Chapter 1
Viscous Flow Due to Moving Continuous Surfaces

Soon after the inception of boundary-layer concept, introduced by Prandtl [1], the first formal attempt toward the understanding of boundary-layer character was made by Blasius [2]. Blasius considered steady-state two-dimensional boundary-layer flow past a semi-infinite flat plate at zero incidence (see Fig. 1.1). He assumed a constant potential flow approaching the leading edge of the flat plate, with continued motion past the plate surface, and studied the flow within the so-formed boundary-layer at the plate surface. The Blasius equation was further investigated by Bairstow [3], Goldstein [4], Töpfer [5], Howarth [6], and Meksyn [7] under different circumstances. Experimental investigations, in order to confirm the theory, were conducted by Burgers [8], van der Hegge Zijnen [9], Hansen [10], and the Nikuradse [11]. The novelty of the idea and the curiosity of the flow character within the boundary-layer attracted several renowned scientists of the time who greatly contributed to the topic and raised the topic to the heights where it is seen today. A detailed account to this topic has been given in the glorious book by Schlichting [12] where all the major contributions have possibly been cited. The other important notable contributions to this topic are due to Goldstein [13], Rosenhead [14], and Batchelor [15] where huge fundamental knowledge has been gathered under one cover. The subject then went on developing day by day, but all the research concerning the flat-plate boundary-layers was limited to the situation when the fixed plate is attacked by a stream of potential flow, till 1961. In 1961, Sakiadis [16] introduced the viscous flow, owing the boundary-layer character caused due to the motion of a continuous solid surface within a quiescent fluid otherwise at rest.

The correct reasons for ignoring this flow, for a long time, are unknown to the author, but it seems that the pioneers of the boundary-layer theory had been more inclined toward the applications of boundary-layer theory in aerodynamics for the calculation of surface drag, in particular. In this perspective, the consideration of Blasius' like situation served as an appropriate theoretical model for the two-dimensional flow on the straight wing of an air craft and having great resemblance with the situation established in the wind tunnel experiments. The extension

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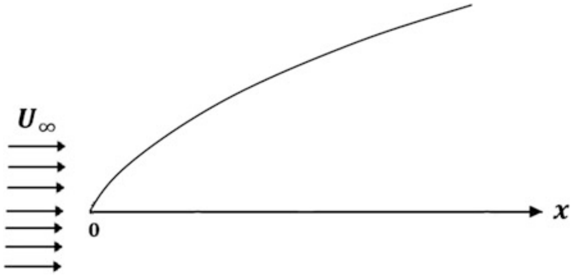
Fig. 1.1 Schematic of the Blasius flow



of the Blasius' model to the three-dimensional case applies to the situations when the potential flow attacks the swept-back wing, for example. With these sophisticated initiations, the development in the boundary-layer theory provided a very sound base to the modern aerodynamics. However, the theory of boundary-layers is not limited to the aerodynamic phenomena only but applies, in general, to every physical flow following the boundary-layer character. It is, sometimes, also misunderstood that the boundary-layer exists in the external flows only. This is, however, not the case; the boundary-layer is also formed in the internal flows such as in pipes and ducts, and also in the free surface flows, such as in free jets.

1.1 Sakiadis Flow

Let us consider the schematic of the Blasius flow once again (Fig. 1.1). Notice that the constant stream of potential flow attacks the leading edge and hence forms the boundary-layer starting right from the leading edge and growing downstream. In contrast, let us assume that the potential flow is absent and the plate moves with a constant velocity U_0 in the $-$ve x-direction as shown in Fig. 1.2.

In this case, as the plate continues to move, the fluid experiences disturbance right from the leading edge and is continuously being disturbed at the intermediate locations on the moving plate. The boundary-layer then develops in the direction of increasing x as the plate penetrates in the fluid in −x-direction. In this case too, the boundary-layer develops in the direction from the leading edge to the trailing edge, as shown in Fig. 1.2, similar to the Blasius flow. Therefore, the flow situations described in Figs. 1.1 and 1.2 can be regarded as equivalent where one can easily be transformed to the other by the use of simple Galilean transformations. In these two situations, the boundary-layer is actually formed due to the leading edge of the plate. Now the question arises: How the "moving plate boundary-layers" are different if the two flows, shown in Figs. 1.1 and 1.2, are the same? The answer is: when the moving surface moves in the + ve x-direction instead of –ve x-direction having no edges. Then, the two flows are entirely different from each other, and it becomes impossible to recover one from the other. Such a situation exists in diverse practical applications; example can be given of polymer industry.

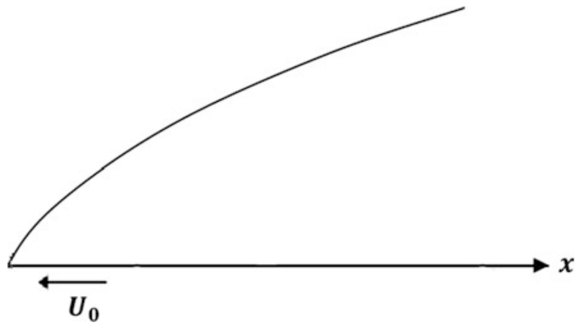


Fig. 1.2 Reverse of Blasius flow situation shown schematically

In the manufacturing of polymer sheets, the polymer melt issues continuously from a slit and travels a wind-up roll as shown in Fig. 1.3. In such a course of sliding past a wind-up roll, the sheet is, sometimes, also being stretched to attain the desired thickness and is cooled simultaneously. In order to obtain the final product of desired characteristic, the process of stretching and cooling requires to be controlled, which in turn gives rise to a fluid mechanics problem. Notice that, as the polymer sheet filament issues from the slit and travels downstream, the disturbance starts penetrating in the ambient fluid right from the slit and continues to grow subsequently downstream, hence forming the boundary-layer on the moving surface. This situation is also referred to as the "moving plate" in a fluid, but is quite different from the situation shown in Fig. 1.2 because of the absence of any leading edge. In such a senior the boundary-layer starts developing from the slit and grows in the direction of motion of the moving surface.

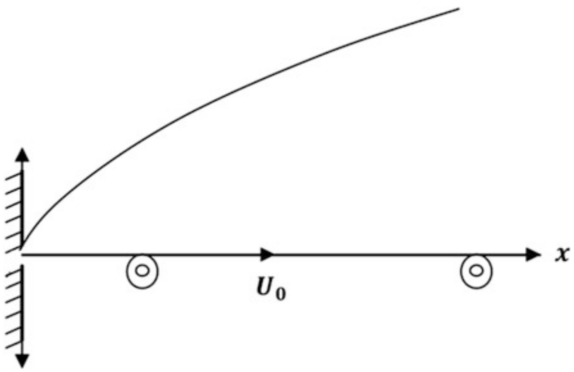
Unlike Fig. 1.2, the continued issuance of polymer filament from the slit provides the reason for the boundary-layer to develop downstream in the direction of motion of the moving sheet or the thread. Based on the similar reasoning, Sakiadis [16] introduced the boundary-layer flow due to a moving surface and essentially associated the word "continuous" as the prefix to the word "surface" just to clear the absence of any leading edge. With these assumptions, his equations of motion for two-dimensional flow due to a moving continuous surface are exactly the same as those of Blasius, i.e.,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1.1a}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}, \tag{1.1b}$$

where the involved quantities bear their usual meanings. Like Blasius, Sakiadis was also lucky enough to get the self-similar solution for this flow. Fortunately, the self-similar formulation of Blasius is also applicable to this case and Eqs. (1.1a, 1.1b) readily transform to the Blasius equation in dimensionless form, given by

Fig. 1.3 Schematic of moving continuous sheet



$$\frac{1}{2}ff'' + f''' = 0, \tag{1.2}$$

where $\eta = \sqrt{\frac{U_0}{\nu x}}y$ and $f(\eta) = \frac{1}{\sqrt{\nu x U_0}}\psi(x, y)$ are the Blasius' similarity variables. However, the boundary conditions are entirely different in this case and read as

$$\left.\begin{array}{lll} \text{at}\,\eta = 0, & f = 0, & f' = 1 \\ \text{at}\,\eta = \infty, & & f' = 0 \end{array}\right\}, \tag{1.3}$$

which makes the Sakiadis flow sufficiently different from the Blasius flow admitting the boundary conditions of the form

$$\left.\begin{array}{lll} f = 0, & f' = 0, & \text{at}\,\eta = 0 \\ f' = 1, & & \text{at}\,\eta = \infty \end{array}\right\}. \tag{1.4}$$

Sakiadis [17, 18] showed that the moving (continuous) surface boundary-layers contribute a new class of boundary-layers where the results of finite (semi-infinite) plate boundary-layer flow do not apply anyway. After his name, the boundary-layer flow due to a moving continuous surface with a uniform speed is referred to as the Sakiadis flow. In the rest of the text, we shall not always write the word "continuous" but occasionally, and the flow due to a moving continuous surface will then be written as moving surface/plate boundary-layer flow.

1.2 Stretching Sheet Flow

Sakiadis considered constant wall velocity $u_w = U_0$ in [17] and [18] while discussing the two-dimensional and axially symmetric cases, respectively. However, the moving plate boundary-layers are not limited to the constant wall velocity only;

rather, one may also take the variable wall velocity, such as $u_w = u_w(x)$. The variable wall velocity of the solid surface is then interpreted as the stretching/shrinking wall velocity, depending upon the sign of $u_w(x)$.

1.2.1 Crane's Flow

This has already been mentioned, in the preface, that despite the novelty and interesting features of the Sakiadis flow, it stayed deprived almost for a decade and was not attracted by the renowned scientists of the time till 1970. Lawrence J. Crane [19] was the first who extended the Sakiadis flow of constant wall velocity to variable wall velocity by taking $u_w = u_w(x) = ax$, where a denotes the constant stretching rate having the dimension of T^{-1}. Crane utilized the same equation as by Sakiadis, namely Blasius' Eqs. (1.1a, 1.1b), and introduced the similarity variables of the form

$$\eta = \sqrt{\frac{a}{\nu}}y, \quad u = axf'(\eta), \quad v = -\sqrt{a\nu}f(\eta). \tag{1.5}$$

Consequently, his equation of motion in dimensionless form came out of the form

$$f''' + ff'' - f'^2 = 0, \tag{1.6}$$

subject to the boundary conditions (1.3). Equation (1.6) is totally different from Eq. (1.2), but is exactly the same as that of Falkner and Skan [20] for $m = 1$. Again the difference between the Crane's and Falkner–Skan flow is the boundary conditions (1.3) and (1.4). It is important to start noting the similarity between the Falkner–Skan and the moving sheet flow; the Sakiadis flow is governed by the Blasius' equation which is actually the Falkner–Skan equation (for $m = 0$), and the Crane's flow is also governed by the Falkner–Skan' equation (for $m = 1$). So far, the difference between the two flows, due to moving continuous surface and on the finite surface, is only due to the boundary conditions. Crane reported a closed form solution to his problem and calculated the heat transfer coefficient analytically.

1.2.2 Power-Law and Exponential Stretching Velocities

After Crane [19], thirteen more years of ignorance passed and the moving plate boundary-layers enjoyed no significant advancement. In 1983, Banks [21] introduced the power-law stretching velocity of the Falkner–Skan form $u_w = ax^m$ and reported similarity solution to this case. Banks utilized the similarity transformations of the form

$$\eta = \sqrt{\frac{a}{\nu}} x^{\frac{m-1}{2}} y, \quad u = ax^m f'(\eta), \quad v = -\sqrt{a\nu} x^{\frac{m-1}{2}} \left(\frac{m+1}{2} f + \frac{m-1}{2} \eta f' \right), \tag{1.7}$$

and transformed Eqs. (1.1a, 1.1b) to the form

$$f''' + \frac{m+1}{2} ff'' - mf'^2 = 0, \tag{1.8}$$

subject to the boundary conditions (1.3). Again, in this case too, self-similar Eq. (1.8) is the same as that of Falkner–Skan [20] for the potential flow $u_\infty(x) = ax^m$, but the only difference is of the boundary conditions. At this stage, it has now become clear that the two flows are governed by the same self-similar equation but own different boundary conditions. The similarity transformations applicable to the cases of finite surfaces are equally applicable to the corresponding cases of continuous surface flows. Later in 1999, Magyari and Keller [22] introduced the self-similar flow due to an exponentially stretching continuous surface. They assumed the wall velocity of the form $u_w = U_0 e^{\frac{x}{L}}$ and utilized the following similarity transformations

$$\eta = \left(\frac{Re_L}{2} \right)^{1/2} \frac{y}{L} e^{x/2L}, \quad u = U_0 e^{x/L} f'(\eta), \quad v = -\frac{\nu}{L} \left(\frac{Re_L}{2} \right)^{1/2} e^{x/2L} (f + \eta f'), \tag{1.9}$$

to reduce Eqs. (1.1a, 1.1b) to the self-similar form

$$f''' + ff'' - 2f'^2 = 0. \tag{1.10}$$

In addition to the two-dimensional case, the axially symmetric and the three-dimensional cases regarding the boundary-layer flow on continuous surfaces have also been reported in the literature. The axially symmetric case includes the flow due to a uniformly moving/stretching cylinder [18, 23] and uniformly stretching circular flat disk [24]. The credit of stretching disk case also goes, indirectly, to Crane who extended the idea of stretching surface flow to the axisymmetric case in 1975. The three-dimensional flow due to a stretching sheet has been investigated by stretching the sheet uniformly [25] or exponentially [26] in the two lateral directions. The case of three-dimensional flow due to linear bilateral stretching of the sheet was considered by Wang [25] in 1984, whereas the exponential form of stretching wall velocity, in three-dimensional flow, was considered by Liu et al. [26] in 2013.[1]

[1]However, the author is not sure if the three-dimensional flow due to exponentially stretching sheet was actually first introduced by [26].

In second part of this book, it will be shown that the two-dimensional case of stretching sheet flow has almost been explored completely,[2] in the existing literature, with some deficiency in the exponential stretching case. However, a very little has been done for the axially symmetric and three-dimensional cases and a big class of self-similar solutions associated with these cases is yet unexplored. The complete[3] self-similar criterion for the three cases, namely the two-dimensional, three-dimensional and axially symmetric flows, has been derived in detail in Chap. 5.

1.3 Shrinking Sheet Flow

In addition to the stretching sheet flows, there is another important, perhaps, interesting class of self-similar flows which is commonly referred to as the shrinking sheet flows. These flows correspond to the situations when the stretching wall velocity (discussed in the previous section) is given the '–ve' sign. Such a shrinking sheet flow was first introduced by Miklavcic and Wang [27] in 2006 where they assumed the wall velocities of the form

$$u = -ax, \quad v = -a(M-1)y, \tag{1.11}$$

with $a > 0$ for a steady three-dimensional flow due to bilateral motion (shrinking) of the flexible sheet and M being 0 or 1. For this flow, they developed the self-similar momentum equation of the form

$$f''' + Mff'' - f'^2 = 0, \tag{1.12}$$

which is the same as for the corresponding stretching sheet flow, but the only difference arose in the boundary conditions: For this case, they obtained $f'(0) = -1$ instead of $f'(0) = 1$ and the ambient condition stayed the same as it does in the stretching sheet flow. With this modeling, they reached a conclusion that the solution to this problem does not exist in the absence of sufficient wall suction.

Following Miklavcic and Wang [27], number of researchers got impressed by this flow and went involved in studying this flow for various flow situations. Consequently, they contributed a great number of research papers in the last decade on this flow. But there happened a very big misfortune with this case that the pioneer authors [27] committed a little mistake in the dimensionless self-similar

[2]Although the developed procedures of finding the self-similar solutions, either systematic or ad hoc, are actually based on the group theoretical approach explained in Chap 3, it is, however, mentioned there that finding some similarity solutions does not mean that one has explored all the similarity solutions and the existence of any other self-similar solution can never be denied.

[3]Complete in a sense, and this completeness does not deny the existence of any other self-similar solution.

formation of this flow and also a little bit mishandling, which was further followed by the other researchers in toto. Consequently, they obtained wrong results and tried to justify them with the help of non-physical reasoning. This mistake committed by the authors of [27] and the subsequent authors[4] is explained in detail in Chap. 7 where the correct self-similar formulation to this case with appropriate interpretation of the wall velocities is also presented.

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[4]Here we intentionally avoid the citation of any such paper because the literature on shrinking surface flow is great in number and it is impossible to cite all those studies over here. Therefore, the author is of the opinion that citing few of them is as bad as ignoring the rest of them.

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