. We also include the layout of a true $\mathscr{O}(B^3 \log^2 B)$ fast SGL Fourier transform and inverse.

2 Spherical Gauss-Laguerre (SGL) Basis Functions

As mentioned above, the SGL basis functions of Definition 1.2 arise from a particular construction approach by Ritchie and Kemp [25]. This approach comprises multiple steps. The first step is the introduction of *spherical coordinates*. We define these as *radius* $r \in [0, \infty)$, *polar angle* $\vartheta \in [0, \pi]$, and *azimuthal angle* $\varphi \in [0, 2\pi)$, being connected to Cartesian coordinates x, y, and z, via

$$\begin{aligned} x &= r \sin\vartheta \cos\varphi, \\ y &= r \sin\vartheta \sin\varphi, \\ z &= r \cos\vartheta. \end{aligned}$$

In the following, with a slight abuse of notation, we write $f(\boldsymbol{x}) = f(r, \vartheta, \varphi)$ if (r, ϑ, φ) are the spherical coordinates of the point $\boldsymbol{x} = (x, y, z) \in \mathbb{R}^3$, in which case we simply write $\boldsymbol{x} = (r, \vartheta, \varphi)$. This allows the inner product (1) to be rewritten as

$$\langle f, g\rangle_H = \int_0^\infty \left\{ \int_0^\pi \int_0^{2\pi} f(r, \vartheta, \varphi)\, \overline{g(r, \vartheta, \varphi)}\, \mathrm{d}\varphi \, \sin\vartheta \, \mathrm{d}\vartheta \right\} r^2 \mathrm{e}^{-r^2} \mathrm{d}r, \quad f, g \in H. \tag{4}$$

Note that the integration range $[0, \pi] \times [0, 2\pi)$ of the two inner integrals above can be identified with the unit sphere $\mathbb{S}^2$.

The next step is a *separation of variables*. In particular, Ritchie and Kemp make the product ansatz

$$p(\boldsymbol{x}) = R(r)\, S(\vartheta, \varphi), \qquad \boldsymbol{x} = (r, \vartheta, \varphi) \in \mathbb{R}^3, \tag{5}$$

for each orthogonal polynomial p to be constructed.

Of course, the radial part R and the spherical part S should be *polynomial* on $[0, \infty)$ and $\mathbb{S}^2$ (by which we mean the restriction of a polynomial on $\mathbb{R}^3$ to $\mathbb{S}^2$), respectively. Furthermore, it is desirable that each two orthogonal polynomials p_j and p_k satisfy separate orthogonality relations with respect to the radius and on the sphere,

$$\int_0^\infty R_j(r)\overline{R_k(r)}\, r^2 \mathrm{e}^{-r^2} \mathrm{d}r = \delta_{jk}, \tag{6}$$

$$\int_0^\pi \int_0^{2\pi} S_j(\vartheta, \varphi)\, \overline{S_k(\vartheta, \varphi)}\, \mathrm{d}\varphi \, \sin\vartheta \, \mathrm{d}\vartheta = \delta_{jk}, \tag{7}$$

denoting by δ_{jk} the standard *Kronecker symbol*, being 1 if $j = k$ and 0 otherwise. The property $\langle p_j, p_k \rangle_H = \delta_{jk}$, i.e., the orthonormality of the SGL basis functions, then follows by (4).

The above separation approach allows the radial part R and the spherical part S in (5) to be constructed almost independently from each other. We begin with the spherical part S, for which solely the spherical harmonics are required.

Definition 2.1. The *spherical harmonic* of *degree* $l \in \mathbb{N}_0$ and *order* $m \in \{-l, \ldots, l\}$ is defined as

$$Y_{lm} : \mathbb{S}^2 \to \mathbb{C}, \qquad Y_{lm}(\vartheta, \varphi) := \sqrt{\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{lm}(\cos\vartheta)\, \mathrm{e}^{\mathrm{i}m\varphi}, \tag{8}$$

where P_{lm} denotes the *associated Legendre polynomial* of degree l and order m [1, Eqs. 8.6.6 and 8.6.18]:

$$P_{lm} : [-1,1] \to \mathbb{R}, \qquad P_{lm}(t) := \frac{(-1)^m}{2^l l!} (1-t^2)^{m/2} \frac{\mathrm{d}^{l+m}}{\mathrm{d}t^{l+m}} (t^2-1)^l.$$

The associated Legendre polynomials satisfy a *three-term recurrence relation* [1, Eq. 8.5.3]:

$$\begin{aligned}(l+1-m)\, P_{l+1,m}(t) &= (2l+1)\, t\, P_{lm}(t) \\ &\quad - (l+m)\, P_{l-1,m}(t), \qquad t \in [-1,1], \quad |m| \le l \in \mathbb{N}.\end{aligned} \tag{9}$$

In our context, the most important properties of the spherical harmonics are the following; for a detailed introduction to the related theory, refer to Freeden et al. [12] or Dai and Xu [7], for example.

Theorem 2.2. *The spherical harmonics constitute an orthonormal basis of the space $L^2(\mathbb{S}^2)$ of square-integrable functions on the unit sphere $\mathbb{S}^2$, endowed with the standard inner product*

$$\langle f, g \rangle_{\mathbb{S}^2} := \int_0^\pi \int_0^{2\pi} f(\vartheta, \varphi)\, \overline{g(\vartheta, \varphi)}\, \mathrm{d}\varphi\, \sin\vartheta\, \mathrm{d}\vartheta, \qquad f, g \in L^2(\mathbb{S}^2). \tag{10}$$

Furthermore, the spherical harmonics of degree at most N span the space of all (complex-valued) polynomials of (total) degree at most N on $\mathbb{S}^2$ ($N \in \mathbb{N}_0$).

With this knowledge, it is clear that the spherical harmonics are a good choice for the spherical part S in (5); the orthogonality relation (7) is thus satisfied (compare with (10)).

In a next step, the spherical harmonics are extended radially in order to regain polynomials on $\mathbb{R}^3$. To this end, Ritchie and Kemp borrow the following result from the theory of orthogonal polynomials in the univariate setting:

Theorem 2.3 (Szegő [28, Sect. 5.1]). *For every fixed real number $\alpha > -1$, there exists exactly one set of polynomials on the positive half-line $[0, \infty)$ satisfying the orthogonality relation*

$$\int_0^\infty L_j^{(\alpha)}(t)\, L_k^{(\alpha)}(t)\, t^\alpha \mathrm{e}^{-t}\, \mathrm{d}t \;=\; \frac{\Gamma(k+\alpha+1)}{k!}\,\delta_{jk}, \qquad j,k \in \mathbb{N}_0, \tag{11}$$

where Γ denotes the gamma function.

These polynomials are called *generalized* (or *associated*) *Laguerre polynomials*. Each generalized Laguerre polynomial $L_k^{(\alpha)}$ is of degree k, and possesses the closed-form expression

$$L_k^{(\alpha)}(t) \;=\; \sum_{j=0}^{k} \frac{(-1)^j}{j!} \binom{k+\alpha}{k-j} t^j, \qquad t \in [0, \infty) \tag{12}$$

[28, Eq. 5.1.6]. As the associated Legendre polynomials, the generalized Laguerre polynomials satisfy a three-term recurrence relation [28, Eq. 5.1.10]:

$$\begin{aligned}(k+1)\, L_{k+1}^{(\alpha)}(t) \;&=\; (2k+\alpha+1-t)\, L_k^{(\alpha)}(t)\\ &\quad -\; (k+\alpha)\, L_{k-1}^{(\alpha)}(t), \qquad t \in [0,\infty), \quad k \in \mathbb{N}.\end{aligned} \tag{13}$$

Inspired by the solution to Schrödinger's equation for the hydrogen atom (cf. [2, Sect. 7.4]), Ritchie and Kemp now make the ansatz

$$R(r) \;=\; R_k^{(\alpha)}(r) \;:=\; N_k^{(\alpha)} r^\alpha L_k^{(\alpha+1/2)}(r^2), \quad \alpha > -1, \quad k \in \mathbb{N}_0,$$

for the radial part R in (5). By setting $\alpha := l$, where l is the order of the spherical harmonic Y_{lm} to be extended, and substituting r^2 for t in (11), this ansatz results in the orthogonality relation

$$\int_0^\infty R_j^{(l)}(r)\, R_k^{(l)}(r)\, r^2 \mathrm{e}^{-r^2}\, \mathrm{d}r \;=\; \frac{\Gamma(k+l+3/2)}{2k!} \left\{N_k^{(l)}\right\}^2 \delta_{jk}, \qquad j,k \in \mathbb{N}_0. \tag{14}$$

This immediately entails setting

$$N_k^{(l)} \;:=\; \sqrt{\frac{2k!}{\Gamma(k+l+3/2)}},$$

so that the orthogonality relation (6) is satisfied. Observe that the polynomials $R_k^{(l)}$ are real.

At this point, it is important to note that one cannot expect to obtain a polynomial on $\mathbb{R}^3$ by extending a spherical harmonic Y_{lm} by an arbitrary polynomial in r.

However, the above combinations of $R_k^{(l)}$ and Y_{lm} are, in fact, polynomials of (total) degree $2k+l$ on $\mathbb{R}^3$. This is due to the fact that $r^l Y_{lm}$ is a polynomial of degree l on $\mathbb{R}^3$, while $L_k^{(l+1/2)}(r^2)$ is a polynomial of degree $2k$ on $\mathbb{R}^3$. By some further working with the closed-form expression (12) of the generalized Laguerre polynomials, Ritchie and Kemp found that by setting $k := n - l - 1$, $n > l$, the arising combinations of $R_{nl} := R_{n-l-1}^{(l)}$ and Y_{lm} actually span the space of polynomials on $\mathbb{R}^3$. This establishes the final form of the SGL basis functions H_{nlm} of Definition 1.2. The notion "basis functions" is justified by Theorem 1.1.

Finally, note that the degree of the SGL basis functions is not to be confused with their polynomial degree: The SGL basis function H_{nlm} is of degree n in the sense of Definition 1.2, but of polynomial degree $2n - l - 2$.

3 Fast Fourier Transforms for SGL Basis Functions

In this section, we develop fast Fourier transforms for the SGL basis functions of Definition 1.2. To this end, we first derive an SGL sampling theorem for *bandlimited functions*. For a fixed *bandlimit* $B \in \mathbb{N}$, these are functions $f \in H$ for which $\hat{f}_{nlm} = 0$ if $n > B$. By construction of the SGL basis functions, with increasing bandlimit B, these spaces exhaust the entire class of polynomials on $\mathbb{R}^3$; recall, however, that these spaces do not coincide with the classical polynomial spaces of $\mathbb{R}^3$. This is why we introduce a different notion here. The SGL sampling theorem enables us to compute the SGL Fourier coefficients of such bandlimited functions in a *discrete* way, that is, with a finite number of computation steps. This immediately results in a first discrete SGL Fourier transform and corresponding inverse. By a separation-of-variables technique and the employment of a fast spherical Fourier transform, we then unveil a whole class of fast SGL Fourier transforms and inverses. We close this section by a linear-algebraic description and comparison of our transforms.

3.1 SGL Sampling Theorem

To derive an SGL sampling theorem, we make use of two auxiliary results: an *equiangular quadrature rule* for the unit sphere $\mathbb{S}^2$, which is a classical construct of Driscoll and Healy [8], and a *Gauss-Hermite quadrature rule* for the positive half-line $[0, \infty)$. We begin with the former.

Theorem 3.1 (Driscoll and Healy [8, Theorem 3]). *Let g be a polynomial of degree $L - 1$ on $\mathbb{S}^2$, i.e., $g \in \operatorname{span}\{Y_{lm} : |m| \le l < L\}$, $L \in \mathbb{N}$. Then the spherical Fourier coefficients of g obey the quadrature rule*

$$\langle g, Y_{lm} \rangle_{\mathbb{S}^2} = \sum_{j,k=0}^{2L-1} b_j \, g(\vartheta_j, \varphi_k) \overline{Y_{lm}(\vartheta_j, \varphi_k)}, \qquad |m| \le l < L, \tag{15}$$

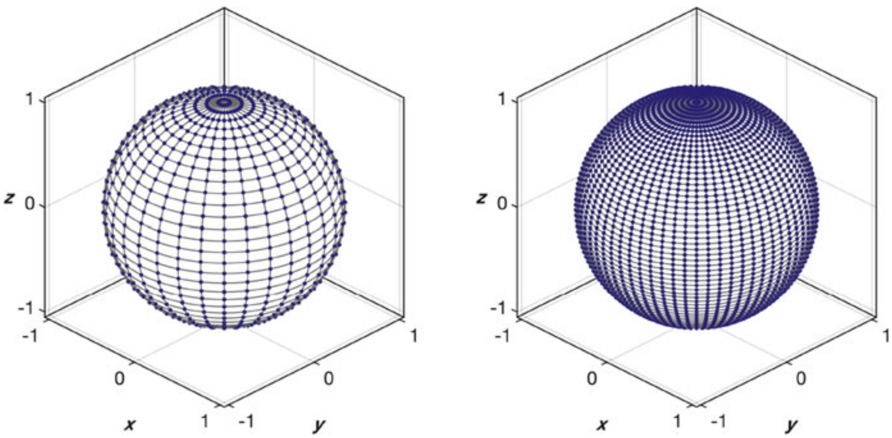

Fig. 1 Sampling angles (ϑ_j, φ_k), plotted as points on the unit sphere $\mathbb{S}^2$, for **(left)** $L = 16$ and **(right)** $L = 32$. Note that the sampling angles are denser near the poles than near the equator.

where the sampling angles are defined as $\vartheta_j := (2j+1)\pi/4L$ *and* $\varphi_k := k\pi/L$, *resulting in the closed-form expression*

$$b_j = \sin\left((2j+1)\frac{\pi}{4L}\right)\frac{2}{L}\sum_{l=0}^{L-1}\frac{1}{2l+1}\sin\left((2j+1)(2l+1)\frac{\pi}{4L}\right)$$

for the quadrature weights.

We call L the *order* of the respective spherical quadrature rule. Note that the quadrature weights b_j are real, and do not depend on the azimuthal sampling angles φ_k. This is due to the special choice of the sampling angles ϑ_j and φ_k. Figure 1 shows the sampling angles ϑ_j and φ_k for the orders $L = 16$ and $L = 32$, respectively, plotted as points on the unit sphere $\mathbb{S}^2$. Figure 2 shows the corresponding quadrature weights b_j.

As it turns out, the weights b_j are *positive*. Since we are not aware of a proof of this feature having been given in this context, we include a direct proof here.

Lemma 3.2. *The quadrature weights* b_j *are positive.*

Proof. Let $L \in \mathbb{N}$ be given. Firstly, we note that $0 < (2j+1)\pi/4L < \pi$ and thus $0 < \sin((2j+1)\pi/4L)$ for $j = 0, \ldots, 2L-1$. Set $\gamma_j := (2j+1)\pi/4$, $j \in \{0, \ldots, 2L-1\}$. We derive

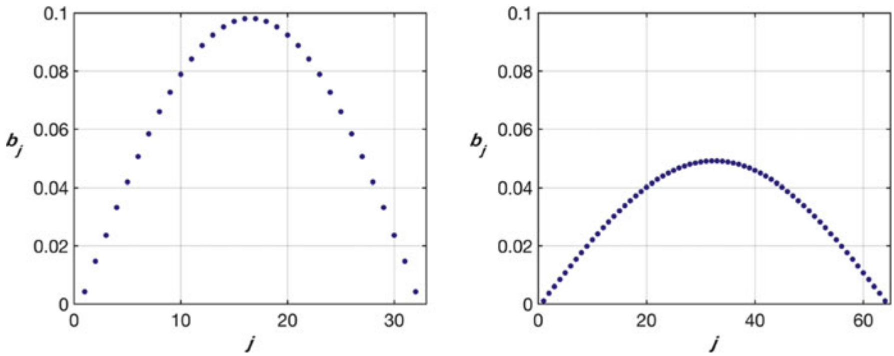

Fig. 2 Spherical quadrature weights b_j, plotted for **(left)** $L = 16$ and **(right)** $L = 32$. Note how the weights compensate for the higher density of sampling angles near the poles of $\mathbb{S}^2$ (cf. Figure 1): the higher the density of sampling angles gets, the smaller the corresponding weights become.

$$
\begin{aligned}
\sum_{l=0}^{L-1} \frac{1}{2l+1} \sin\left((2l+1)\frac{\gamma_j}{L}\right) &= \mathfrak{Im} \sum_{l=0}^{L-1} \frac{1}{2l+1}\, \mathrm{e}^{\mathrm{i}(2l+1)\gamma_j/L} \\
&= \mathfrak{Im} \sum_{l=0}^{L-1} \left\{ \frac{\mathrm{i}}{L} \int_0^{\gamma_j} \mathrm{e}^{\mathrm{i}(2l+1)t/L}\, \mathrm{d}t + \frac{1}{2l+1} \right\} \\
&= \frac{1}{L}\, \mathfrak{Re} \int_0^{\gamma_j} \mathrm{e}^{\mathrm{i}t/L} \sum_{l=0}^{L-1} \mathrm{e}^{\mathrm{i}2lt/L}\, \mathrm{d}t \\
&= \frac{1}{L}\, \mathfrak{Re} \int_0^{\gamma_j} \frac{\mathrm{e}^{2\mathrm{i}t} - 1}{\mathrm{e}^{\mathrm{i}t/L} - \mathrm{e}^{-\mathrm{i}t/L}}\, \mathrm{d}t \\
&= \frac{1}{2L} \int_0^{\gamma_j} \frac{\sin(2t)}{\sin(t/L)}\, \mathrm{d}t. \qquad (16)
\end{aligned}
$$

Substituting $u/2$ for t on the right-hand side of (16), we arrive at

$$
\sum_{l=0}^{L-1} \frac{1}{2l+1} \sin\left((2j+1)(2l+1)\frac{\pi}{4L}\right) = \frac{1}{4L} \int_0^{(j+1/2)\pi} \frac{\sin u}{\sin(u/2L)}\, \mathrm{d}u. \qquad (17)
$$

To show the positivity of the right-hand side of (17), we distinguish between four different cases: $j < L$ or $j \geq L$, j being even or uneven, respectively. The reason for the first distinction is that the denominator $\sin(\cdot/2L)$ is strictly increasing on the interval $[0, L\pi)$ and strictly decreasing on the interval $(L\pi, 2L\pi]$. Furthermore, $\sin(\cdot/2L)$ is non-negative on the integration range $[0, (j+1/2)\pi] \subset [0, 2L\pi]$ which allows all cases to be treated in a straightforward manner.

Let now $j < L$ and set $\kappa_j := 0$ if j is even and $\kappa_j := 1$ if j is odd. Two simple estimations reveal

$$\begin{aligned}\int_0^{(j+1/2)\pi} \frac{\sin u}{\sin(u/2L)}\,\mathrm{d}u &> \int_0^{(j+\kappa_j)\pi} \frac{\sin u}{\sin(u/2L)}\,\mathrm{d}u \\ &> \sum_{k=0}^{(j+\kappa_j)/2-1} \frac{1}{\sin((2k+1)\pi/2L)} \left\{ \int_{2k\pi}^{(2k+1)\pi} + \int_{(2k+1)\pi}^{2(k+1)\pi} \right\} \sin u\,\mathrm{d}u \\ &= 0.\end{aligned}$$

If, on the other hand, $j \geq L$, we make use of the identity

$$\int_0^{(j+1/2)\pi} \frac{\sin u}{\sin(u/2L)}\,\mathrm{d}u = \int_0^{(2L-j-1/2)\pi} \frac{\sin u}{\sin(u/2L)}\,\mathrm{d}u$$

and proceed in the same sense. □

In a next step towards our SGL sampling theorem, we introduce the *half-range Gauss-Hermite quadrature*, i.e., a Gaussian quadrature rule for the Hermite weight $\exp(-r^2)$ on the positive half-line $[0, \infty)$. We add the term "half-range" here because the Hermite weight is usually considered on the entire real line $\mathbb{R}$, leading to other quadrature rules.

Theorem 3.3 (Gautschi [13, Sects. 3.2.2 & 3.2.3]). *Let p be a polynomial of degree at most $2N-1$, $N \in \mathbb{N}$. Furthermore, let $r_0 < \cdots < r_{N-1}$ denote the simple, positive roots of the Nth orthogonal polynomial p_N with respect to the weight function* $\exp(-r^2)$ *on* $[0, \infty)$. *Then equality holds in the Gaussian quadrature formula*

$$\int_0^{\infty} p(r)\,\mathrm{e}^{-r^2}\,\mathrm{d}r = \sum_{i=0}^{N-1} a_i\, p(r_i), \tag{18}$$

where the quadrature weights a_i are real, positive, and satisfy the equation

$$a_i = \int_0^{\infty} \frac{p_N(r)}{(r-r_i)\,p_N'(r_i)}\,\mathrm{e}^{-r^2}\,\mathrm{d}r.$$

As in the spherical quadrature rules introduced in Theorem 3.1, we call N the *order* of the respective quadrature rule.

We do not want to go into detail regarding the numerical aspects of Theorem 3.3. We only mention that Steen et al. [26, Sect. 2] have developed special recurrence relations to compute the coefficients of the three-term recurrence relation satisfied by the orthogonal polynomials p_n, $n \in \mathbb{N}_0$. This, in turn, allows the sampling points r_i in (18) to be computed by a standard approach (see [13, Sect. 3.2.2, (v)]). It is

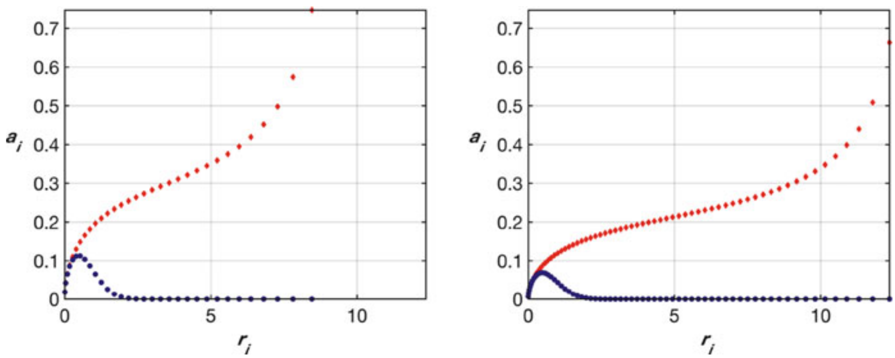

Fig. 3 Sampling points r_i, corresponding quadrature weights a_i (●), and scaled weights $a_i \exp(r_i^2)$ (♦, cf. Section 3.2) of the half-range Gauss-Hermite quadrature rule of order **(left)** $N = 32$ and **(right)** $N = 64$. The points r_i are used as sampling radii which, combined with the sampling angles ϑ_j and φ_k shown in Figure 1, constitute the sampling points of our SGL sampling theorem (Theorem 3.4) for the bandlimits (left) $B = 16$ and (right) $B = 32$.

then also possible to compute the corresponding quadrature weights a_i with desired precision by Steen et al. [26, Eq. 2.1]. This approach is used in the numerical experiments of the upcoming Section 4. Figure 3 shows the sampling points r_i and corresponding weights a_i for the orders $N = 32$ and $N = 64$.

We now combine Theorems 3.1 and 3.3 to obtain our SGL sampling theorem. For this, let f be bandlimited with bandlimit $B \in \mathbb{N}$. The function f thus possesses the unique SGL decomposition

$$f = \sum_{n=1}^{B} \sum_{l=0}^{n-1} \sum_{m=-l}^{l} \hat{f}_{nlm} H_{nlm}. \tag{19}$$

Recalling that $H_{nlm}(r, \vartheta, \varphi) = N_{nl} R_{nl}(r) Y_{lm}(\vartheta, \varphi)$, we see that $f(r, \cdot, \cdot)$ is a linear combination of spherical harmonics of degree $l < B$ for every fixed $r \in [0, \infty)$. Hence, using the spherical quadrature rule of Theorem 3.1 of order $L = B$, we get for $|m| \leq l < n \leq B$

$$\begin{aligned}
\hat{f}_{nlm} &= N_{nl} \int_0^\infty \Big\{ \int_0^\pi \int_0^{2\pi} f(r, \vartheta, \varphi) \overline{Y_{lm}(\vartheta, \varphi)} \, \mathrm{d}\varphi \, \sin\vartheta \, \mathrm{d}\vartheta \Big\} R_{nl}(r) \, r^2 \, \mathrm{e}^{-r^2} \mathrm{d}r \\
&= N_{nl} \int_0^\infty \Big\{ \sum_{j,k=0}^{2B-1} b_j f(r, \vartheta_j, \varphi_k) \overline{Y_{lm}(\vartheta_j, \varphi_k)} \Big\} R_{nl}(r) \, r^2 \, \mathrm{e}^{-r^2} \mathrm{d}r. \qquad (20)
\end{aligned}$$

Considering again the SGL decomposition (19), we verify that the integrand in (20) is a polynomial in r of degree at most $4B - 2$, multiplied by the Hermite weight. Therefore, using the half-range Gauss-Hermite quadrature rule of

Theorem 3.3 of order $N = 2B$, we obtain

$$\int_0^\infty \Big\{ \sum_{j,k=0}^{2B-1} b_j f(r, \vartheta_j, \varphi_k) \overline{Y_{lm}(\vartheta_j, \varphi_k)} \Big\} R_{nl}(r)\, r^2 \mathrm{e}^{-r^2} \mathrm{d}r \;=\; \sum_{i,j,k=0}^{2B-1} a_i r_i^2 b_j f(r_i, \vartheta_j, \varphi_k) R_{nl}(r_i) \overline{Y_{lm}(\vartheta_j, \varphi_k)}. \tag{21}$$

Combining (20) and (21) now yields our SGL sampling theorem:

Theorem 3.4 (SGL sampling theorem). *Let f be a bandlimited function with bandlimit $B \in \mathbb{N}$. Then the SGL Fourier coefficients of f obey the quadrature rule*

$$\hat{f}_{nlm} = \sum_{i,j,k=0}^{2B-1} a_i r_i^2 b_j f(r_i, \vartheta_j, \varphi_k) \overline{H_{nlm}(r_i, \vartheta_j, \varphi_k)}, \qquad |m| \le l < n \le B, \tag{22}$$

where the sampling radii $r_i > 0$ and weights $a_i > 0$ are those of the half-range Gauss-Hermite quadrature rule of order $2B$ (Theorem 3.3), while the sampling angles (ϑ_j, φ_k) and weights $b_j > 0$ are those of the equiangular spherical quadrature rule of order B (Theorem 3.1, Lemma 3.2).

Note that the sampling angles shown in Figure 1 are radially extended by precisely the sampling points shown in Figure 3 to obtain the sampling points of Theorem 3.4 for the bandlimits $B = 16$ and $B = 32$, respectively.

3.2 Discrete SGL Fourier Transforms

Based on the results of the previous section, we are now able to give a rigorous definition of the term "discrete SGL Fourier transform."

Definition 3.5 (DSGLFT/iDSGLFT). Let $B \in \mathbb{N}$. Any method for the computation of the SGL Fourier coefficients of bandlimited functions with bandlimit B by means of (22) is called a *discrete SGL Fourier transform* (DSGLFT). Correspondingly, any method for reconstruction of function values of functions with bandlimited B at the respective sampling nodes $(r_i, \vartheta_j, \varphi_k)$ is referred to as an *inverse discrete SGL Fourier transform* (iDSGLFT).

Let $\tilde{a}_i := a_i \exp(r_i^2)\, r_i^2$. We state a simple DSGLFT as Algorithm 1. The SGL Fourier coefficients of a bandlimited function f are here computed one after another, evaluating the corresponding triple sum every single time. We introduce the factor $\exp(r_i^2)$ to compensate for the fast decay of the quadrature weights a_i (cf. Figure 3). This modification is accounted for by weighting the SGL basis function samples $H_{nlm}(r_i, \vartheta_j, \varphi_k)$ by the factor $\exp(-r_i^2)$ (see also Section 5).

Algorithm 1: Naive DSGLFT

Data: Sample values $f(r_i, \vartheta_j, \varphi_k)$; $i, j, k = 0, \ldots, 2B-1$, of a function f with bandlimit $B \in \mathbb{N}$

Result: SGL Fourier coefficients $\hat{f}_{nlm}$, $|m| \leq l < n \leq B$

for $n = 1$ **to** B **do**
  for $l = 0$ **to** $n - 1$ **do**
    for $m = -l$ **to** l **do**
      Compute $\hat{f}_{nlm} = \sum_{i=0}^{2B-1} \sum_{j=0}^{2B-1} \sum_{k=0}^{2B-1} \{\tilde{a}_i b_j f(r_i, \vartheta_j, \varphi_k)\} \{\overline{H_{nlm}(r_i, \vartheta_j, \varphi_k)}\, \mathrm{e}^{-r_i^2}\}$;
    end
  end
end

Algorithm 2: Naive iDSGLFT

Data: SGL Fourier coefficients $\hat{f}_{nlm}$, $|m| \leq l < n \leq B$, of a function f with bandlimit $B \in \mathbb{N}$

Result: Function values $f(r_i, \vartheta_j, \varphi_k)$; $i, j, k = 0, \ldots, 2B - 1$

for $i = 0$ **to** $2B - 1$ **do**
  for $j = 0$ **to** $2B - 1$ **do**
    for $k = 0$ **to** $2B - 1$ **do**
      Compute $f(r_i, \vartheta_j, \varphi_k) = \sum_{n=1}^{B} \sum_{l=0}^{n-1} \sum_{m=-l}^{l} \hat{f}_{nlm} H_{nlm}(r_i, \vartheta_j, \varphi_k)$;
    end
  end
end

In this work, we use the standard complexity model in which a single operation is defined as a complex multiplication and a subsequent complex addition. To state the asymptotic complexity of Algorithm 1, we make the assumption that the (modified) quadrature weights $\tilde{a}_i$ and b_j, as well as the sampling points $(r_i, \vartheta_j, \varphi_k)$, are stored and readily available during runtime. Using the three-term recurrence relations (9) and (13) of the associated Legendre polynomials and the generalized Laguerre polynomials, we can evaluate any SGL basis function H_{nlm} at an arbitrary sampling node $(r_i, \vartheta_j, \varphi_k)$ in $\mathcal{O}(B)$ steps. Algorithm 1 has, thus, an asymptotic complexity of $\mathcal{O}(B^7)$: the total number of summands of the triple sum scales with B^3, just as the total number of iterations of the three "for" loops.

We state a simple iDSGLFT as Algorithm 2. The function values of a bandlimited function f are here reconstructed at each sampling node $(r_i, \vartheta_j, \varphi_k)$ by directly summing up the SGL basis function values $H_{nlm}(r_i, \vartheta_j, \varphi_k)$, weighted by the respective SGL Fourier coefficient $\hat{f}_{nlm}$. Simple considerations show that this algorithm also has an asymptotic complexity of $\mathcal{O}(B^7)$.

3.3 Fast SGL Fourier Transforms

At this point, we are naturally faced with the task to develop discrete SGL Fourier transforms and corresponding inverse transforms with an asymptotic complexity of less than $\mathscr{O}(B^7)$. This motivates:

Definition 3.6 (FSGLFT/iFSGLFT). We call any DSGLFT (iDSGLFT) with an asymptotic complexity of less than $\mathscr{O}(B^7)$ a *fast (inverse) SGL Fourier transform*, abbreviated FSGLFT (iFSGLFT, respectively).

In this section, we design such fast transforms and, simultaneously, corresponding fast inverse transforms in three main steps: 1) We separate the above naive DSGLFT/iDSGLFT (Algorithm 1 and 2, respectively) into a radial and a spherical subtransform. 2) Subsequently, we employ a fast spherical Fourier transform and inverse to reduce the complexity of the spherical subtransform. 3) We introduce our new discrete R transform, a tool to compute the collection of sums

$$N_{nl}\sum_{i=0}^{2B-1} a_i r_i^2 R_{nl}(r_i)\, s_i = \sum_{i=0}^{2B-1}\left\{N_{nl} R_{nl}(r_i)\,\mathrm{e}^{-r_i^2}\right\}\left\{\tilde{a}_i s_i\right\}, \qquad n = l+1,\ldots,B, \tag{23}$$

for a fixed $0 \leq l < B$, $[s_i]_{i=0,\ldots,2B-1}$ being an input vector of length $2B$, to reduce the complexity of the radial subtransform. For this purpose, we also introduce a corresponding inverse discrete R transform. In the following, we consistently use the notation $[a_\nu]_{\nu=0,\ldots,N-1}$ to denote a (column) vector of length $N \in \mathbb{N}$ with (complex-valued) elements a_ν.

Let $B \in \mathbb{N}$ and a function f with bandlimit B be given. In a first step, we rearrange the triple sum in Algorithm 1 to obtain

$$\hat{f}_{nlm} = \sum_{i=0}^{2B-1}\left\{N_{nl} R_{nl}(r_i)\,\mathrm{e}^{-r_i^2}\right\}\left\{\tilde{a}_i \sum_{j,k=0}^{2B-1} b_j f(r_i,\vartheta_j,\varphi_k)\,\overline{Y_{lm}(\vartheta_j,\varphi_k)}\right\}, \quad |m| \leq l < n \leq B. \tag{24}$$

Note that even without a fast algorithm here, the above separation of variables allows the complexity of Algorithms 1 and 2 to be reduced to $\mathscr{O}(B^6)$ by a simple rearrangement of the computation steps: Precomputation of the inner sum in (24) for all $|m| \leq l < B$ and $i = 0,\ldots,2B-1$ can be done in $\mathscr{O}(B^6)$ steps. Subsequent evaluation of the outer sum for all $|m| \leq l < n \leq B$ can be done in $\mathscr{O}(B^5)$ steps. The costs for evaluating R_{nl} and Y_{lm} are taken into account as $\mathscr{O}(B)$, respectively. The computation steps of the inverse transform may be rearranged in the same sense. We maintain this strategy, and optimize the substeps.

Since f is bandlimited with bandlimit B, we conclude that $f(r_i,\cdot,\cdot)$ is a polynomial of degree at most $B-1$ on $\mathbb{S}^2$ for each i. By Theorem 2.2, this implies that $f(r_i,\cdot,\cdot) \in \operatorname{span}\{Y_{lm} : |m| \leq l < B\}$ (we have already made use of this fundamental feature of bandlimited functions in the derivation of the SGL sampling theorem, Theorem 3.4). Therefore, by the spherical quadrature rule of order B in Theorem 3.1, the inner sum in (24) equals the spherical Fourier coefficient $\langle f(r_i, \cdot, \cdot), Y_{lm}\rangle_{\mathbb{S}^2}$. The computation of these inner sums thus amounts to the computation of all spherical Fourier coefficients of f restricted to the sphere of radius r_i for each i.

The fast spherical Fourier transforms described by Healy et al. [14] are a suitable means to solve this task. At the same time, the corresponding fast inverse transforms allow the function values $f(r_i, \vartheta_j, \varphi_k)$ to be reconstructed from the spherical Fourier coefficients $\langle f(r_i, \cdot, \cdot), Y_{lm}\rangle_{\mathbb{S}^2}$ for each sampling radius r_i. This constitutes the spherical part of our FSGLFTs and iFSGLFTs. We include a brief discussion on fast spherical Fourier transforms based on the spherical quadrature rule of Theorem 3.1 in the upcoming Section 3.3.1.

In order to compute the SGL Fourier coefficients $\hat{f}_{nlm}$ from the precomputed spherical Fourier coefficients $\langle f(r_i, \cdot, \cdot), Y_{lm}\rangle_{\mathbb{S}^2}$, that is, to evaluate the outer sum in (24), we use the above-mentioned discrete R transform, running through all pairs of m and l with $|m| \leq l < B$ (cf. (23) and (24)). This new transform is presented in the upcoming Section 3.3.2. The inverse discrete R transform, also presented in Section 3.3.2, allows the spherical Fourier coefficients $\langle f(r_i, \cdot, \cdot), Y_{lm}\rangle_{\mathbb{S}^2}$ to be reconstructed from the SGL Fourier coefficients $\hat{f}_{nlm}$ with the same asymptotic complexity as the forward transform. The discrete R transform and its inverse thus make up the radial part of our FSGLFTs and iFSGLFTs.

3.3.1 Fast Equiangular Spherical Fourier Transforms

Let g be a polynomial of degree $L-1$ on $\mathbb{S}^2$, i.e., $g \in \operatorname{span}\{Y_{lm} : |m| \leq l < L\}$, $L \in \mathbb{N}$. *Fast equiangular spherical Fourier transforms* based on Theorem 3.1 allow the spherical Fourier coefficients $\langle g, Y_{lm}\rangle_{\mathbb{S}^2}$ of g to be computed with an asymptotic complexity of less than $\mathcal{O}(L^5)$, which is associated with the naive approach (cf. (15)). A large class of such fast transforms was derived and thoroughly tested by Healy et al. [14]. In that work, the authors also presented corresponding fast inverse transforms with the same asymptotic complexity. This is a major advantage of their approach as compared with the preceding work by Driscoll and Healy [8].

The fast spherical Fourier transforms of Healy et al. were developed in several steps, which has led to different variants of the basic algorithm with different asymptotic complexities, ranging from $\mathcal{O}(L^4)$, when using a separation of variables only, to $\mathcal{O}(L^2 \log^2 L)$, when using all techniques presented. We include the derivation of one particular variant, the *seminaive* algorithm and its inverse, here. These seminaive algorithms are later used in the numerical experiment of Section 4.

By (8), a rearrangement of the right-hand side of (15) yields

$$\langle g, Y_{lm}\rangle_{\mathbb{S}^2} = M_{lm} \sum_{j=0}^{2L-1} b_j P_{lm}(\cos \vartheta_j) \sum_{k=0}^{2L-1} g(\vartheta_j, \varphi_k)\, \mathrm{e}^{-\mathrm{i}m\varphi_k}, \qquad |m| \leq l < L, \tag{25}$$

denoting by M_{lm} the normalization constant of the spherical harmonic Y_{lm}. This separation of variables reduces the asymptotic complexity of the naive spherical Fourier transform from $\mathscr{O}(L^5)$ to $\mathscr{O}(L^4)$, as indicated above.

Precomputation of the inner sum in (25) for all $-L < m < L$ can be done in $\mathscr{O}(L \log L)$ steps for each j by using a standard Cooley-Tukey FFT (see [6, Sect. 30.2], for example). This results in an asymptotic complexity of $\mathscr{O}(L^2 \log L)$ for this first step, while the total asymptotic complexity of $\mathscr{O}(L^4)$ remains the same.

The central tool in the fast transforms of Healy et al. is a fast *discrete Legendre transform* (DLT), i.e., a tool to compute the collection of sums

$$M_{lm} \sum_{j=0}^{2L-1} b_j P_{lm}(\cos \vartheta_j)\, t_j, \qquad l = |m|, \ldots, L-1, \tag{26}$$

for a fixed $-L < m < L$, $[t_j]_{j=0,\ldots,2L-1}$ being arbitrary input data. Such fast DLT can be used to evaluate the outer sum in (25), running through all $-L < m < L$.

In the seminaive algorithm, the asymptotic complexity of the naive DLT is reduced by a fast *discrete cosine transform* (DCT). The general DCT is defined as follows (cf. [15, Sect. 5.6]): Let $\boldsymbol{u} := [u_j]_{j=0,\ldots,N-1}$ be some data of length $N \in \mathbb{N}$ and set, in this subsection only, $\vartheta_j := (2j+1)\pi/2N$. Any method for computation of the matrix-vector product

$$\underbrace{\begin{bmatrix} \sqrt{1/N} & & & \\ & \sqrt{2/N} & & \\ & & \ddots & \\ & & & \sqrt{2/N} \end{bmatrix} \cdot \begin{bmatrix} 1 & \cdots & 1 \\ \cos \vartheta_0 & \cdots & \cos \vartheta_{N-1} \\ \vdots & & \vdots \\ \cos((N-1)\vartheta_0) & \cdots & \cos((N-1)\vartheta_{N-1}) \end{bmatrix}}_{=: C_N} \cdot \boldsymbol{u} \tag{27}$$

is referred to as a DCT. Such computation is apparently associated with an asymptotic complexity of $\mathscr{O}(N^2)$ if no fast algorithm is used. By a factorization of the DCT matrix C_N, the asymptotic complexity of the naive DCT can be reduced to $\mathscr{O}(N \log N)$ (see [27] or [15, Sect. 5.6], for example).

Two properties of such DCT are particularly important in our context: Firstly, the DCT matrix C_N is *orthogonal*. If $\boldsymbol{v}$ and $\boldsymbol{w}$ are two vectors of length N, and $\langle \cdot, \cdot \rangle_{\mathbb{C}^N}$ denotes the standard Euclidean inner product, this means that $\langle \boldsymbol{v}, \boldsymbol{w} \rangle_{\mathbb{C}^N} = \langle C_N \boldsymbol{v}, C_N \boldsymbol{w} \rangle_{\mathbb{C}^N}$. Secondly, if $\boldsymbol{p} := [p(\vartheta_j)]_{j=0,\ldots,N-1}$, p being an arbitrary trigonometric polynomial of degree at most N, then the elements $[C_N \boldsymbol{p}]_j$ vanish for $j > \deg(p)$.

We now consider the case $m = 0$ in (26); all other cases can be treated similarly. Choose $N = 2L$ above, and set

$$\boldsymbol{t} := [b_j t_j]_{j=0,\ldots,2L-1} \quad \text{and} \quad \boldsymbol{P}_l := M_{l,0} \cdot [P_{l,0}(\cos \vartheta_j)]_{j=0,\ldots,2L-1}, \quad l < L.$$

Computation of the collection of sums (26) then amounts to the computation of the inner product $\langle \boldsymbol{t}, \boldsymbol{P}_l \rangle_{\mathbb{C}^{2L}}$ for each l. Since $P_{l,0}(\cos \vartheta)$ is a trigonometric polynomial of degree l, we have that

$$\langle \boldsymbol{t}, \boldsymbol{P}_l \rangle_{\mathbb{C}^{2L}} = \langle C_{2L}\boldsymbol{t}, C_{2L}\boldsymbol{P}_l \rangle_{\mathbb{C}^{2L}} = \sum_{j=0}^{2L-1} [C_{2L}\boldsymbol{t}]_j \, [C_{2L}\boldsymbol{P}_l]_j = \sum_{j=0}^{l} [C_{2L}\boldsymbol{t}]_j \, [C_{2L}\boldsymbol{P}_l]_j. \quad (28)$$

Equation (28) shows that the inner product $\langle \boldsymbol{t}, \boldsymbol{P}_l \rangle_{\mathbb{C}^{2L}}$ can be computed in l steps instead of $2L-1$, if the vectors $C_{2L}\boldsymbol{t}$ and $C_{2L}\boldsymbol{P}_l$ are readily available. When this approach is used for all m, this does not yet change the asymptotic complexity, but reduces the total required computation work significantly for sufficiently large L. A truly *fast* DLT can now be obtained in the following way: Let $\boldsymbol{P}_{lm} := M_{lm} \cdot [P_{lm}(\cos\vartheta_j)]_{j=0,\ldots,2L-1}$, $|m| \le l < L$. Since the vectors $C_{2L}\boldsymbol{P}_{lm}$ do not depend on the input data, and a significantly large part of their elements are zero, we may assume them to be stored and readily available during runtime. This results in a fast DLT with an asymptotic complexity of $\mathcal{O}(L^2)$ instead of $\mathcal{O}(L^3)$, and, hence, in a fast spherical Fourier transform with an asymptotic complexity of $\mathcal{O}(L^3)$ instead of $\mathcal{O}(L^4)$.

To close this subsection, we revise the derivation of the inverse seminaive spherical Fourier transform. The task is to reconstruct the function values $g(\vartheta_j, \varphi_k)$ from the spherical Fourier coefficients $\langle g, Y_{lm} \rangle_{\mathbb{S}^2}$. For this purpose, Healy et al. again use a separation of variables to get

$$g(\vartheta_j, \varphi_k) = \sum_{m=-L+1}^{L-1} \mathrm{e}^{\mathrm{i}m\varphi_k} \sum_{l=|m|}^{L-1} M_{lm} P_{lm}(\cos\vartheta_j) \langle g, Y_{lm} \rangle_{\mathbb{S}^2}, \quad j,k = 0,\ldots,2L-1. \quad (29)$$

It is clear that the function values $g(\vartheta_j, \varphi_k)$ can be reconstructed from the collection of inner sums in (29) by means of a standard iFFT, which has the same asymptotic complexity $\mathcal{O}(L \log L)$ as the forward transform (again, see [6, Sect. 30.2]).

We again consider the case $m = 0$ only. The collection of inner sums in (29) can then be written as the matrix-vector product

$$\underbrace{\begin{bmatrix} P_{0,0}(\cos\vartheta_0) & \cdots & P_{L-1,0}(\cos\vartheta_0) \\ \vdots & & \vdots \\ P_{0,0}(\cos\vartheta_{2L-1}) & \cdots & P_{L-1,0}(\cos\vartheta_{2L-1}) \end{bmatrix} \cdot \begin{bmatrix} M_{0,0} & & \\ & \ddots & \\ & & M_{L-1,0} \end{bmatrix}}_{=:P_0^{\mathrm{T}}} \cdot [\langle g, Y_{l,0} \rangle_{\mathbb{S}^2}]_{l=0,\ldots,L-1}.$$

Due to Theorem 3.1 and the orthonormality of the spherical harmonics, the non-transposed matrix P_0 is associated with the forward transform (we encounter the same phenomenon in the upcoming Sections 3.3.2 and 3.4). In particular, P_0 represents the forward DLT when dropping the weights b_j (cf. (26)). The above discussion shows that P_0 possesses the factorization

$$P_0 = [C_{2L}\boldsymbol{P}_{0,0}, \ldots, C_{2L}\boldsymbol{P}_{L-1,0}]^{\mathrm{T}} \cdot C_{2L}.$$

This immediately reveals

$$P_0^{\mathrm{T}} = C_{2L}^{\mathrm{T}} \cdot [C_{2L}\boldsymbol{P}_{0,0}, \ldots, C_{2L}\boldsymbol{P}_{L-1,0}]$$

for the inverse transform.

At this point, we recall the identity $C_{2L}^{\mathrm{T}} = C_{2L}^{-1}$, i.e., the orthogonality of the DCT matrix C_{2L}. By the use of a fast inverse DCT (iDCT) with an asymptotic complexity of $\mathscr{O}(L \log L)$ (again, see [27] or [15, Sect. 5.6]), the same ideas as above now easily yield the inverse seminaive spherical Fourier transform of Healy et al., which has the same asymptotic complexity $\mathscr{O}(L^3)$ as the forward transform.

3.3.2 Completion: Discrete *R* Transforms

We now discuss our discrete R transform (DRT) and its inverse (iDRT) to finalize the above-described class of fast SGL Fourier transforms. To this end, let $\boldsymbol{s} := [s_i]_{i=0,\ldots,2B-1}$ be some input data. For a fixed l, we bring the right-hand side of (23) into matrix-vector notation:

$$\underbrace{\left\{\begin{bmatrix} N_{l+1,l} & & \\ & \ddots & \\ & & N_{B,l} \end{bmatrix} \begin{bmatrix} R_{l+1,l}(r_0) & \cdots & R_{l+1,l}(r_{2B-1}) \\ \vdots & & \vdots \\ R_{B,l}(r_0) & \cdots & R_{B,l}(r_{2B-1}) \end{bmatrix}\right.}_{=: R_l} \overbrace{\left.\begin{bmatrix} \mathrm{e}^{-r_0^2} & & \\ & \ddots & \\ & & \mathrm{e}^{-r_{2B-1}^2} \end{bmatrix}\right\}}^{=: E} \left\{\operatorname{diag}[\tilde{a}_i]_{i=0,\ldots,2B-1} \cdot \boldsymbol{s}\right\}. \tag{30}$$

Running the forward DRT of order l, which we now rigorously define as the evaluation of (30), can be done in $\mathscr{O}(B^2)$ steps by using the *Clenshaw algorithm* [4]. This is possible, since the Laguerre polynomial $L_{n-l-1}^{(l+1/2)}$ is included in the radial part R_{nl} of the SGL basis functions as a factor, and the radial functions R_{nl} thus satisfy the following three-term recurrence relation:

Lemma 3.7. *Let* $0 \leq l < B$ *and* $r \in [0, \infty)$ *be given. Then*

$$R_{n+1,l}(r) = \frac{2n - l - 1/2 - r^2}{\sqrt{(n+1/2)(n-l)}} R_{nl}(r) - \sqrt{\frac{(n-1/2)}{(n+1/2)} \frac{(n-l-1)}{(n-l)}} R_{n-1,l}(r)$$

for $n > l$, *and*

$$R_{l+1,l}(r) = \sqrt{\frac{2}{\Gamma(l+3/2)}}\, r^l, \qquad R_{ll} \equiv 0.$$

In view of the iDRT, we observe that for all $0 \leq m, n \leq B - l$,

$$\begin{aligned}\left[R_l \cdot \mathrm{diag}[a_i r_i^2]_{i=0,\dots,2B-1} \cdot R_l^{\mathrm{T}}\right]_{mn} &= \sum_{k=0}^{2B-1} a_k r_k^2 N_{l+m,l} \overbrace{R_{l+m,l}}^{\deg \leq 2B-2}(r_k) N_{l+n,l} \overbrace{R_{l+n,l}}^{\deg \leq 2B-2}(r_k) \\ &= \int_0^\infty N_{l+m,l} R_{l+m,l}(r) N_{l+n,l} R_{l+n,l}(r)\, r^2 \mathrm{e}^{-r^2} \mathrm{d}r \\ &= \delta_{mn},\end{aligned}$$

by Theorem 3.3 and the orthogonality of the polynomials $R_{l+m,l}$ and $R_{l+n,l}$ (cf. Section 2). Note that we dropped the factors $\exp(r_i^2)$ and $\exp(-r_i^2)$ here (again, see Section 5). Hence,

$$R_l^{\mathrm{T}} = \left\{R_l \cdot \mathrm{diag}[a_i r_i^2]_{i=0,\dots,2B-1}\right\}^{-1}.$$

For a given data vector $\boldsymbol{t} := [t_k]_{k=0,\dots,B-l}$, we thus define the iDRT of order l as the evaluation of the product $R_l^{\mathrm{T}} \cdot \boldsymbol{t}$. This can be done in $\mathscr{O}(B^2)$ steps, just as one run of the forward transform (cf. Section 3.4).

Summarizing all of the above, we state the layout of our FSGLFTs and iFSGLFTs as Algorithm 3 and 4, respectively. Note that all of these transforms have an asymptotic complexity of $\mathscr{O}(B^4)$ instead of the naive $\mathscr{O}(B^7)$.

Algorithm 3: Prototypical FSGLFT

Data: Sample values $f(r_i, \vartheta_j, \varphi_k)$; $i, j, k = 0, \dots, 2B-1$, of a function f with bandlimit $B \in \mathbb{N}$

Result: SGL Fourier coefficients $\hat{f}_{nlm}$, $|m| \leq l < n \leq B$

for $i = 0$ **to** $2B - 1$ **do**

  Compute Fourier coefficients $\langle f(r_i, \cdot, \cdot), Y_{lm}\rangle_{\mathbb{S}^2}$, $|m| \leq l < B$, from function samples $f(r_i, \vartheta_j, \varphi_k)$; $j, k = 0, \dots, 2B - 1$, by using a fast spherical Fourier transform;

end

for $m = 1 - B$ **to** $B - 1$ **do**

  for $l = |m|$ **to** $B - 1$ **do**

    Compute SGL Fourier coefficients $[\hat{f}_{nlm}]_{n=l+1,\dots,B}$ by using the DRT in

$$[\hat{f}_{nlm}]_{n=l+1,\dots,B} = \{R_l \cdot E\} \cdot \left\{\mathrm{diag}[\tilde{a}_i]_{i=0,\dots,2B-1} \cdot [\langle f(r_i, \cdot, \cdot), Y_{lm}\rangle_{\mathbb{S}^2}]_{i=0,\dots,2B-1}\right\};$$

  end

end

Algorithm 4: Prototypical iFSGLFT

Data: SGL Fourier coefficients $\hat{f}_{nlm}$, $|m| \leq l < n \leq B$, of a function f with bandlimit $B \in \mathbb{N}$
Result: Function values $f(r_i, \vartheta_j, \varphi_k)$; $i, j, k = 0, \ldots, 2B-1$
for $m = 1-B$ **to** $B-1$ **do**
  for $l = |m|$ **to** $B-1$ **do**
    Reconstruct spherical Fourier coefficients $\langle f(r_i, \cdot, \cdot), Y_{lm}\rangle_{\mathbb{S}^2}$, $i = 0, \ldots, 2B-1$, by using the iDRT in

$$[\langle f(r_i, \cdot, \cdot), Y_{lm}\rangle_{\mathbb{S}^2}]_{i=0,\ldots,2B-1} = R_l^{\mathrm{T}} \cdot [\hat{f}_{nlm}]_{n=l+1,\ldots,B};$$

  end
end
for $i = 0$ **to** $2B-1$ **do**
  Reconstruct function values $f(r_i, \vartheta_j, \varphi_k)$; $j, k = 0, \ldots, 2B-1$, from Fourier coefficients $\langle f(r_i, \cdot, \cdot), Y_{lm}\rangle_{\mathbb{S}^2}$, $|m| \leq l < B$, by using a fast inverse spherical Fourier transform;
end

3.4 Matrix-Vector Notation of the Transforms

In this section, we give a description of the transforms presented in Sections 3.2 and 3.3 in terms of matrix-vector products. We shall use the standard *Kronecker product*, denoted by $\otimes$. To simplify the notation further, for a fixed bandlimit $B \in \mathbb{N}$, we introduce the linear indices

$$\begin{aligned}
\mu &= \mu(j,k) := 2Bj + k, \\
\psi &= \psi(i,j,k) := 4B^2 i + 2Bj + k, \\
\nu &= \nu(l,m) := l(l+1) + m, \\
\omega &= \omega(n,l,m) := n(n-1)(2n-1)/6 + l(l+1) + m,
\end{aligned}$$

so that μ enumerates the sampling angles $\boldsymbol{a}_\mu := (\vartheta_j, \varphi_k)$ and corresponding weights $c_\mu := b_j$; $j, k = 0, \ldots, 2B-1$, ψ enumerates the sampling points $\boldsymbol{x}_\psi := (r_i, \vartheta_j, \varphi_k)$ and corresponding weights $w_\psi := \tilde{a}_i b_j$; $i, j, k = 0, \ldots, 2B-1$, ν enumerates the spherical harmonics $Y_\nu := Y_{lm}$, $|m| \leq l < B$, while ω enumerates the SGL basis functions $H_\omega := H_{nlm}$, $|m| \leq l < n \leq B$. The indices of the rows and columns of a matrix shall be separated by a semicolon. Because the following considerations are mainly theoretical, we omit all brackets solely relevant in practice, i.e., those specifying the order of operations.

Let a function f with bandlimit $B \in \mathbb{N}$ and SGL Fourier coefficients $\hat{f}_\omega := \hat{f}_{nlm}$ be given. Furthermore, let $\Psi := 8B^3$ denote the total number of sample points $\boldsymbol{x}_\psi$, and let $\Omega := B(B+1)(2B+1)/6$ denote the total number of SGL Fourier coefficients $\hat{f}_\omega$ to be computed. We define the sample vector $\boldsymbol{f} := [f(\boldsymbol{x}_\psi)]_{\psi=0,\ldots,\Psi-1}$ and the vector $\hat{\boldsymbol{f}} := [\hat{f}_\omega]_{\omega=0,\dots,\Omega-1}$ containing the SGL Fourier coefficients of f. The naive DSGLFT/iDSGLFT of Section 3.2 can thus be restated as

$$\hat{\boldsymbol{f}} = [\overline{H_\omega(\boldsymbol{x}_\psi)}\exp(-|\boldsymbol{x}_\psi|^2)]_{\omega=0,\dots,\Omega-1;\,\psi=0,\dots,\Psi-1} \cdot \operatorname{diag}[w_\psi]_{\psi=0,\dots,\Psi-1} \cdot \boldsymbol{f},$$

$$\boldsymbol{f} = [\overline{H_\omega(\boldsymbol{x}_\psi)}]^{\mathrm{H}}_{\omega=0,\dots,\Omega-1;\,\psi=0,\dots,\Psi-1} \cdot \hat{\boldsymbol{f}},$$

respectively. Note that the quadrature weights w_ψ appear only in the forward transform. Furthermore, it becomes apparent that the inverse transform is represented by precisely the Hermitian transpose of the matrix associated with the forward transform, when the weighting of the SGL basis function samples $H_\omega(\boldsymbol{x}_\psi)$ and of the function samples $\boldsymbol{f}$ is dropped (compare also Sections 3.3.1 and 3.3.2). This is a consequence of the orthonormality of the SGL basis functions and our SGL sampling theorem. Recall that the computation of each element of the transformation matrices requires $\mathscr{O}(B)$ steps. Because the size of these matrices is both $B(B+1)(2B+1)/6 \times 8B^3$, this gives the total asymptotic complexity of $\mathscr{O}(B^7)$ of both forward and inverse transform.

Let the matrices R_l, $0 \le l < B$, and E be defined as in (30). Set further

$$Y := [\overline{Y_\nu(\boldsymbol{a}_\mu)}]_{\nu=0,\dots,B^2-1;\,\mu=0,\dots,4B^2-1},$$

$$A := \operatorname{diag}[\tilde{a}_i]_{i=0,\dots,2B-1},$$

$$C := \operatorname{diag}[c_\mu]_{\mu=0,\dots,4B^2-1},$$

and let I_N denote the $N \times N$ identity matrix ($N \in \mathbb{N}$). Note that B^2 is the total number of spherical harmonics Y_{lm}, $|m| \le l < B$, while $4B^2$ is the total number of sampling angles (ϑ_j, φ_k); $j,k = 0,\dots,2B-1$. After the separation of variables described in Section 3.3, we find that

$$\hat{\boldsymbol{f}} = P \cdot \overbrace{\begin{bmatrix} \boxed{\tilde{R}_{1-B}} & & \\ & \ddots & \\ & & \boxed{\tilde{R}_{B-1}} \end{bmatrix}}^{2B-1 \text{ blocks (see below)}} \cdot \overbrace{\{I_{B^2} \otimes E\}}^{\text{diagonal}} \cdot \overbrace{\{I_{B^2} \otimes A\}}^{\text{diagonal}} \cdot Q \cdot \overbrace{\begin{bmatrix} \boxed{Y} & & \\ & \ddots & \\ & & \boxed{Y} \end{bmatrix}}^{2B \text{ blocks of size } B^2 \times 4B^2} \cdot \overbrace{\{I_{2B} \otimes C\}}^{\text{diagonal}} \cdot \boldsymbol{f},$$

$$\boldsymbol{f} = \begin{bmatrix} \boxed{Y^{\mathrm{H}}} & & \\ & \ddots & \\ & & \boxed{Y^{\mathrm{H}}} \end{bmatrix} \cdot Q^{\mathrm{T}} \cdot \begin{bmatrix} \boxed{\tilde{R}^{\mathrm{T}}_{1-B}} & & \\ & \ddots & \\ & & \boxed{\tilde{R}^{\mathrm{T}}_{B-1}} \end{bmatrix} \cdot P^{\mathrm{T}} \cdot \hat{\boldsymbol{f}},$$

where the matrices $\tilde{R}_{1-B}, \ldots, \tilde{R}_{B-1}$ have again a block structure,

$$\tilde{R}_m = \underbrace{\begin{bmatrix} \boxed{R_{|m|}} & & \\ & \ddots & \\ & & \boxed{R_{B-1}} \end{bmatrix}}_{B-|m| \text{ blocks } R_l \text{ of size } B-l \times 2B},$$

while P and Q are suitable permutation matrices. We introduce these permutation matrices here for an improved structural depiction; they do not change the asymptotic complexity. Note that the separation variables results in a factorization of the transformation matrices. The asymptotic complexity is reduced to $\mathcal{O}(B^6)$ by evaluating the matrix-vector products successively. The factorization of the matrix of the inverse transform is obtained by taking the Hermitian transpose of the factorized matrix of the forward transform and dropping the diagonal weight matrices (compare with Section 3.3.1).

Finally, as explained in Section 3.3, a fast spherical Fourier transform and inverse can be used to reduce the asymptotic complexity to $\mathcal{O}(B^5)$. This amounts to a factorization of the matrices Y and Y^{H}. Employing the Clenshaw algorithm in the DRT and iDRT further reduces the asymptotic complexity to $\mathcal{O}(B^4)$. This amounts to a factorization of the matrices R_l and R_l^{T}, $0 \le l < B$.

4 Numerical Experiment

We realized the naive DSGLFT and iDSGLFT of Section 3.2, as well as the FSGLFT and iFSGLFT of Section 3.3 in MathWorks' Matlab R2015a. For the spherical subtransform in the FSGLFT/iFSGLFT, we implemented the seminaive spherical Fourier transform and inverse of Healy et al. [14], described in Section 3.3.1, using the built-in FFT and inverse as well as the built-in fast DCT and inverse. No parallelization was done.

In both the DSGLFT/iDSGLFT and the FSGLFT/iFSGLFT, we precomputed the sampling radii r_i and transformed sampling angles $(\cos\vartheta_j, \varphi_k)$ for the bandlimits listed below with high precision in Wolframs' Mathematica 10, and stored them double format. We did the same for the corresponding spherical quadrature weights b_j and the modified radial quadrature weights $\tilde{a}_i$. For the seminaive spherical Fourier transform and inverse, we precomputed the transformed vectors $C_{2B}\boldsymbol{P}_{lm}$ (cf. Section 3.3.1) for all bandlimits below in Matlab, and stored them in double format.

The actual testruns were performed on a Unix system with a 3.40 GHz Intel Core i7-3770 CPU. We iterated through the bandlimits $B = 2, 4, 8, 16, 32$. For each bandlimit, we generated random SGL Fourier coefficients $\hat{f}_{nlm}$. Both the real part and the imaginary part were uniformly distributed between -1 and 1. We then performed the iDSGLFT as well as the iFSGLFT on these Fourier coefficients to reconstruct the corresponding function values $f(r_i, \vartheta_j, \varphi_k)$. Subsequently, we transformed the function values back into SGL Fourier coefficients $\hat{f}^{\circ}_{nlm}$, using the DSGLFT and FSGLFT, respectively. We measured the total runtime of one forward and subsequent inverse transform, and the absolute and relative (maximum) transformation error,

$$\max_{|m| \leq l < n \leq B} |\hat{f}_{nlm} - \hat{f}^{\circ}_{nlm}| \quad \text{and} \quad \max_{|m| \leq l < n \leq B} \frac{|\hat{f}_{nlm} - \hat{f}^{\circ}_{nlm}|}{|\hat{f}_{nlm}|},$$

respectively. We repeated the above procedure 10 times, and determined the average runtime, the average absolute transformation error, and the average relative transformation error for each bandlimit. We then performed the entire testrun again for $B = 64$ with the fast transforms.

Table 1 shows the results of the error measurement. The results of the runtime measurement are listed in Table 2.

Table 1 **(First row)** average absolute and **(second row)** average relative transformation error of one inverse and subsequent forward DSGLFT/FSGLFT, respectively.

B	iDSGLFT/DSGLFT	iFSGLFT/FSGLFT
2	(5.57 ± 1.67) E -16	(3.85 ± 1.08) E -16
	(7.70 ± 1.69) E -16	(4.64 ± 1.36) E -16
4	(1.35 ± 0.25) E -15	(8.45 ± 1.23) E -16
	(3.32 ± 2.10) E -15	(2.23 ± 1.60) E -15
8	(5.45 ± 0.63) E -15	(1.66 ± 0.18) E -15
	(2.30 ± 2.18) E -14	(4.51 ± 1.11) E -15
16	(2.01 ± 0.33) E -14	(3.96 ± 0.51) E -15
	(1.99 ± 1.25) E -13	(2.98 ± 1.31) E -14
32	(6.39 ± 0.89) E -14	(6.36 ± 0.55) E -15
	(6.82 ± 1.75) E -13	(1.79 ± 1.41) E -13
64	-	(3.50 ± 0.41) E -14
	-	(8.45 ± 2.87) E -13

Table 2 Average runtime of one inverse and subsequent forward DSGLFT/FSGLFT, respectively.

B	iDSGLFT/DSGLFT	iFSGLFT/FSGLFT
2	2.34 E -2 s	7.86 E -3 s
4	3.80 E -1 s	2.93 E -2 s
8	1.56 E $+1$ s	1.44 E -1 s
16	9.10 E $+2$ s	8.68 E -1 s
32	5.27 E $+4$ s	6.00 E $+0$ s
64	-	4.85 E $+1$ s

5 Discussion, Conclusions, and Future Developments

As mentioned in Section 1, the SGL basis functions are nowadays used extensively in the simulation of *biomolecular recognition processes*, such as protein-protein or protein-ligand docking. This is due to the existence of an elaborate machinery of fast *SGL matching algorithms* (see [24, 25], and the references contained therein). All of these algorithms are spectral methods, i.e., they require the computation of the SGL Fourier coefficients of so-called *affinity functions* prior to the actual (docking) simulation. This task is currently accomplished by sampling the affinity function f of interest onto a regular Cartesian grid and using a *midpoint method* for numerical integration:

$$\hat{f}_{nlm} \approx \sum_k f(\boldsymbol{x}_k)\,\overline{H_{nlm}(\boldsymbol{x}_k)}\,\Delta V,$$

where $\boldsymbol{x}_k$ is the midpoint of the kth cell, and ΔV is the cell volume.

While this approach is easily realized and useful for moderate problem sizes, it does benefit from the special structure (2) of the SGL basis functions, and there is no guarantee for exactness. Our fast SGL Fourier transforms, on the other hand, crucially benefit from the special structure of the SGL basis functions and guarantee exactness in the sense of our SGL sampling theorem (Theorem 3.4). Specifically, the special structure of the SGL basis function allows to separate our discrete SGL Fourier transforms into a spherical and a radial subtransform, and, thus, to avoid computational redundancy to a large extent.

The results in Tables 1 and 2 clearly show that the FSGLFT and iFSGLFT tested in Section 4 work very well for all bandlimits considered: The absolute and relative transformation errors are significantly smaller than those of the naive DSGLFT and iDSGLFT. This is due to the smaller total number of operations in the fast transforms, resulting in less round-off error. The total runtime of FSGLFT and iFSGLFT was significantly lower in all cases – even for the smallest bandlimits, which is typically not the case. Since the bandlimits $B \leq 32$ are of most practical relevance, an interesting question for further research is how the SGL matching algorithms of Ritchie et al. perform in combination with these fast transforms.

In the spherical subtransform of the fast SGL Fourier transforms tested in Section 4, we used the seminaive fast spherical Fourier transform and inverse of Healy et al. [14]. The seminaive variant appears to be the optimal choice for bandlimits $B \leq 128$. For larger bandlimits, there are other variants described by Healy et al., which should be considered. Note, however, that all of these variants result in the same asymptotic complexity $\mathscr{O}(B^4)$ of our fast transforms.

The spherical quadrature rules of Driscoll and Healy [8] (Theorem 3.1) and the Gaussian quadrature rules (Theorem 3.3) yield an asymptotically optimal relation between the number of SGL Fourier coefficients ($\mathscr{O}(B^3)$) and sampling points on $\mathbb{R}^3$ ($\mathscr{O}(B^3)$). Spherical quadrature rules with a lower total number of sampling points and corresponding fast spherical Fourier transforms are described in McEwen and Wiaux [20]. These fast transforms can easily be used in our framework as well, leaving the total asymptotic complexity again untouched.

In Section 3.2, we introduced the factor $\exp(r_i^2)$ to compensate for the fast decay of the quadrature weights a_i. This modification was accounted for by weighting the SGL basis function samples $H_{nlm}(r_i, \vartheta_j, \varphi_k)$ by the factor $\exp(-r_i^2)$, which was done during runtime in Section 4. We found that such adjustment is essential for bandlimits $B \geq 64$, when working with double precision. For bandlimits $B < 64$, the above modification results in slightly lower transformation errors, which is why we used this adjustment consistently for all bandlimits. Of course, our approach requires the precomputation of the modified weights $\tilde{a}_i$ using a high precision; it does not affect the asymptotic complexity. Due to the absence of the weights a_i, there is generally no modification required in the inverse transforms.

For all bandlimits considered in this paper, the storage requirements of the precomputed data are not an issue. In the case $B = 64$, for example, the precomputed data for the FSGLFT and iFSGLFT of Section 4 require approximately 25 MB of free disk space. For completeness, we record that the storage complexity of these fast transforms is $\mathcal{O}(B^3)$ for both disk space and memory. This due to the precomputed data $C_{2B}\boldsymbol{P}_{lm}$ of Section 3.3.1, which are stored on the disk and loaded during runtime, and the data in memory being processed. In general, the disk space requirements of our fast transforms are essentially the same as those of the particular spherical Fourier transforms employed.

As a more theoretical remark, we note that it is possible to obtain a true $\mathcal{O}(B^3 \log^2 B)$ FSGLFT/iFSGLFT by using an $\mathcal{O}(B^2 \log^2 B)$ variant of the spherical Fourier transforms of Healy et al., and interchanging the Clenshaw algorithm in our discrete R transform of Section 3.3.2 with an $\mathcal{O}(B \log^2 B)$ *fast discrete polynomial transform* (refer to Driscoll et al. [9], for instance). Note, however, that the smaller asymptotic complexity has to be traded with an increased storage complexity and a larger constant prefactor in runtime. Therefore, we do not expect a benefit of this approach for the bandlimits currently used in practice.

Another interesting task for future research is to extend our fast SGL Fourier transforms in such a way that they can be applied to *scattered* (i.e., non-gridded) data. This has already been achieved successfully in the classical FFT (see, e.g., [22]), as well as in other generalized FFTs (see [17, 23], for example).

Finally, we would like to emphasize that our fast SGL Fourier transforms are *polynomial transforms*. With little adaptions, our approach can also be used for similar combinations of spherical harmonics and generalized Laguerre polynomials, such as those stated in Dunkl and Xu [11, Sect. 5.1.3], or the radially scaled SGL basis functions of Ritchie and Kemp. Moreover, the underlying domain $\mathbb{R}^3$ of our transforms is non-compact. Notably, the experiments of Section 4 are one of the first performances of generalized FFTs on a non-compact domain. Other examples in this direction can be found in Chirikjian and Kyatkin [3].

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Multiscale Radial Basis Functions

Holger Wendland

1 Introduction

For the past few decades, radial basis functions have been established as one of the main tools in multivariate approximation theory. They allow the user to build high-order, meshfree approximation spaces and provide an extremely flexible tool for the reconstruction of functions from scattered data, see [6, 15, 46].

The main ingredient are, of course, radial functions, though many of the results are also true for non-radial functions.

Definition 1. A radial function is a function $\Phi : \mathbb{R}^d \to \mathbb{R}$ of the form $\Phi(\mathbf{x}) = \phi(\|\mathbf{x}\|_2)$, $\mathbf{x} \in \mathbb{R}^d$, where $\phi : [0,\infty) \to \mathbb{R}$ is a univariate function and $\|\mathbf{x}\|_2 = (x_1^2 + \ldots + x_d^2)^{1/2}$ denotes the Euclidean norm of $\mathbf{x} \in \mathbb{R}^d$.

Sometimes, the univariate function ϕ in the above definition is referred to as the radial function but we will not do this here. We will however make the following general assumption.

Remark 1. We will always assume that the univariate function $\phi : [0,\infty) \to \mathbb{R}$ is defined on all of $\mathbb{R}$ by even extension, i.e. by $\phi(-r) := \phi(r)$ for $r > 0$.

We will say that such a $\phi : [0,\infty)$ *generates* or *induces* a multivariate function on $\mathbb{R}^d$. This induced function is simply defined to be $\Phi(\mathbf{x}) := \phi(\|\mathbf{x}\|_2)$, $\mathbf{x} \in \mathbb{R}^d$.

In this article, we will predominantly be dealing with radial functions $\Phi : \mathbb{R}^d \to \mathbb{R}$ having a compact support, i.e. with functions for which

$$\text{supp}\Phi := \overline{\{\mathbf{x} \in \mathbb{R}^d : \Phi(\mathbf{x}) \neq 0\}}$$

H. Wendland (✉)
Department of Mathematics, University of Bayreuth, 95440 Bayreuth, Germany
e-mail: holger.wendland@uni-bayreuth.de

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is bounded. In general, we will simply assume that the support of Φ is given by the unit ball. In that case, we can introduce scaled versions, i.e.

$$\Phi_\delta(\mathbf{x}) := \delta^{-d}\Phi(\mathbf{x}/\delta)$$

with $\delta > 0$, which obviously have support in the ball about the origin with radius δ. Besides scaling we can also shift a scaled basis function. This allows us to define our basic approximation spaces. To this end let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain and let $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subseteq \Omega$ be a set of distinct points, the *data sites*. Associated to such a point set are two geometric quantities, the *fill distance* or *mesh norm* $h_{X,\Omega}$ and the separation distance q_X, which are defined to be

$$h_{X,\Omega} := \sup_{\mathbf{x}\in\Omega} \min_{1\le j\le N} \|\mathbf{x} - \mathbf{x}_j\|_2,$$

$$q_X := \min_{1\le j\ne k\le N} \|\mathbf{x}_j - \mathbf{x}_k\|_2,$$

respectively. The fill distance describes how well the scattered points X "cover" the domain Ω in the following sense. The fill distance $h_{X,\Omega}$ gives the radius of the largest ball completely contained in Ω without a data site within its interior. The fill distance is important for understanding approximation orders. The separation distance defines the smallest distance between the two closest data sites. It becomes important when looking at stability issues.

A scaled, compactly supported kernel and a set of data sites are enough to define a first approximation space via

$$W_X = W_{X,\Phi_\delta} := \operatorname{span}\{\Phi_\delta(\cdot - \mathbf{x}) : \mathbf{x} \in X\}. \tag{1}$$

We will refer to this as a *local approximation space* since we will not deal with only one data set and one associated approximation space, but with a sequence of data sets $X_1, X_2, \dots$ with mesh norms $h_j = h_{X_j,\Omega}$, which are supposed to be monotonically decreasing. To ensure a certain uniformity in decrease, we will assume that $h_{j+1} \approx \mu h_j$ for some fixed $\mu \in (0,1)$. For some of our results we will require that the sequence is *quasi-uniform*, which means that there is a constant c_q such that, with $q_j = q_{X_j}$,

$$q_j \le h_j \le c_q q_j.$$

Having this sequence of data sets X_j, it is clear that we can build a local approximation space $W_j = W_{X_j}$ for each X_j. To this end, we will define a kernel $\Phi_j : \Omega \times \Omega \to \mathbb{R}$ for each level. In our application this kernel will be given by the scaled version of a fixed translation invariant radial basis function. To be more precise, we assume that there is a compactly supported function $\Phi : \mathbb{R}^d \to \mathbb{R}$ with support in the unit ball $B(\mathbf{0}, 1)$ and that, for each level, there is a scaling parameter $\delta_j > 0$ such that we can define

$$\Phi_j(\mathbf{x}, \mathbf{y}) = \delta_j^{-d}\Phi((\mathbf{x} - \mathbf{y})/\delta_j).$$

If we consider $\mathbf{y}$ to be fixed, it follows that the function $\Phi_j(\cdot, \mathbf{y})$ has support in $B(\mathbf{y}, \delta_j)$, the ball with radius δ_j and centre $\mathbf{y}$.

As we assume that the data sets become denser and denser, we will also assume that the support radii become smaller and smaller, usually in the same way, i.e., we will assume that $\delta_j = \nu h_j$ with a constant $\nu > 0$, which we will link to μ later on. Note that this also leads to $\delta_{j+1} \approx \mu \delta_j$.

With the data sets and the associated kernels at hand, we can build approximation spaces of the form

$$W_j = \text{span}\{\Phi_j(\cdot, \mathbf{x}) : \mathbf{x} \in X_j\}. \tag{2}$$

representing details on level j. Thus, the approximation of our function will come from the sum of these spaces, i.e., we have to investigate the approximation power of

$$V_n := W_1 + W_2 + \cdots + W_n$$

for $n \to \infty$.

The rest of the paper is organised as follows. In the next section, we will discuss the relevant material on optimal recovery, reproducing kernel Hilbert spaces and Sobolev spaces. After that, we will discuss compactly supported radial basis functions. In the fourth section, we will then address multiscale interpolation and approximation.

2 Optimal Recovery and Reproducing Kernel Hilbert Spaces

2.1 The Reconstruction Problem

Our main tool in using discrete approximation spaces W_X of the form (1) will be optimal recovery and interpolation, which turns out to be the same in this context. Throughout this section, we will consider only local approximation spaces W_{X,Φ_δ}. In particular, we want to keep the scaling factor δ fixed but might vary the data set X.

To understand why radial basis functions, or, more generally, positive definite functions are natural tools in multivariate approximation theory, we start with an abstract result from linear algebra.

To this end, let H be a Hilbert space and denote its dual by H^*. Suppose $\Lambda = \{\lambda_1, \dots, \lambda_N\} \subseteq H^*$ is a set of linearly independent functionals on H and $f_1, \dots, f_N \in \mathbb{R}$ are certain given values. Then a generalised recovery problem would ask for finding a function $s \in H$ so that $\lambda_j(s) = f_j$, $1 \leq j \leq N$. We will call s a *generalised interpolant*. Usually, there are an infinite number of possible generalised interpolants and hence there is need to distinguish one of them. Moreover, if the data contain noise, interpolation is not an appropriate way of reconstructing the unknown function f. Hence, we will also consider a method, which was originally introduced in the context of splines (see [42]).

Definition 2. The *norm-minimal generalised interpolant* is the element $s_0 \in H$ satisfying

$$\|s_0\| = \min\{\|s\| : s \in H, \lambda_j(s) = f_j, 1 \le j \le N\}. \tag{3}$$

Given a *smoothing parameter* $\varepsilon > 0$, the *optimal recovery* or *smoothing spline* is the element $s_\varepsilon \in H$ satisfying

$$\min\left\{\sum_{j=1}^{N} |\lambda_j(s) - f_j|^2 + \varepsilon\|s\|_H^2\right\}. \tag{4}$$

The solutions to both problems (3) and (4) are unique. A proof of the following result can be found in [42, 46].

Theorem 1. *Let H be a Hilbert space, $\lambda_1, \dots, \lambda_N \in H^*$ linearly independent functionals with Riesz representers v_j, $1, \le j \le N$, and let $f_1, \dots, f_N \in \mathbb{R}$ be given. The unique solutions s_0 of (3) and s_ε of (4) can be written as*

$$s_\varepsilon = \sum_{j=1}^{N} \alpha_j v_j, \tag{5}$$

$\varepsilon \ge 0$, *where the coefficients $\{\alpha_j\}$ are the solution of the linear system*

$$(A + \varepsilon I)\boldsymbol{\alpha} = \mathbf{f} \tag{6}$$

where $A = (\lambda_i(v_j))$ and $\mathbf{f} = (f_1, \dots, f_N)^T$.

Hence, the optimal recovery problem can constructively be solved once the Riesz representer of the functionals are known. For general Hilbert spaces of functions this is not an easy task. However, for *reproducing kernel Hilbert spaces* this becomes simple once the reproducing kernel is known.

Definition 3. A Hilbert space H of functions $f : \Omega \to \mathbb{R}$ is a reproducing kernel Hilbert space if there is a kernel $\Phi : \Omega \times \Omega \to \mathbb{R}$ with

1. $\Phi(\cdot, \mathbf{x}) \in H$ for all $\mathbf{x} \in \Omega$,
2. $f(\mathbf{x}) = \langle f, \Phi(\cdot, \mathbf{x})\rangle_H$ for all $\mathbf{x} \in \Omega$ and all $f \in H$.

In such a reproducing kernel Hilbert space, the Riesz representer of any functional $\lambda \in H^*$ is simply given by applying λ to one of the arguments of the kernel, i.e. $v_\lambda = \lambda^{\mathbf{y}}\Phi(\cdot, \mathbf{y})$. In this paper, we will exclusively be concerned with point evaluation functionals, i.e. $\lambda_j(f) = f(\mathbf{x}_j)$. Hence, the optimal recovery takes the simple form

$$s_\varepsilon = \sum_{j=1}^{N} \alpha_j \Phi(\cdot, \mathbf{x}_j)$$

and the matrix A has thus entries $\Phi(\mathbf{x}_i, \mathbf{x}_j)$.

2.2 Sobolev Spaces

The reproducing kernel Hilbert spaces we will be concerned with are Sobolev spaces. For an integer $\sigma = k \in \mathbb{N}_0$ the Sobolev space $H^k(\Omega)$ consists of all those functions $u \in L_2(\Omega)$ which have weak derivatives $D^{\boldsymbol{\alpha}} u$ in $L_2(\Omega)$ for all $|\boldsymbol{\alpha}| \leq k$. For non-integer $\sigma > 0$, the Sobolev space $H^\sigma(\Omega)$ can be defined using interpolation theory, see, for example, [1].

The Sobolev embedding theorem guarantees that $H^\sigma(\Omega)$ is a reproducing kernel Hilbert space for any $\sigma > d/2$. However, since the reproducing kernel is uniquely determined by the inner product and also depends upon Ω in our setting, it is in general not possible to give an explicit formula for the reproducing kernel. We will circumvent this problem by two means. First of all, we will assume that we can extend our functions defined on Ω to all of $\mathbb{R}^d$, see [4].

Proposition 1. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary. Then, there is a universal extension operator $E : H^\sigma(\Omega) \to H^\sigma(\mathbb{R}^d)$ satisfying*

1. $Ef|\Omega = f$
2. $\|Ef\|_{H^\sigma(\mathbb{R}^d)} \leq C_\sigma \|f\|_{H^\sigma(\Omega)}$

for all $f \in H^\sigma(\Omega)$ and all $\sigma \geq 0$.

It is important that though the constant $C_\sigma > 0$ might depend on σ, the operator E itself does not. The existence of such an operator means that we can equip $H^\sigma(\Omega)$ with an equivalent norm $f \mapsto \|Ef\|_{H^\sigma(\mathbb{R}^d)}$ and thus we can concentrate on determining the reproducing kernel of $H^\sigma(\mathbb{R}^d)$ for $\sigma > d/2$. This becomes particularly easy if we write the norm on $H^\sigma(\mathbb{R}^d)$ using the Fourier transform $\widehat{f}$ of $f \in H^\sigma(\mathbb{R}^d)$, $\sigma > d/2$ defined by

$$\widehat{f}(\boldsymbol{\omega}) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-i\mathbf{x}^T \boldsymbol{\omega}} d\mathbf{x}.$$

With this, we can define the norm on $H^\sigma(\mathbb{R}^d)$ to be

$$\|f\|^2_{H^\sigma(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\widehat{f}(\boldsymbol{\omega})|^2 (1 + \|\boldsymbol{\omega}\|_2^2)^\sigma d\boldsymbol{\omega}. \tag{7}$$

The reproducing kernel can then be written in translation-invariant form, i.e. it satisfies $\Phi(\mathbf{x}, \mathbf{y}) = \Phi_0(\mathbf{x} - \mathbf{y})$ with $\Phi_0 : \mathbb{R}^d \to \mathbb{R}$ for which we will simply write $\Phi(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y})$ in an abuse of notation. The function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is then simply the inverse Fourier transform of one over the weight function, i.e.

$$\widehat{\Phi}(\boldsymbol{\omega}) = (1 + \|\boldsymbol{\omega}\|_2^2)^{-\sigma},$$

which is a consequence of the following general result, see [46].

Proposition 2. *Suppose $\Phi \in L_1(\mathbb{R}^d)$ has a Fourier transform satisfying*

$$c_1(1 + \|\boldsymbol{\omega}\|_2^2)^{-\sigma} \leq \widehat{\Phi}(\boldsymbol{\omega}) \leq c_2(1 + \|\boldsymbol{\omega}\|_2^2)^{-\sigma} \tag{8}$$

Then, Φ also defines a reproducing kernel on $H^\sigma(\mathbb{R}^d)$ with respect to the inner product defined by

$$\langle f, g\rangle_\Phi := \int_{\mathbb{R}^d} \frac{\widehat{f}(\boldsymbol{\omega})\overline{\widehat{g}(\boldsymbol{\omega})}}{\widehat{\Phi}(\boldsymbol{\omega})} d\boldsymbol{\omega}.$$

The norm associated to this inner product is equivalent to the Sobolev norm (7).

This also opens the door to the construction of a variety of kernels, which are also reproducing kernels in $H^\sigma(\mathbb{R}^d)$. We will only consider compactly supported Φ's having a Fourier transform with such a decay. However, since for $t \geq 0$ and $\sigma \geq 0$, we have

$$1 + t^\sigma \leq (1 + t)^\sigma \leq 2^\sigma(1 + t^\sigma)$$

we see that the decay property (8) is equivalent to the decay property

$$c_1(1 + \|\boldsymbol{\omega}\|_2^{2\sigma})^{-1} \leq \widehat{\Phi}(\boldsymbol{\omega}) \leq c_2(1 + \|\boldsymbol{\omega}\|_2^{2\sigma})^{-1}, \tag{9}$$

which is slightly easier to handle in what follows. An immediate consequence of these observations is the following one.

Corollary 1. *Let $\Phi \in L_1(\mathbb{R}^d)$ be a reproducing kernel of $H^\sigma(\mathbb{R}^d)$, $\sigma > d/2$, i.e. its Fourier transform satisfies (9). For $\delta \in (0, 1]$ let $\Phi_\delta := \delta^{-d}\Phi(\cdot/\delta)$. Then, Φ_δ is also a reproducing kernel of $H^\sigma(\mathbb{R}^d)$ and for every $g \in H^\sigma(\mathbb{R}^d)$, we have the norm equivalence*

$$c_1^{1/2}\|g\|_{\Phi_\delta} \leq \|g\|_{H^\sigma(\mathbb{R}^d)} \leq c_2^{1/2}\delta^{-\sigma}\|g\|_{\Phi_\delta},$$

Proof. This follows immediately from $\widehat{\Phi_\delta} = \widehat{\Phi}(\delta\cdot)$. □

It is important to note that the Hilbert space in which the scaled kernel Φ_δ is the reproducing kernel is always $H^\sigma(\mathbb{R}^d)$, i.e. the space itself does not depend on the scaling parameter. However, the norm associated to the inner product in which Φ_δ is the reproducing kernel does depend on $\delta > 0$. Also the equivalence constants depend on δ. While we always have $\|f\|_{\Phi_\delta} \leq C\|f\|_{H^\sigma(\mathbb{R}^d)}$ with $C > 0$ being independent of $\delta > 0$, the constant in the other estimate $\|f\|_{H^\sigma(\mathbb{R}^d)} \leq C\delta^{-\sigma}\|f\|_{\Phi_\delta}$ tends to infinity with $\delta \to 0$. This is not surprising at all, since $\|\cdot\|_{\Phi_\delta} \to \|\cdot\|_{L_2(\mathbb{R}^d)}$ for $\delta \to 0$ and the constant has to balance this loss of derivative in the norm.

3 Compactly Supported Radial Basis Functions

In this section we will review recent results on compactly supported radial basis functions. We are particularly interested in the construction of such basis functions and in those compactly supported RBFs which are reproducing kernels of Sobolev spaces.

Much of the general theory in this section can be found in [48]. Newer results are based on [8, 9, 22, 35, 37, 43, 50, 51].

3.1 Construction of Compactly Supported RBFs

A famous result of Bochner states that a continuous, integrable function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is *positive definite* and hence the reproducing kernel of its associated reproducing kernel Hilbert space, if and only if it possesses a non-negative, non-vanishing Fourier transform

$$\widehat{\Phi}(\boldsymbol{\omega}) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} \Phi(\mathbf{x}) e^{-i\mathbf{x}^T\boldsymbol{\omega}} d\mathbf{x}, \qquad \boldsymbol{\omega} \in \mathbb{R}^d,$$

see [46] for a proof. For a radial, positive definite function, it is well-known that the Fourier transform is also radial and can be reduced to a one-dimensional Hankel transform. This will be essential in what we now want to do. To formulate this, we need to introduce Bessel functions of the first kind.

Definition 4. The Bessel function of the first kind of order $\nu \in \mathbb{C}$ is defined by

$$J_\nu(z) := \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}z)^{2m+\nu}}{m! \Gamma(\nu + m + 1)}, \qquad z \in \mathbb{C}.$$

This gives the desired representation in the case of a positive definite, radial function.

Theorem 2. *Suppose $\Phi \in L_1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ is radial, i.e. $\Phi(\mathbf{x}) = \phi(\|\mathbf{x}\|_2)$, $\mathbf{x} \in \mathbb{R}^d$. Then its Fourier transform $\widehat{\Phi}$ is also radial, i.e. $\widehat{\Phi}(\boldsymbol{\omega}) = \mathscr{F}_d\phi(\|\boldsymbol{\omega}\|_2)$ with*

$$\mathscr{F}_d\phi(r) = r^{-\frac{d-2}{2}} \int_0^\infty \phi(t) t^{\frac{d}{2}} J_{\frac{d-2}{2}}(rt) dt. \tag{10}$$

Hence, Φ is positive definite if and only if $\mathscr{F}_d\phi \geq 0$ and $\mathscr{F}_d\phi \not\equiv 0$.

An interesting consequence of (10) is that the operator $\mathscr{F}_d$ cannot only be defined for $d \in \mathbb{N}$ but actually for all $d \in [0, \infty)$, which we will from now on do. However, the connection to a multivariate Fourier transform is, of course, only given if $d \in \mathbb{N}$. Also, the existence of the integral has to be ensured.

Next, we introduce the key operators to construct compactly supported radial basis functions.

Definition 5. 1. Let ϕ be given so that $t \mapsto \phi(t)t$ is in $L_1[0,\infty)$, then we define for $r \geq 0$,

$$(\mathscr{I}\phi)(r) = \int_r^\infty t\phi(t)dt.$$

2. For even $\phi \in C^2(\mathbb{R})$ we define for $r \geq 0$,

$$(\mathscr{D}\phi)(r) = -\frac{1}{r}\phi'(r).$$

In both cases the resulting functions should be seen as even functions by even extension.

Obviously, these operators are inverse to each other in the sense that $\mathscr{I}\mathscr{D} = I$ and $\mathscr{D}\mathscr{I} = I$ whenever these operations are well-defined, where I denotes the identity operator.

The importance of these operators comes from the fact that they nicely work together with the Fourier transform operator $\mathscr{F}_d$. We will formulate this in a moment but before that, we want to generalise them.

Definition 6. Let $\alpha > 0$ and assume that $t \mapsto t\phi(t)$ is integrable over $[0,\infty)$. Then, we define for $r \geq 0$,

$$\mathscr{I}_\alpha\phi(r) := \frac{1}{2^{\alpha-1}\Gamma(\alpha)}\int_r^\infty \phi(t)t(t^2-r^2)^{\alpha-1}dt$$

and for $r < 0$ the function $\mathscr{I}_\alpha\phi$ is defined by even extension.

Note that we can rewrite the integral in the definition as

$$\int_r^\infty \phi(t)t(t+r)^{\alpha-1}(t-r)^{\alpha-1}dt.$$

Hence, for $\alpha \in (0,1)$, the singularity at $t = r$ is integrable. Since we will be merely be concerned with continuous compactly supported functions in this paper, the operator $\mathscr{I}_\alpha$ can always be applied to such a function for any $\alpha > 0$.

We also see that for $\alpha = 1$ we have $\mathscr{I}_1 = \mathscr{I}$, i.e. both definitions coincide. Finally, we will particularly be interested in the case $\alpha = 1/2$. In this case the operator reduces to

$$\mathscr{I}_{1/2}\phi(r) = \sqrt{\frac{2}{\pi}}\int_r^\infty \phi(t)t(t^2-r^2)^{-1/2}dt.$$

Since we are merely interested in compactly supported functions, it is important for us to note that $\mathscr{I}_\alpha$ respects a compact support in the following sense.

Lemma 1. *Assume that the even function $\phi : \mathbb{R} \to \mathbb{R}$ has a compact support contained in the interval $[-R, R]$. Then, for each $\alpha > 0$ the function $\mathscr{I}_\alpha \phi$ has also compact support contained in $[-R, R]$.*

We are now going to extend the range of possible operators $\mathscr{I}_\alpha$ by defining them also for $\alpha \le 0$. We will do this first formally in the following definition.

Definition 7. 1. We define $\mathscr{I}_0$ to be the identity operator, i.e. $\mathscr{I}_0\phi = \phi$ for all even functions $\phi : \mathbb{R} \to \mathbb{R}$.
2. For $n \in \mathbb{N}$ we define $\mathscr{I}_{-n} := \mathscr{D}^n$.
3. For $\alpha > 0$ we let $k := \lceil \alpha \rceil$ and define $\mathscr{I}_{-\alpha} := \mathscr{I}_{-k}\mathscr{I}_{k-\alpha}$.

To see to what functions ϕ we can apply these operators, we restrict ourselves to compactly supported and even functions. If such a function possesses $\ell + 1$ continuous derivatives then we first note that the derivatives must satisfy $\phi^{(j)}(r) = (-1)^j \phi^{(j)}(-r)$. In particular, we see that for odd j the derivative at zero has to vanish. The Taylor expansion about the origin is thus

$$\begin{aligned} \phi(r) &= \sum_{j=0}^{\ell} \frac{\phi^{(j)}(0)}{j!} r^j + \frac{1}{\ell!}\int_0^r (r-t)^\ell \phi^{(\ell+1)}(t)dt \\ &= \sum_{j=0}^{\lfloor \ell/2 \rfloor} \frac{\phi^{(2j)}(0)}{(2j)!} r^{2j} + \frac{1}{\ell!}\int_0^r (r-t)^\ell \phi^{(\ell+1)}(t)dt. \end{aligned}$$

Applying $\mathscr{I}_{-1} = \mathscr{D}$ to this expansion yields

$$\begin{aligned} \mathscr{I}_{-1}\phi(r) &= -\frac{1}{r}\frac{d}{dr}\phi(r) \\ &= -\sum_{j=1}^{\lfloor \ell/2 \rfloor} \frac{\phi^{(2j)}(0)}{(2j-1)!} r^{2j-2} - \frac{1}{(\ell-1)!}\frac{1}{r}\int_0^r (r-t)^{\ell-1} \phi^{(\ell+1)}(t)dt. \end{aligned}$$

The critical term on the right-hand side behaves like

$$\lim_{r\to 0} \frac{1}{r}\int_0^r (r-t)^{\ell-1}\phi^{(\ell+1)}(t)dt = \lim_{r\to 0}(\ell-1)\int_0^r (r-t)^{\ell-2}\phi^{(\ell+1)}(t)dt = 0$$

as long as $\ell \ge 2$ and like $\phi^{(2)}(0)$ for $\ell = 1$. Hence, for even $\phi \in C^2(\mathbb{R})$ with compact support $\mathscr{I}_{-1}\phi$ is well-defined and continuous. We can iterate this process to derive the following result.

Corollary 2. *Let $\alpha > 0$ and $k = \lceil \alpha \rceil$. If $\phi \in C^{2k}(\mathbb{R})$ is even and compactly supported, then $\mathscr{I}_{-\alpha}\phi$ is well-defined, continuous and compactly supported.*

However, more important to us is the following observation, which can be derived from the above corollary but also straight-forward by induction.

Corollary 3. *If $k \in \mathbb{N}$, then for every even $\phi \in C(\mathbb{R})$ with compact support, $\psi := \mathscr{I}_k\phi$ and $\mathscr{I}_{-k}\psi$ are well-defined and satisfy*

$$\mathscr{I}_{-k}\mathscr{I}_k\phi = \phi.$$

Proof. We already know that ψ is well-defined. To see that $\mathscr{I}_{-k}\psi$ is also well-defined, we first recall that the fundamental theorem of calculus immediately yields $\mathscr{I}_{-1}\mathscr{I}_1\phi = \phi$. For $k \geq 2$, we first compute

$$\begin{aligned}
\mathscr{I}_{-1}\mathscr{I}_k\phi(r) &= -\frac{1}{2^{k-1}(k-1)!}\frac{1}{r}\frac{d}{dr}\int_r^\infty \phi(t)t(t^2-r^2)^{k-1}dt \\
&= \frac{2r(k-1)}{2^{k-1}(k-1)!}\frac{1}{r}\int_r^\infty \phi(t)t(t^2-r^2)^{k-2}dt \\
&= \frac{1}{2^{k-2}(k-2)!}\int_r^\infty \phi(t)t(t^2-r^2)^{k-2}dt \\
&= \mathscr{I}_{k-1}\phi(r)
\end{aligned}$$

and the rest follows by induction. □

To understand the interaction of the operators $\mathscr{I}_\alpha$ and $\mathscr{F}_d$, we need the following relation on Bessel functions.

Lemma 2. *For $\mu > -1/2$ and $\nu > -1$ we have*

$$J_{\mu+\nu+1}(t) = \frac{t^{\nu+1}}{2^\nu \Gamma(\nu+1)}\int_0^1 J_\mu(tu)u^{\mu+1}(1-u^2)^\nu du, \qquad t > 0.$$

The proof of this lemma starts with the integral on the right-hand side. It uses the definition of the Bessel function as an infinite series, exchanges integration and summation, integrates the resulting terms and interprets the result as a Bessel function again. Details can be found in [38, Lemma 4.13].

Theorem 3. *For each $d, \alpha > 0$ we have*

$$\mathscr{F}_d\mathscr{I}_\alpha\phi = \mathscr{F}_{d+2\alpha}\phi \qquad (11)$$

Hence, ϕ defines a positive definite function on $\mathbb{R}^{d+2\alpha}$ if and only if $\mathscr{I}_\alpha$ defines a positive definite function on $\mathbb{R}^d$.

Proof. We compute $\mathscr{F}_d\mathscr{I}_\alpha$ using Theorem 2 and Fubini's theorem

$$\begin{aligned}\mathscr{F}_d \mathscr{I}_\alpha \phi(r) &= \frac{1}{2^{\alpha-1}\Gamma(\alpha)} r^{-\frac{d-2}{2}} \int_0^\infty \int_t^\infty \phi(s)s(s^2-t^2)^{\alpha-1} t^{\frac{d}{2}} J_{\frac{d-2}{2}}(rt)dsdt \\ &= \frac{1}{2^{\alpha-1}\Gamma(\alpha)} r^{-\frac{d-2}{2}} \int_0^\infty \phi(s)s \int_0^s (s^2-t^2)^{\alpha-1} t^{\frac{d}{2}} J_{\frac{d-2}{2}}(rt)dtds.\end{aligned}$$

After a change of variables $u := t/s$, the inner integral can be computed using Lemma 2:

$$\begin{aligned}\int_0^s (s^2-t^2)^{\alpha-1} t^{\frac{d}{2}} J_{\frac{d-2}{2}}(rt)dt &= \int_0^1 (s^2-u^2s^2)^{\alpha-1}(us)^{\frac{d}{2}} J_{\frac{d-2}{2}}(rsu)sdu \\ &= s^{2\alpha-1+\frac{d}{2}} \int_0^1 (1-u^2)^{\alpha-1} u^{\frac{d}{2}} J_{\frac{d-2}{2}}(rsu)du \\ &= s^{2\alpha-1+\frac{d}{2}} 2^{\alpha-1}\Gamma(\alpha)(rs)^{-\alpha} J_{\frac{d-2}{2}+\alpha}(rs).\end{aligned}$$

Inserting this into the above formula for $\mathscr{F}_d\mathscr{I}_\alpha$ immediately yields

$$\begin{aligned}\mathscr{F}_d \mathscr{I}_\alpha \phi(r) &= r^{-\frac{d-2}{2}-\alpha} \int_0^\infty \phi(s)ss^{\alpha-1+\frac{d}{2}} J_{\frac{d-2}{2}+\alpha}(rs)ds \\ &= r^{-\frac{d+2\alpha-2}{2}} \int_0^\infty \phi(s)s^{\frac{d+2\alpha}{2}} J_{\frac{d+2\alpha-2}{2}}(rs)ds \\ &= \mathscr{F}_{d+2\alpha}\phi(r).\end{aligned}$$

□

This means particularly for the operators $\mathscr{I} = \mathscr{I}_1$ and $\mathscr{I}_{1/2}$ the following.

Corollary 4. • *A function ϕ induces a positive definite function on $\mathbb{R}^{d+2}$ if and only if $\mathscr{I}\phi$ induces a positive definite function on $\mathbb{R}^d$.*
- *A function ϕ induces a positive definite function on $\mathbb{R}^{d+1}$ if and only if $\mathscr{I}_{1/2}\phi$ induces a positive definite function on $\mathbb{R}^d$.*

However, (11) has another important consequence. If $\phi \in C(\mathbb{R})$ has compact support, then we can interpret the classical Fourier transform of ϕ also in the L_2 sense. In the L_2-sense, ϕ is uniquely determined by its Fourier transform and hence we have the following result.

Corollary 5. *Let $\alpha, \beta > 0$. Assume that $\phi \in C(\mathbb{R})$ is even and has compact support then we have*

$$\mathscr{I}_\alpha \mathscr{I}_\beta \phi = \mathscr{I}_{\alpha+\beta}\phi = \mathscr{I}_\beta \mathscr{I}_\alpha \tag{12}$$

and

$$\mathscr{I}_{-\alpha}\mathscr{I}_\alpha \phi = \phi. \tag{13}$$

Proof. Using (11) yields

$$\mathscr{F}_d \mathscr{I}_\alpha \mathscr{I}_\beta \phi = \mathscr{F}_{d+2\alpha} \mathscr{I}_\beta \phi = \mathscr{F}_{d+2\alpha+2\beta} \phi.$$

The other expressions in the first identity have the same Fourier transform and hence all of the stated functions must be the same.

To prove the second equality, let $k = \lceil \alpha \rceil$. By definition, we have $\mathscr{I}_{-\alpha} = \mathscr{I}_{-k} \mathscr{I}_{k-\alpha}$. Hence,

$$\mathscr{I}_{-\alpha} \mathscr{I}_\alpha \phi = \mathscr{I}_{-k} \mathscr{I}_{k-\alpha} \mathscr{I}_\alpha \phi = \mathscr{I}_{-k} \mathscr{I}_k \phi = \phi,$$

where we first have used (12) and then Corollary 3. □

3.2 Wendland Functions and Generalised Wendland Functions

We will now describe a popular class of compactly supported radial basis functions, which is widely used in applications. The starting point is the cut-off power function

$$\phi_\nu(r) = (1 - r)_+^\nu \tag{14}$$

which is known to induce a positive definite function on $\mathbb{R}^d$ if $\nu \geq (d + 1)/2$, see [3, 46].

Definition 8. The Wendland function of smoothness $2k$ for space dimension d is defined by

$$\begin{aligned} \phi_{d,k}(r) &= \mathscr{I}_k \phi_{\lfloor d/2 \rfloor + k + 1}(r) \\ &= \frac{1}{2^{k-1}(k-1)!} \int_r^1 (1-t)^{\lfloor d/2 \rfloor + k + 1} t (t^2 - r^2)^{k-1} dt, \qquad 0 \leq r \leq 1. \end{aligned}$$

The definition shows that we will have the same function for even $d = 2n$ and odd $d = 2n + 1$. We will address this issue later on. However, it is also clear from this definition that $\phi_{d,k}$ restricted to $[0, 1]$ is a polynomial of degree $\lfloor d/2 \rfloor + 3k + 1$, which can easily be computed. Some of them are, up to a constant factor, listed in Table 1.

The reason why these functions have become quite popular is summarised in the following theorem.

Theorem 4. *The functions $\phi_{d,k}$ are positive definite on $\mathbb{R}^d$ and are of the form*

$$\phi_{d,k}(r) = \begin{cases} p_{d,k}(r), & 0 \leq r \leq 1, \\ 0, & r > 1, \end{cases}$$

Table 1 Wendland functions.

Space dimension	Function	Smoothness
$d=1$	$\phi_{1,0}(r)=(1-r)_+$	C^0
	$\phi_{1,1}(r)\doteq(1-r)_+^3(3r+1)$	C^2
	$\phi_{1,2}(r)\doteq(1-r)_+^5(8r^2+5r+1)$	C^4
$d\le 3$	$\phi_{3,0}(r)=(1-r)_+^2$	C^0
	$\phi_{3,1}(r)\doteq(1-r)_+^4(4r+1)$	C^2
	$\phi_{3,2}(r)\doteq(1-r)_+^6(35r^2+18r+3)$	C^4
	$\phi_{3,3}(r)\doteq(1-r)_+^8(32r^3+25r^2+8r+1)$	C^6
$d\le 5$	$\phi_{5,0}(r)=(1-r)_+^3$	C^0
	$\phi_{5,1}(r)\doteq(1-r)_+^5(5r+1)$	C^2
	$\phi_{5,2}(r)\doteq(1-r)_+^7(16r^2+7r+1)$	C^4

with a univariate polynomial $p_{d,k}$ of degree $\lfloor d/2\rfloor+3k+1$. They possess continuous derivatives up to order $2k$. The degree of $p_{d,k}$ is minimal for a given space dimension d and smoothness $2k$ and are up to a constant factor uniquely determined by this setting.

The above defined functions have been generalised in the following way, see [8, 9, 22, 35].

Definition 9. Let $\alpha > 0$ and $\nu > -1$ such that $\alpha+\nu > 0$. The generalised Wendland functions are defined to be

$$\psi_{\nu,\alpha}(r) := \mathscr{I}_\alpha\phi_\nu(r) = \frac{1}{2^{\alpha-1}\Gamma(\alpha)}\int_r^1 (1-t)^\nu t(t^2-r^2)^{\alpha-1}dt.$$

Obviously, we have

$$\phi_{d,k} = \psi_{\lfloor d/2\rfloor+k+1,k}$$

so that the new functions are indeed a generalisation of the old once. However, for arbitrary α and ν, we can neither expect $\psi_{\nu,\alpha}$ to be positive definite on $\mathbb{R}^d$ nor will $\psi_{\nu,\alpha}$ in general be representable by a univariate polynomial within its support.

Nonetheless, using the machinery so far, we can compute the Fourier transform of these functions as

$$\begin{aligned}\mathscr{F}_d\psi_{\nu,\alpha}(r) &= \mathscr{F}_{d+2\alpha}\phi_\nu(r) = r^{-(d-2)/2-\alpha}\int_0^1 s^{d/2+\alpha}(1-s)^\nu J_{(d-2)/2+\alpha}(rs)ds\\ &= r^{-d-2\alpha-\nu}\int_0^r t^{d/2+\alpha}(r-t)^\nu J_{(d-2)/2+\alpha}(t)dt.\end{aligned}$$

The latter integral has intensively been studied and can be expressed using hypergeometric series. Recall that such a series is formally defined to be

$$
\begin{aligned}
{}_pF_q(z) &\equiv {}_pF_q[a_1,\ldots,a_p;b_1,\ldots,b_q;z] \\
&:= \sum_{n=0}^{\infty} \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\frac{z^n}{n!} =: \sum_{n=0}^{\infty}\frac{(\mathbf{a})_n}{(\mathbf{b})_n}\frac{z^n}{n!},
\end{aligned}
$$

where we assume that neither of the $b_1,\ldots b_q$ is a non-positive integer and use the Pochhammer symbol defined as $(a)_n := 1$ for $n=0$ and $(a)_n := a(a+1)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ and $(\mathbf{a})_n := (a_1)_n\cdots(a_p)_n$. Such a series is known to converge point-wise if $p \le q$, which is the case we are interested in.

A few simple properties and the connection to the integrals we are interested in are collected in the next lemma.

Lemma 3. *1. If $a_p = b_q$, then ${}_pF_q = {}_{p-1}F_{q-1}$.*
2. For $\beta+\mu > -1$ and $\lambda > -1$ we have

$$
\begin{aligned}
\int_0^r (r-t)^\lambda t^\mu J_\beta(t)dt &= \frac{\Gamma(\lambda+1)\Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\beta+\lambda+\mu+2)} 2^{-\beta} r^{\beta+\lambda+\mu+1} \\
&\cdot {}_2F_3\left[\frac{\beta+\mu+1}{2},\frac{\beta+\mu+2}{2};\beta+1,\frac{\beta+\lambda+\mu+2}{2},\frac{\beta+\lambda+\mu+3}{2};-r^2/4\right].
\end{aligned}
$$

3. The derivatives of the hyper-geometric functions satisfy for $n \in \mathbb{N}$:

$$
\begin{aligned}
&\frac{d^n}{dz^n}{}_pF_q[a_1,\ldots,a_p;b_1,\ldots,b_q;z] \\
&= \frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}{}_pF_q[a_1+n,\ldots a_p+n;b_1+n,\ldots,b_q+n;z].
\end{aligned}
$$

4. For $r > 0$ we have

$$
{}_1F_2\left[a;a+\frac{b}{2},a+\frac{b+1}{2};-r^2/4\right] > 0
$$

if a and b satisfy $b \ge 2a \ge 0$ or $b \ge a \ge 1$ or $0 \le a \le 1, b \ge 1$. The hypergeometric series cannot be strictly positive if $0 \le b < a$ or if $a = b \in (0,1)$.

Proof. The first statement is obvious, the third statement can simply be checked by differentiation under the sum. The second statement can be found, for example, in [11, 13.1(56)]. The final statement is a consequence of the findings in [30]. □

Hence, we can continue to compute our Fourier transform. Setting $\mu := d/2+\alpha$, $\lambda := \nu$ and $\beta := (d-2)/2+\alpha$, we see that the second statement of the lemma yields

$$\begin{aligned}
\mathscr{F}_d\psi_{\nu,\alpha}(r) &= r^{-d-2\alpha-\nu}\int_0^r t^{d/2+\alpha}(r-t)^\nu J_{(d-2)/2+\alpha}(t)dt \\
&= \frac{\Gamma(\nu+1)\Gamma(d+2\alpha)}{2^{d/2+\alpha-1}\Gamma(d/2+\alpha)\Gamma(d+2\alpha+\nu+1)} \\
&\quad \cdot {}_2F_3\left[\frac{d}{2}+\alpha,\frac{d+1}{2}+\alpha;\frac{d}{2}+\alpha,\frac{d+\nu+1}{2}+\alpha,\frac{d+\nu+2}{2}+\alpha;-\frac{r^2}{4}\right] \\
&= C^d_{\nu,\alpha}\,{}_1F_2\left[\frac{d+1}{2}+\alpha;\frac{d+\nu+1}{2}+\alpha,\frac{d+\nu+2}{2}+\alpha;-\frac{r^2}{4}\right].
\end{aligned}$$

with

$$C^d_{\nu,\alpha} := \frac{\Gamma(\nu+1)\Gamma(d+2\alpha)}{2^{d/2+\alpha-1}\Gamma(d/2+\alpha)\Gamma(d+2\alpha+\nu+1)}. \tag{15}$$

This allows us to make the following general statement on the generalised Wendland functions, which comes from [8].

Theorem 5. *Let $\alpha > 0$ and $\nu > -1$ with $\alpha + \nu > 0$. Then, the d-variate Fourier transform of the generalised Wendland function is given by*

$$\mathscr{F}_d\psi_{\nu,\alpha}(r) = C^d_{\nu,\alpha}\ {}_1F_2\left[\frac{d+1}{2}+\alpha;\frac{d+\nu+1}{2}+\alpha,\frac{d+\nu+2}{2}+\alpha;-\frac{r^2}{4}\right].$$

Hence, $\psi_{\nu,\alpha}$ induces a positive definite function on $\mathbb{R}^d$ if

$$\nu \ge \frac{d+1}{2}+\alpha. \tag{16}$$

Furthermore, the Fourier transform satisfies

$$\frac{d}{dr}\mathscr{F}_d\psi_{\nu,\alpha}(r) = -r\mathscr{F}_d\psi_{\nu,\alpha+1}(r). \tag{17}$$

Proof. Setting $a = \frac{d+1}{2}+\alpha$ and $b = \nu$ shows that (16) is equivalent to $b \ge a \ge 1$. Hence, we can conclude from Lemma 3 that $\psi_{\nu,\alpha}$ is positive definite. To see (17), we also use Lemma 3, which yields

$$\begin{aligned}
\frac{d}{dr}\mathscr{F}_d\psi_{\nu,\alpha}(r) &= C^d_{\nu,\alpha}\frac{d}{dr}{}_1F_2\left[\frac{d+1}{2}+\alpha;\frac{d+\nu+1}{2}+\alpha,\frac{d+\nu+2}{2}+\alpha;-\frac{r^2}{4}\right] \\
&= -\frac{r}{2}C^d_{\nu,\alpha}\frac{\frac{d+1}{2}+\alpha}{\left(\frac{d+\nu+1}{2}+\alpha\right)\left(\frac{d+\nu+2}{2}+\alpha\right)}
\end{aligned}$$

$$\cdot {}_1F_2\left[\frac{d+1}{2}+\alpha+1;\frac{d+\nu+1}{2}+\alpha+1\frac{d+\nu+2}{2}+\alpha+1;-\frac{r^2}{4}\right]$$

$$= -r\frac{C^d_{\nu,\alpha}}{C^d_{\nu,\alpha+1}}\frac{d+2\alpha+1}{(d+\nu+2\alpha+1)(d+\nu+2\alpha+2)}\mathscr{F}_d\psi_{\nu,\alpha+1}(r)$$

$$= -r\mathscr{F}_d\psi_{\nu,\alpha+1}(r),$$

where the constant expression in the last but one line can be simplified to 1 using the recurrence formula of the Γ-function. □

Note that the last property can also be expressed as $\mathscr{I}_{-1}\mathscr{F}_d\psi_{\nu,\alpha} = \mathscr{F}_d\psi_{\nu,\alpha+1}$. It also has the following interesting consequence.

Corollary 6. *The Fourier transform of the generalised Wendland function $\psi_{\nu,\alpha}$ is monotonically decreasing if*

$$\nu \geq \frac{d+1}{2}+\alpha+1. \tag{18}$$

Since the classical Wendland functions $\phi_{d,k}$ are a special case of the generalised Wendland functions $\psi_{\nu,\alpha}$, using the parameters $\nu = \lfloor d/2\rfloor + k + 1$ and $\alpha = k$, we see that (16) is satisfied, i.e. we have an alternative proof for them being positive definite. However, for $d \geq n$, the function $\phi_{d,k}$ induces not only a positive definite function on $\mathbb{R}^d$ but also on all $\mathbb{R}^n$ with $n \leq d$ and the monotony condition (18) becomes

$$\lfloor d/2\rfloor + k + 1 \geq \frac{n+1}{2} + k + 1$$

or

$$\lfloor d/2\rfloor \geq \frac{n+1}{2}.$$

Corollary 7. *The Wendland function $\phi_{d,k}$ induces a positive definite function on $\mathbb{R}^n$ with a monotonically decreasing Fourier transform for all $n \leq 2\lfloor d/2\rfloor - 1$.*

In [9], there is also a discussion of the decay of the Fourier transform of the generalised Wendland functions. This generalises earlier results from [44].

Theorem 6. *The d-dimensional Fourier transform $\mathscr{F}_d\psi_{\mu,\alpha}$ of the generalised Wendland function $\psi_{\nu,\alpha}$ with $\nu \geq \alpha + \frac{d+1}{2}$ and $\alpha > 0$ satisfies*

$$c_1(1+r^2)^{-\alpha-\frac{d+1}{2}} \leq \mathscr{F}_d\psi_{\nu,\alpha}(r) \leq c_2(1+r^2)^{-\alpha-\frac{d+1}{2}}, \qquad r \in \mathbb{R}$$

with certain constants $c_1, c_2 > 0$ independent of r. Hence, $\psi_{\nu,\alpha}$ defines a reproducing kernel of

$$H^{\alpha+\frac{d+1}{2}}(\mathbb{R}^d)$$

As mentioned above, this recovers and generalises the decay rate established in [44] for $\mathscr{F}_d\phi_{d,k}$, showing that they are reproducing kernels for $H^\sigma(\mathbb{R}^d)$ with $\sigma = \frac{d+1}{2}+k$. While these are integer order Sobolev spaces in odd space dimensions, they are of order integer plus a half in even dimensions. But we are now also in the situation to explicitly state compactly supported RBFs which are reproducing kernels in Sobolev spaces of integer order in even dimensions. We only have to work with $\alpha = k - 1/2$, $k \in \mathbb{N}$, and $\nu \ge k + d/2$ to have a kernel for $H^{k+d/2}(\mathbb{R}^d)$. These kernels have a more complicated structure than the original Wendland kernels but are still easily computable. They can best be described by introducing the elementary functions

$$S(r) := \sqrt{1-r^2}, \qquad L(r) := \log\left(\frac{r}{1+S(r)}\right), \qquad r \in (0,1).$$

Then, some of the functions together with the space dimension d and the order σ of the Sobolev space in which they are reproducing kernels are, up to a constant factor, given in Table 2. We have always chosen $\nu = k + d/2 = \sigma$, since the decay of the Fourier transform and hence the Sobolev space is independent of ν as long as $\nu \ge \alpha + (d+1)/2 = k + d/2$.

Yet another consequence is the existence of compactly supported reproducing kernels in Sobolev spaces.

Corollary 8. *Each $H^\sigma(\mathbb{R}^d)$ with $\sigma > d/2$ possesses a compactly supported, radial reproducing kernel.*

Proof. The results of Theorem 6 show that $H^\sigma(\mathbb{R}^d)$ has such a reproducing kernel, namely $\psi_{\nu,\alpha}$ as long as $\sigma = \alpha - \frac{d+1}{2} > \frac{d+1}{2}$. The case of $\sigma \in (d/2, (d+1)/2)$ cannot be handled with this technique but follows from another construction technique in [23]. □

Table 2 The missing Wendland functions.

Space dimension	Function	Sobolev
$d=2$	$\psi_{2,1/2}(r) \doteq 3r^2L(r) + (2r^2+1)S(r)$	$H^2(\mathbb{R}^2)$
	$\psi_{3,3/2}(r) \doteq -15r^4(r^2+6)L(r) - (81r^4+28r^2-4)S(r)$	$H^3(\mathbb{R}^2)$
	$\psi_{4,5/2}(r) \doteq 315r^6(3r^2+8)L(r) + (256r^8+2639r^6+690r^4-136r^2+16)S(r)$	$H^4(\mathbb{R}^2)$
$d=4$	$\psi_{3,1/2}(r) \doteq 3r^2(r^2+4)L(r) + (13r^2+2)S(r)$	$H^3(\mathbb{R}^4)$
	$\psi_{4,3/2}(r) \doteq -105r^4(r^2+2)L(r) - (32r^6+247r^4+40r^2-4)S(r)$	$H^4(\mathbb{R}^4)$
	$\psi_{5,5/2}(r) \doteq 315r^6(3r^4+60r^2+80)L(r) + (9295r^8+31670r^6+4704r^4-688r^2+64)S(r)$	$H^5(\mathbb{R}^4)$
$d=6$	$\psi_{4,1/2}(r) = 15r^2(3r^2+4)L(r) + (16r^4+83r^2+6)S(r)$	$H^4(\mathbb{R}^6)$
	$\psi_{5,3/2}(r) \doteq -105r^4(r^4+16r^2+16)L(r) - (919r^6+2346r^4+216r^2-16)S(r)$	$H^5(\mathbb{R}^6)$

As a matter of fact, in [23] Johnson constructed compactly supported radial functions ϕ having a d-variate Fourier transform satisfying (8) for a $\sigma = k \in \mathbb{N}$ with $k \geq d/4$ if d is even and $k \geq \max((d+1)/4, 2)$ if d is odd.

For $k \in [d/4, d/2]$ for even d and $k \in [(d+1)/4, d/2]$ for odd d, this seems at first to be problematic, since it is well-known that an integrable, continuous function with a non-negative Fourier transform has automatically an integrable Fourier transform (see [46, Corollary 6.12]). The resolution of this seeming contradiction is quite simple, for such values of k the constructed ϕ in [23] is not continuous on all of $\mathbb{R}$ any more, it might even have a pole at the origin.

When it comes to the actual computation of the Fourier transform of a generalised Wendland function, it is better to reduce the d-variate Fourier transform to a univariate one:

$$\begin{aligned}\mathscr{F}_d\psi_{\nu,\alpha}(r) &= \mathscr{F}_1\mathscr{I}_{\alpha+\frac{d-1}{2}}\phi_\nu(r)\\ &= \frac{\sqrt{2/\pi}}{2^{\alpha-1+\frac{d-1}{2}}\Gamma(\alpha+\frac{d-1}{2})}\int_0^1\int_s^1(1-t)^\nu t(t^2-s^2)^{\alpha-1+\frac{d-1}{2}}\cos(rs)dtds.\end{aligned}$$

Specifying this to the case of $\phi_{d,k}$ shows that we will naturally have a different Fourier transform for d even or d odd. The Fourier transform for odd d is easily calculated for the classical Wendland functions. As an example, we give the three-dimensional Fourier transform of

$$\phi_{d,1}(r) = \frac{1}{20}(1-r)^4(4r+1).$$

Lemma 4. *The Fourier transform of $\phi_{d,1}$ for $d = 3$ is given as*

$$\mathscr{F}_3\phi_{3,1}(r) = -\frac{6\sqrt{2}}{r^8\sqrt{\pi}}\left(r^2-24)\cos r - 9r\sin r - 4r^2 + 24\right).$$

Clearly, we see that the Fourier transform decays like $\mathscr{O}(r^{-6})$, so that the corresponding Sobolev space is $H^3(\mathbb{R}^3)$ as predicted. However, note that the numerical evaluation near the origin is highly ill-conditioned because of severe cancellation.

3.3 Other Compactly Supported Radial Basis Functions

Besides the functions introduced and discussed in the previous sections, there are plenty of others which can be found in the literature. There is the construction by Buhmann in [5], which produces also "one piece" radial functions, but the decay of their Fourier transform is unknown. There are the earlier constructions by Wu in [50], which are also one piece polynomials but with a higher degree than those mentioned above, having also a Fourier transform with isolated zeros. Then, there is the construction by Johnson in [2, 23], which produces functions with multiple radial pieces. Each of these pieces is poly-harmonic. The functions are again reproducing kernels in integer order Sobolev spaces.

4 Multiscale Interpolation and Approximation

We are now coming to the second main part of this paper, the discussion of multiscale approximation using compactly supported radial basis functions.

Let us first point out why we need a multiscale approach when working with compactly supported radial basis functions. We start with the following negative result (see, for example, [47, 49]).

Recall that the norm-minimal interpolant $s = s_0 = I_{X,\Phi_\delta}$ to a function $f \in C(\Omega)$, which is based upon the data sites $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subseteq \Omega$ and uses the basis function $\Phi_\delta : \mathbb{R}^d \to \mathbb{R}$ can be written as

$$I_{X,\Phi_\delta} f(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \Phi_\delta(\mathbf{x} - \mathbf{x}_j), \qquad \mathbf{x} \in \Omega.$$

where the coefficients $\boldsymbol{\alpha} \in \mathbb{R}^N$ are determined by the linear system $A\boldsymbol{\alpha} = \mathbf{f}$ with the interpolation matrix $A = (\Phi_\delta(\mathbf{x}_i - \mathbf{x}_j))$. From now on, we will refer to this norm-minimal interpolant just as *the interpolant*.

Theorem 7. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary. Let Φ be a reproducing kernel of $H^\sigma(\mathbb{R}^d)$ with $\sigma > d/2$, i.e. $\widehat{\Phi}$ satisfies (9). Let $\Phi_\delta = \delta^{-d}\Phi(\cdot/\delta)$. Finally, let $X = \{\mathbf{x}_1, \dots, \mathbf{x}_N\} \subseteq \Omega$ be given. Then, there is a constant $C > 0$ such that the error between any $f \in H^\sigma(\Omega)$ and its interpolant $s = I_{X,\Phi_\delta} f$ can be bounded by*

$$\|f - I_{X,\Phi_\delta} f\|_{L_2(\Omega)} \le C \left(\frac{h_{X,\Omega}}{\delta}\right)^{\sigma} \|f\|_{H^\sigma(\Omega)}. \tag{19}$$

Consequently, if we denote the interpolant to a function f from our local approximation spaces W_j from (2) as s_j and choose the support radii δ_j proportional to the fill distances $h_j = h_{X_j,\Omega}$, we cannot expect s_j to converge to f with $j \to \infty$. We can only expect convergence if $h_j/\delta_j \to 0$ for $j \to \infty$, which means that we essentially lose the advantage of the compact support, since the interpolation matrices become more and more dense.

4.1 Quasi-Interpolation, Principle Shift-Invariant Spaces and the Strang-Fix Conditions

Even in the more ideal situation of infinite, regular data sites and even if the interpolation process is replaced by a best approximation process the above negative result remains true. This follows from the following short discussion on quasi-interpolation in principle shift-invariant spaces.

For a compactly supported, continuous function $\Phi : \mathbb{R}^d \to \mathbb{R}$ and $h > 0$, we define the space

$$S_h := \operatorname{span}\left\{\Phi(h^{-1} \cdot -\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathbb{Z}^d\right\} = \operatorname{span}\{\Phi_h(\cdot - h\boldsymbol{\alpha}) : \boldsymbol{\alpha} \in \mathbb{Z}^d\}.$$

Such spaces are called *principal shift-invariant spaces* and they mimic our local approximation spaces in the case of the data sites being a regular grid of grid size h. These spaces have extensively been studied and their approximation properties are intrinsically connected to polynomial reproduction, i.e. to the question whether $\pi_m(\mathbb{R}^d) \subseteq S_h$, or alternatively, to the so-called *Strang-Fix* conditions.

Definition 10. An integrable function $\Phi : \mathbb{R}^d \to \mathbb{R}$ satisfies the Strang-Fix conditions of order m if $\widehat{\Phi}(0) \neq 0$ and $D^{\boldsymbol{\beta}}\widehat{\Phi}(2\pi\boldsymbol{\alpha}) = 0$ for all $\boldsymbol{\alpha} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ and $\boldsymbol{\beta} \in \mathbb{N}_0^d$ with $|\boldsymbol{\beta}| \leq m$.

The following result summarises these ideas. Its proof can be found in [39].

Theorem 8. *Let $\Phi : \mathbb{R}^d \to \mathbb{R}$ be a compactly supported, continuous function. Then, the following statements are equivalent:*

1. *Φ satisfies the Strang-Fix conditions of order m.*
2. *For $\boldsymbol{\beta} \in \mathbb{N}_0^d$ with $|\boldsymbol{\beta}| \leq m$ the function*

$$\mathbf{x} \mapsto \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} \boldsymbol{\alpha}^{\boldsymbol{\beta}} \Phi(\mathbf{x} - \boldsymbol{\alpha})$$

is a polynomial from $\pi_{|\boldsymbol{\alpha}|}(\mathbb{R}^d)$.

3. *For each $f \in H^{m+1}(\mathbb{R}^d)$ and each $h > 0$ there are weights $w_{\boldsymbol{\alpha}}^h$ such that as $h \to 0$,*

$$\left\| f - \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} w_{\boldsymbol{\alpha}}^h \Phi(h^{-1} \cdot -\boldsymbol{\alpha}) \right\|_{H^s(\mathbb{R}^d)} \leq C_s h^{m+1-s} \|f\|_{H^{s+1}(\mathbb{R}^d)}, \qquad 0 \leq s \leq p$$

$$\sum_{\boldsymbol{\alpha} \in \mathbb{Z}^d} |w_{\boldsymbol{\alpha}}^h|^2 \leq K \|f\|_{L_2(\mathbb{R}^d)}^2.$$

The constants C_s and K are independent of f.

Hence, in our terminology, at least if working on an infinite uniform grid, we could get away with just one of the approximation spaces $S_h = W_{h\mathbb{Z}^d}$ if our compactly supported function was satisfying the Strang-Fix conditions.

However, as we will see now, if the kernel is in addition radial, it will not satisfy the Strang-Fix conditions. To prove this negative result, which goes back to Wu [51], we need two auxiliary results.

The first one is concerned with a question about the density of functions of the form $\exp(imx)$ in the continuous functions. It was first stated by Pólya as a problem in 1931 ([34]) and then solved by Szász in 1933 ([40]).

Lemma 5 (Pólya). *Let the real numbers* $m_1, m_2, \cdots$ *have the properties* $0 < m_1 < m_2 < \cdots$ *and*

$$\liminf_{n\to\infty} \frac{n}{m_n} > \frac{b-a}{2\pi} > 0.$$

Furthermore, let f *be continuous in the closed interval* $[a,b]$*. Then it will follow from*

$$\int_a^b f(x)\cos(m_a x)dx = \int_a^b f(x) sin(m_n x)dx = 0$$

that f *vanishes identically on* $[a,b]$*.*

The second auxiliary result comes from number theory. It deals with the question which natural numbers can be represented as the sum of two squares, see, for example, [24].

Lemma 6. *Let* a_n *be the nth natural number which can be expressed as a sum of two integer squares. There are constants* $c_1, c_2 > 0$ *and* $n_0 \in \mathbb{N}$ *such that*

$$c_1 n < \frac{a_n}{\sqrt{\log(n)}} < c_2 n, \qquad n \geq n_0.$$

Using the operators introduced in Section 3.1, we are now able to show that there are no compactly supported radial functions, which satisfy the Strang-Fix conditions in dimensions larger than one.

Theorem 9. *For* $d \geq 2$*, there is no non-vanishing, continuous, radial, compactly supported function* $\Phi : \mathbb{R}^d \to \mathbb{R}$*, which satisfies the Strang-Fix conditions.*

Proof. Assume that $\Phi : \mathbb{R}^d \to \mathbb{R}$ is such a function, i.e. $\Phi = \phi(\|\cdot\|_2)$ with an even, compactly supported and continuous function $\phi : \mathbb{R} \to \mathbb{R}$. Then, its Fourier transform is given by

$$\widehat{\Phi}(\boldsymbol{\omega}) = \mathscr{F}_d \phi(\|\boldsymbol{\omega}\|_2) = \mathscr{F}_1 \mathscr{I}_{\frac{d-1}{2}} \phi(\|\omega\|_2) =: \mathscr{F}_1 \psi(\|\boldsymbol{\omega}\|_2).$$

with a new compactly supported, continuous function $\psi := \mathscr{I}_{\frac{d-1}{2}}\phi$. If Φ would satisfy the Strang-Fix condition, then we must therefore have

$$0 = \mathscr{F}_1\psi(2\pi\|\boldsymbol{\alpha}\|_2), \qquad \boldsymbol{\alpha} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}.$$

Next, let a_n be the nth natural number, which can be represented by two squared integers. Since $d \geq 2$, for each such a_n we can therefore pick an $\boldsymbol{\alpha}_n \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ such that $a_n = \|\boldsymbol{\alpha}_n\|_2^2$. Thus, if we define $m_n := 2\pi\|\boldsymbol{\alpha}_n\|_2 = 2\pi\sqrt{a_n}$, then Lemma 6 tells us that

$$\frac{n}{m_n} = \frac{1}{2\pi}\frac{n}{\sqrt{a_n}} > \frac{1}{2\pi\sqrt{c_2}}\frac{n}{\sqrt{n}(\log n)^{1/4}}.$$

This particularly means that $\liminf_{n\to\infty}(n/m_n) = \infty$ and hence that according to Lemma 5, we must have that $\psi = \mathscr{I}_{(d-1)/2}\phi \equiv 0$. But from (13) we can finally conclude that $\phi = \mathscr{I}_{-(d-1)/2}\psi = 0$. □

4.2 Multilevel Interpolation

In this section, we want to discuss and analyse the simplest case of a multilevel algorithm, which produces a global approximant from the space V_n. Let us recall the general setting. We assume that we have a sequence of increasingly finer and finer finite point sets

$$X_1, X_2, \ldots, X_n, \ldots$$

and a decreasing sequence of support radii

$$\delta_1 \geq \delta_2 \geq \ldots \geq \delta_n \geq \ldots$$

Then, using a compactly supported RBF $\Phi : \mathbb{R}^d \to \mathbb{R}$ and its scaled versions

$$\Phi_j(\mathbf{x}, \mathbf{y}) := \delta_j^{-d}\Phi((\mathbf{x} - \mathbf{y})/\delta_j), \tag{20}$$

we build, as mentioned in the introduction, local approximation spaces

$$W_j = \operatorname{span}\{\Phi_j(\cdot, \mathbf{x}) : \mathbf{x} \in X_j\}. \tag{21}$$

and global approximation spaces

$$V_n := W_1 + W_2 + \ldots + W_n. \tag{22}$$

Algorithm 1: Multilevel Approximation

Input : Right-hand side f, Number of levels n
Output : Approximate solution $f_n \in W_1 + \ldots + W_n$

Set $f_0 = 0$, $e_0 = f$
for $j = 1, 2, \ldots, n$ **do**
  Determine a local approximant $s_j \in W_j$ to e_{j-1}.
  Set $f_j = f_{j-1} + s_j$.
  Set $e_j = e_{j-1} - s_j$.

In this situation, the simplest possible algorithm to compute an approximation $f_n \in V_n$ to a function $f \in H$ is a residual correction scheme as described in Algorithm 1.

We will now analyse the approximation properties of this algorithm. To this end, we need a general *sampling inequality*. The following result comes from [32].

Lemma 7. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Let $\sigma > d/2$. Let $X \subseteq \Omega$ be a finite point set with sufficiently small mesh norm $h_{X,\Omega}$. Then, there is a constant $C > 0$, independent of X, such that for all $f \in H^\sigma(\Omega)$ vanishing on X, we have*

$$\|f\|_{H^\mu(\Omega)} \leq C h_{X,\Omega}^{\sigma-\mu} \|f\|_{H^\sigma(\Omega)}.$$

for $0 \leq \mu \leq \sigma$.

Using $s_j = I_{X_j,\Phi_j} e_{j-1}$ as the interpolant to e_{j-1} on X_j with kernel Φ_j, we have the following theorem.

Theorem 10. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. Let $X_1, X_2, \ldots$ be a sequence of point sets in Ω with mesh norms $h_1, h_2, \ldots$ satisfying $h_{j+1} = \mu h_j$ for $j = 1, 2, \ldots$ with fixed $\mu \in (0, 1)$ and $h_1 = \mu$ sufficiently small. Let Φ be a kernel generating $H^\sigma(\mathbb{R}^d)$, i.e. satisfying (9) and let Φ_j be defined by (20) with scale factor $\delta_j = \nu h_j$. Let the target function f belong to $H^\sigma(\Omega)$. Then, there exists a constant $C_1 > 0$ such that*

$$\|Ee_j\|_{\Phi_{j+1}} \leq C_1 \left(\mu^\sigma + \nu^{-\sigma}\right) \|Ee_{j-1}\|_{\Phi_j} \qquad \text{for } j = 1, 2, 3, \ldots \tag{23}$$

and hence there exists a constant $C > 0$ such that

$$\|f - f_n\|_{L_2(\Omega)} \leq C \left[C_1 \mu^\sigma + C_1 \nu^{-\sigma}\right]^n \|f\|_{H^\sigma(\Omega)} \qquad \text{for } n = 1, 2, \ldots,$$

provided that $\mu\nu \geq \gamma > 0$ with a constant $\gamma > 0$. Thus there are constants $\mu_0 \in (0, 1)$ and $\nu_0 > 1$ such that the multiscale approximation f_n converges linearly to f in the L_2 norm for all $\mu \leq \mu_0$ and $\nu \geq \nu_0$ with $\mu\nu \geq \gamma$.

Proof. The proof of this theorem can be found in [49]. We will thus only review its critical steps. The first, important auxiliary observation is that the interpolant at X_j to e_{j-1} is the same as the interpolant to Ee_{j-1} since both functions coincide on $X_j \subseteq \Omega$. Here, E denotes the extension operator from Proposition 1. From this, it follows that

$$\begin{aligned}
\|e_j\|_{H^\sigma(\Omega)} &= \|e_{j-1} - I_{X_j,\delta_j} e_{j-1}\|_{H^\sigma(\Omega)} \\
&= \|Ee_{j-1} - I_{X_j,\delta_j} Ee_{j-1}\|_{H^\sigma(\Omega)} \\
&\le \|Ee_{j-1} - I_{X_j,\delta_j} Ee_{j-1}\|_{H^\sigma(\mathbb{R}^d)} \\
&\le C\delta_j^{-\sigma} \|Ee_{j-1} - I_{X_j,\delta_j} Ee_{j-1}\|_{\Phi_j} \\
&\le C\delta_j^{-\sigma} \|Ee_{j-1}\|_{\Phi_j},
\end{aligned} \tag{24}$$

where we have used Lemma 1 and the fact that the interpolant is norm-minimal with respect to the Φ_j-norm.

Then, we have

$$\|Ee_j\|^2_{\Phi_{j+1}} \le \frac{1}{c_1} \int_{\mathbb{R}^d} |\widehat{Ee_j}(\boldsymbol{\omega})|^2 (1 + (\delta_{j+1}\|\boldsymbol{\omega}\|_2)^{2\sigma}) d\boldsymbol{\omega} =: \frac{1}{c_1}(I_1 + I_2)$$

with

$$\begin{aligned}
I_1 &:= \int_{\mathbb{R}^d} |\widehat{Ee_j}(\boldsymbol{\omega})|^2 d\boldsymbol{\omega}, \\
I_2 &:= \delta_{j+1}^{2\sigma} \int_{\mathbb{R}^d} |\widehat{Ee_j}(\boldsymbol{\omega})|^2 \|\boldsymbol{\omega}\|_2^{2\sigma} d\boldsymbol{\omega}.
\end{aligned}$$

Obviously, using Plancharel's theorem, the properties of the extension operator and (19), we see that the first integral can be bounded by

$$\begin{aligned}
I_1 &= \|Ee_j\|^2_{L_2(\mathbb{R}^d)} \le c\|e_j\|^2_{L_2(\Omega)} = c\|e_{j-1} - s_j\|^2_{L_2(\Omega)} \\
&\le c\left(\frac{h_j}{\delta_j}\right)^{2\sigma} \|e_{j-1}\|^2_{H^\sigma(\Omega)} \le c\nu^{-2\sigma} \|Ee_{j-1}\|^2_{\Phi_j}.
\end{aligned}$$

Similarly, for the second integral, we can use (24) to derive the bound

$$\begin{aligned}
I_2 &= \delta_{j+1}^{2\sigma} \int_{\mathbb{R}^d} |\widehat{Ee_j}(\boldsymbol{\omega})|^2 \|\boldsymbol{\omega}\|_2^{2\sigma} d\boldsymbol{\omega} \le \delta_{j+1}^{2\sigma} \int_{\mathbb{R}^d} |\widehat{Ee_j}(\boldsymbol{\omega})|^2 (1 + \|\boldsymbol{\omega}\|_2^{2\sigma}) d\boldsymbol{\omega} \\
&= \delta_{j+1}^{2\sigma} \|Ee_j\|^2_{H^\sigma(\mathbb{R}^d)} \le \delta_{j+1}^{2\sigma} \|e_j\|^2_{H^\sigma(\Omega)} \le C\left(\frac{\delta_{j+1}}{\delta_j}\right)^{2\sigma} \|Ee_{j-1}\|^2_{\Phi_j} \\
&\le C\mu^{2\sigma} \|Ee_{j-1}\|^2_{\Phi_j}.
\end{aligned}$$

Piecing these bounds together gives estimate (23). The rest then more or less follows by applying (23) iteratively and the following observation. Since $e_n = f - f_n$ vanishes on X_n, we have by Lemma 7 and Lemma 1 that

$$\begin{aligned}\|f - f_n\|_{L_2(\Omega)} &= \|e_n\|_{L_2(\Omega)} \le Ch_n^{\sigma}\|e_n\|_{H^{\sigma}(\Omega)} \le Ch_n^{\sigma}\|Ee_n\|_{H^{\sigma}(\mathbb{R}^d)} \\ &\le Ch_n^{\sigma}\delta_{n+1}^{-\sigma}\|Ee_n\|_{\Phi_{n+1}} = C\|Ee_n\|_{\Phi_{n+1}},\end{aligned} \tag{25}$$

since $h_n/\delta_{n+1} = h_n/(\nu h_{n+1}) \le= \frac{1}{\nu\mu} \le \frac{1}{\gamma}$. □

Though we cannot directly determine μ_0 and ν_0, equation (23) gives some insight into the influence of the two critical parameters μ and ν. On the one hand, the parameter μ determines how much we have to refine our data set from level to level. Hence, the smaller μ the more points we have to use in the next level.

On the other hand, the parameter ν determines the relation between the support radius and the fill distance. Here, a larger ν means that we have more non-zero entries per row in the interpolation matrix, which increases the computational cost. Nonetheless, increasing ν is less critical than decreasing μ.

But there is yet another consequence of this theorem. For simplicity, let us eliminate one of the parameters by setting $\nu = \mu^{-1}$ so that (23) becomes

$$\|Ee_j\|_{\Phi_{j+1}} \le C_1\mu^{\sigma}\|Ee_{j-1}\|_{\Phi_j} \tag{26}$$

and we have convergence for all $\mu > 0$ with $C_1\mu^{\sigma} < 1$. However, we even have convergence if for an arbitrary $\sigma > \varepsilon > 0$ we have $C_1\mu^{\varepsilon} \le 1$. In this case, (26) becomes

$$\|Ee_j\|_{\Phi_{j+1}} \le \mu^{\sigma-\varepsilon}\|Ee_{j-1}\|_{\Phi_j}. \tag{27}$$

We can also revisit (25) by choosing an $0 \le \beta \le \sigma - \varepsilon$. Then, Lemma 7 applied in the derivation of (25) yields

$$\|f - f_n\|_{H^{\beta}(\Omega)} \le Ch_n^{\sigma-\beta}\|e_n\|_{H^{\sigma}(\Omega)} \le Ch_n^{\sigma-\beta}\delta_{n-1}^{-\sigma}\|Ee_n\|_{\Phi_{n+1}} = Ch_n^{-\beta}\|Ee_n\|_{\Phi_{n+1}}.$$

Using (27) n times yields the estimate

$$\|f - f_n\|_{H^{\beta}(\Omega)} \le Ch_n^{-\beta}\mu^{n(\sigma-\varepsilon)}\|f\|_{H^{\sigma}(\Omega)}.$$

Finally, the fact that $h_1 = \mu$ and $h_{j+1} = \mu h_j$ shows that $h_n = \mu^n$ so that we can rephrase the last estimate as

$$\|f - f_n\|_{H^{\beta}(\Omega)} \le Ch_n^{\sigma-\varepsilon-\beta}\|f\|_{H^{\sigma}(\Omega)},$$

i.e., we have not only derived an estimate also for derivatives but have expressed the error in terms of the fill-distance of the finest data set. The exponent is almost optimal.

Corollary 9. *Under the assumptions of Theorem 10 with* $\nu = 1/\mu$ *and* $0 \le \beta \le \sigma - \varepsilon$ *we have the error bound*

$$\|f - f_n\|_{H^\beta(\Omega)} \le C h_n^{\sigma-\varepsilon-\beta} \|f\|_{H^\sigma(\Omega)},$$

provided that μ *is sufficiently small.*

There are several possible extensions to this theorem. First of all, the condition $h_{j+1} = \mu h_j$ can be relaxed to something like $c\mu h_j \le h_{j+1} \le \mu h_j$ with fixed $\mu, c \in (0, 1)$ without altering the result. Secondly, the algorithm also converges if the target function f is rougher, say $f \in H^\tau(\Omega)$ with $d/2 < \tau < \sigma$. Details can be found in [49].

From a numerical point of view, the multilevel scheme is extremely efficient. Once, the neighbourhood information are known, i.e. once we know for each level ℓ and each data site $\mathbf{x}_j^{(\ell)}$ those data sites $\mathbf{x}_k^{(\ell)}$ which are relevant for the computation, i.e. those with $\|\mathbf{x}_k^{(\ell)} - \mathbf{x}_j^{(\ell)}\|_2 \le \delta_\ell$, we have the following computational cost.

Corollary 10. *If the data sets* X_j *are quasi-uniform, i.e* $q_j \approx h_j$*, then we have for the involved linear systems:*

- *The number of non-zero entries per row is independent of the level.*
- *The condition number is independent of the level.*
- *The number of steps required by a non-preconditioned CG method is independent of the level.*
- *The computational cost is linear in each level.*

The neighbourhood information can be assembled in $\mathscr{O}(N_j \log N_j)$ time using tree-based search structures. Finally, we also have to compute the residuals. If the data sets are nested, we can restrict ourselves to compute residuals only on the finest level. If they are not nested, then we have to do this in step j for the remaining point sets $X_{j+1}, \dots, X_n$. Again, the neighbourhood information can be collected in $\mathscr{O}(N_j \log N_j)$ time for level j. Moreover, the number of levels is, because of the uniform refinement, at most $\mathscr{O}(\log N_n)$.

Before we come to numerical results, we want to discuss briefly two versions of this algorithm, one which discards unnecessary coefficients and one which only considers necessary data sites.

The first one was introduced in [25] as one of two discarding strategies for multilevel algorithms on the sphere. The general idea is that after computing the local interpolant at level j, which has a representation of the form

$$s_j = \sum_{k=1}^{N_j} \boldsymbol{\alpha}_k^{(j)} \Phi_j(\cdot, \mathbf{x}_k^{(j)}),$$

to discard all coefficients $\boldsymbol{\alpha}_k^{(j)}$ which have an absolute value smaller than a given threshold. This threshold can and should be level dependent. Since the discarding

Algorithm 2: Multilevel Approximation with Dynamical Discarding

Input : Right-hand side f, Number of levels n
Output : Approximate solution $f_n \in V_n = W_1 + \cdots + W_n$

Set $\widetilde{f}_0 = 0, \widetilde{e}_0 = f$;
for $j = 1, 2, \ldots, n$ **do**
  Determine the local interpolant $s_j \in W_j$ to $\widetilde{e}_{j-1}$.
  Drop all coefficients $|\boldsymbol{\alpha}_k^{(j)}| \le \text{eps}_j$ in s_j to define $\widetilde{s}_j$.
  Set $\widetilde{f}_j = \widetilde{f}_{j-1} + \widetilde{s}_j$.
  Set $\widetilde{e}_j = \widetilde{e}_{j-1} - \widetilde{s}_j$.

is done during each level, it was named *discarding dynamically* in contrast to the strategy of discarding after all steps have been computed. The Algorithm is formally given in Algorithm 2.

It is possible to show convergence of this algorithm in a very similar way as it has been done in the proof of Theorem 10. Details can be found in [25].

Theorem 11. *Let $\varepsilon > 0$ be given. Assume that the assumption of Theorem 10 are satisfied. Let $\varepsilon_j \le c_1 \varepsilon \delta_j^{d/2}$ with a constant $c_1 > 0$. Finally, let $\alpha := C_1(\mu^\sigma + \nu^{-\sigma})$. Then there is a constant $C > 0$ such that*

$$\|\widetilde{e}_j\|_{\Phi_{j+1}} \le \alpha \|\widetilde{e}_{j-1}\|_{\phi_j} + C\varepsilon.$$

Hence, after n steps the error can be bounded by

$$\|f - \widetilde{f}_n\|_{L_2(\Omega)} \le C\alpha^n \|f\|_{H^\sigma(\Omega)} + C\varepsilon \frac{1-\alpha^n}{1-\alpha}.$$

The second variation of our standard multilevel interpolation algorithm is an adaptive version. After computing the local interpolant s_j on X_j to e_{j-1}, we can check the error $e_j = e_{j-1} - s_j$ on the upcoming data set X_{j+1}. Then, instead of interpolating this error on all of X_{j+1}, we will actually only use those points of X_{j+1} on which e_j has an absolute value larger than a given threshold $\varepsilon_j > 0$. To describe this algorithm in more detail let us denote the initial point sets by $\widetilde{X}_j$ and let us denote the adaptive point sets which are actually used by X_j. Then, the algorithm can be stated as in Algorithm 3.

An error analysis of this algorithm is more problematic since it would require to know e_j on all of $\widetilde{X}_j$ but we only know e_j on $X_j \subseteq \widetilde{X}_j$. We could avoid this problem by creating an inner loop in which we check e_j on $\widetilde{X}_j$ and add those points for which e_j is still too large. However, in practice this does not seem to be necessary.

Algorithm 3: Adaptive Multilevel Approximation

Input : Right-hand side f, Number of levels n, thresholds ε_i, point sets $\widetilde{X}_1, \dots, \widetilde{X}_n$
Output : Approximate solution $f_n \in W_1 + \dots + W_n$

Set $f_0 = 0$, $e_0 = f$, $X_1 = \widetilde{X}_1$
for $j = 1, 2, \dots, n$ **do**
  Determine $s_j = I_{X_j,\Phi_j} e_{j-1} \in W_j$.
  Set $f_j = f_{j-1} + s_j$.
  Set $e_j = e_{j-1} - s_j$.
  for $\mathbf{x} \in \widetilde{X}_{j+1}$ **do**
    if $|e_j(\mathbf{x})| > \varepsilon_j$ **then**
      $X_{j+1} = X_{j+1} \cup \{\mathbf{x}\}$

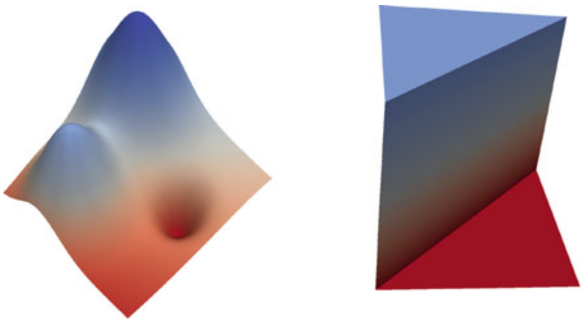

Fig. 1 Franke function (left), step function (right).

4.3 *Numerical Examples*

We will now look at two examples. In the first example, we are interested in understanding the importance of the parameters ν, which is responsible for the relation between the support radius and the fill distance on each level, and the parameter μ, which is responsible for the refinement of the data sets from level to level.

In this example, we will only be interested in varying the parameter ν. We will work on $\Omega = [0,1]^2$ and use the Franke function, see, for example, [20] and the left half of Figure 1, as our target function. The RBF is given by $\phi(r) = (1-r)_+^4(4r+1)$, i.e. the C^2-Wendland function in $\mathbb{R}^2$. We will work only on equidistant grids X_j with grid size $q_j = 2^{-j-2}$, $1 \le j \le 8$, which is also the separation distance of the data set X_j and hence equivalent to its fill distance. This means that we fix the refinement factor to be $\mu = 1/2$. We then vary the overlap factor ν, by defining $\delta_j = \widetilde{\nu} q_j$ and changing $\widetilde{\nu}$. Finally, we measure the error on a fine grid of grid size $q = 2^{-11}$.

A typical result can be found in Table 3, where we have chosen the overlap factor to be $\widetilde{\nu} = 3$, which means that we have at most 25 non-zero entries per row in our matrix, independent of the level. The table contains the number of points per

Table 3 Approximation of the Franke function. Basis function $\phi_{2,1} \in C^2, \widetilde{\nu} = 3$.

level	N	l_2	l_∞	l_2 order	l_∞ order	CG
1	81	1.087e-02	7.201e-02			27
2	289	2.464e-03	2.724e-02	2.14	1.40	38
3	1089	5.420e-04	8.947e-03	2.18	1.61	40
4	4225	1.218e-04	2.994e-03	2.15	1.60	40
5	16641	2.775e-05	1.015e-03	2.13	1.56	40
6	66049	6.360e-06	3.466e-04	2.13	1.55	39
7	263169	1.458e-06	1.151e-04	2.12	1.59	39
8	1050625	3.207e-07	3.605e-05	2.19	1.68	39
expected				2.5	1.5	

Table 4 Approximation of the Franke function. Basis function $\phi_{2,1} \in C^2$, various $\widetilde{\nu}$.

	$\widetilde{\nu} = 5$		$\widetilde{\nu} = 7$		$\widetilde{\nu} = 9$		$\widetilde{\nu} = 11$	
level	l_2	order	l_2	order	l_2	order	l_2	order
1	6.44e-3		5.45e-3		5.13e-3		5.02e-3	
2	8.33e-4	2.95	4.82e-4	3.50	3.47e-4	3.89	2.81e-4	4.16
3	1.53e-4	2.45	8.12e-5	2.57	5.34e-5	2.70	4.04e-5	2.80
4	2.90e-5	2.40	1.43e-5	2.51	8.89e-6	2.59	6.45e-6	2.65
5	5.56e-6	2.38	2.54e-6	2.49	1.49e-6	2.57	1.04e-6	2.63
6	1.07e-6	2.37	4.53e-7	2.48	2.52e-7	2.57	1.69e-7	2.62
7	2.07e-7	2.37	8.10e-8	2.48	4.26e-8	2.57	2.74e-8	2.63
8	3.82e-8	2.44	1.38e-8	2.55	6.86e-9	2.63	4.23e-9	2.69

level, the discrete ℓ_2- and ℓ_∞-error and the approximation order computed from comparing two successive level. Since $\phi_{2,1}$ is a reproducing kernel in $H^\sigma(\mathbb{R}^d)$ with $\sigma = k + (d+1)/2 = 2.5$, which means we would expect an L_2-approximation order $\sigma = 2.5$ and an L_∞-approximation order of $\sigma - d/2 = 1.5$. Finally, the table also contains the number of steps an unpreconditioned CG method requires to compute an approximate solution of the linear system with a relative accuracy of 10^{-6} a higher accuracy does not lead to a significant change in the solutions but only to a larger number CG steps.

In Table 4 we have collected the error and convergence order estimates for increasing overlaps. It seems that the expected order of 2.5 cannot only be matched but is actually exceeded. The number of non-zero entries per row increases to at most 373 in the case of $\widetilde{\nu} = 11$. The example also indicates that halving the fill distance, i.e. choosing $\mu = 1/2$ seems to be an appropriate choice and that the convergence order can be achieved solemnly by increasing the initial support radius.

In our second example, we also want to test the other two algorithms. To this end, we keep the general set-up of our first example but change the test function to a C^∞ step function given by

$$f(\mathbf{x}) = \frac{1}{2}\tanh(2000(x_2 + x_1 - 1)), \qquad \mathbf{x} \in [0,1]^2, \tag{28}$$

Table 5 Approximation of the step function (28). Basis function $\phi_{2,1} \in C^2, \widetilde{\nu} = 3$.

level	N	l_2	l_∞	l_2 order	l_∞ order	CG
1	81	1.329e-01	4.935e-01			19
2	289	9.378e-02	4.869e-01	0.50	0.02	36
3	1089	6.563e-02	4.756e-01	0.51	0.03	41
4	4225	4.520e-02	4.528e-01	0.54	0.07	44
5	16641	3.018e-02	4.203e-01	0.58	0.11	44
6	66049	1.883e-02	3.548e-01	0.68	0.24	44
7	263169	9.965e-03	2.505e-01	0.92	0.50	44
8	1050625	3.398e-03	1.226e-01	1.55	1.03	44
expected				2.5	1.5	

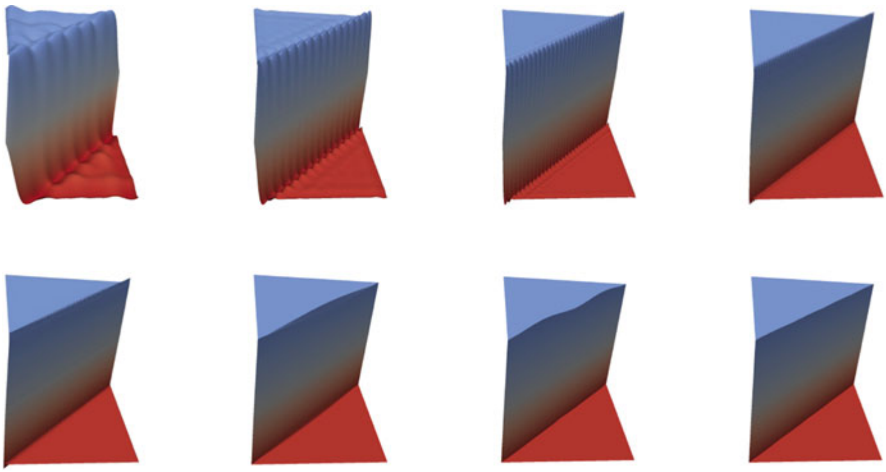

Fig. 2 Approximation of the step function, levels 0 (upper left) to 7 (lower right).

which is shown in Figure 1 (right). We keep the overlap fixed at $\widetilde{\nu} = 3$. The results of the standard multilevel algorithm are given in Table 5 and Figure 2. Clearly, convergence is much slower. In particular we have typical overshoots near the edge of the step function, at least in the first few levels.

The result of the dynamically discarding algorithm, Algorithm 2, is given in Table 6. For these results we have set the overall threshold to $\varepsilon = 10^{-5}$ and have on level j then discarded all the coefficients with absolute value less than $\varepsilon\delta_j$. Clearly, the results are very similar to those of the standard algorithm, Algorithm 1. But the total number of point information used reduces from 1,402,168 in Algorithm 1 to just 31,506 in Algorithm 2, i.e. we use only 2.25% of the original points. As we can see from Figure 3, the reason for this is that only those basis functions centred at data sites near the edge of the step function or near the boundary of the domain have large coefficients.

Table 6 Approximation of the step function with dynamical discarding. Basis function $\phi_{2,1} \in C^2, \widetilde{\nu} = 3$.

level	N	l_2	l_∞	l_2 order	l_∞ order	CG
1	72	1.329e-01	4.935e-01			19
2	268	9.378e-02	4.869e-01	0.50	0.02	36
3	1018	6.563e-02	4.756e-01	0.51	0.03	41
4	2080	4.520e-02	4.528e-01	0.54	0.07	44
5	2880	3.018e-02	4.208e-01	0.58	0.11	44
6	4774	1.884e-02	3.548e-01	0.68	0.24	44
7	7134	9.974e-03	2.505e-01	0.92	0.50	44
8	13280	3.453e-03	1.226e-01	1.53	1.03	44
expected				2.5	1.5	

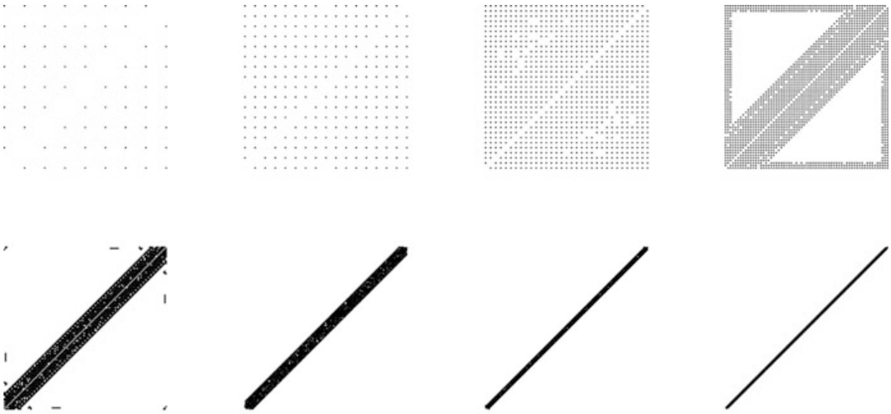

Fig. 3 Approximation of the step function with dynamical discarding, used data sites for levels 0 (upper left) to 7 (lower right).

The results of the adaptive algorithm are given in Table 7. Here we have used the thresholding strategy that we only considered points on the next level where the error of the current level is larger than 10^{-3} times the maximum error of the current interpolant on the next level. Again, the errors are comparable, though slightly worse. The total number of points used is now $74,029$ out of $1,402,168$, or 5.3%. Figure 4 shows the data sites which are actually used. The pattern is similar to the one created by the dynamically discarding algorithm, though more points are kept towards the boundary of the computational domain.

Table 7 Adaptive Approximation of the step function. Basis function $\phi_{2,1} \in C^2, \widetilde{\nu} = 3$.

level	N	l_2	l_∞	l_2 order	l_∞ order	CG
1	81	1.329e-01	4.935e-01			19
2	200	1.010e-01	4.882e-01	0.40	0.02	26
3	812	6.977e-02	4.786e-01	0.53	0.03	30
4	2318	4.828e-02	4.588e-01	0.53	0.06	31
5	5514	3.249e-02	4.283e-01	0.57	0.10	31
6	10094	2.066e-02	3.699e-01	0.65	0.21	30
7	18480	1.159e-02	2.656e-01	0.83	0.48	30
8	36530	4.934e-03	1.441e-01	1.23	0.88	31
expected				2.5	1.5	

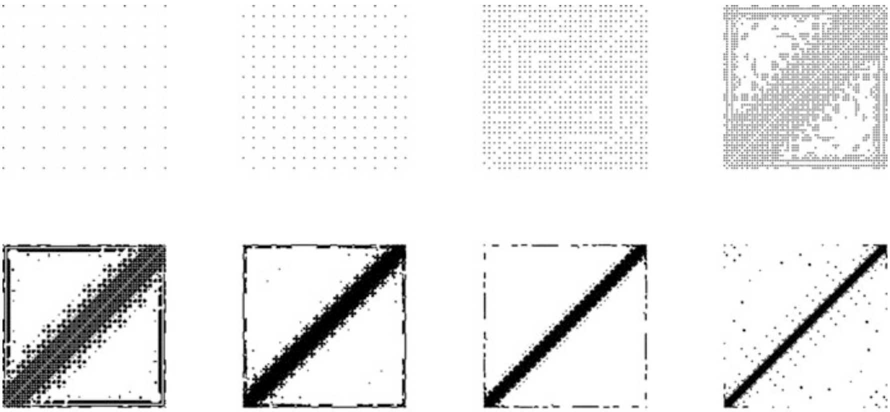

Fig. 4 Adaptive approximation of the step function, used data sites for levels 0 (upper left) to 7 (lower right).

5 Other Multilevel Schemes

So far, we have discussed multilevel and multiscale methods for interpolation problems. It is straight-forward to extend the result to the optimal recovery or smoothing spline approach (4). The results of Theorem 10 and Corollary 9 remain valid if the smoothing parameter ε is chosen level dependent satisfying $\varepsilon_j \leq c(h_j/\delta_j)^{2\sigma}$, see [49].

In [41], the multilevel scheme is analysed for target functions being zero on the boundary. The centres and the support radii on each level are chosen such that the multilevel interpolant is satisfying zero boundary conditions, as well, i.e. the supports of the basis functions are always contained in the domain Ω.

In [13], a multilevel scheme for vector-valued, divergence-free functions is analysed. Here, matrix-valued kernels are employed. In this situation, it is not possible anymore to keep the ratio between fill distance and support radius constant.

The paper [29] follows closely the original papers [26, 27] and discusses convergence orders for interpolation on the sphere, similar to those derived in Corollary 9. It also deals with inverse estimates, i.e. statements that show that a certain convergence rate must lead to a certain smoothness of the target function. These are mainly based upon the results from [36].

Shortly after the introduction of compactly supported RBFs and the initial, purely numerical papers on multilevel interpolation schemes such as [18, 19] and the first theoretical results such as [21, 31], which all differ substantially from the theoretical results given in this paper, also first attempts based upon collocation for solving PDEs numerically with such a multilevel scheme have been derived, see, for example, [14, 16] and [7]. However, only recently, the numerical observations could be underpinned with theoretical proofs in [12, 28]. There have also been first papers on the solution of PDEs based upon a weak formulation and a Galerkin method, see [10, 45]. The first of these two papers is purely numerically, while the second one contains theoretical results, which seem, however, to be too optimistic.

Finally, multiscale schemes like those discussed here have been used in computer graphics (see [33]), in the context of neural networks (see [17]) and in the context of machine learning (see [52]).

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Part IV
Frames in Spaces of Functions on Manifolds and Groups

Orthogonal Wavelet Frames on Manifolds Based on Conformal Mappings

Swanhild Bernstein and Paul Keydel

1 Introduction

Wavelets have been proven to be very useful in signal analysis and function approximation as well as in a lot of practical applications. There are many wavelets, curvelets, and shearlets on $\mathbb{R}^n$. But many real life signals live on curved manifolds, such as those on a sphere or spheroid, a hyperboloid or a cone. Unfortunately there is no simple way to construct wavelets on general manifolds. There are different approaches which are close to the classical theory on $\mathbb{R}^n$. We start with a simple definition of a wavelet transform. The wavelet transform is a generalized windowed Fourier transform. We denote by T_b the translation operator $T_b f(x) = f(x-b)$, $b \in \mathbb{R}$, and by D_a the dilation operator $D_a f(x) = \frac{1}{\sqrt{a}} f(\frac{x}{a})$, $a \in \mathbb{R}_+$. Then the wavelet transform is defined as

$$\begin{aligned} Wf(a;b) &:= \int_{-\infty}^{\infty} f(t) \frac{1}{\sqrt{a}} \overline{\psi\left(\frac{t-b}{a}\right)}\, dt \\ &= \int_{-\infty}^{\infty} f(t) \overline{D_a \psi\,(t-b)}\, dt = \langle f,\, D_a T_b \psi \rangle_{L^2} \\ &= \int_{-\infty}^{\infty} f(t) * \overline{D_a T_b \check{\psi}(t)}\, dt. \end{aligned}$$

with an admissible mother wavelet ψ (cf. Definition 6). A continuous wavelet transform is an integral transform, a convolution-type operator and a singular integral operator. The inversion of the wavelet transform is a result of the Calderon

S. Bernstein (✉) • P. Keydel
Institute of Applied Analysis, TU Bergakademie Freiberg, Freiberg, Germany
e-mail: swanhild.bernstein@math.tu-freiberg.de

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reproducing formula. Group theory shows that continuous wavelets can be interpreted as coherent states [1]. Unfortunately, this approach breaks down on the sphere. There had been several different approaches to defining successfully continuous wavelets on the sphere by [2, 3, 15, 16, 18]. There are also discrete wavelets on the sphere and these wavelets form a multiresolution analysis [4].

But even in that case there are problems to obtain good, i.e. smooth, orthogonal, locally supported wavelet basis on $\mathbb{S}^2$. Here is a list of known approaches:

(1) Discretization of the continuous wavelet transform. The advantage of this approach is that it is easy to implement if the mother wavelet is given explicitly, and there is much freedom in choosing a smooth wavelet. The disadvantages are that we get only frames and not bases. There is a problem of finding an appropriate discretization grid which leads to good frames.
(2) One can consider diffusive wavelets on manifolds or wavelets on spheres constructed by using spherical harmonic kernels. These wavelets are smooth and there are no poles, but they are not localized, the support covers the whole sphere/manifold.
(3) Let a sphere be centered at the origin and assume it contains a convex polyhedron with triangular faces also centered at the origin. Then wavelets can be constructed by radial projection from the polyhedron to the sphere.

But there seem to be easier ways to obtain "wavelets" on manifolds. In many cases it is possible to transform wavelets in $\mathbb{R}^n$ to the manifold as a whole or locally by using finite or countable covering of the manifold. To some extent, the papers by Antoine and Vandergeynst [3, 5] contain this idea for the case spheres. Another approach by Holschneider [22] and Freeden and Windheuser [17] uses the so-called harmonic dilations instead of stereographic ones. A specific projection approach to wavelets was also done by D. Roşca [6, 27, 28] which is based on area preserving projections, which are not conformal. Area preserving projections preserve the scalar product which is not the case for conformal mappings, but there are many more conformal mappings compared to area preserving which widens the range of applications considerably.

The sphere seems to be the most studied compact manifold. In [8, 14] diffusion wavelets on manifolds are discussed. These wavelets are a continuous analogue to the discrete diffusion wavelets [9, 12]. Another way to define wavelets and frames on manifolds is based on splines [25, 26]. [19] contains a construction of Parseval (tight) bandlimited and strongly localized frames on general compact homogeneous manifolds by using harmonic kernels on them.

The aim of this paper is to construct continuous wavelet transforms, discrete wavelet transforms, and frames on manifolds which are locally conformally flat. These wavelets are called lifted wavelets and they are defined in Definition 9. Our main results are formulated in Theorems 4, 5, and 6 in which we assume that there exists a conformal map between $\mathscr{M}$ and $\mathbb{R}^2$ which does not have singular points. In section 5.1 we consider the case when a conformal map between $\mathscr{M}$ and $\mathbb{R}^2$ has singularities.

The chapter is organized as follows. After the introduction in Section 2 we define conformal and locally conformal flat mappings, their properties and give some examples. Continuous wavelets transforms are considered in Section 3. We present the group theoretic approach and define continuous wavelets in $\mathbb{R}^2$ accordingly. Then we explain why the group theoretic approach fails for the sphere and define the lifted continuous wavelet transform. In Section 4 a multiresolution analysis for $L^2(\mathbb{R}^2)$ is defined and the lifted discrete wavelet transform is constructed. In Section 5 in contrast to previous sections we assume that corresponding conformal maps can have singular points. Some examples are given in Section 6, and Section 7 contains some remarks.

2 Conformal Maps and Their Properties

Let us consider Riemannian manifolds [7], which provide the basis for our work. We want to define a conformal map on these manifolds:

Definition 1 ([20]). A map $f : (\mathscr{M}; g) \to (\tilde{\mathscr{M}}; \tilde{g})$ between Riemannian manifolds is called conformal[1] if the induced metric has the form $f^*\tilde{g} = e^{2u}g$ with some function $u : \mathscr{M} \to \mathbb{R}$.

With the aid of g one may define angles on the manifold and the most important property of conformal mappings for our purpose follows.

Lemma 1. *Let $f : (\mathscr{M}; g) \to (\tilde{\mathscr{M}}; \tilde{g})$ be a map between Riemannian manifolds. Then f is conformal if and only if f is angle-preserving.*

Here are some examples of conformal mappings of $\mathbb{S}^2$.

Example 1 ([20]).

- The stereographic projections $\sigma_\pm$, defined in terms of their inverses by

$$\sigma_\pm^{-1} : \mathbb{R}^n \to \mathbb{S}^n \subset \mathbb{R}^{n+1} = \mathbb{R} \times \mathbb{R}^n, \quad p \mapsto \frac{1}{1+|p|^2}(\pm(1-|p|^2), 2p),$$

  are conformal.
- The Mercator map of the world (2-sphere), given by the parametrization

$$(u, v) \mapsto \left(\frac{1}{\cosh(u)}\cos(v), \frac{1}{\cosh(u)}\sin(v), \tanh(u)\right),$$

  is conformal.

[1] By our definition the conformal factor $e^{2u} > 0$ is strictly positive. If a conformal factor is allowed to have zeros, then the map is called "weakly conformal."

- The Archimedes-Lambert projection (projection of the sphere to the cylinder along lines perpendicular to the axis) is *area-preserving but not conformal*. The system of meridians and parallels of latitude still forms an orthogonal net.

Remark 1. No map from the sphere of the plane $\mathbb{R}^2$ can be both conformal and area-preserving.

Example 2 ([20]). The metric

$$\frac{1}{p_0^2}(dp_0^2 + dp^2), \quad p_0 \in (0, \infty), p \in \mathbb{R}^n,$$

on $(0, \infty) \times \mathbb{R}^n$ is clearly conformally equivalent to the standard Euclidean metric on that set as a subset of $\mathbb{R}^{n+1}$. This metric has constant sectional curvature -1. This is the Poincarè half-space model of hyperbolic space.

2.1 Conformal Flatness

It is essential to know when a Riemannian manifold can be mapped to a flat space by a conformal transformation.

Definition 2. A C^1 locally conformally flat manifold $(\mathscr{M}; g)$ is a Riemannian manifold so that $\mathscr{M}$ is of class C^1, the metric is of class C^0 and every point has a C^1 coordinate system $x_1, \ldots, x_n$ so that in this coordinate patch the metric has the form $g = f(x)((dx^1)^2 + \ldots + (dx^n)^2)$ for a function $f > 0$.

The definition is equivalent to

Theorem 1. *A Riemannian manifold is C^1 locally conformally flat if and only if $\mathscr{M}$ has a cover by open sets $\{U_\alpha\}$ which are the domain of C^1 diffeomorphisms $\varphi_\alpha : U_\alpha \to \varphi_\alpha[U_\alpha] \subset \mathbb{R}^n$ onto open sets so that the transition functions $\Phi_{\alpha,\beta} := \varphi_\alpha \circ \varphi_\beta^{-1}\Big|_{\varphi_\beta[U_\alpha \cap U_\beta]} : \varphi_\beta[U_\alpha \cap U_\beta] \to \varphi_\alpha[U_\alpha \cap U_\beta]$ are of class C^1 and if g_0 is the flat metric on $\mathbb{R}^n$ then $\varphi_\alpha^* g_0 = \lambda_\alpha g$ for some positive continuous function λ_α defined on U_α.*

Theorem 2 (Kuiper). *Let $(\mathscr{M}; g)$ be a simply connected, locally conformally flat manifold of class C^1. Then there is a conformal immersion[2] $f : \mathscr{M} \to \mathbb{S}^n$. If $\mathscr{M}$ is compact, then this map is a conformal diffeomorphism of $\mathscr{M}$ with $\mathbb{S}^n$.*

Remark 2. The definition of a locally conformally flat manifold is local, but there are global conformal maps, for example the stereographic projection. Further, due

[2]An immersion is a differentiable function between differentiable manifolds whose derivative is everywhere injective.

to the properties of the covering and the transition functions it is possible to define a covering map that is conformally flat and defined for the whole manifold.

We would like to restrict our consideration in this section to isothermic surfaces.

Definition 3 (Isothermic surface). A surface $f : \mathscr{M}^2 \to \mathbb{S}^n$ is called isothermic if, around each (nonumbilic[3]) point, there exist conformal curvature line coordinates (x, y), i.e.

$$\left.\begin{array}{ll} |f_x|^2 &= |f_y|^2 \\ f_x \cdot f_y &= 0 \end{array}\right\} \iff f_x f_y^{-1} + f_y f_x^{-1} = 0 \quad \text{and} \quad f_{xy} \in \operatorname{span}\{f_x, f_y\}.$$

The importance of locally conformally flat manifolds is explained in the following theorem.

Theorem 3 (Korn-Lichtenstein). *Any two-dimensional Riemannian manifold is locally conformally flat.*

This follows from the existence of isothermal coordinates. We will construct them explicitly for surfaces of revolution.

Example 3 (Square Clifford torus). Equip the Minkowski space $\mathbb{R}^5_1$ with an orthonormal basis $\{e_0, e_1, \ldots, e_4\}$, that is, $\langle e_i, e_j \rangle = \pm\delta_{ij}$ with -1 for $i = 0$ and $+1$ otherwise. Then

$$\mathbb{R}^2/2\pi\mathbb{Z}^2 \ni (u, v) \mapsto \left(1, \tfrac{1}{\sqrt{2}} \cos u, \tfrac{1}{\sqrt{2}} \sin u, \tfrac{1}{\sqrt{2}} \cos v, \tfrac{1}{\sqrt{2}} \sin v\right)$$

parametrizes the square Clifford torus.

Example 4 (Ellipsiod). A curvature line parametrization of the ellipsoid with semi-axes $0 < a < b < c$ using Jacobi elliptic functions $\mathscr{M} \to \mathbb{R}^3$ is

$$(x, y) \mapsto (ap\, \mathrm{cd}_\mathrm{p}(x)\, \mathrm{nd}_\mathrm{q}(y),\ bpq\, \mathrm{sd}_\mathrm{p}(x)\, \mathrm{sd}_\mathrm{q}(y,)\ cq\, \mathrm{nd}_\mathrm{p}(x)\, \mathrm{cd}_\mathrm{q}(y)),$$

where sd_p, cd_p, and nd_p denote the Jacobi elliptic functions of pole type d and module p, and q is the comodule, $p^2 + q^2 = 1$. Note that the metric degenerates exactly at the points where $\mathrm{nd}_\mathrm{p}(x) = \mathrm{nd}_\mathrm{q}(y) = 1$: There are the points that give the umbilics (singularities).

Example 5 (Catenoid). A curvature line parametrization of the catenoid with $0 < a < b < c$ using Jacobi elliptic functions $\mathscr{M} \to \mathbb{R}^3$ is

$$(x, y) \mapsto \left(-\tfrac{a}{p} \operatorname{artanh}\left(\tfrac{\mathrm{cd}_\mathrm{p}(x)}{\mathrm{nd}_\mathrm{q}(y)}\right), -\tfrac{b}{pq} \arctan\left(\tfrac{\mathrm{sd}_\mathrm{p}(x)}{\mathrm{sd}_\mathrm{q}(y)}\right), -\tfrac{c}{q} \operatorname{artanh}\left(\tfrac{\mathrm{cd}_\mathrm{q}(y)}{\mathrm{nd}_\mathrm{p}(x)}\right)\right).$$

[3]In the differential geometry of surfaces, umbilics or umbilical points are points on a surface that are locally spherical. At such points the normal curvature in all directions are equal, hence, both principal curvatures are equal, and every tangent vector is a principal direction. The name "umbilic" comes from the Latin umbilicus - navel [29].

This definition works globally. Nevertheless, since isothermic surfaces need not to be analytic, arbitrarily unpleasant configurations of umbilics can occur. Therefore it is better to work with umbilic-free surface patches or with surface patches that carry regular curvatures line parameters.

2.2 Conformal Maps on Rotationally Symmetric Surfaces

We want to construct a conformal map from a surface of revolution onto the plane. We suppose a surface $\mathscr{S} \subset \mathbb{R}^3$ which is formed by rotating a curve $\{(x, z) = (u(\theta), v(\theta)), 0 \leqslant \theta \leqslant \pi\}$ in the x-z-plane around the z axis (u, v are differentiable and non-negative functions). We obtain

$$\mathscr{S} = \left\{(x, y, z) \in \mathbb{R}^3 : (x, y, z)^T = X(\phi, \theta), \quad \phi \in [0, 2\pi], \quad \theta \in [0, \pi]\right\}$$

with the parametrization $X(\phi, \theta) = \begin{pmatrix} \cos(\phi)u(\theta) \\ \sin(\phi)u(\theta) \\ v(\theta) \end{pmatrix}$. One should note that the parameter θ does not need to be a real angle.

Using the fact that we have a surface of revolution we can obtain a projection

$$p = (f \circ X)(\phi, \theta) = r(\theta) \begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \end{pmatrix}, \quad r > 0.$$

That means, that all points on the great circle $\mathscr{S} \cap \{\phi = \text{const}\}$ are mapped onto a half-line

$$\{t(\cos\phi, \sin\phi)^T,\ t \geqslant 0\}$$

in $\mathbb{R}^2$ with respect to a rescaling function $r = r(\theta) > 0$. A straightforward calculation yields an ordinary differential equation for r and the solution reads as

$$r(\theta) = C \exp\left(\int^{\theta} \frac{\sqrt{u'(\nu)^2 + v'(\nu)^2}}{u(\nu)} d\nu\right), \qquad C \in \mathbb{R}^+$$

The inverse projection p^{-1} is given by

$$p^{-1} : \mathbb{R}^2 \ni \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} \phi \\ \theta \end{pmatrix} = \begin{pmatrix} \arctan 2(y, x) \\ r^{-1}\left(\sqrt{x^2 + y^2}\right) \end{pmatrix} \in \mathscr{S}$$

where r^{-1} is the inverse of r and $\arctan 2(y, x)$ is the modified $\arctan(y/x)$. The hard part is obviously to find r^{-1}.

Example 6 (Stereographic projection and the sphere). First we consider the unit sphere $\mathbb{S}^2$ around the center $(0,0,1)^T \in \mathbb{R}^3$. We have

$$\mathbb{S}^2 = \{(\eta_1, \eta_2, \eta_3) \in \mathbb{R}^3 : \eta_1^2 + \eta_2^2 + (\eta_3 - 1)^2 = 1\}$$

The standard parametrization is obviously given by

$$\begin{aligned} \eta_1 &= \cos(\phi)\sin(\theta) \\ \eta_2 &= \sin(\phi)\sin(\theta), \qquad \theta \in [0,\pi), \phi \in [0, 2\pi) \\ \eta_3 &= 1 - \cos(\theta) \end{aligned}$$

The so-called stereographic projection consists in the following. For any point P on the sphere exists a unique line through the same point and the North pole $N = (0,0,2)^T$. This connection line intersects with the x-y-plane and the image P' of the stereographic projection is defined by the resulting intersection. With $r(\theta)$ we want to denote the distance between the point P' and the South pole S. With the aid of the Intercept theorem we get immediately

$$\frac{2}{r(\theta)} = \frac{2 - (1 - \cos(\theta))}{\sin(\theta)} = \cot\left(\frac{\theta}{2}\right)$$

Thus, the stereographic projection $p : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$ from North pole N onto tangent plane at South pole is given by

$$p(\theta, \phi) = 2\tan\frac{\theta}{2}\begin{pmatrix}\cos(\phi)\\ \sin(\phi)\end{pmatrix}$$

or equivalently

$$p(\eta) = \frac{2}{2 - \eta_3}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}$$

where $\eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{S}^2$.

We could have gotten the stereographic projection also with $u(\theta) = \sin(\theta)$, $v(\theta) = 1 - \cos(\theta)$ and $C = 2$ from our previous considerations:

$$r(\theta) = 2\exp\left(\int^{\theta} \frac{\sqrt{\cos(\nu)^2 + \sin(\nu)^2}}{\sin(\nu)} d\nu\right) = 2\tan\frac{\theta}{2}$$

An easy calculation yields the inversion $p^{-1} : \mathbb{R}^2 \to \mathbb{S}^2$.

$$p^{-1}(x,y) = (\eta_1, \eta_2, \eta_3) = \left(\frac{4x}{x^2 + y^2 + 4}, \frac{4y}{x^2 + y^2 + 4}, \frac{2x^2 + 2y^2}{x^2 + y^2 + 4}\right)$$

Since we should use adjusted coordinates (ϕ, θ) we obtain

$$p^{-1}(x, y) = (\phi, \theta) = \left(\arctan 2(y, x), 2\arctan\left(\frac{\sqrt{x^2+y^2}}{2}\right)\right)$$

The weight factors read

$$\begin{aligned}\nu &= \frac{r(\theta)}{u(\theta)} = \frac{1}{\cos^2(\theta/2)}\\ \rho &= \frac{u\left(r^{-1}\left(\sqrt{x^2+y^2}\right)\right)}{\sqrt{x^2+y^2}} = \frac{4}{4+x^2+y^2}\end{aligned}$$

Remark 3. It is possible to generalize from the spherical projection to an ellipsoidal projection. Any ellipsoid has the form

$$X(\phi, \theta) = \begin{pmatrix} a\cos(\phi)\sin(\theta)\\ b\sin(\phi)\sin(\theta)\\ 1-\cos(\theta)\end{pmatrix}$$

In the case $a = b = 1$ we have a sphere and for $a = 1$ or $b = 1$ we have an ellipsoid of revolution. But for $a \neq 1$ and $b \neq 1$ we obtain the most complex case. Then we call the ellipsoid triaxial and a big effort is necessary to find a conformal map from the triaxial ellipsoid to $\mathbb{R}^2$, because we cannot use the ellipsoidal stereographic projection p:

$$p: \quad (\phi, \theta) \mapsto 2\sqrt{a^2\cos^2(\phi) + b^2\sin^2(\phi)}\tan\frac{\theta}{2}\begin{pmatrix}\cos(\phi)\\ \sin(\phi)\end{pmatrix}$$

It is easy to see that this map is not conformal for $a \neq 1$ and $b \neq 1$.

Example 7 (Rotated cardioid). In the previous section we considered the sphere and the stereographic projection. That is a comparatively slight geometry and we want to make it a bit more difficult by creating a more "realistic" surface. We choose a so-called cardioid in the x-z-plane, its parametrization is given by

$$\begin{aligned}u &= x = 2(1-\cos(\theta))\sin(\theta) = 2\sin(\theta) - \sin(2\theta)\\ v &= z = 2(1-\cos(\theta))\cos(\theta) + 4 = 2\cos(\theta) - \cos(2\theta) + 3\end{aligned}$$

where $\theta \in [0, 2\pi]$. Rotating the curve around the z-axis yields an apple-like surface of revolution. We get the new parametrization

$$x = 2\cos(\phi)\sin(\theta)(1-\cos(\theta))$$

$$y = 2\sin(\phi)\sin(\theta)(1-\cos(\theta))$$
$$z = 2\cos(\theta) - \cos(2\theta) + 3$$

that leads to

$$r(\theta) = C\exp\left(\int^{\theta} \frac{\sqrt{4(\cos\nu - \cos 2\nu)^2 + 4(\sin\nu - \sin 2\nu)^2}}{2\sin(\theta) - \sin(2\theta)} d\nu\right)$$
$$= C\exp\left(\int^{\theta} \frac{\sqrt{8 - 8\cos\nu}}{2\sin(\theta) - \sin(2\theta)} d\nu\right) = C\exp\left(-\frac{1}{\sin(\theta/2)}\right)\tan\left(\frac{\pi+\theta}{4}\right)$$

where $C \in \mathbb{R}^+$ is an arbitrary positive constant.

The map as a projection is bijective. The inversion itself has to be done numerically because we can't get a closed form for r^{-1}. Only ϕ is always reconstructable because of $\phi = \arctan 2(y, x)$.

The weight factors are given by

$$\nu = \frac{r(\theta)}{u(\theta)} = \frac{C\exp\left(-\frac{1}{\sin(\theta/2)}\right)\tan\left(\frac{\pi+\theta}{4}\right)}{2\sin(\theta) - \sin(2\theta)}$$
$$\rho = \frac{u\left(r^{-1}\left(\sqrt{x^2+y^2}\right)\right)}{\sqrt{x^2+y^2}}$$

Example 8 (Rotated nephroid). The planar nephroid is given by the parametrization

$$u = x = 3\sin(\theta) - \sin(3\theta)$$
$$v = z = 3\cos(\theta) - \cos(3\theta) + 2\sqrt{2}$$

where $\theta \in [0, 2\pi]$, and we obtain

$$r(\theta) = C\exp\left(\int^{\theta} \frac{\sqrt{9(\cos\nu - \cos 3\nu)^2 + 9(\sin\nu - \sin 3\nu)^2}}{2\sin(\theta) - \sin(2\theta)} d\nu\right)$$
$$= C\exp\left(\int^{\theta} \frac{\sqrt{36\sin^2\nu}}{3\sin(\theta) - \sin(3\theta)} d\nu\right) = C\exp\left(-\frac{3\cot(\theta)}{2}\right)$$

As we can see, the nephroid is one of the rare examples where we have a closed-form inverse projection and we do not need numerical algorithms to compute p^{-1}. We get immediately:

$$p^{-1}(x,y) = (\phi, \theta) = \left(\arctan 2(y,x), \operatorname{arccot}\left(-\frac{\ln(x^2+y^2)}{3}\right)\right)$$

for $C = 1$ and finally

$$\begin{aligned} \nu &= \frac{r(\theta)}{u(\theta)} = \frac{C\exp\left(-\frac{3\cot(\theta)}{2}\right)}{3\sin(\theta) - \sin(3\theta)} \\ \rho &= \frac{u\left(r^{-1}\left(\sqrt{x^2+y^2}\right)\right)}{\sqrt{x^2+y^2}} \\ &= \frac{1}{\sqrt{x^2+y^2}}\left[3\sin\left(\operatorname{arccot}\left(-\frac{\ln(x^2+y^2)}{3}\right)\right) - \sin\left(3\operatorname{arccot}\left(-\frac{\ln(x^2+y^2)}{3}\right)\right)\right] \end{aligned}$$

3 Continuous Wavelet Transforms

We will explain the group theoretic approach in detail.

3.1 Group Theoretic Approach

Definition 4. A group $\mathscr{G} = (G, \circ)$ is called a *topological group* if it is equipped with a topology, such that the group multiplication $G \times G \to G$, $(a, b) \mapsto a \circ b$ and the inversion $G \to G$, $a \mapsto a^{-1}$ are continuous mappings. A topological group is *compact* if it is compact as topological (Hausdorff-)space and *locally compact* if for every group element $a \in G$ there exists a neighborhood with compact closure.

Definition 5. Let $\mathscr{G} = (G, \circ)$ be a topological group, H a linear space, $\pi : G \to \mathscr{L}(H)$ a homomorphic (structure preserving) mapping into the group $\mathscr{L}(H)$ of continuous linear operators from H into itself, i.e. we require

$$\pi(g_1 \circ g_2) = \pi(g_1)\pi(g_2) \quad \text{and} \quad \pi(e) = id_H$$

Then the pair (H, π) is called *group representation* of $\mathscr{G}$ in H.

Some specific definitions about the properties of group representations in Hilbert-spaces H with an inner product $\langle \cdot, \cdot \rangle_H$:

- π is said to be *unitary* if for any $g \in G$ the operator $\pi(g)$ is unitary.
- The representation is called *continuous* if for any $u \in H$ the map $G \ni g \mapsto \pi(g)u \in H$ is continuous on G.
- We call the representation *irreducible* if there is no proper subspace of H which is invariant under $\pi(g)$ for all $g \in G$.

Definition 6. Let π be a continuous unitary representation of the group $\mathcal{G}$ in a Hilbert-space H with an inner product $\langle\cdot,\cdot\rangle_H$. An element $v \in H$ is called an *admissible vector*, if

$$\int_G |\langle v, \pi(g)v\rangle_H|^2 \, d\mu(g) < \infty$$

If there exists an admissible nontrivial element v and π is irreducible, we call π *square-integrable*.

Putting all this together leads to the group theoretical definition of the wavelet transform:

Definition 7. Let $\mathcal{G} = (G, \circ)$ be a locally compact, topological group and π a square-integrable unitary representation of $\mathcal{G}$ in a Hilbert-space H. Further let $f \in H$. An element $\psi \in H \setminus \{0\}$ is called *wavelet* if ψ is an admissible vector. The *(left) wavelet transform* of f is defined by

$$L_\psi : H \to L^2(G), \quad L_\psi(f)(g) = \langle f, \pi(g)\psi\rangle_H$$

This approach works well for $\mathbb{R}^2$ and will be explained in the following subsection.

3.2 *Construction of the Continuous Wavelet Transform on* $\mathbb{R}^2$

There are many books on wavelets, for example [13, 21, 23, 24]. Continuous wavelets on $\mathbb{R}^2$ (and analogously on $\mathbb{R}^n$) are defined by the Euclidean group $\mathcal{G} = (\mathbb{R}^2, +) \rtimes (\mathbb{R}^+, \cdot) \times (\mathbb{R}, +) =: (\mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}, *)$ and consist of *motions* $T_b : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ (translations), $R_\alpha : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ (rotations), and the *dilation* $D_a : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$, where

$$\begin{array}{ll} T_b f(x) = f(x-b), & b \in \mathbb{R}^2 \\ R_\alpha f(x) = f(r_{-\alpha}x), & r_{-\alpha} \in SO(2) \\ D_a f(x) = a^{-1} f(a^{-1}x), & a \in \mathbb{R}^+ := (0,\infty) \end{array} .$$

The group law in $\mathcal{G}$ reads as

$$(b_1, a_1, \alpha_1) * (b_1, a_1, \alpha_1) = (b_1 + a_1 r_{\alpha_1} b_2, a_1 a_2, \alpha_1 + \alpha_2)$$

The left invariant Haar measure on $\mathcal{G}$ is $d\mu_L = a^{-3} d^2b \, da \, d\alpha$ and

$$(a, b, \alpha) =: g \mapsto \pi(g) = \pi(b, a, \alpha) = T_b D_a R_\alpha$$

is a square-integrable and unitary representation of the (affine) group $\mathscr{G}$ in the Hilbert-space $L^2(\mathbb{R}^2)$.

Therefore wavelets can be defined in the following way:

Definition 8 (2D continuous wavelet transform (CWT)). A complex-valued function $\psi \in L^2(\mathbb{R}^2)$ is a (mother-)wavelet, if

$$\int_G |\langle\psi, \pi(b,a,\alpha)\psi\rangle_H|^2 \, d\mu_L(g) = \|L_\psi(\psi)\|^2_{L^2(G)} < \infty$$

and the left wavelet transform $L_\psi : L^2(\mathbb{R}^2) \to L^2(G, d\mu_L)$ is given by

$$L_\psi(f)(g) = \langle f, \pi(g)\psi\rangle_{L^2(\mathbb{R}^2)} = a^{-1} \int_{\mathbb{R}^2} \overline{\psi(a^{-1} r_{-\alpha}(x-b))} f(x) d^2x$$

But this construction fails on manifolds $\mathscr{M}$. The irreducibility of the representation ensures the invertibilty of the wavelet transform on its range, but unitary representations of compact groups in infinite dimensional spaces such as $L^2(\mathscr{M})$ are always reducible.

There are ways to overcome this difficulties. In [2, 3] quasi-regular representations of the Lorentz group $SO(n+1,1)$ were used to define rotations and dilations on the sphere $\mathbb{S}^n \simeq SO(n+1,1)/SO(n,1)$. To ensure square-integrability, one considers only the admissible sections of the Lorentz group $SO(n+1,1)$. In some sense that might be interpreted as a "point-wise" projection of wavelets on $\mathbb{R}^n$ to the sphere $\mathbb{S}^n$.

An alternative approach by [18] goes back to definition of a continuous wavelet transform by singular integrals. Translations are defined as rotations of the sphere and the wavelets can be defined as a convolution with an approximate identity. Typical kernels are the Gauß-Weierstraß kernel and the Abel-Poisson kernel. That idea can be generalized to so-called diffusive wavelets [8, 14]. Here the translation is the group action on the manifold and the dilations are defined by a diffusion kernel.

The discrete analogue of these are diffusion wavelets by [12].

3.3 Projections and Inner Products in $L^2(\mathscr{M})$

We already know that a strict group theoretical approach fails. Our goal is to lift the wavelet theory in $\mathbb{R}^2$ onto a C^1-surface $\mathscr{M}$.

Let $\mathscr{M} \subset \mathbb{R}^3$ be a C^1-surface defined by the parametrization

$$\mathscr{M} = \{(\xi_1,\xi_2,\xi_3) \in \mathbb{R}^3 : \ (\xi_1,\xi_2,\xi_3)^T = X(\phi,\theta), \ \phi \in I_\phi \wedge \theta \in I_\theta\}$$

where I_ϕ and I_θ are open intervals and $X \in C^1(I_\phi \times I_\theta)$. The length of the normal is represented as a function $L : I_\phi \times I_\theta \to \mathbb{R}^+ \cup \{0\}$,

$$L(\phi, \theta) = \|X_\phi \times X_\theta\|_2 = \left\| \frac{\partial X}{\partial \phi} \times \frac{\partial X}{\partial \theta} \right\|_2,$$

and the area element on $\mathscr{M}$ is $d\mu(\xi) = d\mu(\phi, \theta) = L(\phi, \theta) d\phi\, d\theta$. Next, we consider a global bijective projection $p : \mathscr{M} \to \mathbb{R}^2$. We immediately obtain the following relations between $d\mu$ and the Lebesgue measure dx in the plane:

$$d\boldsymbol{x} = \underbrace{\frac{|\det J_p(\phi, \theta)|}{L(\phi, \theta)}}_{=:\nu(\phi,\theta)} d\mu \quad \text{and} \quad d\mu = \underbrace{\frac{(L \circ p^{-1})(x, y)}{|\det(J_p \circ p^{-1})(x, y)|}}_{=:\rho(x,y)} d\boldsymbol{x}$$

The functions $\nu : \mathscr{M} \to \mathbb{R}^+ \cup \{0\}$ and $\rho : \mathbb{R}^2 \to \mathbb{R}^+ \cup \{0\}$ are *weights* and J_p denotes the Jacobian of p.

In $L^2(\mathscr{M})$ we define the inner product $\langle \cdot, \cdot \rangle_*$ as

$$\langle \tilde{f}, \tilde{g} \rangle_* := \langle \tilde{f} \circ p^{-1}, \tilde{g} \circ p^{-1} \rangle_{L^2(\mathbb{R}^2)} \quad \text{for all } \tilde{f}, \tilde{g} \in L^2(\mathscr{M})$$

This definition yields further

$$\langle f \circ p, g \circ p \rangle_* = \langle f \circ p \circ p^{-1}, g \circ p \circ p^{-1} \rangle_{L^2(\mathbb{R}^2)} = \langle f, g \rangle_{L^2(\mathbb{R}^2)}$$

and leads us finally to the following

Proposition 1. *Let $\Pi : L^2(\mathscr{M}) \to L^2(\mathbb{R}^2, d\boldsymbol{x})$ be an operator defined by*

$$\Pi \tilde{f} := \rho \cdot (\tilde{f} \circ p^{-1}), \quad \textit{for all } \tilde{f} \in L^2(\mathscr{M})$$

Its inverse reads then $\Pi^{-1} f = \nu \cdot (f \circ p)$, $\quad f \in L^2(\mathbb{R}^2, d\boldsymbol{x})$.

Moreover Π (Π^{-1}, respectively) is a unitary operator.

Consequently the orthogonality in $L^2(\mathbb{R}^2, d\boldsymbol{x})$ is transferred to $L^2(\mathscr{M})$! That will allow us to use our known orthogonal wavelet bases in order to construct orthogonal wavelet bases on $\mathscr{M}$.

Remark 4. If there exist constants $0 < m \leqslant M < \infty$ such that

$$m \leqslant \nu(\xi) \leqslant M \quad \text{for all } \xi \in \mathscr{M}$$

then the norm $\| \cdot \|_* = \sqrt{\langle \cdot, \cdot \rangle_*}$, induced by the above defined inner product, is equivalent to the usual 2-norm of $L^2(\mathscr{M}, d\mu)$.

Remark 5. For specific projections see Example 6, page 309, Example 7, page 310, Example 8, page 311, and Remark 3, page 310.

3.4 The Lifted Continuous Wavelet Transform

Following the paragraph before we aim for a “projected” continuous wavelet transform, that is, we lift the usual transform from $\mathbb{R}^2$ to our surface $\mathscr{M}$. Therefore our starting point is the left CWT:

$$L_\psi : L^2(\mathbb{R}^2) \to L^2(G), \quad L_\psi(f)(g) = \langle f, \pi(g)\psi\rangle_{L^2(\mathbb{R}^2)}$$

where $\mathscr{G}$ is the Euclidean group (cf. Section 3.2), i.e.

$$\mathscr{G} = (\mathbb{R}^2, +) \rtimes ((\mathbb{R}^+, \cdot) \times (\mathbb{R}, +))$$

and $\pi(g) = \pi(b, a, \alpha) = T_b D_a R_\alpha$ is the known group representation, that is, $\pi(g)\psi = a^{-1}\psi(a^{-1}r_{-\alpha}(\cdot - b))$

Now we consider a conformal, global bijective projection $p : \mathscr{M} \to \mathbb{R}^2$ between our surface and the plane. That leads us to lifted/projected wavelets and signals:

$$L^2(\mathbb{R}^2) \ni \psi(x, y) \overset{p}{\rightsquigarrow} \tilde{\psi}(\phi, \theta) := (\psi \circ p)(\phi, \theta) \in L^2(\mathscr{M})$$

$$L^2(\mathbb{R}^2) \ni f(x, y) \overset{p}{\rightsquigarrow} \tilde{f}(\phi, \theta) := (f \circ p)(\phi, \theta) \in L^2(\mathscr{M})$$

However we require that $\psi(x, y)$ is an admissible vector in $L^2(\mathbb{R}^2)$ that means ψ satisfies the admissibility condition

$$\int_G |\langle \psi, \pi(g)\psi\rangle_{L^2(\mathbb{R}^2)}|^2 d\mu_L(g) < \infty$$

with the known Haar measure $d\mu_L = a^{-3}d^2b\, da\, d\alpha$.

Definition 9. For $\tilde{f} \in L^2(\mathscr{M})$ the lifted continuous wavelet transform $L^{\mathscr{M}}_{\tilde{\psi}} : L^2(\mathscr{M}) \to L^2(G)$ is defined by

$$L^{\mathscr{M}}_{\tilde{\psi}}(\tilde{f})(g) := \langle \tilde{f}, \tilde{\psi}_g\rangle_* = \int_{\mathscr{M}} \overline{\tilde{\psi}_g(\xi)}\tilde{f}(\xi)\nu^2(\xi)d\omega(\xi)$$

$$= \int_{\mathbb{R}^2} \overline{(\tilde{\psi}_g \circ p^{-1})(x, y)}(\tilde{f} \circ p^{-1})(x, y)dx\, dy = L_{\tilde{\psi}\circ p^{-1}}(\tilde{f} \circ p^{-1})(g),$$

where

$$\tilde{\psi}_g := (\pi(g)(\tilde{\psi} \circ p^{-1})) \circ p = (\pi(g)\psi) \circ p = \psi_g \circ p = a^{-1}\psi(a^{-1}r_{-\alpha}(p(\psi, \theta) - b)) \in L^2(\mathscr{M})$$

Remark 6. It should be mentioned that $\tilde{\psi} = \psi \circ p$ is strictly speaking no admissible vector in the Hilbert-space $L^2(\mathscr{M})$. Although $\langle \tilde{\psi}, \tilde{\psi}_g\rangle_* = \langle \psi, \psi_g\rangle_{L^2(\mathbb{R}^2)}$, we obtain

$$\int_{G_{\mathscr{M}}} |\langle \tilde{\psi}, \tilde{\psi}_g\rangle_*|^2 d\mu_L(g) \neq \int_G |\langle \psi, \psi_g\rangle_{L^2(\mathbb{R}^2)}|^2 d\mu_L(g)$$

since the parameter space G depends on the manifold and it holds generally $G_{\mathscr{M}} \neq G$.

Due to $L_{\tilde{\psi} \circ p^{-1}}(\tilde{f} \circ p^{-1})(g) = L^{\mathscr{M}}_{\tilde{\psi}}(\tilde{f})(g)$ it is easily seen that all properties of the wavelet transform are transferred by the global bijective projection p. More precisely we can state:

Theorem 4. *The lifted continuous wavelet transform has the following properties:*

1. *Plancherel identity: For* $\tilde{f}_1, \tilde{f}_2 \in L^2(\mathscr{M})$ *we have*

$$\int_G L^{\mathscr{M}}_{\tilde{\psi}}(\tilde{f}_1)(g)\overline{L^{\mathscr{M}}_{\tilde{\psi}}(\tilde{f}_2)(g)}d\mu_L(g) = \tilde{c}_{\tilde{\psi}}\langle \tilde{f}_1, \tilde{f}_2\rangle_*$$

   with $\tilde{c}_{\tilde{\psi}} = \|C(\tilde{\psi} \circ p^{-1})\|^2_{L^2(\mathbb{R}^2)}$ *and* C *is a uniquely determined operator.*
2. *Unitarity and invertibility: The wavelet operator* $W^{\mathscr{M}}_{\tilde{\psi}} := (\tilde{c}_{\tilde{\psi}})^{-1/2} L^{\mathscr{M}}_{\tilde{\psi}}$ *is unitary and thus invertible. Further*

$$(L^{\mathscr{M}}_{\tilde{\psi}})^{-1} = (W^{\mathscr{M}}_{\tilde{\psi}})^* = \frac{1}{\tilde{c}_{\tilde{\psi}}} \int_G w(g)\tilde{\psi}_g d\mu_L(g), \quad w \in L^2(G, d\mu_L).$$

Remark 7. The same results can be obtained for the lifted right continuous wavelet transform.

Proof. We have

$$\int_G L^{\mathscr{M}}_{\tilde{\psi}}(\tilde{f}_1)(g)\overline{L^{\mathscr{M}}_{\tilde{\psi}}(\tilde{f}_2)(g)}d\mu_L(g) = \int_G L_{\psi}(f_1)(g)\overline{L_{\psi}(f_2)(g)}d\mu_L(g).$$

For the wavelet transform in $\mathbb{R}^2$ there exists a uniquely determined operator C with

$$\int_G L_{\psi}(f_1)(g)\overline{L_{\psi}(f_2)(g)}d\mu_L(g) = \overline{\langle C\psi_1, C\psi_2\rangle_{L^2(\mathbb{R}^2)}}\,\langle f_1, f_2\rangle_{L^2(\mathbb{R}^2)}$$

$$= ||C(\tilde{\psi} \circ p^{-1})||^2_{L^2(\mathbb{R}^2}\langle \tilde{f}_1, \tilde{f}_2\rangle_*.$$

Further, the map $f \mapsto \frac{1}{\sqrt{\tilde{c}_{\tilde{\psi}}}} L^{\mathscr{M}}_{\tilde{\psi}}$ is an isometry between $L^2(\mathscr{M})$ and $L^2(G, d\mu_L)$. That implies that $W^{\mathscr{M}}_{\tilde{\psi}} = (\tilde{c}_{\tilde{\psi}})^{-1/2} L^{\mathscr{M}}_{\tilde{\psi}}$ is unitary and the inverse transform is given by $(L^{\mathscr{M}}_{\tilde{\psi}})^{-1} = (W^{\mathscr{M}}_{\tilde{\psi}})^*$. □

4 Multiresolution Analysis of $L^2(\mathscr{M})$

Let D be an $n \times n$ nonsingular matrix with the properties:

1. D has only integer entries, i.e. $D\mathbb{Z}^2 \subset \mathbb{Z}^2$
2. $\lambda \in \sigma(D) \Rightarrow |\lambda| > 0$

Then D is called *dilation matrix*.

Definition 10 (Multiresolution Analysis of $\mathbb{R}^2$). A *multi-resolution* analysis of $L^2(\mathbb{R}^n)$ is an increasing sequence of closed subspaces

$$\ldots \subset \boldsymbol{V}_{-2} \subset \boldsymbol{V}_{-1} \subset \boldsymbol{V}_0 \subset \boldsymbol{V}_1 \subset \boldsymbol{V}_2 \subset \ldots$$

where we require

$$\bigcap_{j\in\mathbb{Z}} \boldsymbol{V}_j = \{\boldsymbol{0}\} \text{ and } \overline{\bigcup_{j\in\mathbb{Z}} \boldsymbol{V}_j} = L^2(\mathbb{R}^2)$$

together with

1. $f(\boldsymbol{x}) \in \boldsymbol{V}_j \Leftrightarrow f(D\boldsymbol{x}) \in \boldsymbol{V}_{j+1}, \quad x \in \mathbb{R}^2.$
2. There exists a function $\phi \in \boldsymbol{V}_0$ such that $\{\phi(\boldsymbol{x}-\boldsymbol{k}), \boldsymbol{k} \in \mathbb{Z}^2\}$ is an orthonormal basis of $\boldsymbol{V}_0$.

A particular case is that of tensor product wavelets. Then we obtain a 2D-multi-resolution simply by taking the direct product of two one-dimensional structures ([5]). We take a multi-resolution analysis of $L^2(\mathbb{R})$, namely $\{V_j, j \in \mathbb{Z}\}$, and define by

$$\left\{{}^{(2)}V_j := V_j \otimes V_j, j \in \mathbb{Z}\right\}$$

the resulting multi-resolution analysis associated to the dilation matrix $D = \operatorname{diag}(2,2)$ in $L^2(\mathbb{R}^2)$. The orthogonal complement of ${}^{(2)}V_j$ in ${}^{(2)}V_{j+1}$ is

$${}^{(2)}V_{j+1} = {}^{(2)}V_j \oplus {}^{(2)}W_j$$

and

$$\begin{aligned}
{}^{(2)}V_{j+1} &= V_{j+1}^{(x)} \otimes V_{j+1}^{(y)} \\
&= (V_j^{(x)} \oplus W_j^{(x)}) \otimes (V_j^{(y)} \oplus W_j^{(y)}) \\
&= (V_j^{(x)} \otimes V_j^{(y)}) \oplus \underbrace{(V_j^{(x)} \otimes W_j^{(y)})}_{=:{}^{(2)}W_j^h} \oplus \underbrace{(W_j^{(x)} \otimes V_j^{(y)})}_{=:{}^{(2)}W_j^v} \oplus \underbrace{(W_j^{(x)} \otimes W_j^{(y)})}_{=:{}^{(2)}W_j^d} \\
&= {}^{(2)}V_j \oplus {}^{(2)}W_j
\end{aligned}$$

where ${}^{(2)}W_j$ decomposes in three subspaces ${}^{(2)}W_j = {}^{(2)}W_j^h \oplus {}^{(2)}W_j^v \oplus {}^{(2)}W_j^d$.

The two-dimensional multi-resolution analysis requires only one scaling function

$$\Phi(x,y) := \phi(x)\phi(y) \in {}^{(2)}V_0,$$

but three wavelet functions which detect either horizontal and vertical edges or oblique edges, as the notation suggests (by horizontal edges we mean discontinuities in the vertical direction and vertical edges are discontinuities in the horizontal direction):

$$\begin{aligned}&\Psi^h(x,y) := \phi(x)\psi(y) \in {}^{(2)}W_0^h, \quad \Psi^v(x,y) := \psi(x)\phi(y) \in {}^{(2)}W_0^v,\\ &\Psi^d(x,y) := \psi(x)\psi(y) \in {}^{(2)}W_0^d.\end{aligned}$$

The usual convention

$$\begin{aligned}\Phi^j_{k,l}(x,y) &:= \phi_{j,k}(x)\phi_{j,l}(y), \quad \Psi^{j,h}_{k,l}(x,y) := \phi_{j,k}(x)\psi_{j,l}(y),\\ \Psi^{j,v}_{k,l}(x,y) &:= \psi_{j,k}(x)\phi_{j,l}(y), \quad \Psi^{j,d}_{k,l}(x,y) := \psi_{j,k}(x)\psi_{j,l}(y),\end{aligned}$$

leads immediately to filter-based representations, i.e.

$$\begin{aligned}\Phi(x,y) &= 2\sum_{k,l\in\mathbb{Z}} h_k h_l \phi_{1,k}(x)\phi_{1,l}(y) = 2\sum_{k,l\in\mathbb{Z}} h_{k,l}\Phi^1_{k,l}(x,y) \quad h_{k,l} := h_k h_l,\\ \Psi^h(x,y) &= 2\sum_{k,l\in\mathbb{Z}} h_k g_l \phi_{1,k}(x)\phi_{1,l}(y) = 2\sum_{k,l\in\mathbb{Z}} g^h_{k,l}\Phi^{1,h}_{k,l}(x,y), \quad g^h_{k,l} := h_k g_l,\\ \Psi^v(x,y) &= 2\sum_{k,l\in\mathbb{Z}} g_k h_l \phi_{1,k}(x)\phi_{1,l}(y) = 2\sum_{k,l\in\mathbb{Z}} g^v_{k,l}\Phi^{1,v}_{k,l}(x,y), \quad g^v_{k,l} := g_k h_l,\\ \Psi^d(x,y) &= 2\sum_{k,l\in\mathbb{Z}} g_k g_l \phi_{1,k}(x)\phi_{1,l}(y) = 2\sum_{k,l\in\mathbb{Z}} g^d_{k,l}\Phi^{1,d}_{k,l}(x,y), \quad g^d_{k,l} := g_k g_l,\end{aligned}$$

and we may choose $g_k = (-1)^k\overline{h_{1-k}}$.

Furthermore it can easily be seen that the sets $\{\Phi^j_{k,l},\ k,l \in \mathbb{Z}\}$ together with $\{\Psi^{j,h}_{k,l},\ k,l\in\mathbb{Z}\} \cup \{\Psi^{j,v}_{k,l},\ k,l\in\mathbb{Z}\} \cup \{\Psi^{j,d}_{k,l},\ k,l\in\mathbb{Z}\}$ form orthonormal bases in ${}^{(2)}V_j$ and ${}^{(2)}W_j$.

Thus, a given signal $f \in L^2(\mathbb{R}^2)$ is represented at resolution 2^j by

$$f_j = \sum_{k,l\in\mathbb{Z}} c_{j,k,l}\Phi^j_{k,l} + \sum_{\alpha\in\{h,v,d\}}\sum_{k,l\in\mathbb{Z}} d^\alpha_{j,k,l}\Psi^{j,\alpha}_{k,l},$$

where $c_{j,k,l} = \langle f, \Phi^j_{k,l}\rangle_{L^2(\mathbb{R}^2)}$ and $d^\alpha_{j,k,l} = \langle f, \Psi^{j,\alpha}_{k,l}\rangle_{L^2(\mathbb{R}^2)}$.

4.1 The Lifted Discrete Wavelet Transform

We lift a multi-resolution analysis of $L^2(\mathbb{R}^2)$ to $L^2(\mathscr{M})$. In order to be able to transfer the results easily, we choose again the particular case of tensor product wavelets, i.e. $D = \mathrm{diag}(2,2)$:

$$\left\{ {}^{(2)}\mathscr{V}_j := \left\{ f \circ p : f \in V_j^{(x)} \otimes V_j^{(y)} \right\} \right\}_{j\in\mathbb{Z}}$$

Thus, we obtain ${}^{(2)}\mathscr{V}_{j+1} = {}^{(2)}\mathscr{V}_j \oplus {}^{(2)}\mathscr{W}_j^h \oplus {}^{(2)}\mathscr{W}_j^v \oplus {}^{(2)}\mathscr{W}_j^d$.

Moreover the lifting leads to a family of scaling functions $\tilde{\Phi}^j_{k,l}$ and to three families of wavelet functions $\tilde{\Psi}^{j,\alpha}_{k,l}$, $\alpha \in \{h, v, d\}$:

$$\begin{aligned}
\tilde{\Phi}^j_{k,l}(\phi,\theta) &:= (\Phi^j_{k,l} \circ p)(\phi,\theta) = (\phi_{j,k} \circ p_1)(\phi,\theta) \cdot (\phi_{j,l} \circ p_2)(\phi,\theta),\\
\tilde{\Psi}^{j,h}_{k,l}(\phi,\theta) &:= (\Psi^{j,h}_{k,l} \circ p)(\phi,\theta) = (\phi_{j,k} \circ p_1)(\phi,\theta) \cdot (\psi_{j,l} \circ p_2)(\phi,\theta),\\
\tilde{\Psi}^{j,v}_{k,l}(\phi,\theta) &:= (\Psi^{j,v}_{k,l} \circ p)(\phi,\theta) = (\psi_{j,k} \circ p_1)(\phi,\theta) \cdot (\phi_{j,l} \circ p_2)(\phi,\theta),\\
\tilde{\Psi}^{j,d}_{k,l}(\phi,\theta) &:= (\Psi^{j,d}_{k,l} \circ p)(\phi,\theta) = (\psi_{j,k} \circ p_1)(\phi,\theta) \cdot (\psi_{j,l} \circ p_2)(\phi,\theta),
\end{aligned}$$

where p_1 and p_2 are the components of the projection p, that is, $(x,y) = (p_1(\phi,\theta), p_2(\phi,\theta))$. For each $j \in \mathbb{Z}$, $\{\Psi^{j,\alpha}_{k,l} : k,l \in \mathbb{Z}, \alpha \in \{h,v,d\}\}$ is an orthonormal basis for $\mathscr{W}_j$.

Next, we want to obtain scaling and wavelet filters $\tilde{h}_{k,l}$ and $\tilde{g}_{k,l}$. The defining equations are

$$\begin{aligned}
\tilde{\Phi} &:= \tilde{\Phi}^0_{0,0} = 2\sum_{k,l\in\mathbb{Z}} \tilde{h}_{k,l}\tilde{\Phi}^1_{k,l} \quad \text{(two-scale relation)},\\
\tilde{\Psi}^\alpha &:= \tilde{\Psi}^{0,\alpha}_{0,0} = 2\sum_{k,l\in\mathbb{Z}} \tilde{g}_{k,l}\tilde{\Psi}^{1,\alpha}_{k,l} \quad \alpha \in \{h,v,d\}.
\end{aligned}$$

The projection does not have an influence on these filters since

$$\tilde{h}_{k,l} = \langle \tilde{\Phi}, \tilde{\Phi}^j_{k,l} \rangle_* = \langle \Phi, \Phi^j_{k,l} \rangle_{L^2(\mathbb{R}^2)} = h_{k,l}$$

and

$$4\sum_{k,l\in\mathbb{Z}} h_{k,l}\overline{\tilde{g}^\alpha_{k,l}} = \langle \tilde{\Phi}, \tilde{\Psi}^\alpha \rangle_* = \langle \Phi, \Psi^\alpha \rangle_{L^2(\mathbb{R}^2)} = 4\sum_{k,l\in\mathbb{Z}} h_{k,l}\overline{g^\alpha_{k,l}}.$$

That also implies $\tilde{g}^\alpha_{k,l} = g^\alpha_{k,l}$, $\alpha \in \{h,v,d\}$.

Moreover, the lifting leads to

$$\tilde{\Phi}^j_{k,l}(\phi,\theta) := (\Phi^j_{k,l}\circ p)(\phi,\theta) \quad \text{and} \quad \tilde{\Psi}^{j,\alpha}_{k,l}(\phi,\theta) := (\Psi^{j,\alpha}_{k,l}\circ p)(\phi,\theta).$$

The lifted DWT gives again a recursion to calculate the orthogonal projection of a signal $\tilde{f}\in L^2(\mathscr{M})$ onto the spaces ${}^{(2)}\mathscr{V}_j$ and ${}^{(2)}\mathscr{W}_j$:

$$\begin{aligned}
\langle \tilde{f}, \tilde{\Phi}^j_{k,l}\rangle_* &= \sum_{m,n\in\mathbb{Z}} h_{m-2k,n-2l}\langle \tilde{f}, \tilde{\Phi}^{j+1}_{m,n}\rangle_* \\
&= \sum_{m,n\in\mathbb{Z}} h_{m-2k,n-2l}\langle \tilde{f}\circ p^{-1}, \Phi^{j+1}_{m,n}\rangle_{L^2(\mathbb{R}^2)}, \\
\langle \tilde{f}, \tilde{\Psi}^{j,\alpha}_{k,l}\rangle_* &= \sum_{m,n\in\mathbb{Z}} g^{\alpha}_{m-2k,n-2l}\langle \tilde{f}, \tilde{\Phi}^{j+1}_{m,n}\rangle_* \\
&= \sum_{m,n\in\mathbb{Z}} g^{\alpha}_{m-2k,n-2l}\langle \tilde{f}\circ p^{-1}, \Phi^{j+1}_{m,n}\rangle_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

In general we obtain

Theorem 5. *Let $p:\mathscr{M}\to\mathbb{R}^2$ be the projection and $\{\mathbf{V}_j, j\in\mathbb{Z}\}$ be a general multi-resolution analysis of $L^2(\mathbb{R}^2)$ with dilation matrix $D\in\mathbb{R}^{2\times 2}$. We define $\mathscr{V}_j := \{f\circ p, f\in \mathbf{V}_j\}$. Then the family of sets $\{\mathscr{V}_j, j\in\mathbb{Z}\}$ yields a multi-resolution analysis of $L^2(\mathscr{M})$ with the same dilation matrix.*

Proof. We must check five conditions:

1. increasing sequence of closed subspaces: $\mathscr{V}_j\subset\mathscr{V}_{j+1}$ for all $j\in\mathbb{Z}$.
   The sequence is obviously increasing. Using the unitarity of the map $\pi : L^2(\mathscr{M})\to L^2(\mathbb{R}^2)$ it is obvious that $\mathscr{V}_j$ is a closed subspace of $L^2(\mathscr{M})$.
2. $\bigcap_{j\in\mathbb{Z}}\mathscr{V}_j = \{0\}$ This follows immediately from the definition.
3. The closure $\mathrm{cl}_{L^2(\mathscr{M})}\bigcup_{j\in\mathbb{Z}}\mathscr{V}_j = L^2(\mathscr{M})$.
   Again, that follows immediately from the definition.
4. $\tilde{f}(\xi)\in\mathscr{V}_j \Leftrightarrow \tilde{f}(D\xi)\in\mathscr{V}_{j+1}$ for all $j\in\mathbb{Z}$.
   We have $f(\boldsymbol{x})\in\mathbf{V}_j$ if and only if $f(D\boldsymbol{x})\in\mathbf{V}_{j+1}$. Thus we obtain $f(p(\xi))\in\mathscr{V}_j$ if and only if $f(D\cdot p(\xi))\in\mathscr{V}_{j+1}$.
5. $\exists\tilde{\phi}\in\mathscr{V}_0 :\ \{\tilde{\phi}(\xi-\boldsymbol{k}), \boldsymbol{k}\in\mathbb{Z}^2\}$ is an orthonormal basis in $\mathscr{V}_0$.
   We have $f(\boldsymbol{x})\in\mathbf{V}_j$ if and only if $f(D\boldsymbol{x})\in\mathbf{V}_{j+1}$. Thus we obtain $f(p(\xi))\in\mathscr{V}_j$ if and only if $f(D\cdot p(\xi))\in\mathscr{V}_{j+1}$.

□

5 Frame Theory

We will briefly recall definitions and properties of frames. See, for example, [11]. Let H be a separable Hilbert space. A family $\{f_i\}_{i\in I}$ is a frame for H, if there exist $0 < A \leq B < \infty$ such that for all $f \in H$,

$$A||f||^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B||f||^2.$$

The constants A and B are called lower and upper frame bound for the frame. Those sequences which satisfy only the upper inequality are called Bessel sequences. A frame is tight, if $A = B$, and a Parseval frame if $A = B = 1$.

Two frames $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$ for H are equivalent, if there exists an invertible operator $U : H \to H$ satisfying $Uf_i = g_i$ for all $i \in I$. If U is an unitary operator, $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are called unitarily equivalent.

The synthesis operator $T : \ell^2(I) \to H$ is defined by $T(c) = \sum_{i\in I} c_i f_i$. The adjoint operator $T^* : H \to \ell^2(I)$, the so-called analysis operator is given by $T^*(f) = \{\langle f_i, f\rangle\}_{i\in I}$.

Then the frame operator $S(f) = TT^*(f) = \sum_{i\in I}\langle f, f_i\rangle f_i$ is a bounded, invertible, and positive operator mapping H onto itself. This provides the reconstruction formula

$$f = S^{-1}S(f) = \sum_{i\in I}\langle f, f_i\rangle \tilde{f}_i = \sum_{i\in I}\langle f, \tilde{f}_i\rangle f_i, \quad \tilde{f}_i = S^{-1}f_i.$$

For practical purposes one must discretize the continuous wavelet transform. We emphasize the following:

Theorem 6. *Let* $\{\psi_{m_0,m_1,j,l} : (m_0, m_1, j, l) \in \mathbb{Z}^2 \times \mathbb{Z} \times \mathbb{Z}_{L_0}\}$ *be a frame for the Hilbert-space* $L^2(\mathbb{R}^2)$*, where the set* $\mathbb{Z}^2 \times \mathbb{Z} \times \mathbb{Z}_{L_0}$ *is the discretized parameter space. Moreover, let* $p : \mathscr{M} \to \mathbb{R}^2$ *be the projection. Then the set*

$$\left\{\tilde{\psi}_{m_0,m_1,j,l} := \psi_{m_0,m_1,j,l} \circ p : (m_0, m_1, j, l) \in \mathbb{Z}^2 \times \mathbb{Z} \times \mathbb{Z}_{L_0}\right\} \subset L^2(\mathscr{M})$$

is a frame for $L^2(\mathscr{M})$ *with the same frame bounds.*

Proof. The frame $\{\psi_{m_0,m_1,j,l} : (m_0, m_1, j, l) \in \mathbb{Z}^2 \times \mathbb{Z} \times \mathbb{Z}_{L_0}\}$ satisfies the frame condition, that is

$$A\|f\|^2_{L^2(\mathbb{R}^2)} \leqslant \sum_{\mathbb{Z}^2\times\mathbb{Z}\times\mathbb{Z}_{L_0}} |\langle f, \psi_{m_0,m_1,j,l}\rangle_{L^2(\mathbb{R}^2)}|^2 \leqslant B\|f\|^2_{L^2(\mathbb{R}^2)}$$

Due to $\|\tilde{f}\|^2_* = \|f\|^2_{L^2(\mathbb{R}^2)}$ and $\langle \tilde{f}, \tilde{\psi}\rangle_* = \langle f, \psi\rangle_{L^2(\mathbb{R}^2)}$ we obtain immediately

$$A\|\tilde{f}\|^2_* \leqslant \sum_{\mathbb{Z}^2\times\mathbb{Z}\times\mathbb{Z}_{L_0}} |\langle \tilde{f}, \tilde{\psi}_{m_0,m_1,j,l}\rangle_*|^2 \leqslant B\|\tilde{f}\|^2_* \quad \forall \tilde{f} \in L^2(\mathscr{M})$$

□

5.1 Frames for Locally Conformally Flat Manifolds

A conformal mappings can have umbilic points (singularities). These singularities create problems for the projection wavelets, just think of the north pole of the stereographic projection. On the other hand, a locally conformal manifold can be covered in many different ways by countable family of sets U_i, where the U_i can be conformally mapped to a compact set in $\mathbb{R}^2$ and the mapping has no singularities.

Example 9. The sphere can be covered by a finite set of spherical caps of fixed size. Now, we project each cap by a stereographic projection where center of the cap is in the south pole. That means there are no singularities.

The new problem that arises is that we don't know how our projected wavelets on different sets fit to each other. Due to the different conformal projections we don't get a global wavelet transform. We will build a new Hilbert space on the union of all coverings and prove that the union of all projected frames is a frame for this new Hilbert space.

This idea fits the general concept known as frames of subspaces which are discussed in [10]. Based on the covering we define a Hilbert space

$$H = \oplus_{i\in I} L^2(V_i, d\mu_i) = \{\{f_i\}_{i\in I} : f_i \in L^2(V_i, d\mu_i) \text{ and } \sum_{i\in I} ||f_i||^2_{L^2(V_i,d\mu_i)} < \infty\}$$

with inner product

$$\langle \{f_i\}_{i\in I}, \{g_i\}_{i\in I}\rangle = \sum_{i\in I} \langle f_i, g_i\rangle_* = \int_{V_i} f_i(x) g_i(x)\, d\mu_i,$$

where $d\mu_i = \nu^2(\xi) d\omega(\xi)$ is determined by the (local) conformal transform. There can be different Hilbert spaces for the same manifold $(\mathscr{M}; g)$.

Remark 8. The covering $\{V_i\}$ of $\mathscr{M}$ should be chosen such that each point of $\mathscr{M}$ is an inner point of at least one subset V_i, because singularities have usually their support on set of measure zero and the boundary of V_i is a set of measure zero which is neglected in $L^2(V_i, d\mu_i)$.

Definition 11 ([10]). The family of closed subspaces $\{W_i\}_{i\in I}$ of the Hilbert space H is a frame of subspaces for H, if there exist constants $0 < C \le D < \infty$ such that

$$C||f||^2 \le \sum_{i\in I} ||\pi_{W_i}(f)||^2 \le D||f||^2 \quad \text{for all } f \in H.$$

We have already frames for $L^2(V_i, d\mu_i) = W_i$ and we want to build a frame for H from these wavelet frames.

Definition 12. Let $\{L^2(V_i, d\mu_i)\}_{i\in I}$ be a frame of subspaces for H. Then the synthesis operator H is the operator

$$T_V : \oplus_{i\in I} L^2(V_i, d\mu_i) \to H$$

defined by

$$T_V(f) = \sum_{i\in I} f_i \quad \text{for all } f = \{f_i\} = \sum_i f_i \chi_{V_i} \in \oplus_{i\in I} L^2(V_i, d\mu_i).$$

Theorem 7 ([10]). *For each $i \in I$ let $\{f_{ij}\}_{j\in J_i}$ be a sequence of frames in H with frame bounds A_i and B_i. Define $L^2(V_i, d\mu_i) = \overline{\text{span}}_{j\in J_i}\{f_{ij}\}$ for all $i \in I$ and choose an orthonormal basis $\{e_{ij}\}_{j\in J_i}$ for each subspace $L^2(V_i, d\mu_i)$. Suppose that $0 < A = \inf_{i\in I} A_i \leq B = \sup_{i\in I} B_i < \infty$. The following conditions are equivalent:*

- *$\{f_{ij}\}_{i\in I, j\in J_i}$ is a frame for H.*
- *$\{e_{ij}\}_{i\in I, j\in J_i}$ is a frame for H.*
- *$\{L^2(V_i, d\mu_i)\}_{i\in I}$ is a frame of subspaces for H.*

Remark 9. In what follows we denote by π_{V_i} the projection onto $L^2(V_i, d\mu_i)$.

The following results are adaptions of results from [10] to our case.

Proposition 2. *Let $\{L^2(V_i, d\mu_i)\}_{i\in I}$ be a frame of subspaces for H. Then the analysis operator $T_V^* : H \to \oplus_{i\in I} L^2(V_i, d\mu_i)$ is given by*

$$T_V^*(f) = \{\pi_{V_i}(f)\}_{i\in I}.$$

Proof. Let $f \in H$ and $g = \{g_i\}_{i\in I} \in \oplus_{i\in I} L^2(V_i, d\mu_i)$. By definition we get

$$\begin{aligned}\langle T_V^*(f), g\rangle &= \langle f, T_V(g)\rangle = \langle f, \sum_{i\in I} g_i\rangle = \sum_{i\in I}\langle f, g_i\rangle \\ &= \sum_{i\in I}\langle \pi_{V_i}(f), g_i\rangle_* = \langle \{\pi_{V_i}(f)\}_{i\in I}, \{g_i\}_{i\in I}\rangle.\end{aligned}$$

□

The standard frame theory can be extended to our case.

Theorem 8. *Let $\{L^2(V_i, d\mu_i)\}_{i\in I}$ be a family of subspaces in H. Then the following conditions are equivalent:*

(i) $\{L^2(V_i, d\mu_i)\}_{i\in I}$ is a frame of subspaces.
(ii) The synthesis operator T_V is bounded, linear, and onto.
(iii) The analysis operator T_V^* is a (possibly into) isomorphism.

Proof. $(ii) \iff (iii)$ is a general result for operators on a Hilbert space. The equivalence $(i) \iff (iii)$ follows immediately from

$$||T_V^*(f)||^2 = ||\{\pi_{V_i}(f)\}_{i\in I}||^2 = \sum_{i\in I} ||\pi_{V_i}(f)||^2.$$

□

The next step is to define equivalence classes of frames of subspaces.

Definition 13. Let $\{L^2(V_i, d\mu_i)\}_{i\in I}$ and $\{L^2(\widetilde{V}_i, d\tilde{\mu}_i\}_{i\in I}$ be frames of subspaces. Then the following two conditions are equivalent:

(i) $\{L^2(V_i, d\mu_i)\}_{i\in I}$ and $\{L^2(\widetilde{V}_i, d\tilde{\mu}_i\}_{i\in I}$ are equivalent.
(ii) There exists an invertible operator U on H such that $T_V = U^{-1}T_{\tilde{V}}U$.

If the operator U is unitary, the frames are called unitarily equivalent.

The frame can be built from the frames for each subspaces.

Definition 14. Let $\{L^2(V_i, d\mu_i)\}_{i\in I}$ be a frame of subspaces. Then the frame operator S_V is defined by

$$S_V(f) = T_V T_V^*(f) = T_V(\{\pi_{V_i}(f)\}_{i\in I}) = \sum_{i\in I} \pi_{V_i}(f).$$

Theorem 9. *Let $\{L^2(V_i, d\mu_i)\}_{i\in I}$ be a frame of subspaces with frame bounds C and D. Then the frame operator S_V is a positive, self-adjoint, invertible operator on H with $CI \le S_V \le DI$. Further,*

$$f = \sum_{i\in I} S_V^{-1}\pi_{V_i}(f) \quad \textit{for all } f \in H.$$

Proof. The operator S_v is positive because for any $f \in H$ we have

$$\langle S_V(f), f\rangle = \langle \sum_{i\in I} \pi_{V_i}(f), f\rangle = \sum_{i\in I} \langle \pi_{V_i}(f), f\rangle = \sum_{i\in I} ||\pi_{V_i}(f)||^2.$$

Further, we obtain

$$\langle Cf, f\rangle = C||f||^2 \le \sum_{i\in I} ||\pi_{V_i}(f)||^2 = \langle S_V(f), f\rangle \le \langle Df, f\rangle$$

and we conclude that $CI \le S_V \le DI$ and hence S_V is an invertible operator on H. For $f, g \in H$ we have

$$\langle S_V(f), g\rangle = \sum_{i\in I} \langle \pi_{V_i}(f), g\rangle = \sum_{i\in I} \langle f, \pi_{V_i}(g)\rangle = \langle f, S_V(g)\rangle.$$

Thus S_V is self-adjoint. The reconstruction formula follows from

$$f = S_V^{-1} S_V(f) = \sum_{i \in I} S_V^{-1} \pi_{V_i}(f).$$

□

6 Examples

In the last section we present three different examples of projection wavelets, where the last two examples are variations of the first.

6.1 *Example 1*

Our aim is the illustration of a discrete wavelet transform on the sphere and on the rotated nephroid.

We consider the function $f(\phi, \theta + \theta_0) \in L^2(\mathscr{M})$, which was also used in [28], defined by

$$f(\phi, \theta) := \begin{cases} 1, & \theta \geqslant \frac{\pi}{2} \\ (1 + 3\cos^2\theta)^{-1/2}, & \theta < \frac{\pi}{2} \end{cases}$$

where $\theta_0 \in [0, \pi]$ is an arbitrary constant. It can easily be seen that the function $f(\phi, \theta + \theta_0)$ and its gradient ∇f are continuous. However, the second partial derivative with respect to θ has a discontinuity on the circle $\theta = \frac{\pi}{2} - \theta_0$ since

$$\lim_{\theta \to \pi/2 - \theta_0 - 0} \partial_\theta^2 f(\phi, \theta) = -3 \quad \text{but} \quad \lim_{\theta \to \pi/2 - \theta_0 + 0} \partial_\theta^2 f(\phi, \theta) = 0.$$

Figure 1a,b illustrates the data where the green dashed line represents the circle $\theta = \frac{\pi}{2} - \theta_0$:

That jump discontinuity is supposed to be detected by a discrete wavelet transform, i.e. the discontinuity should be seen in the wavelet coefficients. Therefore, we need appropriate wavelets. To find such wavelets we expand $f_0 := f\big|_{\{const.\} \times [0,\pi]}$ into its Taylor series around point θ_0:

$$f_0(\theta) = \underbrace{f_0(\theta_0) + \frac{\partial f_0}{\partial \theta}(\theta_0)(\theta - \theta_0)}_{\text{continuous for all } \theta_0 \in [0,\pi]} + \frac{1}{2}\frac{\partial^2 f_0}{\partial \theta^2}(\theta_0)(\theta - \theta_0)^2 + \dots$$

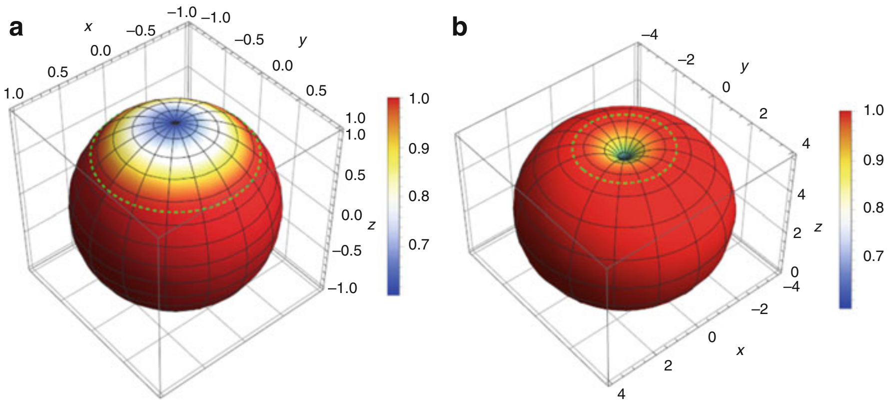

Fig. 1 (**a**) Data on the sphere with $\theta_0 = 40°$. (**b**) Data on the rotated nephroid with $\theta_0 = 40°$.

Since we have a discontinuous second derivative and a continuous function and first derivative the affine-linear term is continuous for all θ_0. That means that the left term does not indicate the discontinuity and all necessary information are contained in the right sum. Consequently we must suppress the constant and the linear term, what can be achieved by using a wavelet with two vanishing moments at least. We use the Daubechies D6 wavelet that has three vanishing moments.

It can easily be seen that scaling and wavelet filters do not depend on the manifold and, furthermore, that a given signal on the plane $\mathbb{R}^2$ or on the manifold $\mathscr{M}$ can be represented by the same matrix $F = (f_{i,j})$ (its discretization). After generating the discrete data set

$$F := \left(f\left(k\frac{2\pi}{K}, \theta_0 + l\frac{\pi}{L}\right)\right)_{k,l=0}^{K,L} \in \mathbb{R}^{(K+1)\times(L+1)}$$

we obtain the discrete wavelet transform of F. Detecting the discontinuity is then equivalent with detecting either a horizontal edge or vertical edge in the matrix F, depending on how one chooses the axes. Since crucial information for edge detection are stored in the wavelet spaces $\mathscr{W}_j$ we are interested in the coefficients $\tilde{d}^{j,\alpha}_{k,l}$ where $\alpha = v$ or $\alpha = h$. Moreover, it becomes apparent that a one-level decomposition is already sufficient and we obtain finally

$$\tilde{d}^{-1,\alpha}_{k,l} = \sum_{m,n} g^{\alpha}_{m-2k,n-2l} F_{m,n}$$

where $F_{m,n}$ is the matrix entry of F at position (m, n) and $\alpha \in \{v, h\}$.

Figure 2 shows the mentioned wavelet coefficients as a discrete plot where the angle ϕ is fixed. The horizontal axis represents the polar angle θ. We see that the critical point $\theta = 90° - 40° = 50°$ was detected and the chosen Daubechies D6 wavelet does the job easily as expected.

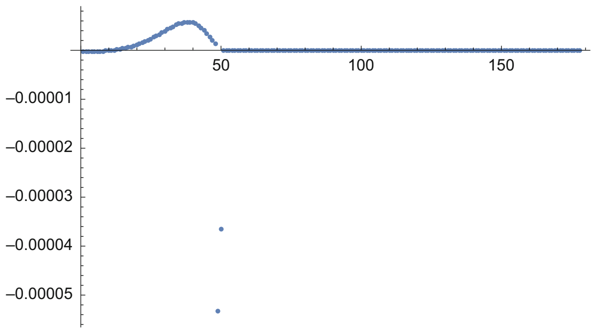

Fig. 2 Discrete plot with $\theta_0 = 40°$.

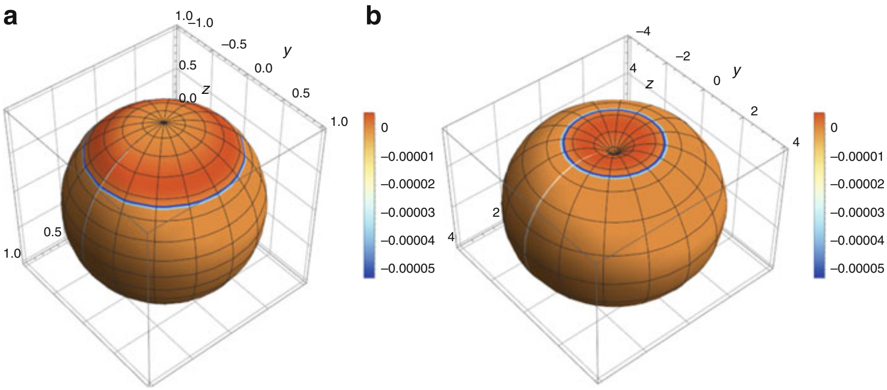

Fig. 3 (**a**) DWT on the sphere, $\theta_0 = 40°$. (**b**) DWT on the rotated nephroid, $\theta_0 = 40°$.

The next step is to lift everything back onto the manifold. The corresponding results are illustrated in Figure 3.

The critical circle $\theta = \frac{\pi}{2} - \theta_0$ is distinctly recognizable.

6.2 *Example 2*

So far, we have dealt with rather simple manifolds. But our implementation of the discrete wavelet transform is not limited to these manifolds. In order to demonstrate this we extend the previous example and consider more "distorted" surfaces. Such surfaces can be obtained by combining the original parametrization with an additional trigonometrical term. In our case we replace the two-sphere from the first example with a modified sphere $\mathscr{M}$ whose parametrization is given by

$$\begin{cases} x = \cos\phi\left(\sin\theta + \frac{1}{30}\sin(19\theta)\right) \\ y = \sin\phi\left(\sin\theta + \frac{1}{30}\sin(19\theta)\right) \quad \phi \in [0, 2\pi],\ \theta \in [0, \pi] \\ z = 1 - \cos\theta - \frac{1}{30}\cos(19\theta) \end{cases}$$

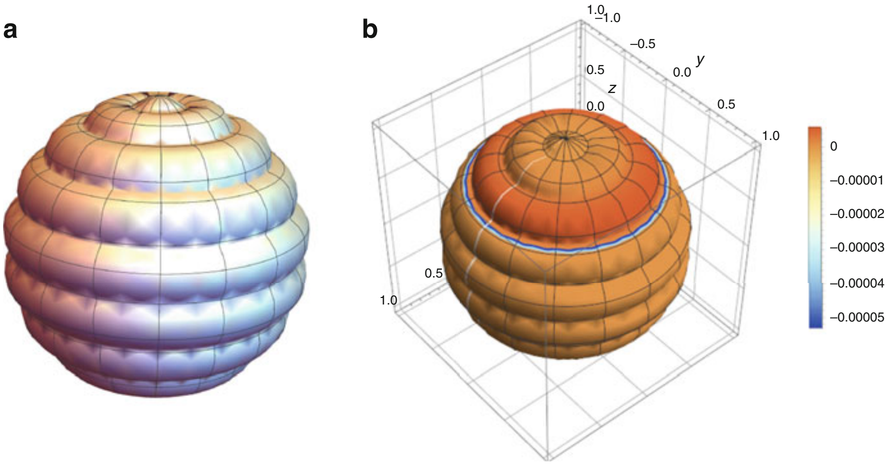

Fig. 4 (**a**) The modified sphere. (**b**) DWT on the modified sphere, $\theta_0 = 35°$.

Now, let us apply the method from Example 1 to the modified sphere. The modified manifold does not affect numerical results since all filters and the discretization of the signal are independent of the chosen surface (Figure 4).

The function $f \in L^2(\mathscr{M})$ remains unchanged, but is now defined on the modified sphere. The green dashed line represents again the critical circle $\theta = \frac{\pi}{2} - \theta_0$.

As we have already mentioned, there are no quantitative differences between the wavelet coefficients. We can proceed as described in Example 1 (with adjusted θ_0), but at the end we lift our results onto $\mathscr{M}$ instead of lifting them onto the two-sphere.

The jump discontinuity is just as well detected.

6.3 *Example 3*

In the last example we want to construct a more complicated function, but our aim is again a discrete wavelet transform on the two-sphere. Let us start by modifying the function $f \in L^2(\mathscr{M})$ such that the occurring jump discontinuities of the second partial derivative (with respect to θ) are no longer on a circle $\theta = const$. We require that all points of discontinuity vary sinusoidally from the equatorial angle ϕ. That leads to

$$f(\phi, \theta) := \begin{cases} 1, & \theta \geqslant \frac{\pi}{2} + \sin(4\phi)/7 \\ \left(1 + 3\cos^2\left(\frac{\pi}{\pi + 2\sin(4\phi)/7}\theta\right)\right)^{-1/2}, & \theta < \frac{\pi}{2} + \sin(4\phi)/7 \end{cases}$$

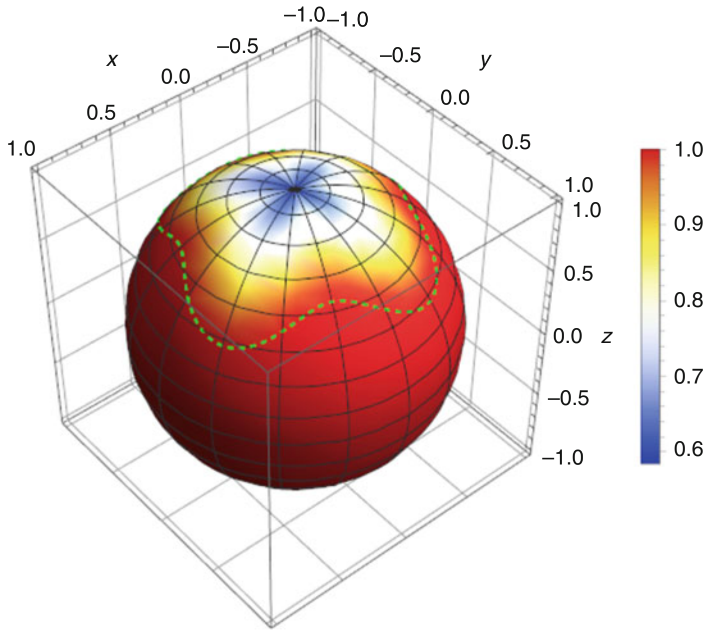

Fig. 5 The modified function on the two-sphere with $\theta_0 = 40°$.

The function $f(\phi, \theta + \theta_0)$, $\theta_0 \in [0, \pi]$, and the gradient ∇f are still continuous, but the second partial derivative with respect to θ has all discontinuities located on the critical line

$$\theta = \frac{\pi}{2} + \frac{\sin(4\phi)}{7} - \theta_0.$$

Figure 5 illustrates the effect on the two-sphere (the dashed line represents the critical line):

We generate the new discrete data set

$$F = \left(f \left(k\frac{2\pi}{K}, \theta_0 + l\frac{\pi}{L} \right) \right)_{k,l=0}^{K,L} \in \mathbb{R}^{(K+1)\times(L+1)}$$

But we do not know which wavelet space $\mathscr{W}_{-1}^{\alpha}$ contains the most significant information to detect the discontinuity. In the first example the decision was quite easy since we had to detect either a horizontal edge or a vertical edge. Here we have a curved edge and it might be that even all three wavelet spaces are qualified for detecting. Hence we must experiment and choose coefficients which are related to two different wavelet spaces:

1. In order to achieve comparability we choose the same wavelet that we have already used in example 1 ($\alpha = v$ or $\alpha = h$, depending on the axes)
2. Furthermore, we consider the case $\alpha = d$ since the diagonal wavelet detects normally oblique edges very well.

It turns out that the diagonal wavelet detects indeed only the oblique part and is not able to detect the discontinuity completely.

7 Final Comments

We have demonstrated that projection wavelets can be defined on a conformal manifold. Any 2-dimensional Riemannian manifold is locally conformally flat which gives a wide range of applications to the projection method.

Although a locally conformally flat manifold can have a global conformal mapping, this mapping might have singularities. To avoid these singularities a patch of locally conformally flat manifolds can be used. Similar problems occur in the case of a simple grid of $\mathbb{R}^2$ that becomes distorted on the manifold.

The overall advantage of projections methods is that they are simple to apply to a large number of manifolds. The main examples are the two-dimensional isothermic manifolds. But the method is not restricted to two-dimensional manifolds.

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Quasi Monte Carlo Integration and Kernel-Based Function Approximation on Grassmannians

Anna Breger, Martin Ehler, and Manuel Gräf

1 Introduction

The present paper is dedicated to numerical experiments concerning two classical problems in numerical analysis, numerical integration and function approximation. The novelty of our experiments is that we work on the Grassmannian manifold as an example of a compact Riemannian manifold illustrating theoretical results in the recent literature.

Indeed, recent data analysis methodologies involve kernel-based approximation of functions on manifolds and other measure spaces, cf. [25, 26] and [17–19]. The kernels are build up by what is known as diffusion polynomials, which are eigenfunctions of elliptic differential operators, commonly chosen as the Laplace-Beltrami operator when dealing with compact Riemannian manifolds.

Numerical implementations of the approximation schemes require pointwise evaluation of the eigenfunctions. However, explicit formulas for eigenfunctions are only known in few special cases. If the manifold is the unit sphere $\mathbb{S}^{d-1}$, for instance, then the eigenfunctions of the spherical Laplacian are the spherical harmonics, which are indeed polynomials in the usual sense. The corresponding kernels for the sphere have been computed explicitly in [23, 24].

The kernel-based approximation requires the computation of the corresponding integral operator, see (7) in Section 3. In the realm of numerical integration, the integral itself is usually approximated by a weighted sum over sample values, see also [15, 16]. The latter fits well to the common scenario when the target function needs to be approximated from a finite sample in the first place.

A. Breger • M. Ehler (✉) • M. Gräf
Department of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
e-mail: anna.breger@univie.ac.at; martin.ehler@univie.ac.at; manuel.graef@univie.ac.at

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Numerical integration on Euclidean spaces is a classical problem in numerical analysis. Recently, Quasi Monte Carlo (QMC) numerical integration on compact Riemannian manifolds has been studied in [6] from a theoretical point of view, see also [28]. If more and more samples are used, then the smoothness parameter of Bessel potential spaces steers the decay of the integration error. QMC integration has been introduced for the sphere in [7], where many explicit examples are provided and extensive numerical experiments illustrate the theoretical claims.

The major aim of the present paper is to provide numerical experiments for the above integration and approximation schemes when the manifold is the Grassmannian, i.e., the collection of k-dimensional subspaces in $\mathbb{R}^d$, naturally identified with the collection $\mathcal{G}_{k,d}$ of rank-k orthogonal projectors on $\mathbb{R}^d$, cf. [8, Chapter 1].

Therefore, we require explicit formulas of the kernels used in [25, 26]. Indeed, the degree of a diffusion polynomial, by definition, relates to the magnitude of the corresponding eigenvalues. We check that diffusion polynomials of degree at most $2t/\sqrt{k}$ are indeed usual multivariate polynomials of degree t restricted to the Grassmannian. The explicit formula for the kernel is derived through generalized Jacobi polynomials. By computing cubature formulas on Grassmannians through some numerical minimization process, we are able to provide numerical experiments for the approximation of functions on Grassmannians and for the QMC integration on Grassmannians supporting the theoretical results in [6, 25, 26].

The outline is as follows: In Section 2 we recall QMC integration from [6, 7], and we recall the approximation scheme from [25, 26] in the special case of the Grassmannian in Section 3. Section 4 provides feasible formulations for numerical experiments. Indeed, Section 4.1 is devoted to derive explicit formulas for the involved kernel by means of generalized Jacobi polynomials. We check the relations between diffusion polynomials and ordinary polynomials restricted to the Grassmannian in Section 4.2, and we provide the framework for numerically computing cubatures in Grassmannians in Section 4.3. The numerical experiments are provided in Section 5.

2 Quasi Monte Carlo Integration

We identify the Grassmannian, the collection of k-dimensional subspaces in $\mathbb{R}^d$, with the set of orthogonal projectors on $\mathbb{R}^d$ of rank k, denoted by

$$\mathcal{G}_{k,d} := \{P \in \mathbb{R}^{d\times d}_{\text{sym}} : P^2 = P;\ \operatorname{tr}(P) = k\}.$$

Here, $\mathbb{R}^{d\times d}_{\text{sym}}$ is the set of symmetric matrices in $\mathbb{R}^{d\times d}$ and $\operatorname{tr}(P)$ denotes the trace of P. The dimension of the Grassmannian is $\dim(\mathcal{G}_{k,d}) = k(d-k)$. The canonical Riemannian measure on $\mathcal{G}_{k,d}$ is denoted by $\mu_{k,d}$, which we assume to be normalized to one. Without loss of generality, we assume $k \le \frac{d}{2}$ throughout since $\mathcal{G}_{d-k,d}$ can be identified with $\mathcal{G}_{k,d}$.

As a classical problem in numerical analysis, we aim to approximate the integral over a continuous function $f : \mathcal{G}_{k,d} \to \mathbb{C}$ by a finite sum over weighted samples, i.e., we consider points $\{P_j\}_{j=1}^n \subset \mathcal{G}_{k,d}$ and nonnegative weights $\{\omega_j\}_{j=1}^n$ such that

$$\sum_{j=1}^{n} \omega_j f(P_j) \approx \int_{\mathcal{G}_{k,d}} f(P)\mathrm{d}\mu_{k,d}(P).$$

In order to quantify the error by means of the smoothness of f, we shall define Bessel potential spaces on $\mathcal{G}_{k,d}$, for which we need some preparation. Let $\{\varphi_\ell\}_{\ell=0}^\infty$ be the collection of orthonormal eigenfunctions of the Laplace-Beltrami operator Δ on $\mathcal{G}_{k,d}$, and $\{-\lambda_\ell\}_{\ell=0}^\infty$ are the corresponding eigenvalues arranged, so that $0 = \lambda_0 \le \lambda_1 \le \ldots$. Without loss of generality, we choose each φ_ℓ to be real-valued, in particular, $\varphi_0 \equiv 1$. The Fourier transform of $f \in L_p(\mathcal{G}_{k,d})$, where $1 \le p \le \infty$, is defined by

$$\hat{f}(\ell) := \int_{\mathcal{G}_{k,d}} f(P)\varphi_\ell(P)\mathrm{d}\mu_{k,d}(P), \qquad \ell = 0, 1, 2, \ldots.$$

Essentially following [6, 26], we formally define $(I - \Delta)^{s/2} f$ to be the distribution on $\mathcal{G}_{k,d}$, such that $\langle (I - \Delta)^{s/2} f, \varphi_\ell\rangle = (1 + \lambda_\ell)^{s/2}\langle f, \varphi_\ell\rangle$, for all $\ell = 0, 1, 2, \ldots$. The Bessel potential space $H_p^s(\mathcal{G}_{k,d})$, for $1 \le p \le \infty$ and $s \ge 0$, is

$$H_p^s(\mathcal{G}_{k,d}) := \{f \in L_p(\mathcal{G}_{k,d}) : \|f\|_{H_p^s} < \infty\}, \quad \text{where}$$
$$\|f\|_{H_p^s} := \|(I - \Delta)^{s/2} f\|_{L_p},$$

i.e., $f \in H_p^s(\mathcal{G}_{k,d})$ if and only if $f \in L_p(\mathcal{G}_{k,d})$ and $(I - \Delta)^s f \in L_p(\mathcal{G}_{k,d})$. Note that this definition is indeed consistent with [6, 26], see [6, Theorem 2.1, Definition 2.2] in particular. For $s > k(d-k)/p$ with $1 \le p \le \infty$, the space $H_p^s(\mathcal{G}_{k,d})$ is embedded into the space of continuous functions on $\mathcal{G}_{k,d}$, see, for instance, [6]. For $1 < p < \infty$, this embedding also follows from results on Bessel potential spaces on general Riemannian manifolds with bounded geometry, cf. [32, Theorem 7.4.5, Section 7.4.2], and on $\mathbb{R}^d$ with $1 \le p \le \infty$, see [31, Chapter V, 6.11]. According to [6], for any sequence of points $\{P_j^t\}_{j=1}^{n_t} \subset \mathcal{G}_{k,d}$, $t = 0, 1, 2, \ldots$, and positive weights $\{\omega_j^t\}_{j=1}^{n_t} \subset \mathbb{R}$ with $n_t \to \infty$, there is a function $f \in H_p^s(\mathcal{G}_{k,d})$ such that[†]

$$\left| \int_{\mathcal{G}_{k,d}} f(P)\mathrm{d}\mu_{k,d}(P) - \sum_{j=1}^{n_t} \omega_j^t f(P_j^t)\right| \gtrsim n_t^{-\frac{s}{k(d-k)}} \|f\|_{H_p^s}, \tag{1}$$

[†]We use the notation $\gtrsim$, meaning the right-hand side is less or equal to the left-hand side up to a positive constant factor. The symbol $\lesssim$ is used analogously, and $\asymp$ means both hold, $\lesssim$ and $\gtrsim$. If not explicitly stated, the dependence or independence of the constants shall be clear from the context.

where the constant does not depend on t. Thus, we cannot do any better than the rate

$$n_t^{-\frac{s}{k(d-k)}}.$$

In order to quantify the quality of weighted point sequences $\{(P_j^t, \omega_j^t)\}_{j=1}^{n_t}$, $t = 0, 1, 2, \ldots$, for numerical integration, we make the following definition, whose analogous formulation on the sphere (with constant weights) is due to [7].

Definition 1. Given $s > k(d-k)/p$, a sequence $\{(P_j^t, \omega_j^t)\}_{j=1}^{n_t}$, $t = 0, 1, 2, \ldots$, of n_t points in $\mathcal{G}_{k,d}$ and positive weights with $n_t \to \infty$ is called a *sequence of Quasi Monte Carlo (QMC) systems for* $H_p^s(\mathcal{G}_{k,d})$ if

$$\Big| \int_{\mathcal{G}_{k,d}} f(P) \mathrm{d}\mu_{k,d}(P) - \sum_{j=1}^{n_t} \omega_j^t f(P_j^t) \Big| \lesssim n_t^{-\frac{s}{k(d-k)}} \|f\|_{H_p^s}$$

holds for all $f \in H_p^s(\mathcal{G}_{k,d})$.

In case $p = 2$, given $s > k(d-k)/2$, any sequence of QMC systems $\{(P_j^t, \omega_j^t)\}_{j=1}^{n_t}$ for $H_2^s(\mathcal{G}_{k,d})$ is also a sequence of QMC systems for $W_2^{s'}(\mathcal{G}_{k,d})$, for all s' satisfying $s \geq s' > k(d-k)/2$, cf. [6].

Especially for the integration of smooth functions, random points lack quality when compared to QMC systems.

Proposition 1. *For $s > k(d-k)/2$, suppose $P_1, \ldots, P_n$ are random points on $\mathcal{G}_{k,d}$, independently identically distributed according to $\mu_{k,d}$ then it holds*

$$\sqrt{\mathbb{E}\Big[\sup_{\substack{f \in H_2^s(\mathcal{G}_{k,d}) \\ \|f\|_{H_2^s} \leq 1}} \Big| \int_{\mathcal{G}_{k,d}} f(P) \mathrm{d}\mu_{k,d}(P) - \frac{1}{n} \sum_{j=1}^{n} f(P_j) \Big|^2 \Big]} = cn^{-\frac{1}{2}}$$

with $c^2 = \sum_{\ell=1}^{\infty} (1 + \lambda_\ell)^{-s}$.

Note that the condition $s > k(d-k)/2$ implies that $\frac{s}{k(d-k)} > \frac{1}{2}$, so that on average QMC systems indeed perform better than random points for smooth functions. The proof of Proposition 1 is derived by following the lines in [7]. In fact, the result is already contained in [20, Corollary 2.8], see also [27], within a more general setting.

In order to derive QMC systems, we shall have a closer look at cubature points, for which we need the space of *diffusion polynomials* of degree at most t, defined by

$$\Pi_t := \operatorname{span}\{\varphi_\ell : \lambda_\ell \leq t^2\}, \tag{2}$$

see [26], and references therein.

Definition 2. For $\{P_j\}_{j=1}^n \subset \mathcal{G}_{k,d}$ and positive weights $\{\omega_j\}_{j=1}^n$, we say that $\{(P_j, \omega_j)\}_{j=1}^n$ is a *cubature for* Π_t if

$$\int_{\mathcal{G}_{k,d}} f(P)\mathrm{d}\mu_{k,d}(P) = \sum_{j=1}^{n} \omega_j f(P_j), \quad \text{for all } f \in \Pi_t. \tag{3}$$

The number t refers to the *strength* of the cubature.

In the following result, cf. [6, Theorem 2.12], the cubature error is bounded by the cubature strength t, not the number of points.

Theorem 1. *Suppose $s > k(d-k)/p$ and assume that $\{(P_j^t, \omega_j^t)\}_{j=1}^{n_t}$ is a cubature for Π_t. Then we have, for $f \in H_p^s(\mathcal{G}_{k,d})$,*

$$\Big| \int_{\mathcal{G}_{k,d}} f(P)\mathrm{d}\mu_{k,d}(P) - \sum_{j=1}^{n_t} \omega_j^t f(P_j^t) \Big| \lesssim t^{-s} \|f\|_{H_p^s}.$$

Weyl's estimates on the spectrum of an elliptic operator yield

$$\dim(\Pi_t) \asymp t^{k(d-k)},$$

cf. [21, Theorem 17.5.3]. This implies that any sequence of cubatures $\{(P_j^t, \omega_j^t)\}_{j=1}^{n_t}$ of strength t, respectively, must obey $n_t \gtrsim t^{k(d-k)}$ asymptotically in t, cf. [10]. There are indeed sequences of cubatures $\{(P_j^t, \omega_j^t)\}_{j=1}^{n_t}$ of strength t, respectively, satisfying

$$n_t \asymp t^{k(d-k)}, \tag{4}$$

cf. [10]. In this case, Theorem 1 leads to

$$\Big| \int_{\mathcal{G}_{k,d}} f(P)\mathrm{d}\mu_{k,d}(P) - \sum_{j=1}^{n_t} \omega_j^t f(P_j^t) \Big| \lesssim n_t^{-\frac{s}{k(d-k)}} \|f\|_{H_p^s}, \tag{5}$$

so that we have settled that QMC systems do exist, for any $s > k(d-k)/p$, and can be derived via cubatures.

Remark 1. Cubature points $\{P_j\}_{j=1}^n$ for Π_t with constant weights $\omega_j = \frac{1}{n}$ are called t-designs. For all $t = 1, 2, \ldots$, there exist t-designs, cf. [30]. The results in [5] imply that there are t-designs satisfying (4) provided that $k = 1$. By invoking the results in [14], the latter also extends to $2 \leq k \leq d/2$.

3 Approximation by Diffusion Kernels

One example, where integrals over the Grassmannian are replaced with weighted finite sums, is the approximation of a function $f : \mathcal{G}_{k,d} \to \mathbb{C}$ from finitely many samples. The approximation scheme developed in [25, 26] works for manifolds and metric measure spaces in general, but we shall restrict the presentation to the Grassmannian.

For a function $f \in L_p(\mathcal{G}_{k,d})$, we denote the (polynomial) *best approximation error* by

$$\operatorname{dist}(f, \Pi_t)_{L_p} := \inf_{g \in \Pi_t} \|f - g\|_{L_p},$$

where $t \geq 0$. It is possible to quantify the best approximation error in dependence of the function's smoothness, see [26, Proposition 5.3] for the following result.

Theorem 2. *If $f \in H^s_p(\mathcal{G}_{k,d})$, then*

$$\operatorname{dist}(f, \Pi_t)_{L_p} \lesssim t^{-s} \|f\|_{H^s_p}.$$

Given $f \in H^s_p(\mathcal{G}_{k,d})$ we now construct a particular sequence of functions $\sigma_t(f) \in \Pi_t$, $t = 1, 2, \ldots$, that realizes this best approximation rate. Note that, since the collection $\{\varphi_\ell\}_{\ell=0}^{\infty}$ is an orthonormal basis for $L_2(\mathcal{G}_{k,d})$, any function $f \in L_2(\mathcal{G}_{k,d})$ can be expanded as a Fourier series by

$$f = \sum_{\ell=0}^{\infty} \hat{f}(\ell) \varphi_\ell.$$

The approach in [25, 26] makes use of a smoothly truncated Fourier expansion of f,

$$\sigma_t(f) := \sum_{\ell=0}^{\infty} h(t^{-2}\lambda_\ell) \hat{f}(\ell) \varphi_\ell \in \Pi_t,$$

where $h : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is an infinitely often differentiable and nonincreasing function with $h(x) = 1$, for $x \leq 1/2$, and $h(x) = 0$, for $x \geq 1$. Using the kernel K_t on $\mathcal{G}_{k,d} \times \mathcal{G}_{k,d}$ defined by

$$K_t(P, Q) = \sum_{\ell=0}^{\infty} h(t^{-2}\lambda_\ell) \varphi_\ell(P) \varphi_\ell(Q) \tag{6}$$

we arrive after interchanging summation and integration at the following alternative representation

$$\sigma_t(f) = \int_{\mathcal{G}_{k,d}} f(P) K_t(P, \cdot) \mathrm{d}\mu_{k,d}(P). \tag{7}$$

Note, that the function $\sigma_t(f)$ is well-defined for general $f \in L_p(\mathcal{G}_{k,d})$, $1 \le p \le \infty$, and it turns out that $\sigma_t(f)$ approximates f up to a constant as good as the best approximation from Π_t, cf. [26, Proposition 5.3].

Theorem 3. *If $f \in H^s_p(\mathcal{G}_{k,d})$, then*

$$\|f - \sigma_t(f)\|_{L_p} \lesssim t^{-s}\|f\|_{H^s_p}.$$

If f needs to be approximated from a finite sample, then $\sigma_t(f)$ in (7) cannot be determined directly and is replaced with a weighted finite sum in [26]. Indeed, for sample points $\{P_j\}_{j=1}^n \subset \mathcal{G}_{k,d}$ and weights $\{\omega_j\}_{j=1}^n$, we define

$$\sigma_t(f, \{(P_j, \omega_j)\}_{j=1}^n) := \sum_{j=1}^n \omega_j f(P_j) K_t(P_j, \cdot). \tag{8}$$

Note that we must now consider functions f in Bessel potential spaces, for which point evaluation makes sense. We shall observe in the following that if samples and weights satisfy some cubature type property, then the approximation rate is still preserved when using $\sigma_t(f, \{(P_j, \omega_j)\}_{j=1}^n)$ in place of $\sigma_t(f)$. However, we need an additional technical assumption on the points $\{P_j\}_{j=1}^n$, for which we denote the geodesic distance between $P, Q \in \mathcal{G}_{k,d}$ by

$$\rho(P, Q) = \sqrt{\theta_1^2 + \ldots + \theta_k^2},$$

where $\theta_1, \ldots \theta_k$ are the principal angles between the associated subspaces of P and Q, respectively, i.e.,

$$\theta_i = \arccos(\sqrt{y_i}), \quad i = 1, \ldots, k,$$

and $y_1, \ldots, y_k$ are the k largest eigenvalues of the matrix PQ.

$$\mathbb{B}_r(P) := \{Q \in \mathcal{G}_{k,d} : \rho(P, Q) \le r\}.$$

The following approximation from finitely many sample points is due to [26, Proposition 5.3].

Theorem 4. *For $t = 1, 2, \ldots$, suppose we are given a sequence of point sets $\{P^t_j\}_{j=1}^{n_t} \subset \mathcal{G}_{k,d}$ and positive weights $\{\omega^t_j\}_{j=1}^{n_t}$ such that*

$$\int_{\mathcal{G}_{k,d}} g_1(P) g_2(P) \mathrm{d}\mu_{k,d}(P) = \sum_{j=1}^{n_t} \omega^t_j g_1(P^t_j) g_2(P^t_j), \qquad g_1, g_2 \in \Pi_t. \tag{9}$$

Then the approximation error, for $f \in H^s_\infty(\mathcal{G}_{k,d})$, is bounded by

$$\|f - \sigma_t(f, \{(P^t_j, \omega^t_j)\}_{j=1}^{n_t})\|_{L_\infty} \lesssim t^{-s}(\|f\|_{L_\infty} + \|f\|_{H^s_\infty}). \tag{10}$$

Note that the original result stated in [26] requires an additional regularity condition on the samples. This condition is satisfied since we restrict us to positive weights, cf. [16, Theorem 5.5 (a)]. The assumption (9) is a cubature type condition, for which our results in Section 4.2 shall provide further clarification. It indeed turns out that there are sequences $\{(P^t_j, \omega^t_j)\}_{j=1}^{n_t}$ satisfying (9) with $n_t \asymp t^{k(d-k)}$, in which case (10) becomes

$$\|f - \sigma_t(f, \{(P^t_j, \omega^t_j)\}_{j=1}^{n_t})\|_{L_\infty} \lesssim n_t^{-\frac{s}{k(d-k)}}(\|f\|_{L_\infty} + \|f\|_{H^s(L_\infty)}). \tag{11}$$

Note that the approximation rate in (11) matches the one in (5) for the integration error. The proof of Theorem 4 in [26] is indeed based on Theorem 3 and on the approximation of the integral $\sigma_t(f)$ in (7) by the weighted finite sum $\sigma_t(f, \{(P^t_j, \omega^t_j)\}_{j=1}^{n})$ in (8). For related results on local smoothness and approximation, we refer to [13].

4 Numerically Feasible Formulations

This section is dedicated to turn the approximation schemes presented in the previous sections into numerically feasible expressions. In other words, we determine explicit expressions for the kernel K_t in (6) and provide an optimization method for the numerical computation of cubature points or QMC systems on the Grassmannian.

4.1 Diffusion Kernels on Grassmannians

The probability measure $\mu_{k,d}$ is invariant under orthogonal conjugation and induced by the Haar (probability) measure $\mu_{\mathcal{O}(d)}$ on the orthogonal group $\mathcal{O}(d)$, i.e., for any $Q \in \mathcal{G}_{k,d}$ and measurable function f, we have

$$\int_{\mathcal{G}_{k,d}} f(P) \mathrm{d}\mu_{k,d}(P) = \int_{\mathcal{O}(d)} f(OQO^\top) \mathrm{d}\mu_{\mathcal{O}(d)}(O).$$

By the orthogonal invariance of the Laplace-Beltrami operator Δ on $\mathcal{G}_{k,d}$ it is convenient for the description of the eigenfunctions to recall the irreducible decomposition of $L_2(\mathcal{G}_{k,d})$ with respect to the orthogonal group. Given a non-negative integer t, a partition of t is an integer vector $\pi = (\pi_1, \ldots, \pi_t)$ with

$\pi_1 \geq \ldots \geq \pi_t \geq 0$ and $|\pi| = t$, where $|\pi| := \sum_{i=1}^{t} \pi_i$ is the size of π. The length $l(\pi)$ is the number of nonzero parts of π. The space $L_2(\mathcal{G}_{k,d})$ decomposes into

$$L_2(\mathcal{G}_{k,d}) = \bigoplus_{l(\pi)\leq k} H_\pi(\mathcal{G}_{k,d}), \qquad H_\pi(\mathcal{G}_{k,d}) \perp H_{\pi'}(\mathcal{G}_{k,d}), \quad \pi \neq \pi', \tag{12}$$

where $H_\pi(\mathcal{G}_{k,d})$ is equivalent to $\mathcal{H}^d_{2\pi}$, the irreducible representation of $\mathcal{O}(d)$ associated to the partition $2\pi = (2\pi_1, \ldots, 2\pi_t)$, cf. [3, 22].

By orthogonal invariance the spaces $H_\pi(\mathcal{G}_{k,d})$ are eigenspaces of the Laplace-Beltrami operator Δ on $\mathcal{G}_{k,d}$ where, according to [22, Theorem 13.2], the associated eigenvalues are

$$\lambda(\pi) = 2|\pi|d + 4\sum_{i=1}^{k} \pi_i(\pi_i - i). \tag{13}$$

Note, for a given eigenvalue λ_ℓ the corresponding eigenspace can decompose into more than one irreducible subspace $H_\pi(\mathcal{G}_{k,d})$.

Note that the following results are translations from representation theory used in [22], see also [2, 3], into the terminology of reproducing kernels, where we have only adapted the scaling of the kernels. The space $H_\pi(\mathcal{G}_{k,d})$ equipped with the L_2 inner product is a finite dimensional reproducing kernel Hilbert space, and its reproducing kernel K_π is given by

$$K_\pi(P, Q) = \sum_{\varphi_\ell \in H_\pi(\mathcal{G}_{k,d})} \varphi_\ell(P)\varphi_\ell(Q). \tag{14}$$

Moreover, K_π is zonal, i.e., the value $K_\pi(P, Q)$ only depends on the k largest eigenvalues

$$y_1(P, Q), \ldots, y_k(P, Q),$$

of the matrix PQ counted with multiplicities, see [22]. It follows that the kernel K_t in (6) is also zonal since it can be written as

$$K_t(P, Q) = \sum_{l(\pi)\leq k} h(t^{-2}\lambda(\pi))K_\pi(P, Q). \tag{15}$$

According to [22], the kernels K_π are in one-to-one correspondence with generalized Jacobi polynomials. For parameters $\alpha, \beta \in \mathbb{R}$ satisfying $\frac{1}{2}(m-1) < \alpha < \beta - \frac{1}{2}(m-1)$, the generalized Jacobi polynomials, $J^{\alpha,\beta}_\pi : [0, 1]^m \to \mathbb{R}$ with $l(\pi) \leq m$, are symmetric polynomials of degree $|\pi|$ and form a complete orthogonal system with respect to the density

$$w_{\alpha,\beta}(y_1,\ldots,y_m) := \prod_{i=1}^{m} \left(y_i^{\alpha-\frac{1}{2}(m+1)}(1-y_i)^{\beta-\alpha-\frac{1}{2}(m+1)}\right) \prod_{j=i+1}^{m} |y_i - y_j|, \tag{16}$$

where $0 < y_1, \ldots y_m < 1$, cf. [9, 11]. For the special parameters $\alpha = \frac{k}{2}$ with $k \leq \frac{d}{2}$ and $\beta = \frac{d}{2}$, and the normalization $J_\pi^{\frac{k}{2},\frac{d}{2}}(1,\ldots,1) = \dim(H_\pi(\mathcal{G}_{k,d}))$ the generalized Jacobi polynomials in $m = k$ variables can be identified with the reproducing kernels K_π of $H_\pi(\mathcal{G}_{k,d})$, i.e.,

$$K_\pi(P,Q) = J_\pi^{\frac{k}{2},\frac{d}{2}}(y_1(PQ),\ldots,y_k(PQ)), \qquad P,Q \in \mathcal{G}_{k,d}. \tag{17}$$

Now, (15) and (17) yield that the expression for the kernel K_t in (6) can be computed explicitly by

$$K_t(P,Q) = \sum_{l(\pi)\leq k} h(t^{-2}\lambda(\pi)) J_\pi^{\frac{k}{2},\frac{d}{2}}(y_1(PQ),\ldots,y_k(PQ)).$$

Thus, avoiding the actual computation of $\{\varphi_\ell\}_{\ell=0}^{\infty}$, we have derived the expression of K_t by means of generalized Jacobi polynomials.

4.2 Diffusion Polynomials on Grassmannians

This section is dedicated to investigate on the relations between diffusion polynomials Π_t and multivariate polynomials of degree t restricted to the Grassmannian. Indeed, the space of polynomials on $\mathcal{G}_{k,d}$ of degree at most t is defined as restrictions of polynomials by

$$\mathrm{Pol}_t(\mathcal{G}_{k,d}) := \{f|_{\mathcal{G}_{k,d}} : f \in \mathbb{C}[X]_t\}, \tag{18}$$

where $\mathbb{C}[X]_t$ is the collection of multivariate polynomials of degree at most t with d^2 many variables arranged as a matrix $X \in \mathbb{C}^{d\times d}$. Here, $f|_{\mathcal{G}_{k,d}}$ denotes the restriction of f to $\mathcal{G}_{k,d}$. It turns out that $\mathrm{Pol}_t(\mathcal{G}_{k,d})$ is a direct sum of eigenspaces of the Laplace-Beltrami operator, i.e.,

$$\mathrm{Pol}_t(\mathcal{G}_{k,d}) = \bigoplus_{\substack{|\pi|\leq t\\ l(\pi)\leq k}} H_\pi(\mathcal{G}_{k,d}), \tag{19}$$

cf. [22, Corollary in Section 11] and also [4, Section 2], which enables us to relate diffusion polynomials to regular polynomials restricted to the Grassmannian. For $k = 1$, the eigenvalues (13) directly lead to

$$\Pi_{\sqrt{4t^2+2t(d-2)}} = \mathrm{Pol}_t(\mathcal{G}_{1,d}).$$

For general k, the situation is more complicated and needs some preparation.

Lemma 1. *Let $d, k, t \in \mathbb{N}$ with $k \leq \frac{d}{2}$ be fixed. Then for any partition π with $l(\pi) \leq k$ and $|\pi| \geq t$ it holds*

$$\lambda(\pi) \geq \lceil \tfrac{4}{k}t^2 + 2t(d-k-1)\rceil. \tag{20}$$

Note that the right-hand side of (20), up to the square root and the ceiling function, is (13) with $\pi_i = t/k$, for $i = 1, \ldots, k$.

Proof. In view of (13), let us define

$$f(x_1, \ldots, x_k) := 2d\sum_{i=1}^{k} x_i + 4\sum_{i=1}^{k} x_i(x_i - i).$$

For partitions π with $|\pi| \geq t$, we obtain the lower bound (20) by solving the following convex optimization problem

$$\min_{x\in\mathbb{R}^k} f(x_1, \ldots, x_k) \quad \text{such that} \quad g_i(x) \leq 0, \quad i = 0, \ldots, k,$$

where $g_0(x) = t - \sum_{i=1}^{k} x_i$ and

$$g_i(x) = x_{i+1} - x_i, \quad i = 1, \ldots, k-1, \qquad g_k(x) = -x_k.$$

Indeed, we shall verify that the minimum is attained at $x^* := (\frac{t}{k}, \ldots, \frac{t}{k})$ with

$$f(x^*) = \tfrac{4}{k}t^2 + 2(d-k-1)t$$

by checking the Karush-Kuhn-Tucker (KKT) conditions

$$\nabla f(x^*) + \sum_{i=0}^{k} \mu_i \nabla g_i(x^*) = 0,$$

$$g_i(x^*) \leq 0, \quad \mu_i \geq 0, \quad \mu_i g_i(x^*) = 0, \qquad i = 0, \ldots, k,$$

with $\mu_0 = 8\frac{t}{k} + 2d - 2(k+1)$ and $\mu_i = 2i(k-i)$, for $i = 1, \ldots, k$. More precisely, denoting the canonical basis in $\mathbb{R}^k$ by $\{e_i\}_{i=1}^{k}$, we obtain

$$
\begin{aligned}
-\sum_{i=0}^{k}\mu_i\nabla g_i(x^*) &= \mu_0\sum_{i=1}^{k}e_i - \sum_{i=1}^{k-1}\mu_i(e_{i+1}-e_i)\\
&= (\mu_0+\mu_1)e_1 + (\mu_0-\mu_{k-1})e_k + \sum_{i=2}^{k-1}(\mu_0+\mu_i-\mu_{i-1})e_i\\
&= (\mu_0+2(k-1))e_1 + (\mu_0-2(k-1))e_k\\
&\quad + \sum_{i=2}^{k-1}(\mu_0-4(i-1)+2(k-1))e_i\\
&= \sum_{i=1}^{k}(\mu_0-4i+2(k+1))e_i\\
&= \sum_{i=1}^{k}(8\tfrac{t}{k}+2d-4i)e_i = \nabla f(x^*)
\end{aligned}
$$

and conclude that the KKT-conditions are satisfied. Hence, (20) holds.

Theorem 5. *Polynomials and diffusion polynomials on the Grassmannian $\mathcal{G}_{k,d}$ satisfy the relation*

$$\Pi_{s(t+1)-\epsilon} \subset \mathrm{Pol}_t(\mathcal{G}_{k,d}) \subset \Pi_{\sqrt{4t^2+2t(d-2)}}, \quad \textit{for all } 0<\epsilon<2s(t+1),$$

where $s(t) = \sqrt{\lceil \frac{4}{k}t^2 + 2t(d-k-1)\rceil}$.

Proof. Due to (12), we are only dealing with partitions π satisfying $l(\pi) \le k$. For $|\pi| \le t$, we derive

$$
\begin{aligned}
\lambda(\pi) = 2|\pi|d + 4\sum_{i=1}^{k}\pi_i(\pi_i-i) &\le 2|\pi|d + 4\sum_{i=1}^{k}\pi_i^2 - 4\sum_{i=1}^{k}\pi_i\\
&\le 4t^2+2t(d-2),
\end{aligned}
$$

which yields the second set inclusion.

Lemma 1 yields that $\lambda(\pi) < s^2(t+1)$ implies $|\pi| < t+1$, the latter being equivalent to $|\pi| \le t$ since both $|\pi|$ and t are integers. The range of ϵ yields $(s(t+1)-\epsilon)^2 < s^2(t+1)$, so that we deduce the first set inclusion.

Asymptotically in t, diffusion polynomials of order $\frac{2}{\sqrt{k}}t$ are indeed polynomials of degree at most t, and Theorem 3 yields, for $f \in H_p^s(\mathcal{G}_{k,d})$,

$$\mathrm{dist}\,(f, \mathrm{Pol}_t(\mathcal{G}_{k,d}))_{L_p} \asymp \mathrm{dist}(f, \Pi_t)_{L_p} \lesssim t^{-s}\|f\|_{H_p^s}.$$

For related further studies on $\mathrm{dist}(f, \mathrm{Pol}_t(\mathcal{G}_{k,d}))$, see [29].

In view of Theorem 5, we shall also define cubatures for $\mathrm{Pol}_t(\mathcal{G}_{k,d})$.

Definition 3. For $\{P_j\}_{j=1}^n \subset \mathcal{G}_{k,d}$ and positive weights $\{\omega_j\}_{j=1}^n$, we say that $\{(P_j,\omega_j)\}_{j=1}^n$ is a *cubature for* $\mathrm{Pol}_t(\mathcal{G}_{k,d})$ if

$$\int_{\mathcal{G}_{k,d}} f(P)\mathrm{d}\mu_{k,d}(P) = \sum_{j=1}^{n} \omega_j f(P_j), \quad \text{for all } f \in \mathrm{Pol}_t(\mathcal{G}_{k,d}). \tag{21}$$

We say that the points $\{P_j\}_{j=1}^n \subset \mathcal{G}_{k,d}$ are a t-design for $\mathrm{Pol}_t(\mathcal{G}_{k,d})$ if (21) holds for constant weights $\omega_1 = \ldots = \omega_n = 1/n$.

It turns out that the numerical construction of cubature points and t-designs for $\mathrm{Pol}_t(\mathcal{G}_{k,d})$ is somewhat easier than for Π_t directly, which is outlined in the subsequent section.

Remark 2. Since $\mathrm{Pol}_t(\mathcal{G}_{k,d})$ are restrictions of ordinary polynomials, we observe $\mathrm{Pol}_{t_1}(\mathcal{G}_{k,d}) \cdot \mathrm{Pol}_{t_2}(\mathcal{G}_{k,d}) \subset \mathrm{Pol}_{t_1+t_2}(\mathcal{G}_{k,d})$. Thus, $\{(P_j^t,\omega_j^t)\}_{j=1}^{n_t}$ being cubatures for $\mathrm{Pol}_{2t}(\mathcal{G}_{k,d})$ yields that (9) is satisfied when Π_t is replaced with $\Pi_{s(t+1)-\epsilon}$. The latter implies that we must then also replace $\sigma_t(f,\{(P_j^t,\omega_j^t)\}_{j=1}^{n_t})$ with $\sigma_{s(t+1)-\epsilon}(f,\{(P_j^t,\omega_j^t)\}_{j=1}^{n_t})$ in Theorem 4.

For general k, the second set inclusion in Theorem 5 is sharp because $\lambda(t,0,\ldots,0) = 4t^2 + 2t(d-2)$. The first set inclusion in Theorem 5 may only be optimal for t being a multiple of k. To prepare for our numerical experiments later, we shall investigate on $\mathcal{G}_{2,d}$ more closely.

Theorem 6. *For $k = 2$, we obtain*

$$\Pi_{s(t+1)-\epsilon} \subset \mathrm{Pol}_t(\mathcal{G}_{2,d}), \quad \textit{for all } 0 < \epsilon < 2s(t+1),$$

where $s(t) = \sqrt{2t^2 + 2t(d-3) + 2(1+(-1)^{t+1})}$.

Note that $s(t)$ in Theorem 6 satisfies $s^2(t) = \lambda(\lceil \frac{t}{2} \rceil, \lfloor \frac{t}{2} \rfloor)$. It matches the definition in Theorem 5 provided that t is even. For odd t, $s(t)$ in Theorem 6 is indeed larger than in Theorem 5, and the difference of the squares is 4.

Proof. Any partition π of length $k = 2$ with $|\pi| = t$ can be parameterized by $\pi(r) = (t-r, r)$, $r = 0,\ldots,\lfloor \frac{t}{2} \rfloor$. We have checked that $\lambda(\pi(r))$ is a quadratic function in r, which is strictly decreasing in r. Observing furthermore that $\lambda(\pi') \geq \lambda(\pi)$ if $\pi'_i \geq \pi_i$, $i = 1,\ldots,k$, we infer that $|\pi| \geq t$ implies

$$\lambda(\pi) \geq \lambda(\lceil \tfrac{t}{2} \rceil, \lfloor \tfrac{t}{2} \rfloor) = s^2(t). \tag{22}$$

Therefore, $\lambda(\pi) < s^2(t+1)$ implies $|\pi| \leq t$ since $|\pi|$ and t are integers.

Example 1. The particular case $\mathcal{G}_{2,4}$ yields

$$\Pi_{s(t+1)-\epsilon} \subset \mathrm{Pol}_t(\mathcal{G}_{2,4}), \quad \text{for all } 0 < \epsilon < 2s(t+1),$$

where $s(t) := \sqrt{2t^2 + 2t + 2(1 + (-1)^{t+1})}$, which implies

$$\Pi_{s(t+1)-\epsilon} \cdot \Pi_{s(t+1)-\epsilon} \subset \mathrm{Pol}_{2t}(\mathcal{G}_{2,4}).$$

Thus, given a cubature for $\mathrm{Pol}_{2t}(\mathcal{G}_{2,4})$, the condition (9) in Theorem 4 is satisfied with respect to $\Pi_{s(t+1)-\epsilon}$.

4.3 Worst Case Error of Integration on Grassmannians

Given some subspace $\mathcal{H}$ of continuous functions on $\mathcal{G}_{k,d}$ the *worst case error* of integration (with respect to some norm $\|\cdot\|$ on $\mathcal{H}$) for points $\{P_j\}_{j=1}^n \subset \mathcal{G}_{k,d}$ and weights $\{\omega_j\}_{j=1}^n$ is defined by

$$\mathrm{wce}_{\mathcal{H},\|\cdot\|}(\{(P_j,\omega_j)\}_{j=1}^n) := \sup_{\substack{f\in\mathcal{H}\\ \|f\|=1}} \Big| \int_{\mathcal{G}_{k,d}} f(P)\mathrm{d}\mu_{k,d}(P) - \sum_{j=1}^n \omega_j f(P_j)\Big|,$$

see also [20, 27]. If $\mathcal{H} = H_K$ is a reproducing kernel Hilbert space, whose reproducing kernel K is

$$K(P,Q) = \sum_{l(\pi)\le k} r_\pi K_\pi(P,Q) = \sum_{l(\pi)\le k} r_\pi \sum_{\lambda_\ell=\lambda(\pi)} \varphi_\ell(P)\varphi_\ell(Q), \qquad P,Q\in\mathcal{G}_{k,d},$$

with $r_\pi \ge 0$, $|\pi| \ge 0$, and sufficient decay of the coefficients, then the associated inner product is

$$(f,g)_K = \sum_{\substack{l(\pi)\le k\\ r_\pi>0}} r_\pi^{-1} \sum_{\lambda_\ell=\lambda(\pi)} \hat{f}(\ell)\overline{\hat{g}(\ell)},$$

and the Riesz representation theorem yields

$$\begin{aligned}\mathrm{wce}_{H_K,\|\cdot\|_K}(\{(P_j,\omega_j)\}_{j=1}^n)^2 &= \sum_{l(\pi)\le k} r_\pi \sum_{\lambda_\ell=\lambda(\pi)} \Big| \int_{\mathcal{G}_{k,d}} \varphi_\ell(P)\mathrm{d}\mu_{k,d}(P) - \sum_{j=1}^n w_j\varphi_\ell(P_j)\Big|^2\\ &= r_{(0)} - 2r_{(0)}\sum_{j=1}^n \omega_j + \sum_{i,j=1}^n \omega_i\omega_j K(P_i,P_j).\end{aligned} \tag{23}$$

Note that the worst case error is a weighted ℓ_2-average of the integration errors of the basis functions $\varphi_0, \varphi_1, \varphi_2, \ldots$.

Recall, for instance, $H_2^s(\mathcal{G}_{k,d})$ is a Hilbert space with inner product

$$\langle f, g\rangle_{H_2^s} = \sum_{\ell=0}^{\infty}(1+\lambda_\ell)^s \hat{f}(\ell)\overline{\hat{g}(\ell)}, \quad f, g \in H_2^s(\mathcal{G}_{k,d}), \tag{24}$$

and the Bessel kernel on the Grassmannian is $K_B^s : \mathcal{G}_{k,d} \times \mathcal{G}_{k,d} \to \mathbb{R}$ with

$$\begin{aligned} K_B^s(P,Q) &= \sum_{\ell=0}^{\infty}(1+\lambda_\ell)^{-s}\varphi_\ell(P)\varphi_\ell(Q) \\ &= \sum_{l(\pi)\le k}^{\infty}(1+\lambda(\pi))^{-s} J_\pi^{\frac{k}{2},\frac{d}{2}}(y_1(P,Q),\ldots,y_k(P,Q)). \end{aligned}$$

If $s > k(d-k)/2$, then it is easily checked that K_B^s is the reproducing kernel for $H_2^s(\mathcal{G}_{k,d})$ with respect to the inner product (24), see also [6].

Note that the polynomial space $\mathrm{Pol}_t(\mathcal{G}_{k,d})$ is also a reproducing kernel Hilbert space. Indeed, given a partition π with $|\pi| \le t$ and $l(\pi) \le k$, the reproducing kernel of $H_\pi(\mathcal{G}_{k,d})$ with respect to the L_2 inner product is K_π in (14). Due to (19), the reproducing kernels for $\mathrm{Pol}_t(\mathcal{G}_{k,d})$ are exactly

$$R_t(P,Q) = \sum_{\substack{|\pi|\le t\\ l(\pi)\le k}} r_\pi K_\pi(P,Q) = \sum_{\substack{|\pi|\le t\\ l(\pi)\le k}} r_\pi \sum_{\lambda_\ell=\lambda(\pi)} \varphi_\ell(P)\varphi_\ell(Q), \qquad P,Q \in \mathcal{G}_{k,d},$$

with $r_\pi > 0$, $|\pi| \ge 0$. Note that R_t is indeed reproducing as a finite linear combination with nonnegative coefficients of reproducing kernels, and it reproduces $\mathrm{Pol}_t(\mathcal{G}_{k,d})$ because of (19) and none of the coefficients vanish. Now, by Definition 3 any cubature for $\mathrm{Pol}_t(\mathcal{G}_{k,d})$ has zero worst case error independent of the chosen norm, and thus independent of R_t. A particularly simple reproducing kernel for $\mathrm{Pol}_t(\mathcal{G}_{k,d})$ is

$$R_t(P,Q) = \mathrm{tr}(PQ)^t, \qquad P,Q \in \mathcal{G}_{k,d},$$

see, for instance, [12]. Hence, formula (23) provides us with a simple method to numerically compute cubature points by some minimization method. In particular, t-designs $\{(P_j, \frac{1}{n})\}_{j=1}^n$ are constructed by minimizing

$$\frac{1}{n^2}\sum_{i,j=1}^{n} \mathrm{tr}(P_i,P_j)^t \ge \int_{\mathcal{G}_{k,d}}\int_{\mathcal{G}_{k,d}} \mathrm{tr}(P,Q)^t \mathrm{d}\mu_{k,d}(P)\mathrm{d}\mu_{k,d}(Q)$$

and checking for equality, which implies $\mathrm{wce}_{\mathrm{Pol}_t,\|\cdot\|_{R_t}}(\{(P_j,\omega_j)\}_{j=1}^n) = 0$.

5 Numerical Experiments

We now aim to illustrate theoretical results of the previous sections. The projective space $\mathcal{G}_{1,d}$ can be dealt with approaches for the sphere by identifying x and $-x$. The space $\mathcal{G}_{d-1,d}$ can be identified with $\mathcal{G}_{1,d}$, so that the first really new example to be considered here is $\mathcal{G}_{2,4}$.

We computed points $\{P_j^t\}_{j=1}^{n_t} \subset \mathcal{G}_{2,4}$, for $t = 1, \dots, 14$, with worst case error

$$\mathrm{wce}_{\mathrm{Pol}_t, \|\cdot\|_{L_2}}(\{(P_j^t, 1/n_t)\}_{j=1}^{n_t}) < 10^{-7}$$

by a nonlinear conjugate gradient method on manifolds, cf. [20, Section 3.3.1], see also [1, Section 8.3]. Although the worst case error may not be zero exactly, we shall refer to $\{P_j^t\}_{j=1}^{n_t}$ in the following simply as t-designs. Note that $\mathcal{G}_{2,4}$ has dimension $\dim(\mathcal{G}_{2,4}) = 4$, so that the number of cubature points must satisfy $n_t \gtrsim t^4$. Indeed we chose

$$n_t := \left\lfloor \frac{1}{3} \dim(\mathrm{Pol}_t(\mathcal{G}_{2,4})) \right\rfloor = \left\lfloor \tfrac{1}{3}(t+1)^2(1+t+\tfrac{1}{2}t^2) \right\rfloor.$$

We emphasize that for $t = 14$ we computed $n_{14} = 8.475$ projection matrices which almost perfectly integrate 25.425 polynomial basis functions.

5.1 Integration

In what follows we consider two positive definite kernels

$$K_1(P, Q) = \sqrt{(2 - \mathrm{tr}(PQ))^3} + 2\,\mathrm{tr}(PQ),$$
$$K_2(P, Q) = \exp(\mathrm{tr}(PQ) - 2).$$

It can be checked by comparison to the Bessel kernel, cf. [6], that the reproducing kernel Hilbert space H_{K_1} equals the Bessel potential space $H_2^{\frac{7}{2}}(\mathcal{G}_{2,4})$, i.e., the corresponding norms are comparable. In contrast, the reproducing kernel Hilbert space H_{K_2} is contained in the Bessel potential space $H_2^s(\mathcal{G}_{2,4})$ for any $s > 2$. The worst case errors can be computed by

$$\mathrm{wce}_{H_{K_1}, \|\cdot\|_{K_1}}(\{(P_j, \frac{1}{n})\}_{j=1}^n)^2 = \frac{1}{n^2}\sum_{i,j=1}^n K_1(P_i, P_j) - \big(2 + \frac{74}{75}\sqrt{2} - \frac{2}{5}\log(1+\sqrt{2})\big),$$

$$\mathrm{wce}_{H_{K_2}, \|\cdot\|_{K_2}}(\{(P_j, \frac{1}{n})\}_{j=1}^n)^2 = \frac{1}{n^2}\sum_{i,j=1}^n K_2(P_i, P_j) - \exp(-1)\mathrm{Shi}(1),$$

where $\mathrm{Shi}(x) = \int_0^x \frac{\sinh(t)}{t}\mathrm{d}t$ is the hyperbolic sine integral.

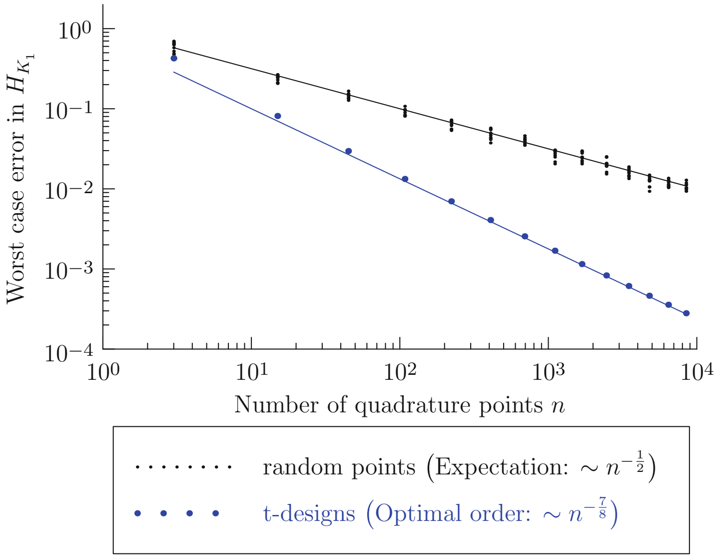

Fig. 1 Random sampling according to $\mu_{2,4}$ vs. integration by t-designs.

In view of illustrating Proposition 1, note that random $P \in \mathcal{G}_{k,d}$ distributed according to $\mu_{k,d}$ can be derived by $P := Z(Z^\top Z)^{-1}Z^\top$, where $Z \in \mathbb{R}^{d\times k}$ with entries that are independently and identically standard normally distributed, cf. [8, Theorem 2.2.2].

Figure 1 depicts clearly the superior integration quality of the computed cubature points over random sampling. Moreover, it can be seen that the theoretical results in Proposition 1 and Theorem 1 with (5) are in perfect accordance with the numerical experiment, i.e., the integration errors of the random points scatter around the expected integration error and cubature points achieve the optimal rate of $n^{-\frac{7}{8}}$ for functions in $H_2^{\frac{7}{2}}(\mathcal{G}_{2,4})$.

In Figure 2 we aim to show the contrast between the integration of functions in H_{K_1} and H_{K_2} by using the computed t-designs. We know by Theorem 1 that the sequence of t-designs with a number of cubature points $n_t \asymp t^4$ is a QMC system for any $s > 2$. Since H_{K_2} is contained in any Bessel potential space $H_2^s(\mathcal{G}_{2,4})$, for $s > 2$, we expect a super linear behavior in our logarithmic plots. Indeed, Figure 2 confirms our expectations. For $t \geq 11$, the effect of the accuracy of the t-designs used becomes significant for integration of smooth functions. For that reason, we added the dashed red line, which represents the accuracy 10^{-7} of the computed t-designs.

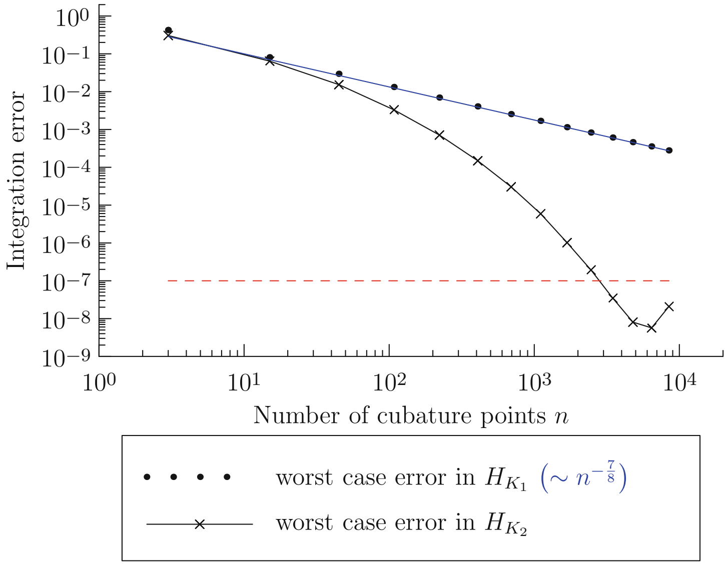

Fig. 2 Integration of smooth vs. integration of nonsmooth functions.

5.2 Approximation

Similar, as in the previous section we aim to approximate a smooth and a nonsmooth function, namely

$$f_1(P) = K_1(I_2, P), \qquad f_2(P) = K_2(I_2, P), \qquad P \in \mathcal{G}_{2,4},$$

where K_1, K_2 are from the previous section and I_2 is a projection matrix with 2 ones on the upper left diagonal. This time we observe that the function f_1 is contained in $H^3_\infty(\mathcal{G}_{2,4})$ but $f_1 \notin H^{3+\epsilon}_\infty(\mathcal{G}_{2,4})$, for all $\epsilon > 0$. For the smooth function f_2, we have $f_2 \in H^s_\infty(\mathcal{G}_{2,4})$, for any $s > 0$.

Since the computed t-designs are with respect to $\mathrm{Pol}_t(\mathcal{G}_{2,4})$ and not $\Pi_t(\mathcal{G}_{2,4})$, we need an additional scaling of $s(t)$ in σ_t. According to Example 1, the choice

$$s(t) = \sqrt{2(t^2 + 3t + 3 + (-1)^t)} - \epsilon \asymp t\sqrt{2},$$

for small $\epsilon > 0$, yields $\Pi_{s(t)} \cdot \Pi_{s(t)} \subset \mathrm{Pol}_{2t}(\mathcal{G}_{2,4})$. For numerical experiments, we take ϵ to be smaller than the machine precision, so that it is effectively zero. Hence, in accordance with Theorem 4, we use the following kernel-based approximation

$$\sigma_{s(t)}(f, X_{2t}) = \frac{1}{n_{2t}} \sum_{j=1}^{n_{2t}} f(P_j) \sum_{l(\pi)\le 2} h(s(t)^{-2}\lambda(\pi)) K_\pi(P_j, \cdot),$$

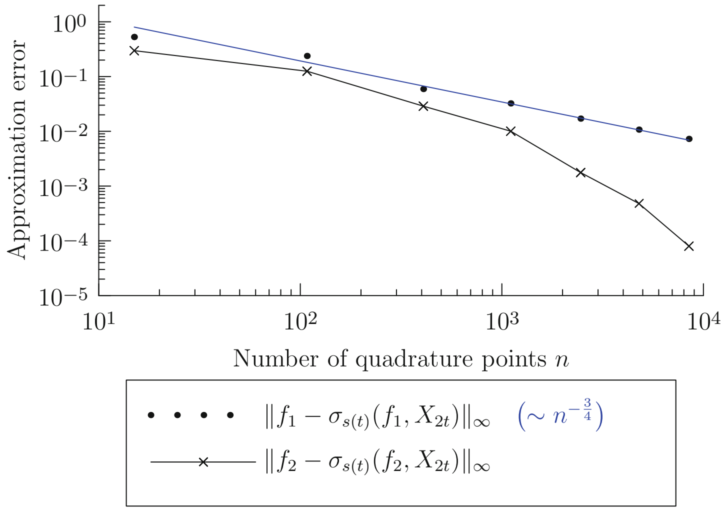

Fig. 3 Approximation of a smooth vs. approximation of a nonsmooth function.

where

$$h(x) = \begin{cases} \left(1 + \exp(\frac{3-4x}{2-6x+4x^2})\right)^{-1}, & 1/2 < x < 1, \\ 1, & x \leq 1/2, \\ 0, & \text{otherwise.} \end{cases}$$

The approximation error is determined by randomly sampling altogether 50000 points. The first 25000 are pseudo random according to $\mu_{2,4}$. Since f_1 has a nonsmooth point at I_2 the maximal error is expected around this point. Therefore, we sampled the other 25000 from normally distributed points around that point I_2 with variance 0.15 and 0.5 in the matrix entries, i.e., we choose $Z \in \mathbb{R}^{4\times 4}$ with independent and identically distributed entries according to a normal distribution with mean zero and variance 0.15 and 0.5, respectively, and then project $I_2 + Z$ onto $\mathcal{G}_{2,4}$, which we accomplished by a QR-decomposition in Matlab.

In Figure 3, we can observe the predicted decay in Theorem 4 for the function $f_1 \in H^3_\infty(\mathcal{G}_{2,4})$. Furthermore, as expected for the smooth function f_2, the error appears to decrease super linearly.

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Construction of Multiresolution Analysis Based on Localized Reproducing Kernels

K. Nowak and M. Pap

1 Introduction

We will consider Hilbert spaces over either the field of real numbers, $\mathbb{R}$, or of complex numbers, $\mathbb{C}$. We will use F to denote either $\mathbb{R}$ or $\mathbb{C}$, so that when we wish to state a definition or result that is true for either the real or complex numbers, we will use F. Given a set X, if we equip the set of all functions from X to F, $\mathscr{F}(X, F)$ with the usual operations of addition, and scalar multiplication, then $\mathscr{F}(X, F)$ is a vector space over F.

Definition 1. Given a set X, we will say that H is a reproducing kernel Hilbert space (RKHS) on X over F, provided that:

1. H is a vector subspace of $\mathscr{F}(X, F)$,
2. H is endowed with an inner product, $\langle ., . \rangle$, making it into a Hilbert space,
3. for every $y \in X$, the linear evaluation functional, $E_y : H \to F$, defined by $E_y(f) = f(y)$, is bounded.

If H is a RKHS on X, then since every bounded linear functional is given by the inner product with a unique vector in H, we have that for every $y \in X$, there exists a unique vector, $k_y \in H$, such that for every $f \in H, f(y) = \langle f, k_y \rangle$.

Definition 2. The function k_y is called the *reproducing kernel* for the point y. The 2-variable function defined by $K(x, y) = k_y(x)$ is called the reproducing kernel for H.

K. Nowak (✉)
Department of Computer Science, Drexel University, 3141 Chestnut Street, Philadelphia, PA 19104, USA
e-mail: knowak1248@gmail.com

M. Pap
Institute of Mathematics and Informatics, University of Pécs, Ifjúság útja 6, 7634 Pécs, Hungary
e-mail: papm@gamma.ttk.pte.hu

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K completely determines the space H and characterizes the functions that are the kernel functions of some RKHS.

Theorem 1. *Let H be a RKHS on the set X with kernel K. Then the linear span of the functions, $k_y(\cdot) = K(\cdot, y), y \in X$ is dense in H.*

A collection of the most important results connected to the reproducing kernel Hilbert spaces can be found, for example, in [Aronszajn 1950] [1], [Paulsen, Raghupati] [42], [Zhu 1997] [52].

The countable set $\{y_n \in X, n \in \mathbb{N}\}$ is a *sampling set* for a RKHS space H if every $f \in H$ can be reconstructed uniquely from the measurements $\{f(y_n), n \in \mathbb{N}\}$.

Because of the reproducing property $f(y_n) = \langle f, k_{y_n} \rangle$ this is equivalent to the fact that the countable set of localized reproducing kernels $\{k_{y_n}(\cdot) = K(\cdot, y_n), n \in \mathbb{N}\}$ is dense in H.

How to choose the sampling points and how to construct the related multiresolution analysis are two important questions of applied science. It turns out that the solution of this problem is dependent on the properties of the considered Hilbert space H.

In this survey paper we consider the case of the Hardy space and the Bergman space of the unit disc. In both cases we choose properly the sampling set, and construct the corresponding multiresolution analysis out of it, via the means of localized reproducing kernels. Our constructions lead to new reconstruction algorithms. It is still an open problem to what degree the approach presented in current paper can be adapted to other classes of reproducing kernel Hilbert space, and to weighted Bergman spaces in particular.

2 Wavelet Transform and Its Generalization to Voice Transforms

In order to define the continuous version of the wavelet transform let us start from a basic function $\psi \in L^2(\mathbb{R})$, called mother wavelet, and use dilation and translation to obtain the collection of functions

$$\psi_{pq}(x) = \frac{\psi((x-q)/p)}{\sqrt{p}} \qquad (x \in \mathbb{R}, (p,q) \in \mathbf{L} := (0,\infty) \times \mathbb{R}).$$

By means of this kernel function we can construct an integral operator

$$(\mathscr{W}_\psi f)(p,q) := \frac{1}{\sqrt{p}} \int_{\mathbb{R}} f(x)\overline{\psi}((x-q)/p)\, dx = \langle f, \psi_{pq} \rangle \qquad ((p,q) \in \mathbf{L}, f \in L^2(\mathbb{R}))$$

called *wavelet transform.* It is known that under general conditions made on ψ the function f can be reconstructed from its wavelet transform and the analogue of the

Plancherel formula, in other words the energy conservation principle holds for it [Daubechies 1992] [10], [Heil, Walnut 1989] [26], [Meyer 1990] [32].

Similarly to the Fourier transform there can be given a group theoretical interpretation for W_ψ by means of the collection $\mathfrak{L}$ of affine maps

$$\ell_{\mathfrak{a}}(x) := px + q \; (x \in \mathbb{R}, \mathfrak{a} = (p,q) \in \mathbf{L}).$$

The function set $\mathfrak{L}$ is closed for composition $\circ$. The group $(\mathfrak{L}, \circ)$ is called *affine group.* Introducing the group operation

$$\mathfrak{a}_1 \circ \mathfrak{a}_2 := (p_1 p_2, q_1 + p_1 q_2) \;\; (\mathfrak{a}_j := (p_j, q_j) \in \mathbf{L}, j = 1,2)$$

on $\mathbf{L}$ we obtain the group $(\mathbf{L}, \circ)$ which is isomorphic with the affine group, and $\ell_{\mathfrak{a}} = \ell_{\mathfrak{a}_1} \circ \ell_{\mathfrak{a}_2}$. The group operations are continuous with respect to the usual topology in $\mathbf{L}$, therefore $(\mathbf{L}, \circ)$ is a (noncommutative, locally compact) topological group.

The wavelet transform can be described by the family of operators

$$U_{\mathfrak{a}}\psi := \frac{1}{\sqrt{p}} \psi \circ \ell_{\mathfrak{a}}^{-1} \qquad (\mathfrak{a} = (p,q) \in \mathbf{L}, \psi \in L^2(\mathbb{R}))$$

as

$$(\mathscr{W}_\psi f)(\mathfrak{a}) = \langle f, U_{\mathfrak{a}}\psi \rangle \qquad (\mathfrak{a} = (p,q) \in \mathbf{L}, f, \psi \in L^2(\mathbb{R})). \tag{1}$$

It is easy to show that the operators $U_{\mathfrak{a}} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ $(\mathfrak{a} \in \mathbf{L})$ form a *unitary representation* of $(\mathbf{L}, \circ)$ on the space $L^2(\mathbb{R})$, i.e.

$$i) \quad \|U_{\mathfrak{a}}\psi\| = \|\psi\|, \;\; ii) \quad U_{\mathfrak{a}_1}(U_{\mathfrak{a}_1}\psi) = U_{\mathfrak{a}_1 \circ \mathfrak{a}_2}\psi \qquad (\mathfrak{a}, \mathfrak{a}_1, \mathfrak{a}_2 \in \mathbf{L}, \psi \in L^2(\mathbb{R})).$$

Moreover the representation is continuous in the following sense: For every function $\psi \in L^2(\mathbb{R})$ we have

$$iii) \quad \|T_{\mathfrak{a}_n}\psi - T_{\mathfrak{a}}\psi\| \to 0, \;\; if \quad \mathfrak{a}_n \to \mathfrak{a} \qquad (n \to \infty).$$

Taking the discrete subgroup $(\mathbf{L}_0, \circ), \mathbf{L}_0 := \{(2^{-n}, k2^{-n}) : k, n \in \mathbb{Z}\}$ instead of $(\mathbf{L}, \circ)$ we obtain, as a generalization of Haar-Fourier coefficients, the discrete version of the wavelet transform

$$(\mathscr{W}_\psi f)(2^{-n}, k2^{-n}) = \sqrt{2^n} \int_{\mathbb{R}} f(x) \overline{\psi}(2^n x - k)\, dx \qquad (k, n \in \mathbb{Z}).$$

Referring to the relation with the affine group the map $\mathscr{W}_\psi$ is usually called *affine wavelet transform.*

There is a rich bibliography of the affine wavelet theory (see, for example, [Grossman, Morlet 1984] [21], [Grossman, Morlet, Paul 1985] [22], [Daubechies

1988] [9], [Mayer 1990] [32], etc.). One important question is the construction of the discrete version, i.e., to find ψ so that the discrete translates and dilates

$$\psi_{n,k} = 2^{-n/2}\psi(2^{-n}x - k)$$

form a (orthonormal) basis in $L^2(\mathbb{R})$ which generate a multiresolution (see [Daubechies 1988] [9], [Heil, Walnut 1989] [26], [Mallat 1989] [30], etc.). In the classical theory of affine wavelets the definition of multiresolution analysis is the following:

Definition 3. Let $V_j,\ j \in \mathbb{Z}$ be a sequence of subspaces of $L^2(\mathbb{R})$. The collections of spaces $\{V_j,\ j \in \mathbb{Z}\}$ is called a *multiresolution analysis* with wavelet function ϕ if the following conditions hold:

1. (nested) $V_j \subset V_{j+1}$
2. (density) $\overline{\cup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R})$
3. (separation) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$
4. (basis) The function ϕ belongs to V_0 and the set $\{2^{n/2}\phi(2^n x - k),\ k \in \mathbb{Z}\}$ is a (orthonormal) basis in V_n.

The dilation is used to obtain a higher resolution level ($f(x) \in V_n \Leftrightarrow f(2x) \in V_{n+1}$) and applying the translation we remain on the same level of the resolution.

The simplest example is due to [Alfréd Haar 1909] [24] the multiresolution generated by the Haar-wavelets. The Haar-wavelets can be derived from the following function using the dilation and translation:

$$h(x) = \begin{cases} 1 & (x \in [0, 1/2)) \\ -1 & (x \in [1/2, 1)) \\ 0 & (x \in \mathbb{R} \setminus [0, 1)), \end{cases}$$

$$h_0(x) = h(x),\ h_{nk}(x) := 2^{-n/2}h(2^n x - k)$$

$$(x \in [0, 1),\ n, k \in \mathbb{N}).$$

The Haar-system is orthogonal in the Hilbert space $L^2 := L^2([0, 1))$ with respect to the usual scalar product, and the Haar-Fourier series of a function $f \in L^1([0, 1))$ converges to the function in both norm and almost everywhere. In particular, if the function is continuous, then the convergence is uniform. In this respect the Haar wavelet system is essentially different from the trigonometric system.

The fact that the members of the system are not continuous make them inappropriate for approximating smooth functions. Starting from 1980 Y. Meyer, I. Daubechies, P. Auscher among others started to construct smooth orthonormal systems, so-called wavelets from a single function φ called mother wavelet, of the form

$$\varphi_{n,k}(x) = 2^{n/2}\varphi(2^n x - k)\ \ (x \in \mathbb{R}, \varphi \in L^2(\mathbb{R}), \|\varphi\|_2 = 1).$$

Except for the Haar system the construction of such systems is a hard task. Then the Fourier transform $\widehat{\varphi}$ instead of the mother wavelet φ itself turned to be a good starting point. Despite the fact that φ cannot be given in an explicit form generally the wavelet Fourier series enjoy nice convergence and approximation properties. The kernel functions of the partial sums can be well estimated and the wavelet Fourier coefficients can be calculated by a fast algorithm.

This model can serve as an example for the construction of useful function transformations. Instead of the affine group one may take a locally compact topological group $(\mathbf{G}, \cdot)$ and a unitary representation $V_g : H \to H$ $(g \in \mathbf{G})$ of it. Then similarly to

$$(\mathscr{V}_\psi f)(g) := \langle f, V_g \psi \rangle \quad (g \in \mathbf{G}, f, \psi \in H)$$

will be a bounded linear operator from the Hilbert space H to the space of bounded continuous functions $C(\mathbf{G})$ defined on $\mathbf{G}$. According to [P Goupillaud, A Grossmann, J Morlet 1984] [23], and [Feichtinger, Gröchenig] [14], the map $\mathscr{V}_\psi$ is called *voice-transform* generated by the representation $(V_g, g \in \mathbf{G})$. We say that the representation is *irreducible* if it has no proper closed invariant subspace, i.e. $V_g \psi$ $(g \in \mathbf{G})$ is a closed system in H for any $\psi \in H, \psi \neq \theta$. It can be shown that if the representation is irreducible then the voice transform is injective. Let the left invariant Haar measure on the group $\mathbf{G}$ be denoted by m, and the Hilbert space generated by the measure m on $\mathbf{G}$ by $L^2_m(\mathbf{G})$. The elements $\psi \in H$ for which $\mathscr{V}_\psi(H) \subset L^2_m(\mathbf{G})$ are called *admissible elements.*

The set H_0 of admissible elements is dense in H. Moreover, $\psi \in H_0, \psi \neq \theta$ if and only if $\mathscr{V}_\psi \psi \in L^2_m(\mathbf{G})$. The classical result due to [Duflo and Moore 1976] [11], later applied to the current context by a slightly different argument by [Grossman, Morlet, Paul, 1985], [22], [Heil, Walnut 1989] [26], shows that there is positive definite quadratic form $C : H_0 \to \mathbb{R}_+$ for which

$$\langle \mathscr{V}_{\psi_1} f_1, \mathscr{V}_{\psi_2} f_2 \rangle_{L^2_m(\mathbf{G})} = C(\psi_1, \psi_2) \langle f_1, f_2 \rangle_H \qquad (f_1, f_2 \in H, \psi_1, \psi_2 \in H_0).$$

This can be considered as the analogue of the Plancherel-theorem for voice transforms. In particular, if the group $\mathbf{G}$ is unimodular, i.e. every left invariant measure is right invariant as well, then there is an absolute constant C_0 for which the equality

$$\|\mathscr{V}_\psi f\|_{L^2_m(\mathbf{G})} = C_0 \|\psi\|_H \|f\|_H \qquad (f \in H, \psi \in H_0)$$

holds. Consequently, with $\|\psi\|_H = 1/C_0$ the voice transform becomes unitary. This means not only that the analogue of the Plancherel theorem holds true under a very general condition but also it explains the special form of the formula in the particular cases by enlightening the role of the group $\mathbf{G}$.

Atomic decompositions of Coifman-Rochberg proceeded the developments of affine wavelet theory (see [Rochberg 1985] [45] for a comprehensive summary of results). For many delicate analytic estimates orthogonality is not critical, e.g. weak type estimates for singular values of commutators on weighted Bergman spaces of [Nowak 1991] [33] follow from the atomic decompositions of Coifman-Rochberg. It should be mentioned, however, that orthogonality reduces redundancy to the minimum, and it is of critical importance in many applications.

The transform introduced by Dénes Gábor in 1946 can be understood also as a special voice transform by taking a special representation of the *Heisenberg group*. This explains that the *Gábor-transform* is also called as *Weyl-Heisenberg wavelet transform*. Another special voice transform which is important from the point of view of the applications is the *shearlet transform* (see [Labate, Lim, Kutyniok, Weiss, 2005] [29], [Kutyniok, Sauer, 2007] [28], [Kutyniok, Labate, 2007], [27] etc.).

If the representation is a unitary, irreducible, and square integrable, normalizing the vector $g \in H^2$ if necessary, the voice transform $V_g : H \to L^2_m(\mathbf{G})$ will be isometric, i.e.,

$$[V_g f, V_g h] = \langle f, h\rangle \ , \qquad (f, h \in H), \tag{2}$$

where the left-hand side is the scalar product generated by the left Haar measure of the group $\mathbf{G}$.

An important consequence of this is the following reproducing formula: For convenient normalized $g \in H^2$ we have the following convolution relation (on $\mathbf{G}$):

$$V_g f = V_g f * V_g g, \quad f \in H. \tag{3}$$

In the papers [Feichtinger, Gröchenig] [15, 16], [Gröchenig, 1991] [19] stronger integrability condition on U is imposed in order to handle other spaces than Hilbert spaces. Let us consider a positive, continuous *submultiplicative weight* w on $\mathbf{G}$, i.e., $w(xy) \leq w(x)w(y)$, $w(x) \geq 1$, $\forall x, y \in \mathbf{G}$. Assume that the representation is *integrable*, i.e., the set of analyzing vectors is not trivial:

$$\mathscr{A}_w = \{g \in H : \ V_g g \in L^1_w(\mathbf{G})\} \neq \{0\}. \tag{4}$$

With this assumption the reproducing formula given by the convolution (3) can be discretized. Let us define the simplest Banach space where atomic decompositions can be obtained:

$$\mathscr{H}^1_w = \{f \in H : \ V_g f \in L^1_w(\mathbf{G})\}. \tag{5}$$

The definition of $\mathscr{H}^1_w$ is independent of the choice of $g \in \mathscr{A}_w$.

In [Feichtinger, Gröchenig, 1988] [14], [Feichtinger, Gröchenig, 1989] [15, 16], [Gröchenig, 1991] [19] it was described a unified approach to atomic decomposition through integrable group representations. The simplest result is for the space $\mathscr{H}^1_w$:

For any $g \in \mathscr{A}_w \setminus \{0\}$ there exists a collection of points $\{x_i\} \subset G$ such that any $f \in \mathscr{H}^1_w$ can be written

$$f = \sum \lambda_i(f) U_{x_i} g, \quad \textit{with} \quad \sum_i |\lambda_i(f)| w(x_i) \leq C_0 \|f\|_{\mathscr{H}^1_w} \tag{6}$$

where the sum is absolutely convergent in $\mathscr{H}_w^1$.

This atomic decomposition result was extended also for more general Banach spaces: for the coorbit spaces.

In the last period the coorbit theory was developed for non-integrable representations too. The requirement is to have some $L^p(\mathbf{G}), p > 1$ condition instead of the integrability of the representation, see [Dahlke, Steidel, Teschke, 2004][5], [Dahlke, Teschke, 2007][4], [Dahlke, Kutyniok, Steidl, Teschke, 2007][6], [Dahlke, Fornasier, Rauhut, Steidel, Teschke, 2008] [7], [Christensen, Olafson, 2011] [2], [Dahlke, De Mari, De Vito, Labate, Teschke, Vigogna, 2014] [8].

The classical theory of atomic decompositions can be applied in the case of the Gábor transform, but not in the case of the affine wavelet transform, because in the affine case the integrability condition is not satisfied.

Beside the affine wavelet transform there are other cases when the integrability or the square integrability conditions are not satisfied by the special voice transforms. In these cases it is a natural question to analyze, whether it is possible, to achieve the discretization in a similar way like in the affine wavelet case, by constructing a multiresolution analysis.

Avoiding the Fourier technique used in the construction of the affine multiresolution, we show through two examples how can be constructed multiresolution by means of localized reproducing kernels. In the case of the Hardy and Bergman spaces, in [Pap 2011] [34] and [Pap 2013] [36], based on the discretization of the two special voice transforms of the Blaschke group, it was introduced the analogue of the multiresolution analysis in these spaces.

3 The Blaschke Group

Let us denote by $\mathbb{D}$ the open unit disc and by $\mathbb{T}$ the unit circle. Instead of linear functions let us consider the following rational linear functions:

$$B_a(z) := \epsilon \frac{z-b}{1-\bar{b}z} \quad (z \in \mathbb{C},\ \bar{b}z \neq 1)$$

the so-called *Blaschke functions*. Let us denote the set of the parameters $\mathbf{B} := \mathbb{D} \times \mathbb{T}$ and $a = (b, \epsilon) \in \mathbb{B}$. If $a \in \mathbf{B}$, then B_a is a 1-1 map on $\mathbb{T}$ and $\mathbb{D}$, respectively.

The Blaschke functions play an important role not only in system identification but also in factorization of functions belonging to Hardy spaces. For instance, they can be used to represent the congruences in the Poincaré model of the Bolyai-Lobachevsky geometry. This suggests that in the construction of analytic wavelets and multiresolution in the Hardy respectively Bergman spaces Blaschke functions can be taken instead of the affine transforms in $\mathbb{R}$ and the *hyperbolic wavelets transforms* induced by representations of the Blaschke group on these spaces.

The disc $\mathbb{D}$ with the *pseudohyperbolic metric*

$$\rho(z_1, z_2) := \frac{|z_1 - z_2|}{|1 - z_1\bar{z}_2|} = |B_{(z_2,1)}(z_1)| \ \ (z_1, z_2 \in \mathbb{D})$$

is a complete metric space. This metric is invariant with respect to Blaschke-functions:

$$\rho(B_{(b,1)}(z_1), B_{(b,1)}(z_2)) = \rho(z_1, z_2) \ \ (z_1, z_2 \in \mathbb{D},\ b \in \mathbb{D}).$$

This property characterizes the Blaschke-functions. Namely, for every f which is analytic and bounded in $\mathbb{D}$ with $\|f\|_\infty \le 1$ we have $\rho(f(z_1), f(z_2)) \le \rho(z_1, z_2)$, and equality holds if and only if f is a Blaschke-function.

The restrictions of the Blaschke functions on the set $\mathbb{D}$ or on $\mathbb{T}$ with the operation $(B_{a_1} \circ B_{a_2})(z) := B_{a_1}(B_{a_2}(z))$ form a group. In the set of the parameters $\mathbf{B} := \mathbb{D} \times \mathbb{T}$ let us define the operation induced by the function composition in the following way: $B_{a_1} \circ B_{a_2} = B_{a_1 \circ a_2}$. The set of the parameters $\mathbf{B}$ with the induced operation is called the *Blaschke group*. The Blaschke group $(\mathbf{B}, \circ)$ will be isomorphic with the group $(\{B_a, a \in \mathbf{B}\}, \circ)$.

If we use the notations $a_j := (b_j, \epsilon_j)$, $j \in \{1, 2\}$ and $a := (b, \epsilon) =: a_1 \circ a_2$, then

$$b = \frac{b_1\bar{\epsilon}_2 + b_2}{1 + b_1\bar{b}_2\bar{\epsilon}_2} = B_{(-b_2,1)}(b_1\bar{\epsilon}_2), \quad \epsilon = \epsilon_1\frac{\epsilon_2 + b_1\bar{b}_2}{1 + \epsilon_2\bar{b}_1 b_2} = B_{(-b_1\bar{b}_2,\epsilon_1)}(\epsilon_2).$$

The neutral element of the group $(\mathbb{B}, \circ)$ is $e := (0, 1) \in \mathbb{B}$ and the inverse element of $a = (b, \epsilon) \in \mathbb{B}$ is $a^{-1} = (-b\epsilon, \bar{\epsilon})$.

The integral of the function $f : \mathbf{B} \to \mathbb{C}$ with respect to the left invariant Haar-measure m of the group $(\mathbf{B}, \circ)$ can be expressed as

$$\int_{\mathbf{B}} f(a)\, dm(a) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{\mathbb{D}} \frac{f(b, e^{it})}{(1 - |b|^2)^2}\, db_1 db_2 dt,$$

where $a = (b, e^{it}) = (b_1 + ib_2, e^{it}) \in \mathbb{D} \times \mathbb{T}$.

It can be shown that this integral is invariant with respect to the left translation $a \to a_0 \circ a$ and under the inverse transformation $a \to a^{-1}$. Consequently the group $(\mathbf{B}, \circ)$ is unimodular.

The one parameter subgroups

$$\mathbf{B}_1 := \{(r, 1) : r \in (-1, 1)\},\ \mathbf{B}_2 := \{(0, \epsilon) : \epsilon \in \mathbb{T}\}$$

generate $\mathbf{B}$, i.e. if $\bar{\epsilon}_1$ is the conjugate of ϵ_1, we have

$$\mathbf{a} = (0, \epsilon_2) \circ (0, \epsilon_1) \circ (r, 1) \circ (0, \bar{\epsilon}_1) \ \ (\mathbf{a} = (r\epsilon_1, \epsilon_2),\ r \in [0, 1), \epsilon_1, \epsilon_2 \in \mathbb{T}).$$

$\mathbf{B}_1$ is the analogue of the group of dilatation, $\mathbf{B}_2$ is the analogue of the group of translations.

4 Construction of Multiresolution in the Hardy Spaces Based on Localized Reproducing Kernels

4.1 Hardy Spaces

Let us denote by $\mathscr{A}(\mathbb{D})$ the set of analytic functions on the unit disc $\mathbb{D}$. Taking the integral means

$$\|f_r\|_2 := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{it})|^2\, dt\right)^{1/2}$$

of a function $f \in \mathscr{A}(\mathbb{D})$ we define the *Hardy space of the unit disc* $H^2(\mathbb{D})$ as the class of functions in $\mathscr{A}(\mathbb{D})$ for which $\sup_{0<r<1} \|f_r\|_2 < \infty$. It is known that the boundary function $f(e^{it}) := \lim_{r\to 1} f(re^{it})$ exists a.e. for every $f \in H^2(\mathbb{D})$ and f belongs to $L^2(\mathbb{T})$ on $\mathbb{T}$. Moreover, $\|f\|_{H^2} = \|f\|_{L^2(\mathbb{T})}$. The *Hardy space of the unit circle* $H^2(\mathbb{T})$ is a Hilbert space and contains the boundary values of the functions from $\mathscr{H}^2(\mathbb{D})$. The space $H^\infty(\mathbb{D})$ is the collection of functions $f \in \mathscr{A}(\mathbb{D})$ for which $\|f\|_{H^\infty} := \sup_{z\in\mathbb{D}} |f(z| < \infty$. The *disc algebra* $A(\mathbb{D})$, i.e. the set of functions analytic on $\mathbb{D}$ and continuous on its closure is a closed subspace of $H^\infty(\mathbb{D})$.

The Hardy spaces are applied intensively not only in the theories of complex functions and Fourier series but as it turned out in the 1960s that, in particular $H^2(\mathbb{D})$, and $H^\infty(\mathbb{D})$, are the proper Banach spaces for mathematical modeling of problems in control and operator theories (see for ex. [Chui, Chen, 1992] [3], [Ward, Partington, 1996] [50], [Partington, 1997] [41].

The transfer function f of a discrete linear time invariant system belongs to $H^2(\mathbb{T})$ or to $H^\infty(\mathbb{D})$. The main problem is to give a good approximation of f from some measurements made on the unit circle or in the unit disc.

The linear space $H^2(\mathbb{T})$ is a Hilbert space with the scalar product $\langle f, g\rangle := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}\, dt$.

The reproducing kernel of this space is

$$K(z,w) = k_w(z) = \sum_{n=0}^{\infty} \overline{w^n} z^n = \frac{1}{1-\overline{w}z} \quad z, w \in \mathbb{D}.$$

This function is called the Szegő or Cauchy kernel on the disk.

4.2 Multiresolution in $H^2(\mathbb{T})$

The representation of the Blaschke group on $H^2(\mathbb{T})$ is defined by

$$(U_{a^{-1}}f)(z) := \frac{\sqrt{e^{i\theta}(1-|b|^2)}}{(1-\overline{b}z)} f\left(\frac{e^{i\theta}(z-b)}{1-\overline{b}z}\right) \quad \big(z = e^{it} \in \mathbb{T}, a = (b, e^{i\theta}) \in \mathbf{B}\big), \tag{7}$$

where we take the principal rank of the square root.

Pap and Schipp in [37–39, 46] studied the properties of this representation and the properties of the induced voice transform, the so-called *hyperbolic wavelet transform*:

$$(V_g f)(a^{-1}) := \langle f, U_{a^{-1}} g\rangle \quad (f, g \in H^2(\mathbb{T})). \tag{8}$$

For this hyperbolic wavelet transform the square integrability required for the discretization theory developed by Feichtinger, Gröchenig is not satisfied (see [37]). But in [34] Pap introduced a multiresolution in $H^2(\mathbb{T})$. Using the localized Cauchy kernels corresponding to a discrete countable subset A of the unit disc, based on the discretization of hyperbolic voice transform, a multiresolution analysis in $H^2(\mathbb{T})$ was introduced. This can be used for $H^2(\mathbb{D})$ identification if we measure the values of function on this set. It has been showed that the resolution levels are spanned by a special rational analytic orthonormal wavelet system, i.e., by the Malmquist-Takenaka system with a special localization of the poles. In this subsection we give an overview of the construction given in [34]. In [37] it was proved that the levels of the multiresolution form a complete model set for the disc algebra of the unit disc, and it was given an estimation of the error them for the proposed approximation process.

Let us remind that in the construction of affine wavelet multiresolutions the dilatation is used to obtain a higher resolution level ($f(x) \in V_n \Leftrightarrow f(2x) \in V_{n+1}$), and applying the translation we remain on the same level of resolution. If we want to construct a multiresolution in $H^2(\mathbb{T})$, we have to find the analogue of dilation by 2 and the analogue of translation. The analogue of dilation will be the action of the representation through a discrete subgroup $\mathbf{B}_1$ of the Blaschke group.

Let us consider the following discrete subgroup of the Blaschke group

$$\mathbf{B}_1 = \left\{ (r_k, 1) : \ r_k = \frac{2^k - 2^{-k}}{2^k + 2^{-k}}, \ k \in \mathbb{Z} \right\}. \tag{9}$$

It can be proved that $(r_k, 1) \circ (r_n, 1) = (r_{k+n}, 1)$ and

$$\rho(r_k, r_n) := \frac{|r_k - r_n|}{|1 - r_k \overline{r_n}|} = \left| \frac{\frac{2^k-2^{-k}}{2^k+2^{-k}} - \frac{2^n-2^{-n}}{2^n+2^{-n}}}{1 - \frac{2^k-2^{-k}}{2^k+2^{-k}} \frac{2^n-2^{-n}}{2^n+2^{-n}}} \right| = |r_{k-n}|.$$

As a consequence we get that the sequence $(r_k,\ k \in \mathbb{N})$ forms an equidistant division of the interval $[0, 1)$ in the pseudohyperbolic metric.

Let us consider the following discrete subset in the unit disc:

$$A = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{2^{2k}}}, \quad \ell = 0, 1, \cdots, 2^{2k} - 1, \quad k = 0, 1, 2, \cdots, \infty\} \tag{10}$$

and for a fixed $k \in \mathbb{N}$ let the level k be the collection of the points from circle with radius r_k

$$A_k = \{z_{k\ell} = r_k e^{i\frac{2\pi\ell}{2^{2k}}}, \ \ell \in \{0, 1, \cdots, 2^{2k} - 1\}\}. \tag{11}$$

The points of A determine a similar decomposition to the Whitney cube decomposition of the unit disc (see, e.g., [Partington, 1997] [41], pp.80). For our purpose it is more convenient to choose $(r_k,\ n \in \mathbb{N})$ as radius of the concentric circles because they are related to the Blaschke group operation, i.e., $(r_k, 1) \circ (r_n, 1) = (r_{k+n}, 1)$, and from this property we can derive the analogue property of the dilatation.

Let us consider the scaling function $\varphi = 1$. Let us define the 0th resolution level by: $V_0 = \{c\varphi,\ c \in \mathbb{C}\}$. Let us consider the non-orthogonal wavelets on the n-th level the localized and normalized Cauchy kernels corresponding to points $\cup_{k=0}^{n} A_k$, given by

$$\varphi_{k\ell}(z) = \frac{\sqrt{(1-r_k^2)}}{(1-\overline{z_{k\ell}}z)}, \quad k = 0, \cdots, n, \quad \ell = 0, 1, \cdots, 2^{2k} - 1, \tag{12}$$

which can be obtained from φ using the representation $U_{(r_n,1)^{-1}}$, and the translations

$$\varphi_{k\ell}(e^{it}) = (U_{((r_k,1)^{-1}}\varphi)(e^{i(t-\frac{2\pi\ell}{2^{2k}})}).$$

Let us define the n-th resolution level by

$$V_n = \{f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{2^{2k}-1} c_{k\ell}\varphi_{k\ell},\ c_{k,\ell} \in \mathbb{C}\ \}. \tag{13}$$

In [34] it has been proved that the collections of spaces $\{V_j,\ j \in \mathbb{N}\}$ satisfy analogue conditions of the affine multiresolution, i.e.:

1. (nested) $V_j \subset V_{j+1}$,
2. (density) $\overline{\cup V_j} = H^2(T)$
3. (analog of dilatation) $U_{(r_1,1)^{-1}}(V_j) \subset V_{j+1}$
4. (basis) There exist $\{\psi_{k\ell},\ k = 0, \cdots, n,\ \ell = 0, 1, \cdots, 2^{2k} - 1\}$ (orthonormal) bases in V_n.

In order to construct the orthonormal bases $\{\psi_{k\ell},\ k = 0, \cdots, n, \ell = 0, 1, \cdots, 2^{2k} - 1\}$ in V_n we apply the Gram-Schmidt orthogonalization to the following non-orthogonal basis in V_n:

$$\left\{ \frac{1}{1-\overline{z_{k\ell}}z},\ \ell = 0, 1, \cdots, 2^{2k} - 1,\ k = 0, 1, \cdots, n. \right\}.$$

The result of the Gram-Schmidt orthogonalization for this set of analytic linearly independent functions can be written in closed form. As a result we obtain the Malmquist–Takenaka system corresponding to the set $\cup_{k=0}^{n} A_k$ (see [31, 49]):

$$\psi_{m\ell}(z) = \frac{\sqrt{1-r_m^2}}{1-\overline{z_{m\ell}}z} \prod_{k=0}^{m-1} \prod_{j=0}^{2^{2k}-1} \frac{z-z_{kj}}{1-\overline{z_{kj}}z} \prod_{j'=0}^{\ell-1} \frac{z-z_{mj'}}{1-\overline{z_{mj'}}z} \tag{14}$$

$$(\ell = 0, 1, \cdots, 2^{2m}-1, \ \ m = 0, 1, 2, \cdots, n).$$

In this way we have constructed an analytic rational orthonormal wavelet system on the resolution level V_n, i.e.,

$$\langle \psi_{m\ell}, \psi_{m'\ell'} \rangle = \frac{1}{2\pi} \int_0^{2\pi} \psi_{m\ell}(e^{it}) \overline{\psi_{m'\ell'}(e^{it})} dt = \delta_{mm'} \delta_{\ell\ell'}, \tag{15}$$

$$(\ell = 0, 1, \cdots, 2^{2k}-1, \ \ k = 0, 1, 2, \cdots, \infty).$$

From the Gram-Schmidt orthogonalization process it follows that:

$$V_n = span\{\psi_{k\ell}, \ \ell = 0, 1, \cdots, 2^{2k}-1, \ k = 0, \cdots, n\}.$$

Because the points of the set A satisfy the non-Blaschke condition

$$\sum_{k,\ell} (1 - |z_{k\ell}|) = \sum_k 2^{2k} \left(1 - \frac{2^k - 2^{-k}}{2^k + 2^{-k}}\right) = \sum_k \frac{2 \cdot 2^k}{2^k + 2^{-k}} = \infty, \tag{16}$$

the Malmquist–Takenaka system corresponding to the set A is a basis in $H^2(\mathbb{T})$, i.e.

$$\overline{\bigcup_{n \in \mathbb{N}} V_n} = H^2(\mathbb{T})$$

in $H^2(\mathbb{T})$ norm, consequently the density property is also valid.

In signal processing and system identification the Malmquist–Takenaka system is more efficient than the trigonometric system in the determination of the transfer functions.

The wavelet space W_n is the orthogonal complement of V_n in V_{n+1}. In [34] it has been proved that

$$W_n = span\{\psi_{n+1\ell}, \ \ \ell = 0, 1, \cdots, 2^{2n+2}-1\}.$$

Consequently we have generated a multiresolution in $H^2(\mathbb{T})$ and we have constructed an analytic rational orthogonal wavelet system. According to the results obtained in [34] and [37] we can conclude the following advantages of the constructed multiresolution in the Hardy space of the unit disc comparing with the classical affine multiresolution :

1. The levels of the multiresolution are finite dimensional, which makes easier to find a basis on every level, but in the same time the density condition remains valid.

2. We have constructed analytic orthonormal rational wavelet bases on the resolution levels given by an explicit formula.

3. We can compute the wavelet coefficients exactly measuring the values of the function f at the points of the set $A = \bigcup_{k=0}^{n} A_k \subset \mathbb{D}$. We can write exactly the projection operator $(P_n f,\ n \in \mathbb{N})$ on the n-th resolution level which is convergent in $H^2(\mathbb{T})$ norm on the unit circle to f, and $P_n f(z) \to f(z)$ uniformly on every compact subset of the unit disc (see [34]).

4. At the same time $P_n f(z)$ is the best approximant interpolation operator on the set $\bigcup_{k=0}^{n} A_k$ inside the unit circle for the analytic continuation of f.

Similar multiresolution construction can be made also in the Hardy space of the upper-half plane, see [17]. Here we mention that for the upper half plane I. Pesenson and H. Feichtinger developed reconstruction algorithms based on frames for band limited signals from Paley-Wiener spaces based on measurements (see [18, 43, 44]).

5 Construction of Multiresolution in the Bergman Space Based on Localized Reproducing Kernels

5.1 The Bergman Space

We present some basic results connected to the Bergman spaces. For more detailed exposition, see, for example, in [12, 53]. Recall that if $z = x + iy \in \mathbb{D}$, then the normalized area measure is $dA(z) = \frac{1}{\pi} dxdy$. An analytic function $f : \mathbb{D} \to \mathbb{C}$ belongs to the Bergman space $A^2(\mathbb{D})$ if

$$\int_{\mathbb{D}} |f(z)|^2 \, dA(z) < \infty.$$

The scalar product in $A^2 = A^2(\mathbb{D})$ is given by

$$\langle f, g \rangle := \int_{\mathbb{D}} f(z)\overline{g(z)} \, dA(z).$$

The Bergman space $A^2(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D})$. For each $z \in \mathbb{D}$ the point-evaluation map is a bounded linear functional on $A^2(\mathbb{D})$, consequently $A^2(\mathbb{D})$ is a reproducing kernel Hilbert space.

The function

$$K : \mathbb{D} \times \mathbb{D} \to \mathbb{C} \quad \textit{with} \quad K(., z) \in A^2(\mathbb{D})$$

$$K(\xi, z) = \frac{1}{(1 - \overline{z}\xi)^2}.$$

is called the Bergman kernel for $\mathbb{D}$. The explicit formula for the kernel function shows that we have the following reproducing formula:

$$f(z) = \frac{1}{\pi}\int_{\mathbb{D}} f(\xi)\frac{1}{(1-\bar{\xi}z)^2}d\xi_1 d\xi_2 \quad (f \in A^2(\mathbb{D}),\ z, \xi \in \mathbb{D},\ \xi = \xi_1 + i\xi_2).$$

A sequence of points $\Gamma = \{z_k : k \in \mathbb{N}\}$ of points in the unit disc is sampling sequence for A^2, if there exist positive constants A and B such that

$$A||f||^2 \le \sum_{k=1}^{\infty} |f(z_k)|^2(1-|z_k|^2)^2 \le B||f||^2, \quad f \in A^p.$$

This is equivalent to the following inequalities:

$$A||f||^2 \le \sum_{k=1}^{\infty} |\langle f, \varphi_k\rangle|^2 \le B||f||^2, \quad f \in A^2,$$

where $\varphi_k(z) = K(z, z_k)/\|K(z, z_k)\|$ are the normalized Bergman and localized kernels in z_k. This last inequality shows that these functions will constitute a frame for A^2 if and only if $\Gamma = \{z_k : k \in \mathbb{N}\}$ is a sampling set for A^2. Some explicit examples for sampling sequences in the Bergman space are due to Seip, Duren, Schuster, Horowitz, Luecking (see, e.g., in [12, 48]). An A^2 sampling sequence is never an A^2 zero-set. A total characterization of sampling sequences is due to Kristian Seip and can be given with the uniformly discrete property and lower density of the set (see [12, 47]). But the computation of the lower density of a set in general is a difficult task. Duren, Schuster, and Vukotic in [13] gave for sampling sufficient conditions based on the pseudohyperbolic metric, that are relatively easy to verify.

5.2 The Continuous Voice Transform on Bergman Space

The voice transform on Bergman space is induced by a unitary representation of the Blaschke group on the Bergman space. Results connected to the voice transform on Bergman space were published in [40]. Let us consider the following set of functions:

$$F_{\mathbf{a}}(z) := \frac{\sqrt{\epsilon(1-|b|^2)}}{1-\bar{b}z} \quad (\mathbf{a} = (b, \epsilon) \in \mathbf{B}, z \in \overline{\mathbb{D}}).$$

These functions induce a unitary representation on the space A^2. Namely let us define

$$U_{\mathbf{a}}f := [F_{a^{-1}}]^2 f \circ B_{\mathbf{a}}^{-1} \quad (\mathbf{a} \in \mathbf{B}, f \in A^2),$$

then this is a representation of Blaschke group on the A^2, i.e.:

- $U_{x\circ y} = U_x \circ U_y \quad (x, y \in \mathbf{B})$,
- $\mathbf{B} \ni x \to U_x f \in A^2$ is continuous for all $f \in A^2$.

In [40] it was proved that $U_{\mathbf{a}}(\mathbf{a} \in \mathbf{B})$ is a unitary, irreducible square integrable representation of the group $\mathbf{B}$ on the Hilbert space A^2.

The *voice transform* of $f \in A^2$ generated by the representation $U_{\mathbf{a}}$ and by the parameter $g \in A^2$ is the (complex-valued) function on $\mathbf{B}$ defined by

$$(V_g f)(\mathbf{a}) := \langle f, U_{\mathbf{a}} g \rangle \quad (\mathbf{a} \in \mathbf{B}, f, g \in A^2).$$

In a recent paper (see [35]) it is shown that the integrability condition in the Bergman space is not satisfied. This motivated the construction of a multiresolution in the Bergman space. The first step was the construction of a sampling set which is connected to the Blaschke group and the second to generate a multiresolution analysis based on localized Bergman kernels on this set in the Bergman space $A^2(\mathbb{D})$.

In [36] it was considered again the following discrete subgroup of $(\mathbf{B}, \circ)$

$$\mathbf{B}_1 = \left\{ (r_k, 1) : \; r_k = \frac{2^k - 2^{-k}}{2^k + 2^{-k}}, \; k \in \mathbb{Z} \right\}.$$

with composition rule: $(r_k, 1) \circ (r_n, 1) = (r_{k+n}, 1)$.

Let $N = N(2,k) = 2^{2k+3}, k \geq 1, N(2,0) := 1$, and consider the following set of points $z_{00} := 0$,

$$\mathscr{A} = \{ z_{k\ell} = r_k e^{i\frac{2\pi\ell}{N}}, \quad \ell = 0, 1, \ldots, N-1, \quad k = 0, 1, 2, \ldots \}$$

and for a fixed $k \in \mathbb{N}$ let the level k be

$$\mathscr{A}_k = \{ z_{k\ell} = r_k e^{i\frac{2\pi\ell}{N}}, \; \ell \in \{0, 1, \ldots, N-1\} \}.$$

Due to Theorem 2.1 of [36] this is be a sampling set in the Bergman space.

This implies that the set of normalized kernels

$$\left\{ \varphi_{kl}(z) = \frac{(1 - r_k^2)}{(1 - \overline{z_{k\ell}} z)^2}, \; k = 0, 1, \cdots, \; \ell = 0, 1, \cdots N-1 \right\}$$

will constitute a frame system for A^2. From the frame theory (see, for example, in [20]), or from the atomic decomposition results (see Theorem 3 of [51]), follows that every function f from A^2 can be represented

$$f(z) = \sum_{(k,\ell)} c_{k\ell} \varphi_{kl}(z)$$

for some $\{c_{k\ell}\} \in \ell^2$, with the series converging in A^2 norm. The determination of the coefficients is related to the construction of the inverse frame operator (see [20]), which is not an easy task in general. This is the reason why in [35] it was proposed other approximation process for $f \in A^2$ such that the determination of the coefficients follows an exactly defined algorithmic scheme. Here follows a short exposition of the construction.

Let us consider the function $\varphi_{00} = 1$ and let $V_0 = \{c,\ c \in \mathbb{C}\}$. Let us consider the nonorthogonal wavelets on the n-th level, $n \geq 1$ given by

$$\varphi_{n,\ell}(z) = (U_{(z_{n\ell},1)^{-1}} p_0)(z) = \frac{(1 - r_n^2)}{(1 - \overline{z_{n\ell}} z)^2}, \quad \ell = 0, 1, \ldots, N - 1.$$

Let us define the n-th resolution level by

$$V_n = \left\{ f : \mathbb{D} \to \mathbb{C},\ f(z) = \sum_{k=0}^{n} \sum_{\ell=0}^{N-1} c_{k,\ell} \varphi_{k,\ell},\ c_{k,\ell} \in \mathbb{C} \right\}$$

The closed subset V_n is spanned by

$$\{\varphi_{k,\ell},\ \ell = 0, 1, \ldots, N - 1,\ k = 0, \ldots, n\}.$$

Continuing this procedure we obtain a sequence of closed, nested subspaces of A^2 for $z \in \mathbb{D}$

$$V_0 \subset V_1 \subset V_2 \subset \ldots..V_n \subset \ldots .A^2.$$

Due to Theorem 2.1 of [36] the normalized kernels $\{\varphi_{kl}(z) = \frac{(1-r_k^2)}{(1-\overline{z_{k\ell}}z)^2},\ k = 0, 1, \cdots,\ \ell = 0, 1, \cdots N - 1\}$ form a frame system for A^2 which implies that this is a complete and a closed set in norm, i.e.,

$$\overline{\bigcup_{n \in \mathbb{N}} V_n} = A^2,$$

consequently the density property is satisfied.

We observe that in the case of the Bergman space we have to divide the n circle in $N = 2^{2n+3}$ points, more than in the case of the Hardy space, where it was enough 2^{2n} points to achieve the density property of the multiresolution levels.

If a function $f \in V_n$, then $U_{(r_1,1)^{-1}} f \in V_{n+1}$. This is the analogue of the dilation. For this it is sufficient to show that

$$U_{(r_1,1)^{-1}}(\varphi_{k,\ell})(z) = U_{(r_1,1)^{-1}}[(U_{(r_k,1)^{-1}} p_0)](ze^{-i\frac{2\pi\ell}{2^{2k+3}}}) =$$

$$= [(U_{(r_{k+1},1)^{-1}} p_0)](ze^{-i\frac{2\pi 4\ell}{2^{2(k+1)+3}}}) \in V_{n+1}, \quad k = 1, .., n,\ \ell = 1, \ldots 2^{2k+3} - 1.$$

Since the set $\{z_{k\ell},\ \ell = 0, 1, \ldots, N-1,\ k = 0, \ldots, \infty\}$ is a sampling set, it follows that is a set of uniqueness for A^2. This means that every function $f \in A^2$ is uniquely determined by the values $\{f(z_{k\ell})\ \ell = 0, 1, \ldots, N-1,\ k = 0, \ldots, \infty\}$. In the paper of Kehe Zhu [52] is described in general, how can be recaptured a function from a Hilbert space when the values of the function on a set of uniqueness are known and is developed in detail this process in the Hardy space. At the beginning we will follow the steps of the recapitulation process but we will combine this with the multiresolution analysis. The elements of the set

$$\left\{\frac{1}{(1-\overline{z_{k\ell}}z)^2},\ \ell = 0, 1, \ldots, 2^{2k+3}-1,\ k = 0, 1, \ldots, n.\right\}$$

are linearly independent and constitute a nonorthogonal basis in V_n.

Using Gram-Schmidt orthogonalization process they can be orthogonalized, but in this case the result of the process cannot be written in close form like in the case of the Hardy space of the unit disc. Denote by $\psi_{k,\ell}$ the resulting functions. They can be seen as the analogue of the Malmquist -Takenaka system in the Hardy space. This function can be obtained using the following two methods. The first arises from the orthogonalization procedure. To describe the method let us reindex the points of the set $\mathscr{A}$ as follows: $a_1 = z_{00}, a_2 = z_{10}, a_3 = z_{11}, a_4 = z_{12}, \ldots, a_{33} = z_{1,31}, a_{34} = z_{2,0}, \cdots a_m = z_{k\ell} \cdots, k = 0, 1, \ldots.., \ell = 0, 1, \ldots, 2^{2k+3} - 1$, and denote by $K(z, z_{k\ell}) = \frac{1}{(1-\overline{z_{k\ell}}z)^2} := K(z, a_m)$, then the resulted orthonormal system is

$$\phi_{00}(z) = \phi(a_1, z) = \frac{K(z, a_1)}{\|K(., a_1)\|},\ \phi_{k\ell}(z) = \phi(a_1, a_2, \ldots, a_m, z) =$$

$$K(z, a_m) - \sum_{i=1}^{m-1} \phi(a_1, a_2, \ldots, a_i, z) \frac{\langle K(., a_m), \phi(a_1, a_2, \ldots, a_i, .)\rangle}{\|\phi(a_1, a_2, \ldots, a_i, .)\|^2}.$$

Thus the normalized functions

$$\left\{\psi_{k\ell}(z) = \frac{\phi_{k\ell}(z)}{\|\phi_{k\ell}\|},\ k = 1, 2, \cdots, \ell = 0, 1, \cdots 2^{2k+3}\right\}$$

became an orthonormal system. Applying similar construction in Hardy space we get in this way the Malmquist-Takenaka system. They can be written in a nice closed form using the Blaschke products. Unfortunately in our situation this is not the case and the properties of the system cannot be seen from the previous construction.

Another approach is given by Zhu in [52]. He proved that the result of the Gram-Schmidt process is connected to some reproducing kernels and the contractive divisors on Bergman spaces. Let denote $A_m = \{a_1, a_2, \cdots a_m\}$ a set of distinct points in the unit disc. Let H_{A_m} be the subspace of A^2 consisting of all functions in A^2 which vanish on A_m. H_{A_m} is a closed subspace of A^2 and denote by K_{A_m} the reproducing kernel of H_{A_m}. These reproducing kernels satisfy the following recursion formula:

$$K_{A_{m+1}}(z,w) = K_{A_m}(z,w) - \frac{K_{A_m}(z,a_{m+1})K_{A_m}(a_{m+1},w)}{K_{A_m}(a_{m+1},a_{m+1})}, m \geq 0,$$

$$K_{A_0} := K(z,w) = \frac{1}{(1-\overline{w}z)^2}.$$

The result of the Gram-Schmidt process can be expressed as

$$\frac{K(z,a_1)}{\sqrt{K(a_1,a_1)}}, \frac{K_{A_1}(z,a_2)}{\sqrt{K_{A_1}(a_2,a_2)}}, \cdots \frac{K_{A_{m-1}}(z,a_m)}{\sqrt{K_{A_{m-1}}(a_m,a_m)}}, \cdots .$$

Then

$$\psi_{k\ell}(z) = \frac{K_{A_{m-1}}(z,a_m)}{\sqrt{K_{A_{m-1}}(a_m,a_m)}}$$

and is the solution of the following problem

$$\sup\{Ref(a_m) : f \in H_{A_{m-1}}, \|f\| \leq 1\}.$$

This extremal functions in the context of the Bergman spaces have been studied extensively in recent years by Hedenmalm [25]. The main result in [25] is that the function

$$\frac{K_{A_{m-1}}(z,a_m)}{\sqrt{K_{A_{m-1}}(a_m,a_m)}}$$

is a contractive divisor on the Bergman space, vanishes on A_{m-1}, and if $\mathscr{A}$ is not a zero set for A^2, as is in our case, the functions converge to 0 as $m \to \infty$. In Hardy space the partial products of a Blaschke product corresponding to a nonzero set own all these nice properties.

From the Gram-Schmidt orthogonalization process it follows that

$$V_n = span\{\psi_{k,\ell},\ \ell = 0,1,\ldots,2^{2k+3}-1,\ k = \overline{0,n}\}.$$

The wavelet space W_n is the orthogonal complement of V_n in V_{n+1}. We will prove that

$$W_n = span\{\psi_{n+1,\ell},\ \ \ell = 0,1,\ldots,2^{2n+5}-1\}.$$

The constructed multiresolution in the Bergman space (see [36]) and the projection on the resolution levels have all the nice properties of the multiresolution constructed in the case of the Hardy space of the unit disc.

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Regular Sampling on Metabelian Nilpotent Lie Groups: The Multiplicity-Free Case

Vignon S. Oussa

1 Introduction

It is a well-established fact that a function on the real line with its Fourier transform vanishing outside of an interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$ can be reconstructed by the Whittaker-Kotel'nikov-Shannon sampling series from its values at points in the lattice $\mathbb{Z}$ (see [10]). This series expansion takes the form

$$\mathbf{f}(t) = \sum_{k \in \mathbb{Z}} \mathbf{f}(k) \left(\frac{\sin(\pi(t-k))}{\pi(t-k)} \right)$$

with convergence in $L^2(\mathbb{R})$ as well as convergence in $L^\infty(\mathbb{R})$. A relatively novel problem in harmonic analysis has been to find analogues of Whittaker-Kotel'nikov-Shannon sampling series for non-commutative groups. Since $\mathbb{R}$ is a commutative nilpotent Lie group, it is natural to investigate if it is possible to extend Whittaker-Kotel'nikov-Shannon's theorem to nilpotent Lie groups which are not commutative.

Let G be a locally compact group and Γ a discrete subset of G. Let $\mathbf{H}$ be a left-invariant closed subspace of $L^2(G)$ consisting of continuous functions. We say that $\mathbf{H}$ is a **sampling space** with respect to the set Γ [6] if the following conditions are satisfied. Firstly, the restriction map $\mathbf{f} \mapsto \mathbf{f}|_\Gamma$ defines a constant multiple of an isometry of $\mathbf{H}$ into the Hilbert space of square-summable sequences defined over Γ. In other words, there exists a positive constant $c_\mathbf{H}$ such that

$$\sum_{\gamma \in \Gamma} |\mathbf{f}(\gamma)|^2 = c_\mathbf{H} \|\mathbf{f}\|_2^2 \tag{1}$$

V.S. Oussa (✉)
Bridgewater State University, 131 Summer Street, Bridgewater, MA, USA
e-mail: voussa@bridgew.edu

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for all $\mathbf{f}$ in $\mathbf{H}$. Secondly, there exists a vector $\mathbf{s}$ in $\mathbf{H}$ such that an arbitrary element $\mathbf{f}$ in the given Hilbert space has the expansion

$$\mathbf{f}(x) = \sum_{\gamma \in \Gamma} \mathbf{f}(\gamma)\, \mathbf{s}\left(\gamma^{-1} x\right) \tag{2}$$

with convergence in the norm of $L^2(G)$. If Γ is a discrete subgroup of G, we say that $\mathbf{H}$ is a **regular sampling space** with respect to Γ. Also, if $\mathbf{H}$ is a sampling space with respect to Γ and if the restriction mapping $\mathbf{f} \mapsto \mathbf{f}|_{\Gamma} \in l^2(\Gamma)$ is surjective, then we say that $\mathbf{H}$ has the **interpolation property** with respect to Γ. This notion of sampling space is taken from [6] and is analogous to Whittaker-Kotel'nikov-Shannon's theorem. In [7], the authors used a less restrictive definition. They defined a sampling space to be a left-invariant closed subspace of $L^2(G)$ consisting of continuous functions with the additional requirement that the restriction map $\mathbf{f} \mapsto \mathbf{f}|_{\Gamma}$ is a topological embedding of $\mathbf{H}$ into $l^2(\Gamma)$ in the sense that there exist positive real numbers $a \leq b$ such that $a \|\mathbf{f}\|_2^2 \leq \sum_{\gamma \in \Gamma} |\mathbf{f}(\gamma)|^2 \leq b \|\mathbf{f}\|_2^2$ for all $\mathbf{f} \in \mathbf{H}$. The positive number b/a is called the tightness of the sampling set. Notice that in (1) the tightness of the sampling is required to be equal to one. Using oscillation estimates, the authors in [7] provide general but precise results on the existence of sampling spaces on locally compact groups. The band-limited vectors in [7] are functions that belong to the range of a spectral projection of a self-adjoint positive definite operator on $L^2(G)$ called the sub-Laplacian. This notion of band-limitation is essentially due to Pesenson [14] and does not rely on the group Fourier transform.

We shall employ in this work a different concept of band-limitation which in our opinion is consistent with the classical one (the Whittaker-Kotel'nikov-Shannon band-limitation), and the main objective of the present work is to prove that under reasonable assumptions (see Condition 1) Whittaker-Kotel'nikov-Shannon Theorem naturally extends to a large class of non-commutative nilpotent Lie groups.

Let G be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. A subspace $\mathbf{H}$ of $L^2(G)$ is said to be a **band-limited space** with respect to the group Fourier transform if there exists a bounded subset E of the unitary dual of the group G such that E has positive Plancherel measure, and $\mathbf{H}$ consists of vectors whose group Fourier transforms are supported on the bounded set E. In this work, we address the following.

Problem 1. Let N be a simply connected and connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ with rational structure constants. Are there conditions on the Lie algebra $\mathfrak{n}$ under which there exists a uniform discrete subgroup $\Gamma \subset N = \exp \mathfrak{n}$ such that $L^2(N)$ admits a band-limited (in terms of the group Fourier/Plancherel transform) sampling subspace with respect to Γ?

Here are some partial answers to Problem 1.

- If $N = \mathbb{R}^d$, then Γ can be taken to be an integer lattice, and the Hilbert space of functions vanishing outside the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^d$ is a sampling space which enjoys the interpolation property with respect to $\mathbb{Z}^d$.

- Put

$$N = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} : x, y, z \in \mathbb{R} \right\} \text{ and } \Gamma = \left\{ \begin{bmatrix} 1 & m & k \\ 0 & 1 & l \\ 0 & 0 & 1 \end{bmatrix} : k, l, m \in \mathbb{Z} \right\}.$$

  Then N is the three-dimensional Heisenberg Lie group and Γ is a discrete uniform subgroup of N. The unitary dual of N is up to a null set parametrized by the punctured line $\mathbb{R}\setminus\{0\}$, and the Plancherel measure is the weighted Lebesgue measure $|\lambda|\, d\lambda$. It is shown in [4, 6] that there exist band-limited subspaces with respect to the group Fourier transform of $L^2(N)$ which are sampling subspaces with respect to Γ.
- Let N be a step-two nilpotent Lie group with Lie algebra $\mathfrak{n}$ of dimension n with center $\mathfrak{z}$ such that $\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{b} \oplus \mathfrak{a}$ where $[\mathfrak{a}, \mathfrak{b}] \subseteq \mathfrak{z}$, $\mathfrak{a}, \mathfrak{b}$ are commutative Lie algebras,

$$\mathfrak{a} = \sum_{k=1}^{d} \mathbb{R}X_k,\ \mathfrak{b} = \sum_{k=1}^{d} \mathbb{R}Y_k,\ \mathfrak{z} = \sum_{k=1}^{n-2d} \mathbb{R}Z_k \qquad (d \geq 1, n > 2d)$$

  and

$$\det \begin{bmatrix} [X_1, Y_1] & \cdots & [X_1, Y_d] \\ \vdots & \ddots & \vdots \\ [X_d, Y_1] & \cdots & [X_d, Y_d] \end{bmatrix} \tag{3}$$

  is a non-vanishing polynomial in the variables $Z_1, \cdots, Z_{n-2d}$. The map

$$\mathfrak{n} = \mathfrak{z} \oplus \mathfrak{b} \oplus \mathfrak{a} \ni Z + Y + X \mapsto \exp(Z + Y)\exp(X) \in N$$

  defines an analytic diffeormorphism between $\mathfrak{n}$ and N which is used to endow N with a coordinate system. Moreover, given $Z, Z' \in \mathfrak{z}$, $Y, Y' \in \mathfrak{b}$ and $X, X' \in \mathfrak{a}$, the multiplication law for N is given by

$$(\exp(Z+Y)\exp(X))\left(\exp\left(Z'+Y'\right)\exp\left(X'\right)\right) = \exp\left(Z + Z' + Y + e^{adX'}Y'\right)\exp\left(X + X'\right)$$

  where the linear operator adX is defined as follows: $adX(Y) = [X, Y]$. The unitary dual of N is up to a null set parametrized by

$$\{\lambda \in \mathfrak{z}^* : \det \mathbf{B}(\lambda) \neq 0\} \text{ where } \mathbf{B}(\lambda) = \begin{bmatrix} \langle \lambda, [X_1, Y_1] \rangle & \cdots & \langle \lambda, [X_1, Y_d] \rangle \\ \vdots & \ddots & \vdots \\ \langle \lambda, [X_d, Y_1] \rangle & \cdots & \langle \lambda, [X_d, Y_d] \rangle \end{bmatrix}$$

and the corresponding Plancherel measure is the weighted Lebesgue measure $|\det \mathbf{B}(\lambda)| \, d\lambda$. It is proved in [13] that $L^2(N)$ admits band-limited sampling subspaces with respect to the discrete subset

$$\Gamma = \exp\left(\sum_{k=1}^{n-2d} \mathbb{Z}Z_k + \sum_{k=1}^{d} \mathbb{Z}Y_k\right) \exp\left(\sum_{k=1}^{d} \mathbb{Z}X_k\right) \subset N.$$

In the case where the structure constants of the Lie algebra are integers then Γ is a discrete uniform subgroup of N.

The main objective of the present work is to exploit well-known representation theoretic tools for nilpotent Lie groups to establish in a unified manner the existence of regular sampling spaces on a large class of nilpotent Lie groups (see Condition 1). It is worth pointing out that the class of groups considered here contains nilpotent Lie groups of arbitrary step.

1.1 Overview of the Paper

Let us start by fixing notation and by recalling some relevant concepts.

- Let Q be a linear operator acting on an n-dimensional real vector space V. The norm of the matrix Q induced by the max-norm of the vector space V is given by

$$\|Q\|_\infty = \sup\{\|Qv\|_{\max} : v \in V \text{ and } \|v\|_{\max} = 1\}$$

and the max-norm of an arbitrary vector is given by $\|v\|_{\max} = \max\{|v_k| : 1 \leq k \leq n\}$. Next, letting $[Q]$ be the matrix representation of Q with respect to a fixed basis, the transpose of this matrix is denoted $[Q]^T$. Additionally, the **adjoint** of a linear operator Q is denoted Q^*.
- Given a countable sequence $(f_i)_{i\in I}$ of vectors in a Hilbert space $\mathbf{H}$, we say that $(f_i)_{i\in I}$ forms a **frame** [1, 9, 15] if and only if there exist strictly positive real numbers a, b such that for any vector $f \in \mathbf{H}$, $a\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq b\|f\|^2$. In the case where $a = b$, the sequence $(f_i)_{i\in I}$ is called a **tight frame**. If $a = b = 1$, $(f_i)_{i\in I}$ is called a **Parseval frame**.
- Let π be a unitary representation of a locally compact group G acting on a Hilbert space $\mathbf{H}_\pi$. We say that the representation π is **admissible** [6] if there exists a vector h in $\mathbf{H}_\pi$ such that the linear map V_h^π given by

$$V_h^\pi f(x) = \langle f, \pi(x) h\rangle \tag{4}$$

defines an isometry of the Hilbert space $\mathbf{H}_\pi$ into $L^2(G)$. In this case, the vector h is called an **admissible vector** for the representation π.

- Let $(A, \mathcal{M})$ be a measurable space. A family $(\mathbf{H}_a)_{a\in A}$ of Hilbert spaces indexed by the set A is called a field of Hilbert spaces over A [5]. An element f of $\Pi_{a\in A}\mathbf{H}_a$ is a vector-valued function $a \mapsto f(a) \in \mathbf{H}_a$ defined on the set A. Such a map is called a **vector field** on A. A **measurable field** of Hilbert spaces defined on a measurable set A is a field of Hilbert spaces together with a countable set $(e_j)_{j\in J}$ of vector fields such that the functions $a \mapsto \langle e_j(a), e_k(a)\rangle_{\mathbf{H}_a}$ are measurable for all $j, k \in J$, and the linear span of $\{e_j(a)\}_{j\in J}$ is dense in $\mathbf{H}_a$ for each a. A vector field f is called a **measurable vector field** if $a \mapsto \langle f(a), e_j(a)\rangle_{\mathbf{H}_a}$ is a measurable function for each index j.
- Let $\mathfrak{n}$ be a nilpotent Lie algebra of dimension n, and let $\mathfrak{n}^*$ be the dual vector space of $\mathfrak{n}$. A **polarizing subalgebra** $\mathfrak{p}(\lambda)$ subordinated to a linear functional $\lambda \in \mathfrak{n}^*$ (see [2, 12]) is a maximal algebra satisfying $[\mathfrak{p}(\lambda), \mathfrak{p}(\lambda)] = \text{Span-}\{[X, Y] \in \mathfrak{n} : X, Y \in \mathfrak{p}(\lambda)\} \subseteq \ker(\lambda)$.
- The coadjoint action on the dual of $\mathfrak{n}$ is the dual of the adjoint action of $N = \exp \mathfrak{n}$ on $\mathfrak{n}$. In other words, for $X \in \mathfrak{n}$, and a linear functional $\lambda \in \mathfrak{n}^*$, the coadjoint action is defined as follows:

$$(\exp X \cdot \lambda)(Y) = \left\langle \left(e^{-ad(X)}\right)^* \lambda, Y \right\rangle = \left[\left(e^{-ad(X)}\right)^* \lambda\right](Y). \tag{5}$$

The following is a concept which is central to our results.

Definition 1. Let $\mathfrak{p}$ be a subalgebra (or ideal) of $\mathfrak{n}$. We say that $\mathfrak{p}$ is a **constant polarization subalgebra (or ideal)** of $\mathfrak{n}$ if there exists a Zariski open set $\Omega \subset \mathfrak{n}^*$ which is invariant under the coadjoint action of N and $\mathfrak{p}$ is a polarization subalgebra subordinated to every linear functional in Ω.

In other words, $\mathfrak{p}$ is a constant polarization subalgebra of $\mathfrak{n}$ if $\mathfrak{p}$ is a polarization algebra for all linear functionals in general position, and it can then be shown (see Proposition 2) that $\mathfrak{p}$ is necessarily commutative.

1.1.1 Summary of Main Results

Let us suppose that $N = P \rtimes M = \exp(\mathfrak{p}) \rtimes \exp(\mathfrak{m})$ is a simply connected, connected non-commutative nilpotent Lie group with Lie algebra $\mathfrak{n} = \mathfrak{p} + \mathfrak{m}$ such that

Condition 1.

1. *$\mathfrak{p}$ is a constant polarization ideal of $\mathfrak{n}$ (thus commutative) $\mathfrak{m}$ is commutative as well, $p = \dim \mathfrak{p}$, $m = \dim \mathfrak{m}$ and $p - m > 0$.*
2. *There exists a strong Malcev basis $\{Z_1, \cdots, Z_p, A_1, \cdots, A_m\}$ for $\mathfrak{n}$ such that $\{Z_1, \cdots, Z_p\}$ is a basis for $\mathfrak{p}$, $\{A_1, \cdots, A_m\}$ is a basis for $\mathfrak{m}$ and $\Gamma = \exp(\mathbb{Z}Z_1 + \cdots + \mathbb{Z}Z_p)\exp(\mathbb{Z}A_1 + \cdots + \mathbb{Z}A_m)$ is a discrete uniform subgroup of N. This is equivalent to the fact that $\mathfrak{n}$ has rational structure constants (see Chapter 5, [2]).*

First of all, we observe that the mapping $\mathfrak{p}+\mathfrak{m}\ni Z+A\mapsto\exp(Z)\exp(A)\in N$ defines an analytic diffeomorphism between $\mathfrak{n}$ and N which provides a coordinate system on the Lie group N. This diffeomorphism takes the canonical Lebesgue measure on the algebra $\mathfrak{n}$ to a Haar measure on the Lie group N which is both left and right invariant. The multiplication law on N is given by

$$(\exp(Z)\exp(A))\left(\exp\left(Z'\right)\exp\left(A'\right)\right)=\exp\left(Z+e^{ad(A)}Z'\right)\exp\left(A+A'\right)$$

where $Z,Z'\in\mathfrak{p}$ and $A,A'\in\mathfrak{m}$.

In order to properly introduce the concept of band-limitation with respect to the group Fourier transform, we appeal to Kirillov's theory [2] which states that the unitary irreducible representations of N are parametrized by orbits of the coadjoint action of N on the dual of its Lie algebra and can be modeled as acting in the Hilbert space $L^2(\mathbb{R}^m)$. Let Σ be a parameterizing set for the unitary dual of N. In other words, Σ is a cross-section for the coadjoint orbits in an N-invariant Zariski open set $\Omega\subset\mathfrak{n}^*$. If the ideal $\mathfrak{p}$ is a constant polarization subalgebra for $\mathfrak{n}$, then the orbits in general position are $2m$-dimensional submanifolds of $\mathfrak{n}^*$ and we shall (this is a slight abuse of notation) regard Σ as a Zariski open subset of $\mathbb{R}^{p-m}=\mathbb{R}^{n-2m}$. Next, let L be the left regular representation of N acting on $L^2(N)$ by left translations. Let $\mathcal{P}:L^2(N)\longrightarrow L^2\left(\Sigma,L^2(\mathbb{R}^m)\otimes L^2(\mathbb{R}^m),d\mu(\lambda)\right)$ be the Plancherel transform which defines a unitary map on $L^2(N)$ (see Subsection 2.2.2). The Plancherel transform intertwines the left regular representation with a direct integral of irreducible representations of N. The measure used in the decomposition is the so-called Plancherel measure: $d\mu$; which is a weighted Lebesgue measure on Σ. More precisely $d\mu(\lambda)$ is equal to $|\mathbf{P}(\lambda)|\,d\lambda$. $\mathbf{P}(\lambda)$ is a polynomial defined over Σ and $d\lambda$ is the Lebesgue measure on Σ (see Lemma 8.) Given a μ-measurable bounded set $\mathbf{A}\subset\Sigma$, and a fixed measurable field of unit vectors $(\mathbf{u}(\lambda))_{\lambda\in\mathbf{A}}$ in $L^2(\mathbb{R}^m)$, we define the Hilbert space $\mathbf{H_A}$ as follows. $\mathbf{H_A}$ consists of vectors $f\in L^2(N)$ such that

$$\mathcal{P}f(\lambda)=\begin{cases}\mathbf{v}(\lambda)\otimes\mathbf{u}(\lambda)\ \text{if}\ \lambda\in\mathbf{A}\\ 0\otimes 0\ \text{if}\ \lambda\notin\mathbf{A}\end{cases}$$

and $(\mathbf{v}(\lambda)\otimes\mathbf{u}(\lambda))_{\lambda\in\mathbf{A}}$ is a measurable field of rank-one operators. Consequently, $\mathbf{H_A}$ is a left-invariant multiplicity-free band-limited subspace of $L^2(N)$ which is naturally identified with the function space $L^2(\mathbf{A}\times\mathbb{R}^m)$ (see Subsection 2.3.) Conjugating the operators $L(x)$ by the Plancherel transform, we obtain that $[\mathcal{P}\circ L(\cdot)\circ\mathcal{P}^{-1}](\mathbf{v}(\lambda)\otimes\mathbf{u}(\lambda))_{\lambda\in\Sigma}=([\sigma_\lambda(\cdot)\mathbf{v}(\lambda)]\otimes\mathbf{u}(\lambda))_{\lambda\in\Sigma}$ where σ_λ is the unitary irreducible representation corresponding to the linear functional $\lambda\in\Sigma$, and the action of σ_λ on $L^2(\mathbb{R}^m)$ is given by (see Lemma 7)

$$\sigma_\lambda(\exp T)\mathbf{f}(x)=\begin{cases}\mathbf{f}(x-a) & \text{if } T=\sum_{j=1}^m a_jA_j\\ e^{2\pi i\left\langle\lambda,e^{-ad(x_1A_1+\cdots+x_mA_m)}\sum_{j=1}^m z_jZ_j\right\rangle}\mathbf{f}(x) & \text{if } T=\sum_{j=1}^m z_jZ_j\end{cases}.$$

Let $L_{\mathbf{H_A}}$ be the representation induced by the action of the left regular representation on the Hilbert space $\mathbf{H_A}$. It can be shown that if the spectral set $\mathbf{A}$ satisfies precise conditions specified in Theorem 2 then the restriction of $L_{\mathbf{H_A}}$ to the discrete group Γ is unitarily equivalent with a subrepresentation of the left regular representation of Γ acting on $l^2(\Gamma)$. More precisely if $\mathbf{A}$ satisfies the conditions described in Theorem 2, it can be proved that there exists a vector $\eta \in \mathbf{H_A}$ such that $\mathbf{H_A} \ni F \mapsto \left(\left\langle F, L\left(\gamma^{-1}\right)\eta\right\rangle\right)_{\gamma\in\Gamma}$ defines an isometric embedding of $\mathbf{H_A}$ into $l^2(\Gamma)$ which intertwines the representation $L_{\mathbf{H_A}}$ with a subrepresentation of the right regular representation of Γ. Let $\text{proj}_{\mathfrak{p}^*} : \mathfrak{n}^* \to \mathfrak{p}^*$ be the restriction mapping given by

$$\text{proj}_{\mathfrak{p}^*}\left(\sum_{j=1}^{p}\left(\lambda_j Z_j^*\right) + \sum_{j=1}^{m}\left(\xi_j A_j^*\right)\right) = \sum_{j=1}^{n}\lambda_j Z_j^* \tag{6}$$

where $\left\{Z_1^*, \cdots, Z_p^*, A_1^*, \cdots, A_m^*\right\}$ is a dual basis for $\left\{Z_1, \cdots, Z_p, A_1, \cdots, A_m\right\}$. Next define the map $\beta : \Sigma \times \mathbb{R}^m \to \mathbb{R}^p$ by

$$\beta(\lambda, t) = \text{proj}_{\mathfrak{p}^*}\left(\exp\left(\sum_{j=1}^{m} t_j A_j\right)\cdot\lambda\right) = \text{proj}_{\mathfrak{p}^*}\left(\left(e^{-ad\sum_{j=1}^{m} t_j A_j}\right)^*\lambda\right) \tag{7}$$

where $\cdot$ denotes the coadjoint action of N on the dual of its Lie algebra and $t = (t_1, \cdots, t_m)$. Under the assumptions listed in Condition 1, we prove in Lemma 6 that β is a diffeomorphism with rational inverse, and the following holds true.

Theorem 2. *Let $N = P \rtimes M = \exp(\mathfrak{p}) \rtimes \exp(\mathfrak{m})$ be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ satisfying Condition 1. Let $\mathbf{A}$ be a μ-measurable bounded subset of Σ.*

1. *If $\beta(\mathbf{A} \times [0,1)^m)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is contained in a fundamental domain of $\mathbb{Z}^p$, then there exists a vector $\eta \in \mathbf{H_A}$ such that $V_\eta^L(\mathbf{H_A})$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to Γ.*
2. *If $\beta(\mathbf{A} \times [0,1)^m)$ is equal to a fundamental domain of $\mathbb{Z}^p$, then there exists a vector $\eta \in \mathbf{H_A}$ such that $V_\eta^L(\mathbf{H_A})$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with the interpolation property with respect to Γ.*

Let $s = (s_1, s_2, \cdots, s_m)$ be an element of $\mathbb{R}^m$ and define $A(s)$ to be the restriction of the linear map $ad\left(-\sum_{j=1}^{m} s_j A_j\right)$ to the ideal $\mathfrak{p} \subset \mathfrak{n}$. Let $[A(s)]$ be the matrix representation of the linear map $A(s)$ with respect to the basis $\left\{Z_1, \cdots, Z_p\right\}$. Let $e^{[A(s)]}$ be the matrix obtained by exponentiating $[A(s)]$. Since $s \mapsto \left\|e^{[A(s)]^T}\right\|_\infty$ is a continuous function of s, it is bounded over any compact set and in particular over the cube $[0,1]^m$. As such, letting ε be a positive real number satisfying

$$\varepsilon \le \delta = \frac{1}{2}\left(\sup\left\{\left\|e^{[A(s)]^T}\right\|_\infty : s \in [0,1)^m\right\}\right)^{-1} < \infty, \tag{8}$$

we shall prove that under the assumptions provided in Condition 1, the set $\beta\left((-\varepsilon,\varepsilon)^{n-2m}\times[0,1)^m\right)$ has positive Lebesgue measure and is contained in a fundamental domain of $\mathbb{Z}^p$. Appealing to Theorem 2, we are then able to establish the following result which provides us with a concrete formula for the bandwidth of various sampling spaces.

Corollary 1. *Let $N = PM = \exp(\mathfrak{p})\exp(\mathfrak{m})$ be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ satisfying Condition 1. For any positive number ε satisfying (8) there exists a band-limited vector $\eta = \eta_\varepsilon$ in the Hilbert space $\mathbf{H}_{(-\varepsilon,\varepsilon)^{n-2m}}$ such that $V_\eta^L\left(\mathbf{H}_{(-\varepsilon,\varepsilon)^{n-2m}}\right)$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to Γ.*

Next, we exhibit several examples to illustrate that the class of groups under consideration is fairly large.

Example 1.

1. Let N be a simply connected, connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ of dimension four or less. Then, there is a basis for the Lie algebra $\mathfrak{n}$ with respect to which there exists a uniform discrete subgroup $\Gamma \subset N$ such that $L^2(N)$ admits a band-limited sampling subspace with respect to Γ. Additionally, the Heisenberg Lie group admits a sampling space which has the interpolation property with respect to a uniform discrete subgroup (this result for the Heisenberg group was originally proved by Currey and Mayeli in [4]).
2. (Nilpotent Lie groups of the type $\mathbb{R}^p \rtimes \mathbb{R}$) Let N be a simply connected, connected nilpotent Lie group with Lie algebra spanned by $Z_1, Z_2, \cdots, Z_p, A_1$, the vector space generated by $Z_1, Z_2, \cdots, Z_p$ is a commutative ideal, $[adA_1]|_{\mathfrak{p}} = A$ is a nonzero rational upper triangular nilpotent matrix of order p, and $e^A\mathbb{Z}^p \subseteq \mathbb{Z}^p$. Then $L^2(N)$ admits a band-limited sampling subspace with respect to the discrete uniform subgroup $\exp\left(\mathbb{Z}Z_1 + \cdots + \mathbb{Z}Z_p\right)\exp\left(\mathbb{Z}A_1\right)$.
3. Let N be a simply connected, connected nilpotent Lie group with Lie algebra spanned by the vectors $Z_1, Z_2, \cdots, Z_p, A_1, \cdots, A_m$ where $p = m + 1$, the vector space generated by $Z_1, Z_2, \cdots, Z_p$ is a commutative ideal, the vector space generated by $A_1, \cdots A_m$ is commutative and the matrix representation of $ad\left(\sum_{k=1}^m t_k A_k\right)$ restricted to $\mathfrak{p}$ is given by

$$A(t) = \left[ad \sum_{k=1}^m t_k A_k\right]\Bigg|\mathfrak{p} = m! \begin{bmatrix} 0 & t_1 & t_2 & \cdots & t_{m-1} & t_m \\ & 0 & t_1 & t_2 & \ddots & t_{m-1} \\ & & 0 & t_1 & \ddots & \vdots \\ & & & 0 & \ddots & t_2 \\ & & & & \ddots & t_1 \\ & & & & & 0 \end{bmatrix}. \tag{9}$$

Then $L^2(N)$ admits a band-limited sampling subspace with respect to the discrete uniform subgroup $\exp\left(\mathbb{Z}Z_1 + \cdots + \mathbb{Z}Z_p\right)\exp\left(\mathbb{Z}A_1 + \cdots + \mathbb{Z}A_m\right)$.

The work is organized as follows. In Section 2, we present general well-known results of harmonic analysis on nilpotent Lie groups. Section 3 contains intermediate results leading to the proofs of Theorem 2, Corollary 1, and Example 1 which are given in Section 4.

2 Harmonic Analysis on Nilpotent Lie Groups

2.1 Parametrization of Coadjoint Orbits

Let $\mathfrak{n}$ be a finite-dimensional nilpotent Lie algebra of dimension n. We say that $\mathfrak{n}$ has a rational structure [2] if there is a real basis $\{Z_1, \cdots, Z_n\}$ for the Lie algebra $\mathfrak{n}$ having rational structure constants. Moreover, the rational span $\mathfrak{n}_{\mathbb{Q}}\ \{Z_1, \cdots, Z_n\}$ provides a rational structure such that $\mathfrak{n}$ is isomorphic to the vector space $\mathfrak{n}_{\mathbb{Q}} \otimes \mathbb{R}$. Let $\mathfrak{B} = \{Z_1, \cdots, Z_n\}$ be a basis for the Lie algebra $\mathfrak{n}$ such that for any $Z_i, Z_j \in \mathfrak{B}$, we have: $\left[Z_i, Z_j\right] = \sum_{k=1}^{n} c_{ijk} Z_k$ and $c_{ijk} \in \mathbb{Q}$. We say that $\mathfrak{B}$ is a **strong Malcev basis** (see Page 10, [2]) if and only if for each $1 \leq j \leq n$ the real span of $\left\{Z_1, Z_2, \cdots, Z_j\right\}$ is an ideal of $\mathfrak{n}$. Now, let N be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{n}$ having a rational structure. The following result is taken from Corollary 5.1.10, [2]. Let $\{Z_1, \cdots, Z_n\}$ be a strong Malcev basis for the Lie algebra $\mathfrak{n}$. There exists a suitable integer q such that $\Gamma_q = \exp\left(q\mathbb{Z}Z_1\right) \cdots \exp\left(q\mathbb{Z}Z_n\right)$ is a discrete uniform subgroup of N (there is a compact set $K \subset G$ such that $\Gamma K = N$). Setting $X_k = qZ_k$ for $1 \leq k \leq n$, from now on, we fix $\{X_1, \cdots, X_n\}$ as a strong Malcev basis for the Lie algebra $\mathfrak{n}$ such that $\Gamma = \exp\left(\mathbb{Z}X_1\right) \cdots \exp\left(\mathbb{Z}X_n\right)$ is a discrete uniform subgroup of N.

We shall next discuss the Plancherel theory for N. This theory is well exposed in [2] for nilpotent Lie groups. Let $\mathfrak{s}$ be a subset of $\mathfrak{n} = \log(N)$. For each linear functional $\lambda \in \mathfrak{n}^*$, we define the corresponding set

$$\mathfrak{s}(\lambda) = \{Z \in \mathfrak{n} :\ \langle \lambda, [Z, X]\rangle = 0 \text{ for every } X \in \mathfrak{s}\}.$$

Next, we consider a fixed strong Malcev basis $\mathcal{B}' = \{X_1, \cdots, X_n\}$ and we construct a sequence of ideals $\mathfrak{n}_1 \subseteq \mathfrak{n}_2 \subseteq \cdots \subseteq \mathfrak{n}_{n-1} \subseteq \mathfrak{n}$ where each ideal $\mathfrak{n}_k$ is spanned by $\{X_1, \cdots, X_k\}$. It is easy to see that the differential of the coadjoint action on λ at the identity is given by the matrix $\left[\left\langle \lambda, \left[X_j, X_k\right]\right\rangle\right]_{1 \leq j,k \leq n} = \left[\lambda\left(\left[X_j, X_k\right]\right)\right]_{1 \leq j,k \leq n}$. Defining the skew-symmetric matrix-valued function

$$\lambda \mapsto \mathbf{M}(\lambda) = \begin{bmatrix} \lambda\left[X_1, X_1\right] & \cdots & \lambda\left[X_1, X_n\right] \\ \vdots & \ddots & \vdots \\ \lambda\left[X_n, X_1\right] & \cdots & \lambda\left[X_n, X_n\right] \end{bmatrix} \tag{10}$$

on $\mathfrak{n}^*$, it is worth noting that $\mathfrak{n}(\lambda)$ is equal to the null-space of $\mathbf{M}(\lambda)$, if $\mathbf{M}(\lambda)$ is regarded as a matrix with respect to the ordered basis $\mathcal{B}'$ acting on $\mathfrak{n}$ [12]. According to the orbit method [2], the unitary dual of N is in one-to-one correspondence with the set of coadjoint orbits in the dual of the Lie algebra. For each $\lambda \in \mathfrak{n}^*$ we define

$$\mathbf{e}(\lambda) = \{1 \leq k \leq n : \mathfrak{n}_k \not\subseteq \mathfrak{n}_{k-1} + \mathfrak{n}(\lambda)\}. \tag{11}$$

The set $\mathbf{e}(\lambda)$ collects all basis elements $\{X_i : i \in \mathbf{e}(\lambda)\} \subset \mathcal{B}'$ such that if $\mathbf{e}(\lambda) = \{\mathbf{e}_1(\lambda) < \cdots < \mathbf{e}_{2m}(\lambda)\}$ then the dimension of the manifold $\exp\left(\mathbb{R}X_{\mathbf{e}_1(\lambda)}\right) \cdots \exp\left(\mathbb{R}X_{\mathbf{e}_{2m}(\lambda)}\right) \cdot \lambda$ is equal to the dimension of the N-orbit of λ. Each element of the set $\mathbf{e}(\lambda)$ is called a **jump index** and clearly the cardinality of the set of jump indices $\mathbf{e}(\lambda)$ must be equal to the dimension of the coadjoint orbit of λ.

Referring to Theorem 3.1.6, [2], the following holds true.

Proposition 1. *For each subset $\mathbf{e}^\circ \subseteq \{1, 2, \cdots, n\}$, the set $\Omega_{\mathbf{e}^\circ} = \{\lambda \in \mathfrak{n}^* : \mathbf{e}(\lambda) = \mathbf{e}^\circ\}$ is algebraic and N-invariant [3]. Moreover, there exists a set of jump indices $\mathbf{e}$ such that $\Omega_{\mathbf{e}} = \Omega$ is a Zariski open set in $\mathfrak{n}^*$ which is invariant under the action of N.*

Put $\Omega = \Omega_{\mathbf{e}}$. We recall that a polarization subalgebra subordinated to the linear functional λ is a maximal subalgebra $\mathfrak{p}(\lambda)$ of $\mathfrak{n}^*$ satisfying the condition $[\mathfrak{p}(\lambda), \mathfrak{p}(\lambda)] \subseteq \ker \lambda$. Notice that if $\mathfrak{p}(\lambda)$ is a polarization subalgebra associated with the linear functional λ then $\chi_\lambda(\exp X) = e^{2\pi i \lambda(X)}$ defines a character on $\exp(\mathfrak{p}(\lambda)$. It is also well-known that $\dim(\mathfrak{n}(\lambda)) = n - 2m$ and $\dim(\mathfrak{n}/\mathfrak{p}(\lambda)) = m$. Finally, the algebra defined by the formula $\mathfrak{p}(\lambda) = \sum_{k=1}^{n} \mathfrak{n}_k(\lambda|\mathfrak{n}_k)$ (see Page 30, [2] and [12]) is called a Vergne polarization.

Proposition 2. *If $\mathfrak{p}$ is a constant polarization subalgebra for $\mathfrak{n}$, then $\mathfrak{p}$ is commutative.*

Proof. Let Ω be a Zariski open and N-invariant subset of $\mathfrak{n}^*$ such that $\mathfrak{p}$ is a subalgebra subordinated to every linear functional $\lambda \in \Omega$. Next, let $\{W_1, \cdots, W_r\}$ be a basis for $[\mathfrak{p}, \mathfrak{p}]$. Define $[\mathfrak{p}, \mathfrak{p}]^*$ to be the vector subspace of $\mathfrak{n}^*$ spanned by the elements $\{W_1^*, \cdots, W_r^*\}$. Here W_k^* is the unique element in $\mathfrak{n}^*$ satisfying the property $\langle W_k^*, W_j \rangle = \delta_{k,j}$. Observe that $\Omega \cap [\mathfrak{p}, \mathfrak{p}]^*$ is open in $[\mathfrak{p}, \mathfrak{p}]^*$. Next, for arbitrary $\ell \in \Omega \cap [\mathfrak{p}, \mathfrak{p}]^*$, by assumption $[\mathfrak{p}, \mathfrak{p}]$ is contained in the kernel of ℓ. Thus, $[\mathfrak{p}, \mathfrak{p}]$ must be a trivial vector space and it follows that $\mathfrak{p}$ is commutative.

The following result is established in Theorem 3.1.9, [2]

Proposition 3. *A cross-section for the coadjoint orbits in Ω is given by*

$$\Sigma = \{\lambda \in \Omega : \lambda(Z_k) = 0 \text{ for all } k \in \mathbf{e}\}. \tag{12}$$

2.2 Unitary Dual and Plancherel Theory

The setting in which we are studying sampling spaces requires the following ingredients:

1. An explicit description of the irreducible representations occurring in the decomposition of the left regular representation of N.
2. The Plancherel measure and a formula for the group Fourier/Plancherel transform.
3. A description of left-invariant multiplicity-free spaces.

2.2.1 A Realization of the Irreducible Representations of N

The following discussion is taken from Chapter 6, [5]. Let G be a locally compact group, and let K be a closed subgroup of G. Let us define $q : G \to G/K$ to be the canonical quotient map and let φ be a unitary representation of the group K acting in some Hilbert space which we call $\mathbf{H}$. Next, let $\mathbf{K}_1$ be the set of continuous $\mathbf{H}$-valued functions f defined over G satisfying the following properties:

- The image of the support of f under the quotient map q is compact.
- $f(gk) = \left[\varphi(k)^{-1} f\right](g)$ for $g \in G$ and $k \in K$.

Clearly, G acts on the set $\mathbf{K}_1$ by left translation. Now, to simplify the presentation, let us suppose that G/K admits a G-invariant measure (this assumption is not always true.) However, since we are mainly dealing with unimodular groups, the assumption holds. First, we endow $\mathbf{K}_1$ with the following inner product: $\langle f, f' \rangle = \int_{G/K} \langle f(g), f'(g) \rangle_{\mathbf{H}}\, d(gK)$ for $f, f' \in \mathbf{K}_1$. Second, let $\mathbf{K}$ be the Hilbert completion of the space $\mathbf{K}_1$ with respect to this inner product. The translation operators extend to unitary operators on $\mathbf{K}$ inducing the unitary representation $\mathrm{Ind}_K^G(\varphi)$ which acts on $\mathbf{K}$ as follows: $\left[\mathrm{Ind}_K^G(\varphi)(x) f\right](g) = f\left(x^{-1} g\right)$ for $f \in \mathbf{K}$. We notice that if φ is a character, then the Hilbert space $\mathbf{K}$ can be naturally identified with $L^2(G/K)$. The reader who is not familiar with these notions is invited to refer to Chapter 6 of the book of Folland [5] for a thorough presentation.

For each linear functional in the set Σ (see (12)), there is a corresponding unitary irreducible representation of N which is realized as acting in $L^2(\mathbb{R}^m)$ as follows. Define a character χ_λ on the normal subgroup $\exp(\mathfrak{p}(\lambda))$ such that $\chi_\lambda(\exp X) = e^{2\pi i \lambda(X)}$ for $X \in \mathfrak{p}(\lambda)$. In order to realize an irreducible representation corresponding to the linear functional λ, we induce the character χ_λ as follows:

$$\sigma_\lambda = \mathrm{Ind}_{P_\lambda}^N(\chi_\lambda), \text{ where } P_\lambda = \exp(\mathfrak{p}(\lambda)). \tag{13}$$

The induced representation σ_λ acts by left translations on the Hilbert space

$$\mathbf{H}_\lambda = \Big\{ f : N \longrightarrow \mathbb{C} : f(xy) = \chi_\lambda(y)^{-1} f(x) \text{ for } y \in P_\lambda \text{ and } \int_{N/P_\lambda} |f(x)|^2\, d(xP_\lambda) < \infty \Big\}, \tag{14}$$

which is endowed with the following inner product: $\langle f, f' \rangle = \int_{N/P_\lambda} f(n) \overline{f'(n)} d(nP_\lambda)$. Picking a cross-section in N (which we may identify with $\mathbb{R}^{\dim N}$) for N/P_λ, since χ_λ is a character there is an obvious identification between $\mathbf{H}_\lambda$ and the Hilbert space $L^2(N/P_\lambda) = L^2(\mathbb{R}^m)$.

2.2.2 The Plancherel Measure and the Plancherel Transform

For a linear functional $\lambda \in \Omega$, put $\mathbf{e} = \{\mathbf{e}_1 < \mathbf{e}_2 < \cdots < \mathbf{e}_{2m}\}$ and define

$$B(\lambda) = \left[\langle \lambda, \left[X_{\mathbf{e}_i}, X_{\mathbf{e}_j}\right]\rangle\right]_{1 \le i,j \le 2m} = \left[\lambda\left[X_{\mathbf{e}_i}, X_{\mathbf{e}_j}\right]\right]_{1 \le i,j \le 2m}. \tag{15}$$

Then $B(\lambda)$ is a skew-symmetric invertible matrix of rank $2m$. Let $d\lambda$ be the Lebesgue measure on Σ which is parametrized by a Zariski subset of $\mathbb{R}^{n-2m}$. Put $d\mu(\lambda) = |\det B(\lambda)|^{1/2} d\lambda$. It is proved in Section 4.3 [2] that up to multiplication by a constant, the measure $d\mu(\lambda)$ is the Plancherel measure for N, which can be understood as follows. The group Fourier transform $\mathcal{F}$ is an operator-valued bounded operator which is weakly defined on $L^2(N) \cap L^1(N)$ as follows:

$$\sigma_\lambda(f) = \mathcal{F}(f)(\lambda) = \int_N f(n)\, \sigma_\lambda\left(n^{-1}\right) dn \text{ where } f \in L^2(N) \cap L^1(N). \tag{16}$$

$\sigma_\lambda(f)$ is weakly defined as follows. Given $\mathbf{u}, \mathbf{v} \in L^2(\mathbb{R}^m)$, $\langle \sigma_\lambda(f)\mathbf{u}, \mathbf{v}\rangle = \int_N f(n) \left\langle \sigma_\lambda\left(n^{-1}\right)\mathbf{u}, \mathbf{v}\right\rangle dn$. Next, the Plancherel transform is a unitary operator

$$\mathcal{P} : L^2(N) \to L^2\left(\Sigma, L^2(\mathbb{R}^m) \otimes L^2(\mathbb{R}^m), d\mu(\lambda)\right)$$

which is obtained by extending the Fourier transform to $L^2(N)$. This extension induces the equality $\|f\|^2_{L^2(N)} = \int_\Sigma \left\|\widehat{f}(\sigma_\lambda)\right\|^2_{\mathcal{HS}} d\mu(\lambda)$ where $\widehat{f}(\sigma_\lambda) = \mathcal{P}f(\lambda)$ and $\|\cdot\|_{\mathcal{HS}}$ stands for the Hilbert-Schmidt norm. Let L be the left regular representation of the nilpotent group N. It is easy to check that for almost every $\lambda \in \Sigma$ (with respect to the Plancherel measure)

$$\left(\mathcal{P}L(n)\,\mathcal{P}^{-1}A\right)(\lambda) = \sigma_\lambda(n) \circ A(\lambda).$$

In other words, the Plancherel transform intertwines the regular representation with a direct integral of irreducible representations of N. The irreducible representations occurring in the decomposition are parametrized up to a null set by the manifold Σ and each irreducible representation occurs with infinite multiplicities in the decomposition.

2.3 Bandlimited Multiplicity-Free Spaces

Given a Lebesgue measurable set $\mathbf{A} \subseteq \Sigma$, it is easily checked that the Hilbert space

$$\mathcal{P}^{-1}\left(L^2\left(\mathbf{A}, L^2\left(\mathbb{R}^m\right) \otimes L^2\left(\mathbb{R}^m\right)\right), d\mu\left(\lambda\right)\right)$$

is a left-invariant subspace of $L^2(N)$. Let us suppose that $\mathbf{A}$ is a **bounded** subset of Σ of positive Plancherel measure. Letting $|\mathbf{A}|$ be the Lebesgue measure of the set $\mathbf{A}$

$$\mu\left(\mathbf{A}\right) = \int_{\mathbf{A}} \left|\det B\left(\lambda\right)\right|^{1/2} d\lambda \leq |\mathbf{A}| \times \sup\left\{\left|\det B\left(\lambda\right)\right|^{1/2} : \lambda \in \mathbf{A}\right\}. \tag{17}$$

Next, since $\lambda \mapsto |\det B(\lambda)|^{\frac{1}{2}}$ is a continuous function then $\mu(\mathbf{A})$ is finite. Fixing a measurable field $\left(\mathbf{u}\left(\lambda\right)\right)_{\lambda\in\mathbf{A}}$ of unit vectors in $L^2\left(\mathbb{R}^m\right)$, we define

$$\mathbf{H}_{\mathbf{A}} = \left\{ f \in L^2\left(N\right) : \mathcal{P}f\left(\lambda\right) = \begin{cases} \mathbf{v}\left(\lambda\right) \otimes \mathbf{u}\left(\lambda\right) \text{ if } \lambda \in \mathbf{A} \\ 0 \otimes 0 \text{ if } \lambda \notin \mathbf{A} \end{cases} \text{ and } \begin{array}{c} \left(\mathbf{v}\left(\lambda\right) \otimes \mathbf{u}\left(\lambda\right)\right)_{\lambda\in\mathbf{A}} \text{ is a measurable} \\ \text{field of rank-one operators} \end{array} \right\}. \tag{18}$$

Then $\mathbf{H}_{\mathbf{A}}$ is a left-invariant, band-limited and multiplicity-free subspace of $L^2(N)$. Let $h \in \mathbf{H}_{\mathbf{A}}$ such that the Plancherel transform of h is a measurable field of projections. More precisely, let us assume that

$$\mathcal{P}h\left(\sigma_\lambda\right) = \widehat{h}\left(\sigma_\lambda\right) = \begin{cases} \mathbf{u}\left(\lambda\right) \otimes \mathbf{u}\left(\lambda\right) \text{ if } \lambda \in \mathbf{A} \\ 0 \otimes 0 \text{ if } \lambda \notin \mathbf{A} \end{cases} \tag{19}$$

and

$$\left[V_h^L\left(f\right)\right]\left(\exp\left(X\right)\right) = \langle f, L\left(\exp\left(X\right)\right) h\rangle = f * h^*\left(\exp\left(X\right)\right)$$

where $h^*\left(x\right) = \overline{h\left(x^{-1}\right)}$ and $*$ is the convolution operator given by $f * g\left(n\right) = \int_N f\left(m\right) g\left(m^{-1}n\right) dm.$

Proposition 4. *If* $\mathbf{A}$ *is a bounded subset of* Σ *of positive Plancherel measure and if* h *is as given in (19), then* h *is an admissible vector for the representation* $\left(L, \mathbf{H}_{\mathbf{A}}\right)$.

Proof. To check that h is an element of the Hilbert space $\mathbf{H}_{\mathbf{A}}$, it is enough to verify that

$$\|h\|_{L^2(N)}^2 = \int_{\mathbf{A}} \|\mathbf{u}\left(\lambda\right) \otimes \mathbf{u}\left(\lambda\right)\|_{\mathcal{HS}}^2 \, d\mu\left(\lambda\right) = \mu\left(\mathbf{A}\right)$$

is finite. Next, for any vector $f \in \mathbf{H}_{\mathbf{A}}$, the square of the norm of the image of f under the map V_h^L is computed as follows:

$$\left\| V_h^L(f) \right\|_{L^2(N)}^2 = \int_{\mathbf{A}} \left\| \widehat{f}(\sigma_\lambda)(\mathbf{u}(\lambda) \otimes \mathbf{u}(\lambda)) \right\|_{\mathcal{HS}}^2 d\mu(\lambda)$$
$$= \int_{\mathbf{A}} \left\langle \widehat{f}(\sigma_\lambda)\mathbf{u}(\lambda), \widehat{f}(\sigma_\lambda)\mathbf{u}(\lambda) \right\rangle_{L^2(\mathbb{R}^m)} d\mu(\lambda).$$

Letting $\widehat{f}(\sigma_\lambda) = \mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda)$ where $\mathbf{v}(\lambda)$ is in $L^2(\mathbb{R}^m)$, it follows that

$$\left\| V_h^L(f) \right\|_{L^2(N)}^2 = \int_{\mathbf{A}} \langle \mathbf{v}(\lambda), \mathbf{v}(\lambda) \rangle_{L^2(\mathbb{R}^m)} \, d\mu(\lambda) = \|f\|_{L^2(N)}^2.$$

In other words, the map V_h^L defines an isometry from $\mathbf{H}_{\mathbf{A}}$ into $L^2(N)$ and the representation $(L, \mathbf{H}_{\mathbf{A}})$ which is a subrepresentation of the left regular representation of N is admissible. Thus, the vector h is an admissible vector.

Remark 1. It is worth noting that h is convolution idempotent in the sense that $h = h * h^* = h^*$. Next, $V_h^L(\mathbf{H}_{\mathbf{A}})$ is a left-invariant vector subspace of $L^2(N)$ consisting of continuous functions. Moreover, the projection onto the Hilbert space $V_h^L(\mathbf{H}_{\mathbf{A}})$ is given by right convolution in the sense that $V_h^L(\mathbf{H}_{\mathbf{A}}) = L^2(N) * h$.

In order to simplify our presentation, we shall naturally identify the Hilbert space $\mathcal{P}(\mathbf{H}_{\mathbf{A}})$ with the function space $L^2(\mathbf{A} \times \mathbb{R}^m, d\mu(\lambda)\, dt)$. This identification is given by the map

$$(\mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda))_{\lambda \in \mathbf{A}} \mapsto [\mathbf{v}(\lambda)](t) := \mathbf{v}(\lambda, t)$$

for any measurable field of rank-one operators $(\mathbf{v}(\lambda) \otimes \mathbf{u}(\lambda))_{\lambda \in \mathbf{A}}$.

Lemma 1. *Let π be a unitary representation of a group N acting in a Hilbert space $\mathbf{H}_\pi$. Assume that π is admissible, and let h be an admissible vector for π. Furthermore, suppose that $\pi(\Gamma)h$ is a tight frame with frame bound C_h. Then the vector space $V_h(\mathbf{H}_\pi)$ is a left-invariant closed subspace of $L^2(N)$ consisting of continuous functions and $V_h(\mathbf{H}_\pi)$ is a sampling space with sinc-type function $\frac{1}{C_h} V_h(h)$.*

Lemma 1 is proved in Proposition 2.54, [6]. This result establishes a connection between admissibility and sampling theories. This connection will play a central role in the proof of our main results. The following result is a slight extension of Proposition 2.61 [6], and the proof given here is essentially inspired by the one given in the Monograph [6].

Lemma 2. *Let Γ be a discrete subgroup of N with positive co-volume. Let π be a unitary representation of N acting in a Hilbert space $\mathbf{H}_\pi$. If the restriction of π to the discrete subgroup Γ is unitarily equivalent to a subrepresentation of the left regular representation of Γ, then there exists a subspace of $L^2(N)$ which is a* *sampling space with respect to Γ. Moreover, if π is equivalent to the left regular representation of Γ, then there exists a subspace of $L^2(N)$ which is a sampling space with the interpolation property with respect to Γ.*

Proof. Let $T : \mathbf{H}_\pi \to \mathbf{H} \subset l^2(\Gamma)$ be a unitary map which is intertwining the restricted representation of π to Γ with a representation which is a subrepresentation of the left regular representation of the lattice Γ. Since Γ is a discrete group, the left regular representation of Γ is admissible. To see this, let κ be the sequence which is equal to one at the identity of Γ and zero everywhere else. By shifting the sequence κ by elements in Γ, we generate an orthonormal basis for the Hilbert space $l^2(\Gamma)$. Now, let $P : l^2(\Gamma) \to \mathbf{H}$ be an orthogonal projection. Next, we claim that the vector $\eta = T^{-1}(P(\kappa))$ is an admissible vector for $\pi|_\Gamma$ as well. We recall that $V_\eta^\pi(f) = \langle f, \pi(\cdot)\eta\rangle$. Let $N = A\Gamma$ where A is a set of finite measure with respect to the Haar measure of N. Without loss of generality, let us assume that a Haar measure for N is fixed so that $|A| = 1$. Then

$$\left\| V_\eta^\pi(f) \right\|_{L^2(N)}^2 = \int_N |\langle f, \pi(x)\eta\rangle|^2\, dx = \int_A \sum_{\gamma\in\Gamma} \left|\left\langle \pi\left(x^{-1}\right) f, \pi(\gamma)\eta\right\rangle\right|^2 dx.$$

Next, since $\pi(\Gamma)\eta$ is a Parseval frame in $\mathbf{H}_\pi$, $\sum_{\gamma\in\Gamma} \left|\left\langle \pi\left(x^{-1}\right) f, \pi(\gamma)\eta\right\rangle\right|^2 = \left\|\pi\left(x^{-1}\right) f\right\|_{\mathbf{H}_\pi}^2$ and it follows that

$$\left\| V_\eta^\pi(f) \right\|_{L^2(N)}^2 = \int_A \left\|\pi\left(x^{-1}\right) f\right\|_{\mathbf{H}_\pi}^2 dx = \|f\|_{\mathbf{H}_\pi}^2 |A| = \|f\|_{\mathbf{H}_\pi}^2 .$$

Thus η is a continuous wavelet for the representation π, $\pi(\Gamma)(\eta)$ is a Parseval frame, and the Hilbert space $V_\eta^\pi(\mathbf{H}_\pi)$ is a sampling space of $L^2(N)$ with respect to the lattice Γ. Now, for the second part, if we assume that π is equivalent to the left regular representation, then the operator P described above is just the identity map. Next, $\pi(\Gamma)\eta = \pi(\Gamma)\left(T^{-1}(\kappa)\right)$ is an orthonormal basis of $\mathbf{H}_\pi$ and $V_\eta(\eta)$ is a sinc-type function. It follows from Theorem 2.56 [6] that $V_\eta(\mathbf{H}_\pi)$ is a sampling space of $L^2(N)$ which has the interpolation property with respect to the lattice Γ.

Remark 2. Let $\mathbf{H}_\mathbf{A}$ be the Hilbert space of band-limited functions as described in (18). We recall that $\Gamma = \exp(\mathbb{Z}X_1)\cdots\exp(\mathbb{Z}X_n)$ is a discrete uniform subgroup of N. Since $\mathbf{H}_\mathbf{A}$ is left-invariant, the regular representation of N admits a subrepresentation obtained by restricting the action of the left regular representation to the Hilbert space $\mathbf{H}_\mathbf{A}$. Let us denote such a representation by $L_{\mathbf{H}_\mathbf{A}}$. Furthermore, let $L_{\mathbf{H}_\mathbf{A},\Gamma}$ be the restriction of $L_{\mathbf{H}_\mathbf{A}}$ to Γ. If the representation $L_{\mathbf{H}_\mathbf{A},\Gamma}$ is unitarily equivalent to a subrepresentation of the left regular representation of the discrete group Γ, then according to arguments used in the proof of Lemma 2, there exists a vector η such that $V_\eta^L(\mathbf{H}_\mathbf{A})$ is a sampling space of $L^2(N)$ with respect to the discrete uniform group Γ. In the present work, we are aiming to find conditions on the spectral set $\mathbf{A}$ which guarantees that $L_{\mathbf{H}_\mathbf{A},\Gamma}$ is unitarily equivalent to a subrepresentation of the left regular representation of the discrete group Γ. We shall also prove that under the assumptions given in Condition 1, it is possible to find $\mathbf{A}$ such that $V_{\eta}^{L}(\mathbf{H_A})$ is a sampling subspace of $L^2(N)$ with respect to the discrete uniform group Γ.

3 Intermediate Results

Let us now fix assumptions and specialize the theory of harmonic analysis of nilpotent Lie groups to the class of groups being considered here (see Condition 1)

Let N be a simply connected, connected non-commutative nilpotent Lie group with Lie algebra $\mathfrak{n}$ with rational structure constants such that $N = P \rtimes M = \exp(\mathfrak{p})\exp(\mathfrak{m})$ where $\mathfrak{p}$ and $\mathfrak{m}$ are commutative Lie algebras, and $\mathfrak{p}$ is an ideal of $\mathfrak{n}$. We fix a strong Malcev basis

$$\left\{Z_1, \cdots, Z_p, A_1, \cdots, A_m\right\} \tag{20}$$

for $\mathfrak{n}$ such that $\left\{Z_1, \cdots, Z_p\right\}$ is a basis for $\mathfrak{p}$ and $\{A_1, \cdots, A_m\}$ is a basis for $\mathfrak{m}$. Moreover it is assumed that

$$\Gamma = \exp\left(\sum_{k=1}^{p} \mathbb{Z} Z_k\right) \exp\left(\sum_{k=1}^{m} \mathbb{Z} A_k\right)$$

is a discrete uniform subgroup of N. Indeed, in order to ensure that Γ is a discrete uniform group, it is enough to pick $\{A_1, \cdots, A_m\}$ such that the matrix representation of $e^{adA_k}|\mathfrak{p}$ with respect to the basis $\left\{Z_1, \cdots, Z_p\right\}$ has entries in $\mathbb{Z}$. Let $\mathbf{M}(\lambda)$ be the skew-symmetric matrix defined in (10). Regarding $\mathbf{M}(\lambda)$ as the matrix representation of a linear operator acting on $\mathfrak{n}$, we recall that the null-space of $\mathbf{M}(\lambda)$ corresponds to $\mathfrak{n}(\lambda)$. Now, let $\mathbf{M}_k(\lambda)$ be the matrix obtained by retaining the first k columns of $\mathbf{M}(\lambda)$ (see illustration below)

$$\mathbf{M}(\lambda) = \underbrace{\left[\begin{array}{cccccc} 0 & \cdots & 0 & \lambda[Z_1, A_1] & \cdots & \lambda[Z_1, A_m] \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda[Z_p, A_1] & \cdots & \lambda[Z_p, A_m] \\ \lambda[A_1, Z_1] & \cdots & \lambda[A_1, Z_p] & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \lambda[A_m, Z_1] & \cdots & \lambda[A_m, Z_p] & 0 & \cdots & 0 \end{array}\right]}_{\mathbf{M}(\lambda)=\mathbf{M}_n(\lambda)}.$$

with the first column braced as $\mathbf{M}_1(\lambda)$, the first p columns as $\mathbf{M}_p(\lambda)$, and the first $p+1$ columns as $\mathbf{M}_{p+1}(\lambda)$.

For convenience of notation, we use the convention $\text{rank}(\mathbf{M}_0(\lambda)) = 0$. Put

$$X_1 = Z_1, \cdots, X_p = Z_p, X_{p+1} = A_1, \cdots, X_{n-1} = A_{m-1}, X_n = A_m.$$

Lemma 3. *Given* $\lambda \in \mathfrak{n}^*$, *the following holds true.*

$$\{1 \leq k \leq n : \text{rank}\,(\mathbf{M}_k\,(\lambda)) > \text{rank}\,(\mathbf{M}_{k-1}\,(\lambda))\} = \{1 \leq k \leq n : \mathfrak{n}_k \not\subseteq \mathfrak{n}_{k-1} + \mathfrak{n}\,(\lambda)\} = \mathbf{e}\,(\lambda)\,.$$

Proof. First, assume that the rank of $\mathbf{M}_i(\lambda)$ is greater than the rank of $\mathbf{M}_{i-1}(\lambda)$. Then it is clear that X_i cannot be in the null-space of the matrix $\mathbf{M}(\lambda)$. Thus, $\mathfrak{n}_i = \mathfrak{n}_{i-1} + \mathbb{R}X_i \not\subseteq \mathfrak{n}_{i-1} + \mathfrak{n}(\lambda)$. Next, if $\mathfrak{n}_i \not\subseteq \mathfrak{n}_{i-1} + \mathfrak{n}(\lambda)$ and $\mathfrak{n}_i = \mathfrak{n}_{i-1} + \mathbb{R}X_i$ since the basis element X_i cannot be in $\mathfrak{n} + \mathfrak{n}(\lambda)$. Thus the rank of $\mathbf{M}_i(\lambda)$ is greater than the rank of $\mathbf{M}_{i-1}(\lambda)$ and the stated result is established.

It is proved in Theorem 3.1.9, [2] that there exist a Zariski open subset Ω of $\mathfrak{n}^*$ and a fixed set $\mathbf{e} \subset \{1, 2, \cdots, n\}$ such that the map $\lambda \mapsto \{1 \leq k \leq n : \text{rank}\,(\mathbf{M}_k\,(\lambda)) > \text{rank}\,(\mathbf{M}_{k-1}\,(\lambda))\} = \mathbf{e}$ is constant, Ω is invariant under the coadjoint action of N and $\Sigma = \{\lambda \in \Omega : \lambda\,(X_k) = 0 \text{ for all } k \in \mathbf{e}\}$ is an algebraic set which is a cross-section for the coadjoint orbits of N in Ω, as well as a parameterizing set for the unitary dual of N.

Lemma 4. *If* $\mathfrak{p}$ *is a constant polarization for* $\mathfrak{n}$, *then the set* $\{p + 1, p + 2, \cdots, \}$ $\{p + m = n\}$ *is contained in* $\mathbf{e}$ *and* card $(\mathbf{e}) = 2m$.

Proof. Let $\lambda \in \Sigma$. Let us suppose that there exists one index k such that $k \in \{p + 1, p + 2, \cdots, p + m = n\}$ and k is not an element of the set $\mathbf{e}\,(\lambda) = \mathbf{e}$. Then the dimension of the coadjoint orbit of the linear functional λ is equal to $2(m - 1)$. However, the dimension of any polarization algebra subordinated to λ must be equal to $n-(m-1) = n-m+1$. This contradicts the fact that the ideal $\mathfrak{p}$ is a polarization algebra subordinated to λ and $\dim \mathfrak{p} = n - m$. The second part of the lemma is true because $\mathbf{M}(\lambda)$ is a skew symmetric rank of positive rank.

Remark 3. From now on, we shall assume that $\mathfrak{p}$ is a constant polarization ideal for $\mathfrak{n}$. Since $P = \exp \mathfrak{p}$ is normal in N, it is clear that the dual of the Lie algebra $\mathfrak{p}$ is invariant under the coadjoint action of the commutative group M.

We recall that $\beta : \Sigma \times \mathbb{R}^m \to \mathbb{R}^p$ and

$$\beta\,(\lambda, t) = \text{proj}_{\mathfrak{p}^*}\left(\exp\left(\sum_{j=1}^{m} t_j A_j\right)\cdot\lambda\right) = \text{proj}_{\mathfrak{p}^*}\left(\left(e^{-ad\sum_{j=1}^{m} t_j A_j}\right)^*\lambda\right). \qquad (21)$$

Remark 4. In vector-form, $\beta\,(\lambda, t_1, \cdots, t_m)$ is easily computed as follows. Let $\mathfrak{P}\,(A(t))$ be the transpose of the matrix representation of $e^{-ad\left(\sum_{k=1}^{m} t_k A_k\right)|\mathfrak{p}}$ with respect to the ordered basis $\{Z_k : 1 \leq k \leq p\}$. We write $\mathfrak{P}\,(A(t)) = \left[e^{-ad\left(\sum_{k=1}^{m} t_k A_k\right)|\mathfrak{p}}\right]^T$ and $\beta\,(\lambda, t_1, \cdots, t_m)$ is identified with

$$\mathfrak{P}\,(A(t))\begin{bmatrix} f_1 \\ \vdots \\ f_p \end{bmatrix} \text{ where } \lambda = \sum_{k=1}^{p} f_k Z_k^*$$

$\left\{Z_k^* : 1 \leq k \leq p\right\}$ is a dual basis to $\{Z_k : 1 \leq k \leq p\}$. We shall generally make no distinction between linear functionals and their representations as either row or column vectors. Secondly for any linear functional $\lambda = \sum_{k=1}^{p} f_k Z_k^* \in \Sigma$ since $\mathfrak{P}(A(t))$ is a unipotent matrix, the components of $\beta(\lambda, t_1, \cdots, t_m)$ are polynomials in the variables f_k where $k \notin \mathbf{e}$ and $t_1, \cdots t_m$.

The following result is proved in [2], Proposition 1.3.6

Lemma 5. *Let $\lambda \in \Sigma$. $\mathfrak{p}$ is a polarizing ideal subordinated to the linear functional λ if and only if for any given $X \in \mathfrak{n}$, $\mathfrak{p}$ is also a polarizing ideal subordinated to the linear functional* $\exp(X) \cdot \lambda$.

Lemma 6. *If for each $\lambda \in \Sigma$, $\mathfrak{p} = \mathbb{R}$-span $\left\{Z_1, \cdots, Z_p\right\}$ is a commutative polarizing ideal which is subordinated to the linear functional λ, then β defines a diffeomorphism between $\Sigma \times \mathbb{R}^m$ and its range. Moreover, the inverse of β is a rational map.*

Proof. In order to prove this result, it is enough to show that β is a bijective smooth map with constant full rank (see Proposition 6.5, [11]). First, we need a formula for the coadjoint action. Let us define $\beta^{\mathbf{e}} : \Sigma \times \mathbb{R}^{2m}$ such that $\beta^{\mathbf{e}}(\lambda, t_{\mathbf{e}_1}, \cdots, t_{\mathbf{e}_{2m}}) = \exp(t_{\mathbf{e}_1} X_{\mathbf{e}_1}) \cdots \exp(t_{\mathbf{e}_{2m}} X_{\mathbf{e}_{2m}}) \cdot \lambda$ and $\{\mathbf{e}_1 < \mathbf{e}_2 < \cdots < \mathbf{e}_{2m}\} = \mathbf{e}$. For a fixed linear functional λ in the cross-section Σ, the map $(t_{\mathbf{e}_1}, \cdots, t_{\mathbf{e}_{2m}}) \mapsto \exp(t_{\mathbf{e}_1} X_{\mathbf{e}_1}) \cdots \exp(t_{\mathbf{e}_{2m}} X_{\mathbf{e}_{2m}}) \cdot \lambda$ defines a diffeomorphism between $\mathbb{R}^{2m}$ and the N-orbit of λ which is a closed submanifold of the dual of the Lie algebra $\mathfrak{n}$. Next, since all orbits in Ω are $2m$-dimensional manifolds, the map $\beta^{\mathbf{e}}$ is a bijection with constant full-rank. Thus, $\beta^{\mathbf{e}}$ defines a diffeomorphism between $\Sigma \times \mathbb{R}^{2m}$ and Ω. Next, there exist indices $i_1, \cdots, i_m \leq p$ such that for $g = \exp(t_{\mathbf{e}_1} X_{\mathbf{e}_1}) \cdots \exp(t_{\mathbf{e}_{2m}} X_{\mathbf{e}_{2m}})$ we have $g \cdot \lambda = \exp\left(\sum_{k=1}^{m} t_{i_k} Z_{i_k}\right) \exp\left(\sum_{j=1}^{m} s_j A_j\right) \cdot \lambda$. In order to compute the coadjoint action of N on the linear functional λ, it is quite convenient to identify $\mathfrak{n}^*$ with $\mathfrak{p}^* \times \mathfrak{m}^*$ via the map $\iota : \mathfrak{n}^* = \mathfrak{p}^* + \mathfrak{m}^* \to \mathfrak{p}^* \times \mathfrak{m}^*$ given by

$$\iota(f_1 + f_2) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \text{ where } f_1 \in \mathfrak{p}^* \text{ and } f_2 \in \mathfrak{m}^*. \tag{22}$$

Thus, for any linear functional $\lambda \in \Sigma$, $\iota(\lambda) = \begin{bmatrix} f \\ 0 \end{bmatrix}$ for some $f \in \mathfrak{p}$. Put $A(s) = \sum_{j=1}^{m} \left(s_j A_j\right) \in \mathfrak{m}$, $Z(t) = \sum_{k=1}^{m} (t_{i_k} Z_{i_k}) \in \mathfrak{p}$ where $s = (s_1, \cdots, s_m), t = (t_{i_1}, \cdots, t_{i_m})$. Clearly, with the fixed choice of the Malcev basis described in (20), it is easy to check that (see Remark 4)

$$\iota(\exp(A(s)) \cdot \lambda) = \begin{bmatrix} \mathfrak{P}(A(s)) f \\ 0 \end{bmatrix} \text{ and } \iota(\exp(Z(t)) \cdot \lambda) = \begin{bmatrix} f \\ \sigma(t, f) \end{bmatrix} \tag{23}$$

where $\sigma(t,f)$ is an $m \times 1$ vector which depends on t and f. Next, (23) gives $\exp(Z(t))\exp(A(s))\cdot\lambda = \begin{bmatrix} \mathfrak{P}(A(s))f \\ \sigma(t,\mathfrak{P}(A(s))f) \end{bmatrix}$. Now, we consider the map

$$(f,s,t) \mapsto \begin{bmatrix} \mathfrak{P}(A(s))f \\ \sigma(t,\mathfrak{P}(A(s))f) \end{bmatrix}. \tag{24}$$

The Jacobian of (24) takes the form

$$\mathbf{T}(f,s,t) = \begin{bmatrix} \mathbf{A}(f,s) & 0 \\ \mathbf{C}(f,s,t) & \mathbf{D}(f,s,t) \end{bmatrix}$$

where $\mathbf{A}(f,s)$ is a matrix of order p obtained by taking the partial derivatives of the components of $\mathfrak{P}(A(s))f$ with respect to the coordinates corresponding to $(f,s) \in \mathbb{R}^p$. $\mathbf{C}(f,s,t)$ is an $m \times p$ matrix obtained by computing the partial derivatives of the components of $\sigma(t,\mathfrak{P}(A(s))f)$ with respect to (f,s). Finally $\mathbf{D}(f,s,t)$ is a matrix of order m which is given by computing the partial derivatives of the components of $\sigma(t,\mathfrak{P}(A(s))f)$ with respect to the coordinates of $s \in \mathbb{R}^m$. Observe that $\det\mathbf{T}(f,s,t) = \det(\mathbf{A}(f,s))\det(\mathbf{D}(f,s,t)) \neq 0$. Since the Jacobian of β is the submatrix $\mathbf{A}(f,s)$, clearly $\det(\mathbf{A}(f,s)) \neq 0$. It follows that the Jacobian of the map β has constant full rank as well. Since $\iota(\beta(\lambda,s)) = \mathfrak{P}(A(s))f$ and (24) is a bijection then it is clear that β is a smooth bijection with constant full-rank. Thus, β is a diffeomorphism. For the second part we write $\beta(\lambda,t) = (\beta_1(\lambda,t),\cdots,\beta_p(\lambda,t))$ where the range of each β_k is a subset of $\mathbb{R}$. Since N is a nilpotent Lie group, for each index k the function β_k is a polynomial function in the variables λ and t. Next we consider the system of equations $s_k = \beta_k(\lambda,t)$ where the variables are given by the coordinates of (λ,t). Since each $\beta_k(\lambda,t)$ is a polynomial, the solutions for the equations above are given by some rational functions $R_1(s),\cdots,R_p(s)$ in the variables s and $\beta^{-1}(s) = (R_1(s),\cdots,R_p(s))$.

Remark 5. For $\lambda \in \Sigma$ there exist real numbers f_k such that $\lambda = \sum_{k\notin\mathbf{e}} f_k Z_k^* = f_{k_1}Z_{k_1}^* + \cdots + f_{k_{n-2m}}Z_{k_{n-2m}}^*$. Thus, defining $\mathrm{j} : f_{k_1}Z_{k_1}^* + \cdots + f_{k_{n-2m}}Z_{k_{n-2m}}^* \mapsto (f_{k_1},\cdots,f_{k_{n-2m}})$ it is clear that the unitary dual of the Lie group N is parametrized by $\mathrm{j}(\Sigma)$ which is a Zariski open subset of $\mathbb{R}^{\dim\Sigma}$. Although this is an abuse of notation, we shall make no distinction between $\mathrm{j}(\Sigma)$ and Σ.

Lemma 7. *Given a linear functional $\lambda \in \Sigma$ and a vector $h \in L^2(\mathbb{R}^m)$, the irreducible representation σ_λ of N acts on $L^2(\mathbb{R}^m)$ as follows*

$$[\sigma_\lambda(\exp(Z(t))\exp(A(a)))h](x) = e^{2\pi i\left\langle \lambda, e^{-ad(x_1A_1+\cdots+x_mA_m)}Z(t)\right\rangle} h(x_1-a_1,\cdots,x_m-a_m). \tag{25}$$

Proof. For each $\lambda \in \Sigma$, the corresponding unitary irreducible representation σ_λ of N (see (13)) is obtained by inducing the character χ_λ of the normal subgroup P

which is defined as follows: $\chi_\lambda\left(\exp\left(t_1Z_1+\cdots+t_pZ_p\right)\right)=e^{2\pi i\left\langle\lambda,t_1Z_1+\cdots+t_pZ_p\right\rangle}=e^{2\pi i\lambda\left(t_1Z_1+\cdots+t_pZ_p\right)}$. Put $a=(a_1,\cdots,a_m)$, $x=(x_1,\cdots,x_m)$, $t=\left(t_1,\cdots,t_p\right)$. Define $A(a)=a_1A_1+\cdots+a_mA_m$, and $Z(t)=t_1Z_1+\cdots+t_pZ_p$. We observe that $\exp(A(a))^{-1}\exp A(x)=\exp(A(x-a))$ and $(\exp Z(t))^{-1}\exp A(x)=\exp A(x)\exp\left(e^{adA(-x)}Z(-t)\right)$. Since $\exp(\mathbb{R}A_1+\cdots+\mathbb{R}A_m)$ is a cross-section for N/P in N, we shall realize the unitary representation σ_λ as acting on the Hilbert space $L^2(N/P)$ which we identify with $L^2(\mathbb{R}^m)$. Following the discussion given in Subsection 2.2.1, if $h\in L^2(\mathbb{R}^m)$, then for every linear functional $\lambda\in\Sigma$

$$[\sigma_\lambda(\exp(Z(t))\exp(A(a)))h](x)=e^{2\pi i\left\langle\lambda,e^{-adA(x)}Z(t)\right\rangle}h(x_1-a_1,\cdots,x_m-a_m). \tag{26}$$

We remark that although the Plancherel measure for an arbitrary nilpotent Lie group has already been computed in general form in the book of Corwin and Greenleaf [2], in order to prove the main results stated in the introduction, we will need to establish a connection between the Plancherel measure of N and the determinant of the Jacobian of the map β. To make this connection as transparent as possible, we shall need the following lemma.

Lemma 8. *Let $J_\beta(\lambda,s_1,\cdots,s_m)$ be the Jacobian of the smooth map β defined in (20). The Plancherel measure of N is up to multiplication by a constant equal to*

$$d\mu(\lambda)=\left|\det J_\beta(\lambda,0)\right|d\lambda \tag{27}$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}^{\dim\Sigma}$.

Proof. Let $N=P\rtimes M$ as defined above. Since the set of smooth functions of compact support is dense in $L^2(N)$, it suffices to show that for any smooth function $\mathbf{F}$ of compact support on the group N,

$$\int_\Sigma\left\|\widehat{\mathbf{F}}(\sigma_\lambda)\right\|^2_{\mathcal{HS}}\left|\det J_\beta(\lambda,0)\right|d\lambda=\|\mathbf{F}\|^2_{L^2(N)}.$$

In order to simplify our presentation, we shall identify the set $N=PM$ with $\mathbb{R}^p\times\mathbb{R}^m$ via the map

$$\exp\left(t_1Z_1+\cdots+t_pZ_p\right)\exp(a_1A_1+\cdots+a_mA_m)\mapsto\left(t_1,\cdots,t_p,a_1,\cdots,a_m\right)=(t,a).$$

As such the Haar measure on the group N is taken to be the Lebesgue measure on $\mathbb{R}^p\times\mathbb{R}^m$. For any smooth function $\mathbf{F}$ of compact support on the group N, the operator $\widehat{\mathbf{F}}(\sigma_\lambda)$ (see (16)) is defined as acting on $L^2(\mathbb{R}^m)$ as follows. For $\phi\in L^2(\mathbb{R}^m)$ we have

$$\left[\widehat{\mathbf{F}}(\sigma_\lambda)\phi\right](x)=\int_{\mathbb{R}^p}\int_{\mathbb{R}^m}\mathbf{F}(t,a)\left[(\sigma_\lambda(t,a))\phi\right](x)\,dadt \tag{28}$$

$$= \int_{\mathbb{R}^p} \int_{\mathbb{R}^m} \mathbf{F}(t,a) \, e^{2\pi i \left\langle \lambda, e^{-adA(x)} Z(t) \right\rangle} \phi(x-a) \, dadt \tag{29}$$

$$= \int_{\mathbb{R}^p} \int_{\mathbb{R}^m} \mathbf{F}(t,a) \, e^{2\pi i \langle \exp(A(x)) \cdot \lambda, Z(t) \rangle} \phi(x-a) \, dadt. \tag{30}$$

Next, we recall that $\mathfrak{P}(A(x))f = \left[e^{-adA(x)|\mathfrak{p}}\right]^T f$ and

$$\iota(\exp(A(x)) \cdot \lambda) = \begin{bmatrix} \mathfrak{P}(A(x))f \\ 0 \end{bmatrix} \text{ where } \iota(\lambda) = \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

Next,

$$\begin{aligned} \left[\widehat{\mathbf{F}}(\sigma_\lambda)\phi\right](x) &= \int_{\mathbb{R}^p} \int_{\mathbb{R}^m} \mathbf{F}(t,a) \, e^{2\pi i \langle \exp(A(x)) \cdot \lambda, Z(t) \rangle} \phi(x-a) \, dadt \\ &= \int_{\mathbb{R}^p} \int_{\mathbb{R}^m} \mathbf{F}(t,a) \, e^{2\pi i \langle \beta(\lambda,x), Z(t) \rangle} \phi(x-a) \, dadt \\ &= \int_{\mathbb{R}^p} \int_{\mathbb{R}^m} \mathbf{F}(t,x-a) \, e^{2\pi i \langle \beta(\lambda,x), Z(t) \rangle} \phi(a) \; da \, dt \\ &= \int_{\mathbb{R}^p} \left(\int_{\mathbb{R}^m} \mathbf{F}(t,x-a) \, e^{2\pi i \langle \beta(\lambda,x), Z(t) \rangle} \, dt \right) \phi(a) \, da. \end{aligned}$$

Thus, $\widehat{\mathbf{F}}(\sigma_\lambda)$ is an integral operator on $L^2(\mathbb{R}^m)$ with kernel $K_{\lambda,\mathbf{F}}$ given by

$$K_{\lambda,\mathbf{F}}(x,a) = \int_{\mathbb{R}^p} \mathbf{F}(t,x-a) \, e^{2\pi i \langle \beta(\lambda,x), Z(t) \rangle} dt. \tag{31}$$

Now, let $\mathfrak{F}_1$ be the partial Euclidean Fourier transform in the direction of t. It is clear that $K_{\lambda,\mathbf{F}}(x,a) = [\mathfrak{F}_1\mathbf{F}](\beta(\lambda,x), x-a)$. Additionally, the square of the Hilbert-Schmidt norm of the operator $\widehat{\mathbf{F}}(\sigma_\lambda)$ is given by

$$\begin{aligned} \left\| \widehat{\mathbf{F}}(\sigma_\lambda) \right\|_{\mathcal{HS}}^2 &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |K_{\lambda,\mathbf{F}}(x,a)|^2 \, dx \, da \\ &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |\mathfrak{F}_1\mathbf{F}(\beta(\lambda,x), x-a)|^2 \, dx \, da \\ &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} |\mathfrak{F}_1\mathbf{F}(\beta(\lambda,x), a)|^2 \, dx \, da. \end{aligned}$$

Observing that $\iota(\beta(\lambda, x+t)) = \mathfrak{P}(A(x))\mathfrak{P}(A(t))f$, the components of $\beta(\lambda, x+t)$ may be computed by multiplying a unipotent matrix by the matrix representation of $\beta(\lambda,t)$, the determinant of the Jacobian of the map β at (λ, x) is then given by

$$\det J_\beta\left((\lambda, x+t)\right) = \det\left(\left[e^{(-adx_1A_1-\cdots-adx_mA_m)|_{\mathfrak{p}}}\right]\right)\det J_\beta\left(\lambda, t\right)$$
$$= 1 \times \det J_\beta\left(\lambda, t\right).$$

It follows that $\det J_\beta(\lambda, x) = \det J_\beta(\lambda, 0) = \mathbf{P}(\lambda)$ where $\mathbf{P}(\lambda)$ is a polynomial in the coordinates of λ. Next,

$$\begin{aligned}
&\int_\Sigma \left\|\widehat{\mathbf{F}}(\sigma_\lambda)\right\|^2_{\mathcal{HS}} \left|\det J_\beta(\lambda,0)\right| d\lambda \\
&= \int_\Sigma \left(\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \left|(\mathfrak{F}_1\mathbf{F})(\beta(\lambda,x),a)\right|^2 da\, dx\, \left|\det J_\beta(\lambda,0)\right|\right) d\lambda \\
&= \int_\Sigma \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \left|(\mathfrak{F}_1\mathbf{F})(\beta(\lambda,x),a)\right|^2 da\, dx\, \left|\det J_\beta(\lambda,0)\right|\, d\lambda \\
&= \int_{\Sigma\times\mathbb{R}^m}\int_{\mathbb{R}^m} \left|(\mathfrak{F}_1\mathbf{F})(\beta(\lambda,x),a)\right|^2 da\, \left|\det J_\beta(\lambda,0)\right|\, d(\lambda,x) \\
&= \int_{\Sigma\times\mathbb{R}^m}\int_{\mathbb{R}^m} \left|(\mathfrak{F}_1\mathbf{F})(\beta(\lambda,x),a)\right|^2 da\, d(\beta(\lambda,x)) \\
&= \int_\Omega\int_{\mathbb{R}^m} \left|(\mathfrak{F}_1\mathbf{F})(z,a)\right|^2 da\, dz.
\end{aligned}$$

Finally, appealing to the Plancherel Theorem for $\mathbb{R}^m$

$$\int_\Sigma \left\|\widehat{\mathbf{F}}(\sigma_\lambda)\right\|^2_{\mathcal{HS}} \left|\det J_\beta(\lambda,0)\right| d\lambda = \int_{\mathbb{R}^p}\int_{\mathbb{R}^m} \left|\mathbf{F}(z,a)\right|^2 dadz = \|\mathbf{F}\|^2_{L^2(N)}.$$

We are now interested in finding conditions on the spectral set $\mathbf{A}$ which will allow us to isometrically embed the Hilbert space $L^2(\mathbf{A}\times\mathbb{R}^m, d\mu(\lambda)\,dt)$ into the sequence space $l^2(\Gamma)$. Moreover, this map should also intertwine the operators $\mathcal{P}\circ L(\gamma)\circ\mathcal{P}^{-1}$ with the right shift operators acting on $l^2(\Gamma)$ ($\gamma\in\Gamma$). Let $\mathbf{A}$ be a Lebesgue measurable subset of $\mathbb{R}^{n-2m}$. Define (in a formal way) the map

$$J_{\mathbf{A}} : L^2(\mathbf{A}\times\mathbb{R}^m, d\mu(\lambda)\,dt) \to \text{Set of sequences over } \Gamma$$

such that for $l = (l_1, \cdots, l_m) \in \mathbb{Z}^m$ we have

$$[J_{\mathbf{A}}F]\left(\exp\left(\sum_{j=1}^p k_jZ_j\right)\exp\left(\sum_{j=1}^m l_jA_j\right)\right) = \int_{\mathbf{A}}\int_{[0,1)^m} F(\lambda, t-l)\, e^{2\pi i\left\langle\beta(\lambda,t),\sum_{j=1}^p k_jZ_j\right\rangle}\left|\det J_\beta(\lambda,0)\right| dt\, d\lambda \tag{32}$$

where $\exp\left(\sum_{j=1}^p k_jZ_j\right)\exp\left(\sum_{j=1}^m l_jA_j\right) \in \Gamma$. Next, we define $Z(k) = \sum_{j=1}^p k_jZ_j$ and $A(l) = \sum_{j=1}^m l_jA_j$. Let 1_X denotes the indicator function for a given set X.

Lemma 9. *For any $\gamma \in \Gamma$, and $F \in L^2(\mathbf{A} \times \mathbb{R}^m, d\mu(\lambda)\, dt)$,*

$$|[J_{\mathbf{A}}F](\gamma)| \leq \|F\| \times \int_{\mathbf{A}} \int_{[0,1)^m} \left|\det J_\beta(\lambda, 0)\right| dt\, d\lambda.$$

Proof. For $\gamma \in \Gamma$ such that $\gamma = \exp(Z(k)) \exp(A(l))$ we obtain:

$$\begin{aligned}
[J_{\mathbf{A}}F](\gamma) &= \int_{\mathbf{A}} \int_{[0,1)^m} F(\lambda, t-l)\, e^{2\pi i \left\langle \beta(\lambda,t), \sum_{j=1}^{p} k_j Z_j \right\rangle} \left|\det J_\beta(\lambda, 0)\right| dt\, d\lambda \\
&= \int_{\mathbf{A}} \underbrace{\int_{\mathbb{R}^m} \left[F(\lambda, t-l)\, e^{2\pi i \left\langle \beta(\lambda,t), \sum_{j=1}^{p} k_j Z_j \right\rangle}\right] \times \left(1_{\mathbf{A}\times[0,1)^m}(\lambda, t)\right) dt}_{= \left\langle \sigma_\lambda(\gamma) F(\lambda,\cdot), 1_{\mathbf{A}\times[0,1)^m}(\lambda,\cdot)\right\rangle_{L^2(\mathbb{R}^m)}} \left|\det J_\beta(\lambda, 0)\right| d\lambda \\
&= \int_{\mathbf{A}} \left\langle \sigma_\lambda(\gamma) F(\lambda, \cdot), 1_{\mathbf{A}\times[0,1)^m}(\lambda, \cdot)\right\rangle_{L^2(\mathbb{R}^m)} \left|\det J_\beta(\lambda, 0)\right| d\lambda \\
&= \left\langle L(\gamma) \mathcal{P}^{-1}\left((F(\lambda, \cdot) \otimes \mathbf{u}(\lambda))_{\lambda\in\mathbf{A}}\right), \mathcal{P}^{-1}\left(\left(1_{\mathbf{A}\times[0,1)^m}(\lambda, \cdot) \otimes \mathbf{u}(\lambda)\right)_{\lambda\in\mathbf{A}}\right)\right\rangle_{L^2(N)}.
\end{aligned}$$

Applying Cauchy–Schwarz inequality

$$\begin{aligned}
|[J_{\mathbf{A}}F](\gamma)| &\leq \|F\| \cdot \left\|\mathcal{P}^{-1}\left(\left(1_{\mathbf{A}\times[0,1)^m}(\lambda, \cdot) \otimes \mathbf{u}(\lambda)\right)_{\lambda\in\mathbf{A}}\right)\right\| \\
&= \|F\| \times \int_{\mathbf{A}} \int_{[0,1)^m} \left|\det J_\beta(\lambda, 0)\right| dt\, d\lambda.
\end{aligned}$$

Recall that Σ is identified with a Zariski open subset of $\mathbb{R}^{n-2m}$.

Proposition 5. *Let $\mathbf{A}$ be a $d\mu$-measurable bounded subset of Σ. If $H \in L^2(\mathbf{A} \times \mathbb{R}^m, d\mu(\lambda)\, dt)$ is a smooth function of compact support, then $J_{\mathbf{A}}H \in l^2(\Gamma)$.*

Proof. First, observe that

$$[J_{\mathbf{A}}H](\exp(Z(k)) \exp(A(l))) = \int_{\mathbf{A}} \int_{[0,1)^m} H(\lambda, t-l)\; e^{2\pi i \langle \beta(\lambda,t), Z(k)\rangle} dt\; \left|\det J_{\beta(\lambda,0)}\right| d\lambda$$

and for fixed $Z(k)$, the sequence $([J_{\mathbf{A}}H](\exp Z(k) \exp A(l)))_{l\in\mathbb{Z}^m}$ has bounded support. Making the change of variable $s = t - l$, we obtain that

$$[J_{\mathbf{A}}H](\exp Z(k) \exp A(l)) = \int_{\mathbf{A}} \int_{[0,1)^m - l} H(\lambda, s)\; e^{2\pi i \langle \beta(\lambda,s+l), Z(k)\rangle} ds\; \left|\det J_{\beta(\lambda,0)}\right| d\lambda.$$

Next, since $\beta(\lambda, s+l) = \exp A(s) \exp A(l) \cdot \lambda$ it follows that

$$\begin{aligned}
[J_{\mathbf{A}}H](\exp Z(k) \exp A(l)) &= \int_{\mathbf{A}\times([0,1)^m - l)} H(\lambda, s)\; e^{2\pi i \left\langle \beta(\lambda,s+l), Z(k)\right\rangle} \left|\det J_{\beta(\lambda,0)}\right|\; d(\lambda, s) \\
&= \int_{\mathbf{A}\times([0,1)^m - l)} H(\lambda, s)\; e^{2\pi i \left\langle \exp(A(s)) \exp(A(l))\cdot\lambda, Z(k)\right\rangle} \left|\det J_{\beta(\lambda,0)}\right|\; d(\lambda, s)
\end{aligned}$$

$$= \int_{\mathbf{A}\times([0,1)^m - l)} H(\lambda, s)\; e^{2\pi i \left\langle \exp(A(s))\cdot\lambda, e^{-adA(l)} Z(k)\right\rangle} \left|\det J_{\beta(\lambda,0)}\right| \, d(\lambda, s)$$

$$= \int_{\mathbf{A}\times([0,1)^m - l)} H(\lambda, s)\; e^{2\pi i \left\langle \beta(\lambda,s), e^{-adA(l)} Z(k)\right\rangle} \left|\det J_{\beta(\lambda,0)}\right| \, d(\lambda, s).$$

Next, the change of variable $r = \beta(\lambda, s)$ yields

$$[J_{\mathbf{A}} H](\exp(Z(k))\exp(A(l))) = \int_{\beta(\mathbf{A}\times([0,1)^m - l))} \left(H \circ \beta^{-1}\right)(r)\; e^{2\pi i \left\langle r, e^{-adA(l)} Z(k)\right\rangle} \, dr$$

$$= \int_{\beta(\mathbf{A}\times([0,1)^m - l)) \,\cap\, \text{support}\left(H\circ\beta^{-1}\right)} \left(H \circ \beta^{-1}\right)(r)\; e^{2\pi i \left\langle r, e^{-adA(l)} Z(k)\right\rangle} \, dr$$

where dr is the Lebesgue measure on $\mathbb{R}^p$. Next for each fixed $l \in \mathbb{Z}^m$, we write

$$\mathbf{E} = \beta\left(\mathbf{A}\times\left([0,1)^m - l\right)\right) \cap \left(\text{support } \left(H \circ \beta^{-1}\right)\right)$$

as a finite disjoint union of subsets of $\mathbb{R}^p$; each contained in a fundamental domain of $\mathbb{Z}^p$ as follows:

$$\mathbf{E} = \dot{\bigcup_{j \in J(H,l)}} \left(K_{\mathbf{A},l} + j\right) \quad \text{where } J(H,l) \text{ is a finite subset of } \mathbb{Z}^p.$$

In the disjoint union given above, each $K_{\mathbf{A},l}$ is a measurable subset of $\mathbb{R}^p$ contained in some fundamental domain of $\mathbb{Z}^p$. Letting $1_{K_{\mathbf{A},l}+j}$ be the indicator function of the set $K_{\mathbf{A},l} + j$, we obtain

$$[J_{\mathbf{A}} H](\exp(Z(k))\exp(A(l))) = \sum_{j \in J(H,l)} \int_{K_{\mathbf{A},l}+j} \left(H \circ \beta^{-1}\right)(r)\; e^{2\pi i \left\langle r, e^{-adA(l)} Z(k)\right\rangle} \, dr.$$

Since β^{-1} is a diffeomorphism and since H is smooth and has compact support then $H \circ \beta^{-1}$ is smooth and has compact support as well. Put $\Gamma_{\mathbb{Z}^p} = \sum_{k=1}^{p} \mathbb{Z} Z_k \simeq \mathbb{Z}^p$. Since $\det\left[e^{-adA(l)}\middle|\,\mathfrak{p}\right] = 1$ and the entries of the matrix $\left[e^{-adA(l)}\middle|\,\mathfrak{p}\right]$ are all integers, it is clear that $e^{-adA(l)}\left(\Gamma_{\mathbb{Z}^p}\right) = \Gamma_{\mathbb{Z}^p}$. Let $\Phi \subset \mathbb{R}^p$ be a fundamental domain of $\mathbb{Z}^p$. We define the Fourier transform on $L^2(\Phi)$ as follows:

$$\left(\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}} \mathbf{F}\right)(Z(k)) = \int_{\Phi} \mathbf{F}(r)\; e^{2\pi i \langle r, Z(k)\rangle} dr \qquad \text{where } \mathbf{F} \in L^2(\Phi).$$

For a fixed $l \in \mathbb{Z}^m$, we have

$$[J_{\mathbf{A}} H](\exp Z(k) \exp A(l)) = \sum_{j \in J(H,l)} \left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\left(H \circ \beta^{-1}\right) \times 1_{K_{\mathbf{A},l}+j}\right)\right]\left(e^{-adA(l)} Z(k)\right).$$

In order to avoid cluster of notation, we set $e^{-adA(l)}Z(k) = a_l(k)$ and $\gamma_{k,l} = \exp(Z(k))\exp(A(l))$. Then

$$\|[J_{\mathbf{A}}H]\|^2_{l^2(\Gamma)} = \sum_{k\in\mathbb{Z}^p}\sum_{l\in\mathbb{Z}^m}|J_{\mathbf{A}}H(\gamma_{k,l})|^2$$

$$= \sum_{k\in\mathbb{Z}^p}\sum_{l\in F}\left|\sum_{j\in J(H,l)}\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\left(H\circ\beta^{-1}\right)\times 1_{K_{\mathbf{A},l}+j}\right)\right](a_l(k))\right|^2$$

where F is a finite subset of $\mathbb{Z}^m$ (because H has compact support) and

$$\|[J_{\mathbf{A}}H]\|^2_{l^2(\Gamma)} = \sum_{l\in F}\sum_{k\in\mathbb{Z}^p}\left|\sum_{j\in J(H,l)}\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\left(H\circ\beta^{-1}\right)\times 1_{K_{\mathbf{A},l}+j}\right)\right](a_l(k))\right|^2.$$

Letting $\Theta_{H,l,j} = \left(H\circ\beta^{-1}\right)\times 1_{K_{\mathbf{A},l}+j} \in L^2(K_{\mathbf{A},l}+j)$ and applying the Triangle Inequality, since $J(H,l)$ is a finite set, it follows that

$$\|[J_{\mathbf{A}}H]\|^2_{l^2(\Gamma)} \leq \sum_{l\in F}\sum_{k\in\mathbb{Z}^p}\left(\sum_{j\in J(H,l)}\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\Theta_{H,l,j}\right](a_l(k))\right|\right)^2.$$

and

$$\begin{aligned}
&\|[J_{\mathbf{A}}H]\|^2_{l^2(\Gamma)}\\
&\leq \sum_{l\in F}\sum_{k\in\mathbb{Z}^p}\sum_{j\in J(H,l)}\sum_{j'\in J(H,l)}\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\Theta_{H,l,j}\right](a_l(k))\right|\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\Theta_{H,l,j'}\right](a_l(k))\right|\\
&= \sum_{l\in F}\sum_{j\in J(H,l)}\sum_{j'\in J(H,l)}\left(\sum_{k\in\mathbb{Z}^p}\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\Theta_{H,l,j}\right)\right](a_l(k))\right|\times\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\Theta_{H,l,j'}\right)\right](a_l(k))\right|\right)\\
&= \sum_{l\in F}\sum_{j\in J(H,l)}\sum_{j'\in J(H,l)}\left\langle\left(\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\Theta_{H,l,j}\right)\right](a_l(k))\right|\right)_{k\in\mathbb{Z}^p},\left(\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\Theta_{H,l,j'}\right)\right](a_l(k))\right|\right)_{k\in\mathbb{Z}^p}\right\rangle_{l^2(\mathbb{Z}^p)}.
\end{aligned}$$

By Cauchy–Schwarz inequality on $l^2(\mathbb{Z}^p)$,

$$\left\langle\left(\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\Theta_{H,l,j}\right)\right](a_l(k))\right|\right)_{k\in\mathbb{Z}^p},\left(\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\Theta_{H,l,j'}\right)\right](a_l(k))\right|\right)_{k\in\mathbb{Z}^p}\right\rangle_{l^2(\mathbb{Z}^p)}$$

is less than or equal to

$$\left\|\left(\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\Theta_{H,l,j}\right)\right](a_l(k))\right|\right)_{k\in\mathbb{Z}^p}\right\|_{l^2(\mathbb{Z}^p)}\times\left\|\left(\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\Theta_{H,l,j'}\right)\right](a_l(k))\right|\right)_{k\in\mathbb{Z}^p}\right\|_{l^2(\mathbb{Z}^p)}$$

and it follows that

$$\begin{aligned}
&\|[J_{\mathbf{A}}H]\|^2_{l^2(\Gamma)} \\
&\leq \sum_{l\in F}\sum_{j\in J(H,l)}\sum_{j'\in J(H,l)} \left\|\left(\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\Theta_{H,l,j}\right)\right](a_l(k))\right|\right)_{k\in\mathbb{Z}^p}\right\|_{l^2(\mathbb{Z}^p)} \times \left\|\left(\left|\left[\mathfrak{F}_{\Gamma_{\mathbb{Z}^p}}\left(\Theta_{H,l,j'}\right)\right](a_l(k))\right|\right)_{k\in\mathbb{Z}^p}\right\|_{l^2(\mathbb{Z}^p)} \\
&= \sum_{l\in F}\sum_{j\in J(H,l)}\sum_{j'\in J(H,l)} \left\|\Theta_{H,l,j}\right\|_{L^2(K_{\mathbf{A},l}+j)} \times \left\|\Theta_{H,l,j'}\right\|_{L^2(K_{\mathbf{A},l}+j')}.
\end{aligned}$$

The last equality above is due to Parseval equality. Next, appealing to the fact that each $\Theta_{H,l,j} = \left(H\circ\beta^{-1}\right)\times 1_{K_{\mathbf{A},l}+j}$ is square-integrable over $K_{\mathbf{A},l}+j$, we obtain

$$\|[J_{\mathbf{A}}H]\|^2_{l^2(\Gamma)} \leq \sum_{l\in F}\sum_{j\in J(H,l)}\sum_{j'\in J(H,l)} \left\|\left(H\circ\beta^{-1}\right)\times 1_{K_{\mathbf{A},l}+j}\right\|_{L^2(K_{\mathbf{A},l}+j)} \times \left\|\left(H\circ\beta^{-1}\right)\times 1_{K_{\mathbf{A},l}+j}\right\|_{L^2(K_{\mathbf{A},l}+j')}$$

Thus, $\|[J_{\mathbf{A}}H]\|^2_{l^2(\Gamma)}$ is a finite sum of finite quantities. Therefore, $\|[J_{\mathbf{A}}H]\|_{l^2(\Gamma)}$ is finite and this completes the proof.

Let τ be the unitary representation of Γ which acts on the Hilbert space $\mathcal{S}_{\mathbf{A}} = L^2(\mathbf{A}\times\mathbb{R}^m, d\mu(\lambda)\,dt)$ as follows:

$$[\tau(\gamma)F](\lambda,t) = \sigma_\lambda(\gamma)F(\lambda,t) = e^{2\pi i\left\langle\beta(\lambda,t),\sum_{j=1}^p k_jZ_j\right\rangle}F(\lambda,t-l)$$

where $\gamma = \exp Z(k)\exp A(l) \in \Gamma$. We shall prove the following three important facts.

- $J_{\mathbf{A}}$ intertwines τ with the right regular representation of the discrete uniform group Γ (Lemma 10)
- If $|\beta(\mathbf{A}\times[0,1)^m)| > 0$ and if $\beta(\mathbf{A}\times[0,1)^m)$ is contained in a fundamental domain of $\mathbb{Z}^p$ then $J_{\mathbf{A}}$ defines an isometry on a dense subset of $\mathcal{S}_{\mathbf{A}}$ into $l^2(\Gamma)$ which extends uniquely to an isometry of $\mathcal{S}_{\mathbf{A}}$ into $l^2(\Gamma)$ (Lemma 11)
- If $|\beta(\mathbf{A}\times[0,1)^m)| > 0$ and if $\beta(\mathbf{A}\times[0,1)^m)$ is up to a null set equal to a fundamental domain of $\mathbb{Z}^p$ then $J_{\mathbf{A}}$ defines a unitary map (Lemma 12)

Lemma 10. *The map $J_{\mathbf{A}}$ intertwines τ with R (the right regular representation of Γ).*

Proof. Let $F\in\mathcal{S}_{\mathbf{A}}$ and $\gamma\in\Gamma$. Then clearly, $[J_{\mathbf{A}}F](\gamma) = \left\langle\tau(\gamma)F, 1_{\mathbf{A}\times[0,1)^m}\right\rangle$. Next, given $\alpha\in\Gamma$ and letting R denotes the right regular representation of Γ, we obtain

$$[J_{\mathbf{A}}\tau(\alpha)F](\gamma) = \left\langle\tau(\gamma\alpha)F, 1_{\mathbf{A}\times[0,1)^m}\right\rangle = [J_{\mathbf{A}}F](\gamma\alpha) = R(\alpha)[J_{\mathbf{A}}F](\gamma).$$

Lemma 11. *If $\beta(\mathbf{A}\times[0,1)^m)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is contained in a fundamental domain of $\mathbb{Z}^p$ then $J_{\mathbf{A}}$ defines an isometry on a dense subset of $\mathcal{S}_{\mathbf{A}}$ into $l^2(\Gamma)$ which extends uniquely to an isometry of $\mathcal{S}_{\mathbf{A}}$ into $l^2(\Gamma)$.*

Proof. Let $F \in \mathcal{S}_{\mathbf{A}}$. Furthermore, let us assume that F is smooth with compact support in $\mathbf{A} \times \mathbb{R}^m$. Computing the norm of F, we obtain

$$\begin{aligned} \|F\|^2_{L^2(\mathbf{A}\times\mathbb{R}^m, d\mu(\lambda))} &= \int_{\mathbf{A}} \int_{\mathbb{R}^m} |F(\lambda, t)|^2 \, dt \, d\mu(\lambda) \\ &= \int_{\mathbf{A}} \sum_{l\in\mathbb{Z}^m} \int_{[0,1)^m} |F(\lambda, t-l)|^2 \, dt \, d\mu(\lambda). \end{aligned} \tag{33}$$

Letting $G_l(\lambda, t) = F(\lambda, t-l)$, making the change of variable $s = \beta(\lambda, t)$ and using the fact that β is a diffeomorphism (see Lemma 6) we obtain

$$\begin{aligned} \int_{\mathbf{A}} \sum_{l\in\mathbb{Z}^m} \int_{[0,1)^m} |F(\lambda, t-l)|^2 \, dt \, d\mu(\lambda) &= \sum_{l\in\mathbb{Z}^m} \int_{[0,1)^m} \int_{\mathbf{A}} |G_l(\lambda, t)|^2 \, d\mu(\lambda) \, dt \\ &= \sum_{l\in\mathbb{Z}^m} \int_{\mathbf{A}\times[0,1)^m} |G_l(\lambda, t)|^2 \left|\det J_\beta(\lambda, 0)\right| d(\lambda, t) \\ &= \sum_{l\in\mathbb{Z}^m} \int_{\beta(\mathbf{A}\times[0,1)^m)} \left|G_l\beta^{-1}(s)\right|^2 \left|\det J_\beta(\lambda, 0)\right| d\left(\beta^{-1}(s)\right) \\ &= \sum_{l\in\mathbb{Z}^m} \int_{\beta(\mathbf{A}\times[0,1)^m)} \left|G_l\beta^{-1}(s)\right|^2 ds. \end{aligned}$$

Since $\beta(\mathbf{A} \times [0,1)^m)$ is contained in a fundamental domain of $\mathbb{Z}^p$ then

$$\begin{aligned} \int_{\mathbf{A}} \sum_{l\in\mathbb{Z}^m} \int_{[0,1)^m} |F(\lambda, t-l)|^2 \, dt \, d\mu(\lambda) &= \sum_{l\in\mathbb{Z}^m} \left\| s \mapsto G_l\beta^{-1}(s) \right\|^2 \\ &= \sum_{l\in\mathbb{Z}^m} \left\| \mathfrak{F}_{\Gamma_{\mathbb{Z}^p}} \left(G_l\beta^{-1}\right) \right\|^2 \\ &= \sum_{l\in\mathbb{Z}^m} \sum_{k\in\mathbb{Z}^p} \left| \mathfrak{F}_{\Gamma_{\mathbb{Z}^p}} \left(G_l\beta^{-1}\right)(k) \right|^2 \\ &= \sum_{l\in\mathbb{Z}^m} \sum_{k\in\mathbb{Z}^p} \left| \int_{\beta(\mathbf{A}\times[0,1)^m)} \left(G_l\beta^{-1}\right)(s) \exp(2\pi i \langle s, k\rangle) \, ds \right|^2. \end{aligned}$$

Thus

$$\|F\|^2_{L^2(\mathbf{A}\times\mathbb{R}^m, d\mu(\lambda))} = \sum_{l\in\mathbb{Z}^m} \sum_{k\in\mathbb{Z}^p} \left| \int_{\beta(\mathbf{A}\times[0,1)^m)} G_l\beta^{-1}(s) \exp(2\pi i \langle s, k\rangle) \, ds \right|^2. \tag{34}$$

Next, since $\beta(\mathbf{A}\times [0,1)^m)$ is contained in a fundamental domain of $\mathbb{Z}^p$, the trigonometric system

$$\left\{ \chi_{\beta(\mathbf{A}\times[0,1)^m)}(s) \times \exp(2\pi i \langle s, k\rangle) : k \in \mathbb{Z}^p \right\}$$

forms a Parseval frame in $L^2(\beta(\mathbf{A}\times[0,1)^m), ds)$. Clearly this is true because the orthogonal projection of an orthonormal basis is always a Parseval frame. Letting

$$\widehat{G_l\beta^{-1}} = \mathfrak{F}_{L^2(\beta(\mathbf{A}\times[0,1)^m))}\left(s \mapsto G_l\left(\beta^{-1}(s)\right)\right)$$

be the Fourier transform of the function $s \mapsto G_l\left(\beta^{-1}(s)\right) \in L^2(\beta(\mathbf{A}\times[0,1)^m), ds)$ it follows that

$$\sum_{l\in\mathbb{Z}^m}\sum_{k\in\mathbb{Z}^p}\left|\int_{\beta(\mathbf{A}\times[0,1)^m)} G_l\left(\beta^{-1}(s)\right)\exp\left(2\pi i\langle s,k\rangle\right) ds\right|^2 = \sum_{l\in\mathbb{Z}^m}\sum_{k\in\mathbb{Z}^p}\left|\widehat{G_l\beta^{-1}}(k)\right|^2.$$

Next,

$$\sum_{l\in\mathbb{Z}^m}\sum_{k\in\mathbb{Z}^p}\left|\widehat{G_l\beta^{-1}}(k)\right|^2 = \sum_{l\in\mathbb{Z}^m}\left\|\widehat{G_l\beta^{-1}}\right\|^2_{l^2(\mathbb{Z}^p)} = \sum_{l\in\mathbb{Z}^m}\int_{\beta(\mathbf{A}\times[0,1)^m)}\left|G_l\left(\beta^{-1}(s)\right)\right|^2 ds.$$

Now substituting (λ, t) for $\beta^{-1}(s)$,

$$\begin{aligned}&\sum_{l\in\mathbb{Z}^m}\sum_{k\in\mathbb{Z}^p}\left|\int_{\beta(\mathbf{A}\times[0,1)^m)} G_l\left(\beta^{-1}(s)\right)\exp\left(2\pi i\langle s,k\rangle\right) ds\right|^2\\ &= \sum_{l\in\mathbb{Z}^m}\int_{\mathbf{A}}\int_{[0,1)^m}|G_l(\lambda,t)|^2\left|\det J_\beta(\lambda,0)\right| dt\, d\lambda \qquad (35)\\ &= \sum_{l\in\mathbb{Z}^m}\int_{\mathbf{A}}\int_{[0,1)^m}|F(\lambda,t-l)|^2\, dt\, d\mu(\lambda). \qquad (36)\end{aligned}$$

Equation (33) together with (36) gives

$$\sum_{l\in\mathbb{Z}^m}\sum_{k\in\mathbb{Z}^p}\left|\int_{\beta(\mathbf{A}\times[0,1)^m)} G_l\left(\beta^{-1}(s)\right)\exp\left(2\pi i\langle s,k\rangle\right)\ ds\right|^2 = \|F\|^2_{L^2(\mathbf{A}\times\mathbb{R}^m, d\mu(\lambda))}.$$

Finally, we obtain $\|F\|_{L^2(\mathbf{A}\times\mathbb{R}^m, d\mu(\lambda)dt)} = \|J_{\mathbf{A}}F\|_{l^2(\Gamma)}$. Now, since the set of smooth functions of compact support is dense in $\mathcal{S}_{\mathbf{A}}$ and since $J_{\mathbf{A}}$ defines an isometry on a dense set, then J extends uniquely to an isometry on $\mathcal{S}_{\mathbf{A}}$.

The proof given for Lemma 11 can be easily modified to establish the following result

Lemma 12. *If $\beta(\mathbf{A}\times[0,1)^m)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is equal to a fundamental domain of $\mathbb{Z}^p$ then $J_{\mathbf{A}}$ defines an isometry on a dense subset of $\mathcal{S}_{\mathbf{A}}$ into $l^2(\Gamma)$ which extends uniquely to a unitary map of $\mathcal{S}_{\mathbf{A}}$ into $l^2(\Gamma)$.*

Remark 6. Suppose that $\beta(\mathbf{A}\times[0,1)^m)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is contained in a fundamental domain of $\mathbb{Z}^p$. We have shown that $J_{\mathbf{A}}$ is an isometry. Moreover, the image of $\mathcal{S}_{\mathbf{A}}$ under the map $J_{\mathbf{A}}$ is stable under the action of the right regular representation of Γ. Now, let Φ be the orthogonal projection of $l^2(\Gamma)$ onto the Hilbert space $J_{\mathbf{A}}(\mathcal{S}_{\mathbf{A}})$ and let κ be the indicator sequence of the singleton containing the identity element in Γ. Identifying $\mathcal{S}_{\mathbf{A}}$ with $\mathcal{P}(\mathbf{H}_{\mathbf{A}})$, it is clear that $\mathcal{P}^{-1}\left(J_{\mathbf{A}}^*(\Phi\kappa)\right) \in \mathbf{H}_{\mathbf{A}} \subset L^2(N)$ and $L(\Gamma)\left(\mathcal{P}^{-1}\left(J_{\mathbf{A}}^*(\Phi\kappa)\right)\right)$ is a Parseval frame for the band-limited Hilbert space $\mathbf{H}_{\mathbf{A}}$. We remark that the vector κ could be replaced by any other vector which generates an orthonormal basis or a Parseval frame under the action of the right regular representation of Γ. Additionally, we observe that

$$[J_{\mathbf{A}}F](\gamma) = \left\langle L(\gamma)\mathcal{P}^{-1}\left((F(\lambda,\cdot)\otimes \mathbf{u}(\lambda))_{\lambda\in\mathbf{A}}\right), \mathcal{P}^{-1}\left(\left(1_{\mathbf{A}\times[0,1)^m}(\lambda,\cdot)\otimes \mathbf{u}(\lambda)\right)_{\lambda\in\mathbf{A}}\right)\right\rangle_{L^2(N)}.$$

Letting

$$\mathbf{F} = \mathcal{P}^{-1}\left((F(\lambda,\cdot)\otimes \mathbf{u}(\lambda))_{\lambda\in\mathbf{A}}\right) \text{ and } \mathbf{S}_{\mathbf{A}} = \mathcal{P}^{-1}\left(\left(1_{\mathbf{A}\times[0,1)^m}(\lambda,\cdot)\otimes \mathbf{u}(\lambda)\right)_{\lambda\in\mathbf{A}}\right)$$

then $[J_{\mathbf{A}}F](\gamma) = \left\langle \mathbf{F}, L\left(\gamma^{-1}\right)\mathbf{S}_{\mathbf{A}}\right\rangle = \mathbf{F} * \mathbf{S}_{\mathbf{A}}^*\left(\gamma^{-1}\right)$. Thus, if $\beta\left(\mathbf{A}\times[0,1)^m\right)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is contained in a fundamental domain of $\mathbb{Z}^p$ then $L(\Gamma)\mathbf{S}_{\mathbf{A}}$ is a Parseval frame for $\mathbf{H}_{\mathbf{A}} \subset L^2(N)$. Next since $\|\mathbf{S}_{\mathbf{A}}\| \leq 1$ it follows that

$$\|\mathbf{S}_{\mathbf{A}}\|^2 = \int_{\mathbf{A}}\int_{[0,1)^m}\left|\det J_\beta(\lambda,0)\right| dt\, d\lambda = \int_{\mathbf{A}}\left|\det J_\beta(\lambda,0)\right|\, d\lambda \leq 1$$

and $\int_{\mathbf{A}}\left|\det J_\beta(\lambda,0)\right|\, d\lambda \leq 1$. Thus if $\int_{\mathbf{A}}\left|\det J_\beta(\lambda,0)\right|\, d\lambda > 1$ then $J_{\mathbf{A}}$ cannot be an isometry.

4 Proof of Main Results

4.1 Proof of Theorem 2

First, we observe that the right regular and left regular representations of Γ are unitarily equivalent ([5], Page 69). To prove Part 1, we appeal to Lemma 11, and Lemma 10. Assuming that $\beta\left(\mathbf{A}\times[0,1)^m\right)$ has positive Lebesgue measure in $\mathbb{R}^p$ and is contained in a fundamental domain of $\mathbb{Z}^p$, the restriction of the representation $(L, \mathbf{H}_{\mathbf{A}})$ to the discrete group Γ is equivalent to a subrepresentation of the left regular representation of Γ. Appealing to Lemma 2, there exists a vector η such that $V_\eta^L(\mathbf{H}_{\mathbf{A}})$ is a sampling space with respect to Γ. For Part 2, Lemma 11, Lemma 10 together with the assumption that $\beta\left(\mathbf{A}\times[0,1)^m\right)$ is equal to a fundamental domain of $\mathbb{Z}^p$ imply that the restriction of the representation $(L, \mathbf{H}_{\mathbf{A}})$ to the discrete group Γ is equivalent to the left regular representation of Γ. Finally, there exists a vector $\eta \in \mathbf{H}_{\mathbf{A}}$ such that $V_\eta^L(\mathbf{H}_{\mathbf{A}})$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with the interpolation property with respect to Γ.

4.2 *Proof of Corollary 1*

For $s = (s_1, \cdots, s_m) \in \mathbb{R}^m$, let $A(s) = s_1 A_1 + \cdots + s_m A_m \in \mathfrak{m}$. Since the linear operators $adA_1, \cdots, adA_m$ are pairwise commutative and nilpotent, and $e^{-adA(s)|\mathfrak{p}}$ is unipotent, there is a unit vector which is an eigenvector for $e^{-adA(s)|\mathfrak{p}}$ with corresponding eigenvalue 1. So, it is clear that $\left\| \left[e^{-adA(s)|\mathfrak{p}} \right]^T \right\|_\infty \geq 1$ and $\sup \left\{ \left\| \left[e^{-adA(s)|\mathfrak{p}} \right]^T \right\|_\infty : s \in \mathbf{E} \right\} \geq 1$ for any nonempty $\mathbf{E} \subseteq \mathbb{R}^m$. We recall again that

$$\mathfrak{P}(A(s)) = \left[e^{ad\left(-\sum_{j=1}^m s_j A_j\right)|\mathfrak{p}} \right]^T. \tag{37}$$

Lemma 13. *Let* $\mathbf{E}$ *be an open bounded subset of* $\mathbb{R}^m$*. If* ε *is a positive number satisfying* $\varepsilon \leq \delta = (2 \sup \{ \|\mathfrak{P}(A(s))\|_\infty : s \in \mathbf{E} \})^{-1}$ *then* $\beta\left(\left((-\varepsilon, \varepsilon)^{\dim \Sigma} \cap \Sigma\right) \times \mathbf{E}\right)$ *is open in* $\mathbb{R}^p$ *and is contained in a fundamental domain of* $\mathbb{Z}^p$*.*

Proof. Since the map β is a diffeomorphism (see Lemma 6) and since the set $\left((-\varepsilon, \varepsilon)^{\dim \Sigma} \cap \Sigma\right) \times \mathbf{E}$ is an open set in $\Sigma \times \mathbb{R}^m$, it is clear that its image under the map β is also open in $\mathbb{R}^p$. Next, it remains to show that it is possible to find a positive real number δ such that if $0 < \varepsilon \leq \delta$ then $\beta\left(\left((-\varepsilon, \varepsilon)^{\dim \Sigma} \cap \Sigma\right) \times \mathbf{E}\right)$ is an open set contained in a fundamental domain of $\mathbb{Z}^p$. Let $\lambda \in \Sigma$. Then there exists a linear functional f in the dual of the ideal $\mathfrak{p}$ such that $\iota(\lambda) = \begin{bmatrix} f \\ 0 \end{bmatrix}$ and

$$\iota\left(\exp\left(\sum_{j=1}^m s_j A_j\right) \cdot \lambda\right) = \begin{bmatrix} \mathfrak{P}(A(s)) f \\ 0 \end{bmatrix}. \tag{38}$$

Moreover, it is worth noting that $\left\| \exp\left(\sum_{j=1}^m s_j A_j\right) \cdot \lambda \right\|_{\max} = \|\mathfrak{P}(A(s)) f\|_{\max}$. Let δ be a positive real number defined as follows:

$$\delta = (2 \sup \{ \|\mathfrak{P}(A(s))\|_\infty : s \in \mathbf{E} \})^{-1}. \tag{39}$$

If $f \in (-\varepsilon, \varepsilon)^{\dim \Sigma} \subseteq (-\delta, \delta)^{\dim \Sigma}$ and if $s \in \mathbf{E}$ then

$$\|\mathfrak{B}(A(s)) f\|_\infty \leq \|f\|_{\max} \times \sup \{ \|\mathfrak{P}(A(s))\|_\infty : s \in \mathbf{E} \} = \frac{1}{2} \times \frac{\|f\|_{\max}}{\delta}.$$

Now since $\|f\|_{\max} < \delta$, it follows that $\|\mathfrak{P}(A(s)) f\|_{\max} < \frac{1}{2}$. As a result, $\beta\left(\left((-\varepsilon, \varepsilon)^{\dim \Sigma} \cap \Sigma\right) \times \mathbf{E}\right) \subseteq \left(-\frac{1}{2}, \frac{1}{2}\right)^p$ and clearly $\left(-\frac{1}{2}, \frac{1}{2}\right)^p$ is contained in a fundamental domain of $\mathbb{Z}^p$.

Appealing to Lemma 11, and Lemma 13 the following is immediate

Proposition 6. *If* $0 < \varepsilon \leq \delta = \left(2\sup\left\{\|\mathfrak{P}(A(s))\|_{\infty} : s \in [0,1)^m\right\}\right)^{-1}$ *then* $J_{(-\varepsilon,\varepsilon)^{n-2m}\cap\Sigma}$ *defines an isometry between* $L^2\left(\left((-\varepsilon,\varepsilon)^{n-2m}\cap\Sigma\right)\times\mathbb{R}^m, d\mu(\lambda)\right)$ *and* $l^2(\Gamma)$.

4.2.1 Proof of Corollary 1

Let δ be a positive number defined by $\delta = \left(2\sup\left\{\|\mathfrak{P}(A(s))\|_{\infty} : s \in [0,1)^m\right\}\right)^{-1}$. We want to show that for $\varepsilon \in (0,\delta]$ there exists a band-limited vector $\eta = \eta^{\varepsilon} \in \mathbf{H}_{(-\varepsilon,\varepsilon)^{n-2m}}$ such that the Hilbert space $V_{\eta}^{L}\left(\mathbf{H}_{(-\varepsilon,\varepsilon)^{n-2m}}\right)$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to Γ. According to Lemma 13 the set $\beta\left(\left((-\varepsilon,\varepsilon)^{\dim\Sigma}\cap\Sigma\right)\times[0,1)^m\right)$ has positive Lebesgue measure and is contained in a fundamental domain of $\mathbb{Z}^p$. The desired result follows immediately from Theorem 2.

4.3 Proof of Example 1 Part 1

The case of a commutative simply connected, and connected nilpotent Lie group is already known to be true. Thus, to prove this result, it remains to focus on the non commutative algebras. According to the classification of four-dimensional nilpotent Lie algebras [8] there are three distinct cases to consider. Indeed if $\mathfrak{n}$ is a non-commutative nilpotent Lie algebra of dimension three, then $\mathfrak{n}$ must be isomorphic with the three-dimensional Heisenberg Lie algebra. If $\mathfrak{n}$ is four-dimensional then up to isomorphism either $\mathfrak{n}$ is the direct sum of the Heisenberg Lie algebra with a one-dimensional algebra, or there is a basis Z_1, Z_2, Z_3, A_1 for $\mathfrak{n}$ with the following non-trivial Lie brackets $[A_1, Z_2] = 2Z_1, [A_1, Z_3] = 2Z_2$.

Case 1 (The Heisenberg Lie algebra) Let N be the simply connected, connected Heisenberg Lie group with Lie algebra $\mathfrak{n}$ which is spanned by Z_1, Z_2, A_1 with non-trivial Lie brackets $[A_1, Z_2] = Z_1$.We check that $N = PM$ where $P = \exp(\mathbb{R}Z_1 + \mathbb{R}Z_2)$ and $M = \exp(\mathbb{R}A_1)$. Put $\Gamma = \exp(\mathbb{Z}Z_1)\exp(\mathbb{Z}Z_2)\exp(\mathbb{Z}A_1)$. It is easily checked that

$$\mathbf{M}(\lambda) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\lambda(Z_1) \\ 0 & \lambda(Z_1) & 0 \end{bmatrix}.$$

Next, since $\mathbf{e}(\lambda) = \begin{cases} \emptyset \text{ if } \lambda(Z_1) = 0 \\ \{2,3\} \text{ if } \lambda(Z_1) \neq 0 \end{cases}$ we obtain that $\mathbf{e} = \{2,3\}$. It follows that $\Omega_{\mathbf{e}} = \{\lambda \in \mathfrak{n}^* : \lambda(Z_1) \neq 0\}$. Next, the unitary dual of N is parametrized by $\Sigma = \{\lambda \in \Omega_{\mathbf{e}} : \lambda(Z_2) = \lambda(A_1) = 0\}$ which we identify with the punctured line: $\mathbb{R}^*$. It is not hard to check that

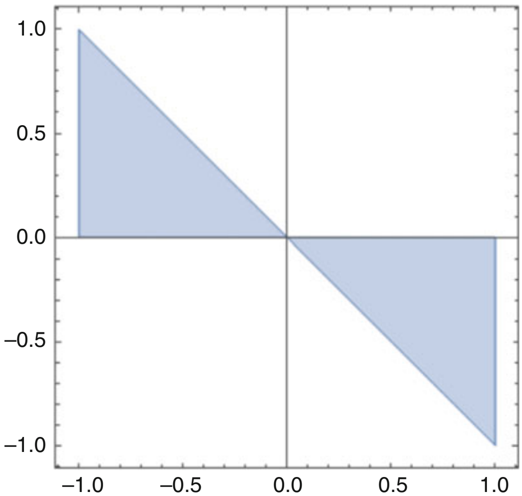
$$\delta^{-1} = 2 \sup \left\{ \left\| \begin{bmatrix} 1 & 0 \\ -s & 1 \end{bmatrix} \right\|_\infty : s \in [0,1) \right\} = 4.$$

So, there exists a band-limited vector $\eta \in \mathbf{H}_{(-\frac{1}{4},\frac{1}{4})}$ such that $V_\eta^L \left(\mathbf{H}_{(-\frac{1}{4},\frac{1}{4})}\right)$ is a sampling space with respect to Γ.

To prove that the Heisenberg group admits sampling spaces with the interpolation property with respect to Γ, we claim that the set

$$B(1) = \beta\left((-1,1) \times [0,1)\right) = \left\{ \begin{bmatrix} f \\ -sf \end{bmatrix} : f \in (-1,1)\,, s \in [0,1) \right\}$$

is up to a null set equal to a fundamental domain of $\mathbb{Z}^2$ (see illustration of $B(1)$ below)



To prove this we write $\beta((-1,1) \times [0,1)) = \beta((0,1) \times [0,1)) \cup \beta((-1,0) \times [0,1))$. Next, it is easy to check that $\left(\beta\left((0,1) \times [0,1)\right) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \cup \left(\beta\left((-1,0) \times [0,1)\right) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ is up to a null set equal to the unit square $[0,1)^2$. Thus the set $\beta\left((-1,1) \times [0,1)\right)$ is up to a null set equal to a fundamental domain of $\mathbb{Z}^2$. Appealing to Theorem 2, the following result confirms the work proved in [4, 13]. There exists a band-limited vector $\eta \in \mathbf{H}_{(-1,1)}$ such that $V_\eta^L \left(\mathbf{H}_{(-1,1)}\right)$ is a sampling space with respect to Γ which also enjoys the **interpolation property**.

Case 2 (Four-dimensional and step two) Assume that $\mathfrak{n}$ is the direct sum of the Heisenberg Lie algebra with $\mathbb{R}$. Let us suppose that the Lie algebra $\mathfrak{n}$ is spanned by

Z_1, Z_2, Z_3, A_1 with non-trivial Lie brackets $[A_1, Z_2] = Z_1$. We check that

$$\mathbf{M}(\lambda) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -(Z_1) \\ 0 & 0 & 0 & 0 \\ 0 & \lambda(Z_1) & 0 & 0 \end{bmatrix}$$

and $\mathbf{e}(\lambda) = \begin{cases} \emptyset \text{ if } \lambda(Z_1) = 0 \\ \{2,4\} \text{ if } \lambda(Z_1) \neq 0 \end{cases}$. Fix $\mathbf{e} = \{2,4\}$ such that $\Omega_{\mathbf{e}} = \{\lambda \in \mathfrak{n}^* : \lambda(Z_1)\}$ $\{\neq 0\}$. The unitary dual of N is parametrized by $\Sigma = \{\lambda \in \Omega_{\mathbf{e}} : \lambda(Z_2) = \lambda(A_1) = 0\}$. For any linear functional $\lambda \in \Sigma$, the ideal spanned by Z_1, Z_2, Z_3 is a polarization algebra subordinated to λ and

$$\delta = \left(2 \sup \left\{ \left\| \begin{bmatrix} 1 & 0 & 0 \\ -s & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\|_\infty : s \in [0,1) \right\} \right)^{-1} = \frac{1}{4}.$$

So, there exists a band-limited vector $\eta \in \mathbf{H}_{\left(-\frac{1}{4}, \frac{1}{4}\right)}$ such that $V_\eta^L \left(\mathbf{H}_{\left(-\frac{1}{4}, \frac{1}{4}\right)} \right)$ is a sampling space with respect to $\Gamma = \exp(\mathbb{Z}Z_1 + \mathbb{Z}Z_2 + \mathbb{Z}Z_3)\exp(\mathbb{Z}A_1)$.

Case 3 (Four-dimensional and three step) Assume that $\mathfrak{n}$ is a four-dimensional Z_1, Z_2, Z_3, A_1 such that $[A_1, Z_2] = 2Z_1, [A_1, Z_3] = 2Z_2$. With respect to the ordered basis Z_1, Z_2, Z_3, we have

$$[adA_1] | \mathfrak{p} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \exp [adA_1] | \mathfrak{p} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Next, we check that

$$\delta = \left(2 \sup \left\{ \left\| \begin{bmatrix} 1 & 0 & 0 \\ -2s & 1 & 0 \\ 2s^2 & -2s & 1 \end{bmatrix} \right\|_\infty : s \in [0,1) \right\} \right)^{-1}$$
$$= \frac{1}{2} \left(\max \left\{ 1, 1 + 2|s|, 1 + 2|s| + 2|s|^2 : s \in [0,1) \right\} \right)^{-1} = \frac{1}{10}.$$

Indeed, the set $\beta \left(\left(-\frac{1}{10}, \frac{1}{10}\right)^2 \times [0,1) \right) \subset \left(-\frac{1}{2}, \frac{1}{2}\right)^3$ is contained in a fundamental domain of $\mathbb{Z}^3$. Thus, there exists a band-limited vector $\eta \in \mathbf{H}_{\left(-\frac{1}{10}, \frac{1}{10}\right)}$ such that $V_\eta^L \left(\mathbf{H}_{\left(-\frac{1}{10}, \frac{1}{10}\right)} \right)$ is a sampling space with respect to $\Gamma = \exp(\mathbb{Z}Z_1 + \mathbb{Z}Z_2 + \mathbb{Z}Z_3)\exp(\mathbb{Z}A_1)$.

4.4 Proof of Example 1 Part 2

Let N be a simply connected, connected nilpotent Lie group with Lie algebra spanned by $Z_1, Z_2, \cdots, Z_p, A_1$ such that $[adA_1]|_{\mathfrak{p}} = A$ is a nonzero rational upper triangular nilpotent matrix of order p such that $e^A \mathbb{Z}^p \subseteq \mathbb{Z}^p$ and the algebra generated by $Z_1, Z_2, \cdots, Z_p$ is commutative. Then N is isomorphic to a semi-direct product group $\mathbb{R}^p \rtimes \mathbb{R}$ with multiplication law given by $(x,t)(x',t') = \left(x + e^{tA}x', t + t'\right)$. Clearly since A is not the zero matrix then $\max\{\text{rank}(\mathbf{M}(\lambda)) : \lambda \in \mathfrak{n}^*\} = 2$ and the unitary dual of N is parametrized by a Zariski open subset of $\mathbb{R}^{p-1}$. Finally, let

$$\delta = \left(2 \times \sup\left\{\left\|\sum_{k=0}^{m-1} \frac{\left(-sA^T\right)^k}{k!}\right\|_\infty : s \in [0,1)\right\}\right)^{-1} > 0.$$

For $\varepsilon \in (0, \delta]$ there exists a band-limited vector $\eta = \eta^\varepsilon \in \mathbf{H}_{(-\varepsilon,\varepsilon)^{p-1}}$ such that the Hilbert space $V_\eta^L\left(\mathbf{H}_{(-\varepsilon,\varepsilon)^{p-1}}\right)$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to Γ.

4.5 Proof of Example 1 Part 3

Let $A(t)$ be as given in (9). Then $\exp A(t) = \sum_{k=0}^{m+1} \frac{A(t)^k}{k!}$ and N is a nilpotent Lie group of step $p = m + 1$. Moreover, the unitary dual of N is parametrized by the manifold:

$$\Sigma = \{\lambda \in \mathfrak{n}^* : \lambda(Z_1) \neq 0 \\ \text{and } \lambda(Z_{k+1}) = \lambda(A_k) = 0 \text{ for } 1 \leq k \leq m\} \simeq \mathbb{R}^*$$

and the Plancherel measure is up to multiplication by a constant given by $|\lambda|^m \, d\lambda$. The positive number δ described in Corollary 1 is equal to

$$\delta = \left(\sup\left\{2\left\|\sum_{k=0}^{m+1} \frac{\left(-A(t)^T\right)^k}{k!}\right\|_\infty : t \in [0,1)^m\right\}\right)^{-1}.$$

Thus, for $\varepsilon \in (0, \delta]$ there exists a band-limited vector $\eta = \eta^\varepsilon \in \mathbf{H}_{(-\varepsilon,\varepsilon)}$ such that the Hilbert space $V_\eta^L\left(\mathbf{H}_{(-\varepsilon,\varepsilon)}\right)$ is a left-invariant subspace of $L^2(N)$ which is a sampling space with respect to Γ.

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Parseval Space-Frequency Localized Frames on Sub-Riemannian Compact Homogeneous Manifolds

Isaac Pesenson

1 Introduction

The objective of this chapter is to describe a construction of Parseval bandlimited and localized frames in L_2-spaces on a class of sub-Riemannian compact homogeneous manifolds.

The chapter begins with a brief review in section 2 of some results obtained in [4] where a construction of Parseval bandlimited and localized frames was performed in $L_2(\mathbf{M}), \mathbf{M}$ being a compact homogeneous manifold equipped with a natural Riemannian metric.

In section 3 we are using a sub-Riemannian structure on the two-dimensional standard unit sphere $\mathbf{S}^2$ to explain the main differences between Riemannian and sub-Riemannian settings. Each of these structures is associated with a distinguished second-order differential operator which arises from a metric. These operators are self-adjoint with respect to the usual normalized invariant (with respect to rotations) measure on $\mathbf{S}^2$. The major difference between these operators is that in the case of Riemannian metric the operator is elliptic (the Laplace-Beltrami operator $\mathbf{L}$) and in the sub-Riemannian case it is not (the sub-Laplacian $\mathscr{L}$). As a result, the corresponding Sobolev spaces which are introduced as domains of powers of these operators are quite different. In the elliptic case one obtains the regular Sobolev spaces and in sub-elliptic one obtains function spaces (sub-elliptic Sobolev spaces) in which functions have variable smoothness (compared to regular (elliptic) Sobolev smoothness).

In section 4 we describe a class of sub-Riemannian structures on compact homogeneous manifolds and consider a construction of Parseval bandlimited and

I. Pesenson (✉)
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA
e-mail: pesenson@temple.edu

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localized frames associated with such structures. Leaving a detailed description of sub-Riemannian structures for later sections we will formulate our main result now.

We consider compact homogeneous manifolds $\mathbf{M}$ equipped with the so-called sub-Riemannian metric $\mu(x, y),\ x, y \in \mathbf{M}$ (see Definition 5). To formulate our main result we need a definition of a sub-Riemannian lattice on a manifold $\mathbf{M}$. The precise definitions of all the notions used below will be given in the text.

Lemma 1. *Let* $\mathbf{M}$ *be a compact sub-Riemannian manifold and* $\mu(x, y),\ x, y \in \mathbf{M}$ *be a sub-Riemannian metric. Let* $B^{\mu}(x, r)$ *be a ball in this metric with center* $x \in \mathbf{M}$ *and radius* r*. There exists a natural number* $N_{\mathbf{M}}^{\mu}$ *such that for any sufficiently small* $r > 0$ *there exists a set of points* $\mathscr{M}_r^{\mu} = \{x_i\}$ *with the following properties:*

1. *the balls* $B^{\mu}(x_i, r/4)$ *are disjoint,*
2. *the balls* $B^{\mu}(x_i, r/2)$ *form a cover of* $\mathbf{M}$*,*
3. *every point of* $\mathbf{M}$ *is covered by not more than* $N_{\mathbf{M}}^{\mu}$ *balls* $B^{\mu}(x_i, r)$*.*

Definition 1. A set $\mathscr{M}_r^{\mu} = \{x_i\}$ constructed in the previous lemma will be called a metric r-lattice.

The meaning of this definition is that points $\{x_i\}$ are distributed over $\mathbf{M}$ "almost uniformly" in the sense of the metric μ.

We will consider compact homogeneous manifolds $\mathbf{M} = \mathbf{G}/\mathbf{H}$ where $\mathbf{G}$ is a compact Lie group and $\mathbf{H} \subset \mathbf{G}$ is a closed subgroup. Let dx be an invariant (with respect to natural action of $\mathbf{G}$ on $\mathbf{M}$) measure on $\mathbf{M}$ and $L_2(\mathbf{M}) = L_2(\mathbf{M}, dx)$ the corresponding Hilbert space of complex-valued functions on $\mathbf{M}$ with the inner product

$$\langle f, g\rangle = \int_{\mathbf{M}} f\overline{g}dx.$$

The notation $|B^{\mu}(x, r)|$ will be used for the volume of the ball with respect to the measure dx. An interesting feature of sub-Riemann structures is that balls of the same radius may have essentially different volumes (in contrast to the case of the Riemann metric and Riemann measure).

In the next theorem we will mention a sub-elliptic operator (sub-Laplacian) $\mathscr{L}$ (see the precise definition in (26)) which is hypoelliptic [6], self-adjoint, and non-negative in $L_2(\mathbf{M})$. This operator is a natural analog of a Laplace-Beltrami operator in the case of a Riemannian manifold.

Theorem 1. *We assume that* $\mathbf{M}$ *is a compact homogeneous manifold equipped with a sub-Riemann metric* μ *(see section 4). Set* $r_j = 2^{-j-1},\ j = 0, 1, 2, \ldots,$ *and let* $\mathscr{M}_{r_j}^{\mu} = \{x_k^j\}_{k=1}^{m_j},\ \ x_k^j \in \mathbf{M},\ \ j = 0, 1, 2, ..$ *be a sequence of metric lattices.*

With every point x_k^j *one can associate a function* Θ_k^j *such that:*

1. *every* Θ_k^j *is bandlimited in the sense that* Θ_k^j *belongs to the space* $\mathbf{E}_{[2^{2j-2}, 2^{2j+2}]}(\mathscr{L})$ *which is the span of all eigenfunctions of* $\mathscr{L}$ *whose corresponding eigenvalues belong to the interval* $[2^{2j-2}, 2^{2j+2})$*,*

2. *every* Θ_k^j *is essentially supported around* x_k^j *in the sense that for any* $N > 0$ *there exists a constant* $C(N) > 0$ *such that for all* j, k *one has*

$$\left|\Theta_k^j(y)\right| \leq C(N) \left|B^\mu\left(x_k^j, 2^{-j}\right)\right|^{-1/2} \left(1 + 2^j \mu(x_k^j, y)\right)^{-N}, \tag{1}$$

3. $\{\Theta_k^j\}$ *is a Parseval frame, i.e. for all* $f \in L_2(\mathbf{M})$

$$\sum_{j \geq 0} \sum_{1 \leq k \leq m_j} \left|\left\langle f, \Theta_k^j \right\rangle\right|^2 = \|f\|^2_{L_2(\mathbf{M})}, \tag{2}$$

and as a consequence of the Parseval property one has the following reconstruction formula:

4.

$$f = \sum_{j \geq 0} \sum_{1 \leq k \leq m_j} \left\langle f, \Theta_k^j \right\rangle \Theta_k^j.$$

In Theorem 9 this frame is used to obtain characterization of sub-elliptic Besov spaces in terms of the frame coefficients.

2 Parseval Localized Frames on Riemannian Compact Homogeneous Manifolds

2.1 Hilbert Frames

Frames in Hilbert spaces were introduced in [2].

Definition 2. A set of vectors $\{\psi_v\}$ in a Hilbert space $\mathscr{H}$ is called a frame if there exist constants $A, B > 0$ such that for all $f \in \mathscr{H}$

$$A\|f\|_2^2 \leq \sum_v |\langle f, \psi_v \rangle|^2 \leq B\|f\|_2^2. \tag{3}$$

The largest A and smallest B are called lower and upper frame bounds.

The set of scalars $\{\langle f, \psi_v \rangle\}$ represents a set of measurements of a signal f. To synthesize the signal f from this set of measurements one has to find another (dual) frame $\{\Psi_v\}$ and then a reconstruction formula is

$$f = \sum_v \langle f, \psi_v \rangle \Psi_v. \tag{4}$$

Dual frames are not unique in general. Moreover it is difficult to find a dual frame. However, for frames with $A = B = 1$ the decomposition and synthesis of functions can be done with the same frame. In other words

$$f = \sum_{v} \langle f, \psi_v \rangle \, \psi_v. \tag{5}$$

Such frames are known as Parseval frames. For example, three vectors in $\mathbf{R}^2$ with angles $2\pi/3$ between them whose lengths are all $\sqrt{2/3}$ form a Parseval frame.

2.2 Compact Homogeneous Manifolds

The basic information about compact homogeneous manifolds can be found in [7, 8]. A homogeneous compact manifold $\mathbf{M}$ is a C^∞-compact manifold on which a compact Lie group $\mathbf{G}$ acts transitively. In this case $\mathbf{M}$ is necessarily of the form $\mathbf{G}/\mathbf{H}$, where $\mathbf{H}$ is a closed subgroup of $\mathbf{G}$. The notation $L_2(\mathbf{M})$ is used for the usual Hilbert spaces, where dx is the normalized invariant measure on $\mathbf{M}$.

The best known example of such manifold is a unit sphere $\mathbf{S}^n$ in $\mathbf{R}^{n+1}$: $\mathbf{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n) = \mathbf{G}/\mathbf{H}$.

If $\mathbf{g}$ is the Lie algebra of a compact Lie group $\mathbf{G}$, then there exists a such choice of basis $X_1, \ldots, X_d$ in $\mathbf{g}$, for which the operator

$$-\mathbf{L} = X_1^2 + X_2^2 + \ \ldots + X_d^2, \ d = dim\,\mathbf{G} \tag{6}$$

is a bi-invariant operator on $\mathbf{G}$. Here X_j^2 is $X_j \circ X_j$ where we identify each X_j with a left-invariant vector field on $\mathbf{G}$. We will use the same notation for its image under differential of the quasi-regular representation of $\mathbf{G}$ in $L_2(\mathbf{M})$. This operator $\mathbf{L}$, which is known as the Casimir operator is elliptic. There are situations in which the operator $\mathbf{L}$ is, or is proportional to, the Laplace-Beltrami operator of an invariant metric on $\mathbf{M}$. This happens for example, if $\mathbf{M}$ is an n-dimensional torus, a compact semi-simple Lie group, or a compact symmetric space of rank one.

Since $\mathbf{M}$ is compact and the operator $\mathbf{L}$ is elliptic it has a discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots\ldots$ which goes to infinity without any accumulation points and there exists a complete family $\{u_j\}$ of orthonormal eigenfunctions which form a basis in $L_2(\mathbf{M})$.

The elliptic differential self-adjoint (in $L_2(\mathbf{M})$) operator $\mathbf{L}$ and its powers $\mathbf{L}^{s/2}, \ k \in \mathbf{R}_+$, can be extended from $C^\infty(\mathbf{M})$ to distributions. The family of Sobolev spaces $W_p^s(\mathbf{M}), \ 1 \leq p < \infty, \ s \in \mathbf{R}$, can be introduced as subspaces of $L_p(\mathbf{M})$ with the norm

$$\|f\|_p + \|\mathbf{L}^{s/2} f\|_p. \tag{7}$$

One can show that when $s = k$ is a natural number this norm is equivalent to the norm

$$|||f|||_{k,p} = \|f\|_p + \sum_{1\le i_1,\ldots,i_k\le d} \|X_{i_1}\ldots X_{i_k} f\|_p, \; 1 \le p < \infty. \tag{8}$$

We assume now that $\mathbf{M}$ is equipped with a $\mathbf{G}$-invariant Riemann metric ρ. The Sobolev spaces can also be introduced in terms of local charts [23]. We fix a finite cover $\{B^\rho(y_\nu, r_0)\}$ of $\mathbf{M}$

$$\mathbf{M} = \bigcup_\nu B^\rho(y_\nu, r_0), \tag{9}$$

where $B^\rho(y_\nu, r_0)$ is a ball centered at $y_\nu \in \mathbf{M}$ of radius r_0 contained in a coordinate chart. Let consider $\Psi = \{\psi_\nu\}$ be a partition of unity $\Psi = \{\psi_\nu\}$ subordinate to this cover. The Sobolev spaces $W^k_p(\mathbf{M}), k \in \mathbf{N}, 1 \le p < \infty$, are introduced as the completion of $C^\infty(\mathbf{M})$ with respect to the norm

$$\|f\|_{W^k_p(\mathbf{M})} = \left(\sum_\nu \|\psi_\nu f\|^p_{W^k_p(B^\rho(y_\nu, r_0))}\right)^{1/p}. \tag{10}$$

Remark 1. Spaces $W^k_p(\mathbf{M})$ are independent of the choice of elliptic self-ajoint second order differential operator. For every choice of such operators corresponding norms (7) will be equivalent. Also, any two norms of the form (10) are equivalent [23].

The Besov spaces can be introduced via the formula

$$\mathscr{B}^\alpha_{p,q}(\mathbf{M}) := \left(L_p(\mathbf{M}), W^r_p(\mathbf{M})\right)^K_{\alpha/r,q}, \tag{11}$$

where $0 < \alpha < r \in \mathbf{N}$, $1 \le p < \infty$, $1 \le q \le \infty$. Here K is the Peetre's interpolation functor.

An explicit norm in these spaces was given in [11–14, 16]. For the same operators as above $\{X_1, \ldots, X_d\}$, $d = dim\ \mathbf{G}$, let $T_1, \ldots, T_d$ be the corresponding one-parameter groups of translation along integral curves of the corresponding vector fields, i.e.

$$T_j(\tau)f(x) = f(\exp \tau X_j \cdot x), x \in \mathbf{M}, \tau \in \mathbf{R}, f \in L_2(\mathbf{M}); \tag{12}$$

here $\exp \tau X_j \cdot x$ is the integral curve of the vector field X_j which passes through the point $x \in \mathbf{M}$. The modulus of continuity is introduced as

$$\Omega^r_p(s,f) = \sum_{1\le j_1,\ldots,j_r\le d} \sup_{0\le\tau_{j_1}\le s} \ldots \sup_{0\le\tau_{j_r}\le s} \|\left(T_{j_1}(\tau_{j_1}) - I\right)\ldots\left(T_{j_r}(\tau_{j_r}) - I\right)f\|_{L_p(\mathbf{M})}, \tag{13}$$

where $f \in L_p(\mathbf{M}),\ r \in \mathbf{N}$, and I is the identity operator in $L_p(\mathbf{M})$. We consider the space of all functions in $L_p(\mathbf{M})$ for which the following norm is finite:

$$\|f\|_{L_p(\mathbf{M})} + \left(\int_0^\infty (s^{-\alpha}\Omega_p^r(s,f))^q \frac{ds}{s}\right)^{1/q}, 1 \le p < \infty, 1 \le q \le \infty, \tag{14}$$

with the usual modifications for $q = \infty$.

Theorem 2. *The norm of the Besov space $B_p^{\alpha q}(\mathbf{M}) = (L_p(\mathbf{M}), W_p^r(\mathbf{M}))^K_{\alpha/r,q},\ 0 < \alpha < r \in \mathbf{N},\ 1 \le p < \infty, 1 \le q \le \infty$, is equivalent to the norm (14). Moreover, the norm (14) is equivalent to the norm*

$$\|f\|_{W_p^{[\alpha]}(\mathbf{M})} + \sum_{1 \le j_1,\dots,j_{[\alpha]} \le d} \left(\int_0^\infty \left(s^{[\alpha]-\alpha}\Omega_p^1(s, X_{j_1}\dots X_{j_{[\alpha]}}f)\right)^q \frac{ds}{s}\right)^{1/q} \tag{15}$$

if α is not integer ($[\alpha]$ is its integer part). If $\alpha = k \in \mathbf{N}$ is an integer, then the norm (14) is equivalent to the norm (Zygmund condition)

$$\|f\|_{W_p^{k-1}(\mathbf{M})} + \sum_{1 \le j_1,\dots,j_{k-1} \le d} \left(\int_0^\infty \left(s^{-1}\Omega_p^2(s, X_{j_1}\dots X_{j_{k-1}}f)\right)^q \frac{ds}{s}\right)^{1/q}. \tag{16}$$

Definition 3. The space of ω-bandlimited functions $\mathbf{E}_\omega(\mathbf{L})$ is defined as the span of all eigenfunctions of $\mathbf{L}$ whose eigenvalues are not greater than ω.

To describe our construction of frames we need the notion of a lattice on a manifold $\mathbf{M}$ equipped with a Riemann metric ρ. This notion is similar to the corresponding notion introduced in Lemma 1.

Lemma 2. *If $\mathbf{M}$ is a compact Riemannian manifold then there exists a natural $N_{\mathbf{M}}^\rho$ such that for any sufficiently small r there exists a set of points $\mathscr{M}_r^\rho = \{x_i\}$ with the following properties:*

1. *the balls $B^\rho(x_i, r/4)$ are disjoint,*
2. *the balls $B^\rho(x_i, r/2)$ form a cover of $\mathbf{M}$,*
3. *the height of the cover by the balls $B^\rho(x_i, r)$ is not greater than $N_{\mathbf{M}}^\rho$.*

The meaning of this definition is that points $\{x_k\}$ distributed over $\mathbf{M}$ almost uniformly.

In [4] the following theorem was proved for compact homogeneous manifolds considered with invariant Riemann metric.

Theorem 3. *Set $r_j = 2^{-j-1},\ j = 0, 1, 2, \dots$, and let $\mathscr{M}_{r_j}^\rho = \{x_k^j\}_{k=1}^{m_j},\ x_k^j \in \mathbf{M},\ j = 0, 1, 2, ..$ be a sequence of metric lattices.*

With every point x_k^j we associate a function Ψ_k^j such that:

1. *every* Ψ_k^j *is bandlimited in the sense that* Ψ_k^j *belongs to the space* $\mathbf{E}_{[2^{2j-2},2^{2j+2}]}(\mathbf{L})$ *which is the span of all eigenfunction of* $\mathbf{L}$ *whose corresponding eigenvalues belong to the interval* $[2^{2j-2}, 2^{2j+2})$*,*
2. *every* Ψ_k^j *is essentially supported around* x_k^j *in the sense that the following estimate holds for every* $N > n$*:*

$$\left|\Psi_k^j(y)\right| \leq C(N)2^{jn}\left(1 + 2^j\rho\,(x_k^j, y)\right)^{-N}, \quad dim\,\mathbf{M} = n, \tag{17}$$

3. $\{\Psi_k^j\}$ *is a Parseval frame, i.e. for all* $f \in L_2(\mathbf{M})$

$$\sum_{j\geq 0}\sum_{1\leq k\leq m_j}\left|\left\langle f, \Psi_k^j\right\rangle\right|^2 = \|f\|^2_{L_2(\mathbf{M})}, \tag{18}$$

and

$$f = \sum_{j\geq 0}\sum_{1\leq k\leq m_j}\left\langle f, \Psi_k^j\right\rangle \Psi_k^j. \tag{19}$$

As an important application of Theorem 3 one can describe Besov spaces in terms of the frame coefficients [4].

Theorem 4. *The norm of the Besov space* $\|f\|_{\mathscr{B}^\alpha_{p,q}(\mathbf{M})}, \quad 1 \leq p < \infty, 0 < q \leq \infty$ *is equivalent to the norm*

$$\|\tau(f)\|_{\mathbf{b}^\alpha_{p,q}} = \left(\sum_{j=0}^{\infty} 2^{jq(\alpha - n/p + n/2)}\left(\sum_k |\langle f, \Psi_k^j\rangle|^p\right)^{q/p}\right)^{1/q}.$$

2.3 Example of $\mathbf{S}^2$ with Riemannian Metric

We consider $\mathbf{M} = \mathbf{S}^2$. In this case the Casimir operator coincides with the Laplace-Beltrami operator $\mathbf{L}$ on $\mathbf{S}^2$ and it can be written as a sum of the vector fields on $\mathbf{S}^2$:

$$\mathbf{L} = \sum_{i,j=1;i<j}^{3} X_{i,j}^2 = \sum_{i,j=1;i<j}^{3} (x_i\partial_{x_j} - x_j\partial_{x_i})^2 = \mathbf{L}.$$

Let $\mathscr{P}_l$ denote the space of spherical harmonics of degree l, which are restrictions to $\mathbf{S}^2$ of harmonic homogeneous polynomials of degree l in $\mathbf{R}^3$.

Each $\mathscr{P}_l$ is the eigenspace of $\mathbf{L}$ that corresponds to the eigenvalue $-l(l+1)$. Let $\mathscr{Y}_{n,l}, \ n = 1, \ldots, 2l+1$ be an orthonormal basis in $\mathscr{P}_l$. One has

$$\mathbf{L}\mathscr{Y}_{m,l} = -l(l+1)\mathscr{Y}_{m,l}.$$

Sobolev spaces $W_p^k(\mathbf{L}), 1 \leq p < \infty$, can be introduced as usual by using a system of local coordinates or by using vector fields $X_{i,j}$:

$$\|f\|_{W_p^k(\mathbf{M})} = \|f\|_p + \sum\sum \|X_{i,j}\ldots X_{i,j}f\|_p \tag{20}$$

Corresponding Besov spaces $\mathscr{B}_{p,q}^{\alpha}(\mathbf{L})$ can be described either using local coordinates or in terms of the modules of continuity constructed in terms of one-parameter groups of rotations $e^{\tau X_{i,j}}$ [11–15]. In particular, when $p = 2$ the Parseval identity for orthonormal bases and the theory of interpolation spaces imply descriptions of the norms of $W_2^k(\mathbf{L})$ and $\mathscr{B}_{2,2}^{\alpha}(\mathbf{L})$ in terms of Fourier coefficients:

$$\left(\sum_{l=0}^{\infty}\sum_{n=1}^{2l+1}(l+1)^{2\alpha}|c_{n,l}(f)|^2\right)^{1/2}, \tag{21}$$

where

$$c_{n,l}(f) = \int_{\mathbf{S}^d} f\mathscr{Y}_{n,l}, \quad f \in L_2(\mathbf{S}^d).$$

3 Sphere S^2 with a Sub-Riemannian Structure. A Sub-Laplacian and Sub-Elliptic Spaces on S^2

To illustrate nature of sub-elliptic spaces we will consider the case of two-dimensional sphere $\mathbf{S}^2$. We consider on $\mathbf{S}^2$ two vector fields $Y_1 = X_{2,3}$ and $Y_2 = X_{1,3}$ and the corresponding sub-Laplace operator

$$\mathscr{L} = Y_1^2 + Y_2^2.$$

Note that since the operators Y_1, Y_2 do not span the tangent space to $\mathbf{S}^2$ along a great circle with $x_3 = 0$ the operator $\mathscr{L}$ is not elliptic on $\mathbf{S}^2$. However, this operator is hypoelliptic [6] since $Y_1, \ Y_2$, and their commutator $Y_3 = Y_1Y_2 - Y_2Y_1 = X_{1,2}$ span the tangent space at every point of $\mathbf{S}^2$.

Let's compute its corresponding eigenvalues. In the standard spherical coordinates (φ, ϑ) spherical harmonics $\mathscr{Y}_{m,l}(\varphi, \vartheta)$, $l = 0, 1, \ldots, |m| \leq l$ are proportional to $e^{im\varphi}P_l^m(\cos\vartheta)$, where P_l^m are associated Legendre polynomials. This representation shows that for $Y_3 = X_{1,2}$ one has

$$Y_3^2\mathscr{Y}_{m,l} = -m^2\mathscr{Y}_{m,l}.$$

Since $\mathscr{Y}_{m,l}$ is an eigenfunction of $\mathbf{L}$ with the eigenvalue $-l(l+1)$ we obtain

$$\mathbf{L}\mathscr{Y}_{m,l} = -l(l+1)\mathscr{Y}_{m,l}$$

and

$$\mathscr{L}\mathscr{Y}_{m,l} = \mathbf{L}\mathscr{Y}_{m,l} - Y_3^2\mathscr{Y}_{m,l} = -\left(l(l+1) - m^2\right)\mathscr{Y}_{m,l}.$$

It shows that spherical functions are eigenfunctions of both $\mathscr{L}$ and $\mathbf{L}$.

The graph norm of a fractional power of $\mathscr{L}$ is equivalent to the norm

$$\left(\sum_{l=0}^{\infty}\sum_{|m|\leq l}\left((l+1)^2 - m^2\right)^{\alpha}|c_{m,l}(f)|^2\right)^{1/2},$$

$$c_{m,l}(f) = \int_{\mathbf{S}^d} f\mathscr{Y}_{m,l}, \quad f \in L_2(\mathbf{L}). \tag{22}$$

Note that these spaces $W_2^{\alpha}(\mathscr{L})$ are exactly the Besov spaces $\mathscr{B}_{2,2}^{\alpha}(\mathscr{L})$.

We introduce subelliptic (anisotropic) Sobolev space $W_2^{\alpha}(\mathscr{L})$, $\alpha \geq 0$, as the domain of $\mathscr{L}^{\alpha}$ with the graph norm and define Besov spaces $\mathscr{B}_{2,q}^{\alpha}(\mathscr{L})$ as

$$\mathscr{B}_{2,q}^{\alpha}(\mathscr{L}) = (L_2(\mathbf{S}^2), W_2^r(\mathscr{L}))_{\theta,q}^K, \quad 0 < \theta = \alpha/r < 1, \quad 1 \leq q \leq \infty.$$

where K is the Peetre's interpolation functor.

Note that vector fields Y_1, Y_2 span the tangent space to $\mathbf{S}^2$ at every point away from a great circle $x_3 = 0$. For this reason around such points a function belongs to the domain of $\mathscr{L}$ if and only if it belongs to the regular Sobolev space $W_2(\mathbf{L})$.

At the same time the fields $Y_1,\ Y_2$ do not span the tangent space to $\mathbf{S}^2$ along a great circle with $x_3 = 0$. However, the fields Y_1, Y_2 and their commutator $Y_3 = Y_1Y_2 - Y_2Y_1 = X_{1,2}$ do span the tangent space along $x_3 = 0$. This fact implies that along the circle $x_3 = 0$, functions in the spaces $W_2^r(\mathscr{L})$ and $\mathscr{B}_{2,q}^{\alpha}(\mathscr{L})$ are losing $1/2$ in smoothness compared to their smoothness at other points on $\mathbf{S}^2$. In other words, the following embeddings hold true:

$$W_2^{\alpha}(\mathbf{L}) \subset W_2^{\alpha}(\mathscr{L}) \subset W_2^{\alpha/2}(\mathbf{L}),$$

$$\mathscr{B}_{2,q}^{\alpha}(\mathbf{L}) \subset \mathscr{B}_{2,q}^{\alpha}(\mathscr{L}) \subset \mathscr{B}_{2,q}^{\alpha/2}(\mathbf{L}),$$

which follow from a much more general results in [10, 15, 17, 22].

We would like to stress that subelliptic function spaces are different from the usual (elliptic) spaces. For example, if $W_2^{\alpha}(\mathbf{L})$ is the regular Sobolev space than general theory implies the embeddings

$$W_2^{\alpha}(\mathscr{L}) \subset W_2^{\alpha/2}(\mathbf{L}), \quad \mathscr{B}_{2,q}^{\alpha}(\mathscr{L}) \subset \mathscr{B}_{2,q}^{\alpha/2}(\mathbf{L}).$$

As the following Lemma shows, these embeddings are generally sharp.

Lemma 3. *For every $\alpha > 0$ and $\delta > \alpha/2$ there exists a function that belongs to $W_2^{\alpha}(\mathscr{L})$ but does not belong to $W_2^{\delta}(\mathbf{L})$.*

Proof. For a $\delta > \alpha/2 > 0$ pick any γ that satisfies the inequalities

$$-\frac{1}{2} - \delta < \gamma < -\frac{1}{2} - \frac{\alpha}{2}$$

Let $c_{n,l}$ be a sequence such that $c_{n,l} = 0$ if $n \neq l$ and $c_{l,l} = (2l+1)^{\gamma}$. For a function with such Fourier coefficients the norm (22) is finite since

$$\sum_{l=0}^{\infty}(2l+1)^{\alpha}(2l+1)^{2\gamma} = \sum_{l=0}^{\infty}(2l+1)^{\alpha+2\gamma} < \infty, \quad \alpha + 2\gamma < -1,$$

but the norm (21) is infinite

$$\sum_{l=0}^{\infty}(2l+1)^{2\delta}(2l+1)^{2\gamma} = \sum_{l=0}^{\infty}(2l+1)^{2(\delta+\gamma)}, \quad 2(\delta+\gamma) > -1.$$

4 A Sub-Riemannian Structure and Corresponding Metric on Compact Homogeneous Manifolds

Let $\mathbf{M} = \mathbf{G}/\mathbf{H}$ be a compact homogeneous manifold and $\mathbf{X} = \{X_1, \ldots, X_d\}$ be a basis of the Lie algebra $\mathbf{g}$, the same as in (6). Let

$$\mathbf{Y} = \{Y_1, \ldots, Y_m\} \tag{23}$$

be a subset of $\mathbf{X} = \{X_1, \ldots, X_d\}$ such that $Y_1, \ldots, Y_m$ and all their commutators

$$\begin{aligned} Y_{j,k} &= [Y_j, \ Y_k] = Y_jY_k - Y_kY_j, \\ Y_{j_1,\ldots,j_n} &= [Y_{j_1}, [\ldots[Y_{j_{n-1}}, Y_{j_n}]\ldots]], \end{aligned} \tag{24}$$

of order $n \leq Q$ span the entire algebra $\mathbf{g}$. Let

$$Z_1 = Y_1, Z_2 = Y_2, \ldots, Z_m = Y_m, \quad \ldots \quad , Z_N, \tag{25}$$

be an enumeration of all commutators (24) up to order $n \leq Q$. If a Z_j corresponds to a commutator of length n, we say that $deg(Z_j) = n$.

Images of vector fields (25) under the natural projection $p : \mathbf{G} \to \mathbf{M} = \mathbf{G}/\mathbf{H}$ span the tangent space to $\mathbf{M}$ at every point and will be denoted by the same letters.

Definition 4. A sub-Riemann structure on $\mathbf{M} = \mathbf{G}/\mathbf{H}$ is defined as a set of vectors fields on $\mathbf{M}$ which are images of the vector fields (23) under the projection p. They can also be identified with differential operators in $L_p(\mathbf{M}),\ 1 \leq p < \infty$, under the quasi-regular representation of $\mathbf{G}$.

One can define a non-isotropic metric μ on $\mathbf{M}$ associated with the fields $\{Y_1, \ldots, Y_m\}$.

Definition 5 ([10]). Let $C(\epsilon)$ denote the class of absolutely continuous mappings $\varphi : [0, 1] \to \mathbf{M}$ which almost everywhere satisfy the differential equation

$$\varphi^{'}(t) = \sum_{j=1}^{m} b_j(t) Z_j(\varphi(t)),$$

where $|b_j(t)| < \epsilon^{deg(Z_j)}$. Then we define $\mu(x, y)$ as the lower bound of all such $\epsilon > 0$ for which there exists $\varphi \in C(\epsilon)$ with $\varphi(0) = x,\ \varphi(1) = y$.

The corresponding family of balls in $\mathbf{M}$ is given by

$$B^{\mu}(x, \epsilon) = \{y \in \mathbf{M} : \ \mu(x, y) < \epsilon\}.$$

These balls reflect the non-isotropic nature of the vector fields $Y_1, \ldots, Y_m$ and their commutators. For a small $\epsilon > 0$ ball $B^{\mu}(x, \epsilon)$ is of size ϵ in the directions $Y_1, \ldots, Y_m$, but only of size ϵ^n in the directions of commutators of length n.

It is known [10] that the following property holds for certain $c = c(Y_1, \ldots, Y_m),\ C = C(Y_1, \ldots, Y_m)$:

$$c\rho(x, y) \leq \mu(x, y) \leq C\left(\rho(x, y)\right)^{1/Q}$$

where ρ stands for an $\mathbf{G}$-invariant Riemannian metric on $\mathbf{M} = \mathbf{G}/\mathbf{H}$. We will be interested in the following sub-elliptic operator (sub-Laplacian)

$$-\mathscr{L} = Y_1^2 + \ldots + Y_m^2 \tag{26}$$

which is hypoelliptic [6] self-adjoint and non-negative in $L_2(\mathbf{M})$.

Definition 6. The space of ω-bandlimited functions $\mathbf{E}_{\omega}(\mathscr{L})$ is defined as the span of all eigenfunctions of $\mathscr{L}$ whose eigenvalues are not greater than ω.

Due to the uncertainty principle bandlimited functions in $\mathbf{E}_\omega(\mathscr{L})$ are not localized on $\mathbf{M}$ in the sense that their supports coincide with $\mathbf{M}$.

Using the operator $\mathscr{L}$ we define non-isotropic Sobolev spaces $W_p^k(\mathscr{L}),\ 1 \leq p < \infty$, and non-isotropic Besov spaces $\mathscr{B}^\alpha_{p,q}(\mathscr{L}),\ \ 1 \leq p < \infty,\ 1 \leq q \leq \infty$, by using formulas (7) and (11), respectively.

5 Product Property for Subelliptic Laplace Operator

The results of this section play a crucial role in our construction of the Parseval frames. In what follows we consider previously defined operators

$$-\mathbf{L} = X_1^2 + X_2^2 + \ \ldots + X_d^2,\ d = dim\, \mathbf{G},$$

and

$$-\mathscr{L} = Y_1^2 + \ldots + Y_m^2,\ m < d,$$

as differential operators in $L_2(\mathbf{M})$.

Lemma 4 ([4, 20]). *If $\mathbf{M} = \mathbf{G}/\mathbf{H}$ is a compact homogeneous manifold, then for any f and g in $\mathbf{E}_\omega(\mathbf{L})$, their product fg belongs to $\mathbf{E}_{4d\omega}(\mathbf{L})$, where d is the dimension of the group $\mathbf{G}$.*

Proof. For every X_j one has

$$X_j^2(fg) = f(X_j^2 g) + 2(X_j f)(X_j g) + g(X_j^2 f).$$

Thus, the function $\mathbf{L}^k\,(fg)$ is a sum of $(4d)^k$ terms of the form

$$(X_{i_1} \ldots X_{i_m} f)(X_{j_1} \ldots X_{j_{2k-m}} g).$$

This implies that

$$\left\| \mathbf{L}^k\,(fg) \right\|_\infty \leq (4d)^k \sup_{0 \leq m \leq 2k} \sup_{x,y \in \mathbf{M}} |X_{i_1} \ldots X_{i_m} f(x)| \left| X_{j_1} \ldots X_{j_{2k-m}} g(y) \right|. \qquad (27)$$

Let us show that for all $f, g \in \mathbf{E}_\omega(\mathbf{L})$ the following inequalities hold:

$$\|X_{i_1} \ldots X_{i_m} f\|_{L_2(\mathbf{M})} \leq \omega^{m/2} \|f\|_{L_2(\mathbf{M})} \qquad (28)$$

and

$$\|X_{j_1} \ldots X_{j_{2k-m}} g\|_{L_2(\mathbf{M})} \leq \omega^{(2k-m)/2} \|g\|_{L_2(\mathbf{M})}. \qquad (29)$$

By construction (see (6)) the operator $-\mathbf{L} = X_1^2 + \ \ldots + X_d^2$ commutes with every X_j and the same is true for $(-\mathbf{L})^{1/2}$. From here one can obtain the following equality:

$$\|\mathbf{L}^{s/2}f\|^2_{L_2(\mathbf{M})} = \sum_{1\leq i_1,\ldots,i_s\leq d} \|X_{i_1}\ldots X_{i_s}f\|^2_{L_2(\mathbf{M})},\ s\in\mathbf{N}, \tag{30}$$

which implies the estimates (28) and (29). The formula (27) along with the formula

$$\|\mathbf{L}^{m/2}f\|_{L_2(\mathbf{M})} \leq \omega^{m/2}\|f\|_{L_2(\mathbf{M})}. \tag{31}$$

implies the estimate

$$\|\mathbf{L}^k(fg)\|_{L_2(\mathbf{M})} \leq (4d)^k \sup_{0\leq m\leq 2k} \|X_{i_1}\ldots X_{i_m}f\|_{L_2(\mathbf{M})}\|X_{j_1}\ldots X_{j_{2k-m}}g\|_\infty \leq$$

$$(4d)^k\omega^{m/2}\|f\|_{L_2(\mathbf{M})} \sup_{0\leq m\leq 2k}\|X_{j_1}\ldots X_{j_{2k-m}}g\|_\infty. \tag{32}$$

Using the Sobolev embedding Theorem and the elliptic regularity of $\mathbf{L}$, we obtain for every $s > \frac{dim\mathbf{M}}{2}$

$$\|X_{j_1}\ldots X_{j_{2k-m}}g\|_\infty \leq C(\mathbf{M})\|X_{j_1}\ldots X_{j_{2k-m}}g\|_{W_2^s(\mathbf{M})} \leq$$

$$C(\mathbf{M})\left\{\|X_{j_1}\ldots X_{j_{2k-m}}g\|_{L_2(\mathbf{M})} + \|\mathbf{L}^{s/2}X_{j_1}\ldots X_{j_{2k-m}}g\|_{L_2(\mathbf{M})}\right\}, \tag{33}$$

where $W_2^s(\mathbf{M})$ is the Sobolev space of s-regular functions on $\mathbf{M}$. The estimate (31) gives the following inequality:

$$\|X_{j_1}\ldots X_{j_{2k-m}}g\|_\infty \leq C(\mathbf{M})\left\{\omega^{k-m/2}\|g\|_{L_2(\mathbf{M})} + \omega^{k-m/2+s}\|g\|_{L_2(\mathbf{M})}\right\} \leq$$

$$C(\mathbf{M})\omega^{k-m/2}\left\{\|g\|_{L_2(\mathbf{M})} + \omega^{s/2}\|g\|_{L_2(\mathbf{M})}\right\} = C(\mathbf{M},g,\omega,s)\omega^{k-m/2},\quad s > \frac{dim\,\mathbf{M}}{2}. \tag{34}$$

Finally we have the following estimate:

$$\|\mathbf{L}^k(fg)\|_{L_2(\mathbf{M})} \leq C(\mathbf{M},f,g,\omega,s)(4d\omega)^k,\quad s > \frac{dim\,\mathbf{M}}{2},\ \ k\in\mathbf{N}, \tag{35}$$

which leads to our result.

Lemma 5. *There exist positive $c,\ C$ such that for $\omega > 1$ the following embeddings hold:*

$$\mathbf{E}_\omega(\mathscr{L}) \subset \mathbf{E}_{c\omega^Q}(\mathbf{L}), \tag{36}$$

$$\mathbf{E}_\omega(\mathbf{L}) \subset \mathbf{E}_{C\omega}(\mathscr{L}). \tag{37}$$

Proof. There exists a constant $a = a(\mathbf{L},\ \mathscr{L})$ such that for all f in the Sobolev space $W_2^Q(\mathbf{M})$ [10]

$$\|\mathbf{L}f\| \le a\|(I+\mathscr{L})^Q f\|.$$

Since $\mathbf{L}$ belongs to the center of the enveloping algebra of the Lie algebra $\mathbf{g}$ it commutes with $\mathscr{L}$. Thus one has for sufficiently smooth f:

$$\|\mathbf{L}^l f\| \le a^l \|(I+\mathscr{L})^{Ql} f\|, \quad l \in \mathbf{R}.$$

It implies that if $f \in \mathbf{E}_\omega(\mathscr{L})$, then for $\omega \ge 1$

$$\|\mathbf{L}^l f\| \le a^l \|(I+\mathscr{L})^{Ql} f\| \le \left(a(1+\omega)^Q\right)^l \|f\| \le \left(2a\omega^Q\right)^l \|f\|, \quad l \in \mathbf{R},$$

which shows that $f \in \mathbf{E}_{2a\omega^Q}(\mathbf{L})$. Conversely, since for some $b = b(\mathbf{L}, \mathscr{L})$

$$\|\mathscr{L} f\| \le b\|(I+\mathbf{L})f\|, \quad f \in W_2^2(\mathbf{M}),$$

we have

$$\|\mathscr{L}^l f\| \le b^l \|(I+\mathbf{L})^l f\|, \quad f \in W_2^{2l}(\mathbf{M}),$$

and for $f \in \mathbf{E}_\omega(\mathbf{L})$

$$\|\mathscr{L}^l f\| \le b^l \|(I+\mathbf{L})^l f\| \le (b(1+\omega))^l \|f\| \le (2b\omega)^l \|f\|, \quad f \in W_2^{2l}(\mathbf{M}).$$

The product property of bandlimited functions is described in the following Theorem.

Theorem 5. *There exists a constant $C_0 = C_0(\mathscr{L}) > 0$ such that for any $f, g \in \mathbf{E}_\omega(\mathscr{L})$ the product fg belongs to $\mathbf{E}_{C_0\omega^Q}(\mathscr{L})$.*

Proof. If $f, g \in \mathbf{E}_\omega(\mathscr{L})$, then $f, g \in \mathbf{E}_{c\omega^Q}(\mathbf{L})$. According to Lemma 4 their product fg belongs to $\mathbf{E}_{4dc\omega^Q}(\mathbf{L})$ which implies that for some $C_0 = C_0(\mathscr{L})$ the product fg belongs to $\mathbf{E}_{C_0\omega^Q}(\mathscr{L})$.

6 Positive Cubature Formulas on Sub-Riemannian Manifolds

Now we are going to prove existence of cubature formulas which are exact on $\mathbf{E}_\omega(\mathscr{L})$, and have positive coefficients of the right size.

Let $\mathscr{M}_r = \{x_k\}$ be a r-lattice and $\{B^\mu(x_k, r)\}$ be an associated family of balls that satisfy only properties (1) and (2) of Lemma 1. We define

$$U_1 = B^\mu(x_1, r/2) \setminus \cup_{i,\, i\ne 1} B^\mu(x_i, r/4),$$

and

$$U_k = B^\mu(x_k, r/2) \setminus \left(\cup_{j<k} U_j \cup_{i,\, i\ne k} B^\mu(x_i, r/4)\right). \tag{38}$$

One can verify the following properties.

Lemma 6. *The sets* $\{U_k\}$ *form a disjoint measurable cover (up to a set of measure zero) of* $\mathbf{M}$ *and*

$$B^{\mu}(x_k, r/4) \subset U_k \subset B^{\mu}(x_k, r/2) \tag{39}$$

We have the following Plancherel-Polya inequalities [18, 19].

Theorem 6. *There exist positive constants* $a_1 = a_1(\mathbf{M}, \mathbf{Y}), a_2 = a_2(\mathbf{M}, \mathbf{Y})$*, and* $a_0 = a_0(\mathbf{M}, \mathbf{Y})$ *such that, if for a given* $\omega > 0$ *one has*

$$0 < r < a_0\omega, \tag{40}$$

then for any metric r-lattice $\mathscr{M}_r = \{x_k\}$ *the following inequalities hold:*

$$a_1 \sum_k |U_k||f(x_k)|^2 \leq \|f\|_{L_2(\mathbf{M})} \leq a_2 \sum_k |U_k||f(x_k)|^2, \tag{41}$$

for every $f \in \mathbf{E}_{\omega}(\mathscr{L})$.

Proof. One has

$$|f(x)| \leq |f(x_k)| + |f(x) - f(x_k)|,$$

$$\int_{U_k} |f(x)|^2 dx \leq 2\left(|U_k||f(x_k)|^2 + \int_{U_k} |f(x) - f(x_k)|^2 dx\right),$$

and

$$\|f\|^2 \leq \sum_k \int_{U_k} |f(x)|^2 dx \leq 2\left(\sum_k |U_k||f(x_k)|^2 + \sum_k \int_{U_k} |f(x) - f(x_k)|^2 dx\right).$$

Take an $X \in \mathbf{g}$, $|X| = 1$, for which $\exp\, tX \cdot x_k = x$ for some $t \in \mathbf{R}$. Since every such vector field (as a field on $\mathbf{M}$) is a linear combination of the fields $[Y_{i_1}, \ldots [Y_{i_{l-1}}, Y_{i_l}] \ldots]$, $1 \leq l \leq Q$, $1 \leq i_j \leq m$, the Newton-Leibniz formula applied to a smooth f along the corresponding integral curve joining x and x_k gives

$$|f(x) - f(x_k)|^2 \leq Cr^2 \sum_{l=1}^{Q} \sum_{1 \leq i_1, i_2, \ldots i_l \leq m} \left(\sup_{y \in B^{\mu}(x_k, r/2)} |Y_{i_1} Y_{i_2} \ldots Y_{i_l} f(y)|\right)^2.$$

Applying anisotropic version of the Sobolev inequality [10] we obtain

$$|f(x) - f(x_k)|^2 \leq Cr^2 \sum_{l=1}^{Q} \sum_{1 \leq i_1, i_2, \ldots i_l \leq m} \left(\sup_{y \in B^{\mu}(x_k, r/2)} |Y_{i_1} Y_{i_2} \ldots Y_{i_l} f(y)|\right)^2 \leq$$

$$Cr^2 \sum_{l=0}^{Q} \sum_{1\le i_1,i_2,\dots,i_l\le m} \|Y_{i_1}Y_{i_2}\dots Y_{i_l}f\|^2_{H^{Q/2+\varepsilon}(B^{\mu}(x_k,r/2))},$$

where $x \in U_k, \ \ \varepsilon > 0, \ \ C = C(\varepsilon)$. Next,

$$\sum_k \int_{B^{\mu}(x_k,r/2)} |f(x) - f(x_k)|^2 dx \le$$

$$Cr^{n+2} \sum_{l=0}^{Q} \sum_{1\le i_1,i_2,..i_l\le m} \sum_k \|Y_{i_1}\dots Y_{i_l}f\|^2_{H^{Q/2+\varepsilon}(B^{\mu}(x_k,r/2))} \le$$

$$Cr^{n+2} \sum_{l=0}^{Q} \sum_{1\le i_1,\dots,i_l\le m} \|Y_{i_1}\dots Y_{i_l}f\|^2_{H^{Q/2+\varepsilon}(\mathbf{M})} \le Cr^{n+2}\left(\|f\|^2 + \|\mathscr{L}^Q f\|^2\right).$$

All together we obtain the inequality

$$\|f\|^2 \le 2\sum_k |U_k||f(x_k)|^2 + Cr^{n+2}\left(\|f\|^2 + \|\mathscr{L}^Q f\|^2\right).$$

Note that for $f \in \mathbf{E}_{\omega}(\mathscr{L})$

$$\|\mathscr{L}^Q f\| \le C\omega^Q \|f\|.$$

Thus, if for a given $\omega > 0$ we pick an $r > 0$ a way that

$$Cr^{n+2}(1+\omega)^Q < 1$$

then for a certain $C_1 = C_1(M) > 0$ one obtains the right-hand side of (41)

$$\|f\|^2 \le C_1 \sum_k |U_k||f(x_k)|^2.$$

The left-hand side of (41) follows from the Sobolev and Bernstein inequalities.

The Plancherel-Polya inequalities (41) can be used to prove the so-called sub-elliptic positive cubature formula. The proof goes along the same lines as in [4, 21], (see also [1, 3]).

The precise statement is the following.

Theorem 7. *There exists a constant $a = a(\mathbf{M}, \mathbf{Y}) > 0$ such that for a given $\omega > 0$ if $r = a\omega^{-1}$ then for any r-lattice $\mathscr{M}_r = \{x_k\}$ there exist strictly positive coefficients $\{\alpha_k\}$, for which the following equality holds for all functions in $\mathbf{E}_{\omega}(\mathscr{L})$:*

$$\int_{\mathbf{M}} f dx = \sum_k f(x_k)\alpha_k. \tag{42}$$

Moreover, there exists constants $b_1 > 0$, $b_2 > 0$, *such that the following inequalities hold:*

$$b_1|U_k| \le \alpha_k \le b_2|U_k|, \tag{43}$$

where the sets U_k *are defined in (38).*

7 Space Localization of Kernels

According to the spectral theorem if F is a Schwartz function on the line, then there is a well defined operator $F(\mathscr{L})$ in the space $L_2(\mathbf{M})$ such that for any $f \in L_2(\mathbf{M})$ one has

$$(F(\mathscr{L})f)(x) = \int_{\mathbf{M}} \mathscr{K}^F(x,y)f(y)dy, \tag{44}$$

where dy is the invariant normalized measure on $\mathbf{M}$. If $\{\lambda_j\}$ and $\{u_j\}$ are sets of eigenvalues and eigenfunctions of $\mathscr{L}$ respectively then

$$\mathscr{K}^F(x,y) = \sum_{j=0}^{\infty} F(\lambda_j)u_j(x)\overline{u_j}(y). \tag{45}$$

We will be especially interested in operators of the form $F(t^2\mathscr{L})$, where F is a Schwartz function and $t > 0$. The corresponding kernel will be denoted as $\mathscr{K}_t^F(x,y)$ and

$$\mathscr{K}_t^F(x,y) = \sum_{j=0}^{\infty} F(t^2\lambda_j)u_j(x)\overline{u_j}(y). \tag{46}$$

Note, that variable t here is a kind of scaling parameter.

The following important estimate was proved in [1] in the setting of the so-called Dirichlet spaces. It is a consequence of the main result in [9] that sub-Riemannin manifolds we consider in our article are the Dirichlet spaces.

Theorem 8. *If* $F \in C_0^\infty(\mathbf{R})$ *is even than for every* $N > 2Q$, *there exists a* $C_N = C_N(F, \mathbf{M}, \mathbf{Y}) > 0$ *such that*

$$\left|\mathscr{K}_t^F(x,y)\right| \le C_N \left(|B^\mu(x,t)|\,|B^\mu(y,t)|\right)^{-1/2}\left(1+t^{-1}\mu(x,y)\right)^{-N}, \quad 0 < t \le 1. \tag{47}$$

8 Parseval Space-Frequency Localized Frames on Sub-Riemannian Manifolds and Proof of Theorem 1

Let $g \in C^{\infty}(\mathbf{R}_+)$ be a monotonic function with support in $[0,\ 2^2]$, and $g(s) = 1$ for $s \in [0,\ 1],\ 0 \le g(s) \le 1,\ s > 0$. Setting $G(s) = g(s) - g(2^2 s)$ implies that $0 \le G(s) \le 1,\ s \in supp\, G \subset [2^{-2},\ 2^2]$. Clearly, $supp\, G(2^{-2j}s) \subset [2^{2j-2}, 2^{2j+2}],\ j \ge 1$. For the functions $F_0(s) = \sqrt{g(s)},\ \ F_j(s) = \sqrt{G(2^{-2j}s)},\ \ j \ge 1,$ one has $\sum_{j\ge 0} F_j^2(s) = 1,\ \ s \ge 0$. Using the spectral theorem for $\mathscr{L}$ one can define bounded self-adjoint operators $F_j(\mathscr{L})$ as

$$F_j(\mathscr{L})f(x) = \int_{\mathbf{M}} \mathscr{K}^F_{2^{-j}}(x,y)f(y)dy,$$

where

$$\mathscr{K}^F_{2^{-j}}(x,y) = \sum_{\lambda_m \in [2^{2j-2}, 2^{2j+2}]} F(2^{-2j}\lambda_m)u_m(x)\overline{u_m(y)}. \tag{48}$$

The same spectral theorem implies $\sum_{j\ge 0} F_j^2(\mathscr{L})f = f,\ f \in L_2(\mathbf{M})$, and taking inner product with f gives

$$\|f\|^2 = \sum_{j\ge 0} \left\langle F_j^2(\mathscr{L})f, f\right\rangle = \sum_{j\ge 0} \|F_j(\mathscr{L})f\|^2. \tag{49}$$

Moreover, since the function $F_j(s)$ has its support in $[2^{2j-2},\ \ 2^{2j+2}]$ the functions $F_j(\mathscr{L})f$ are bandlimited to $[2^{2j-2},\ \ 2^{2j+2}]$.

Next, consider the sequence $\omega_j = 2^{2j+2},\ j = 0, 1, \ldots$. By (49) the equality $\|f\|^2 = \sum_{j\ge 0} \|F_j(\mathscr{L})f\|^2$ holds, where every function $F_j(\mathscr{L})f$ is bandlimited to $[2^{2j-2},\ \ 2^{2j+2}]$. Since for every $\overline{F_j(\mathscr{L})f} \in \mathbf{E}_{2^{2j+2}}(\mathscr{L})$ one can use Theorem 5 to conclude that

$$|F_j(\mathscr{L})f|^2 \in \mathbf{E}_{C_0 2^{Q(2j+2)}}(\mathscr{L}).$$

According to Theorem 7 there exists a constant $a = a(\mathbf{M}, \mathbf{Y}) > 0$ such that for all natural j if

$$r_j = b2^{-Q(j+1)},\ \ b = aC_0, \tag{50}$$

then for any r_j-lattice $\mathscr{M}_{r_j}$ one can find positive coefficients $\alpha_{j,k}$ with for which the following exact cubature formula holds:

$$\|F_j(\mathscr{L})f\|_2^2 = \sum_{k=1}^{K_j} \alpha_{j,k} \left|F_j(\mathscr{L})f(x_{j,k})\right|^2, \tag{51}$$

where $x_{j,k} \in \mathscr{M}_{r_j}, k = 1, \ldots, K_j = card(\mathscr{M}_{r_j})$. Using the kernel $\mathscr{K}^F_{2^{-j}}$ of the operator $F_j(\mathscr{L})$ we define the functions

$$\Theta_{j,k}(y) = \sqrt{\alpha_{j,k}}\, \overline{\mathscr{K}^F_{2^{-j}}}(x_{j,k}, y) =$$
$$\sqrt{\alpha_{j,k}} \sum_{\lambda_m \in [2^{2j-2}, 2^{2j+2}]} \overline{F}(2^{-2j}\lambda_m)\overline{u}_m(x_{j,k})u_m(y). \tag{52}$$

One can easily see that for every $f \in L_2(\mathbf{M})$ the equality $\|f\|_2^2 = \sum_{j,k} |\langle f, \Theta_{j,k}\rangle|^2$ holds. Moreover, the first two items of Theorem 1 are also satisfied. Thus, Theorem 1 is proven.

As an application one can obtain description of sub-elliptic Besov spaces $\mathscr{B}^\alpha_{p,q}(\mathscr{L})$, $1 \le p < \infty$, $1 \le q \le \infty$, in terms of the Fourier coefficients with respect to this frame $\{\Theta_{j,k}\}$.

Consider the quasi-Banach space $\mathbf{b}^\alpha_{p,q}$ which consists of sequences $s = \{s^j_k\}$ $(j \ge 0,\ 1 \le k \le \mathscr{K}_j)$ satisfying

$$\|s\|_{\mathbf{b}^\alpha_{p,q}} = \left(\sum_{j\ge 0}^{\infty} 2^{j\alpha q} \left(\sum_k \left| B^\mu(x^j_k,\ 2^{-j}) \right|^{1/p-1/2} |s^j_k|^p \right)^{q/p} \right)^{1/q} < \infty, \tag{53}$$

and introduce the following mappings

$$\tau(f) = \{\langle f, \Theta^j_k\rangle\}, \tag{54}$$

and

$$\sigma(\{s^j_k\}) = \sum_{j\ge 0}^{\infty} \sum_k s^j_k \Theta^j_k. \tag{55}$$

It is not difficult to prove the following result (see [4, 5] for the Riemann case).

Theorem 9. *Let Θ^j_k be the same as above. Then for $1 \le p < \infty, 0 < q \le \infty, \alpha > 0$ the following statements are valid:*

1. *τ in (54) is a well-defined bounded operator $\tau : \mathscr{B}^\alpha_{p,q}(\mathscr{L}) \to \mathbf{b}^\alpha_{p,q}$;*
2. *σ in (55) is a well-defined bounded operator $\sigma : \mathbf{b}^\alpha_{p,q} \to \mathscr{B}^\alpha_{p,q}(\mathscr{L})$;*
3. *$\sigma \circ \tau = id$;*

Moreover, the following norms are equivalent:

$$\|f\|_{\mathscr{B}^\alpha_{p,q}(\mathscr{L})} \asymp \|\tau(f)\|_{\mathbf{b}^\alpha_{p,q}},$$

where

$$\|\tau(f)\|_{\mathbf{b}^{\alpha}_{p,q}} = \left(\sum_{j\geq 0}^{\infty} 2^{j\alpha q} \left(\sum_{k} \left| B^{\mu}(x^{j}_{k},\ 2^{-j}) \right|^{1/p-1/2} |s^{j}_{k}|^{p} \right)^{q/p} \right)^{1/q}.$$

The constants in these norm equivalence relations can be estimated uniformly over compact ranges of the parameters p, q, α.

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