Alternating Hadamard Series and Some Theorems on Strongly Regular Graphs

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Abstract In this paper we consider a strongly regular graph, G, whose adjacency matrix A has three distinct eigenvalues, and a particular real three dimensional Euclidean Jordan subalgebra with rank three of the Euclidean algebra of real symmetric matrices of order n, with the product and the inner product being the Jordan product and the usual trace of matrices, respectively. Next, we compute the unique Jordan frame $\mathscr{B}$ associated to A and we consider particular alternating Hadamard series constructed from the idempotents of $\mathscr{B}$. Finally, by the analysis of the spectra of the sums of these alternating Hadamard series we deduce some theorems over the parameters of a strongly regular graph.

Keywords Graphs and linear algebra · Algebraic combinatorics · Graph theory

1 Introduction

In this chapter we establish some inequalities on the parameters of a strongly regular graph like we have done in the papers [11–13, 16] but recurring to alternating Hadamard series of a matrix.

Euclidean Jordan algebras have a lot of applications to many branches of mathematics, for instance in statistics (see [14]), interior point methods (see [3, 6, 7]

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and combinatorics (see [2, 11–13, 16]). In this paper we apply the theory of Euclidean Jordan algebras to strongly regular graphs to present some theorems over their parameters.

This paper is organized as follows. In Sects. 2 and 3, we present some preliminary definitions and results on Euclidean Jordan algebras and strongly regular graphs, respectively, that will be used throughout this paper. In Sect. 4 we associate a particular real Euclidean Jordan algebra to the adjacency matrix of a strongly regular graph, A, and we consider the unique Jordan frame, $\mathscr{B}$, associated to A. Next, we establish some theorems over the parameters and the spectra of a strongly regular graph by the analysis of the spectra of an alternating Hadamard series of an element of $\mathscr{B}$. Finally, in Sect. 5, we present some experimental results.

2 Finite Dimensional Real Euclidean Jordan Algebras

In this section we present the concepts on Euclidean Jordan algebras which are relevant for our work. A more detailed exposition can be found in the monograph by Faraut and Korányi [5], and in Koecher's lecture notes, [10].

Let $\mathscr{A}$ be a finite dimensional real algebra with a bilinear mapping $(u, v) \mapsto u \cdot v$ from $\mathscr{A} \times \mathscr{A}$ into $\mathscr{A}$. Then $\mathscr{A}$ is a real Jordan algebra if $u \cdot v = v \cdot u$ and $u \cdot (u^2 \cdot v) = u^2 \cdot (u \cdot v)$, where $u^2 = u \cdot u$. From now on we suppose that if $\mathscr{A}$ is a real Jordan algebra, then $\mathscr{A}$ is a finite dimensional real algebra and has a unit element denoted by $\mathbf{e}$.

Example 1 The real vector space of real symmetric matrices of order n, $\mathscr{A} = \text{Sym}_n(\mathbb{R})$, equipped with the bilinear map $u \bullet v = (uv + vu)/2$ is a real Jordan algebra.

Remark 1 Let $\mathscr{A}$ be a finite dimensional associative real algebra with the bilinear map $(u, v) \mapsto u \cdot v$. We introduce on $\mathscr{A}$ a structure of Jordan algebra by considering a new product $\bullet$ defined by $u \bullet v = (u \cdot v + v \cdot u)/2$ for all u and v in $\mathscr{A}$. The product $\bullet$ is called the Jordan product.

A real Jordan algebra is not necessarily an associative algebra. But, a real Jordan algebra is always power associative, that is, is an algebra such that the algebra spanned by any element and the unit is associative.

Let $\mathscr{A}$ be a n-dimensional real Jordan algebra and u in $\mathscr{A}$. The rank of u is the least natural number k such that $\{\mathbf{e}, u, \ldots, u^k\}$ is linearly dependent and we write $\text{rank}(u) = k$. Since $\text{rank}(u) \leq n$ we define the rank of $\mathscr{A}$ as being the natural number $\text{rank}(\mathscr{A}) = \max\{\text{rank}(u) : u \in \mathscr{A}\}$. An element u in $\mathscr{A}$ is regular if $\text{rank}(u) = \text{rank}(\mathscr{A})$. Let u be a regular element of $\mathscr{A}$ and $r = \text{rank}(u)$. Then, there exist real scalars $a_1(u), a_2(u), \ldots, a_{r-1}(u)$ and $a_r(u)$ such that

$$u^r - a_1(u)u^{r-1} + \cdots + (-1)^r a_r(u)\mathbf{e} = 0, \tag{1}$$

where 0 is the null vector of $\mathscr{A}$. Taking into account (1) we conclude that the polynomial

$$p_u(\lambda) = \lambda^r - a_1(u)\lambda^{r-1} + \cdots + (-1)^r a_r(u) \tag{2}$$

is the minimal polynomial of u. When u is not regular the minimal polynomial of u has a degree less than r. The roots of the minimal polynomial of u are the eigenvalues of u.

A real Euclidean Jordan algebra $\mathscr{A}$ is a Jordan algebra with an inner product $< \cdot, \cdot >$ such that $< u \cdot v, w >=< v, u \cdot w >$ for all u, v and w in $\mathscr{A}$.

Example 2 The real vector space $\mathrm{Sym}_n(\mathbb{R})$ is a real Euclidean Jordan algebra when endowed with the Jordan product and with the inner product $< u, v >= \mathrm{tr}(uv)$, where tr denotes the usual trace of matrices.

Let $\mathscr{A}$ be a real Euclidean Jordan algebra with unit element $\mathbf{e}$. An element f in $\mathscr{A}$ is an idempotent if $f^2 = f$. Two idempotents f_1 and f_2 are orthogonal if $f_1 \cdot f_2 = 0$. A *complete system of orthogonal idempotents* of $\mathscr{A}$ is a set $\{f_1, f_2, \ldots, f_k\}$ such that (i) $f_i^2 = f_i, \forall i \in \{1, \ldots, k\}$; (ii) $f_i \circ f_j = 0, \forall i \neq j$ and (iii) $f_1 + f_2 + \cdots + f_k = \mathbf{e}$. An idempotent f is primitive if it is a nonzero idempotent of $\mathscr{A}$ and if it can't be written as a sum of two non-zero orthogonal idempotents. We say that $\{f_1, f_2, \ldots, f_k\}$ is a Jordan frame if $\{f_1, f_2, \ldots, f_k\}$ is a complete system of orthogonal idempotents such that each idempotent is primitive.

Theorem 1 *([5], p. 43) Let $\mathscr{A}$ be a real Euclidean Jordan algebra. Then for u in $\mathscr{A}$ there exists unique real numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$, all distinct, and a unique complete system of orthogonal idempotents $\{f_1, f_2, \ldots, f_k\}$ such that*

$$u = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_k f_k. \tag{3}$$

The numbers λ_j's of (3) are the eigenvalues of u and the decomposition (3) is the first spectral decomposition of u.

Theorem 2 *([5], p. 44) Let $\mathscr{A}$ be a real Euclidean Jordan algebra with rank $(\mathscr{A}) = r$. Then, for each u in $\mathscr{A}$ there exists a Jordan frame $\{f_1, f_2, \ldots, f_r\}$ and real numbers $\lambda_1, \ldots, \lambda_{r-1}$ and λ_r such that*

$$u = \lambda_1 f_1 + \lambda_2 f_2 + \cdots + \lambda_r f_r. \tag{4}$$

The numbers λ_j's (with their multiplicities) are uniquely determined by u.

The decomposition (4) is called the second spectral decomposition of u. Regard that the second spectral decomposition of u is not unique.

3 Preliminaries on Strongly Regular Graphs

Herein we will introduce some relevant preliminaries on the theory of strongly regular graphs. Detailed information can be found in [8].

Along this paper, we consider only non-empty, not complete, undirected, simple graphs. Considering a graph G, we denote its vertex set by $V(G)$ and its edge set by $E(G)$. An edge of G with endpoints x and y is denoted by xy. In this case the vertices are called adjacent or neighbors. The number of vertices of G, $|V(G)|$, is called the order of G. If all vertices of G have k neighbors, then G is a k-regular graph.

Let G be a graph of order n. Then G is an (n, k, a, c)-strongly regular graph if it is k-regular and any pair of adjacent vertices have a common neighbors and any pair of non-adjacent vertices have c common neighbors. The parameters of an (n, k, a, c)-strongly regular graph are not independent and are related by the equality

$$k(k - a - 1) = (n - k - 1)c. \tag{5}$$

The adjacency matrix of G, $A = \left[a_{ij}\right]$, is a matrix of order n such that $a_{ij} = 1$, if the vertex i is adjacent to j and 0 otherwise. The adjacency matrix of a strongly regular graph satisfies the equation $A^2 = kI_n + aA + c(J_n - A - I_n)$, where J_n is the all one matrix of order n and I_n is the identity matrix of order n.

Equation (5) is an example of a condition that must be satisfied by the parameters of any strongly regular graph. Among the most important feasibility conditions there are the Krein conditions obtained in 1973 by Scott Jr [15], and the Absolute Bounds by Seidel, [4]. However, there are still many parameter sets for which we do not know if they correspond to a strongly regular graph. The most notable example is perhaps the four graph of Moore with parameter set (3250, 57, 0, 1). In this work we deduce some new inequalities on the parameters and on the spectra of a strongly regular graph.

4 Alternating Hadamard Series and Some Theorems on Strongly Regular Graphs

Let G be an (n, k, a, c)-strongly regular graph and A be its adjacency matrix with three distinct eigenvalues, namely k, θ and τ, and let $\mathscr{A} = \mathrm{Sym}_n(\mathbb{R})$. We consider the Euclidean Jordan subalgebra of $\mathscr{A}$, $\mathscr{A}^*$, see [8, p. 177], spanned by I_n, and the natural powers of A. Since A has three distinct eigenvalues, then $\mathscr{A}^*$ is a three dimensional real Euclidean Jordan algebra with $\mathrm{rank}(\mathscr{A}^*) = 3$. Let $\mathscr{B} = \{E_1, E_2, E_3\}$ be the unique complete system of orthogonal idempotents of $\mathscr{A}^*$ associated to A, with

$$E_1 = \frac{1}{n} I_n + \frac{1}{n} A + \frac{1}{n}(J_n - A - I_n),$$
$$E_2 = \frac{|\tau| n + \tau - k}{n(\theta - \tau)} I_n + \frac{n + \tau - k}{n(\theta - \tau)} A + \frac{\tau - k}{n(\theta - \tau)}(J_n - A - I_n),$$
$$E_3 = \frac{\theta n + k - \theta}{n(\theta - \tau)} I_n + \frac{-n + k - \theta}{n(\theta - \tau)} A + \frac{k - \theta}{n(\theta - \tau)}(J_n - A - I_n).$$

Let $M_n(\mathbb{R})$ be the set of square matrices of order n with real entries. For $B = [b_{ij}]$, $C = [c_{ij}]$ in $M_n(\mathbb{R})$, we denote by $B \circ C = [b_{ij} c_{ij}]$ the Hadamard product of matrices B and C (see [9]).

For B in $M_n(\mathbb{R})$ and for l in $\mathbb{N}$ we denote by $B^{\circ l}$ the Hadamard power of order l of B, respectively, with $B^{\circ 1} = B$.

Consider the idempotent E_3 given in the previous section. The eigenvalues q_1, q_2 and q_3 of E_3 are given by

$$q_1 = \frac{\theta n + k - \theta}{n(\theta - \tau)} + \frac{-n + k - \theta}{n(\theta - \tau)} k + \frac{k - \theta}{n(\theta - \tau)}(n - k - 1),$$
$$q_2 = \frac{\theta n + k - \theta}{n(\theta - \tau)} + \frac{-n + k - \theta}{n(\theta - \tau)} \theta + \frac{k - \theta}{n(\theta - \tau)}(-\theta - 1),$$
$$q_3 = \frac{\theta n + k - \theta}{n(\theta - \tau)} + \frac{-n + k - \theta}{n(\theta - \tau)} \tau + \frac{k - \theta}{n(\theta - \tau)}(-\tau - 1).$$

From E_3 we build the following partial sum:

$$S_{4l-1} = \sum_{j=1}^{2l} (-1)^{j-1} \frac{(E_3^{\circ 2})^{\circ(2j-1)}}{(2j-1)!} + \frac{1}{3!} \left(\frac{\theta n + k - \theta}{n(\theta - \tau)} \right)^3 \frac{1 - \left(\frac{\theta n + k - \theta}{n(\theta - \tau)} \right)^{4l}}{1 - \left(\frac{\theta n + k - \theta}{n(\theta - \tau)} \right)^4} I_n.$$

Since $\mathscr{A}^*$ is closed under the Hadamard product and $\mathscr{B}$ is a basis of $\mathscr{A}^*$, we can write S_{4l-1} as: $S_{4l-1} = \sum_{i=1}^{3} q^i_{S_{4l-1}} E_i$, where the $q^i_{S_{4l-1}}$ with $i \in \{1, 2, 3\}$, are the eigenvalues of S_{4l-1}. We prove that $q^i_{S_{4l-1}} \geq 0$, $\forall i \in \{1, 2, 3\}$. First, we note the following identity regarding one of the eigenvalues of $E_3^{\circ 2}$:

$$\left(\frac{\theta n + k - \theta}{n(\theta - \tau)} \right)^2 + \left(\frac{-n + k - \theta}{n(\theta - \tau)} \right)^2 k + \left(\frac{(k - \theta)}{n(\theta - \tau)} \right)^2 (n - k - 1) = \frac{\theta n + k - \theta}{n(\theta - \tau)}.$$

Secondly, since all of the eigenvalues of E_3 are smaller in modulus than q_1, then the eigenvalues of the summands of $\sum_{j=1}^{2l} (-1)^{j-1} (E_3^{\circ 2})^{\circ(2j-1)}/(2j-1)!$ when $j - 1$ is odd are smaller, in modulus, than $\frac{1}{3!} \left(\theta n + k - \theta)/(n(\theta - \tau))\right)^{2j-1}$. For this assertion we also use the property $\lambda_{\max}(A_1 \circ \cdots \circ A_i) \leq \lambda_{\max}(A_1) \ldots \lambda_{\max}(A_i)$, where $\lambda_{\max}(A)$ denotes the maximum eigenvalue of the matrix A. Therefore, we conclude that all the eigenvalues of S_{4l-1} are nonnegative. Now we consider the sum $S_\infty = \lim_{l \to \infty} S_{4l-1}$. Therefore we have:

$$S_\infty = \left[\sin\left(\frac{(\theta n + k - \theta)}{n(\theta-\tau)}\right)^2 + \frac{1}{3!}\left(\frac{\theta n + k - \theta}{n(\theta-\tau)}\right)^3 \frac{1}{1-\left(\frac{\theta * n + k - \theta}{n(\theta-\tau)}\right)^4}\right] I_n + \\ + \sin\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2 A + \sin\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2 (J_n - A - I_n).$$

Let q_∞^i, $i \in \{1,2,3\}$ be the eigenvalues of S_∞ such that $S_\infty = \sum_{i=1}^3 q_\infty^i E_i$. Then, since $q_\infty^i = \lim_{l\to} q_{S_{2l-1}}^i$, for $i \in \{1,2,3\}$, and $q_{S_{2l-1}}^i \geq 0, \forall i \in \{1,2,3\}$, we conclude that $q_\infty^i \geq 0, \forall i \in \{1,2,3\}$.

Finally, we consider the new matrix, S_3, obtained as $S_3 = E_3 \circ S_\infty$. The eigenvalues of S_3 are also nonnegative because of the non-negativity of the eigenvalues of E_3 and S_∞ and the property $\lambda_{\min}(A \circ B) \geq \lambda_{\min}(A)\lambda_{\min}(B)$, where $\lambda_{\min}(A)$ denotes the minimum eigenvalue of the matrix A. From the non-negativity of the eigenvalues of S_3 we establish the following result.

Theorem 3 *Let X be a strongly regular graph with parameter set (n,k,a,c) and three distinct eigenvalues, k, θ and τ. If $k < n/3$ and $\theta < |\tau| - 2/3$, then*

$$k \leq \frac{56}{9}\frac{(3\theta+1)^3\theta^4}{32\theta^4-1}. \tag{6}$$

Proof Let q_3^i, $i \in \{1,2,3\}$ be the eigenvalues of S_3 then $S_3 = \sum_{i=1}^3 q_3^i E_i$. We have already proved that all the eigenvalues of S_3 are nonnegative. In particular, we have that $q_3^1 \geq 0$, that is

$$0 \leq \frac{\theta n + k - \theta}{n(\theta-\tau)}\left[\sin\left(\frac{(\theta n + k - \theta)}{n(\theta-\tau)}\right)^2 + \frac{1}{3!}\left(\frac{\theta n + k - \theta}{n(\theta-\tau)}\right)^3 \frac{1}{1-\left(\frac{\theta n + k - \theta}{n(\theta-\tau)}\right)^4}\right] \\ + \frac{-n+k-\theta}{n(\theta-\tau)} \sin\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2 k + \\ + \frac{k-\theta}{n(\theta-\tau)} \sin\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2 (n-k-1). \tag{7}$$

Since, for any strongly regular graph, we have $q_1 = 0$, then inequality (7) can be rewritten as

$$0 \leq \frac{\theta n + k - \theta}{n(\theta-\tau)}\left[\sin\left(\frac{(\theta n + k - \theta)}{n(\theta-\tau)}\right)^2 - \sin\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right] \\ + \frac{1}{3!}\left(\frac{\theta n + k - \theta}{n(\theta-\tau)}\right)^4 \frac{1}{1-\left(\frac{\theta n + k - \theta}{n(\theta-\tau)}\right)^4} +$$

$$+\frac{-n+k-\theta}{n(\theta-\tau)}\left[\sin\left(\frac{-n+k-\theta}{n(\theta-\tau)}\right)^2 k-\sin\left(\frac{k-\theta}{n(\theta-\tau)}\right)^2\right]k. \quad (8)$$

Applying the Mean Value Theorem, see [1, Theorem 4.8.2, p. 308], to (8) to the function sin in the interval $\left[((k-\theta)/(n(\theta-\tau)))^2,\ \ ((n-k+\theta)/(n(\theta-\tau)))^2\right]$ and after making the minorization of cos in this interval, and finally since $\sin\big((\theta n+k-\theta)/(n(\theta-\tau))\big)^2 \le \big((\theta n+k-\theta)/(n(\theta-\tau))\big)^2$ one obtains the equality (9).

$$0 \le (\frac{\theta n+k-\theta}{n(\theta-\tau)})^3+\frac{1}{3!}\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^4\frac{1}{1-\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^4}+$$

$$+\frac{-n+k-\theta}{n(\theta-\tau)}\cos(\frac{(n-k+\theta)^2}{n(\theta-\tau)})^2\frac{1}{\theta-\tau}\frac{n-2k+2\theta}{n(\theta-\tau)}k. \quad (9)$$

Since $\theta<|\tau|-\frac{2}{3}$ implies that $\big((\theta n+k-\theta)/(n(\theta-\tau))\big)/\big(1-\big((\theta n+k-\theta)/(n(\theta-\tau))\big)^4\big)\le 1$ and finally since $\cos\big((n-k+\theta)/(n(\theta-\tau))\big)^2\ge 1-(1/32)(1/\theta^4)$ we obtain from (9) the inequality (10).

$$0 \le \frac{7}{6}\left(\frac{\theta n+k-\theta}{n(\theta-\tau)}\right)^3+\frac{-n+k-\theta}{n(\theta-\tau)}\frac{32\theta^4-1}{32\theta^4}\frac{1}{\theta-\tau}\frac{n-2k+2\theta}{n(\theta-\tau)}k. \quad (10)$$

Using the fact that $k<n/3$ and making an algebraic manipulation on the right member of (10) we obtain $k\le 7\big(3\theta+1\big)^3\big(32\theta^4\big)36\big(32\theta^4-1\big)$. □

From Theorem 3 we obtain the Corollary 1.

Corollary 1 *Let X be an strongly regular with the distinct eigenvalues* θ, τ *and* k. *If* $k>\frac{2n}{3}-1$ *and* $|\tau|<\theta+\frac{4}{3}$ *then*

$$n-k-1\le\frac{56}{9}\frac{(3|\tau|-2)^3(|\tau|-1)^4}{32(|\tau|-1)^4-1}. \quad (11)$$

5 Numerical Results

In this section we present some examples of parameter sets that show the effectiveness of the deduced inequalities (6) and (11).

We present in Table 1 some examples of parameter sets (n,k,a,c) that do not verify the inequality (6) of Theorem 3. We consider the parameter sets $P_1=(64,21,0,3)$, $P_2=(300,92,10,36)$, $P_3=(1156,275,18,80)$, $P_4=(1225,408,59,174)$ and $P_5=(1225,352,24,132)$. For each example we have $k<n/3$ and we present the respective eigenvalues θ, τ and the value of $q_{\theta k}$ defined by

Table 1 Numerical results when $k < n/3$

	P_1	P_2	P_3	P_4	P_5
θ	1	2	3	2	2
τ	−11	−28	−65	−117	−110
$q_{\theta k}$	−8.2	−25.2	−80.5	−342	−286.2

Table 2 Numerical results when $k > 2n/3 - 1$

	P_6	P_7	P_8	P_9	P_{10}
θ	10.0	27.0	64.0	116.0	109.0
τ	−2.0	−3	−4	−3.0	−3
$q_{\tau k}$	−8.2	−25.2	−80.5	−342.2	−285.2

$$q_{\theta k} = \left[56(3\theta + 1)^3\theta^4\right] / \left[9(32\theta^4 - 1)\right] - k.$$

Next, in Table 2, we present some examples of parameter sets (n, k, a, c) that do not verify the inequality (11) of Corollary 1. We consider the parameter sets $P_6 = (64, 42, 30, 22)$, $P_7 = (300, 207, 150, 126)$, $P_8 = (1156, 880, 684, 624)$, $P_9 = (1225, 816, 581, 468)$ and $P_{10} = (1225, 872, 651, 545)$. For each example we have $k > 2n/3 - 1$ and we present the respective data as in Table 1 but in the last line we compute the value of $q_{\tau k} = \left[56(3|\tau| - 2)^3(|\tau| - 1)^4\right] / \left[9(32(|\tau| - 1)^4 - 1)\right] - (n - k - 1)$.

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