Chapter 5
The Criterion of Self-similarity for Wall Velocities

The laminar boundary-layer flows can mainly be subdivided into two subclasses, namely the self-similar flows and the non-similar flows, as mentioned previously. Among these two classes, the self-similar flows had been more popular and studied extensively in the previous decades. The reason behind their wider acceptance is twofold: first, because of their governing boundary-layer equations, which are *pdes* actually, but readily reduce to ordinary differential equations by means of the similarity transformations, thus facilitating greatly toward their solution exploration; second, such self-similar flows help greatly toward the understanding of flow character within the boundary-layer. Because of these advantages, the self-similar boundary-layer flows, especially two-dimensional, have extensively been studied not only for the finite plate but also for the continuous sheet. The theory can almost be considered as complete for the self-similar flows past finite surfaces, but is still pending for the continuous surfaces. In this particular class of flows, the development had neither been quick nor been complete; rather, it had been contributed in bits. For example, Sakiadis [1] first introduced the self-similar solution for the continuous flat surface moving with uniform velocity. Crane [2] introduced the self-similar solution for variable wall velocity by restricting it to the linear form and referred it as the linear wall stretching. Later, the nonlinear stretching wall velocity was introduced by Banks [3] and Magyari and Keller [4] for the power-law and exponential wall velocities, respectively. This development is strictly restricted to the two-dimensional case; in view of these developments, the (two-dimensional) self-similar flows due to power-law wall velocities can be regarded as explored completely,[1] but the flows due to exponential wall velocities still require further exploration. The situation is far more adverse in the cases of three-dimensional and axisymmetric flows.

The available literature on three-dimensional flows is mainly restricted to the linear stretching velocities in two lateral directions. Some self-similar solutions

[1]Complete in a sense (see Sect. 3.1). This statement does not deny the existence of any other self-similar solution.

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corresponding to the exponential bilateral wall velocities have also been reported without covering the whole class. In this regard, three-dimensional self-similar flow due to a stretching sheet was introduced by Wang [5] in which he considered linear stretching in the two lateral directions. The axially symmetric case of self-similar flow due to a stretching cylinder was also introduced by Wang [6] where he again assumed the linear stretching velocity at the surface of the cylinder. Another type of axis-symmetric flow, due to a stretching surface, is the case of stretching disk flow which was introduced by Fang [7] by considering the linear radial stretching of the flexible circular disk. In all the aforementioned cases of three-dimensional and axisymmetric flows, the criterion of self-similarity has not been explored completely, thus requiring a thorough investigation regarding the identification of self-similar wall velocities in these cases. After having done all this, one will be more quite in claiming that the criterion of self-similarity in view of the stretching/shrinking wall velocities has completely been determined for the continuous surfaces. This will in turn help to complete the theory of self-similar laminar boundary-layers due to moving continuous surfaces.

In view of above-cited history regarding the, in bits, development of the self-similar boundary-layer flows due to moving continuous surfaces, a student or a new researcher in this area misleadingly perceives that the identification of such particular wall velocities for which the similarity solution exists is just by luck or due to the hit-and-trial method. This is the reason behind the fact that whenever a new self-similar solution was introduced in this area, it was immediately adopted by almost all the researchers involved with this topic. The same is the fact with the "shrinking sheet flow" which has also been adopted by a huge number of researchers despite the fact that the self-similar modeling introduced by Miklavcic and Wang [8] is, somewhat, wrong. But even then, the involved authors are continuously following the incorrect formulation and are publishing wrong results on this topic. A detail account on this issue is given in Chap. 7.

Therefore, keeping these facts in mind the systematic approach toward the determination of self-similar criterion for any flow situation has been explained and employed to the two-dimensional, three-dimensional, and axisymmetric flow situations in this chapter. The allowed wall velocities in the aforementioned cases for which the self-similar solutions exist have been determined, and the corresponding self-similar governing systems have also been derived.

5.1 Two-Dimensional Flow

Consider a flexible impermeable flat sheet emerging from a slit (situated at the point $(0,0)$) in the positive x-direction with velocity $u_w(x)$. The x-axis has been taken aligned to the sheet, and the y-axis goes deep into the fluid vertically upward by fixing the origin of the coordinate system at the orifice. The fluid is assumed to be viscous and incompressible following the Newton's law of viscosity. Outside the boundary-layer, the fluid velocity is denoted by u_∞ and is assumed to be zero.

Because of the absence of any potential flow and the utility of the boundary-layer assumption, the pressure gradient within the boundary-layer, formed on the moving continuous sheet, is zero. In this situation, the flow is steady and two-dimensional in nature. A schematic of the flow is shown in Fig. 5.1.

Based on the flow assumptions, the appropriate velocity vector reads as:

$$\boldsymbol{V} = [u(x,y), v(x,y), 0]. \tag{5.1}$$

Consequently, the governing boundary-layer system (2.10)–(2.12) reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{5.2}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{5.3}$$

subject to the boundary conditions [due to Eq. (2.17)]

$$\left.\begin{array}{lll} u = u_w(x), v = 0, & \text{at} & y = 0 \\ u = 0, & \text{at} & y = \infty \end{array}\right\}. \tag{5.4}$$

Further simplification to the system (5.2)–(5.4) can be achieved by introducing the stream function $\psi(x,y)$ owing to the following relation with the velocity components:

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}. \tag{5.5}$$

Due to Eq. (5.5), the equation of continuity (5.2) is satisfied identically and Eqs. (5.3) and (5.4) take the form

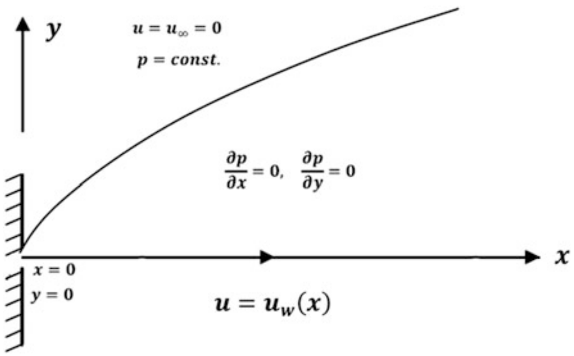


Fig. 5.1 Schematic of the two-dimensional flow and the coordinate system

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}=v\frac{\partial^3\psi}{\partial y^3}, \tag{5.6}$$

$$\left.\begin{array}{ll}\frac{\partial\psi}{\partial y}=u_w(x),\frac{\partial\psi}{\partial x}=0, & \text{at} \quad y=0\\ \frac{\partial\psi}{\partial y}=0, & \text{at} \quad y=\infty\end{array}\right\}, \tag{5.7}$$

respectively. Equations (5.6) and (5.7) are the representative equations for the considered two-dimensional flow with certain possible simplifications to be determined.

These equations shall now be applied the procedure of group theoretical approach in obtaining the similarity transformations, as explained in Sect. 3.3. According to the general theory, we need to introduce the scaling of all the variable quantities involved in the system (5.6)–(5.7). Therefore, we choose a scaling group of the form:

$$\bar{x}=k^{\alpha_1}x,\quad \bar{y}=k^{\alpha_2}y,\quad \bar{\psi}=k^{\alpha_3}\psi,\quad \bar{u}_w=k^{\alpha_4}u_w, \tag{5.8}$$

where k is the scaling parameter and $\alpha_i(i=1,\ldots,4)$ denote the scaling exponents. Utilization of the group (5.8) transforms the system (5.6)–(5.7) in the form of new variables as

$$k^{\alpha_1+2\alpha_2-2\alpha_3}\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{x}\partial\bar{y}}-k^{\alpha_1+2\alpha_2-2\alpha_3}\frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}=v\,k^{3\alpha_2-\alpha_3}\frac{\partial^3\bar{\psi}}{\partial\bar{y}^3}, \tag{5.9}$$

$$\left.\begin{array}{lll}k^{\alpha_2-\alpha_3}\frac{\partial\bar{\psi}}{\partial\bar{y}}=k^{-\alpha_4}\bar{u}_w, & \frac{\partial\bar{\psi}}{\partial\bar{y}}=0, & \text{at} \quad \bar{y}=0\\ \frac{\partial\bar{\psi}}{\partial\bar{y}}=0, & & \text{at} \quad \bar{y}=\infty\end{array}\right\}. \tag{5.10}$$

The restriction of invariance of the system (5.6)–(5.7) under the action of scaling group (5.8) requires that all the constant coefficients in (5.9) must have the same powers of k and the same applies to Eq. (5.10) also. This gives rise to a system of algebraic equations in $\alpha_i(i=1,..,4)$ of the form

$$\alpha_1+2\alpha_2-2\alpha_3=3\alpha_2-\alpha_3,\alpha_2-\alpha_3=-\alpha_4. \tag{5.11}$$

Before we solve the system (5.11), it is important to decide, first, about the variable to be eliminated among the original independent variables x and y. In this case, the obvious choice is x. This gives rise the consideration of two cases regarding the zero and nonzero character of α_1, the scaling exponent of x:

Case I ($\alpha_1\neq 0$)
In this case, the division by α_1 to the system (5.11) is possible and results in the system

$$1+2\frac{\alpha_2}{\alpha_1}-2\frac{\alpha_3}{\alpha_1}=3\frac{\alpha_2}{\alpha_1}-\frac{\alpha_3}{\alpha_1},\frac{\alpha_2}{\alpha_1}-\frac{\alpha_3}{\alpha_1}=-\frac{\alpha_4}{\alpha_1},$$

admitting the non-trivial solution of the form

$$\frac{\alpha_3}{\alpha_1}=1-\frac{\alpha_2}{\alpha_1},\frac{\alpha_4}{\alpha_1}=1-2\frac{\alpha_2}{\alpha_1}.$$

The ratio $\frac{\alpha_2}{\alpha_1}$ has appeared as the free variable and will be treated as an arbitrary constant in the subsequent process. Based on this solution, the new variables are constructed as follows (see Eq. 3.14):

$$\eta=\frac{y}{x^{\alpha_2/\alpha_1}},\quad f(\eta)=\frac{\psi(x,y)}{x^{\alpha_3/\alpha_1}},\quad a\equiv\text{const.}=\frac{u_w(x)}{x^{\alpha_4/\alpha_1}}. \tag{5.12}$$

Let us say

$$1-2\frac{\alpha_2}{\alpha_1}=m(\text{an arbitrary constant}), \tag{5.13}$$

due to which Eq. (5.12) takes the form

$$\eta=x^{\frac{m-1}{2}}y,\quad \psi=x^{\frac{m+1}{2}}f(\eta),\quad u_w=ax^m, \tag{5.14}$$

where η and $f(\eta)$ are the new independent and dependent variables, respectively, and a is a pure constant having suitable dimensions. These variables shall be called the similarity variables if they successfully transform the system (5.6)–(5.7) of partial differential equations to an equivalent system of ordinary differential equations. The utilization of Eq. (5.14) in the system (5.6)–(5.7) immediately gives the self-similar system of the form

$$mf'^2-\left(\frac{m+1}{2}\right)ff''=\nu f''', \tag{5.15}$$

$$f(0)=0,\quad f'(0)=1,\quad f'(\infty)=0, \tag{5.16}$$

where the previous variables have successfully been removed. Therefore, the variables η and $f(\eta)$ (given in Eq. 5.14) can safely be regarded as the similarity variables. Notice that in Eq. (5.14), the wall velocity follows the form $u_w=ax^m$ which is of the power-law type. Thus, the case $\alpha_1\neq 0$ results in the power-law wall velocity criterion of the moving continuous surface for which the resulting system (5.15)–(5.16) is self-similar. Notice that Eq. (5.15) is exactly the same as it is for the Falkner–Skan [9] flow. This can also be written as

$$\frac{2m}{m+1}f'^2 - ff'' = \nu f''', \tag{5.17}$$

if the variable η is scaled by a factor of $\sqrt{\frac{m+1}{2}}$.

Case II $(\boldsymbol{\alpha_1 = 0})$
In this case, the system (5.11) reduces to the form

$$2\alpha_2 - 2\alpha_3 = 3\alpha_2 - \alpha_3, \quad \alpha_2 - \alpha_3 = -\alpha_4,$$

which admits a non-trivial solution, given by

$$\alpha_3 = -\alpha_2, \quad \alpha_4 = -2\alpha_2.$$

Here α_2 serves as the free variable, whereas α_3 and α_4 are determined using α_2. Let us put $-2\alpha_2 = m$, where m is an arbitrary constant and is local to this case, due to which the above solution is modified as

$$\alpha_2 = -\frac{m}{2}, \quad \alpha_3 = \frac{m}{2}, \quad \alpha_4 = m. \tag{5.18}$$

For the case $\alpha_1 = 0$, the similarity variables are constructed as (see Eq. 3.15):

$$\eta = \frac{y}{e^{\alpha_2 x}}, \quad f(\eta) = \frac{\psi(x,y)}{e^{\alpha_3 x}}, \quad a \equiv \text{const.} = \frac{u_w(x)}{e^{\alpha_4 x}}. \tag{5.19}$$

Substituting the values of $\alpha_i (i = 2, 3, 4)$ from Eq. (5.18) into Eq. (5.19), one gets the new variables of the form

$$\eta = y e^{\frac{m}{2}x}, \quad \psi = e^{\frac{m}{2}x} f(\eta), \quad u_w = a e^{mx}, \tag{5.20}$$

which transform the system (5.6)–(5.7) to the form

$$m\left(f'^2 - \frac{1}{2}ff''\right) = \nu f''', \tag{5.21}$$

$$f(0) = 0, \quad f'(0) = 1, \quad f'(\infty) = 0. \tag{5.22}$$

Evidently, Eqs. (5.21) and (5.22) are independent of the previous (original) variables. Therefore, the transformations (5.20) can be regarded as similarity transformations and the system (5.21)–(5.22) as the self-similar one. In this case, the wall velocity came out of the exponential form, i.e., $u_w = ae^{mx}$, where m is an arbitrary constant exponent. Thus, the case $\alpha_1 = 0$ leads to another type of similarity solutions for this problem caused by the exponentially varying motion of the continuous surface.

In the available literature, only the case for $m = 1$ has been discussed in the case of exponentially stretching or shrinking wall velocities. The consideration of other values of m finally completes this class of self-similar flows. This, ignorance of the other values of m, was actually the "incompletion" which we have pointed out in the start of this chapter regarding the two-dimensional flows due to moving continuous surfaces. However, it is again emphasized that, besides the use of the word "complete," we do not claim the nonexistence of any other self-similar solution to this case. The meanings of this completion over here are in the sense that we have explored the self-similar solutions completely, corresponding to the chosen group of scalings.

At the end of this section, it is quite important to summarize that the implementation of group theoretical procedure resulted in two self-similar forms of the two-dimensional boundary-layer equations with the restriction that the wall velocity $u_w(x)$ must either follow the power-law form $(u_w = ax^m)$ or the exponential form $(u_w = a\mathrm{e}^{mx})$. That is, if the wall velocity is taken either of the forms the flow will be self-similar; otherwise, it will be non-similar. Such a restriction on the form of variable wall velocity is actually regarded as the criterion of self-similarity to this case.

5.2 Three-Dimensional Flow

In the continuation of above, two-dimensional case, it is again assumed that the fluid of our interest is viscous and incompressible following the Newton's law of viscosity. A semi-infinite body of fluid is assumed to be occupying the upper half space and bounded by an infinite flexible sheet situated at $y = 0$. The flow is assumed to be caused by the variable motion of the sheet surface in two lateral directions. The flow geometry and the chosen system of coordinates are shown in Fig. 5.2.

This type of flow was first considered by Wang [5] in 1984 where he assumed uniform stretching velocities in the two lateral directions and obtained a self-similar solution. Unsteady case of this flow due to an impulsively started stretching sheet was considered by Takhar et al. [10]. Another three-dimensional flow due to a

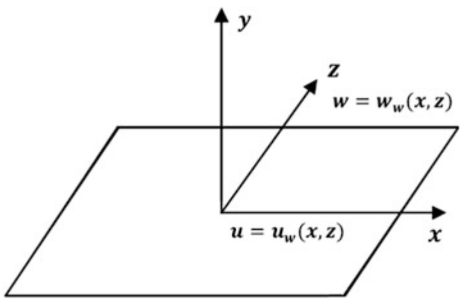


Fig. 5.2 Three-dimensional flow schematic and the associated system of coordinates

stretching sheet was also considered by Wang [11] where he considered unidirectional stretching of the sheet in a rotating fluid. Recently, Liu et al. [12] assumed the exponential stretching wall velocities in the two lateral directions and obtained a self-similar solution to this case. This had been very much unfortunate that the three-dimensional flow due to nonlinear (power-law) wall velocities and the other powers (other than 1) of the exponential wall velocities have not been given any attention, so far. It is therefore important to consider a flow in which the lateral wall velocities have been assumed to be the general functions of x and z coordinates, that is $u_w(x,z)$ and $w_w(x,z)$. The group theoretical procedure will be employed to determine the similarity criterion for the wall velocities by determining the explicit forms of the functions $u_w(x,z)$ and $w_w(x,z)$.

Based on the above assumptions, the flow is essentially three-dimensional owing to the boundary-layer character. Therefore, the velocity vector for such a steady three-dimensional flow reads as

$$\boldsymbol{V} = [u(x,y,z), v(x,y,z), w(x,y,z)]. \tag{5.23}$$

Compatible to this velocity vector, the governing system comprises of Eqs. (2.10)–(2.12) and the appropriate boundary conditions are described as

$$\left.\begin{array}{ll} u = u_w(x,z), v = 0, w = w_w(x,z), & \text{at} \quad y = 0 \\ u = 0,\ w = 0, & \text{at} \quad y = \infty \end{array}\right\}. \tag{5.24}$$

Similar to the previous section, the continuity equation can be made satisfied identically by introducing the two stream functions $\psi(x,y,z)$ and $\phi(x,y,z)$ which have the following relations with velocity components:

$$u = \frac{\partial\psi}{\partial y},\quad w = \frac{\partial\phi}{\partial y},\quad v = -\left(\frac{\partial\psi}{\partial x} + \frac{\partial\phi}{\partial z}\right). \tag{5.25}$$

The above stream functions were first introduced by Moore [13] and were further refined by Geis [14] for the rectangular Cartesian coordinates. Due to (5.25), the equation of continuity (2.10) satisfies identically by reducing the number of unknowns from three to two. Consequently, Eqs. (2.11), (2.12), and (5.24) readily transform to the new form

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \frac{\partial\phi}{\partial z}\frac{\partial^2\psi}{\partial y^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\psi}{\partial y\partial z} = \nu\frac{\partial^3\psi}{\partial y^3}, \tag{5.26}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\phi}{\partial y^2} - \frac{\partial\phi}{\partial z}\frac{\partial^2\phi}{\partial y^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y\partial z} = \nu\frac{\partial^3\phi}{\partial y^3}, \tag{5.27}$$

and

$$\left.\begin{array}{lll} \frac{\partial\psi}{\partial y}=u_w(x,z),\ \ \frac{\partial\phi}{\partial y}=w_w(x,z),\ \ \frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial z}=0, & \text{at} & y=0 \\ \frac{\partial\psi}{\partial y}=0,\ \ \frac{\partial\phi}{\partial y}=0, & \text{at} & y=\infty \end{array}\right\}. \tag{5.28}$$

Let us consider a one-parameter group of scalings, of all the involved independent and dependent variables, of the form

$$\bar{x}=k^{\alpha_1}x,\ \bar{y}=k^{\alpha_2}y,\ \bar{z}=k^{\alpha_3}z,\ \bar{\psi}=k^{\alpha_4}\psi,\ \bar{\phi}=k^{\alpha_5}\phi,\ \bar{u}_w=k^{\alpha_6}u_w,\ \bar{w}_w=k^{\alpha_7}w_w, \tag{5.29}$$

where k is the scaling parameter and $\alpha_i(i=1,\ldots,7)$ are the scaling exponents. In this case too, the procedure is exactly the same as was implemented in the previous section. The details of the procedure can, therefore, be skipped by retaining the major steps. However, the author feels it necessary and useful, especially for the students, to proceed with a bit more detail in order to facilitate the reader. Moreover, this chapter in general and this section in particular include the crux of this book; therefore, a bit more detail seems not that costly. The extra burden of this chapter will be compensated in the forthcoming chapters. The substitution of (5.29) into the system (5.26)–(5.28) results in the following system in bared notation:

$$\begin{aligned} &k^{\alpha_1+2\alpha_2-2\alpha_4}\left(\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{x}\partial\bar{y}}-\frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}\right)-k^{2\alpha_2+\alpha_3-\alpha_4-\alpha_5}\left(\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}^2}-\frac{\partial\bar{\phi}}{\partial\bar{y}}\frac{\partial^2\bar{\psi}}{\partial\bar{y}\partial\bar{z}}\right)\\ &=\nu k^{3\alpha_2-\alpha_4}\frac{\partial^3\bar{\psi}}{\partial\bar{y}^3}, \end{aligned} \tag{5.30}$$

$$\begin{aligned} &k^{\alpha_1+2\alpha_2-\alpha_4-\alpha_5}\left(\frac{\partial\bar{\psi}}{\partial\bar{y}}\frac{\partial^2\bar{\phi}}{\partial\bar{x}\partial\bar{y}}-\frac{\partial\bar{\psi}}{\partial\bar{x}}\frac{\partial^2\bar{\phi}}{\partial\bar{y}^2}\right)-k^{2\alpha_2+\alpha_3-2\alpha_5}\left(\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial^2\bar{\phi}}{\partial\bar{y}^2}-\frac{\partial\bar{\phi}}{\partial\bar{y}}\frac{\partial^2\bar{\phi}}{\partial\bar{y}\partial\bar{z}}\right)\\ &=\nu k^{3\alpha_2-\alpha_5}\frac{\partial^3\bar{\phi}}{\partial\bar{y}^3}, \end{aligned} \tag{5.31}$$

and

$$\left.\begin{array}{ll} k^{\alpha_2-\alpha_4}\frac{\partial\bar{\psi}}{\partial\bar{y}}=k^{-\alpha_6}\bar{u}_w,k^{\alpha_2-\alpha_5}\frac{\partial\bar{\phi}}{\partial\bar{y}}=k^{-\alpha_7}\bar{w}_w,k^{\alpha_1-\alpha_4}\frac{\partial\bar{\psi}}{\partial\bar{x}}+k^{\alpha_3-\alpha_5}\frac{\partial\bar{\phi}}{\partial\bar{z}}=0, & \text{at}\ \ \bar{y}=0 \\ \frac{\partial\bar{\psi}}{\partial\bar{y}}=0,\ \ \frac{\partial\bar{\phi}}{\partial\bar{y}}=0, & \text{at}\ \ \bar{y}=\infty \end{array}\right\}. \tag{5.32}$$

The requirement of invariance of the system (5.26)–(5.28), under the group of scalings (5.29), requires that the system (5.30)–(5.32) must be free from the constant coefficients appearing in the powers of k. This is certainly possible if the following linear system of equations holds:

$$\alpha_1 + 2\alpha_2 - 2\alpha_4 = 2\alpha_2 + \alpha_3 - \alpha_4 - \alpha_5 = 3\alpha_2 - \alpha_4, \tag{5.33}$$

$$\alpha_1 + 2\alpha_2 - \alpha_4 - \alpha_5 = 2\alpha_2 + \alpha_3 - 2\alpha_5 = 3\alpha_2 - \alpha_5, \tag{5.34}$$

$$\alpha_2 - \alpha_4 = -\alpha_6, \alpha_2 - \alpha_5 = -\alpha_7, \alpha_1 - \alpha_4 = \alpha_3 - \alpha_5. \tag{5.35}$$

Note that this case involves three independent variables; in order to transform the system (5.26)–(5.28) to an equivalent system of ordinary differential equations, one must eliminate two independent variables from the original three. Before solving the system (5.33)–(5.35), it is, therefore, quite important to decide for the variable to be eliminated first. Being slack within the boundary-layer, one among the x and z can equally be chosen and this choice will not affect the final result. We prefer to choose x to be eliminated first. In this way, two cases arise corresponding to the zero and nonzero character of the scaling exponent α_1.

Case I ($\alpha_1 \neq 0$)
Dividing the system (5.33)–(5.35) by $\alpha_1 (\neq 0)$ throughout and solving subsequently, one finds

$$\frac{\alpha_2}{\alpha_1} = A, \ \frac{\alpha_3}{\alpha_1} = B, \ \frac{\alpha_4}{\alpha_1} = 1 - A, \ \frac{\alpha_5}{\alpha_1} = B - A, \ \frac{\alpha_6}{\alpha_1} = 1 - 2A, \ \frac{\alpha_7}{\alpha_1} = B - 2A, \tag{5.36}$$

due to which the new variables are constructed as:

$$\xi = \frac{z}{x^{\alpha_3/\alpha_1}}, \quad \eta = \frac{y}{x^{\alpha_2/\alpha_1}}, \quad F(\xi, \eta) = \frac{\psi(x, y, z)}{x^{\alpha_4/\alpha_1}}, \quad G(\xi, \eta) = \frac{\phi(x, y, z)}{x^{\alpha_5/\alpha_1}},$$
$$F_w = \frac{u_w(x, z)}{x^{\alpha_6/\alpha_1}}, \quad G_w = \frac{w_w(x, z)}{x^{\alpha_7/\alpha_1}}. \tag{5.37}$$

Assuming that $1 - 2A = m$ (an arbitrary constant), we finally get

$$\xi = x^{-B} z, \quad \eta = x^{\frac{m-1}{2}} y, \quad \psi = x^{\frac{m+1}{2}} F(\xi, \eta), \quad \phi = x^{B - \frac{m-1}{2}} G(\xi, \eta), \tag{5.38}$$

with

$$u_w(x, z) = x^m F_w(\xi), \quad w_w(x, z) = x^{B-(m-1)} G_w(\xi), \tag{5.39}$$

where B is also an arbitrary constant which can also be chosen equal to zero. For the sake of generality, B will be treated as nonzero in the further proceedings.

At this stage, the variables x has been absorbed in the new independent variables ξ and η which are now $2(= 3 - 1)$ in number. The above new variables (5.38) and (5.39) transform the system (5.26)–(5.28) to the form

$$\begin{aligned} m\left(\frac{\partial F}{\partial \eta}\right)^2 - B\xi\frac{\partial F}{\partial \eta}\frac{\partial^2 F}{\partial \xi \partial \eta} - \left(\frac{m+1}{2}\right)F\frac{\partial^2 F}{\partial \eta^2} + B\xi\frac{\partial F}{\partial \xi}\frac{\partial^2 F}{\partial \eta^2} \\ - \frac{\partial G}{\partial \xi}\frac{\partial^2 F}{\partial \eta^2} + \frac{\partial G}{\partial \eta}\frac{\partial^2 F}{\partial \xi \partial \eta} = v\frac{\partial^3 F}{\partial \eta^3}, \end{aligned} \tag{5.40}$$

$$\begin{aligned} (B+m-1)\frac{\partial F}{\partial \eta}\frac{\partial G}{\partial \eta} - B\xi\frac{\partial F}{\partial \eta}\frac{\partial^2 G}{\partial \xi \partial \eta} - \left(\frac{m+1}{2}\right)F\frac{\partial^2 G}{\partial \eta^2} + B\xi\frac{\partial F}{\partial \xi}\frac{\partial^2 G}{\partial \eta^2} \\ - \frac{\partial G}{\partial \xi}\frac{\partial^2 G}{\partial \eta^2} + \frac{\partial G}{\partial \eta}\frac{\partial^2 G}{\partial \xi \partial \eta} = v\frac{\partial^3 G}{\partial \eta^3}, \end{aligned} \tag{5.41}$$

and

$$\left.\begin{array}{ll} \frac{\partial F}{\partial \eta} = F_w(\xi),\ \ \frac{\partial G}{\partial \eta} = G_w(\xi),\ \ \left(\frac{m+1}{2}\right)F - B\xi\frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \xi} = 0, & \text{at} \quad \eta = 0 \\ \frac{\partial F}{\partial \eta} = 0,\ \ \frac{\partial G}{\partial \eta} = 0, & \text{at} \quad \eta = \infty \end{array}\right\}. \tag{5.42}$$

Evidently, the variable x has been eliminated successfully indicating that the process can be continued further for the elimination of one more independent variable. For doing so, we again assume a one-parameter group of scalings of the variables involved in the system (5.40)–(5.42):

$$\bar{\xi} = k^{\beta_1}\xi,\ \ \bar{\eta} = k^{\beta_2}\eta,\ \ \bar{F} = k^{\beta_3}F,\ \ \bar{G} = k^{\beta_4}G,\ \ \bar{F}_w = k^{\beta_5}F_w,\ \ \bar{G}_w = k^{\beta_6}G_w, \tag{5.43}$$

where $\beta_i (i = 1, \ldots, 6)$ are the scaling exponents to be determined. The group (5.43) transforms the system (5.40)–(5.42) to the form

$$\begin{aligned} k^{2\beta_2-2\beta_3}\left[m\left(\frac{\partial \bar{F}}{\partial \bar{\eta}}\right)^2 - B\bar{\xi}\frac{\partial \bar{F}}{\partial \bar{\eta}}\frac{\partial^2 \bar{F}}{\partial \bar{\xi}\partial \bar{\eta}} - \left(\frac{m+1}{2}\right)\bar{F}\frac{\partial^2 \bar{F}}{\partial \bar{\eta}^2} + B\bar{\xi}\frac{\partial \bar{F}}{\partial \bar{\xi}}\frac{\partial^2 \bar{F}}{\partial \bar{\eta}^2}\right] \\ - k^{\beta_1+2\beta_2-\beta_3-\beta_4}\left[\frac{\partial \bar{G}}{\partial \bar{\xi}}\frac{\partial^2 \bar{F}}{\partial \bar{\eta}^2} - \frac{\partial \bar{G}}{\partial \bar{\eta}}\frac{\partial^2 \bar{F}}{\partial \bar{\xi}\partial \bar{\eta}}\right] = vk^{3\beta_2-\beta_3}\frac{\partial^3 \bar{F}}{\partial \bar{\eta}^3}, \end{aligned} \tag{5.44}$$

$$\begin{aligned} k^{2\beta_2-\beta_3-\beta_4}\left[(B+m-1)\frac{\partial \bar{F}}{\partial \bar{\eta}}\frac{\partial \bar{G}}{\partial \bar{\eta}} - B\bar{\xi}\frac{\partial \bar{F}}{\partial \bar{\eta}}\frac{\partial^2 \bar{G}}{\partial \bar{\xi}\partial \bar{\eta}} - \left(\frac{m+1}{2}\right)\bar{F}\frac{\partial^2 \bar{G}}{\partial \bar{\eta}^2} + B\bar{\xi}\frac{\partial \bar{F}}{\partial \bar{\xi}}\frac{\partial^2 \bar{G}}{\partial \bar{\eta}^2}\right] \\ - k^{\beta_1+2\beta_2-2\beta_4}\left[\frac{\partial \bar{G}}{\partial \bar{\xi}}\frac{\partial^2 \bar{G}}{\partial \bar{\eta}^2} - \frac{\partial \bar{G}}{\partial \bar{\eta}}\frac{\partial^2 \bar{G}}{\partial \bar{\xi}\partial \bar{\eta}}\right] = vk^{3\beta_2-\beta_4}\frac{\partial^3 \bar{G}}{\partial \bar{\eta}^3}, \end{aligned} \tag{5.45}$$

and

$$\left.\begin{array}{ll}\left.\begin{array}{l}k^{\beta_2-\beta_3}\frac{\partial \bar{F}}{\partial \bar{\eta}}=k^{-\beta_5}\bar{F}_w,\quad k^{\beta_2-\beta_4}\frac{\partial \bar{G}}{\partial \bar{\eta}}=k^{-\beta_6}\bar{G}_w,\\ k^{-\beta_3}\left[\left(\frac{m+1}{2}\right)\bar{F}-B\bar{\xi}\frac{\partial \bar{F}}{\partial \bar{\xi}}\right]+k^{\beta_1-\beta_4}\frac{\partial \bar{G}}{\partial \bar{\xi}}=0,\end{array}\right\} & \text{at}\quad \bar{\eta}=0\\ \frac{\partial \bar{F}}{\partial \bar{\eta}}=0,\quad \frac{\partial \bar{G}}{\partial \bar{\eta}}=0, & \text{at}\quad \bar{\eta}=\infty\end{array}\right\}. \tag{5.46}$$

The requirement of invariance of the system (5.40)–(5.42) under the group (5.43) leads to the system of linear algebraic equations in $\beta_i (i = 1, \ldots, 6)$ given as

$$2\beta_2 - 2\beta_3 = \beta_1 + 2\beta_2 - \beta_3 - \beta_4 = 3\beta_2 - \beta_3, \tag{5.47}$$

$$2\beta_2 - \beta_3 - \beta_4 = \beta_1 + 2\beta_2 - 2\beta_4 = 3\beta_2 - \beta_4, \tag{5.48}$$

$$\beta_2 - \beta_3 = -\beta_5, \beta_2 - \beta_4 = -\beta_6, \beta_1 - \beta_4 = -\beta_3. \tag{5.49}$$

Again, before solving the system (5.47)–(5.49) the selection of the leaving variable is mandatory. In view of the expressions of ξ and η given in Eq. (5.38), the natural choice is ξ. In the group of scalings (5.43), β_1 is the scaling exponent of the variable ξ. The zero and nonzero character of β_1 is again of particular importance in the construction of new variables. Before we continue with the solution of the system (5.47)–(5.49), it is worth remembering that we are already proceeding the Case I ($\alpha_1 \neq 0$) and the ongoing process is actually a part of Case I. Therefore, the cases $\beta_1 \neq 0$ and $\beta_1 = 0$ need to be designated as the subcases of Case I. Thus, from now onward the cases $\beta_1 \neq 0$ and $\beta_1 = 0$ shall, respectively, be designated as

Case I; Subcase I ($\alpha_1 \neq 0$; $\beta_1 \neq 0$),
Case I; Subcase II ($\alpha_1 \neq 0$; $\beta_1 = 0$).

Case I; Subcase I ($\alpha_1 \neq 0$; $\beta_1 \neq 0$)
The nonzero character of β_1 allows the division of the system (5.47)–(5.49) by β_1 everywhere. This results in the following system of algebraic equations

$$2\frac{\beta_2}{\beta_1} - 2\frac{\beta_3}{\beta_1} = 1 + 2\frac{\beta_2}{\beta_1} - \frac{\beta_3}{\beta_1} - \frac{\beta_4}{\beta_1} = 3\frac{\beta_2}{\beta_1} - \frac{\beta_3}{\beta_1}, \tag{5.50}$$

$$2\frac{\beta_2}{\beta_1} - \frac{\beta_3}{\beta_1} - \frac{\beta_4}{\beta_1} = 1 + 2\frac{\beta_2}{\beta_1} - 2\frac{\beta_4}{\beta_1} = 3\frac{\beta_2}{\beta_1} - \frac{\beta_4}{\beta_1}, \tag{5.51}$$

$$\frac{\beta_2}{\beta_1} - \frac{\beta_3}{\beta_1} = -\frac{\beta_5}{\beta_1},\quad \frac{\beta_2}{\beta_1} - \frac{\beta_4}{\beta_1} = -\frac{\beta_6}{\beta_1},\quad 1 - \frac{\beta_4}{\beta_1} = -\frac{\beta_3}{\beta_1}, \tag{5.52}$$

which ultimately solves as

$$\frac{\beta_3}{\beta_1} = -\frac{\beta_2}{\beta_1},\quad \frac{\beta_4}{\beta_1} = 1 - \frac{\beta_2}{\beta_1},\quad \frac{\beta_5}{\beta_1} = -2\frac{\beta_2}{\beta_1},\quad \frac{\beta_6}{\beta_1} = 1 - 2\frac{\beta_2}{\beta_1}. \tag{5.53}$$

Having this solution in hand, the new variables are thus constructed as

$$\zeta = \xi^{\frac{n-1}{2}}\eta, \quad F(\xi,\eta) = \xi^{\frac{n-1}{2}}f(\zeta), \quad G(\xi,\eta) = \xi^{\frac{n+1}{2}}g(\zeta), \tag{5.54}$$

where the wall profiles read as

$$F_w(\xi) = a\xi^{n-1}, \quad G_w(\xi) = b\xi^n. \tag{5.55}$$

where n denotes an arbitrary (dimensionless) constant constructed as

$$1 - 2\frac{\beta_2}{\beta_1} = n. \tag{5.56}$$

Furthermore, the constants a and b are also arbitrary, having suitable dimensions which are usually referred to as the stretching or shrinking rates. Consequently, the system (5.40)–(5.42) transforms as:

$$\begin{aligned}(m - B(n-1))f'^2 - \left(\frac{m+1}{2} - B\left(\frac{n-1}{2}\right)\right)ff'' - \left(\frac{n+1}{2}\right)gf'' + (n-1)f'g' \\ = \nu f''',\end{aligned} \tag{5.57}$$

$$(m - 1 - B(n-1))f'g' - \left(\frac{m+1}{2} - B\left(\frac{n-1}{2}\right)\right)fg'' - \left(\frac{n+1}{2}\right)gg'' + ng'^2 = \nu g''', \tag{5.58}$$

and

$$\left.\begin{array}{llll} f' = a, & g' = b, & \left(\frac{m+1}{2} - B\left(\frac{n-1}{2}\right)\right)f + \left(\frac{n+1}{2}\right)g = 0, & \text{at} \quad \zeta = 0 \\ f' = 0, & g' = 0, & & \text{at} \quad \zeta = \infty \end{array}\right\}, \tag{5.59}$$

where the $'$ denotes differentiation with respect to ζ. Clearly, the system (5.57)–(5.59) is a system of ordinary differential equations from where the, absorbed (previous), independent variables have completely been eliminated.

Since the scaling groups of one-parameter transformations have been utilized, therefore, the reduction in the number of independent variables at each step is also one. This is the reason for the elimination of x and z in two steps. However, it is also possible to eliminate more than one variable at once; for doing so, one must utilize the multi-parameter group of scalings. The details of such a procedure can be found in Refs. [15, 16]. After having determined the suitable similarity transformations due to one-parameter group of scalings, it then stays not necessary to transform the original system (5.26)–(5.28) to the self-similar form (5.57)–(5.59) in two steps, essentially. An integrated set of similarity transformations is possible to obtain by combining Eqs. (5.38) and (5.54) as

$$\zeta = z^{\frac{n-1}{2}} x^{\frac{m-1}{2} - B\left(\frac{n-1}{2}\right)} y, \quad \psi = z^{\frac{n-1}{2}} x^{\frac{m+1}{2} - B\left(\frac{n-1}{2}\right)} f(\zeta), \quad \phi = z^{\frac{n+1}{2}} x^{\frac{m-1}{2} - B\left(\frac{n-1}{2}\right)} g(\zeta), \tag{5.60}$$

which can directly transform the system (5.26)–(5.28) to the self-similar form (5.57)–(5.59) in a single step. Similarly, the obtained wall velocities after combining Eqs. (5.39) and (5.55) in their final form read as

$$\left.\begin{aligned} u_w &= a x^{m-B(n-1)} z^{n-1} \\ w_w &= b x^{m-1-B(n-1)} z^{n} \end{aligned}\right\}. \tag{5.61}$$

Case I; Subcase II ($\alpha_1 \neq 0,\ \beta_1 = 0$)
Substitution of $\beta_1 = 0$ does not affect the system (5.47)–(5.49) by any large. The resulting system does admit a non-trivial solution of the form

$$\beta_3 = -\beta_2, \quad \beta_4 = -\beta_2, \quad \beta_5 = -2\beta_2, \quad \beta_6 = -2\beta_2. \tag{5.62}$$

This solution can also be recovered by multiplying Eq. (5.53) by β_1 and substituting $\beta_1 = 0$ subsequently. In this case, the new variables are constructed as

$$\zeta = \frac{\eta}{e^{\beta_2 \xi}}, \quad f(\zeta) = \frac{F(\xi,\eta)}{e^{\beta_3 \xi}}, \quad g(\zeta) = \frac{G(\xi,\eta)}{e^{\beta_4 \xi}}, \tag{5.63}$$

and the wall velocities come out of the form

$$a \equiv \text{const.} = \frac{F_w(\xi)}{e^{\beta_5 \xi}}, \quad b \equiv \text{const.} = \frac{G_w(\xi)}{e^{\beta_6 \xi}}, \tag{5.64}$$

where a and b are constants having suitable dimensions. With the aid of Eq. (5.62), one explicitly finds from Eqs. (5.63) and (5.64) that

$$\zeta = e^{\frac{n}{2}\xi} \eta, \quad F(\xi,\eta) = e^{\frac{n}{2}\xi} f(\zeta), \quad G(\xi,\eta) = e^{\frac{n}{2}\xi} g(\zeta), \tag{5.65}$$

and

$$F_w(\xi) = a e^{n\xi}, \quad G_w(\xi) = b e^{n\xi}, \tag{5.66}$$

where n is an arbitrary (dimensionless) constant defined by

$$-2\beta_2 = n. \tag{5.67}$$

The use of similarity variables (5.65) transforms the system of partial differential Eqs. (5.40) and (5.41) to the system of ordinary differential equations, given by

$$m f'^2 - \left(\frac{m+1}{2}\right) f f'' - \frac{n}{2} g f'' + n f' g' - B n \xi \left(f'^2 - \frac{1}{2} f f''\right) = \nu f''', \tag{5.68}$$

$$(B+m-1)f'g' - \left(\frac{m+1}{2}\right)fg'' - \frac{n}{2}gg'' + ng'^2 - Bn\xi\left(f'g' - \frac{1}{2}fg''\right) = \nu g''', \tag{5.69}$$

from which the previous variable ξ has not been eliminated completely. However, the elimination of ξ can be ensured by choosing the arbitrary constant

$$B = 0. \tag{5.70}$$

This is important to remember that this particular choice of B is particular to this case only and does not apply to other cases in general. By doing so, the self-similar system reads as:

$$mf'^2 - \left(\frac{m+1}{2}\right)ff'' - \frac{n}{2}gf'' + nf'g' = \nu f''', \tag{5.71}$$

$$(m-1)f'g' - \left(\frac{m+1}{2}\right)fg'' - \frac{n}{2}gg'' + ng'^2 = \nu g'''. \tag{5.72}$$

The use of the similarity variables (5.65) and the so determined wall velocity laws (5.66) transform the boundary conditions (5.42) to the form

$$\left.\begin{array}{llll} f' = a, & g' = b, & \left(\frac{m+1}{2}\right)f + \frac{n}{2}g = 0, & \text{at} \quad \zeta = 0 \\ f' = 0, & g' = 0, & & \text{at} \quad \zeta = \infty \end{array}\right\}. \tag{5.73}$$

The unified transformations for this case are obtained by combining (5.38) and (5.65) as

$$\zeta = x^{\frac{m-1}{2}} e^{\frac{n}{2}zx^{-B}} y, \quad \psi = x^{\frac{m+1}{2}} e^{\frac{n}{2}zx^{-B}} f(\zeta), \quad \phi = x^{B+\frac{m-1}{2}} e^{\frac{n}{2}zx^{-B}} g(\zeta),$$

which cannot serve as similarity transformation until $B = 0$. Owing to Eq. (5.70), the above transformation takes the form

$$\zeta = x^{\frac{m-1}{2}} e^{\frac{n}{2}z} y, \quad \psi = x^{\frac{m+1}{2}} e^{\frac{n}{2}z} f(\zeta), \quad \phi = x^{\frac{m-1}{2}} e^{\frac{n}{2}z} g(\zeta), \tag{5.74}$$

which can directly transform original Eqs. (5.26)–(5.28) to the self-similar form (5.71)–(5.73). Accordingly, the compact form of the wall velocity functions can also be obtained by combining Eq. (5.66) with Eq. (5.39). Consequently, after incorporating Eq. (5.70), one finally obtains

$$u_w = ax^m e^{nz}, \quad w_w = bx^{m-1} e^{nz}. \tag{5.75}$$

To this end, the Case I ($\alpha_1 \neq 0$) and the two subcases of it corresponding to $\beta_1 \neq 0$ and $\beta_1 = 0$ which were named as "Case I; Subcase I" and "Case I;

Subcase II," respectively, have been completed. Further proceeding of the procedure requires the reconsideration of the system (5.33)–(5.35) for the case $\alpha_1 = 0$. In the series of "main cases," this case is designated as "Case II ($\alpha_2 = 0$)."

Case II ($\alpha_2 = 0$)
With the substitution $\alpha_2 = 0$, the system (5.33)–(5.35) admits the (non-trivial) solution of the form

$$\alpha_4 = -\alpha_2,\ \alpha_5 = \alpha_3 - \alpha_2,\ \alpha_6 = -2\alpha_2,\ \alpha_7 = -2\alpha_2 + \alpha_3, \tag{5.76}$$

where α_2 and α_3 are the free variables, hence arbitrary. Substituting

$$-2\alpha_2 = m \text{ and } \alpha_3 = B, \tag{5.77}$$

the new variables, constructed in the same way as did in the previous cases, come out to be

$$\xi = \mathrm{e}^{-Bx}z,\ \ \eta = \mathrm{e}^{\frac{m}{2}x}y,\ \ \psi = \mathrm{e}^{\frac{m}{2}x}F(\xi,\eta),\ \ \phi = \mathrm{e}^{\left(B+\frac{m}{2}\right)x}G(\xi,\eta). \tag{5.78}$$

The wall velocities also involve the exponential functions and come out of the form

$$u_w = \mathrm{e}^{mx}F_w(\xi),\ \ w_w = \mathrm{e}^{(B+m)x}G_w(\xi). \tag{5.79}$$

With the aid of these new variables, the original system (5.26)–(5.28) transforms to the following new system involving two independent variables:

$$\begin{aligned} m\left(\frac{\partial F}{\partial \eta}\right)^2 - \frac{m}{2}F\frac{\partial^2 F}{\partial \eta^2} - \frac{\partial G}{\partial \xi}\frac{\partial^2 F}{\partial \eta^2} + \frac{\partial G}{\partial \eta}\frac{\partial^2 F}{\partial \xi \partial \eta} \\ - B\xi\left(\frac{\partial F}{\partial \eta}\frac{\partial^2 F}{\partial \xi \partial \eta} - \frac{\partial F}{\partial \xi}\frac{\partial^2 F}{\partial \eta^2}\right) = \nu\frac{\partial^3 F}{\partial \eta^3}, \end{aligned} \tag{5.80}$$

$$\begin{aligned} (B+m)\frac{\partial F}{\partial \eta}\frac{\partial G}{\partial \eta} - \frac{m}{2}F\frac{\partial^2 G}{\partial \eta^2} - \frac{\partial G}{\partial \xi}\frac{\partial^2 G}{\partial \eta^2} + \frac{\partial G}{\partial \eta}\frac{\partial^2 G}{\partial \xi \partial \eta} \\ - B\xi\left(\frac{\partial F}{\partial \eta}\frac{\partial^2 G}{\partial \xi \partial \eta} - \frac{\partial F}{\partial \xi}\frac{\partial^2 G}{\partial \eta^2}\right) = \nu\frac{\partial^3 G}{\partial \eta^3}, \end{aligned} \tag{5.81}$$

$$\left.\begin{array}{ll} \frac{\partial F}{\partial \eta} = F_w(\xi),\ \frac{\partial G}{\partial \eta} = G_w(\xi),\ \frac{m}{2}F - B\xi\frac{\partial F}{\partial \xi} + \frac{\partial G}{\partial \xi} = 0, & \text{at} \quad \eta = 0 \\ \frac{\partial F}{\partial \eta} = 0,\ \frac{\partial G}{\partial \eta} = 0, & \text{at} \quad \eta = \infty \end{array}\right\}. \tag{5.82}$$

In order to eliminate ξ from the system (5.80)–(5.82), one again requires to follow the same procedure as was performed in Case I. It is worth noting that the names of the variables in (5.80)–(5.82) are, however, exactly the same as they are in (5.40)–(5.42), but are entirely different, in actual. Therefore, for the sake of doing mathematics the similarity in their symbolic names can be utilized in order to avoid the replication of similar things. In this way, staying limited to their symbolic names, we employ the group (5.43) to the above system (5.80)–(5.82). Transforming Eqs. (5.80)–(5.82) to the form of new variables, defined in Eq. (5.43), and imposing the restriction of invariance, one is finally left with the system of following linear equations:

$$2\beta_2 - 2\beta_3 = \beta_1 + 2\beta_2 - \beta_3 - \beta_4 = 3\beta_2 - \beta_3, \tag{5.83}$$

$$2\beta_2 - \beta_3 - \beta_4 = \beta_1 + 2\beta_2 - 2\beta_4 = 3\beta_2 - \beta_4, \tag{5.84}$$

$$\beta_2 - \beta_3 = -\beta_5,\ \beta_2 - \beta_4 = -\beta_6,\ \beta_1 - \beta_4 = -\beta_3. \tag{5.85}$$

Following the previous practice, let us decide to eliminate the variable ξ for which two cases arise regarding the zero and nonzero character of β_1. The cases $\beta_1 \neq 0$ and $\beta_1 = 0$ shall be referred to as the Subcase I and Subcase II, respectively. In the perspective of Case II ($\alpha_1 = 0$), they shall finally be referred to as "Case II; Subcase I" for $\alpha_1 = 0$ & $\beta_1 \neq 0$ and "Case II; Subcase II" for $\alpha_1 = 0$ & $\beta_1 = 0$.

Case II; Subcase I ($\alpha_1 = 0;\ \beta_1 \neq 0$)
Interestingly, the system (5.83)–(5.85) is the same as (5.47)–(5.49). Utilization of the assumption $\beta_1 \neq 0$ in the system (5.83)–(5.85) produces the same non-trivial solution as given in Eq. (5.53) with $\frac{\beta_2}{\beta_1}$ as an arbitrary constant. Therefore, the construction of new variables follows immediately from Eq. (5.54) and the expressions of the wall velocities are also exactly the same as given in Eq. (5.55). Thus, the new variables in this case read as

$$\zeta = \xi^{\frac{n-1}{2}}\eta,\ \ F(\xi,\eta) = \xi^{\frac{n-1}{2}} f(\zeta),\ \ G(\xi,\eta) = \xi^{\frac{n+1}{2}} g(\zeta), \tag{5.86}$$

and the wall velocities come out to be

$$F_w(\xi) = a\xi^{n-1},\ \ G_w(\xi) = b\xi^{n}, \tag{5.87}$$

with a and b serving as (pure) constants having suitable dimensions. In terms of new variables, Eqs. (5.80) and (5.81) take the form

$$(m - B(n-1))f'^2 - \frac{1}{2}(m - B(n-1))ff'' - \left(\frac{n+1}{2}\right)gf'' + (n-1)f'g' = vf''', \tag{5.88}$$

$$(m - B(n-1))f'g' - \frac{1}{2}(m - B(n-1))fg'' - \left(\frac{n+1}{2}\right)gg'' + ng'^2 = vg''', \tag{5.89}$$

and the boundary conditions (5.82), with the aid of Eq. (5.87), transform as

$$\left.\begin{array}{lr} f' = a,\ \ g' = b,\ \ \frac{1}{2}(m - B(n-1))f + \left(\frac{n+1}{2}\right)g = 0, & \text{at} \quad \zeta = 0 \\ f' = 0,\ \ g' = 0, & \text{at} \quad \zeta = \infty \end{array}\right\}. \tag{5.90}$$

Combination of Eq. (5.86) with Eq. (5.78) and of Eq. (5.87) with Eq. (5.79) gives, respectively, the unified form of the similarity variables and the associated wall velocities, as

$$\zeta = z^{\frac{n-1}{2}} e^{\frac{1}{2}(m-B(n-1))x} y,\ \ \psi = z^{\frac{n-1}{2}} e^{\frac{1}{2}(m-B(n-1))x} f(\zeta),\ \ \phi = z^{\frac{n+1}{2}} e^{\frac{1}{2}(m-B(n-1))x} g(\zeta), \tag{5.91}$$

and

$$u_w = a z^{n-1} e^{(m-B(n-1))x},\ \ w_w = b z^n e^{(m-B(n-1))x}. \tag{5.92}$$

It is evident that the system (5.88)–(5.90) is in the self-similar form and can also be recovered by applying the transformation (5.91) directly to the system (5.26)–(5.28).

Case II; Subcase II $(\alpha_1 = 0,\ \beta_1 = 0)$

The choice $\beta_1 = 0$ is the same as that in the case "Case I; Subcase II." Therefore, the system of linear Eqs. (5.83)–(5.85), at $\beta_1 = 0$, recovers to the same results as given in Eq. (5.62). Consequently, the new variables in the present case are exactly the same as those constructed in Eq. (5.65), in symbolic sense. Just to avoid any confusion, we prefer to write here the new variables of the present case given by

$$\zeta = e^{\frac{n}{2}\xi}\eta,\ \ F(\xi,\eta) = e^{\frac{n}{2}\xi} f(\zeta),\ \ G(\xi,\eta) = e^{\frac{n}{2}\xi} g(\zeta), \tag{5.93}$$

which are, of course, of the same form as given in Eq. (5.65), but are entirely different (from 5.65) in physical sense because of the different definitions of ξ and η in these two cases. Similarly, the wall velocity profiles of this case, given by

$$F_w(\xi) = a e^{n\xi},\ \ G_w(\xi) = b e^{n\xi}, \tag{5.94}$$

are also similar to those given in Eq. (5.66), in symbolic sense. According to the variables defined in Eq. (5.93) and the wall velocities given in Eq. (5.94), the system (5.80)–(5.82) readily transforms to the form

$$mf'^2 - \frac{m}{2}ff'' - \frac{n}{2}gf'' + nf'g' - Bn\xi\left(f'^2 - \frac{1}{2}ff''\right) = \nu f''', \tag{5.95}$$

$$(B+m)f'g' - \frac{m}{2}fg'' - \frac{n}{2}gg'' + ng'^2 - Bn\xi\left(f'g' - \frac{1}{2}fg''\right) = \nu g''', \tag{5.96}$$

with

$$\left.\begin{array}{llll} f' = a, & g' = b, & \frac{1}{2}(m - Bn\xi)f + \frac{1}{2}ng = 0, & \text{at} \quad \zeta = 0 \\ f' = 0, & g' = 0, & & \text{at} \quad \zeta = \infty \end{array}\right\}. \tag{5.97}$$

Again, similar to the case "Case I; Subcase II" the system (5.95)–(5.97) is not free from the previous variable ξ. Choosing $B = 0$, the self-similar system for the present case furnishes as

$$mf'^2 - \frac{m}{2}ff'' - \frac{n}{2}gf'' + nf'g' = \nu f''', \tag{5.98}$$

$$mf'g' - \frac{m}{2}fg'' - \frac{n}{2}gg'' + ng'^2 = \nu g'''. \tag{5.99}$$

$$\left.\begin{array}{llll} f' = a, & g' = b, & mf + ng = 0, & \text{at} \quad \zeta = 0 \\ f' = 0, & g' = 0 & & \text{at} \quad \zeta = \infty \end{array}\right\}. \tag{5.100}$$

Unification of Eqs. (5.93) and (5.78) gives the complete set of similarity variables (under the assumption $B = 0$)

$$\zeta = e^{\frac{1}{2}(mx+nz)}y, \quad \psi = e^{\frac{1}{2}(mx+nz)}f(\zeta), \quad \phi = e^{\frac{1}{2}(mx+nz)}g(\zeta), \tag{5.101}$$

due to which Eqs. (5.26)–(5.28) can directly be transformed to the system (5.98)–(5.100) in a single step. Combination of Eqs. (5.94) and (5.79) gives the final form of the velocity functions (after substituting $B = 0$)

$$u_w = ae^{mx+nz}, \quad w_w = be^{mx+nz}. \tag{5.102}$$

In the above calculations, the self-similar equations of the generalized three-dimensional flow, due to the bidirectional motion of the flexible continuous sheet, have been furnished. The utilized group theoretical approach splits itself into four separate cases and produces different self-similar systems for each case. Normally, in a case, when there is only one variable to be eliminated, as in the two-dimensional case, Sect. 5.1, two types of the wall velocity function are obtained, namely the power-law (given in Eq. 5.14) and the exponential form (given in Eq. 5.20). The nonzero case of the scaling exponent of the leaving variable results in the power-law wall velocity, and the vanishing value of such scaling exponent produces the exponential form of the wall velocity in order to end up with a self-similar system. In other words, this can also be rephrased as that these two cases corresponding to zero and nonzero character of the scaling variable promise to end up with a self-similar system provided that the wall velocities must follow either exponential form or the power-law form, respectively. From here, it can easily be perceived that if the self-similar solution to the boundary-layer equations exists, the wall velocities (or reference velocities) must follow either power-law or exponential form, in general. The converse of this does, however, not apply in general. This is because the self-similarity of the involved partial differential equations is strictly associated with the requirement of their reduction to the ordinary differential equations, free from the previous variables completely. From Eqs. (5.68)–(5.69) and Eqs. (5.95)–(5.96), it is obvious that the power-law or exponential form of the wall velocities did not guarantee the self-similarity of the whole system. Thus, in general, the requirement of the wall velocities to follow the power-law or exponential forms is the first ingredient and the requirement of reduction of the governing system of *pdes* to an equivalent system of *odes* by eliminating the previous variables completely is the second ingredient. These two fundamental ingredients of self-similarity are actually executed due to the similarity transformations. Hence, the appropriate wall velocity functions (power-law or exponential) and the suitable similarity transformations along with the condition of thorough elimination of the previous variables from the transformed system in new variables ensure the self-similarity of the boundary-layer equations. If any one of these ingredients is not fully achieved, the self-similarity cannot be guaranteed.

Finally, regarding the forms of wall the velocity functions in the perspective of self-similarity, it is concluded that wall velocities can never take any form other than the power-law or exponential ones. Neither a linear combination of the two families nor a linear combination of any two particular entities of the same family can be taken, in general, in order to obtain the self-similar solution. For example, the self-similar solution exists for the famous Falkner–Skan [9] equation if the potential flow follows the power-law form, i.e., $u_\infty(x) = ax^m$. The value $m = 0$ recovers the Blasius [17] case $u_\infty(x) = U_0 = \text{const.}$, while the value $m = 1 (u_\infty(x) = ax)$ recovers the wedge flow. It is well known that the similarity solution exists in these two cases. However, there is no self-similar solution for the combination of these two, that is, when the potential velocity is a combination of the two $(u_\infty(x) = U_0 \pm ax)$. This type of potential flow is particular to the famous Howarth's [18] non-similar flow.

In the currently treated three-dimensional flow, there were two variables x & z to be eliminated. In this case too, there were two choices for the wall velocities to follow, namely the power-law and the exponential one. In this case, the product of the two families has appeared as another possible form of the wall velocities as can be seen in Eqs. (5.61), (5.75), (5.92), and (5.102). This is because of the fact that the variables have been eliminated successively, and the exponential and the power-law cases of the descendent variables are automatically combined with either the exponential or the power-law cases of the preceding elimination. That is, the power-law in x has been combined by the power-law in z (Eq. 5.61) and exponential in z (Eq. 5.75); similarly, exponential wall velocity in x has been combined with the power-law in z (Eq. 5.92) and exponential in z (Eq. 5.102). Thus, the wall velocity functions defined in Eqs. (5.61), (5.75), (5.92), and (5.102) define the criterion of self-similarity for the wall velocities in three-dimensional flow due to a moving continuous flat surface. If the wall velocities deviate from the forms given in these equations, the self-similarity is not guaranteed.

5.3 Axially Symmetric Flow

In the previous two sections, the planner cases of two- and three-dimensional flows near the flat surfaces have been considered. In the case of axially symmetric flows, the solid surface of interest would be the solid body of revolution either flat or non-flat. In the case of non-flat surfaces, the surface curvature imparts significant effects on the flow characteristics within the boundary-layer. Consequently, the surface curvature also plays an important role while determining the self-similar wall velocities. A long, slim continuous cylinder is a trivial example of the body of revolution involving surface curvature. On the other hand, the circular flat disk is the example where the surface involves no curvature besides being an axially symmetric body of revolution. Thus, the circular cylinder and the circular disk shall be the objects of our interest in this section. Different from the previous two planner cases, some important facts shall be revealed regarding the cylinder and the disk geometries in view of the similarity criterion because of the axially symmetric nature of these flows.

5.3.1 Moving Cylinder

Consider a long continuous solid cylinder, having symmetry about the z-axis, immersed in a viscous and incompressible fluid and moving with velocity $u = u_w(z)$ in the steady state. The circular cylinder might be of constant as well as of variable cross section. Therefore, in general, the radius of the cylinder is taken as $R(z)$, varying in z. The schematic of the flow and the chosen system of coordinates is shown in Fig. 5.3. The governing equations in this case are the same as (2.13) and (2.14) subject to the boundary condition

$$\left.\begin{array}{lll} u = u_w(z),\ v = 0, & \text{at} & r = R(z) \\ u = 0, & \text{at} & r = \infty \end{array}\right\}. \tag{5.103}$$

Introducing the stream function of the form

$$u = \frac{1}{r}\frac{\partial \psi}{\partial r}, \quad v = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \tag{5.104}$$

due to which Eq. (2.13) satisfies identically and Eq. (2.14) transforms to

$$\frac{1}{r}\frac{\partial \psi}{\partial r}\frac{\partial^2 \psi}{\partial r \partial z} + \frac{1}{r^2}\frac{\partial \psi}{\partial r}\frac{\partial \psi}{\partial z} - \frac{1}{r}\frac{\partial \psi}{\partial z}\frac{\partial^2 \psi}{\partial r^2} = \nu\left[\frac{1}{r^2}\frac{\partial \psi}{\partial r} - \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} + \frac{\partial^3 \psi}{\partial r^3}\right]. \tag{5.105}$$

The procedure of finding the self-similarity criterion in this case is the same as implemented in the previous two sections. We, therefore, assume a one-parameter

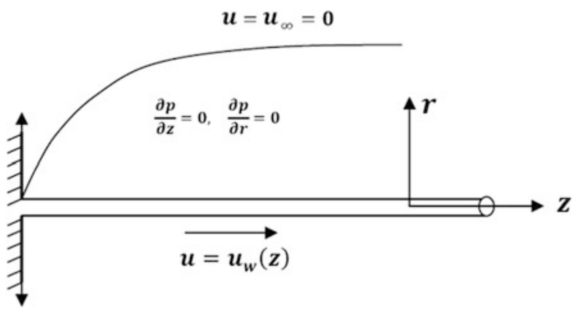


Fig. 5.3 Axisymmetric flow and the associated system of coordinate shown schematically

group of scaling transformations for the variables involved in Eqs. (5.105) and (5.103), of the form

$$\bar{z} = k^{\alpha_1} z, \quad \bar{r} = k^{\alpha_2} r, \quad \bar{\psi} = k^{\alpha_3} \psi, \quad \bar{u}_w = k^{\alpha_4} u_w, \quad \bar{R} = k^{\alpha_5} R. \tag{5.106}$$

The restriction of invariance of the governing system under the group of scalings will be imposed in order to get a system of linear equations in the scaling exponents. The non-trivial solution of this system will lead toward the construction of new variables. In view of the boundary-layer character, the leaving variable must be z. Accordingly, the two cases arise regarding the zero and nonzero character of the scaling exponent α_1. The details of determining the new variables have been omitted in order to avoid the repetition of previously exercised steps. Therefore, the similarity variables for the case $\alpha_1 \neq 0$ are given directly as

$$\eta = r z^{\frac{m-1}{2}}, \quad \psi = z f(\eta). \tag{5.107}$$

The wall velocity and the cylinder radius come out of the form

$$u_w = a z^m, \quad R(z) = R_0 z^{\frac{1-m}{2}}, \tag{5.108}$$

where R_0 denotes the fixed reference radius of the cylinder corresponding to the case $m = 1$.

With the help of transformation (5.107), Eq. (5.105) transforms as

$$m\left(\frac{f'}{\eta}\right)^2 - \frac{f}{\eta}\left(\frac{f''}{\eta} - \frac{f'}{\eta^2}\right) = \nu \frac{1}{\eta}\frac{d}{d\eta}\left(\eta \frac{d}{d\eta}\left(\frac{f'}{\eta}\right)\right), \tag{5.109}$$

and boundary conditions (5.103), in view of Eqs. (5.107)–(5.108), take the form

$$\left.\begin{aligned} f' &= aR_0, \ f = 0, \quad &\text{at} \quad \eta = R_0 \\ f' &= 0, \quad &\text{at} \quad \eta = \infty \end{aligned}\right\}. \tag{5.110}$$

Equations (5.109) and (5.110) are completely free from the previous variables; therefore, they can be regarded as self-similar. Consequently, Eq. (5.107) contributes as the similarity transformations to this case.

Equation (5.108) contains interesting information regarding the similarity criterion of this case. The power-law wall velocity has been obtained for the case $\alpha_1 \neq 0$ as it has also happened in the previous two sections. The different thing in this case is the involvement of z in the expression of R for $m \neq 1$. For $m = 1$, one is left with the linear wall velocity case (see Eq. 5.108) for which the radius of cylinder stays fixed. Corresponding to the other values of m, the particular construction of η does not allow the radius of the cylinder to stay constant. This means that the nonlinear stretching of the cylinder is possible only if the cylinder radius does not stay constant but follows the power-law form (defined in Eq. 5.108), analogous to the boundary-layer thickness. The boundary-layer thickness in the nonlinear stretching flow varies as $z^{\frac{1-m}{2}}$ which actually guides the body contour of the axially symmetric body of revolution to follow the same law, i.e., $z^{\frac{1-m}{2}}$. This simply reflects that the similarity solution is possible for the power-law velocities if the cylindrical surface does also vary in the same manner as does the boundary-layer thickness; otherwise, the solution must be non-similar. Another, worth noting, difference between the planner and the axisymmetric flows is the case of constant wall velocity. In the case of moving sheet, the self-similar Sakiadis flow is recovered immediately, by taking $m = 0$, without imposing any restriction on the sheet's thickness. On the other hand, the self-similar solution is, though recovered, in the cylinder case for $m = 0$ but with a compromise on the thickness of cylinder. If one forces the cylinder's radius to be constant (for $m = 0$), the self-similarity is lost and the classical Sakiadis' non-similar flow due to a moving continuous cylinder is recovered. This was in fact the reason behind the utilization of approximate integral method by Sakiadis to his (non-similar) flow. Very few audiences are aware of the fact that Sakiadis started the cylinder case with the non-similar flow. Hence, the radius of the cylinder can be made to stay constant (only) if the cylinder is being stretched with linear velocity, in order to ensure the existence of self-similar solution. In this way, the case of moving cylinder is much more interesting in comparison with the two-dimensional case and needs to be explored completely. So far, the available literature concerning the continuous cylinder is strictly limited to the cases $m = 0$ and $m = 1$, only.

In the case $\alpha_1 = 0$, the exponential forms of wall velocity and the cylinder radius are obtained, which are given by

$$u_w = a\mathrm{e}^{mz}, \quad R(z) = R_0\mathrm{e}^{-\frac{m}{2}z}. \tag{5.111}$$

The corresponding similarity variables are constructed as

$$\eta = r\mathrm{e}^{\frac{m}{2}z}, \quad \psi = f(\eta), \tag{5.112}$$

which transform Eq. (5.105) to the form

$$m\left(\frac{f'}{\eta}\right)^2 = \nu\frac{1}{\eta}\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{f'}{\eta}\right)\right). \tag{5.113}$$

The boundary conditions (5.103), in view of Eqs. (5.111)–(5.112), transform as

$$\left.\begin{array}{lll} f' = aR_0 & \text{at} & \eta = R_0 \\ f' = 0, & \text{at} & \eta = \infty \end{array}\right\}. \tag{5.114}$$

Thus in the case of continuous cylinder too, the power-law and exponential forms are the ultimate wall velocities in order for the existence of self-similar solution. This fact will also be proved in Chap. 10 while modeling the non-similar flows. Regarding the nonlinear stretching/shrinking of the cylinder, the curvilinear system of coordinates as considered in Chap. 10 is recommended. The conventional cylindrical system of coordinates, however, creates certain ambiguities in the mathematical formulation.

5.3.2 *Radial Motion of Flexible Disk*

Consider a flexible flat circular disk of infinite radius immersed in an incompressible viscous fluid. The disk geometry and the associated system of coordinates are shown in Fig. 5.4. Following the notation convention practiced in the existing literature, particular to the disk flow, r is taken as the radial coordinate and z is taken as the axial coordinate where u and w denote the velocity components along these axes, respectively. The disk is being stretched or shrunk in the radial direction with a velocity $u_w(r)$ as shown in Fig. 5.4. Because of no involvement of any circular motion, the angular component of velocity is zero. Therefore, the suitable velocity vector for this flow in the steady-state form reads as

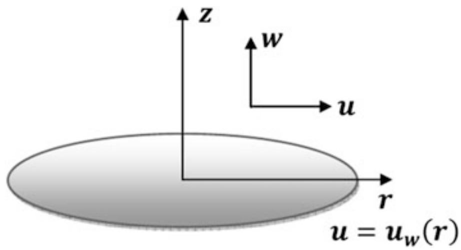
$$\boldsymbol{V} = [u(r,z), 0, w(r,z)], \tag{5.115}$$

due to which the governing equations of this flow are the same as given in Eqs. (2.15) and (2.16). The appropriate boundary conditions read as

$$\left.\begin{array}{llll} u = u_w(r), & w = 0, & \text{at} & z = 0 \\ u = 0, & & \text{at} & z = \infty \end{array}\right\}. \tag{5.116}$$

Introducing the stream function $\psi(r,z)$ which is related to the velocity components as

Fig. 5.4 Disk geometry and the chosen system of coordinates



$$u = \frac{1}{r}\frac{\partial}{\partial z}(r\psi), \quad w = -\frac{1}{r}\frac{\partial}{\partial r}(r\psi). \tag{5.117}$$

Because of Eq. (5.117), the equation of continuity (2.15) is satisfied identically and Eq. (2.16) takes the form

$$\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial r\partial z} - \frac{\psi}{r}\frac{\partial^2\psi}{\partial z^2} - \frac{\partial\psi}{\partial r}\frac{\partial^2\psi}{\partial z^2} = v\frac{\partial^3\psi}{\partial z^3}. \tag{5.118}$$

The group theoretical procedure will be employed to Eqs. (5.118) and (5.116) in order to find the self-similarity criterion for this flow. The variables involved in this system are r, z, ψ and u_w for which the scaling group reads as

$$\bar{r} = k^{\alpha_1} r, \quad \bar{z} = k^{\alpha_2} z, \quad \bar{\psi} = k^{\alpha_3}\psi, \quad \bar{u}_w = k^{\alpha_4} u_w. \tag{5.119}$$

The substitution of Eq. (5.119) into the system (5.118) and (5.116) subject to the condition of invariance under (5.119) gives rise to a system of simultaneous algebraic equations, similar to the previous problems. In this case, we decide to eliminate r due to which two cases arise for the values of α_1, namely $\alpha_1 \neq 0$ and $\alpha_1 = 0$. In the case $\alpha_1 \neq 0$, the similarity variables so constructed are (by omitting the details of their derivation)

$$\eta = r^{\frac{m-1}{2}} z, \quad \psi = r^{\frac{m+1}{2}} f(\eta), \tag{5.120}$$

which successfully transform Eq. (5.118) to the self-similar form, given by

$$mf'^2 - \left(\frac{m+3}{2}\right) ff'' = vf''', \tag{5.121}$$

subject to the restriction that the wall velocity must follow the power-law form defined by

$$u_w = ar^m. \tag{5.122}$$

Accordingly, the boundary conditions (5.116) transform as

$$\left.\begin{array}{llll} f' = a, & f = 0, & \text{at} & \eta = 0 \\ f' = 0, & & \text{at} & \eta = \infty \end{array}\right\}. \tag{5.123}$$

Thus, Eqs. (5.121) and (5.123) are in self-similar form based on the condition that the wall velocity must follow the power-law form given in Eq. (5.122).

In the case $\alpha_1 = 0$, the group theoretical procedure ends up with the transformations involving exponential form, such as

$$\eta = e^{\frac{m}{2}r} z, \quad \psi = e^{\frac{m}{2}r} f(\eta), \tag{5.124}$$

with the wall velocity following the exponential form, given by

$$u_w = a e^{mr}. \tag{5.125}$$

Utilization of the transformations (5.124) transforms Eq. (5.118) to the form

$$mf'^2 - \frac{1}{r} ff'' - \frac{m}{2} ff'' = \nu f'''. \tag{5.126}$$

Obviously, the variable r has not been eliminated completely from the equation after the utilization of Eq. (5.124). This simply reflects that the transformations (5.126) cannot transform the Eq. (5.118) to the self-similar form and implies that the self-similar solution is not possible in this case. Thus, in the case of circular disk, the similarity solutions are limited to the power-law case only and the exponential wall velocities fail to produce the self-similar solution. The mathematical reason behind this fact is the appearance of r as a variable coefficient in the second term of Eq. (5.126) which is impossible to eliminate.

5.4 Restriction on Wall Suction/Injection

This has already been explored in the previous sections that the self-similar solutions are the limited solutions and are possible only if the wall velocities follow certain particular forms. In the case of cylinder, the radius of the cylinder also undergoes certain restrictions in addition to the wall velocities in order to ensure the self-similarity. The similar situation persists for the cases when one also takes into account the wall suction/injection in the boundary-layer. This section is devoted to the determination of those particular wall suction/injection profiles which do not break the self-similarity of the considered flow.

Corresponding to the two-dimensional and three-dimensional cases, the normal wall velocity shall be denoted by $v_w(x)$ and $v_w(x, z)$, respectively. Being a function of x and (x, z), the wall velocities $v_w(x)$ and $v_w(x, z)$ serve as variable quantities in the governing system. Similar to the other variable quantities, either dependent or independent, the considered groups of scalings shall be appended by another scaling transformation of the variable v_w. In doing so, the obtained system of linear equations (due to the restriction of invariance) will also be increased by one more equation resulting from the boundary condition $v = v_w$ at $y = 0$ in Sects. 5.1 and 5.2. Consequently, the non-trivial solution of the system of such algebraic equations in scaling exponents will also include the solution for the scaling exponent of v_w which will subsequently be utilized in the construction of corresponding new variables.

Particular to Sects. 5.1 and 5.2, the scaling group for v_w could be taken of the form

$$\bar{v}_w = k^{\gamma} v_w, \tag{5.127}$$

where γ denotes the scaling exponent. Combining Eq. (5.127) with the group (5.8) and following the subsequent procedure executed in Sect. 5.1, the system (5.11) is appended by an additional linear equation of the form

$$\alpha_1 - \alpha_3 = -\gamma, \tag{5.128}$$

which, for the case $\alpha_1 \neq 0$, admits the solution

$$-\frac{\gamma}{\alpha_1} = \frac{\alpha_2}{\alpha_1}. \tag{5.129}$$

In view of Eq. (5.14), the wall velocity $v_w(x)$ comes out of the form

$$v_w(x) = \mathrm{d}x^{\frac{m-1}{2}}, \tag{5.130}$$

where d denotes a pure constant having suitable dimensions. The positive and negative values of d characterize the injection and suction velocities, respectively, while $d = 0$ designates no suction or injection at the wall. It is important to note that the power-law form of the wall velocity (given in Eq. 5.14) does also require the normal wall velocity to follow the same (power-law) form in order to ensure the self-similarity. The same is the case with exponential wall velocity (5.20) which requires the suction/injection velocity also to follow the exponential form, given by

$$v_w(x) = \mathrm{d}e^{\frac{m}{2}x}. \tag{5.131}$$

The trend follows similarly in Sect. 5.2 where the suction/injection velocity not only follows the power-law and exponential forms but also is a product of the two. The details have, however, been omitted completely for the sake of brevity. Corresponding to the every case of Sect. 5.2, the suction/injection velocity has been obtained as follows:

Case I; Subcase I

$$v_w(x,z) = \mathrm{d}x^{\frac{m-1}{2}-B\left(\frac{n-1}{2}\right)}z^{\frac{n-1}{2}}, \tag{5.132}$$

Case I; Subcase II

$$v_w = \mathrm{d}x^{\frac{m-1}{2}}\mathrm{e}^{\frac{n}{2}z}, \tag{5.133}$$

Case II; Subcase I

$$v_w(x,z) = \mathrm{d}\mathrm{e}^{\frac{1}{2}(m-B(n-1))x}z^{\frac{n-1}{2}}, \tag{5.134}$$

Case II; Subcase II

$$v_w(x,z) = \mathrm{d}\mathrm{e}^{\frac{1}{2}(mx+nz)}. \tag{5.135}$$

Accordingly, the corresponding boundary conditions at the wall also modify as

$$\frac{1}{2}(m+1-B(n-1))f(0)+\frac{n+1}{2}g(0) = -d, \tag{5.136a}$$

$$\left(\frac{m+1}{2}\right)f(0)+\frac{n}{2}g(0) = -d, \tag{5.136b}$$

$$\frac{1}{2}(m-B(n-1))f(0)+\frac{n+1}{2}g(0) = -d, \tag{5.136c}$$

$$\frac{1}{2}(mf(0)+ng(0)) = -d, \tag{5.136d}$$

which, respectively, refer to the cases "Case I; Subcase I," "Case I; Subcase II," "Case II; Subcase I," and "Case II; Subcase II." The corresponding boundary conditions of the two-dimensional case can also be recovered from Eq. (5.136a)–(5.136d).

The cases of axially symmetric flow due to continuous cylinder or circular flexible disk follow in the similar manner. In the case of continuous cylinder, when the surface velocity obeys the power-law profile the second condition in Eq. (5.110) at $\eta = R_0$ modifies as

$$f = -\mathrm{d}R_0. \tag{5.137}$$

In the case of circular flexible disk, the second condition in Eq. (5.124) at $\eta = 0$ modifies as

$$\frac{m+3}{2} f(0) = -d, \tag{5.138}$$

where the wall velocity obeys power-law profile.

By the end of this chapter, the criterion of self-similarity for the planner and the axisymmetric cases has in general been derived. The wall velocities, other than the derived ones, will make the flow non-similar. The power-law, exponential, and a product of the two have been discovered for the three-dimensional flow, whereas the exponential wall velocity has been extended for various powers of the already known exponential wall law in the two-dimensional case. Regarding the axisymmetric flow due to moving cylinder, the nonlinear and exponential stretching or shrinking have been discovered. The case of linear stretching or shrinking has been extended to the nonlinear one in the case of circular flexible disk. In what follows, the determination of self-similarity criterion regarding the wall velocities in the above-named flow situations has completely been discovered.

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