Chapter 3
The Concept of Self-similarity

One of the important classes of boundary-layer flows comprises the self-similar flows. The concept of self-similarity is equally important in mathematical as well as physical point of views. Normally, the boundary-layer flow problems are modeled in the form of partial differential equations (*pdes*) involving two or more independent variables in addition to the involved physical parameters or constants. If, under certain conditions, it becomes possible to reduce the number of independent variables, in a particular problem, to one by combining all the independent variables suitably, then the problem under consideration is called *self-similar* or *auto-model*. Consequently, the governing partial differential equations are transformed to ordinary differential equations *(odes)* which are completely in the form of new variables and free from the previous variables. A useful example could be the steady, two-dimensional flow past a static wedge governed by the system (1.1) which are *pdes* in two independent variables. In this flow, the boundary-layer is formed at the wedge surface due to the presence of external potential flow, having velocity $u_\infty(x)$. The self-similar solutions for this flow exist if the external potential velocity is of the form, but not limited to, $u_\infty(x) = ax^m$ where a is a constant having suitable dimensions and m is a pure number. Consequently, the system (1.1) completely transforms to an ordinary differential equation (Eq. 1.8) and becomes free from previous variables.

Dimensional analysis, based upon the Buckingham Pi-theorem,[1] is one of the fundamental approaches for reducing the number of independent variables, while dealing with the partial differential equations, by combining them suitably to construct the new variables. The criterion of Buckingham Pi-theorem guides in this regard completely and not only tells, exactly, what number of independent variables can be reduced but also guides toward the construction of new variables. Following

[1]Buckingham Pi-theorem is of fundamental importance in dimensional analysis. The interested readers is recommended to consult the Ref. [1].

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its criterion, the dimensional analysis determines suitable new fundamental dimensions which come out to represent the problem in the form of new variables. This actually is done by the suitable scaling of the involved physical quantities.

The existence of self-similarity in a particular flow problem is usually a consequence of the non-existence of a characteristic length along one or more space directions. Such a problem, therefore, have a capacity of absorbing more than one variables into a single one, thus forming the new self-similar (independent) variable. The construction of new variables is, sometimes, also guided/restricted by the involved boundary and initial conditions. However, there is a systematic approach of constructing the new self-similar variables which will be discussed in detail in Sect. 3.3 and will be utilized in Chap. 5. Another important aspect of similarity solutions is that they usually exhibit asymptotic behavior; the same is true with the boundary-layer flows as the velocity within the boundary-layer also exhibits the asymptotic character. This actually is the reason that the family of self-similar solutions to the boundary-layer equations constitutes a big class of important flows in the fluid dynamics.

3.1 In View of Group Theoretic Approach

The application of dimensional analysis does always not result in the reduction of independent variables even if the reduction is possible. Sometimes, it happens that the dimensional analysis fails in finding those new fundamental dimensions which could be used to describe the original problem in self-similar variables. Consequently, the stuck guy is forced to think about any other strategy (method). The way out to this situation comes directly from the notion/concept of invariance of *pdes* under the scaling of variables in the frame work of Group Theoretic approach. A partial differential equation can actually be transformed to an ordinary differential equation, completely, due to the use of some suitable transformations only if the original *pde* is invariant under the Lie group of scaling transformations [1]. A detailed account to this topic can be found in [1], and the interested reader is referred to follow the Chap. 1 of [1].

Hence, the determination of new variables in the framework of dimensional analysis is actually attributed to the invariance property of the original equation(s) under the scaling group of involved variables, whereas the dimensional analysis actually does not implement the criterion of invariance, in complete, to the given *pde* under the utilized group of scaling transformations and thus stays unable to capture the self-similarity in many cases. There are examples, as we already mentioned above, where the dimensional analysis does not find any new fundamental dimensions due to which the original problem cannot be transformed to the self-similar form, but the Group Theoretic approach via one- or multi-parameter group of scaling transformations successfully determines the self-similar variables in the same problem.[2] Therefore, the Group Theoretic approach can be regarded as the most generalized one, which successfully determines the criterion of self-similarity for a particularly chosen problem. However, the determination of self-similarity criterion via a one-parameter Lie group of scaling transformations does not deny the existence of any other self-similar solution(s) because the determined self-similar solutions represent all those concerning to the particularly utilized group of scaling transformations. Besides the Group Theoretic procedure, there are several other, ad hoc, approaches which can be utilized to determine the self-similarity in a particular problem. However, the underlying property, working behind all such approaches, is the requirement of invariance of the original *pde*. Worth mentioning other techniques are the determination of self-similarity through separation of variables and through the conservation laws, etc.

In the above discussion, we have repeatedly been using the word "problem" by which we mean the given partial differential equation(s) and the associated initial and boundary conditions which subsequently will be called as auxiliary data. The author's experience with the 'similarity' reveals that the existence of self-similarity is strongly dependent upon the nature of auxiliary data. For example, if a certain *pde* admits a similarity solution under the constraints of one auxiliary data, it may not be admitting the self-similarity for the other auxiliary data. The existence of dual (or more) similarity variables for certain problems and the fundamental reason behind the non-uniqueness of the similarity variables is basically the nature of auxiliary data.[3] Particular to the boundary-layer flows past flat surfaces, self-similar solutions are possible in those cases where the reference velocities follow the power-law or exponential form as did in the Falkner–Skan flow.[4] The dependence of the self-similarity on the auxiliary data can further be explained due to the following example. Let us consider the Stokes first problem described by the system of equations

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \tag{3.1}$$

$$u(0, y) = 0, \quad y \geq 0 \tag{3.2a}$$

$$u(t, 0) = U_0, \quad t > 0 \tag{3.2b}$$

$$u(t, \infty) = 0. \tag{3.2c}$$

[2]For further detail on this account the reader is referred to follow [1].

[3]As in the Falkner–Skan flow, the similarity variables take different forms for different values of m, though the nature of the flow is the same, that is the potential flow past a wedge.

[4]This fact can be confirmed in Chap. 5 which, however, does not deny the possibility of any other form.

The system (3.1)–(3.2a) admits a similarity solution of the form

$$u = U_0 f(\eta), \quad \eta = \frac{y}{2\sqrt{\nu t}} = \frac{y t^{-1/2}}{2\sqrt{\nu}}, \tag{3.3}$$

which transforms Eq. (3.1) to an ordinary differential equation of the form

$$f'' + 2\eta f' = 0, \tag{3.4}$$

subject to the transformed boundary conditions

$$f(0) = 1, \quad f(\infty) = 0. \tag{3.5}$$

Notice that the original three constraints (the auxiliary conditions) have now been reduced to two in number. Both of these are the boundary conditions, and the initial condition has completely been vanished. To understand this fact the definition of $\eta = \frac{yt^{-1/2}}{2\sqrt{\nu}}$ is important and the fact that the initial and boundary data are described at $t = 0$, $y = 0$ and $y = \infty$ only. In view of the definition of η and the critical values of y and t where the boundary and initial data have been described, we note that $\eta = 0$ at $y = 0$ only but $\eta = \infty$ either at $y = \infty$ or at $t = 0$, simultaneously. This means that in the transformed system the condition $f(\infty) = 0$ at $\eta = \infty$ must, simultaneously, represent the initial and boundary conditions defined at $t = 0$ and $y = \infty$, respectively. This can only be achieved if the said initial and boundary conditions do coalesce, that is

$$u(0, y) = u(t, \infty). \tag{3.6}$$

This means that if such a coalition of the initial and the boundary conditions is not possible, then the similarity variable η, defined above, can never be utilized in order to get the transformed *ode* (Eq. 3.4). Fortunately, the condition (3.6) is met by the auxiliary data (3.2a) due to which the similarity solution exists for this problem. Otherwise, it was impossible to achieve any way. In this perspective, regarding the existence of similarity solutions, few of the auxiliary conditions, defined at different points in the domain of interest, must coalesce to one. Such a situation is only possible if the new variables are constructed from the original variables by raising them to suitable powers. Such a power-law product of the original variables, in the construction of similarity variables, can never be achieved without having the reference velocity of the similar form. This is one of the important reasons behind the fact that the self-similar solutions follow the power-law form of the reference velocities. However, such a coalition is always not necessary, especially in those cases where the auxiliary conditions are already very few.

Based upon the number of auxiliary data, interesting conclusions regarding the number of similarity variables are drawn here:

- If the problem is well-posed[5], then a unique similarity variable will exist provided the problem admits a self-similar solution. For an ill-posed[6] problem, the uniqueness of the similarity variable is not guaranteed; the problem may admit one or more similarity variables if the self-similar solution is possible.
- If the given problem, whether well-posed or ill-posed, does not admit a self-similar solution then the solution will be called non-similar.

3.2 Physical Meanings

The concept of self-similarity is a little bit hard to explain in words, on one hand or seems to be explainable in a single sentence, on the other hand. In view of physical meanings of self-similarity the author is more inclined to the latter opinion because, in words, to-the-point explanation of self-similarity is hard to extend beyond few lines and one ultimately requires the assistance of mathematical language. Based upon the second opinion, we shall start trying to understand the physics of self-similarity from the mathematical view point and will try to become more and more less mathematical, gradually.

In the start of this chapter we explicitly stated that a problem is self-similar if the total number of involved independent variables can be reduced to one. Obviously, this can only be done by a suitable mixing of the original independent variables to form a single new variable, as discussed in the previous section. Ultimately, the resulting (new) single variable is called the similarity variable. Another important ingredient of this discussion is the scaling of variables which is a common base line among the dimensional analysis and the Group Theoretic method. Thus the mathematical sense of developing the similarity variables is based upon the suitable mixing of the original variables or more formally the suitable scaling of the original variables. At this stage it seems very useful to pick a particular example so that the concept of self-similarity can more conveniently be explained. For this purpose the Falkner–Skan flow (1.7)–(1.8) would be the best choice.

Let us concentrate on the definition of η and $f'(\eta)$ in Eq. 1.7 by ignoring the constant coefficient $\sqrt{\frac{a}{\nu}}$, for instance. Such constant coefficients have actually nothing to do with the self-similarity and are present just to non-dimensionalize the system. Notice that the Falkner–Skan problem is defined for $x > 0$ and $0 \leq y < \infty$ with the boundary conditions

[5]If for a given differential equation, sufficient numbers of auxiliary conditions are known to make the unique solution sure and the solution thus obtained depends continuously upon the given auxiliary data.

[6]If for a given differential equation, at least one or more auxiliary conditions are missing, the problem is ill-posed.

$$u(x,0)=0, \quad v(x,0)=0, \tag{3.7}$$

$$u(x,\infty)=u_\infty(x)=ax^m, \tag{3.8}$$

including no condition at any x-location. Since the independent variables are only x and y, the construction of η by combining x and y does not require any boundary conditions to coalesce. In this particular flow $\eta=\sqrt{\frac{a}{\nu}}x^{\frac{1-m}{2}}y$ having the domain [0, ∞) derived directly from the domain of y. Clearly, in the construction of η, y has simply been transformed to become η after a suitable scaling by an appropriate scale factor, namely $\sigma(x)=x^{\frac{1-m}{2}}$. Similarly, the construction of $f'(\eta)$ is also a consequence of suitable scaling of u by an appropriate scale factor which is obviously the external potential velocity $u_\infty(x)$. Hence, the function $\sigma(x)=x^{\frac{1-m}{2}}$ is the suitable scale factor in y and the reference velocity $u_\infty(x)=ax^m$ is the suitable scale factor in u.

Let us assume that we have calculated the velocity $f'(\eta)$ at $\eta=1$ which is $f'(1)=0.3298$ for $m=0$. Notice that, in Fig. 3.1a, the velocity $f'(1)=0.3298$ is the same at the locations x_1 and x_2, but differs in scale factors ax_1^0 and ax_2^0 in u and by $\sqrt{x_1}$ and $\sqrt{x_2}$ in y at the two locations. Mathematically, this fact can be expressed as

$$\frac{u\left(x_1,\frac{y_1}{\sqrt{x_1}}\right)}{ax_1^0}=\frac{u\left(x_2,\frac{y_2}{\sqrt{x_2}}\right)}{ax_2^0}. \tag{3.9}$$

Similar situation can be seen in Fig. 3.1b for the case $m=1$. If η is taken arbitrary for some fixed x_1 and x_2, then y must also be taken as arbitrary. Consequently, Eq. (3.9) modifies as, for $m=0$

$$\frac{u\left(x_1,\frac{y}{\sqrt{x_1}}\right)}{ax_1^0}=\frac{u\left(x_2,\frac{y}{\sqrt{x_2}}\right)}{ax_2^0}, \tag{3.10}$$

and for $m=1$

$$\frac{u(x_1,y)}{ax_1}=\frac{u(x_2,y)}{ax_2}. \tag{3.11}$$

Now, for the arbitrary m, Eqs. (3.9)–(3.11) unify as

$$\frac{u(x_1,y/\sigma(x_1))}{u_\infty(x_1)}=\frac{u(x_2,y/\sigma(x_2))}{u_\infty(x_2)}. \tag{3.12}$$

Thus if the solution is self-similar, then the x-component of velocity differs only by a scale factor in u and y, at any two different x-locations. In other words, the nature of the velocity profile does not depend upon x at all. The reason behind this

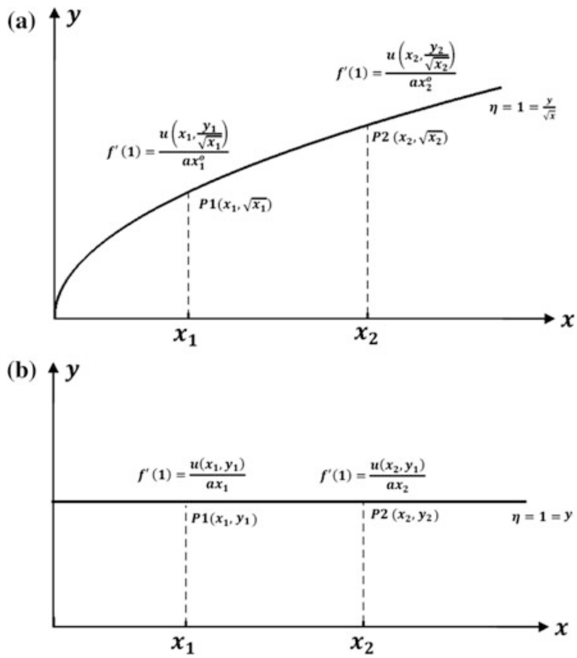

Fig. 3.1 **a** and **b** Graphical description of self-similarity

fact is the allowance, by the governing system, of the construction of η due to x and y in which x and y are so suitably combined. Such an allowance by the governing system is in fact recognized as the invariance of the system. The appropriate mixing of x and y in the construction of η generates a family of curves in the xy-plane. When a numerical code iterates to compute the solution at different η-nodes (say η_i), it actually computes the solution at $y = \eta_i\sqrt{x}$ curves in the physical/actual domain as shown in Fig. 3.2. This property, actually, lifts the requirement of computing the solution at various x-nodes, meaning that the solution is "self-similar" at all x-locations and only differs by a constant scale factor. Hence, if the solution is known at any x-location, the solution at any other x-location can easily be determined from the already known solution. The independence of the velocity from the variable x guarantees the absence of any length scale in the x-direction. In contrast, the Howarth's retarded flow [2] does involve flow separation and thus involves a definite length in x-direction, namely the distance of the point of separation from the leading edge, hence reacting as non-similar in nature.

Here, x-component of velocity has particularly been mentioned and the y-component of velocity has not been named at all. Actually, within the boundary-layer, only the lateral component(s) of velocity constitutes the main flow and the normal component of velocity is usually determined from the equation of continuity in the form of lateral velocity component. Moreover, the variation in velocity across the boundary-layer is more significant than that in the lateral direction. This could also be said a reason for similarity in x because the role of x within the boundary-layer is somewhat like a slack variable; and the slack has to leave the system, ultimately.

3.3 General Theory

The simplification of a given differential equation via some transformation of involved variables based on the criterion of invariance is actually credited to Sophous Lie [3, 4]. Lie introduced the procedure of finding the infinitesimal transformations which leave the given differential equation invariant and result in significant simplification of the original equation, either by reducing the order of the original equation or by reducing the number of independent variables in it. The process of finding the infinitesimal symmetries of a differential equation is quite hectic but is highly algorithm and is easy to implement in a computer code. However, regarding the determination of self-similar solutions one does not need to follow the Lie's algorithm of finding the point symmetries, because the number of independent variables in a *pde* can be reduced through a group of scaling transformations. It is therefore straightforward to utilize the scaling group of transformations directly for finding the similarity solution. Morgan [5] utilized the general theory of invariance and developed a straightforward procedure for constructing the similarity variables. The procedure developed by Morgan was further utilized in fluid dynamics problems by [6–10].

Consider a system of n *pdes*, $E_j = 0$ in n unknowns u_j, $(j = 1, 2, \ldots, n)$ depending upon m number of independent variables x_i $(i = 1, 2, \ldots, m)$. A one-parameter group G of scaling transformations is assumed for the involved variables, of the form

$$\bar{x}_i = k^{\alpha_i} x_i, \bar{u}_j = k^{\gamma_j} u_j \quad (i = 1, 2, \ldots, m), \quad (j = 1, 2, \ldots, n), \tag{3.13}$$

where $k \neq 0$ is the continuous real parameter and the α_i and γ_j are the real exponents to be determined. The group (3.13) is applied to the given system of equations where the condition of invariance of the transformed system results in a system of simultaneous linear equations in the exponents α_i and γ_j. The non-trivial solution of this system is then utilized in the construction of (new) similarity variables which actually are the invariants of the scaling group G. At a time, only one independent variable can be reduced through one-parameter group. Let us assume that the linear system in α_i and γ_j admits a non-trivial solution and the variable x_1 is to be eliminated; there arise two cases for the exponent α_1, namely for $\alpha_1 \neq 0$ or $\alpha_1 = 0$.

Case I $(\alpha_1 \neq 0)$
If $\alpha_1 \neq 0$ the similarity (or new) variables (or invariants of G) are constructed as

$$\eta_r = \frac{x_r}{x_1^{\frac{\alpha_r}{\alpha_1}}}, \quad (r = 2, \ldots, m), \tag{3.14a}$$

and

$$f_j(\eta_2, \eta_3, \ldots, \eta_m) = \frac{u_j(x_1, x_2, \ldots, x_m)}{x_1^{\gamma_j/\alpha_1}}, \quad (j = 1, 2, \ldots, n). \tag{3.14b}$$

Case II $(\alpha_1 = 0)$
If $\alpha_1 = 0$, the similarity variables are constructed as

$$\eta_r = \frac{x_r}{e^{\alpha_r x_1}}, \quad (r = 2, 3, \ldots, m), \tag{3.15a}$$

and

$$f_j(\eta_2, \ldots, \eta_m) = \frac{u_j(x_1, x_2, \ldots, x_m)}{e^{\gamma_j x_1}}, \quad (j = 1, 2, \ldots, n). \tag{3.15b}$$

In the following, this procedure has been employed to a simple problem in fluid dynamics which will further help to understand the utility of the above procedure.

Example Consider the Stokes first problem which we already considered in Sect. 3.1 (Eqs. 3.1–3.2a) in some different context. Equation (3.1) admits a similarity solution for which the corresponding similarity variables are defined in Eq. (3.3). Let us derive Eq. (3.3) with the help of above described general theory of constructing the similarity variables. Consider the scaling group G of the form (for the variables involved in Eqs. (3.1)–(3.2a)).

$$G : \bar{t} = k^{\alpha_1} t, \quad \bar{y} = k^{\alpha_2} y, \quad \bar{u} = k^{\gamma_1} u. \tag{3.16}$$

Substitution of Eq. (3.16) in Eq. (3.1) and the requirement of invariance of the original *pde* result in the following linear equation:

$$\alpha_1 - \gamma_1 = 2\alpha_2 - \gamma_1. \tag{3.17a}$$

The initial condition (3.2a) and the boundary condition (3.2c) contribute nothing to the system, whereas the boundary condition (3.2b) gives

$$\gamma_1 = 0. \tag{3.17b}$$

The non-trivial solution of the simultaneous system (3.17a) reads as

$$\frac{\alpha_2}{\alpha_1} = \frac{1}{2}; \quad \alpha_1 \neq 0. \tag{3.18}$$

Therefore, the similarity variables, according to Eq. (3.14a), are constructed as

$$\eta = \frac{y}{t^{\frac{\alpha_2}{\alpha_1}}} = \frac{y}{t^{1/2}} = yt^{-1/2}, \tag{3.19a}$$

and

$$f(\eta) = \frac{u}{t^{\frac{\gamma_1}{\alpha_1}}} = \frac{u}{t^0} = u. \tag{3.19b}$$

Thus, $u = f(\eta)$ and $\eta = yt^{-1/2}$ are the (new) similarity variables which transform governing Eq. (3.1) to the self-similar form. The presence of the factors U_0 and $\frac{1}{\sqrt{\nu}}$ in Eq. (3.3) with $f(\eta)$ and $yt^{-1/2}$, respectively, is just for the sake of non-dimensionalization; also the presence of a factor $\frac{1}{2}$ in the definition of η (in Eq. 3.3) is simply to manipulate the constant coefficient in the transformed Eq. (3.4). These are actually the niceties and have nothing to do with the process of determining the similarity variables. For further examples and a bit more detail on this topic, the interested reader is referred to follow a very nice book by Ames [11]. The method will be applied to the boundary-layer equations in Chap. 5 where a detailed account on the construction of similarity variables is presented.

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