Chapter 6

Principal Bundles and Characteristic Classes

A principal bundle is a locally trivial family of groups. It turns out that the theory of connections on a vector bundle can be subsumed under the theory of connections on a principal bundle. The latter, moreover, has the advantage that its connection forms are basis-free.

In this chapter we will first give several equivalent constructions of a connection on a principal bundle, and then generalize the notion curvature to a principal bundle, paving the way to a generalization of characteristic classes to principal bundles. Along the way, we also generalize covariant derivatives to principal bundles.

§27 Principal Bundles

We saw in Section 11 that a connection ∇ on a vector bundle E over a manifold M can be represented by a matrix of 1-forms over a framed open set. For any frame $e = [e_1 \ \cdots \ e_r]$ for E over an open set U, the connection matrix ω_e relative to e is defined by

$$\nabla_X e_j = \sum_i (\omega_e)^i_j(X) e_i$$

for all C^∞ vector fields X over U. If $\bar{e} = [\bar{e}_1 \ \cdots \ \bar{e}_r] = ea$ is another frame for E over U, where $a\colon U \to \mathrm{GL}(r, \mathbb{R})$ is a matrix of C^∞ transition functions, then by Theorem 22.1 the connection matrix ω_e transforms according to the rule

$$\omega_{\bar{e}} = a^{-1}\omega_e a + a^{-1}\,da.$$

Associated to a vector bundle is an object called its ***frame bundle*** $\pi\colon \mathrm{Fr}(E) \to M$; the total space $\mathrm{Fr}(E)$ of the frame bundle is the set of all ordered bases in the fibers of the vector bundle $E \to M$, with a suitable topology and manifold structure. A ***section*** of the frame bundle $\pi\colon \mathrm{Fr}(E) \to M$ over an open set $U \subset M$ is a map $s\colon U \to \mathrm{Fr}(E)$

L.W. Tu, *Differential Geometry*, Graduate Texts in Mathematics 275,
DOI 10.1007/978-3-319-55084-8_6

such that $\pi \circ s = \mathbb{1}_U$, the identity map on U. From this point of view a frame $e = [e_1 \ \cdots \ e_r]$ over U is simply a section of the frame bundle $\mathrm{Fr}(E)$ over U.

Suppose ∇ is a connection on the vector bundle $E \to M$. Miraculously, there exists a matrix-valued 1-form ω on the frame bundle $\mathrm{Fr}(E)$ such that for every frame e over an open set $U \subset M$, the connection matrix ω_e of ∇ is the pullback of ω by the section $e\colon U \to \mathrm{Fr}(E)$ (Theorem 29.10). This matrix-valued 1-form, called an ***Ehresmann connection*** on the frame bundle $\mathrm{Fr}(E)$, is determined uniquely by the connection on the vector bundle E and vice versa. It is an intrinsic object of which a connection matrix ω_e is but a local manifestation. The frame bundle of a vector bundle is an example of a principal G-bundle for the group $G = \mathrm{GL}(r, \mathbb{R})$. The Ehresmann connection on the frame bundle generalizes to a connection on an arbitrary principal bundle.

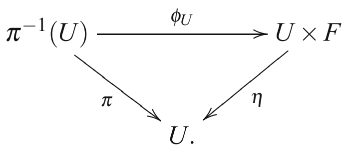
This section collects together some general facts about principal bundles.

Charles Ehresmann

(1905–1979)

27.1 Principal Bundles

Let E, M, and F be manifolds. We will denote an open cover $\mathfrak{U}$ of M either as $\{U_\alpha\}$ or more simply as an unindexed set $\{U\}$ whose general element is denoted by U. A ***local trivialization*** with fiber F for a smooth surjection $\pi\colon E \to M$ is an open cover $\mathfrak{U} = \{U\}$ for M together with a collection $\{\phi_U\colon \pi^{-1}(U) \to U \times F \mid U \in \mathfrak{U}\}$ of fiber-preserving diffeomorphisms $\phi_U\colon \pi^{-1}(U) \to U \times F$:



where η is projection to the first factor. A ***fiber bundle*** with fiber F is a smooth surjection $\pi\colon E \to M$ having a local trivialization with fiber F. We also say that it is ***locally trivial*** with fiber F. The manifold E is the ***total space*** and the manifold M the ***base space*** of the fiber bundle.

The ***fiber*** of a fiber bundle $\pi\colon E \to M$ over $x \in M$ is the set $E_x := \pi^{-1}(x)$. Because π is a submersion, by the regular level set theorem ([21], Th. 9.13, p. 96) each fiber E_x is a regular submanifold of E. For $x \in U$, define $\phi_{U,x} := \phi_U|_{E_x}\colon E_x \to \{x\} \times F$ to be the restriction of the trivialization $\phi_U\colon \pi^{-1}(U) \to U \times F$ to the fiber E_x.

Proposition 27.1. *Let $\pi\colon E \to M$ be a fiber bundle with fiber F. If $\phi_U\colon \pi^{-1}(U) \to U \times F$ is a trivialization, then $\phi_{U,x}\colon E_x \to \{x\} \times F$ is a diffeomorphism.*

Proof. The map $\phi_{U,x}$ is smooth because it is the restriction of the smooth map ϕ_U to a regular submanifold. It is bijective because ϕ_U is bijective and fiber-preserving. Its inverse $\phi_{U,x}^{-1}$ is the restriction of the smooth map $\phi_U^{-1}: U \times F \to \pi^{-1}(U)$ to the fiber $\{x\} \times F$ and is therefore also smooth. □

A smooth ***right action*** of a Lie group G on a manifold M is a smooth map

$$\mu : M \times G \to M,$$

denoted by $x \cdot g := \mu(x,g)$, such that for all $x \in M$ and $g,h \in G$,

(i) $x \cdot e = x$, where e is the identity element of G,
(ii) $(x \cdot g) \cdot h = x \cdot (gh)$.

We often omit the dot and write more simply xg for $x \cdot g$. If there is such a map μ, we also say that G ***acts smoothly on*** M ***on the right***. A ***left action*** is defined similarly. The ***stabilizer*** of a point $x \in M$ under an action of G is the subgroup

$$\operatorname{Stab}(x) := \{g \in G \mid x \cdot g = x\}.$$

The ***orbit*** of $x \in M$ is the set

$$\operatorname{Orbit}(x) := xG := \{x \cdot g \in M \mid g \in G\}.$$

Denote by $\operatorname{Stab}(x)\backslash G$ the set of right cosets of $\operatorname{Stab}(x)$ in G. By the orbit-stabilizer theorem, for each $x \in M$ the map: $G \to \operatorname{Orbit}(x)$, $g \mapsto x \cdot g$ induces a bijection of sets:

$$\operatorname{Stab}(x)\backslash G \longleftrightarrow \operatorname{Orbit}(x),$$
$$\operatorname{Stab}(x)g \longleftrightarrow x \cdot g.$$

The action of G on M is ***free*** if the stabilizer of every point $x \in M$ is the trivial subgroup $\{e\}$.

A manifold M together with a right action of a Lie group G on M is called a ***right G-manifold*** or simply a ***G-manifold***. A map $f : N \to M$ between right G-manifolds is ***right G-equivariant*** if

$$f(x \cdot g) = f(x) \cdot g$$

for all $(x,g) \in N \times G$. Similarly, a map $f : N \to M$ between left G-manifolds is ***left G-equivariant*** if

$$f(g \cdot x) = g \cdot f(x)$$

for all $(g,x) \in G \times N$.

A left action can be turned into a right action and vice versa; for example, if G acts on M on the left, then

$$x \cdot g = g^{-1} \cdot x$$

is a right action of G on M. Thus, if N is a right G-manifold and M is a left G-manifold, we say a map $f : N \to M$ is ***G-equivariant*** if

$$f(x \cdot g) = f(x) \cdot g = g^{-1} \cdot f(x) \tag{27.1}$$

for all $(x,g) \in N \times G$.

A smooth fiber bundle $\pi\colon P \to M$ with fiber G is a smooth ***principal G-bundle*** if G acts smoothly and freely on P on the right and the fiber-preserving local trivializations

$$\phi_U\colon \pi^{-1}(U) \to U \times G$$

are G-equivariant, where G acts on $U \times G$ on the right by

$$(x,h)\cdot g = (x,hg).$$

Example 27.2 (*Product G-bundles*). The simplest example of a principal G-bundle over a manifold M is the product G-bundle $\eta\colon M \times G \to M$. A trivialization is the identity map on $M \times G$.

Example 27.3 (*Homogenous manifolds*). If G is a Lie group and H is a closed subgroup, then the quotient G/H can be given the structure of a manifold such that the projection map $\pi\colon G \to G/H$ is a principal H-bundle. This is proven in [22, Th. 3.58, p. 120].

Example 27.4 (*Hopf bundle*). The group S^1 of unit complex numbers acts on the complex vector space $\mathbb{C}^{n+1}$ by left multiplication. This action induces an action of S^1 on the unit sphere S^{2n+1} in $\mathbb{C}^{n+1}$. The complex projective space $\mathbb{C}P^n$ may be defined as the orbit space of S^{2n+1} by S^1. The natural projection $S^{2n+1} \to \mathbb{C}P^n$ with fiber S^1 turn out to be a principal S^1-bundle. When $n = 1$, $S^3 \to \mathbb{C}P^1$ with fiber S^1 is called the ***Hopf bundle***.

Definition 27.5. Let $\pi_Q\colon Q \to N$ and $\pi_P\colon P \to M$ be principal G-bundles. A ***morphism*** of principal G-bundles is a pair of maps $(\bar{f}\colon Q \to P, f\colon N \to M)$ such that $\bar{f}\colon Q \to P$ is G-equivariant and the diagram

$$\begin{array}{ccc} Q & \xrightarrow{\bar{f}} & P \\ {\scriptstyle \pi_Q}\downarrow & & \downarrow{\scriptstyle \pi_P} \\ N & \xrightarrow[f]{} & M \end{array}$$

commutes.

Proposition 27.6. *If $\pi\colon P \to M$ is a principal G-bundle, then the group G acts transitively on each fiber.*

Proof. Since G acts transitively on $\{x\} \times G$ and the fiber diffeomorphism $\phi_{U,x}\colon P_x \to \{x\} \times G$ is G-equivariant, G must also act transitively on the fiber P_x. □

Lemma 27.7. *For any group G, a right G-equivariant map $f\colon G \to G$ is necessarily a left translation.*

Proof. Suppose that for all $x, g \in G$,

$$f(xg) = f(x)g.$$

Setting $x = e$, the identity element of G, we obtain

$$f(g) = f(e)g = \ell_{f(e)}(g),$$

where $\ell_{f(e)} \colon G \to G$ is left translation by $f(e)$. □

Suppose $\{U_\alpha\}_{\alpha \in \mathrm{A}}$ is a local trivialization for a principal G-bundle $\pi \colon P \to M$. Whenever the intersection $U_{\alpha\beta} := U_\alpha \cap U_\beta$ is nonempty, there are two trivializations on $\pi^{-1}(U_{\alpha\beta})$:

$$U_{\alpha\beta} \times G \xleftarrow{\phi_\alpha} \pi^{-1}(U_{\alpha\beta}) \xrightarrow{\phi_\beta} U_{\alpha\beta} \times G.$$

Then $\phi_\alpha \circ \phi_\beta^{-1} \colon U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G$ is a fiber-preserving right G-equivariant map. By Lemma 27.7, it is a left translation on each fiber. Thus,

$$(\phi_\alpha \circ \phi_\beta^{-1})(x,h) = (x, g_{\alpha\beta}(x)h), \tag{27.2}$$

where $(x,h) \in U_{\alpha\beta} \times G$ and $g_{\alpha\beta}(x) \in G$. Because $\phi_\alpha \circ \phi_\beta^{-1}$ is a C^∞ function of x and h, setting $h = e$, we see that $g_{\alpha\beta}(x)$ is a C^∞ function of x. The C^∞ functions $g_{\alpha\beta} \colon U_{\alpha\beta} \to G$ are called ***transition functions*** of the principal bundle $\pi \colon P \to M$ relative to the trivializing open cover $\{U_\alpha\}_{\alpha \in \mathrm{A}}$. They satisfy the ***cocycle condition***: for all $\alpha, \beta, \gamma \in \mathrm{A}$,

$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \quad \text{if } U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing.$$

From the cocycle condition, one can deduce other properties of the transition functions.

Proposition 27.8. *The transition functions $g_{\alpha\beta}$ of a principal bundle $\pi \colon P \to M$ relative to a trivializing open cover $\{U_\alpha\}_{\alpha \in \mathrm{A}}$ satisfy the following properties: for all $\alpha, \beta \in \mathrm{A}$,*

(i) $g_{\alpha\alpha} =$ *the constant map e,*
(ii) $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ *if $U_\alpha \cap U_\beta \neq \varnothing$.*

Proof. (i) If $\alpha = \beta = \gamma$, the cocycle condition gives

$$g_{\alpha\alpha} g_{\alpha\alpha} = g_{\alpha\alpha}.$$

Hence, $g_{\alpha\alpha} =$ the constant map e.
(ii) if $\gamma = \alpha$, the cocycle condition gives

$$g_{\alpha\beta} g_{\beta\alpha} = g_{\alpha\alpha} = e$$

or

$$g_{\alpha\beta} = g_{\beta\alpha}^{-1} \quad \text{for } U_\alpha \cap U_\beta \neq \varnothing.$$

□

In a principal G-bundle $P \to M$, the group G acts on the right on the total space P, but the transition functions $g_{\alpha\beta}$ in (27.2) are given by left translations by $g_{\alpha\beta}(x) \in G$. This phenomenon is a consequence of Lemma 27.7.

27.2 The Frame Bundle of a Vector Bundle

For any real vector space V, let $\mathrm{Fr}(V)$ be the set of all ordered bases in V. Suppose V has dimension r. We will represent an ordered basis $v_1,\dots,v_r$ by a row vector $v=[v_1 \ \cdots \ v_r]$, so that the general linear group $\mathrm{GL}(r,\mathbb{R})$ acts on $\mathrm{Fr}(V)$ on the right by matrix multiplication

$$\begin{aligned} v\cdot a &= [v_1 \ \cdots \ v_r][a^i_j] \\ &= \left[\sum v_i a^i_1 \ \cdots \ \sum v_i a^i_r\right]. \end{aligned}$$

Fix a point $v\in \mathrm{Fr}(V)$. Since the action of $\mathrm{GL}(r,\mathbb{R})$ on $\mathrm{Fr}(V)$ is clearly transitive and free, i.e., $\mathrm{Orbit}(v)=\mathrm{Fr}(V)$ and $\mathrm{Stab}(v)=\{I\}$, by the orbit-stabilizer theorem there is a bijection

$$\begin{aligned} \phi_v\colon\ \mathrm{GL}(r,\mathbb{R}) = \frac{\mathrm{GL}(r,\mathbb{R})}{\mathrm{Stab}(v)} &\longleftrightarrow \mathrm{Orbit}(v)=\mathrm{Fr}(V), \\ g &\longleftrightarrow vg. \end{aligned}$$

Using the bijection ϕ_v, we put a manifold structure on $\mathrm{Fr}(V)$ in such a way that ϕ_v becomes a diffeomorphism.

If v' is another element of $\mathrm{Fr}(V)$, then $v'=va$ for some $a\in\mathrm{GL}(r,\mathbb{R})$ and

$$\phi_{va}(g)=vag=\phi_v(ag)=(\phi_v\circ\ell_a)(g).$$

Since left multiplication $\ell_a\colon\ \mathrm{GL}(r,\mathbb{R})\to\mathrm{GL}(r,\mathbb{R})$ is a diffeomorphism, the manifold structure on $\mathrm{Fr}(V)$ defined by ϕ_v is the same as the one defined by ϕ_{va}. We call $\mathrm{Fr}(V)$ with this manifold structure the ***frame manifold*** of the vector space V.

Remark 27.9. A linear isomorphism $\phi\colon V\to W$ induces a C^∞ diffeomorphism $\widetilde{\phi}\colon$ $\mathrm{Fr}(V)\to\mathrm{Fr}(W)$ by

$$\widetilde{\phi}[v_1 \ \cdots \ v_r]=[\phi(v_1) \ \cdots \ \phi(v_r)].$$

Define an action of $\mathrm{GL}(r,\mathbb{R})$ on $\mathrm{Fr}(\mathbb{R}^r)$ by

$$g\cdot[v_1 \ \cdots \ v_r]=[gv_1 \ \cdots \ gv_r].$$

Thus, if $\phi\colon\mathbb{R}^r\to\mathbb{R}^r$ is given by left multiplication by $g\in\mathrm{GL}(r,\mathbb{R})$, then so is the induced map $\widetilde{\phi}$ on the frame manifold $\mathrm{Fr}(\mathbb{R}^r)$.

Example 27.10 (*The frame bundle*). Let $\eta: E\to M$ be a C^∞ vector bundle of rank r. We associate to the vector bundle E a C^∞ principal $\mathrm{GL}(r,\mathbb{R})$-bundle $\pi\colon\ \mathrm{Fr}(E)\to M$ as follows. As a set the total space $\mathrm{Fr}(E)$ is defined to be the disjoint union

$$\mathrm{Fr}(E)=\coprod_{x\in M}\mathrm{Fr}(E_x).$$

There is a natural projection map $\pi\colon\ \mathrm{Fr}(E)\to M$ that maps $\mathrm{Fr}(E_x)$ to $\{x\}$.

A local trivialization $\phi_\alpha \colon E|_{U_\alpha} \xrightarrow{\sim} U_\alpha \times \mathbb{R}^r$ induces a bijection

$$\begin{aligned}\widetilde{\phi_\alpha} \colon \operatorname{Fr}(E)|_{U_\alpha} &\xrightarrow{\sim} U_\alpha \times \operatorname{Fr}(\mathbb{R}^r),\\ [v_1 \cdots v_r] \in \operatorname{Fr}(E_x) &\mapsto (x, [\phi_{\alpha,x}(v_1) \cdots \phi_{\alpha,x}(v_r)]).\end{aligned}$$

Via $\widetilde{\phi_\alpha}$ one transfers the topology and manifold structure from $U_\alpha \times \operatorname{Fr}(\mathbb{R}^r)$ to $\operatorname{Fr}(E)|_{U_\alpha}$. This gives $\operatorname{Fr}(E)$ a topology and a manifold structure such that $\pi \colon \operatorname{Fr}(E) \to M$ is locally trivial with fiber $\operatorname{Fr}(\mathbb{R}^r)$. As the frame manifold $\operatorname{Fr}(\mathbb{R}^r)$ is diffeomorphic to the general linear group $\operatorname{GL}(r,\mathbb{R})$, it is easy to check that $\operatorname{Fr}(E) \to M$ is a C^∞ principal $\operatorname{GL}(r,\mathbb{R})$-bundle. We call it the ***frame bundle*** of the vector bundle E.

On a nonempty overlap $U_{\alpha\beta} := U_\alpha \cap U_\beta$, the transition function for the vector bundle E is the C^∞ function $g_{\alpha\beta} \colon U_{\alpha\beta} \to \operatorname{GL}(r,\mathbb{R})$ given by

$$\begin{aligned}\phi_\alpha \circ \phi_\beta^{-1} \colon U_{\alpha\beta} \times \mathbb{R}^r &\to U_{\alpha\beta} \times \mathbb{R}^r,\\ (\phi_\alpha \circ \phi_\beta^{-1})(x,w) &= \big(x, g_{\alpha\beta}(x)w\big).\end{aligned}$$

Since the local trivialization for the frame bundle $\operatorname{Fr}(E)$ is induced from the trivialization $\{U_\alpha, \phi_\alpha\}$ for E, the transition functions for $\operatorname{Fr}(E)$ are induced from the transition functions $\{g_{\alpha\beta}\}$ for E. By Remark 27.9 the transition functions for the open cover $\{\operatorname{Fr}(E)|_{U_\alpha}\}$ of $\operatorname{Fr}(E)$ are the same as the transition functions $g_{\alpha\beta} \colon U_{\alpha\beta} \to \operatorname{GL}(r,\mathbb{R})$ for the vector bundle E, but now of course $\operatorname{GL}(r,\mathbb{R})$ acts on $\operatorname{Fr}(\mathbb{R}^r)$ instead of on $\mathbb{R}^r$.

27.3 Fundamental Vector Fields of a Right Action

Suppose G is a Lie group with Lie algebra $\mathfrak{g}$ and G acts smoothly on a manifold P on the right. To every element $A \in \mathfrak{g}$ one can associate a vector field $\underline{A}$ on P called the ***fundamental vector field on P associated to A***: for p in P, define

$$\underline{A}_p = \left.\frac{d}{dt}\right|_{t=0} p \cdot e^{tA} \in T_pP.$$

To understand this equation, first fix a point $p \in P$. Then $c_p \colon t \mapsto p \cdot e^{tA}$ is a curve in P with initial point p. By definition, the vector $\underline{A}_p$ is the initial vector of this curve. Thus,

$$\underline{A}_p = c_p'(0) = c_{p*}\left(\left.\frac{d}{dt}\right|_{t=0}\right) \in T_pP.$$

As a tangent vector at p is a derivation on germs of C^∞ functions at p, in terms of a C^∞ function f at p,

$$\underline{A}_p f = c_{p*}\left(\left.\frac{d}{dt}\right|_{t=0}\right) f = \left.\frac{d}{dt}\right|_{t=0} f \circ c_p = \left.\frac{d}{dt}\right|_{t=0} f(p \cdot e^{tA}).$$

Proposition 27.11. *For each $A \in \mathfrak{g}$, the fundamental vector field $\underline{A}$ is C^∞ on P.*

Proof. It suffices to show that for every C^∞ function f on P, the function $\underline{A}f$ is also C^∞ on P. Let $\mu\colon P\times G\to P$ be the C^∞ map defining the right action of G on P. For any p in P,

$$\underline{A}_p f = \left.\frac{d}{dt}\right|_{t=0} f(p\cdot e^{tA}) = \left.\frac{d}{dt}\right|_{t=0} (f\circ\mu)(p,e^{tA}).$$

Since e^{tA} is a C^∞ function of t, and f and μ are both C^∞, the derivative

$$\frac{d}{dt}(f\circ\mu)(p,e^{tA})$$

is C^∞ in p and in t. Therefore, $\underline{A}_p f$ is a C^∞ function of p. □

Recall that $\mathfrak{X}(P)$ denotes the Lie algebra of C^∞ vector fields on the manifold P. The fundamental vector field construction gives rise to a map

$$\sigma\colon \mathfrak{g}\to\mathfrak{X}(P),\quad \sigma(A):=\underline{A}.$$

For p in P, define $j_p\colon G\to P$ by $j_p(g)=p\cdot g$. Computing the differential j_{p*} using the curve $c(t)=e^{tA}$, we obtain the expression

$$j_{p*}(A) = \left.\frac{d}{dt}\right|_{t=0} j_p(e^{tA}) = \left.\frac{d}{dt}\right|_{t=0} p\cdot e^{tA} = \underline{A}_p. \tag{27.3}$$

This alternate description of fundamental vector fields, $\underline{A}_p = j_{p*}(A)$, shows that the map $\sigma\colon\mathfrak{g}\to\mathfrak{X}(P)$ is linear over $\mathbb{R}$. In fact, σ is a Lie algebra homomorphism (Problem 27.1).

Example 27.12. Consider the action of a Lie group G on itself by right multiplication. For $p\in G$, the map $j_p\colon G\to G$, $j_p(g)=p\cdot g=\ell_p(g)$ is simply left multiplication by p. By (27.3), for $A\in\mathfrak{g}$, $\underline{A}_p=\ell_{p*}(A)$. Thus, for the action of G on G by right multiplication, the fundamental vector field $\underline{A}$ on G is precisely the left-invariant vector field generated by A. In this sense the fundamental vector field of a right action is a generalization of a left-invariant vector field on a Lie group.

For g in a Lie group G, let $c_g\colon G\to G$ be conjugation by g: $c_g(x)=gxg^{-1}$. The ***adjoint representation*** is defined to be the differential of the conjugation map: $\operatorname{Ad}(g)=(c_g)_*\colon\mathfrak{g}\to\mathfrak{g}$.

Proposition 27.13. *Suppose a Lie group G acts smoothly on a manifold P on the right. Let $r_g\colon P\to P$ be the right translation $r_g(p)=p\cdot g$. For $A\in\mathfrak{g}$ the fundamental vector field $\underline{A}$ on P satisfies the following equivariance property:*

$$r_{g*}\underline{A} = \underline{(\operatorname{Ad} g^{-1})A}.$$

Proof. We need to show that for every p in P, $r_{g*}(\underline{A}_p)=\underline{(\operatorname{Ad} g^{-1})A}_{pg}$. For x in G,

$$(r_g\circ j_p)(x) = pxg = pgg^{-1}xg = j_{pg}(g^{-1}xg) = (j_{pg}\circ c_{g^{-1}})(x).$$

By the chain rule,

$$r_{g*}(\underline{A}_p) = r_{g*}j_{p*}(A) = j_{pg*}(c_{g^{-1}})_*(A) = j_{pg*}\big((\operatorname{Ad} g^{-1})A\big) = \underline{(\operatorname{Ad} g^{-1})A}_{pg}.$$ □

27.4 Integral Curves of a Fundamental Vector Field

In this section suppose a Lie group G with Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ acts smoothly on the right on a manifold P.

Proposition 27.14. *For $p \in P$ and $A \in \mathfrak{g}$, the curve $c_p(t) = p \cdot e^{tA}$, $t \in \mathbb{R}$, is the integral curve of the fundamental vector field $\underline{A}$ through p.*

Proof. We need to show that $c_p'(t) = \underline{A}_{c_p(t)}$ for all $t \in \mathbb{R}$ and all $p \in P$. It is essentially a sequence of definitions:

$$c_p'(t) = \left.\frac{d}{ds}\right|_{s=0} c_p(t+s) = \left.\frac{d}{ds}\right|_{s=0} pe^{tA}e^{sA} = \underline{A}_{pe^{tA}} = \underline{A}_{c_p(t)}. \qquad \square$$

Proposition 27.15. *The fundamental vector field $\underline{A}$ on a manifold P vanishes at a point p in P if and only if A is in the Lie algebra of* $\mathrm{Stab}(p)$.

Proof. $(\Leftarrow)$ If $A \in \mathrm{Lie}\big(\mathrm{Stab}(p)\big)$, then $e^{tA} \in \mathrm{Stab}(p)$, so

$$\underline{A}_p = \left.\frac{d}{dt}\right|_{t=0} p \cdot e^{tA} = \left.\frac{d}{dt}\right|_{t=0} p = 0.$$

$(\Rightarrow)$ Suppose $\underline{A}_p = 0$. Then the constant map $\gamma(t) = p$ is an integral curve of $\underline{A}$ through p, since

$$\gamma'(t) = 0 = \underline{A}_p = \underline{A}_{\gamma(t)}.$$

On the other hand, by Proposition 27.14, $c_p(t) = p \cdot e^{tA}$ is also an integral curve of $\underline{A}$ through p. By the uniqueness of the integral curve through a point, $c_p(t) = \gamma(t)$ or $p \cdot e^{tA} = p$ for all $t \in \mathbb{R}$. This implies that $e^{tA} \in \mathrm{Stab}(p)$ and therefore $A \in \mathrm{Lie}\big(\mathrm{Stab}(p)\big)$. $\square$

Corollary 27.16. *For a right action of a Lie group G on a manifold P, let $p \in P$ and $j_p \colon G \to P$ be the map $j_p(g) = p \cdot g$. Then the kernel* $\ker j_{p*}$ *of the differential of j_p at the identity*

$$j_{p*} = (j_p)_{*,e} \colon \mathfrak{g} \to T_pP$$

is $\mathrm{Lie}\big(\mathrm{Stab}(p)\big)$.

Proof. For $A \in \mathfrak{g}$, we have $\underline{A}_p = j_{p*}(A)$ by (27.3). Thus,

$$\begin{aligned} A \in \ker j_{p*} \quad &\iff \quad j_{p*}(A) = 0 \\ &\iff \quad \underline{A}_p = 0 \\ &\iff \quad A \in \mathrm{Lie}\big(\mathrm{Stab}(p)\big) \quad \text{(by Proposition 27.15).} \end{aligned} \qquad \square$$

27.5 Vertical Subbundle of the Tangent Bundle TP

Throughout this section, G is a Lie group with Lie algebra $\mathfrak{g}$ and $\pi\colon P \to M$ is a principal G-bundle. On the total space P there is a natural notion of vertical tangent vectors. We will show that the vertical tangent vectors on P form a trivial subbundle of the tangent bundle TP.

By the local triviality of a principal bundle, at every point $p \in P$ the differential $\pi_{*,p}\colon T_pP \to T_{\pi(p)}M$ of the projection π is surjective. The ***vertical tangent subspace*** $\mathcal{V}_p \subset T_pP$ is defined to be $\ker \pi_{*,p}$. Hence, there is a short exact sequence of vector spaces

$$0 \to \mathcal{V}_p \longrightarrow T_pP \xrightarrow{\pi_{*,p}} T_{\pi(p)}M \to 0, \tag{27.4}$$

and

$$\dim \mathcal{V}_p = \dim T_pP - \dim T_{\pi(p)}M = \dim G.$$

An element of $\mathcal{V}_p$ is called a ***vertical tangent vector*** at p.

Proposition 27.17. *For any $A \in \mathfrak{g}$, the fundamental vector field $\underline{A}$ is vertical at every point $p \in P$.*

Proof. With $j_p\colon G \to P$ defined as usual by $j_p(g) = p \cdot g$,

$$(\pi \circ j_p)(g) = \pi(p \cdot g) = \pi(p).$$

Since $\underline{A}_p = j_{p*}(A)$ by (27.3), and $\pi \circ j_p$ is a constant map,

$$\pi_{*,p}(\underline{A}_p) = (\pi_{*,p} \circ j_{p*})(A) = (\pi \circ j_p)_*(A) = 0. \qquad \square$$

Thus, in case P is a principal G-bundle, we can refine Corollary 27.16 to show that j_{p*} maps $\mathfrak{g}$ into the vertical tangent space:

$$(j_p)_{*,e}\colon \mathfrak{g} \to \mathcal{V}_p \subset T_pP.$$

In fact, this is an isomorphism.

Proposition 27.18. *For $p \in P$, the differential at e of the map $j_p\colon G \to P$ is an isomorphism of $\mathfrak{g}$ onto the vertical tangent space: $j_{p*} = (j_p)_{*,e}\colon \mathfrak{g} \xrightarrow{\sim} \mathcal{V}_p$.*

Proof. By Corollary 27.16, $\ker j_{p*} = \operatorname{Lie}\big(\operatorname{Stab}(p)\big)$. Since G acts freely on P, the stabilizer of any point $p \in P$ is the trivial subgroup $\{e\}$. Thus, $\ker j_{p*} = 0$ and j_{p*} is injective. By Proposition 27.17, the image j_{p*} lies in the vertical tangent space $\mathcal{V}_p$. Since $\mathfrak{g}$ and $\mathcal{V}_p$ have the same dimension, the injective linear map $j_{p*}\colon \mathfrak{g} \to \mathcal{V}_p$ has to be an isomorphism. $\square$

Corollary 27.19. *The vertical tangent vectors at a point of a principal bundle are precisely the fundamental vectors.*

Let $B_1, \dots, B_\ell$ be a basis for the Lie algebra $\mathfrak{g}$. By the proposition, the fundamental vector fields $\underline{B_1}, \dots, \underline{B_\ell}$ on P form a basis of $\mathcal{V}_p$ at every point $p \in P$. Hence, they span a trivial subbundle $\mathcal{V} := \coprod_{p \in P} \mathcal{V}_p$ of the tangent bundle TP. We call $\mathcal{V}$ the ***vertical subbundle*** of TP.

As we learned in Section 20.5, the differential $\pi_*: TP \to TM$ of a C^∞ map $\pi: P \to M$ induces a bundle map $\tilde{\pi}_*: TP \to \pi^* TM$ over P, given by

$$\begin{array}{ccc} TP & \xrightarrow{\tilde{\pi}_*} & \pi^* TM \\ & \searrow \quad \swarrow & \\ & P & \end{array} \qquad T_pP \ni X_p \mapsto (p, \pi_{*,p} X_p).$$

The map $\tilde{\pi}_*$ is surjective because it sends the fiber T_pP onto the fiber $(\pi^*TM)_p \simeq T_{\pi(p)}M$. Its kernel is precisely the vertical subbundle $\mathcal{V}$ by (27.4). Hence, $\mathcal{V}$ fits into a short exact sequence of vector bundles over P:

$$0 \to \mathcal{V} \longrightarrow TP \xrightarrow{\tilde{\pi}_*} \pi^* TM \to 0. \tag{27.5}$$

27.6 Horizontal Distributions on a Principal Bundle

On the total space P of a smooth principal bundle $\pi: P \to M$, there is a well-defined vertical subbundle $\mathcal{V}$ of the tangent bundle TP. We call a subbundle $\mathcal{H}$ of TP a ***horizontal distribution*** on P if $TP = \mathcal{V} \oplus \mathcal{H}$ as vector bundles; in other words, $T_pP = \mathcal{V}_p + \mathcal{H}_p$ and $\mathcal{V}_p \cap \mathcal{H}_p = 0$ for every $p \in P$. In general, there is no canonically defined horizontal distribution on a principal bundle.

A ***splitting*** of a short exact sequence of vector bundles $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ over a manifold P is a bundle map $k: C \to B$ such that $j \circ k = \mathbb{1}_C$, the identity bundle map on C.

Proposition 27.20. *Let*

$$0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0 \tag{27.6}$$

be a short exact sequence of vector bundles over a manifold P. Then there is a one-to-one correspondence

$$\{\textit{subbundles } H \subset B \mid B = i(A) \oplus H\} \longleftrightarrow \{\textit{splittings } k: C \to B \textit{ of } (27.6)\}.$$

Proof. If H is a subbundle of B such that $B = i(A) \oplus H$, then there are bundle isomorphisms $H \simeq B/i(A) \simeq C$. Hence, C maps isomorphically onto H in B. This gives a splitting $k: C \to B$.

If $k: C \to B$ is a splitting, let $H := k(C)$, which is a subbundle of B. Moreover, if $i(a) = k(c)$ for some $a \in A$ and $c \in C$, then

$$0 = ji(a) = jk(c) = c.$$

Hence, $i(A) \cap k(C) = 0$.

Finally, to show that $B = i(A) + k(C)$, let $b \in B$. Then

$$j\big(b - kj(b)\big) = j(b) - j(b) = 0.$$

By the exactness of (27.6), $b - kj(b) = i(a)$ for some $a \in A$. Thus,

$$b = i(a) + kj(b) \in i(A) + k(C).$$

This proves that $B = i(A) + k(C)$ and therefore $B = i(A) \oplus k(C)$. □

As we just saw in the preceding section, for every principal bundle $\pi\colon P \to M$ the vertical subbundle $\mathcal{V}$ fits into a short exact sequence (27.5) of vector bundles over P. By Proposition 27.20, there is a one-to-one correspondence between horizontal distributions on P and splittings of the sequence (27.5).

Problems

27.1. Lie bracket of fundamental vector fields
Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let P be a manifold on which G acts on the right. Prove that for $A, B \in \mathfrak{g}$,

$$\underline{[A,B]} = [\underline{A}, \underline{B}].$$

Hence, the map $\sigma\colon \mathfrak{g} \to \mathfrak{X}(P)$, $A \mapsto \underline{A}$ is a Lie algebra homomorphism.

27.2.* Short exact sequence of vector spaces
Prove that if $0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0$ is a short exact sequence of finite-dimensional vector spaces, then $\dim B = \dim A + \dim C$.

27.3. Splitting of a short exact sequence
Suppose $0 \to A \xrightarrow{i} B \to C \to 0$ is a short exact sequence of vector bundles over a manifold P. A ***retraction*** of $i\colon A \to B$ is a map $r\colon B \xrightarrow{j} A$ such that $r \circ i = \mathbb{1}_A$. Show that i has a retraction if and only if the sequence has a splitting.

27.4.* The differential of an action
Let $\mu\colon P \times G \to P$ be an action of a Lie group G on a manifold P. For $g \in G$, the tangent space T_gG may be identified with $\ell_{g*}\mathfrak{g}$, where $\ell_g\colon G \to G$ is left multiplication by $g \in G$ and $\mathfrak{g} = T_eG$ is the Lie algebra of G. Hence, an element of the tangent space $T_{(p,g)}(P \times G)$ is of the form $(X_p, \ell_{g*}A)$ for $X_p \in T_pP$ and $A \in \mathfrak{g}$. Prove that the differential

$$\mu_* = \mu_{*,(p,g)}\colon T_{(p,g)}(P \times G) \to T_{pg}P$$

is given by

$$\mu_*(X_p, \ell_{g*}A) = r_{g*}(X_p) + \underline{A}_{pg}.$$

27.5. Fundamental vector field under a trivialization
Let $\phi_\alpha\colon \pi^{-1}U_\alpha \to U_\alpha \times G$

$$\phi_\alpha(p) = \big(\pi(p), g_\alpha(p)\big)$$

be a trivialization of $\pi^{-1}U_{ga}$ in a principal bundle P. Let $A \in \mathfrak{g}$, the Lie algebra of G and $\underline{A}$ the fundamental vector field on P that it induces. Prove that

$$g_{\alpha*}(\underline{A}_p) = \ell_{g_\alpha(p)*}(A) \in T_{g_\alpha(p)}(G).$$

27.6. Trivial principal bundle
Prove that a principal bundle $\pi\colon P \to M$ is trivial if and only if it has a section.

27.7. Pullback of a principal bundle to itself
Prove that if $\pi\colon P \to M$ is a principal bundle, then the pullback bundle $\pi^* P \to P$ is trivial.

27.8. Quotient space of a principal bundle
Let G be a Lie group and H a closed subgroup. Prove that if $\pi P \to M$ is a principal G-bundle, then $P \to P/H$ is a principal H-subbundle.

27.9. Fundamental vector fields
Let N and M be G-manifolds with G acting on the right. If $A \in \mathfrak{g}$ and $f\colon N \to M$ is G-equivariant, then

$$f_*(\underline{A}_{N,q}) = \underline{A}_{M,f(q)}.$$

§28 Connections on a Principal Bundle

Let G be a Lie group with Lie algebra $\mathfrak{g}$. As we saw in the preceding section, on a principal G-bundle $P \to M$, the notion of a vertical tangent vector is well defined, but not that of a horizontal tangent vector. A connection on a principal bundle is essentially the choice of a horizontal complement to the vertical tangent bundle on P. Alternatively, it can be given by a $\mathfrak{g}$-valued 1-form on P. In this section we will study these two equivalent manifestations of a connection:

(i) a smooth right-invariant horizontal distribution on P,
(ii) a smooth G-equivariant $\mathfrak{g}$-valued 1-form ω on P such that on the fundamental vector fields,

$$\omega(\underline{A}) = A \quad \text{for all } A \in \mathfrak{g}. \tag{28.1}$$

Under the identification of $\mathfrak{g}$ with a vertical tangent space, condition (28.1) says that ω restricts to the identity map on vertical vectors.

The correspondence between (i) and (ii) is easy to describe. Given a right-invariant horizontal distribution $\mathcal{H}$ on P, we define a $\mathfrak{g}$-valued 1-form ω on P to be, at each point p, the projection with kernel $\mathcal{H}_p$ from the tangent space to the vertical space. Conversely, given a right-equivariant $\mathfrak{g}$-valued 1-form ω that is the identity on the vertical space at each point $p \in P$, we define a horizontal distribution $\mathcal{H}$ on P to be $\ker \omega_p$ at each $p \in P$.

28.1 Connections on a Principal Bundle

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and let $\pi\colon P \to M$ be a principal G-bundle. A ***distribution*** on a manifold is a subbundle of the tangent bundle. Recall that a distribution $\mathcal{H}$ on P is ***horizontal*** if it is complementary to the vertical subbundle $\mathcal{V}$ of the tangent bundle TP: for all p in P,

$$T_pP = \mathcal{V}_p \oplus \mathcal{H}_p.$$

Suppose $\mathcal{H}$ is a horizontal distribution on the total space P of a principal G-bundle $\pi\colon P \to M$. For $p \in P$, if $j_p\colon G \to P$ is the map $j_p(g) = p \cdot g$, then the vertical tangent space $\mathcal{V}_p$ can be canonically identified with the Lie algebra $\mathfrak{g}$ via the isomorphism $j_{p*}\colon \mathfrak{g} \to \mathcal{V}_p$ (Proposition 27.18). Let $v\colon T_pP = \mathcal{V}_p \oplus \mathcal{H}_p \to \mathcal{V}_p$ be the projection to the vertical tangent space with kernel $\mathcal{H}_p$. For $Y_p \in T_pP$, $v(Y_p)$ is called the ***vertical component*** of Y_p. (Although the vertical subspace $\mathcal{V}_p$ is intrinsically defined, the notion of the vertical component of a tangent vector depends on the choice of a horizontal complement $\mathcal{H}_p$.) If ω_p is the composite

$$\omega_p := j_{p*}^{-1} \circ v\colon T_pP \xrightarrow{v} \mathcal{V}_p \xrightarrow{j_{p*}^{-1}} \mathfrak{g}, \tag{28.2}$$

then ω is a $\mathfrak{g}$-valued 1-form on P. In terms of ω, the vertical component of $Y_p \in T_pP$ is

$$v(Y_p) = j_{p*}\big(\omega_p(Y_p)\big). \tag{28.3}$$

Theorem 28.1. *If $\mathcal{H}$ is a smooth right-invariant horizontal distribution on the total space P of a principal G-bundle $\pi\colon P \to M$, then the $\mathfrak{g}$-valued 1-form ω on P defined above satisfies the following three properties:*

(i) *for any $A \in \mathfrak{g}$ and $p \in P$, we have $\omega_p(\underline{A}_p) = A$;*
(ii) (G-equivariance) *for any $g \in G$, $r_g^*\omega = (\operatorname{Ad} g^{-1})\omega$;*
(iii) *ω is C^∞.*

Proof. (i) Since $\underline{A}_p$ is already vertical (Proposition 27.17), the projection v leaves it invariant, so

$$\omega_p(\underline{A}_p) = j_{p*}^{-1}\left(v(\underline{A}_p)\right) = j_{p*}^{-1}(\underline{A}_p) = A.$$

(ii) For $p \in P$ and $Y_p \in T_pP$, we need to show

$$\omega_{pg}(r_{g*}Y_p) = (\operatorname{Ad} g^{-1})\omega_p(Y_p).$$

Since both sides are $\mathbb{R}$-linear in Y_p and Y_p is the sum of a vertical and a horizontal vector, we may treat these two cases separately.

If Y_p is vertical, then by Proposition 27.18, $Y_p = \underline{A}_p$ for some $A \in \mathfrak{g}$. In this case

$$\begin{aligned}
\omega_{pg}(r_{g*}\underline{A}_p) &= \omega_{pg}\left(\underline{(\operatorname{Ad} g^{-1})A}_{pg}\right) && \text{(by Proposition 27.13)}\\
&= (\operatorname{Ad} g^{-1})A && \text{(by (i))}\\
&= (\operatorname{Ad} g^{-1})\omega_p(\underline{A}_p) && \text{(by (i) again).}
\end{aligned}$$

If Y_p is horizontal, then by the right-invariance of the horizontal distribution $\mathcal{H}$, so is $r_{g*}Y_p$. Hence,

$$\omega_{pg}(r_{g*}Y_p) = 0 = (\operatorname{Ad} g^{-1})\omega_p(Y_p).$$

(iii) Fix a point $p \in P$. We will show that ω is C^∞ in a neighborhood of p. Let $B_1, \dots, B_\ell$ be a basis for the Lie algebra $\mathfrak{g}$ and $\underline{B_1}, \dots, \underline{B_\ell}$ the associated fundamental vector fields on P. By Proposition 27.11, these vector fields are all C^∞ on P. Since $\mathcal{H}$ is a C^∞ distribution on P, one can find a neighborhood W of p and C^∞ horizontal vector fields $X_1, \dots, X_n$ on W that span $\mathcal{H}$ at every point of W. Then $\underline{B_1}, \dots, \underline{B_\ell}, X_1, \dots, X_n$ is a C^∞ frame for the tangent bundle TP over W. Thus, any C^∞ vector field X on W can be written as a linear combination

$$X = \sum a^i \underline{B_i} + \sum b^j X_j$$

with C^∞ coefficients a^i, b^j on W. By the definition of ω,

$$\omega(X) = \omega\left(\sum a^i \underline{B_i}\right) = \sum a^i B_i.$$

This proves that ω is a C^∞ 1-form on W. □

Note that in this theorem the proof of the smoothness of ω requires only that the horizontal distribution $\mathcal{H}$ be smooth; it does not use the right-invariance of $\mathcal{H}$.

Definition 28.2. An ***Ehresmann connection*** or simply a ***connection*** on a principal G-bundle $P \to M$ is a $\mathfrak{g}$-valued 1-form ω on P satisfying the three properties of Theorem 28.1.

A $\mathfrak{g}$-valued 1-form α on P can be viewed as a map $\alpha\colon TP \to \mathfrak{g}$ from the tangent bundle TP to the Lie algebra $\mathfrak{g}$. Now both TP and $\mathfrak{g}$ are G-manifolds: the Lie group G acts on TP on the right by the differentials of right translations and it acts on $\mathfrak{g}$ on the left by the adjoint representation. By (27.1), $\alpha\colon TP \to \mathfrak{g}$ is G-equivariant if and only if for all $p \in P$, $X_p \in T_pP$, and $g \in G$,

$$\alpha(X_p \cdot g) = g^{-1} \cdot \alpha(X_p),$$

or

$$\alpha(r_{g*}X_p) = (\operatorname{Ad} g^{-1})\alpha(X_p).$$

Thus, $\alpha\colon TP \to \mathfrak{g}$ is G-equivariant if and only if $r_g^*\alpha = (\operatorname{Ad} g^{-1})\alpha$ for all $g \in G$. Condition (ii) of a connection ω on a principal bundle says precisely that ω is G-equivariant as a map from TP to $\mathfrak{g}$.

28.2 Vertical and Horizontal Components of a Tangent Vector

As we noted in Section 27.5, on any principal G-bundle $\pi : P \to M$, the vertical subspace $\mathcal{V}_p$ of the tangent space T_pP is intrinsically defined:

$$\mathcal{V}_p := \ker \pi_* : T_pP \to T_{\pi(p)}M.$$

By Proposition 27.18, the map j_{p*} naturally identifies the Lie algebra $\mathfrak{g}$ of G with the vertical subspace $\mathcal{V}_p$.

In the presence of a horizontal distribution on the total space P of a principal bundle, every tangent vector $Y_p \in T_pP$ decomposes uniquely into the sum of a vertical vector and a horizontal vector:

$$Y_p = v(Y_p) + h(Y_p) \in \mathcal{V}_p \oplus \mathcal{H}_p.$$

These are called, respectively, the ***vertical component*** and ***horizontal component*** of the vector Y_p. As p varies over P, this decomposition extends to a decomposition of a vector field Y on P:

$$Y = v(Y) + h(Y).$$

We often omit the parentheses in $v(Y)$ and $h(Y)$, and write vY and hY instead.

Proposition 28.3. *If $\mathcal{H}$ is a C^∞ horizontal distribution on the total space P of a principal bundle, then the vertical and horizontal components $v(Y)$ and $h(Y)$ of a C^∞ vector field Y on P are also C^∞.*

Proof. Let ω be the $\mathfrak{g}$-valued 1-form associated to the horizontal distribution $\mathcal{H}$ by (28.2). It is C^∞ by Theorem 28.1(iii). In terms of a basis $B_1,\dots,B_\ell$ for $\mathfrak{g}$, we can write $\omega = \sum \omega^i B_i$, where ω^i are C^∞ 1-forms on P. If $Y_p \in T_pP$, then by (28.3) its vertical component $v(Y_p)$ is

$$v(Y_p) = j_{p*}(\omega_p(Y_p)) = j_{p*}\left(\sum \omega_p^i(Y_p)B_i\right) = \sum \omega_p^i(Y_p)(\underline{B_i})_p.$$

As p varies over P,

$$v(Y) = \sum \omega^i(Y)\underline{B_i}.$$

Since ω^i, Y, and $\underline{B_i}$ are all C^∞, so is $v(Y)$. Because $h(Y) = Y - v(Y)$, the horizontal component $h(Y)$ of a C^∞ vector field Y on P is also C^∞. □

On a principal bundle $\pi\colon P \to M$, if $r_g\colon P \to P$ is right translation by $g \in G$, then $\pi \circ r_g = \pi$. It follows that $\pi_* \circ r_{g*} = \pi_*$. Thus, the right translation $r_{g*}: T_pP \to T_{pg}P$ sends a vertical vector to a vertical vector. By hypothesis, $r_{g*}\mathcal{H}_p = \mathcal{H}_{pg}$ and hence the right translation r_{g*} also sends a horizontal vector to a horizontal vector.

Proposition 28.4. *Suppose $\mathcal{H}$ is a smooth right-invariant horizontal distribution on the total space of a principal G-bundle $\pi\colon P \to M$. For each $g \in G$, the right translation r_{g*} commutes with the projections v and h.*

Proof. Any $X_p \in T_pP$ decomposes into vertical and horizontal components:

$$X_p = v(X_p) + h(X_p).$$

Applying r_{g*} to both sides, we get

$$r_{g*}X_p = r_{g*}v(X_p) + r_{g*}h(X_p). \tag{28.4}$$

Since r_{g*} preserves vertical and horizontal subspaces, $r_{g*}v(X_p)$ is vertical and $r_{g*}h(X_p)$ is horizontal. Thus, (28.4) is the decomposition of $r_{g*}X_p$ into vertical and horizontal components. This means for every $X_p \in T_pP$,

$$vr_{g*}(X_p) = r_{g*}v(X_p) \quad \text{and} \quad hr_{g*}(X_p) = r_{g*}h(X_p).$$ □

28.3 The Horizontal Distribution of an Ehresmann Connection

In Section 28.1 we showed that a smooth, right-invariant horizontal distribution on the total space of a principal bundle determines an Ehresmann connection. We now prove the converse.

Theorem 28.5. *If ω is a connection on the principal G-bundle $\pi\colon P \to M$, then $\mathcal{H}_p := \ker \omega_p$, $p \in P$, is a smooth right-invariant horizontal distribution on P.*

Proof. We need to verify three properties:

(i) At each point p in P, the tangent space T_pP decomposes into a direct sum $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$.

(ii) For $p \in P$ and $g \in G$, $r_{g*}(\mathcal{H}_p) \subset \mathcal{H}_{pg}$.
(iii) $\mathcal{H}$ is a C^∞ subbundle of the tangent bundle TP.

(i) Since $\mathcal{H}_p = \ker \omega_p$, there is an exact sequence

$$0 \to \mathcal{H}_p \to T_pP \overset{\omega_p}{\to} \mathfrak{g} \to 0.$$

The map $j_{p*}\colon \mathfrak{g} \to \mathcal{V}_p \subset T_pP$ provides a splitting of the sequence. By Proposition 27.20, there is a sequence of isomorphisms

$$T_pP \simeq \mathfrak{g} \oplus \mathcal{H}_p \simeq \mathcal{V}_p \oplus \mathcal{H}_p.$$

(ii) Suppose $Y_p \in \mathcal{H}_p = \ker \omega_p$. By the right-equivariance property of an Ehresmann connection,

$$\omega_{pg}(r_{g*}Y_p) = (r_g^*\omega)_p(Y_p) = (\operatorname{Ad} g^{-1})\omega_p(Y_p) = 0.$$

Hence, $r_{g*}Y_p \in \mathcal{H}_{pg}$.

(iii) Let $B_1, \dots, B_\ell$ be a basis for the Lie algebra $\mathfrak{g}$ of G. Then $\omega = \sum \omega^i B_i$, where $\omega^1, \dots, \omega^\ell$ are smooth $\mathbb{R}$-valued 1-forms on P and for $p \in P$,

$$\mathcal{H}_p = \bigcap_{i=1}^{\ell} \ker \omega_p^i.$$

Since $\omega_p\colon T_pP \to \mathfrak{g}$ is surjective, $\omega^1, \dots, \omega^\ell$ are linearly independent at p.

Fix a point $p \in P$ and let $x^1, \dots, x^m$ be local coordinates near p on P. Then

$$\omega^i = \sum_{j=1}^{m} f_j^i \, dx^j, \quad i = 1, \dots, \ell$$

for some C^∞ functions f_j^i in a neighborhood of p.

Let $b^1, \dots, b^m$ be the fiber coordinates of TP near p, i.e., if $v_q \in T_qP$ for q near p, then

$$v_q = \sum b^j \left.\frac{\partial}{\partial x^j}\right|_q.$$

In terms of local coordinates,

$$\begin{aligned}\mathcal{H}_q = \bigcap_{i=1}^{\ell} \ker \omega_q^i &= \{v_q \in T_qP \mid \omega_q^i(v_q) = 0, i = 1, \dots, \ell\} \\ &= \{(b^1, \dots, b^m) \in \mathbb{R}^m \mid \sum_{j=1}^{m} f_j^i(q) b^j = 0, i = 1, \dots, \ell\}.\end{aligned}$$

Let $F^i(q,b) = \sum_{j=1}^m f_j^i(q) b^j$, $i = 1, \dots, \ell$. Since $\omega^1, \dots, \omega^\ell$ are linearly independent at p, the Jacobian matrix $[\partial F^i / \partial b^j] = [f_j^i]$, an $\ell \times m$ matrix, has rank ℓ at p. Without loss of generality, we may assume that the first $\ell \times \ell$ block of $[f_j^i(p)]$ has rank ℓ. Since having maximal rank is an open condition, there is a neighborhood U_p of p on which the first $\ell \times \ell$ block of $[f^i_j]$ has rank ℓ. By the implicit function theorem, on U_p, $b^1, \dots, b^\ell$ are C^∞ functions of $b^{\ell+1}, \dots, b^m$, say

$$\begin{aligned} b^1 &= b^1(b^{\ell+1}, \dots, b^m), \\ &\vdots \\ b^\ell &= b^\ell(b^{\ell+1}, \dots, b^m). \end{aligned}$$

Let

$$\begin{aligned} X_1 &= \sum_{j=1}^{\ell} b^j(1,0,\dots,0)\frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^{\ell+1}} \\ X_2 &= \sum_{j=1}^{\ell} b^j(0,1,0,\dots,0)\frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^{\ell+2}} \\ &\vdots \\ X_{m-\ell} &= \sum_{j=1}^{\ell} b^j(0,0,\dots,1)\frac{\partial}{\partial x^j} + \frac{\partial}{\partial x^m}. \end{aligned}$$

These are C^∞ vector fields on U_p that span $\mathcal{H}_q$ at each point $q \in U_p$. By the subbundle criterion (Theorem 20.4), $\mathcal{H}$ is a C^∞ subbundle of TP. □

28.4 Horizontal Lift of a Vector Field to a Principal Bundle

Suppose $\mathcal{H}$ is a horizontal distribution on a principal bundle $\pi\colon P \to M$. Let X be a vector field on M. For every $p \in P$, because the vertical subspace $\mathcal{V}_p$ is $\ker \pi_*$, the differential $\pi_*\colon T_pP \to T_{\pi(p)}M$ induces an isomorphism

$$\frac{T_pP}{\ker \pi_*} \xrightarrow{\sim} \mathcal{H}_p \xrightarrow{\sim} T_{\pi(p)}M$$

of the horizontal subspace $\mathcal{H}_p$ with the tangent space $T_{\pi(p)}M$. Consequently, there is a unique horizontal vector $\tilde{X}_p \in \mathcal{H}_p$ such that $\pi_*(\tilde{X}_p) = X_{\pi(p)} \in T_{\pi(p)}M$. The vector field $\tilde{X}$ is called the ***horizontal lift*** of X to P.

Proposition 28.6. *If $\mathcal{H}$ is a C^∞ right-invariant horizontal distribution on the total space P of a principal bundle $\pi\colon P \to M$, then the horizontal lift $\tilde{X}$ of a C^∞ vector field X on M is a C^∞ right-invariant vector field on P.*

Proof. Let $x \in M$ and $p \in \pi^{-1}(x)$. By definition, $\pi_*(\tilde{X}_p) = X_x$. If q is any other point of $\pi^{-1}(x)$, then $q = pg$ for some $g \in G$. Since $\pi \circ r_g = \pi$,

$$\pi_*(r_{g*}\tilde{X}_p) = (\pi \circ r_g)_*\tilde{X}_p = \pi_*\tilde{X}_p = X_p.$$

By the uniqueness of the horizontal lift, $r_{g*}\tilde{X}_p = \tilde{X}_{pg}$. This proves the right-invariance of $\tilde{X}$.

We prove the smoothness of $\tilde{X}$ by proving it locally. Let $\{U\}$ be a trivializing open cover for P with trivializations $\phi_U : \pi^{-1}(U) \xrightarrow{\sim} U \times G$. Define

$$Z_{(x,g)} = (X_x, 0) \in T_{(x,g)}(U \times G).$$

Let $\eta : U \times G \to U$ be the projection to the first factor. Then Z is a C^∞ vector field on $U \times G$ such that $\eta_* Z_{(x,g)} = X_x$, and $Y := (\phi_{U*})^{-1} Z$ is a C^∞ vector field on $\pi^{-1}(U)$ such that $\pi_* Y_p = X_{\pi(p)}$. By Proposition 28.3, hY is a C^∞ vector field on $\pi^{-1}(U)$. Clearly it is horizontal. Because $Y_p = v(Y_p) + h(Y_p)$ and $\pi_* v(Y_p) = 0$, we have $\pi_* Y_p = \pi_* h(Y_p) = X_{\pi(p)}$. Thus, hY lifts X over U. By the uniqueness of the horizontal lift, $hY = \widetilde{X}$ over U. This proves that $\tilde{X}$ is a smooth vector field on P. □

28.5 Lie Bracket of a Fundamental Vector Field

If a principal bundle P comes with a connection, then it makes sense to speak of horizontal vector fields on P; these are vector fields all of whose vectors are horizontal.

Lemma 28.7. *Suppose P is a principal bundle with a connection. Let $\underline{A}$ be the fundamental vector field on P associated to $A \in \mathfrak{g}$.*

(i) *If Y is a horizontal vector field on P, then $[\underline{A}, Y]$ is horizontal.*
(ii) *If Y is a right-invariant vector field on P, then $[\underline{A}, Y] = 0$.*

Proof. (i) A local flow for $\underline{A}$ is $\phi_t(p) = pe^{tA} = r_{e^{tA}}(p)$ (Proposition 27.14). By the identification of the Lie bracket with the Lie derivative of vector fields [21, Th. 20.4, p. 225] and the definition of the Lie derivative,

$$[\underline{A}, Y]_p = (\mathcal{L}_{\underline{A}} Y)_p = \lim_{t \to 0} \frac{(r_{e^{-tA}})_* Y_{pe^{tA}} - Y_p}{t}. \tag{28.5}$$

Since right translation preserves horizontality (Theorem 28.5), both $(r_{e^{-tA}})_* Y_{pe^{tA}}$ and Y_p are horizontal vectors. Denote the difference quotient in (28.5) by $c(t)$. For every t near 0 in $\mathbb{R}$, $c(t)$ is in the vector space $\mathcal{H}_p$ of horizontal vectors at p. Therefore, $[\underline{A}, Y]_p = \lim_{t \to 0} c(t) \in \mathcal{H}_p$.

(ii) If Y is right-invariant, then

$$(r_{e^{-tA}})_* Y_{pe^{tA}} = Y_p.$$

In that case, it follows from (28.5) that $[\underline{A}, Y]_p = 0$. □

Problems

28.1. Maurer–Cartan connection
If θ is the Maurer–Cartan form on a Lie group and $\pi_2 : M \times G \to G$ is the projection to the second factor, prove that $\omega := \pi_2^* \theta$ is a connection on the trivial bundle $\pi_1 : M \times G \to M$. It is called the ***Maurer–Cartan connection***.

28.2. Convex linear combinations of connections
Prove that a convex linear combination ω of connections $\omega_1, \ldots, \omega_n$ on a principal bundle $\pi\colon P \to M$ is again a connection on P. ($\omega = \sum \lambda_i \omega_i$, $\sum \lambda_i = 1$, $\lambda_i \geq 0$.)

28.3. Pullback of a connection
Let $\pi_Q\colon Q \to N$ and $\pi_P\colon P \to M$ be principal G-bundles, and let $(\bar{f}\colon Q \to P, f\colon N \to M)$ be a morphism of principal bundles. Prove that if θ is a connection on P, then $\bar{f}^*\theta$ is a connection on Q.

§29 Horizontal Distributions on a Frame Bundle

In this section we will explain the process by which a connection ∇ on a vector bundle E over a manifold M gives rise to a smooth right-invariant horizontal distribution on the associated frame bundle $\mathrm{Fr}(E)$. This involves a sequence of steps. A connection on the vector bundle E induces a covariant derivative on sections of the vector bundle along a curve. Parallel sections along the curve are those whose derivative vanishes. Just as for tangent vectors in Section 14, starting with a frame e_x for the fiber of the vector bundle at the initial point x of a curve, there is a unique way to parallel translate the frame along the curve. In terms of the frame bundle $\mathrm{Fr}(E)$, what this means is that every curve in M has a unique lift to $\mathrm{Fr}(E)$ starting at e_x representing parallel frames along the curve. Such a lift is called a ***horizontal lift***. The initial vector at e_x of a horizontal lift is a ***horizontal vector*** at e_x. The horizontal vectors at a point of $\mathrm{Fr}(E)$ form a subspace of the tangent space $T_{e_x}\big(\mathrm{Fr}(E)\big)$. In this way we obtain a horizontal distribution on the frame bundle. We show that this horizontal distribution on $\mathrm{Fr}(E)$ arising from a connection on the vector bundle E is smooth and right-invariant. It therefore corresponds to a connection ω on the principal bundle $\mathrm{Fr}(E)$. We then show that ω pulls back under a section e of $\mathrm{Fr}(E)$ to the connection matrix ω_e of the connection ∇ relative to the frame e on an open set U.

29.1 Parallel Translation in a Vector Bundle

In Section 14 we defined parallel translation of a tangent vector along a curve in a manifold with an affine connection. In fact, the same development carries over to an arbitrary vector bundle $\eta\colon E \to M$ with a connection ∇.

Let $c\colon [a,b] \to M$ be a smooth curve in M. Instead of vector fields along the curve c, we consider smooth sections of the pullback bundle c^*E over $[a,b]$. These are called ***smooth sections of the vector bundle E along the curve c***. We denote by $\Gamma(c^*E)$ the vector space of smooth sections of E along the curve c. If $E = TM$ is the tangent bundle of a manifold M, then an element of $\Gamma(c^*TM)$ is simply a vector field along the curve c in M. Just as in Theorem 13.1, there is a unique $\mathbb{R}$-linear map

$$\frac{D}{dt}\colon \Gamma(c^*E) \to \Gamma(c^*E),$$

called the ***covariant derivative*** corresponding to ∇, such that

(i) (Leibniz rule) for any C^∞ function f on the interval $[a,b]$,

$$\frac{D(fs)}{dt} = \frac{df}{dt}s + f\frac{Ds}{dt};$$

(ii) if s is induced from a global section $\tilde{s} \in \Gamma(M,E)$ in the sense that $s(t) = \tilde{s}\big(c(t)\big)$, then

$$\frac{Ds}{dt}(t) = \nabla_{c'(t)}\tilde{s}.$$

Definition 29.1. A section $s \in \Gamma(c^*E)$ is ***parallel along a curve*** $c\colon [a,b] \to M$ if $Ds/dt \equiv 0$ on $[a,b]$.

As in Section 14.5, the equation $Ds/dt \equiv 0$ for a section s to be parallel is equivalent to a system of linear first-order ordinary differential equations. Suppose $c\colon [a,b] \to M$ maps into a framed open set $(U, e_1, \ldots, e_r)$ for E. Then $s \in \Gamma(c^*E)$ can be written as

$$s(t) = \sum s^i(t) e_{i,c(t)}.$$

By properties (i) and (ii) of the covariant derivative,

$$\begin{aligned}
\frac{Ds}{dt} &= \sum_i \frac{ds^i}{dt} e_i + \sum_j s^j \frac{D}{dt} e_{j,c(t)} \\
&= \sum_i \frac{ds^i}{dt} e_i + \sum_j s^j \nabla_{c'(t)} e_j \\
&= \sum_i \frac{ds^i}{dt} e_i + \sum_{i,j} s^j \omega^i_j\big(c'(t)\big) e_i.
\end{aligned}$$

Hence, $Ds/dt \equiv 0$ if and only if

$$\frac{ds^i}{dt} + \sum_j s^j \omega^i_j\big(c'(t)\big) = 0 \text{ for all } i.$$

This is a system of linear first-order differential equations. By the existence and uniqueness theorems of differential equations, it has a solution on a small interval about a give point t_0 and the solution is uniquely determined by its value at t_0. Thus, a parallel section is uniquely determined by its value at a point. If $s \in \Gamma(c^*E)$ is a parallel section of the pullback bundle c^*E, we say that $s(b)$ is the ***parallel transport*** of $s(a)$ along $c\colon [a,b] \to M$. The resulting map: $E_{c(a)} \to E_{c(b)}$ is called ***parallel translation*** from $E_{c(a)}$ to $E_{c(b)}$.

Theorem 29.2. *Let $\eta\colon E \to M$ be a C^∞ vector bundle with a connection ∇ and let $c\colon [a,b] \to M$ be a smooth curve in M. There is a unique parallel translation $\varphi_{a,b}$ from $E_{c(a)}$ to $E_{c(b)}$ along c. This parallel translation $\varphi_{a,b}\colon E_{c(a)} \to E_{c(b)}$ is a linear isomorphism.*

The proof is similar to that of Theorem 14.14.

A ***parallel frame along*** the curve $c\colon [a,b] \to M$ is a collection of parallel sections $\big(e_1(t), \ldots, e_r(t)\big)$, $t \in [a,b]$, such that for each t, the elements $e_1(t), \ldots, e_r(t)$ form a basis for the vector space $E_{c(t)}$.

Let $\pi\colon \operatorname{Fr}(E) \to M$ be the frame bundle of the vector bundle $\eta\colon E \to M$. A curve $\tilde{c}(t)$ in $\operatorname{Fr}(E)$ is called a ***lift*** of the curve $c(t)$ in M if $c(t) = \pi(\tilde{c}(t))$. It is a ***horizontal lift*** if in addition $\tilde{c}(t)$ is a parallel frame along c.

Restricting the domain of the curve c to the interval $[a,t]$, we obtain from Theorem 29.2 that parallel translation is a linear isomorphism of $E_{c(a)}$ with $E_{c(t)}$. Thus, if a collection of parallel sections $\big(s_1(t),\dots,s_r(t)\big) \in \Gamma(c^*E)$ forms a basis at one time t, then it forms a basis at every time $t \in [a,b]$. By Theorem 29.2, for every smooth curve $c\colon [a,b] \to M$ and ordered basis $(s_{1,0},\dots,s_{r,0})$ for $E_{c(a)}$, there is a unique parallel frame along c whose value at a is $(s_{1,0},\dots,s_{r,0})$. In terms of the frame bundle $\mathrm{Fr}(E)$, this shows the existence and uniqueness of a horizontal lift with a specified initial point in $\mathrm{Fr}(E)$ of a curve $c(t)$ in M.

29.2 Horizontal Vectors on a Frame Bundle

On a general principal bundle vertical vectors are intrinsically defined, but horizontal vectors are not. We will see shortly that a connection on a vector bundle E over a manifold M determines a well-defined horizontal distribution on the frame bundle $\mathrm{Fr}(E)$. The elements of the horizontal distribution are the horizontal vectors. Thus, the notion of a horizontal vector on the frame bundle $\mathrm{Fr}(E)$ depends on a connection on E.

Definition 29.3. Let $E \to M$ be a vector bundle with a connection ∇, $x \in M$, and $e_x \in \mathrm{Fr}(E_x)$. A tangent vector $v \in T_{e_x}(\mathrm{Fr}(E))$ is said to be ***horizontal*** if there is a curve $c(t)$ through x in M such that v is the initial vector $\tilde{c}'(0)$ of the unique horizontal lift of $\tilde{c}(t)$ of $c(t)$ to $\mathrm{Fr}(E)$ starting at e_x.

Our goal now is to show that the horizontal vectors at a point e_x of the frame bundle form a vector subspace of the tangent space $T_{e_x}\big(\mathrm{Fr}(E)\big)$. To this end we will derive an explicit formula for $\tilde{c}'(0)$ in terms of a local frame for E. Suppose $c\colon [0,b] \to M$ is a smooth curve with initial point $c(0) = x$, and $\tilde{c}(t)$ is its unique horizontal lift to $\mathrm{Fr}(E)$ with initial point $e_x = (e_{1,0},\dots,e_{r,0})$. Let s be a frame for E over a neighborhood U of x with $s(x) = e_x$. Then $s(c(t))$ is a lift of $c(t)$ to $\mathrm{Fr}(E)$ with initial point e_x, but of course it is not necessarily a horizontal lift (see Figure 29.1). For any $t \in [0,b]$, we have two ordered bases $s(c(t))$ and $\tilde{c}(t)$ for $E_{c(t)}$, so there is a smooth matrix $a(t) \in \mathrm{GL}(r,\mathbb{R})$ such that $s(c(t)) = \tilde{c}(t)a(t)$. At $t = 0$, $s(c(0)) = e_x = \tilde{c}(0)$, so that $a(0) = I$, the identity matrix in $\mathrm{GL}(r,\mathbb{R})$.

Lemma 29.4. *In the notation above, let* $s_*\colon T_x(M) \to T_{e_x}(\mathrm{Fr}(E))$ *be the differential of* s *and* $\underline{a'(0)}$ *the fundamental vector field on* $\mathrm{Fr}(E)$ *associated to* $a'(0) \in \mathfrak{gl}(r,\mathbb{R})$. *Then*

$$s_*\big(c'(0)\big) = \tilde{c}'(0) + \underline{a'(0)}_{e_x}.$$

Proof. Let $P = \mathrm{Fr}(E)$ and $G = \mathrm{GL}(r,\mathbb{R})$, and let $\mu\colon P \times G \to P$ be the right action of G on P. Then

$$s\big(c(t)\big) = \tilde{c}(t)a(t) = \mu\big(\tilde{c}(t), a(t)\big), \tag{29.1}$$

with $c(0) = x$, $\tilde{c}(0) = e_x$, and $a(0) =$ the identity matrix I. Differentiating (29.1) with respect to t and evaluating at 0 gives

$$s_*\big(c'(0)\big) = \mu_{*,(\tilde{c}(0),a(0))}\big(\tilde{c}'(0), a'(0)\big).$$

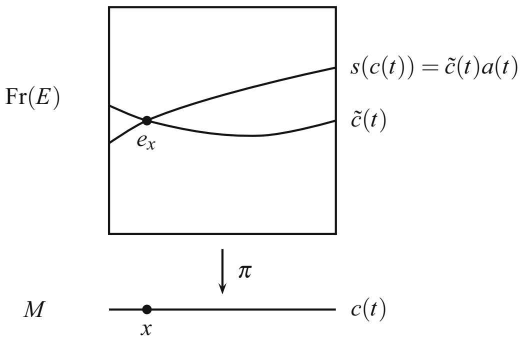


Fig. 29.1. Two liftings of a curve

By the formula for the differential of an action (Problem 27.4),

$$s_*\big(c'(0)\big) = r_{a(0)*}\tilde{c}'(0) + \underline{a'(0)}_{\tilde{c}(0)} = \tilde{c}'(0) + \underline{a'(0)}_{e_x}. \qquad \square$$

Lemma 29.5. *Let $E \to M$ be a vector bundle with a connection ∇. Suppose $s = (s_1, \dots, s_r)$ is a frame for E over an open set U, $\tilde{c}(t)$ a parallel frame over a curve $c(t)$ in U with $\tilde{c}(0) = s(c(0))$, and $a(t)$ the curve in $\mathrm{GL}(r, \mathbb{R})$ such that $s(c(t)) = \tilde{c}(t)a(t)$. If $\omega_s = [\omega^i_j]$ is the connection matrix of ∇ with respect to the frame $(s_1, \dots, s_r)$ over U, then $a'(0) = \omega_s(c'(0))$.*

Proof. Label $c(0) = x$ and $\tilde{c}_i(0) = s_i\big(c(0)\big) = e_{i,x}$. By the definition of the connection matrix,

$$\nabla_{c'(0)} s_j = \sum \omega^i_j\big(c'(0)\big) s_i\big(c(0)\big) = \sum \omega^i_j\big(c'(0)\big) e_{i,x}. \tag{29.2}$$

On the other hand, by the defining properties of the covariant derivative (Section 29.1),

$$\begin{aligned}
\nabla_{c'(t)} s_j &= \frac{D(s_j \circ c)}{dt}(t) = \frac{D}{dt}\sum \tilde{c}_i(t) a^i_j(t) \\
&= \sum (a^i_j)'(t)\tilde{c}_i(t) + \sum a^i_j(t)\frac{D\tilde{c}_i}{dt}(t) \\
&= \sum (a^i_j)'(t)\tilde{c}_i(t) \qquad (\text{since } D\tilde{c}_i/dt \equiv 0).
\end{aligned}$$

Setting $t = 0$ gives

$$\nabla_{c'(0)} s_j = \sum (a^i_j)'(0) e_{i,x}. \tag{29.3}$$

Equating (29.2) and (29.3), we obtain $(a^i_j)'(0) = \omega^i_j(c'(0))$. $\square$

Thus, Lemma 29.4 for the horizontal lift of $c'(0)$ can be rewritten in the form

$$\tilde{c}'(0) = s_*\big(c'(0)\big) - \underline{a'(0)}_{e_x} = s_*\big(c'(0)\big) - \underline{\omega_s\big(c'(0)\big)}_{e_x}. \tag{29.4}$$

Proposition 29.6. *Let $\pi\colon E \to M$ be a smooth vector bundle with a connection over a manifold M of dimension n. For $x \in M$ and e_x an ordered basis for the fiber E_x, the subset $\mathcal{H}_{e_x}$ of horizontal vectors in the tangent space $T_{e_x}(\mathrm{Fr}(E))$ is a vector space of dimension n, and $\pi_*\colon \mathcal{H}_{e_x} \to T_xM$ is a linear isomorphism.*

Proof. In formula (29.4), $\omega_s(c'(0))$ is $\mathbb{R}$-linear in its argument $c'(0)$ because ω_s is a 1-form at $c(0)$. The operation $A \mapsto \underline{A}_{e_x}$ of associating to a matrix $A \in \mathfrak{gl}(r,\mathbb{R})$ a tangent vector $\underline{A}_{e_x} \in T_{e_x}\big(\mathrm{Fr}(E)\big)$ is $\mathbb{R}$-linear by (27.3). Hence, formula (29.4) shows that the map

$$\begin{aligned}\phi\colon T_xM &\to T_{e_x}\big(\mathrm{Fr}(E)\big),\\ c'(0) &\mapsto \tilde{c}'(0)\end{aligned}$$

is $\mathbb{R}$-linear. As the image of a vector space T_xM under a linear map, the set $\mathcal{H}_{e_x}$ of horizontal vectors $\tilde{c}'(0)$ at e_x is a vector subspace of $T_{e_x}\big(\mathrm{Fr}(E)\big)$.

Since $\pi\big(\tilde{c}(t)\big) = c(t)$, taking the derivative at $t = 0$ gives $\pi_*\big(\tilde{c}'(0)\big) = c'(0)$, so π_* is a left inverse to the map ϕ. This proves that $\phi\colon T_xM \to T_{e_x}\big(\mathrm{Fr}(E)\big)$ is injective. Its image is by definition $\mathcal{H}_{e_x}$. It follows that $\phi\colon T_xM \to \mathcal{H}_{e_x}$ is an isomorphism with inverse $\pi_*\colon \mathcal{H}_{e_x} \to T_{e_x}M$. □

29.3 Horizontal Lift of a Vector Field to a Frame Bundle

We have learned so far that a connection on a vector bundle $E \to M$ defines a horizontal subspace $\mathcal{H}_p$ of the tangent space T_pP at each point p of the total space of the frame bundle $\pi\colon P = \mathrm{Fr}(E) \to M$. The horizontal subspace $\mathcal{H}_p$ has the same dimension as M. The vertical subspace $\mathcal{V}_p$ of T_pP is the kernel of the surjection $\pi_*\colon T_pP \to T_{\pi(p)}M$; as such, $\dim \mathcal{V}_p = \dim T_pP - \dim M$. Hence, $\mathcal{V}_p$ and $\mathcal{H}_p$ have complementary dimensions in T_pP. Since $\pi_*(\mathcal{V}_p) = 0$ and $\pi_*\colon \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism, $\mathcal{V}_p \cap \mathcal{H}_p = 0$. It follows that there is a direct sum decomposition

$$T_p(\mathrm{Fr}(E)) = \mathcal{V}_p \oplus \mathcal{H}_p. \tag{29.5}$$

Our goal now is to show that as p varies in P, the subset $\mathcal{H} := \bigcup_{p\in P} \mathcal{H}_p$ of the tangent bundle TP defines a C^∞ horizontal distribution on P in the sense of Section 27.6.

Since $\pi_{*,p}\colon \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism for each $p \in P$, if X is a vector field on M, then there is a unique vector field $\tilde{X}$ on P such that $\tilde{X}_p \in \mathcal{H}_p$ and $\pi_{*,p}(\tilde{X}_p) = X_{\pi(p)}$. The vector field $\tilde{X}$ is called the ***horizontal lift*** of X to the frame bundle P.

Since every tangent vector $X_x \in T_xM$ is the initial vector $c'(0)$ of a curve c, formula (29.4) for the horizontal lift of a tangent vector can be rewritten in the following form.

Lemma 29.7 (Horizontal lift formula). *Suppose ∇ is a connection on a vector bundle $E \to M$ and ω_s is its connection matrix on a framed open set (U,s). For $x \in U$, $p = s(x) \in \mathrm{Fr}(E)$, and $X_x \in T_xM$, let $\tilde{X}_p$ be the horizontal lift of X_x to p in $\mathrm{Fr}(E)$. Then*

$$\tilde{X}_p = s_{*,x}(X_x) - \underline{\omega_s(X_x)}_p.$$

Proposition 29.8. *Let $E \to M$ be a C^∞ rank r vector bundle with a connection and $\pi\colon \mathrm{Fr}(E) \to M$ its frame bundle. If X is a C^∞ vector field on M, then its horizontal lift $\tilde{X}$ to $\mathrm{Fr}(E)$ is a C^∞ vector field.*

Proof. Let $P = \mathrm{Fr}(E)$ and $G = \mathrm{GL}(r, \mathbb{R})$. Since the question is local, we may assume that the bundle P is trivial, say $P = M \times G$. By the right invariance of the horizontal distribution,

$$\tilde{X}_{(x,a)} = r_{a*}\tilde{X}_{(x,1)}. \tag{29.6}$$

Let $s\colon M \to P = M \times G$ be the section $s(x) = (x,1)$. By the horizontal lift formula (Lemma 29.7),

$$\tilde{X}_{(x,1)} = s_{*,x}(X_x) - \underline{\omega_s(X_x)}_{(x,1)}. \tag{29.7}$$

Let $p = (x,a) \in P$ and let f be a C^∞ function on P. We will prove that $\tilde{X}_p f$ is C^∞ as a function of p. By (29.6) and (29.7),

$$\tilde{X}_p f = r_{a*}s_{*,x}(X_x) f - r_{a*}\underline{\omega_s(X_x)}_{(x,1)} f, \tag{29.8}$$

so it suffices to prove separately that $\big(r_{a*}(s_{*,x}X_x)\big) f$ and $\big(r_{a*}\underline{\omega_s(X_x)}_{(x,1)}\big) f$ are C^∞ functions on P.

The first term is

$$\begin{aligned}
\big(r_{a*}s_{*,x}(X_x)\big) f &= X_x(f \circ r_a \circ s)\\
&= X(f \circ r_a \circ s)\big(\pi(p)\big)\\
&= X\big(f\big(s(\pi(p))a\big)\big) = X\big(f\big(\mu(s(\pi(p)),a)\big)\big)\\
&= X\big(f\big(\mu(s(\pi(p)),\pi_2(p))\big)\big),
\end{aligned} \tag{29.9}$$

where $\mu\colon P \times G \to P$ is the action of G on P and $\pi_2\colon P = M \times G \to G$ is the projection $\pi_2(p) = \pi_2(x,a) = a$. The formula (29.9) expresses $\big(r_{a*}s_{*,x}(X_x)\big) f$ as a C^∞ function on P.

By the right equivariance of the connection form ω_s, in (29.8) the second term can be rewritten as

$$\begin{aligned}
r_{a*}\underline{\omega_s(X_x)}_{(x,1)} f &= \underline{(\mathrm{Ad}\, a^{-1})\omega_s(X_x)}_{(x,a)} f\\
&= \underline{\big(\mathrm{Ad}\, \pi_2(p)^{-1}\big)\omega_s\big(X_{\pi(p)}\big)}_p f,
\end{aligned}$$

where $\big(\mathrm{Ad}\, \pi_2(p)^{-1}\big)\omega_s(X_{\pi(p)})$ is a C^∞ function: $P \to \mathfrak{gl}(r,\mathbb{R})$ that we will denote by $A(p)$. The problem now is to show that $p \mapsto \underline{A(p)}_p f$ is a C^∞ function of p.

Let $\mu\colon P \times G \to P$ be the right action of $G = \mathrm{GL}(r,\mathbb{R})$ on $P = \mathrm{Fr}(E)$. Then

$$\underline{A(p)}_p f = \frac{d}{dt}\bigg|_{t=0} f(p \cdot e^{tA(p)}) = \frac{d}{dt}\bigg|_{t=0} f(\mu(p, e^{tA(p)})).$$

Since f, μ, A, and the exponential map are all C^∞ functions, $\underline{A(p)}_p f$ is a C^∞ function of p. Thus, $\tilde{X}_p f$ in (29.8) is a C^∞ function of p. This proves that $\tilde{X}$ is a C^∞ vector field on P. □

Theorem 29.9. *A connection ∇ on a smooth vector bundle $E \to M$ defines a C^∞ distribution $\mathcal{H}$ on the frame bundle $\pi\colon P = \mathrm{Fr}(E) \to M$ such that at any $p \in P$,*

(i) $T_pP = \mathcal{V}_p \oplus \mathcal{H}_p$;
(ii) $r_{g*}(\mathcal{H}_p) = \mathcal{H}_{pg}$ *for any* $g \in G = \mathrm{GL}(r, \mathbb{R})$,

where $r_g\colon P \to P$ is the right action of G on P.

Proof. To prove that $\mathcal{H}$ is a C^∞ subbundle of TP, let U be a coordinate open set in M and $s_1, \dots, s_n$ a C^∞ frame on U. By Proposition 29.8 the horizontal lifts $\widetilde{s_1}, \dots, \widetilde{s_n}$ are C^∞ vector fields on $\tilde{U} := \pi^{-1}(U)$. Moreover, for each $p \in \tilde{U}$, since $\pi_{*,p}\colon \mathcal{H}_p \to T_{\pi(p)}M$ is an isomorphism, $(\widetilde{s_1})_p, \dots, (\widetilde{s_n})_p$ form a basis for $\mathcal{H}_p$. Thus, over $\tilde{U}$ the C^∞ sections $\widetilde{s_1}, \dots, \widetilde{s_n}$ of TP span $\mathcal{H}$. By Theorem 20.4, this proves that $\mathcal{H}$ is a C^∞ subbundle of TP.

Equation (29.5) establishes (i).

As for (ii), let $\tilde{c}'(0) \in \mathcal{H}_p$, where $c(t)$ is a curve in M and $\tilde{c}(t) = [v_1(t) \ \cdots \ v_r(t)]$ is its horizontal lift to P with initial point p. Here we are writing a frame as a row vector so that the group action is simply matrix multiplication on the right. For any $g = [g^i_j] \in \mathrm{GL}(r, \mathbb{R})$,

$$\tilde{c}(t)g = \left[\sum g^i_1 v_i(t) \ \cdots \ \sum g^i_r v_i(t)\right].$$

Since $Dv_i/dt \equiv 0$ by the horizontality of v_i and g^i_j are constants, $D(\sum g^i_j v_i)/dt \equiv 0$. Thus, $\tilde{c}(t)g$ is the horizontal lift of $c(t)$ with initial point $\tilde{c}(0)g$. It has initial tangent vector

$$\left.\frac{d}{dt}\right|_{t=0} \tilde{c}(t)g = r_{g*}\tilde{c}'(0) \in \mathcal{H}_{pg}.$$

This proves that $r_{g*}\mathcal{H}_p \subset \mathcal{H}_{pg}$. Because $r_{g*}\colon \mathcal{H}_p \to \mathcal{H}_{pg}$ has a two-sided inverse $r_{g^{-1}*}$, it is bijective. In particular, $r_{g*}\mathcal{H}_p = \mathcal{H}_{pg}$. □

29.4 Pullback of a Connection on a Frame Bundle Under a Section

Recall that a connection ∇ on a vector bundle E can be represented on a framed open set $(U, e_1, \dots, e_r)$ for E by a connection matrix ω_e depending on the frame. Such a frame $e = (e_1, \dots, e_r)$ is in fact a section $e\colon U \to \mathrm{Fr}(E)$ of the frame bundle. We now use the horizontal lift formula (Lemma 29.7) to prove that the Ehresmann connection ω on the frame bundle $\mathrm{Fr}(E)$ determined by ∇ pulls back under the section e to the connection matrix ω_e.

Theorem 29.10. *Let ∇ be a connection on a vector bundle $E \to M$ and let ω be the Ehresmann connection on the frame bundle $\mathrm{Fr}(E)$ determined by ∇. If $e = (e_1, \dots, e_r)$ is a frame for E over an open set U, viewed as a section $e\colon U \to \mathrm{Fr}(E)|_U$, and ω_e is the connection matrix of ∇ relative to the frame e, then $\omega_e = e^*\omega$.*

Proof. Let $x \in U$ and $p = e(x) \in \mathrm{Fr}(E)$. Suppose X_x is a tangent vector to M at x. If we write $\omega_{e,x}$ for the value of the connection matrix ω_e at the point $x \in U$, then $\omega_{e,x}$ is an $r \times r$ matrix of 1-forms at x and $\omega_{e,x}(X_x)$ is an $r \times r$ matrix of real numbers, i.e., an element of the Lie algebra $\mathfrak{gl}(r, \mathbb{R})$. The corresponding fundamental vector field on $\mathrm{Fr}(E)$ is $\underline{\omega_{e,x}(X_x)}$. By Lemma 29.7, the horizontal lift of X_x to $p \in \mathrm{Fr}(E)$ is

$$\tilde{X}_p = e_* X_x - \underline{\omega_{e,x}(X_x)}_p.$$

Applying the Ehresmann connection ω_p to both sides of this equation, we get

$$\begin{aligned} 0 = \omega_p(\tilde{X}_p) &= \omega_p(e_* X_x) - \omega_p\left(\underline{\omega_{e,x}(X_x)}_p\right) \\ &= (e^*\omega_p)(X_x) - \omega_{e,x}(X_x) \qquad \text{(by Theorem 28.1(i)).} \end{aligned}$$

Since this is true for all $X_x \in T_x M$,

$$e^*\omega_p = (e^*\omega)_x = \omega_{e,x}. \qquad \square$$

§30 Curvature on a Principal Bundle

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Associated to a connection ω on a principal G-bundle is a $\mathfrak{g}$-valued 2-form Ω called its curvature. The definition of the curvature is suggested by the second structural equation for a connection ∇ on a vector bundle E. Just as the connection form ω on the frame bundle $\mathrm{Fr}(E)$ pulls back by a section e of $\mathrm{Fr}(E)$ to the connection matrix ω_e of ∇ with respect to the frame e, so the curvature form Ω on the frame bundle $\mathrm{Fr}(E)$ pulls back by e to the curvature matrix Ω_e of ∇ with respect to e. Thus, the curvature form Ω on the frame bundle is an intrinsic object of which the curvature matrices Ω_e are but local manifestations.

30.1 Curvature Form on a Principal Bundle

By Theorem 11.1 if ∇ is a connection on a vector bundle $E \to M$, then its connection and curvature matrices ω_e and Ω_e on a framed open set $(U,e) = (U,e_1,\ldots,e_r)$ are related by the second structural equation (Theorem 11.1)

$$\Omega_e = d\omega_e + \omega_e \wedge \omega_e.$$

In terms of the Lie bracket of matrix-valued forms (see (21.12)), this can be rewritten as

$$\Omega_e = d\omega_e + \frac{1}{2}[\omega_e, \omega_e].$$

An Ehresmann connection on a principal bundle is Lie algebra-valued. In a general Lie algebra, the wedge product is not defined, but the Lie bracket is always defined. This strongly suggests the following definition for the curvature of an Ehresmann connection on a principal bundle.

Definition 30.1. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose ω is an Ehresmann connection on a principal G-bundle $\pi\colon P \to M$. Then the ***curvature*** of the connection ω is the $\mathfrak{g}$-valued 2-form

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

Recall that frames for a vector bundle E over an open set U are sections of the frame bundle $\mathrm{Fr}(E)$. Let ω be the connection form on the frame bundle $\mathrm{Fr}(E)$ determined by a connection ∇ on E. In the same way that ω pulls back by sections of $\mathrm{Fr}(E)$ to connection matrices, the curvature form Ω of the connection ω on $\mathrm{Fr}(E)$ pulls back by sections to curvature matrices.

Proposition 30.2. *If ∇ is a connection on a vector bundle $E \to M$ and ω is the associated Ehresmann connection on the frame bundle* $\mathrm{Fr}(E)$*, then the curvature matrix Ω_e relative to a frame $e = (e_1,\ldots,e_r)$ for E over an open set U is the pullback $e^*\Omega$ of the curvature Ω on the frame bundle* $\mathrm{Fr}(E)$.

Proof.

$$
\begin{aligned}
e^*\Omega &= e^* d\omega + \frac{1}{2} e^*[\omega,\omega] \\
&= de^*\omega + \frac{1}{2}[e^*\omega, e^*\omega] \quad (e^* \text{ commutes with } d \text{ and } [\ ,\ ] \text{ by Proposition 21.8}) \\
&= d\omega_e + \frac{1}{2}[\omega_e,\omega_e] \quad (\text{by Theorem 29.10}) \\
&= \Omega_e. \quad (\text{by the second structural equation})
\end{aligned}
$$
□

30.2 Properties of the Curvature Form

Now that we have defined the curvature of a connection on a principal G-bundle $\pi\colon P \to M$, it is natural to study some of its properties. Like a connection form, the curvature form Ω is equivariant with respect to right translation on P and the adjoint representation on $\mathfrak{g}$. However, unlike a connection form, a curvature form is horizontal in the sense that it vanishes as long as one argument is vertical. In this respect it acts almost like the opposite of a connection form, which vanishes on horizontal vectors.

Lemma 30.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$ and $\pi\colon P \to M$ a principal G-bundle with a connection ω. Fix a point $p \in P$.*

(i) *Every vertical vector $X_p \in T_pP$ can be extended to a fundamental vector field $\underline{A}$ on P for some $A \in \mathfrak{g}$.*
(ii) *Every horizontal vector $Y_p \in T_pP$ can be extended to the horizontal lift $\tilde{B}$ of a C^∞ vector field B on M.*

Proof. (i) By the surjectivity of $j_{p*}\colon \mathfrak{g} \to \mathcal{V}_p$ (Proposition 27.18) and Equation (27.3),

$$X_p = j_{p*}(A) = \underline{A}_p$$

for some $A \in \mathfrak{g}$. Then the fundamental vector field $\underline{A}$ on P extends X_p.

(ii) Let $x = \pi(p)$ in M and let B_x be the projection $\pi_*(Y_p) \in T_xM$ of the vector Y_p. We can extend B_x to a smooth vector field B on M. The horizontal lift $\tilde{B}$ of B extends Y_p on P. □

By Proposition 28.6, such a horizontal lift $\tilde{B}$ is necessarily right-invariant.

Theorem 30.4. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Suppose $\pi\colon P \to M$ is a principal G-bundle, ω a connection on P, and Ω the curvature form of ω.*

(i) (Horizontality) *For $p \in P$ and $X_p, Y_p \in T_pP$,*

$$\Omega_p(X_p, Y_p) = (d\omega)_p(hX_p, hY_p). \tag{30.1}$$

(ii) (G-equivariance) *For $g \in G$, we have $r_g^*\Omega = (\operatorname{Ad} g^{-1})\Omega$.*
(iii) (Second Bianchi identity) $d\Omega = [\Omega, \omega]$.

Proof. (i) Since both sides of (30.1) are linear in X_p and in Y_p, we may decompose X_p and Y_p into vertical and horizontal components, and so it suffices to check the equation for vertical and horizontal vectors only. There are three cases.

Case 1. Both X_p and Y_p are horizontal. Then

$$\begin{aligned}
\Omega_p(X_p,Y_p) &= (d\omega)_p(X_p,Y_p) + \frac{1}{2}[\omega_p,\omega_p](X_p,Y_p) && \text{(definition of } \Omega\text{)}\\
&= (d\omega)_p(X_p,Y_p)\\
&\quad + \frac{1}{2}\Big([\omega_p(X_p),\omega_p(Y_p)] - [\omega_p(Y_p),\omega_p(X_p)]\Big)\\
&= (d\omega)_p(X_p,Y_p) && (\omega_p(X_p)=0)\\
&= (d\omega)_p(hX_p,hY_p). && (X_p,\ Y_p \text{ horizontal})
\end{aligned}$$

Case 2. One of X_p and Y_p is horizontal; the other is vertical. Without loss of generality, we may assume X_p vertical and Y_p horizontal. Then $[\omega_p,\omega_p](X_p,Y_p) = 0$ as in Case 1.

By Lemma 30.3 the vertical vector X_p extends to a fundamental vector field $\underline{A}$ on P and the horizontal vector Y_p extends to a right-invariant horizontal vector field $\tilde{B}$ on P. By the global formula for the exterior derivative (Problem 21.8)

$$d\omega(\underline{A},\tilde{B}) = \underline{A}(\omega(\tilde{B})) - \tilde{B}(\omega(\underline{A})) - \omega([\underline{A},\tilde{B}]).$$

On the right-hand side, $\omega(\tilde{B}) = 0$ because $\tilde{B}$ is horizontal, and $\tilde{B}\omega(\underline{A}) = \tilde{B}A = 0$ because A is a constant function on P. Being the bracket of a fundamental and a horizontal vector field, $[\underline{A},\tilde{B}]$ is horizontal by Lemma 28.7, and therefore $\omega([\underline{A},\tilde{B}]) = 0$. Hence, the left-hand side of (30.1) becomes

$$\Omega_p(X_p,Y_p) = (d\omega)_p(\underline{A}_p,\tilde{B}_p) = 0.$$

The right-hand side of (30.1) is also zero because $hX_p = 0$.

Case 3. Both X_p and Y_p are vertical. As in Case 2, we can write $X_p = \underline{A}_p$ and $Y_p = \underline{B}_p$ for some $A,B \in \mathfrak{g}$. We have thus extended the vertical vectors X_p and Y_p to fundamental vector fields $X = \underline{A}$ and $Y = \underline{B}$ on P. By the definition of curvature,

$$\begin{aligned}
\Omega(X,Y) &= \Omega(\underline{A},\underline{B})\\
&= d\omega(\underline{A},\underline{B}) + \frac{1}{2}\big([\omega(\underline{A}),\omega(\underline{B})] - [\omega(\underline{B}),\omega(\underline{A})]\big)\\
&= d\omega(\underline{A},\underline{B}) + [A,B]. \qquad (30.2)
\end{aligned}$$

In this sum the first term is

$$\begin{aligned}
d\omega(\underline{A},\underline{B}) &= \underline{A}\big(\omega(\underline{B})\big) - \underline{B}\big(\omega(\underline{A})\big) - \omega\big([\underline{A},\underline{B}]\big)\\
&= \underline{A}(B) - \underline{B}(A) - \omega\big(\underline{[A,B]}\big) && \text{(Problem 27.1)}\\
&= 0 - 0 - [A,B].
\end{aligned}$$

Hence, (30.2) becomes

$$\Omega(X,Y) = -[A,B] + [A,B] = 0.$$

On the other hand,

$$(d\omega)_p(hX_p, hY_p) = (d\omega)_p(0,0) = 0.$$

(ii) Since the connection form ω is right-equivariant with respect to Ad,

$$\begin{aligned}
r_g^*\Omega &= r_g^*\left(d\omega + \frac{1}{2}[\omega,\omega]\right) && \text{(definition of curvature)} \\
&= dr_g^*\omega + \frac{1}{2}[r_g^*\omega, r_g^*\omega] && \text{(Proposition 21.8)} \\
&= d(\operatorname{Ad} g^{-1})\omega + \frac{1}{2}[(\operatorname{Ad} g^{-1})\omega, (\operatorname{Ad} g^{-1})\omega] \\
&= (\operatorname{Ad} g^{-1})\left(d\omega + \frac{1}{2}[\omega,\omega]\right) \\
&= (\operatorname{Ad} g^{-1})\Omega.
\end{aligned}$$

In this computation we used the fact that because $\operatorname{Ad} g^{-1} = (c_{g^{-1}})_*$ is the differential of a Lie group homomorphism, it is a Lie algebra homomorphism.

(iii) Taking the exterior derivative of the definition of the curvature form, we get

$$\begin{aligned}
d\Omega &= \frac{1}{2} d[\omega,\omega] \\
&= \frac{1}{2}([d\omega,\omega] - [\omega, d\omega]) && \text{(Proposition 21.6)} \\
&= [d\omega, \omega] && \text{(Proposition 21.5)} \\
&= \left[\Omega - \frac{1}{2}[\omega,\omega], \omega\right] && \text{(definition of } \Omega) \\
&= [\Omega, \omega] - \frac{1}{2}[[\omega,\omega],\omega] \\
&= [\Omega, \omega]. && \text{(Problem 21.5)}
\end{aligned}$$

□

In case P is the frame bundle $\operatorname{Fr}(E)$ of a rank r vector bundle E, with structure group $\operatorname{GL}(r,\mathbb{R})$, the second Bianchi identity becomes by Proposition 21.7

$$d\Omega = [\Omega, \omega] = \Omega \wedge \omega - \omega \wedge \Omega, \tag{30.3}$$

where the connection and curvature forms ω and Ω are $\mathfrak{gl}(r,\mathbb{R})$-valued forms on $\operatorname{Fr}(E)$. It should not be so surprising that it has the same form as the second Bianchi identity for the connection and curvature matrices relative to a frame e for E (Proposition 22.3). Indeed, by pulling back (30.3) by a frame $e\colon U \to \operatorname{Fr}(E)$, we get

$$\begin{aligned}
e^* d\Omega &= e^*(\Omega \wedge \omega) - e^*(\omega \wedge \Omega), \\
de^*\Omega &= (e^*\Omega) \wedge e^*\omega - (e^*\omega) \wedge e^*\Omega, \\
d\Omega_e &= \Omega_e \wedge \omega_e - \omega_e \wedge \Omega_e,
\end{aligned}$$

which is precisely Proposition 22.3.

Problems

30.1. Curvature of the Maurer–Cartan connection

Let G be a Lie group with Lie algebra $\mathfrak{g}$, and M a manifold. Compute the curvature of the Maurer–Cartan connection ω on the trivial bundle $\pi\colon M \times G \to M$.

30.2. Generalized second Bianchi identity on a frame bundle

Suppose $\mathrm{Fr}(E)$ is the frame bundle of a rank r vector bundle E over M. Let ω be an Ehresmann connection and Ω its curvature form on $\mathrm{Fr}(E)$. These are differential forms on $\mathrm{Fr}(E)$ with values in the Lie algebra $\mathfrak{gl}(r,\mathbb{R})$. Matrix multiplication and the Lie bracket on $\mathfrak{gl}(r,\mathbb{R})$ lead to two ways to multiply $\mathfrak{gl}(r,\mathbb{R})$-valued forms (see Section 21.5). We write Ω^k to denote the wedge product of Ω with itself k times. Prove that $d(\Omega^k) = [\Omega^k, \omega]$.

30.3. Lie bracket of horizontal vector fields

Let $P \to M$ be a principal bundle with a connection, and X, Y horizontal vector fields on P.

(a) Prove that $\Omega(X,Y) = -\omega([X,Y])$.
(b) Show that $[X,Y]$ is horizontal if and only if the curvature $\Omega(X,Y)$ equals zero.

§31 Covariant Derivative on a Principal Bundle

Throughout this chapter, G will be a Lie group with Lie algebra $\mathfrak{g}$ and V will be a finite-dimensional vector space. To a principal G-bundle $\pi\colon P \to M$ and a representation $\rho\colon G \to \mathrm{GL}(V)$, one can associate a vector bundle $P \times_\rho V \to M$ with fiber V. When ρ is the adjoint representation Ad of G on its Lie algebra $\mathfrak{g}$, the associated bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ is called the ***adjoint bundle***, denoted by $\mathrm{Ad}\,P$.

Differential forms on M with values in the associated bundle $P \times_\rho V$ turn out to correspond in a one-to-one manner to certain V-valued forms on P called ***tensorial forms of type*** ρ. The curvature Ω of a connection ω on the principal bundle P is a $\mathfrak{g}$-valued tensorial 2-form of type Ad on P. Under this correspondence it may be viewed as a 2-form on M with values in the adjoint bundle $\mathrm{Ad}\,P$.

Using a connection ω, one can define a covariant derivative D of vector-valued forms on a principal bundle P. This covariant derivative maps tensorial forms to tensorial forms, and therefore induces a covariant derivative on forms on M with values in an associated bundle. In terms of the covariant derivative D, the curvature form is $\Omega = D\omega$, and Bianchi's second identity becomes $D\Omega = 0$.

31.1 The Associated Bundle

Let $\pi\colon P \to M$ be a principal G-bundle and $\rho\colon G \to \mathrm{GL}(V)$ a representation of G on a finite-dimensional vector space V. We write $\rho(g)v$ as $g \cdot v$ or even gv. The ***associated bundle*** $E := P \times_\rho V$ is the quotient of $P \times V$ by the equivalence relation

$$(p,v) \sim (pg, g^{-1} \cdot v) \qquad \text{for } g \in G \text{ and } (p,v) \in P \times V. \tag{31.1}$$

We denote the equivalence class of (p,v) by $[p,v]$. The associated bundle comes with a natural projection $\beta\colon P \times_\rho V \to M$, $\beta([p,v]) = \pi(p)$. Because

$$\beta([pg, g^{-1} \cdot v]) = \pi(pg) = \pi(p) = \beta([p,v]),$$

the projection β is well defined.

As a first example, the proposition below shows that an associated bundle of a trivial principal G-bundle is a trivial vector bundle.

Proposition 31.1. *If $\rho\colon G \to \mathrm{GL}(V)$ is a finite-dimensional representation of a Lie group G, and U is any manifold, then there is a fiber-preserving diffeomorphism*

$$\begin{aligned} \phi\colon (U \times G) \times_\rho V &\xrightarrow{\sim} U \times V, \\ [(x,g),v] &\mapsto (x, g \cdot v). \end{aligned}$$

Proof. The map ϕ is well defined because if h is any element of G, then

$$\phi\big([(x,g)h, h^{-1} \cdot v]\big) = \big(x, (gh) \cdot h^{-1} \cdot v\big) = (x, g \cdot v) = \phi\big([(x,g),v]\big).$$

Define $\psi\colon U \times V \to (U \times G) \times_\rho V$ by

$$\psi(x, v) = [(x, 1), v].$$

It is easy to check that ϕ and ψ are inverse to each other, are C^∞, and commute with the projections. □

Since a principal bundle $P \to M$ is locally $U \times G$, Proposition 31.1 shows that the associated bundle $P \times_\rho V \to M$ is locally trivial with fiber V. The vector space structure on V then makes $P \times_\rho V$ into a vector bundle over M:

$$\begin{aligned} [p, v_1] + [p, v_2] &= [p, v_1 + v_2], \\ \lambda[p, v] &= [p, \lambda v], \quad \lambda \in \mathbb{R}. \end{aligned} \tag{31.2}$$

It is easy to show that these are well-defined operations not depending on the choice of $p \in E_x$ and that this makes the associated bundle $\beta\colon E \to M$ into a vector bundle (Problem 31.2).

Example 31.2. Let $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ be the adjoint representation of a Lie group G on its Lie algebra $\mathfrak{g}$. For a principal G-bundle $\pi\colon P \to M$, the associated vector bundle $\mathrm{Ad}\, P := P \times_{\mathrm{Ad}} \mathfrak{g}$ is called the ***adjoint bundle*** of P.

31.2 The Fiber of the Associated Bundle

If $\pi\colon P \to M$ is a principal G-bundle, $\rho\colon G \to \mathrm{GL}(V)$ is a representation, and $E := P \times_\rho V \to M$ is the associated bundle, we denote by P_x the fiber of P above $x \in M$, and by E_x the fiber of E above $x \in M$. For each $p \in P_x$, there is a canonical way of identifying the fiber E_x with the vector space V:

$$\begin{aligned} f_p\colon V &\to E_x, \\ v &\mapsto [p, v]. \end{aligned}$$

Lemma 31.3. *Let $\pi\colon P \to M$ be a principal G-bundle, $\rho\colon G \to \mathrm{GL}(V)$ a finite-dimensional representation, and $E = P \times_\rho V$ the associated vector bundle. For each point p in the fiber P_x, the map $f_p\colon V \to E_x$ is a linear isomorphism.*

Proof. Suppose $[p, v] = [p, w]$. Then $(p, w) = (pg, g^{-1}v)$ for some $g \in G$. Since G acts freely on P, the equality $p = pg$ implies that $g = 1$. Hence, $w = g^{-1}v = v$. This proves that f_p is injective.

If $[q, w]$ is any point in E_x, then $q \in P_x$, so $q = pg$ for some $g \in G$. It follows that

$$[q, w] = [pg, w] = [p, gw] = f_p(gw).$$

This proves that f_p is surjective. □

The upshot is that every point p of the total space P of a principal bundle gives a linear isomorphism $f_p\colon V \to E_{\pi(p)}$ from V to the fiber of the associated bundle E above $\pi(p)$.

Lemma 31.4. *Let $E = P \times_\rho V$ be the vector bundle associated to the principal G-bundle $P \to M$ via the representation $\rho \colon G \to \mathrm{GL}(V)$, and $f_p \colon V \to E_x$ the linear isomorphism $v \mapsto [p, v]$. If $g \in G$, then $f_{pg} = f_p \circ \rho(g)$.*

Proof. For $v \in V$,

$$f_{pg}(v) = [pg, v] = [p, g \cdot v] = f_p(g \cdot v) = f_p\big(\rho(g)v\big). \qquad \square$$

Example 31.5. Let $\pi \colon P \to M$ be a principal G-bundle. The vector bundle $P \times_\rho V \to M$ associated to the trivial representation $\rho \colon G \to \mathrm{GL}(V)$ is the trivial bundle $M \times V \to M$, for there is a vector bundle isomorphism

$$P \times_\rho V \to M \times V,$$
$$[p, v] = [pg, g^{-1} \cdot v] = [pg, v] \mapsto \big(\pi(p), v\big),$$

with inverse map

$$(x, v) \mapsto [p, v] \quad \text{for any } p \in \pi^{-1}(x).$$

In this case, for each $p \in P$ the linear isomorphism $f_p \colon V \to E_x = V$, $v \mapsto [p, v]$, is the identity map.

31.3 Tensorial Forms on a Principal Bundle

We keep the same notation as in the previous section. Thus, $\pi \colon P \to M$ is a principal G-bundle, $\rho \colon G \to \mathrm{GL}(V)$ a finite-dimensional representation of G, and $E := P \times_\rho V$ the vector bundle associated to P via ρ.

Definition 31.6. A V-valued k-form φ on P is said to be ***right-equivariant of type*** ρ or ***right-equivariant with respect to*** ρ if for every $g \in G$,

$$r_g^* \varphi = \rho(g^{-1}) \cdot \varphi.$$

What this means is that for $p \in P$ and $v_1, \ldots, v_k \in T_pP$,

$$(r_g^* \varphi)_p(v_1, \ldots, v_k) = \rho(g^{-1})\big(\varphi_p(v_1, \ldots, v_k)\big).$$

In the literature (for example, [12, p. 75]), such a form is said to be ***pseudo-tensorial of type*** ρ.

Definition 31.7. A V-valued k-form φ on P is said to be ***horizontal*** if φ vanishes whenever one of its arguments is a vertical vector. Since a 0-form never takes an argument, every 0-form on P is by definition horizontal.

Definition 31.8. A V-valued k-form φ on P is ***tensorial of type*** ρ if it is right-equivariant of type ρ and horizontal. The set of all smooth tensorial V-valued k-forms of type ρ is denoted by $\Omega^k_\rho(P, V)$.

Example. Since the curvature Ω of a connection ω on a principal G-bundle P is horizontal and right-equivariant of type Ad, it is tensorial of type Ad.

The set $\Omega^k_\rho(P,V)$ of tensorial k-forms of type ρ on P becomes a vector space with the usual addition and scalar multiplication of forms. These forms are of special interest because they can be viewed as forms on the base manifold M with values in the associated bundle $E := P \times_\rho V$. To each tensorial V-valued k form $\varphi \in \Omega^k_\rho(P,V)$ we associate a k-form $\varphi^\flat \in \Omega^k(M,E)$ as follows. Given $x \in M$ and $v_1,\dots,v_k \in T_xM$, choose any point p in the fiber P_x and choose lifts $u_1,\dots,u_k$ at p of $v_1,\dots,v_k$, i.e., vectors in T_pP such that $\pi_*(u_i) = v_i$. Then $\varphi^\flat$ is defined by

$$\varphi^\flat_x(v_1,\dots,v_k) = f_p\big(\varphi_p(u_1,\dots,u_k)\big) \in E_x, \tag{31.3}$$

where $f_p\colon V \to E_x$ is the isomorphism $v \mapsto [p,v]$ of the preceding section.

Conversely, if $\psi \in \Omega^k(M,E)$, we define $\psi^\sharp \in \Omega^k_\rho(P,V)$ as follows. Given $p \in P$ and $u_1,\dots,u_k \in T_pP$, let $x = \pi(p)$ and set

$$\psi^\sharp_p(u_1,\dots,u_k) = f_p^{-1}\big(\psi_x(\pi_*u_1,\dots,\pi_*u_k)\big) \in V. \tag{31.4}$$

Theorem 31.9. *The map*

$$\Omega^k_\rho(P,V) \to \Omega^k(M,E),$$
$$\varphi \mapsto \varphi^\flat,$$

is a well-defined linear isomorphism with inverse $\psi^\sharp \leftarrow\!\shortmid \psi$.

Proof. To show that $\varphi^\flat$ is well defined, we need to prove that the definition (31.3) is independent of the choice of $p \in P_x$ and of $u_1,\dots,u_k \in T_pP$. Suppose $u'_1,\dots,u'_k \in T_pP$ is another set of vectors such that $\pi_*(u'_i) = v_i$. Then $\pi_*(u'_i - u_i) = 0$ so that $u'_i - u_i$ is vertical. Since φ is horizontal and k-linear,

$$\begin{aligned}\varphi_p(u'_1,\dots,u'_k) &= \varphi_p(u_1 + \text{vertical},\dots,u_k + \text{vertical})\\ &= \varphi_p(u_1,\dots,u_k).\end{aligned}$$

This proves that for a given $p \in P$, the definition (31.3) is independent of the choice of lifts of $v_1,\dots,v_k$ to p.

Next suppose we choose pg instead of p as the point in the fiber P_x. Because $\pi \circ r_g = \pi$,

$$\pi_*(r_{g*}u_i) = (\pi \circ r_g)_*u_i = \pi_*u_i = v_i,$$

so that $r_{g*}u_1,\dots,r_{g*}u_k$ are lifts of $v_1,\dots,v_k$ to pg. We have, by right equivariance with respect to ρ,

$$\begin{aligned}\varphi_{pg}(r_{g*}u_1,\dots,r_{g*}u_k) &= (r_g^*\varphi_{pg})(u_1,\dots,u_k)\\ &= \rho(g^{-1})\varphi_p(u_1,\dots,u_k).\end{aligned}$$

So by Lemma 31.4,

$$\begin{aligned} f_{pg}\big(\varphi_{pg}(r_{g*}u_1,\dots,r_{g*}u_k)\big) &= f_{pg}\big(\rho(g^{-1})\varphi_p(u_1,\dots,u_k)\big) \\ &= \big(f_p\circ\rho(g)\big)\big(\rho(g^{-1})\varphi_p(u_1,\dots,u_k)\big) \\ &= f_p\big(\varphi_p(u_1,\dots,u_k)\big). \end{aligned}$$

This proves that the definition (31.3) is independent of the choice of p in the fiber P_x.

Let $\psi\in\Omega^k(M,E)$. It is clear from the definition (31.4) that $\psi^\sharp$ is horizontal. It is easy to show that $\psi^\sharp$ is right-equivariant with respect to ρ (Problem 31.4). Hence, $\psi^\sharp\in\Omega^k_\rho(P,V)$.

For $v_1,\dots,v_k\in T_xM$, choose $p\in P_x$ and vectors $u_1,\dots,u_k\in T_pP$ that lift $v_1,\dots,v_k$. Then

$$\begin{aligned} (\psi^{\sharp\flat})_x(v_1,\dots,v_k) &= f_p\big(\psi^\sharp_p(u_1,\dots,u_k)\big) \\ &= f_p\Big(f_p^{-1}\big(\psi_x(\pi_*u_1,\dots,\pi_*u_k)\big)\Big) \\ &= \psi_x(v_1,\dots,v_k). \end{aligned}$$

Hence, $\psi^{\sharp\flat}=\psi$.

Similarly, $\varphi^{\flat\sharp}=\varphi$ for $\varphi\in\Omega^k_\rho(P,V)$, which we leave to the reader to show (Problem 31.5). Therefore, the map $\psi\mapsto\psi^\sharp$ is inverse to the map $\varphi\mapsto\varphi^\flat$. □

Example 31.10 (*Curvature as a form on the base*). By Theorem 31.9, the curvature form Ω of a connection on a principal G-bundle P can be viewed as an element of $\Omega^2(M,\mathrm{Ad}\,P)$, a 2-form on M with values in the adjoint bundle $\mathrm{Ad}\,P$.

When $k=0$ in Theorem 31.9, $\Omega^0_p(P,V)$ consists of maps $f\colon P\to V$ that are right-equivariant with respect to ρ:

$$(r_g^*f)(p)=\rho(g)^{-1}f(p),$$

or

$$f(pg)=\rho(g^{-1})f(p)=g^{-1}\cdot f(p).$$

On the right-hand side of Theorem 31.9,

$$\Omega^0(M,P\times_\rho V)=\Omega^0(M,E)=\text{ sections of the associated bundle } E.$$

Hence, we have the following corollary.

Corollary 31.11. *Let G be a Lie group, $P\to M$ a principal G-bundle, and $\rho\colon G\to$ $\mathrm{Aut}(V)$ a representation of G. There is a one-to-one correspondence*

$$\left\{\begin{matrix}\textit{G-equivariant maps}\\ f\colon P\to V\end{matrix}\right\}\quad\longleftrightarrow\quad\left\{\begin{matrix}\textit{sections of the associated bundle}\\ P\times_\rho V\to M\end{matrix}\right\}.$$

By the local triviality condition, for any principal bundle $\pi\colon P\to M$ the projection map π is a submersion and therefore the pullback map $\pi^*\colon\Omega^*(M)\to\Omega^*(P)$ is an injection. A differential form φ on P is said to be ***basic*** if it is the pullback $\pi^*\psi$

of a form ψ on M; it is ***G*-invariant** if $r_g^*\varphi = \varphi$ for all $g \in G$. More generally, for any vector space V, these concepts apply to V-valued forms as well.

Suppose $\rho\colon G \to \mathrm{GL}(V)$ is the trivial representation $\rho(g) = \mathbb{1}$ for all $g \in G$. Then an equivariant form φ of type ρ on P satisfies

$$r_g^*\varphi = \rho(g^{-1}) \cdot \varphi = \varphi \quad \text{for all } g \in G.$$

Thus, an equivariant form of type ρ for the trivial representation ρ is exactly an invariant form on P. Unravelling Theorem 31.9 for a trivial representation will give the following theorem.

Theorem 31.12. *Let $\pi\colon P \to M$ be a principal G-bundle and V a vector space. A V-valued form on P is basic if and only if it is horizontal and G-invariant.*

Proof. Let $\rho\colon G \to \mathrm{GL}(V)$ be the trivial representation. As noted above, $\Omega_\rho^k(P,V)$ consists of horizontal, G-invariant V-valued k-forms on P.

By Example 31.5, when ρ is the trivial representation, the vector bundle $E = P \times_\rho V$ is the product bundle $M \times V$ over M and for each $p \in P$, the linear isomorphism $f_p\colon V \to E_x = V$, where $x = \pi(p)$, is the identity map. Then the isomorphism

$$\Omega^k(M,E) = \Omega^k(M, M \times V) = \Omega^k(M,V) \to \Omega_\rho^k(P,V),$$
$$\psi \mapsto \psi^\#,$$

is given by

$$\begin{aligned}\psi_p^\#(u_1,\dots,u_k) &= \psi_x(\pi_*u_1,\dots,\pi_*u_k) \qquad \text{(by (31.4))}\\ &= (\pi^*\psi)_p(u_1,\dots,u_k).\end{aligned}$$

Therefore,

$$\psi^\# = \pi^*\psi.$$

This proves that horizontal, G-invariant forms on P are precisely the basic forms. □

31.4 Covariant Derivative

Recall that the existence of a connection ω on a principal G-bundle $\pi\colon P \to M$ is equivalent to the decomposition of the tangent bundle TP into a direct sum of the vertical subbundle $\mathcal{V}$ and a smooth right-invariant horizontal subbundle $\mathcal{H}$. For any vector $X_p \in T_pP$, we write

$$X_p = vX_p + hX_p$$

as the sum of its vertical and horizontal components. This will allow us to define a covariant derivative of vector-valued forms on P. By the isomorphism of Theorem 31.9, we obtain in turn a covariant derivative of forms on M with values in an associated bundle.

Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation of G and let $E := P \times_\rho V$ be the associated vector bundle.

Proposition 31.13. *If $\varphi \in \Omega^k(P,V)$ is right-equivariant of type ρ, then so is $d\varphi$.*

Proof. For a fixed $g \in G$,

$$\begin{aligned} r_g^* d\varphi &= d r_g^* \varphi = d\rho(g^{-1})\varphi \\ &= \rho(g^{-1}) d\varphi, \end{aligned}$$

since $\rho(g^{-1})$ is a constant linear map for a fixed g. □

In general, the exterior derivative does not preserve horizontality. For any V-valued k-form φ on P, we define its ***horizontal component*** $\varphi^h \in \Omega^k(P,V)$ as follows: for $p \in P$ and $v_1, \ldots, v_k \in T_pP$,

$$\varphi_p^h(v_1, \ldots, v_k) = \varphi_p(hv_1, \ldots, hv_k).$$

Proposition 31.14. *If $\varphi \in \Omega^k(P,V)$ is right-equivariant of type ρ, then so is φ^h.*

Proof. For $g \in G$, $p \in P$, and $v_1, \ldots, v_k \in T_pP$,

$$\begin{aligned} r_g^*(\varphi_{pg}^h)(v_1, \ldots, v_k) &= \varphi_{pg}^h(r_{g*}v_1, \ldots, r_{g*}v_k) && \text{(definition of pullback)} \\ &= \varphi_{pg}(hr_{g*}v_1, \ldots, hr_{g*}v_k) && \text{(definition of } \varphi^h\text{)} \\ &= \varphi_{pg}(r_{g*}hv_1, \ldots, r_{g*}hv_k) && \text{(Proposition 28.4)} \\ &= (r_g^*\varphi_{pg})(hv_1, \ldots, hv_k) \\ &= \rho(g^{-1}) \cdot \varphi_p(hv_1, \ldots, hv_k) && \text{(right-equivariance of } \varphi\text{)} \\ &= \rho(g^{-1}) \cdot \varphi_p^h(v_1, \ldots, v_k) \end{aligned}$$

□

Propositions 31.13 and 31.14 together imply that if $\varphi \in \Omega^k(P,V)$ is right-equivariant of type ρ, then $(d\varphi)^h \in \Omega^{k+1}(P,V)$ is horizontal and right-equivariant of type ρ, i.e., tensorial of type ρ.

Definition 31.15. Let $\pi\colon P \to M$ be a principal G-bundle with a connection ω and let V be a real vector space. The ***covariant derivative*** of a V-valued k-form $\varphi \in \Omega^k(P,V)$ is $D\varphi = (d\varphi)^h$.

Let $\rho\colon G \to \mathrm{GL}(V)$ be a finite-dimensional representation of the Lie group G. The covariant derivative is defined for any V-valued k-form on P, and it maps a right-equivariant form of type ρ to a tensorial form of type ρ. In particular, it restricts to a map

$$D\colon \Omega_\rho^k(P,V) \to \Omega_\rho^{k+1}(P,V) \tag{31.5}$$

on the space of tensorial forms.

Proposition 31.16. *Let $\pi\colon P \to M$ be a principal G-bundle with a connection and $\rho\colon G \to \mathrm{GL}(V)$ a representation of G. The covariant derivative*

$$D\colon \Omega_\rho^k(P,V) \to \Omega_\rho^{k+1}(P,V)$$

on tensorial forms of type ρ is an antiderivation of degree $+1$.

Proof. Let $\omega, \tau \in \Omega^*_\rho(P,V)$ be tensorial forms of type ρ. Then

$$\begin{aligned} D(\omega\wedge\tau) &= \big(d(\omega\wedge\tau)\big)^h \\ &= \big((d\omega)\wedge\tau + (-1)^{\deg\omega}\omega\wedge d\tau\big)^h \\ &= (d\omega)^h\wedge\tau^h + (-1)^{\deg\omega}\omega^h\wedge(d\tau)^h \\ &= Dw\wedge\tau^h + (-1)^{\deg\omega}\omega^h\wedge D\tau. \end{aligned}$$

Since τ and ω are horizontal, $\tau^h = \tau$ and $\omega^h = \omega$. Therefore,

$$D(\omega\wedge\tau) = D\omega\wedge\tau + (-1)^{\deg\omega}\omega\wedge D\tau. \qquad \square$$

If $E := P\times_\rho V$ is the associated vector bundle via the representation ρ, then the isomorphism of Theorem 31.9 transforms the linear map (31.5) into a linear map

$$D\colon \Omega^k(M,E)\to\Omega^{k+1}(M,E).$$

Unlike the exterior derivative, the covariant derivative depends on the choice of a connection on P. Moreover, $D^2 \neq 0$ in general.

Example 31.17 (*Curvature of a principal bundle*). By Theorem 30.4 the curvature form $\Omega \in \Omega^2_{\mathrm{Ad}}(P,\mathfrak{g})$ on a principal bundle is the covariant derivative $D\omega$ of the connection form $\omega \in \Omega^1(P,\mathfrak{g})$. Because ω is not horizontal, it is not in $\Omega^1_{\mathrm{Ad}}(P,\mathfrak{g})$.

31.5 A Formula for the Covariant Derivative of a Tensorial Form

Let $\pi\colon P\to M$ be a smooth principal G-bundle with a connection ω, and let $\rho\colon G\to \mathrm{GL}(V)$ be a finite-dimensional representation of G. In the preceding section we defined the covariant derivative of a V-valued k-form φ on P: $D\varphi = (d\varphi)^h$, the horizontal component of $d\varphi$. In this section we derive a useful alternative formula for the covariant derivative, but only for a tensorial form.

The Lie group representation $\rho\colon G\to\mathrm{GL}(V)$ induces a Lie algebra representation $\rho_*\colon \mathfrak{g}\to\mathfrak{gl}(V)$, which allows us to define a product of a $\mathfrak{g}$-valued k-form τ and a V-valued ℓ-form φ on P: for $p\in P$ and $v_1,\dots,v_{k+\ell}\in T_pP$,

$$\begin{aligned} &(\tau\cdot\varphi)_p(v_1,\dots,v_{k+\ell}) \\ &\qquad = \frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}} \operatorname{sgn}(\sigma)\rho_*\big(\tau_p(v_{\sigma(1)},\dots,v_{\sigma(k)})\big)\varphi_p\left(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)}\right). \end{aligned}$$

For the same reason as the wedge product, $\tau\cdot\varphi$ is multilinear and alternating in its arguments; it is therefore a $(k+\ell)$-covector with values in V.

Example 31.18. If $V=\mathfrak{g}$ and $\rho = \mathrm{Ad}\colon G\to\mathrm{GL}(\mathfrak{g})$ is the adjoint representation, then

$$(\tau\cdot\varphi)_p = \frac{1}{k!\ell!}\sum_{\sigma\in S_{k+\ell}}\operatorname{sgn}(\sigma)\left[\tau_p(v_{\sigma(1)},\dots,v_{\sigma(k)}),\varphi_p(v_{\sigma(k+1)},\dots,v_{\sigma(k+\ell)})\right].$$

In this case we also write $[\tau,\varphi]$ instead of $\tau\cdot\varphi$.

Theorem 31.19. *Let $\pi\colon P \to M$ be a principal G-bundle with connection form ω, and $\rho\colon G \to \mathrm{GL}(V)$ a finite-dimensional representation of G. If $\varphi \in \Omega^k_\rho(P,V)$ is a V-valued tensorial form of type ρ, then its covariant derivative is given by*

$$D\varphi = d\varphi + \omega \cdot \varphi.$$

Proof. Fix $p \in P$ and $v_1, \ldots, v_{k+1} \in T_pP$. We need to show that

$$\begin{aligned}(d\varphi)_p(hv_1,\ldots,hv_{k+1}) &= (d\varphi)_p(v_1,\ldots,v_{k+1}) \\ &\quad + \frac{1}{k!}\sum_{\sigma\in S_{k+1}} \operatorname{sgn}(\sigma)\rho_*\left(\omega_p(v_{\sigma(1)})\right)\varphi_p\left(v_{\sigma(2)},\ldots,v_{\sigma(k+1)}\right). \qquad (31.6)\end{aligned}$$

Because both sides of (31.6) are linear in each argument v_i, which may be decomposed into the sum of a vertical and a horizontal component, we may assume that each v_i is either vertical or horizontal. By Lemma 30.3, throughout the proof we may further assume that the vectors $v_1, \ldots, v_{k+1}$ have been extended to vector fields $X_1, \ldots, X_{k+1}$ on P each of which is either vertical or horizontal. If X_i is vertical, then it is a fundamental vector field $\underline{A_i}$ for some $A_i \in \mathfrak{g}$. If X_i is horizontal, then it is the horizontal lift $\tilde{B}_i$ of a vector field B_i on M. By construction, $\tilde{B}_i$ is right-invariant (Proposition 28.6).

Instead of proving (31.6) at a point p, we will prove the equality of functions

$$(d\varphi)(hX_1,\ldots,hX_{k+1}) = \mathrm{I} + \mathrm{II}, \qquad (31.7)$$

where

$$\mathrm{I} = (d\varphi)(X_1,\ldots,X_{k+1})$$

and

$$\mathrm{II} = \frac{1}{k!}\sum_{\sigma\in S_{k+1}} \operatorname{sgn}(\sigma)\rho_*\left(\omega(X_{\sigma(1)})\right)\varphi\left(X_{\sigma(2)},\ldots,X_{\sigma(k+1)}\right).$$

Case 1. *The vector fields $X_1, \ldots, X_{k+1}$ are all horizontal.*
Then $\mathrm{II} = 0$ because $\omega(X_{\sigma(1)}) = 0$ for all $\sigma \in S_{k+1}$. In this case, (31.7) is trivially true.

Case 2. *At least two of $X_1, \ldots, X_{k+1}$ are vertical.*
By the skew-symmetry of the arguments, we may assume that $X_1 = \underline{A_1}$ and $X_2 = \underline{A_2}$ are vertical. By Problem 27.1, $[X_1, X_2] = \underline{[A_1, A_2]}$ is also vertical.

The left-hand side of (31.7) is zero because $hX_1 = 0$. By the global formula for the exterior derivative [21, Th. 20.14, p. 233],

$$\mathrm{I} = \sum_{i=1}^{k+1}(-1)^{i-1}X_i\varphi(\ldots,\widehat{X_i},\ldots) + \sum_{1\le i<j\le k+1}(-1)^{i+j}\varphi([X_i,X_j],\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots).$$

In this expression every term in the first sum is zero because φ is horizontal and at least one of its arguments is vertical. In the second sum at least one of the arguments of φ is X_1, X_2, or $[X_1,X_2]$, all of which are vertical. Therefore, every term in the second sum in I is also zero.

As for II in (31.7), in every term at least one of the arguments of φ is vertical, so $\mathrm{II}=0$.

Case 3. *The first vector field $X_1=\underline{A}$ is vertical; the rest $X_2,\dots,X_{k+1}$ are horizontal and right-invariant.*
The left-hand side of (31.7) is clearly zero because $hX_1=0$.

On the right-hand side,

$$\begin{aligned}\mathrm{I}&=(d\varphi)(X_1,\dots,X_{k+1})\\ &=\sum(-1)^{i+1}X_i\varphi(X_1,\dots,\widehat{X}_i,\dots,X_{k+1})\\ &\qquad\qquad+\sum(-1)^{i+j}\varphi([X_i,X_j],X_1,\dots,\widehat{X}_i,\dots,\widehat{X}_j,\dots,X_{k+1}).\end{aligned}$$

Because φ is horizontal and X_1 is vertical, the only nonzero term in the first sum is

$$X_1\varphi(X_2,\dots,X_{k+1})=\underline{A}\varphi(X_2,\dots,X_{k+1})$$

and the only nonzero terms in the second sum are

$$\sum_{j=2}^{k+1}(-1)^{1+j}\varphi([X_1,X_j],\widehat{X}_1,X_2,\dots,\widehat{X}_j,\dots,X_{k+1}).$$

Since the X_j, $j=2,\dots,k+1$, are right-invariant horizontal vector fields, by Lemma 28.7,

$$[X_1,X_j]=[\underline{A},X_j]=0.$$

Therefore,

$$\mathrm{I}=\underline{A}\varphi(X_2,\dots,X_{k+1}).$$

If $\sigma(i)=1$ for any $i\geq 2$, then

$$\varphi(X_{\sigma(2)},\dots,X_{\sigma(k+1)})=0.$$

It follows that the nonzero terms in II all satisfy $\sigma(1)=1$ and

$$\begin{aligned}\mathrm{II}&=\frac{1}{k!}\sum_{\substack{\sigma\in S_{k+1}\\ \sigma(1)=1}}\operatorname{sgn}(\sigma)\rho_*\big(\omega(X_1)\big)\varphi\left(X_{\sigma(2)},\dots,X_{\sigma(k+1)}\right)\\ &=\frac{1}{k!}\sum_{\substack{\sigma\in S_{k+1}\\ \sigma(1)=1}}\operatorname{sgn}(\sigma)\rho_*(A)\varphi\left(X_{\sigma(2)},\dots,X_{\sigma(k+1)}\right)\\ &=\rho_*(A)\varphi\left(X_2,\dots,X_{k+1}\right)\qquad\text{(because }\varphi\text{ is alternating).}\end{aligned}$$

Denote by f the function $\varphi(X_2, \dots, X_{k+1})$ on P. For $p \in P$, to calculate $\underline{A}_p f$, choose a curve $c(t)$ in G with initial point $c(0) = e$ and initial vector $c'(0) = A$, for example, $c(t) = \exp(tA)$. Then with $j_p \colon G \to P$ being the map $j_p(g) = p \cdot g$,

$$\begin{aligned}\underline{A}_p f &= j_{p*}(A) f = j_{p*}\big(c'(0)\big) f = j_{p*}\left(c_*\left(\frac{d}{dt}\bigg|_{t=0}\right)\right) f \\ &= (j_p \circ c)_*\left(\frac{d}{dt}\bigg|_{t=0}\right) f = \frac{d}{dt}\bigg|_{t=0} (f \circ j_p \circ c).\end{aligned}$$

By the right-invariance of the horizontal vector fields $X_2, \dots, X_{k+1}$,

$$\begin{aligned}(f \circ j_p \circ c)(t) &= f\big(pc(t)\big) \\ &= \varphi_{pc(t)}\left(X_{2,pc(t)}, \dots, X_{k+1,pc(t)}\right) \\ &= \varphi_{pc(t)}\left(r_{c(t)*}X_{2,p}, \dots, r_{c(t)*}X_{k+1,p}\right) \\ &= r_{c(t)}^* \varphi_{pc(t)}\left(X_{2,p}, \dots, X_{k+1,p}\right) \\ &= \rho(c(t)^{-1})\varphi_p\left(X_{2,p}, \dots, X_{k+1,p}\right) \qquad \text{(right-equivariance of } \varphi) \\ &= \rho(c(t)^{-1}) f(p).\end{aligned}$$

Differentiating this expression with respect to t and using the fact that the differential of the inverse is the negative [21, Problem 8.8(b)], we have

$$\underline{A}_p f = (f \circ j_p \circ c)'(0) = -\rho_*(c'(0)) f(p) = -\rho_*(A) f(p).$$

So the right-hand side of (31.7) is

$$\mathrm{I} + \mathrm{II} = \underline{A} f + \rho_*(A) f = -\rho_*(A) f + \rho_*(A) f = 0. \qquad \square$$

If V is the Lie algebra $\mathfrak{g}$ of a Lie group G and ρ is the adjoint representation of G, then $\omega \cdot \varphi = [\omega, \varphi]$. In this case, for any tensorial k-form $\varphi \in \Omega^k_{\mathrm{Ad}}(P, \mathfrak{g})$,

$$D\varphi = d\varphi + [\omega, \varphi].$$

Although the covariant derivative is defined for any V-valued form on P, Theorem 31.19 is true only for tensorial forms. Since the connection form ω is not tensorial, Theorem 31.19 cannot be applied to ω. In fact, by the definition of the curvature form,

$$\Omega = d\omega + \frac{1}{2}[\omega, \omega].$$

By Theorem 30.4, $\Omega = (d\omega)^h = D\omega$. Combining these two expressions for the curvature, one obtains

$$D\omega = d\omega + \frac{1}{2}[\omega, \omega].$$

The factor of $1/2$ shows that Theorem 31.19 is not true when applied to ω.

Since the curvature form Ω on a principal bundle P is tensorial of type Ad, Theorem 31.19 applies and the second Bianchi identity (Theorem 30.4) may be restated as

$$D\Omega = d\Omega + [\omega, \Omega] = 0. \tag{31.8}$$

Problems

Unless otherwise specified, in the following problems G is a Lie group with Lie algebra $\mathfrak{g}$, $\pi\colon P \to M$ a principal G-bundle, $\rho\colon G \to \mathrm{GL}(V)$ a finite-dimensional representation of G, and $E = P \times_\rho V$ the associated bundle.

31.1. Transition functions of an associated bundle
Show that if $\{(U_\alpha, \phi_\alpha)\}$ is a trivialization for P with transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$, then there is a trivialization $\{(U_\alpha, \psi_\alpha)\}$ for E with transition functions $\rho \circ g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}(V)$.

31.2. Vector bundle structure on an associated bundle
Show that the operations (31.2) on $E = P \times_\rho V$ are well defined and make the associated bundle $\beta\colon E \to M$ into a vector bundle.

31.3. Associated bundle of a frame bundle
Let $E \to M$ be a vector bundle of rank r and $\mathrm{Fr}(E) \to M$ its frame bundle. Show that the vector bundle associated to $\mathrm{Fr}(E)$ via the identity representation $\rho\colon \mathrm{GL}(r,\mathbb{R}) \to \mathrm{GL}(r,\mathbb{R})$ is isomorphic to E.

31.4. Tensorial forms
Prove that if $\psi \in \Omega^k(M, P \times_\rho V)$, then $\psi^\sharp \in \Omega^k(P,V)$ is right-equivariant with respect to ρ.

31.5. Tensorial forms
For $\varphi \in \Omega^k_\rho(P,V)$, prove that $\varphi^{\flat\sharp} = \varphi$.

§32 Characteristic Classes of Principal Bundles

To a real vector bundle $E \to M$ of rank r, one can associate its frame bundle $\mathrm{Fr}(E) \to M$, a principal $\mathrm{GL}(r,\mathbb{R})$-bundle. Similarly, to a complex vector bundle of rank r, one can associate its frame bundle, a principal $\mathrm{GL}(r,\mathbb{C})$-bundle and to an oriented real vector bundle of rank r, one can associate its oriented frame bundle, a principal $\mathrm{GL}^+(r,\mathbb{R})$-bundle, where $\mathrm{GL}^+(r,\mathbb{R})$ is the group of all $r \times r$ matrices of positive determinant. The Pontrjagin classes of a real vector bundle, the Chern classes of a complex vector bundle, and the Euler class of an oriented real vector bundle may be viewed as characteristic classes of the associated principal G-bundle for $G = \mathrm{GL}(r,\mathbb{R}), \mathrm{GL}(r,\mathbb{C})$, and $\mathrm{GL}^+(r,\mathbb{R})$, respectively.

In this section we will generalize the construction of characteristic classes to principal G-bundles for any Lie group G. These are some of the most important diffeomorphism invariants of a principal bundle.

32.1 Invariant Polynomials on a Lie Algebra

Let V be a vector space of dimension n and $V^\vee$ its dual space. An element of $\mathrm{Sym}^k(V^\vee)$ is called a ***polynomial*** of degree k on V. Relative to a basis $e_1,\dots,e_n$ for V and corresponding dual basis $\alpha^1,\dots,\alpha^n$ for $V^\vee$, a function $f\colon V \to \mathbb{R}$ is a polynomial of degree k if and only if it is expressible as a sum of monomials of degree k in $\alpha^1,\dots,\alpha^n$:

$$f = \sum a_I \alpha^{i_1} \cdots \alpha^{i_k}. \tag{32.1}$$

For example, if $V = \mathbb{R}^{n\times n}$ is the vector space of all $n \times n$ matrices, then $\mathrm{tr}\, X$ is a polynomial of degree 1 on V and $\det X$ is a polynomial of degree n on V.

Suppose now that $\mathfrak{g}$ is the Lie algebra of a Lie group G. A polynomial $f\colon \mathfrak{g} \to \mathbb{R}$ is said to be $\mathrm{Ad}(G)$-***invariant*** if for all $g \in G$ and $X \in \mathfrak{g}$,

$$f\big((\mathrm{Ad}\, g)X\big) = f(X).$$

For example, if G is the general linear group $\mathrm{GL}(n,\mathbb{R})$, then $(\mathrm{Ad}\, g)X = gXg^{-1}$ and $\mathrm{tr}\, X$ and $\det X$ are $\mathrm{Ad}\, G$-invariant polynomials on the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$.

32.2 The Chern–Weil Homomorphism

Let G be a Lie group with Lie algebra $\mathfrak{g}$, $P \to M$ a principal G-bundle, ω an Ehresmann connection on P, and Ω the curvature form of ω. Fix a basis $e_1,\dots,e_n$ for $\mathfrak{g}$ and dual basis $\alpha^1,\dots,\alpha^n$ for $\mathfrak{g}^\vee$. Then the curvature form Ω is a linear combination

$$\Omega = \sum \Omega^i e_i,$$

where the coefficients Ω^i are real-valued 2-forms on P. If $f\colon \mathfrak{g} \to \mathbb{R}$ is the polynomial $\sum a_I \alpha^{i_1} \cdots \alpha^{i_k}$, we define $f(\Omega)$ to be the $2k$-form

$$f(\Omega) = \sum a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}$$

on P. Although defined in terms of a basis for $\mathfrak{g}$, the $2k$-form $f(\Omega)$ is independent of the choice of a basis (Problem 32.2).

Recall that the ***covariant derivative*** $D\varphi$ of a k-form φ on a principal bundle P is given by

$$(D\varphi)_p(v_1, \dots, v_k) = (d\varphi)_p(hv_1, \dots, hv_k),$$

where $v_i \in T_pP$ and hv_i is the horizontal component of v_i.

Lemma 32.1. *Let $\pi\colon P \to M$ be a principal bundle. If φ is a basic form on P, then $d\varphi = D\varphi$.*

Proof. A tangent vector $X_p \in T_pP$ decomposes into the sum of its vertical and horizontal components:

$$X_p = vX_p + hX_p.$$

Here $h\colon T_pP \to T_pP$ is the map that takes a tangent vector to its horizontal component. Since $\pi_* X_p = \pi_* hX_p$ for all $X_p \in T_pP$, we have

$$\pi_* = \pi_* \circ h.$$

Suppose $\varphi = \pi^* \tau$ for $\tau \in \Omega^k(M)$. Then

$$\begin{aligned}
D\varphi &= (d\varphi) \circ h && \text{(definition of } D\text{)} \\
&= (d\pi^*\tau) \circ h && (\varphi \text{ is basic}) \\
&= (\pi^* d\tau) \circ h && \text{([21, Prop. 19.5])} \\
&= d\tau \circ \pi_* \circ h && \text{(definition of } \pi^*\text{)} \\
&= d\tau \circ \pi_* && (\pi_* \circ h = \pi_*) \\
&= \pi^* d\tau && \text{(definition of } \pi^*\text{)} \\
&= d\pi^*\tau && \text{([21, Prop. 19.5])} \\
&= d\varphi && (\varphi = \pi^*\tau).
\end{aligned}$$

□

The Chern–Weil homomorphism is based on the following theorem. As before, G is a Lie group with Lie algebra $\mathfrak{g}$.

Theorem 32.2. *Let Ω be the curvature of a connection ω on a principal G-bundle $\pi\colon P \to M$, and f an $\mathrm{Ad}(G)$-invariant polynomial of degree k on $\mathfrak{g}$. Then*

(i) *$f(\Omega)$ is a basic form on P, i.e., there exists a $2k$-form Λ on M such that $f(\Omega) = \pi^*\Lambda$.*
(ii) *Λ is a closed form.*
(iii) *The cohomology class $[\Lambda]$ is independent of the connection.*

Proof. (i) Since the curvature Ω is horizontal, so are its components Ω^i and therefore so is $f(\Omega) = \sum a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}$.

To check the G-invariance of $f(\Omega)$, let $g \in G$. Then

$$\begin{aligned} r_g^*\big(f(\Omega)\big) &= r_g^*\big(\sum a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}\big) \\ &= \sum a_I r_g^*(\Omega^{i_1}) \wedge \cdots \wedge r_g^*(\Omega^{i_k}). \end{aligned}$$

Since the curvature form Ω is right-equivariant,

$$r_g^*\Omega = (\operatorname{Ad} g^{-1})\Omega$$

or

$$r_g^*(\sum \Omega^i e_i) = \sum \big((\operatorname{Ad} g^{-1})\Omega\big)^i e_i,$$

so that

$$r_g^*(\Omega^i) = \big((\operatorname{Ad} g^{-1})\Omega\big)^i.$$

Thus,

$$\begin{aligned} r_g^*\big(f(\Omega)\big) &= \sum a_I \big((\operatorname{Ad} g^{-1})\Omega\big)^{i_1} \wedge \cdots \wedge \big((\operatorname{Ad} g^{-1})\Omega\big)^{i_k} \\ &= f\big((\operatorname{Ad} g^{-1})\Omega\big) \\ &= f(\Omega) \qquad \text{(by the } \operatorname{Ad} G\text{-invariance of } f). \end{aligned}$$

Since $f(\Omega)$ is horizontal and G-invariant, by Theorem 31.12, it is basic.

(ii) Since $\pi_*: T_pP \to T_{\pi(p)}M$ is surjective, $\pi^*: \Omega^*(M) \to \Omega^*(P)$ is injective. Therefore, to show that $d\Lambda = 0$, it suffices to show that

$$\pi^* d\Lambda = d\pi^*\Lambda = df(\Omega) = 0.$$

If $f = \sum a_I \alpha^{i_1} \cdots \alpha^{i_k}$, then

$$f(\Omega) = \sum a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}.$$

In this expression, each a_I is a constant and therefore by Lemma 32.1

$$Da_I = da_I = 0.$$

By the second Bianchi identity (31.8), $D\Omega = 0$. Therefore, $D\Omega^i = 0$ for each i. Since the Ω^i are right-equivariant of type Ad and horizontal, they are tensorial forms. By Lemma 32.1 and because D is an antiderivation on tensorial forms (Proposition 31.16)

$$\begin{aligned} d\big(f(\Omega)\big) &= D\big(f(\Omega)\big) = D\big(\sum a_I \Omega^{i_1} \wedge \cdots \wedge \Omega^{i_k}\big) \\ &= \sum_I \sum_j a_I \Omega^{i_1} \wedge \cdots \wedge D\Omega^{i_j} \wedge \cdots \wedge \Omega^{i_{2k}} \\ &= 0. \end{aligned}$$

(iii) Let I be an open interval containing the closed interval $[0,1]$. Then $P \times I$ is a principal G-bundle over $M \times I$. Denote by ρ the projection $P \times I \to P$ to the first factor. If ω_0 and ω_1 are two connections on P, then

$$\tilde{\omega} = (1-t)\rho^*\omega_0 + t\rho^*\omega_1 \tag{32.2}$$

is a connection on $P \times I$ (Check the details). Moreover, if $i_t\colon P \to P \times I$ is the inclusion $p \mapsto (p,t)$, then $i_0^*\tilde{\omega} = \omega_0$ and $i_1^*\tilde{\omega} = \omega_1$.

Let

$$\tilde{\Omega} = d\tilde{\omega} + \frac{1}{2}[\tilde{\omega}, \tilde{\omega}]$$

be the curvature of the connection $\tilde{\omega}$. It pulls back under i_0 to

$$\begin{aligned} i_0^*\tilde{\Omega} &= d i_0^*\tilde{\omega} + \frac{1}{2} i_0^*[\tilde{\omega}, \tilde{\omega}] \\ &= d\omega_0 + \frac{1}{2}[i_0^*\tilde{\omega}, i_0^*\tilde{\omega}] \\ &= d\omega_0 + \frac{1}{2}[\omega_0, \omega_0] \\ &= \Omega_0, \end{aligned}$$

the curvature of the connection ω_0. Similarly, $i_1^*\tilde{\Omega} = \Omega_1$, the curvature of the connection ω_1.

For any $\mathrm{Ad}(G)$-invariant polynomial

$$f = \sum a_I \alpha^{i_1} \cdots \alpha^{i_k}$$

of degree k on $\mathfrak{g}$,

$$\begin{aligned} i_0^* f(\tilde{\Omega}) &= i_0^* \sum a_I \tilde{\Omega}^{i_1} \wedge \cdots \wedge \tilde{\Omega}^{i_k} \\ &= \sum a_I \Omega_0^{i_1} \wedge \cdots \wedge \Omega_0^{i_k} \\ &= f(\Omega_0) \end{aligned}$$

and

$$i_1^* f(\tilde{\Omega}) = f(\Omega_1).$$

Note that i_0 and $i_1\colon P \to P \times I$ are homotopic through the homotopy i_t. By the homotopy axiom of de Rham cohomology, the cohomology classes $[i_0^* f(\tilde{\Omega})]$ and $[i_1^* f(\tilde{\Omega})]$ are equal. Thus, $[f(\Omega_0)] = [f(\Omega_1)]$, or

$$\pi^*[\Lambda_0] = \pi^*[\Lambda_1].$$

By the injectivity of π^*, $[\Lambda_0] = [\Lambda_1]$, so the cohomology class of Λ is independent of the connection. □

Let $\pi\colon P \to M$ be a principal G-bundle with curvature form Ω. To every $\mathrm{Ad}(G)$-invariant polynomial on $\mathfrak{g}$, one can associate the cohomology class $[\Lambda] \in H^*(M)$ such that $f(\Omega) = \pi^*\Lambda$. The cohomology class $[\Lambda]$ is called the ***characteristic class*** of P associated to f. Denote by $\mathrm{Inv}(\mathfrak{g})$ the algebra of all $\mathrm{Ad}(G)$-invariant polynomials on $\mathfrak{g}$. The map

$$\begin{aligned} w\colon \mathrm{Inv}(\mathfrak{g}) &\to H^*(M) \\ f &\mapsto [\Lambda], \text{ where } f(\Omega) = \pi^*\Lambda, \end{aligned} \tag{32.3}$$

that maps each $\mathrm{Ad}(G)$-invariant polynomial to its characteristic class is called the ***Chern–Weil homomorphism***.

Example 32.3. If the Lie group G is $\mathrm{GL}(r,\mathbb{C})$, then by Theorem B.10 the ring of $\mathrm{Ad}(G)$-invariant polynomials on $\mathfrak{gl}(r,\mathbb{C})$ is generated by the coefficients $f_k(X)$ of the characteristic polynomial

$$\det(\lambda I + X) = \sum_{k=0}^{r} f_k(X)\lambda^{r-k}.$$

The characteristic classes associated to $f_1(X),\dots,f_k(X)$ are the ***Chern classes*** of a principal $\mathrm{GL}(r,\mathbb{C})$-bundle. These Chern classes generalize the Chern classes of the frame bundle $\mathrm{Fr}(E)$ of a complex vector bundle E of rank r.

Example 32.4. If the Lie group G is $\mathrm{GL}(r,\mathbb{R})$, then by Theorem B.13 the ring of $\mathrm{Ad}(G)$-invariant polynomials on $\mathfrak{gl}(r,\mathbb{R})$ is also generated by the coefficients $f_k(X)$ of the characteristic polynomial

$$\det(\lambda I + X) = \sum_{k=0}^{r} f_k(X)\lambda^{r-k}.$$

The characteristic classes associated to $f_1(X),\dots,f_k(X)$ generalize the Pontrjagin classes of the frame bundle $\mathrm{Fr}(E)$ of a real vector bundle E of rank r. (For a real frame bundle the coefficients $f_k(\Omega)$ vanish for k odd.)

Problems

32.1. Polynomials on a vector space

Let V be a vector space with bases $e_1,\dots,e_n$ and $u_1,\dots,u_n$. Prove that if a function $f\colon V \to \mathbb{R}$ is a polynomial of degree k with respect to the basis $e_1,\dots,e_n$, then it is a polynomial of degree k with respect to the basis $u_1,\dots,u_n$. Thus, the notion of a polynomial of degree k on a vector space V is independent of the choice of a basis.

32.2. Chern–Weil forms

In this problem we keep the notations of this section. Let $e_1,\dots,e_n$ and $u_1,\dots u_n$ be two bases for the Lie algebra $\mathfrak{g}$ with dual bases $\alpha^1,\dots,\alpha^n$ and $\beta^1,\dots,\beta^n$, respectively. Suppose

$$\Omega = \sum \Omega^i e_i = \sum \Psi^j u_j$$

and

$$f = \sum a_I \alpha^{i_1}\cdots\alpha^{i_k} = \sum b_I \beta^{i_1}\cdots\beta^{i_k}.$$

Prove that

$$\sum a_I \Omega^{i_1}\wedge\cdots\wedge\Omega^{i_k} = \sum b_I \Psi^{i_1}\wedge\cdots\wedge\Psi^{i_k}.$$

This shows that the definition of $f(\Omega)$ is independent of the choice of basis for $\mathfrak{g}$.

32.3. Connection on $P\times I$

Show that the 1-form $\tilde{\omega}$ in (32.2) is a connection on $P\times I$.

32.4. Chern–Weil homomorphism

Show that the map $w\colon \mathrm{Inv}(\mathfrak{g}) \to H^*(M)$ in (32.3) is an algebra homomorphism.