Knowledge and Consequence in AC Semantics for General Rough Sets

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Abstract Antichain based semantics for general rough sets was introduced recently by the present author. In her paper two different semantics, one for general rough sets and another for general approximation spaces over quasi-equivalence relations, were developed. These semantics are improved and studied further from a lateral algebraic logic and an algebraic logic perspective in this research. The framework of granular operator spaces is also generalized. The main results concern the structure of the algebras, deductive systems and the algebraic logic approach. The epistemological aspects of the semantics is also studied in this chapter in some depth and revolve around nature of knowledge representation, Peircean triadic semiotics and temporal aspects of parthood. Examples have been constructed to illustrate various aspects of the theory and applications to human reasoning contexts that fall beyond information systems.

1 Introduction

It is well known that sets of rough objects (in various senses) are quasi or partially orderable. Specifically in classical or Pawlak rough sets [31], the set of roughly equivalent sets has a Quasi Boolean order on it while the set of rough and crisp objects is Boolean ordered. In the classical semantic domain or classical meta level, associated with general rough sets, the set of crisp and rough objects is quasi or partially orderable. Under minimal assumptions on the nature of these objects, many orders with rough ontology can be associated—these necessarily have to do with concepts of discernibility. Concepts of rough objects, in these contexts, depend additionally on approximation operators and granulations used. These were part of the motivations of the development of the concept of granular operator spaces by the present author in [26].

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In quasi or partially ordered sets, sets of mutually incomparable elements are called *antichains* (for basics see [9, 11, 15]). The possibility of using antichains of rough objects for a possible semantics was mentioned in [23, 24, 27] by the present author and developed in [26]. The semantics is applicable for a large class of operator based rough sets including specific cases of RYS [21], other general approaches like [7, 12, 13] and all specific cases of relation based and cover based rough set approaches. In [7], negation like operators are assumed in general and these are not definable operations relative the order related operations/relation. A key problem in many of the latter types of approaches is of closure of possible aggregation and commonality operations [14, 27, 44, 45].

In the present paper, the semantics of [26] is improved and developed further from an algebraic logic point of view (based on defining ternary deduction terms), the concept of knowledge in the settings is also explored in some depth and related interpretations are offered. The basic framework of granular operator spaces used in [26] is generalized in this paper as most of the mathematical parts carry over. The semantics of [26], as improved in the present paper by way of ternary terms, is very general, open ended, extendable and optimal for lateral studies. Most of it applies to *general granular operator spaces*, introduced in a separate paper by the present author. In the same framework, the machinery for isolation of deductive systems is developed and studied from a purely algebraic logic point of view. New results on representation of roughness related objects are also developed. Last but not least, the concept of knowledge as considered in [21, 22, 27, 32] is recast in very different terms for describing the knowledge associated with representation of data by maximal antichains. These representations are also examined for compatibility with triadic semiotics (that is not necessarily faithful to Peirce's ideas) for integration with ontology. Philosophical questions relating to perdurantism and endurantism are also solved in some directions. Illustrative examples that demonstrate applicability to *human reasoning contexts involving approximations but no reasonable data tables* have also been constructed in this chapter.

In the next subsection, relevant background is presented. Granular operator spaces are generalized and nature of parthood is explained in the next section. In the third section, the essential algebraic logic approach used is outlined. In the next section, possible operations on sets of maximal antichains derived from granular operator spaces are considered, AC-algebras are defined and their generation is studied. Representation of antichains derived from the context are also improved and earlier examples are refined. The algebras of quasi-equivalential rough sets formed by related procedures is presented to illustrate key aspects of the semantics in the fifth and sixth sections. Ternary deduction terms in the context of the AC-algebra are explored next and various results are proved. The connections with epistemology and knowledge forms the following section. Further directions are provided in Sect. 9.

Background

Let K be any set and l, u be lower and upper approximation operators on $\mathcal{K} \subseteq \wp(K)$ that satisfy monotonicity and $(\forall a \subseteq K)\, a \subseteq a^u$. An element $a \in \mathcal{K}$ will be said to be *lower definite* (resp. *upper definite*) if and only if $a^l = a$ (resp. $a^u = a$) and *definite*, when it is both lower and upper definite. In this chapter, the operators will be on $\wp(K)$ and not on a proper subset thereof. For possible concepts of rough objects [21, 26] may be consulted. Finiteness of K and granular operator spaces, defined below, will be assumed (though not always essential) unless indicated otherwise.

Let K be any set and l, u be lower and upper approximation operators on $\mathcal{K} \subseteq \wp(K)$ that satisfy monotonicity and $(\forall a \subseteq K)\, a \subseteq a^u$. An element $a \in \mathcal{K}$ will be said to be *lower definite* (resp. *upper definite*) if and only if $a^l = a$ (resp. $a^u = a$) and *definite*, when it is both lower and upper definite. The following are some of the the possible concepts of rough objects considered in the literature, (and all considerations will be restricted as indicated in the next definition):

- A non definite subset of K, that is x is a rough object if and only if $x^l \neq x^u$.
- Any pair of definite subsets of the form (a, b) satisfying $a \subseteq b$.
- Any pair of subsets of the form (x^l, x^u).
- Sets in an interval of the form (x^l, x^u).
- Sets in an interval of the form (a, b) satisfying $a \subseteq b$ and a, b being definite subsets.
- A non-definite element in a RYS, that is an x satisfying $\neg \mathbf{P} x^u x^l$ (x may be a subset and both upper and lower case letters may be used for them).
- An interval of the form, (a, b) satisfying $a \subseteq B$ and a, b being definite subsets.

Set framework with operators will be used as all considerations will require quasi orders in an essential way. The evolution of the operators need not be induced by a cover or a relation (corresponding to cover or relation based systems respectively), but these would be special cases. The generalization to some rough Y-systems RYS (see [21] for definitions), will of course be possible as a result.

Definition 1 A *Granular Operator Space* [26] S will be a structure of the form $S = \langle \underline{S}, \mathcal{G}, l, u \rangle$ with $\underline{S}$ being a set, $\mathcal{G}$ an *admissible granulation* (defined below) over S and l, u being operators $: \wp(\underline{S}) \longmapsto \wp(\underline{S})$ satisfying the following ($\underline{S}$ will be replaced with S if clear from the context. Lower and upper case alphabets will both be used for subsets):

$$a^l \subseteq a \,\&\, a^{ll} = a^l \,\&\, a^u \subset a^{uu}$$
$$(a \subseteq b \longrightarrow a^l \subseteq b^l \,\&\, a^u \subseteq b^u)$$
$$\emptyset^l = \emptyset \,\&\, \emptyset^u = \emptyset \,\&\, \underline{S}^l \subseteq S \,\&\, \underline{S}^u \subseteq S.$$

Here, Admissible granulations are granulations $\mathcal{G}$ that satisfy the following three conditions (Relative RYS [21], $\mathbf{P} = \subseteq$, $\mathbb{P} = \subset$) and t is a term operation formed from set operations):

$$(\forall a \exists b_1, \ldots b_r \in \mathcal{G})\, t(b_1, b_2, \ldots b_r) = a^l$$
$$\text{and } (\forall a)\, (\exists b_1, \ldots b_r \in \mathcal{G})\, t(b_1, b_2, \ldots b_r) = a^u, \quad \text{(Weak RA, WRA)}$$
$$(\forall b \in \mathcal{G})(\forall a \in \wp(\underline{S}))\, (b \subseteq a \longrightarrow b \subseteq (a^l)), \text{ (Lower Stability, LS)}$$
$$(\forall a, b \in \mathcal{G})(\exists z \in \wp(\underline{S}))\, a \subset z, b \subset z \,\&\, z^l = z^u = z, \quad \text{(Full Underlap, FU)}$$

In the present context, these conditions mean that every approximation is somehow representable by granules, that granules are lower definite, and that all pairs of distinct granules are contained in definite objects.

On $\wp(\underline{S})$, the relation $\sqsubset$ is defined by

$$A \sqsubset B \text{ if and only if } A^l \subseteq B^l \,\&\, A^u \subseteq B^u. \tag{1}$$

The rough equality relation on $\wp(\underline{S})$ is defined via $A \approx B$ if and only if $A \sqsubset B$ & $B \sqsubset A$.

Regarding the quotient $\underline{S}| \approx$ as a subset of $\wp(\underline{S})$, the order $\Subset$ will be defined as per

$$\alpha \Subset \beta \text{ if and only if } \alpha^l \subseteq \beta^l \,\&\, \alpha^u \subseteq \beta^u. \tag{2}$$

Here α^l is being interpreted as the lower approximation of α and so on. $\Subset$ will be referred to as the *basic rough order*.

Definition 2 By a *roughly consistent object* will be meant a set of subsets of $\underline{S}$ of the form $H = \{A; (\forall B \in H)\, A^l = B^l, A^u = B^u\}$. The set of all roughly consistent objects is partially ordered by the inclusion relation. Relative this maximal roughly consistent objects will be referred to as *rough objects*. By *definite rough objects*, will be meant rough objects of the form H that satisfy

$$(\forall A \in H)\, A^{ll} = A^l \,\&\, A^{uu} = A^u. \tag{3}$$

Proposition 1 *$\Subset$ is a bounded partial order on $\underline{S}| \approx$.*

Proof Reflexivity is obvious. If $\alpha \Subset \beta$ and $\beta \Subset \alpha$, then it follows that $\alpha^l = \beta^l$ and $\alpha^u = \beta^u$ and so antisymmetry holds.

If $\alpha \Subset \beta$, $\beta \Subset \gamma$, then the transitivity of set inclusion induces transitivity of $\Subset$. The poset is bounded by $0 = (\emptyset, \emptyset)$ and $1 = (S^l, S^u)$. Note that 1 need not coincide with (S, S). □

Theorem 1 *Some known results relating to antichains and lattices are the following:*

1. *Let X be a partially ordered set with longest chains of length r, then X can be partitioned into k number of antichains implies $r \leq k$.*
2. *If X is a finite poset with k elements in its largest antichain, then a chain decomposition of X must contain at least k chains.*
3. *The poset $AC_m(X)$ of all maximum sized antichains of a poset X is a distributive lattice.*
4. *For every finite distributive lattice L and every chain decomposition C of J_L (the set of join irreducible elements of L), there is a poset X_C such that $L \cong AC_m(X_C)$.*

Proof Proofs of the first three of the assertions can be found in in [9, 17] for example. Many proofs of results related to Dilworth's theorems are known in the literature and some discussion can be found in [17] (pp. 126–135).

1. To prove the first, start from a chain decomposition and recursively extract the minimal elements from it to form r number of antichains.
2. This is proved by induction on the size of X across many possibilities.
3. See [9, 17] for details.
4. In [15], the last connection between chain decompositions and representation by antichains reveals important gaps—there are other posets X that satisfy $L \cong AC_m(X)$. Further the restriction to posets is too strong and can be relaxed in many ways [39].

□

If R is a binary relation on a set $\underline{S}$, then the neighborhood generated by an $x \in \underline{S}$ will be

$$[x] = \{y : Ryx\}$$

A binary relation R on a set $\underline{S}$ is said to be a *Quasi-Equivalence* if and only if it satisfies:

$$(\forall x, y)\,([x] = [y] \leftrightarrow Rxy \,\&\, Ryx).$$

It is useful in algebras when it behaves as a good factor relation [2]. But the condition is of interest in rough sets by itself. *Note that Rxy* is a compact form of $(x, y) \in R$.

2 General Granular Operator Spaces (GSP)

Definition 3 A *General Granular Operator Space* (GSP) S shall be a structure of the form $S = \langle \underline{S}, \mathcal{G}, l, u, \mathbf{P} \rangle$ with $\underline{S}$ being a set, $\mathcal{G}$ an *admissible granulation* (defined below) over S, l, u being operators $: \wp(\underline{S}) \longmapsto \wp(\underline{S})$ and $\mathbf{P}$ being a definable binary generalized transitive predicate (for parthood) on $\wp(\underline{S})$ satisfying the following: ($\underline{S}$ will be replaced with S if clear from the context. Lower and upper case alphabets will both be used for subsets):

$$a^l \subseteq a \,\&\, a^{ll} = a^l \,\&\, a^u \subset a^{uu}$$
$$(a \subseteq b \longrightarrow a^l \subseteq b^l \,\&\, a^u \subseteq b^u)$$
$$\emptyset^l = \emptyset \,\&\, \emptyset^u = \emptyset \,\&\, \underline{S}^l \subseteq S \,\&\, \underline{S}^u \subseteq S.$$

Here, the generalized transitivity can be any proper nontrivial generalization of parthood (see [23]) and Admissible granulations are granulations $\mathcal{G}$ that satisfy the following three conditions (In the granular operator space of [26], $\mathbf{P} = \subseteq$, $\mathbb{P} = \subset$ only in that definition), $\mathbb{P}$ is proper parthood (defined via $\mathbb{P}ab$ iff $\mathbf{P}ab \,\&\, \neg \mathbf{P}ba$) and t is a term operation formed from set operations:

$$(\forall x \exists y_1, \ldots y_r \in \mathcal{G})\, t(y_1, y_2, \ldots y_r) = x^l$$
$$\text{and } (\forall x)\, (\exists y_1, \ldots y_r \in \mathcal{G})\, t(y_1, y_2, \ldots y_r) = x^u, \quad \text{(Weak RA, WRA)}$$
$$(\forall y \in \mathcal{G})(\forall x \in \wp(\underline{S}))\, (\mathbf{P}yx \longrightarrow \mathbf{P}yx^l), \text{ (Lower Stability, LS)}$$
$$(\forall x, y \in \mathcal{G})(\exists z \in \wp(\underline{S}))\, \mathbb{P}xz, \,\&\, \mathbb{P}yz \,\&\, z^l = z^u = z, \quad \text{(Full Underlap, FU)}$$

On $\wp(\underline{S})$, if the parthood relation $\mathbf{P}$ is defined via a formula $\boldsymbol{\Phi}$ as per

$$\mathbf{P}ab \text{ if and only if } \boldsymbol{\Phi}(a, b), \tag{4}$$

then the $\boldsymbol{\Phi}$-rough equality would be defined via

$$a \approx_{\boldsymbol{\Phi}} b \text{ if and only if } \mathbf{P}ab \,\&\, \mathbf{P}ba. \tag{5}$$

In a granular operator space, $\mathbf{P}$ is the same as $\sqsubset$ and is defined by

$$a \sqsubset b \text{ if and only if } a^l \subseteq b^l \,\&\, a^u \subseteq b^u. \tag{6}$$

The rough equality relation on $\wp(\underline{S})$ is defined via $a \approx b$ if and only if $a \sqsubset b \,\&\, b \sqsubset a$. Regarding the quotient $\underline{S}| \approx$ as a subset of $\wp(\underline{S})$, the order $\Subset$ will be defined as per

$$\alpha \Subset \beta \text{ if and only if } \boldsymbol{\Phi}(\alpha, \beta) \tag{7}$$

Here $\boldsymbol{\Phi}(\alpha, \beta)$ is an abbreviation for $(\forall a \in \alpha, b \in \beta)\boldsymbol{\Phi}(a, b)$. $\Subset$ will be referred to as the *basic rough order*.

Definition 4 By a *roughly consistent object* will be meant a set of subsets of $\underline{S}$ of the form $H = \{A; (\forall B \in H)\, A^l = B^l, A^u = B^u\}$. The set of all roughly consistent objects is partially ordered by the inclusion relation. Relative this maximal roughly consistent objects will be referred to as *rough objects*. By *definite rough objects*, will be meant rough objects of the form H that satisfy

$$(\forall A \in H)\, A^{ll} = A^l \,\&\, A^{uu} = A^u. \tag{8}$$

Other concepts of rough objects will also be used in this chapter.

Proposition 2 *When S is a granular operator space, $\Subset$ is a bounded partial order on $\underline{S}| \approx$. More generally it is a bounded quasi order.*

Proof The proof of the first part is in [26], the second part is provable analogously.

2.1 Parthood and Frameworks

Many of the philosophical issues relating to mereology take more specific forms in the context of rough sets in general and in the GSP framework. The axioms of parthood that can be seen to be not universally satisfied in all rough contexts include the following:

$$\mathbf{P}ab \,\&\, \mathbf{P}bc \longrightarrow \mathbf{P}ac \qquad \text{(Transitivity)}$$
$$(\mathbf{P}\,ab \leftrightarrow \mathbf{P}ba) \longrightarrow a = b \qquad \text{(Extensionality)}$$
$$(\mathbf{P}ab \,\&\, \mathbf{P}ba \longrightarrow a = b) \qquad \text{(Antisymmetry)}$$

This affords many distinct concepts of *proper parthoods* $\mathbb{P}$:

$$\mathbb{P}ab \text{ if and only if } \mathbf{P}ab \,\&\, a \neq b \qquad \text{(PP1)}$$
$$\mathbb{P}ab \text{ if and only if } \mathbf{P}ab \,\&\, \neg\mathbf{P}ba \qquad \text{(PP2)}$$
$$\mathbb{P}ab \longrightarrow (\exists z)\mathbf{P}zb \,\&\, (\forall w)\neg(\mathbf{P}wa \,\&\, \mathbf{P}wz) \qquad \text{(WS)}$$

PP1 does not follow from PP2 without antisymmetry and WS (weak supplementation) is a kind of proper parthood. All this affords a mereological approach with much variation to abstract rough sets.

3 Deductive Systems

In this section, key aspects of the approach to ternary deductive systems in [4, 5] are presented. These are intended as natural generalizations of the concepts of ideals and filters and classes of congruences that can serve as subsets or subalgebras closed under consequence operations or relations (also see [10]).

Definition 5 Let $\mathbb{S} = \langle S, \Sigma\rangle$ be an algebra, then the set of term functions over it will be denoted by $\mathbf{T}^{\Sigma}(\mathbb{S})$ and the set of r-ary term functions by $\mathbf{T}^{\Sigma}_{r}(\mathbb{S})$. Further let

$$g \in \mathbf{T}^{\Sigma}_{1}(\mathbb{S}),\ z \in S,\ \tau \subset \mathbf{T}^{\Sigma}_{3}(\mathbb{S}), \tag{0}$$
$$g(z) \in \Delta \subset S, \tag{1}$$
$$(\forall t \in \tau)(a \in \Delta \,\&\, t(a, b, z) \in \Delta \longrightarrow b \in \Delta), \tag{2}$$
$$(\forall t \in \tau)(b \in \Delta \longrightarrow t(g(z), b, z) \in \Delta), \tag{3}$$

then Δ is a $(g, z)-$ τ-*deductive system* of $\mathbb{S}$. If further for each k-ary operation $f \in \Sigma$ and ternary $p \in \tau$

$$(\forall a_i, b_i \in S)(\&_{i=1}^{k} p(a_i, b_i, z) \in \Delta \longrightarrow p(f(a_1, \ldots, a_k), f(b_1, \ldots, b_k), x) \in \Delta), \tag{9}$$

then Δ is said to be compatible.

τ is said to be a *g-difference system* for $\mathbb{S}$ if τ is finite and the condition

$$(\forall t \in \tau)t(a, b, c) = g(c) \text{ if and only if } a = b \text{ holds.} \tag{10}$$

A variety $\mathcal{V}$ of algebras is regular with respect to a unary term g if and only if for each $S \in \mathcal{V}$,

$$(\forall b \in S)(\forall \sigma, \rho \in con(S))([g(b)]_\sigma = [g(b)]_\rho \longrightarrow \sigma = \rho). \tag{11}$$

It should be noted that in the above τ is usually taken to be a finite subset and a variety has a g-difference system if and only if it is regular with respect to g.

Proposition 3 *In the above definition, it is provable that*

$$(\forall t \in \tau)(t(g(z), b, z) \in \Delta \longrightarrow b \in \Delta). \tag{12}$$

Definition 6 In the context of Definition 5, $\Theta_{Delta,z}$ shall be a relation induced on S by τ as per the following

$$(a, b) \in \Theta_{\Delta,z} \text{ if and only if } (\forall t \in \tau)\, t(a, b, z) \in \Delta. \tag{13}$$

Proposition 4 *In the context of Definition 6,* $\Delta = [g(z)]_{\Theta_{\Delta,z}}$.

Proposition 5 *Let* $\tau \subset T_3^{\Sigma}(\mathbb{S})$ *with the algebra* $\mathbb{S} = \langle S, \Sigma \rangle$, $v \in T_1^{\Sigma}(\mathbb{S})$, $e \in S$, $K \subseteq S$ *and let* $\Theta_{K,e}$ *be induced by* τ. *If* $\Theta_{K,e}$ *is a reflexive and transitive relation such that* $K = [v(e)]_{Theta_{K,e}}$, *then* K *is a* (v, e)- τ*-deductive system of* $\mathbb{S}$.

Theorem 2 *Let h is a unary term of a variety* $\mathcal{V}$ *and* τ *a h-difference system for* $\mathcal{V}$. *If* $\mathbb{S} \in \mathcal{V}m\ \Theta \in Con(\mathbb{S})$, $z \in S$ *and* $\Delta = [h(z)]_\Theta$, *then* $\Theta_{\Delta,z} = \Theta$ *and* Δ *is a compatible* (h, z)-τ*-deductive system of* $\mathbb{S}$.

The converse holds in the following sense:

Theorem 3 *If h is a unary term of a variety* $\mathcal{V}$, τ *is a h-difference system in it,* $\mathbb{S} \in \mathcal{V}$, $z \in S$ *and if* Δ *is a compatible* (h, z)-τ*-deductive system of* $\mathbb{S}$, *then* $\Theta_{\Delta,z} \in Con(\mathbb{S})$ *and* $\Delta = [g(z)]_{\Theta_{\Delta,z}}$.

When $\mathcal{V}$ is regular relative h, then $\mathcal{V}$ has a h-difference system relative τ and for each $\mathbb{S} \in \mathcal{V}$, $z \in S$ and $\Delta \subset S$, $\Delta = [h(z)]$ if and only if Δ is a (h, z)- τ-deductive system of $\mathbb{S}$.

In each case below, $\{t\}$ is a h-difference system ($x \oplus y = ((x \wedge y^*)^* \wedge (x^* \wedge y)^*)^*$):

$h(z) = z \;\&\; t(a, b, c) = a - b + c$	(Variety of Groups)
$h(z) = z \;\&\; t(a, b, c) = a \oplus b \oplus c$	(Variety of Boolean Algebras)
$h(z) = z^{**} \;\&\; t(a, b, c) = (a + b) + c$	(Variety of p-Semilattices)

4 Anti Chains for Representation

In this section, the main algebraic semantics of [26] is summarized, extended to *AC*-algebras and relative properties are studied. It is also proved that the number of maximal antichains required to generate the AC-algebra is rather small.

Definition 7 $\mathbb{A}, \mathbb{B} \in \underline{S}| \approx$, will be said to be *simply independent* (in symbols $\Xi(\mathbb{A}, \mathbb{B})$)if and only if

$$\neg(\mathbb{A} \Subset \mathbb{B}) \text{ and } \neg(\mathbb{B} \Subset \mathbb{A}). \tag{14}$$

A subset $\alpha \subseteq \underline{S}| \approx$ will be said to be *simply independent* if and only if

$$(\forall \mathbb{A}, \mathbb{B} \in \alpha)\, \Xi(\mathbb{A}, \mathbb{B}) \vee (\mathbb{A} = \mathbb{B}). \tag{15}$$

The set of all simply independent subsets shall be denoted by $\mathcal{SY}(S)$.

A *maximal simply independent subset*, shall be a simply independent subset that is not properly contained in any other simply independent subset. The set of maximal simply independent subsets will be denoted by $\mathcal{SY}_m(S)$. On the set $\mathcal{SY}_m(S)$, $\ll$ will be the relation defined by

$$\alpha \ll \beta \text{ if and only if } (\forall \mathbb{A} \in \alpha)(\exists \mathbb{B} \in \beta)\, \mathbb{A} \Subset \mathbb{B}. \tag{16}$$

Theorem 4 *$\langle \mathcal{SY}_m(S), \ll \rangle$ is a distributive lattice.*

Analogous to the above, it is possible to define essentially the same order on the set of maximal antichains of $\underline{S}| \approx$ denoted by $\mathfrak{S}$ with the $\Subset$ order. This order will be denoted by $\lessdot$ - this may also be seen to be induced by maximal ideals.

Theorem 5 *If $\alpha = \{\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_n, \dots\} \in \mathfrak{S}$, and if L is defined by*

$$L(\alpha) = \{\mathbb{B}_1, \mathbb{B}_2, \dots, \mathbb{B}_n, \dots\} \tag{17}$$

with $X \in \mathbb{B}_i$ if and only if $X^l = \mathbb{A}_i^{ll} = \mathbb{B}_i^l$ and $X^u = \mathbb{A}_i^{lu} = \mathbb{B}_i^u$, then L is a partial operation in general.

Proof The operation is partial because $L(\alpha)$ may not always be a maximal antichain. This can happen in general in which the properties $A^{ll} \subset A^l$ and/or $A^{ul} \subset A$ hold for some elements. The former possibility is not possible by our assumptions, but the latter is scenario is permitted.

Specifically this can happen in bitten rough sets when the bitten upper approximation [19] operator is used in conjunction with the lower approximation. But many more examples are known in the literature (see [21]). □

Definition 8 Let $\chi(\alpha \cap \beta) = \{\xi;\ \xi \text{ is a maximal antichain } \&\ \alpha \cap \beta \subseteq \xi\}$ be the set of all possible extensions of $\alpha \cap \beta$. The function $\delta : \mathfrak{S}^2 \longmapsto \mathfrak{S}$ corresponding to *extension under cognitive dissonance* will be defined as per $\delta(\alpha, \beta) \in \chi(\alpha \cap \beta)$ and (LST means *maximal subject to*)

$$\delta(\alpha, \beta) = \begin{cases} \xi, & \text{if } \xi \cap \beta \text{ is a maximum subject to } \xi \neq \beta \text{ and } \xi \text{is unique,} \\ \xi, & \text{if } \xi \cap \beta \,\&\, \xi \cap \alpha \text{ are LST } \xi \neq \beta, \alpha \text{ and } \xi \text{is unique,} \\ \beta, & \text{if } \xi \cap \beta \,\&\, \xi \cap \alpha \text{ are LST } \& \, \xi \neq \beta, \alpha \text{ but } \xi \text{ is not unique,} \\ \beta, & \text{if } \chi(\alpha \cap \beta) = \{\alpha, \beta\}. \end{cases} \tag{18}$$

Definition 9 In the context of the above definition, the function $\varrho : \mathfrak{S}^2 \longmapsto \mathfrak{S}$ corresponding to *radical extension* will be defined as per $\varrho(\alpha, \beta) \in \chi(\alpha \cap \beta)$ and (MST means *minimal subject to*)

$$\varrho(\alpha, \beta) = \begin{cases} \xi, & \text{if } \xi \cap \beta \text{ is a minimum under } \xi \neq \beta \text{ and } \xi \text{ is unique,} \\ \xi, & \text{if } \xi \cap \beta \,\&\, \xi \cap \alpha \text{ are MST } \xi \neq \beta, \alpha \text{ and } \xi \text{is unique,} \\ \alpha, & \text{if } (\exists \xi)\, \xi \cap \beta \,\&\, \xi \cap \alpha \text{ are MST } \xi \neq \beta, \alpha \,\&\, \xi \text{ is not unique,} \\ \alpha, & \text{if } \chi(\alpha \cap \beta) = \{\alpha, \beta\}. \end{cases} \tag{19}$$

Theorem 6 *The operations ϱ, δ satisfy all of the following:*

$$\varrho, \delta \text{ are groupoidal operations,} \tag{1}$$
$$\varrho(\alpha, \alpha) = \alpha \tag{2}$$
$$\delta(\alpha, \alpha) = \alpha \tag{3}$$
$$\delta(\alpha, \beta) \cap \beta \subseteq \delta(\delta(\alpha, \beta), \beta) \cap \beta \tag{4}$$
$$\delta(\delta(\alpha, \beta), \beta) = \delta(\alpha, \beta) \tag{5}$$
$$\varrho(\varrho(\alpha, \beta), \beta) \cap \beta \subseteq \varrho(\alpha, \beta) \cap \beta. \tag{6}$$

Proof 1. Obviously ϱ, δ are closed as the cases in their definition cover all possibilities. So they are groupoid operations. Associativity can be easily shown to fail through counterexamples.
2. Idempotence follows from definition.
3. Idempotence follows from definition.

For the rest, note that by definition, $\alpha \cap \beta \subseteq \delta(\alpha, \beta)$ holds. The intersection with β of $\delta(\alpha, \beta)$ is a subset of $\delta(\delta(\alpha, \beta), \beta) \cap \beta$ by recursion. □

In general, a number of possibilities (potential non-implications) like the following are satisfied by the algebra: $\alpha \lessdot \beta \,\&\, \alpha \lessdot \gamma \nrightarrow \alpha \lessdot \delta(\beta, \gamma)$. Given better properties of l and u, interesting operators can be induced on maximal antichains towards improving the properties of ϱ and δ. The key constraint hindering the definition of total l, u induced operations can be avoided in the following way:

Definition 10 In the context of Theorem 5, operations $\Box, \Diamond$ can be defined as follows:

- Given $\alpha = \{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n, \ldots\} \in \mathfrak{S}$, form the set $\gamma(\alpha) = \{\mathbb{A}_1^l, \mathbb{A}_2^l, \ldots, \mathbb{A}_n^l, \ldots\}$. If this is an antichain, then α would be said to be *lower pure*.
- Form the set of all relatively maximal antichains $\gamma_+(\alpha)$ from $\gamma(\alpha)$.
- Form all maximal antichains $\gamma_*(\alpha)$ containing elements of $\gamma_+(\alpha)$ and set $\Box(\alpha) = \wedge\gamma_*(\alpha)$.
- For $\Diamond$, set $\pi(\alpha) = \{\mathbb{A}_1^u, \mathbb{A}_2^u, \ldots, \mathbb{A}_n^u, \ldots\}$. If this is an antichain, then α would be said to be *upper pure*.
- Form the set of all relatively maximal antichains $\pi_+(\alpha)$ from $\pi(\alpha)$.
- Form all maximal antichains $\pi_*(\alpha)$ containing elements of $\pi_+(\alpha)$ and set $\Diamond(\alpha) = \vee\pi_*(\alpha)$.

Theorem 7 *In the context of the above definition, the following hold:*

$$\alpha \lessdot \beta \longrightarrow \Box(\alpha) \lessdot \Box(\beta) \,\&\, \Diamond(\alpha) \lessdot \Diamond(\beta)$$
$$\Box(\alpha) \lessdot \alpha \lessdot \Diamond(\alpha),\ \Box(0) = 0 \,\&\, \Diamond(1) = 1$$

Based on the above properties, the following algebra can be defined.

Definition 11 By a *Concrete AC* algebra (AC -algebra) will be meant an algebra of the form $\left\langle \mathfrak{S}, \varrho, \delta, \vee, \wedge, \Box, \Diamond, 0, 1 \right\rangle$ associated with a granular operator space S satisfying all of the following:

- $\langle \mathfrak{S}, \vee, \wedge \rangle$ is a bounded distributive lattice derived from a granular operator space as in the above.
- $\varrho, \delta, \Box, \Diamond$ are as defined above.

The following concepts of ideals and filters are of interest as deductive systems in a natural sense and relate to ideas of rough consequence (detailed investigation will appear separately).

Definition 12 By a *LD-ideal* (resp. LD-filter) K of an AC-algebra $\mathfrak{S}$ will be meant a lattice ideal (resp. filter) that satisfies:

$$(\forall \alpha \in K)\, \Box(\alpha), \Diamond(\alpha) \in K \tag{20}$$

By a *VE-ideal* (resp. VE-filter) K of an AC-algebra $\mathfrak{S}$ will be meant a lattice ideal (resp. filter) that satisfies:

$$(\forall \xi \in \mathfrak{S})(\forall \alpha \in K)\, \varrho(\xi, \alpha), \delta(\xi, \alpha) \in K \tag{21}$$

Proposition 6 *Every VE filter is closed under* $\Diamond$

4.1 Generating AC-Algebras

Now it will be shown below that specific subsets of AC-algebras suffice to generate the algebra itself and that the axioms satisfied by the granulation affect the generation process and properties of AC-algebras and forgetful variants thereof.

An element $x \in \mathfrak{S}$ will be said to be *meet irreducible* (resp. join irreducible) if and only if $\wedge\{x_i\} = x \longrightarrow (\exists i)\, x_i = x$ (resp. $\vee\{x_i\} = x \longrightarrow (\exists i)\, x_i = x$). Let $W(S)$, $J(S)$ be the set of meet and join irreducible elements of $\mathfrak{S}$ and let $l(\mathfrak{S})$ be the length of the distributive lattice.

Theorem 8 *All of the following hold:*

- $(\mathfrak{S}, \vee, \wedge, 0, 1)$ *is a isomorphic to the lattice of principal ideals of the poset of join irreducibles.*
- $l(\mathfrak{S}) = \#(J(S)) = \#(W(S))$.
- $J(S)$ *is not necessarily the set of sets of maximal antichains of granules in general.*
- *When $\mathcal{G}$ satisfies mereological atomicity that is* $(\forall a \in \mathcal{G})(\forall b \in S)(\mathbf{P}ba,\, a^l = a^u = a \longrightarrow a = b)$*, and all approximations are unions of granules, then elements of* $J(S)$ *are proper subsets of* $\mathcal{G}$.
- *In the previous context,* $W(S)$ *must necessarily consist of two subsets of S that are definite and are not parts of each other.*

Proof • The first assertion is a well known.
- Since the lattice is distributive and finite, its length must be equal to the number of elements in $J(S)$ and $W(S)$. For a proof see [29].
- Under the minimal assumptions on $\mathcal{G}$, it is possible for definite elements to be properly included in granules as in esoteric or prototransitive rough sets [18, 23]. These provide the required counterexamples.
- The rest of the assertions follows from the nature of maximal antichains and the constructive nature of approximations. □

Theorem 9 *In the context of the previous theorem if* $R(\Diamond)$, $R(\Box)$ *are the ranges of the operations* $\Diamond, \Box$ *respectively, then these have a induced lattice order on them. Denoting the associated lattice operations by* $\curlyvee, \curlywedge$ *on* $R(\Diamond)$*, it can be shown that*

- $R(\Diamond)$ *can be reconstructed from* $J(R(\Diamond)) \cup W(R(\Diamond))$.
- $R(\Box)$ *can be reconstructed from* $J(R(\Box)) \cup W(R(\Box))$.
- *When $\mathcal{G}$ satisfies mereological atomicity and absolute crispness (i.e.* $(\forall x \in \mathcal{G})\, x^l = x^u = x$*), then* $R(\Diamond)$ *are lattices which are constructible from two sets A, C (with* $A = \{\mathcal{G} \cup \{g_1 \cup g_2\}^u \setminus \{g_1, g_2\};\, g_1, g_2 \in \mathcal{G}\}$ *and C being the set of two element maximal antichains formed by sets that are upper approximations of other sets).*

Proof It is clear that $R(\Diamond)$ is a lattice in the induced order with $J(R(\Diamond))$ and $W(R(\Diamond))$ being the partially ordered sets of join and meet irreducible elements respectively. This holds because the lattice is finite.

The reconstruction of the lattice can be done through the following steps:

- Let $Z = J(R(\Diamond)) \cup W(R(\Diamond))$. This is a partially ordered set in the order induced from $R(\Diamond)$.
- For $b \in J(R(\Diamond))$ and $a \in W(R(\Diamond))$, let $b \prec a$ if and only if $a \neq b$ in $R(\Diamond)$.
- On the new poset Z with $\prec$, form sets including elements of $W(R(\Diamond))$ connected to it.
- The set of union of all such sets including empty set ordered by inclusion would be isomorphic to the original lattice [29].
- Under additional assumptions on $\mathcal{G}$, the structure of Z can be described further.

When the granulation satisfies the properties of crispness and mereological atomicity, then $A = J(R(\Diamond))$ and $C = W(R(\Diamond))$. So the third part holds as well. □

The results motivate this concept of purity: A maximal antichain will be said to *pure* if and only if it is both lower and upper pure.

4.2 Enhancing the Anti Chain Based Representation

An integration of the orders on sets of maximal antichains or antichains and the representation of rough objects and possible orders among them leads to interesting multiple orders on the resulting structure. A major problem is that of defining the orders or partials thereof in the first place among the various possibilities.

Definition 13 By the *rough interpretation of an antichain* will be meant the sequence of pairs obtained by substituting objects in the rough domain in place of objects in the classical perspective. Thus if $\alpha = \{a_1, a_2 \ldots, a_p\}$ is a antichain, then its rough interpretation would be ($\pi(a_i) = (a_i^l, a_i^u)$ for each i)

$$\underline{\alpha} = \{\pi(a_1), \pi(a_2), \ldots, \pi(a_p)\}. \tag{22}$$

Proposition 7 *It is possible that some rough objects are not representable by maximal antichains.*

Proof Suppose the objects represented by the pairs (a, b) and (e, f) are such that $a = e$ and $b \subset f$, then it is clear that any maximal antichain containing (e, f) cannot contain any element from $\{x : x^l = a \,\&\, x^u = b\}$. This situation can happen, for example, in the models of proto transitive rough sets [24, 27]. Concrete counterexamples can be found in the same paper. □

Definition 14 A set of maximal antichains V will be said to be *fluent* if and only if $(\forall x)(\exists \alpha \in V)(\exists (a, b) \in \alpha)\, x^l = a \,\&\, x^u = b$.

It will be said to be *well fluent* if and only if it is fluent and no proper subset of it is fluent.

A related problem is of finding conditions on $\mathcal{G}$, that ensure that V is fluent.

Table 1 Successor neighborhoods

Objects E	a	b	c	e	f
Neighborhoods [E]	$\{a\}$	$\{a, b, e\}$	$\{c, e\}$	$\{c, f\}$	$\{e\}$

4.3 Extended Abstract Example-1

The following example is intended to illustrate some aspects of the intricacies of the inverse problem situation where anti chains may be described. It is done within the relation based paradigm and the assumption that objects are completely determined by their properties.

Let $\S = \{a, b, c, e, f\}$ and let R be a binary relation on it defined via

$$R = \{(a, a), (b, b), (c, c), (a, b), (c, e), (e, f), (e, c), (f, e), (e, b)\}$$

If the formula for successor neighborhoods is

$$[x] = \{z\, ;\, Rzx\},$$

then the table for successor neighborhoods would be as in Table 1.

Using the definitions

$$x^l = \bigcup_{[z]\subseteq x} [z] \;\&\; x^u = \bigcup_{[z]\cap x \neq \emptyset} [z],$$

the approximations and rough objects that follow are in Table 2 (strings of letters of the form *abe* are intended as abbreviation for the subset $\{a, b, e\}$ and $\lrcorner$ is for, among subsets).

Under the rough inclusion order, the bounded lattice of rough objects in Fig. 1 (arrows point towards smaller elements) is the result:

From this ordered structure, maximal antichains can be evaluated by standard algorithms or by a differential process of looking at elements, their order ideals (and order filters) and maximal antichains that they can possibly form. In the figure, for example, elements in the order ideals of 69 cannot form antichains with it. This computation is targeted at representation in terms of relatively exact objects. The direct computation that is likely to come first before representation in practice is presented after the Table 3 in which some of the maximal antichains are computed by representation:

$\{60, 54, 69, 72\}$ is a maximal antichain because no more elements can be added to the set without violating incomparability. Note that the singletons $\{0\}$ and $\{1\}$ are also maximal antichains by definition. A diagram of the associated distributive lattice will not be attempted because of the number of elements.

Table 2 Approximations and rough objects

Rough object x	z^l	z^u	RO identifier
$\{a \lrcorner b \lrcorner ab\}$	$\{a\}$	$\{abe\}$	$\{3\}$
$\{ae \lrcorner abe\}$	$\{a\}$	$\{abce\}$	$\{6\}$
$\{e \lrcorner be\}$	$\{e\}$	$\{abec\}$	$\{9\}$
$\{c\}$	$\{\emptyset\}$	$\{cef\}$	$\{15\}$
$\{f\}$	$\{\emptyset\}$	$\{cf\}$	$\{24\}$
$\{cf\}$	$\{cf\}$	$\{cef\}$	$\{27\}$
$\{bc \lrcorner bf\}$	$\{\emptyset\}$	$\{S\}$	$\{30\}$
$\{ac \lrcorner af \lrcorner abc \lrcorner abf\}$	$\{a\}$	$\{S\}$	$\{33\}$
$\{aef\}$	$\{ae\}$	$\{S\}$	$\{36\}$
$\{ef \lrcorner bef\}$	$\{e\}$	$\{S\}$	$\{42\}$
$\{ec \lrcorner bce\}$	$\{ec\}$	$\{S\}$	$\{45\}$
$\{bcf\}$	$\{fc\}$	$\{S\}$	$\{51\}$
$\{abef\}$	$\{abe\}$	$\{S\}$	$\{54\}$
$\{ace\}$	$\{ace\}$	$\{S\}$	$\{60\}$
$\{acf\}$	$\{acf\}$	$\{S\}$	$\{63\}$
$\{ecf \lrcorner bcef\}$	$\{cef\}$	$\{S\}$	$\{69\}$
$\{abcf\}$	$\{abcf\}$	$\{S\}$	$\{72\}$
$\{abce\}$	$\{abcf\}$	$\{S\}$	$\{78\}$

Table 3 Maximal antichains

Rough object Z	Antichains including Z (differential)
78	$\{78, 69, 72\}$
60	$\{60, 54, 69, 72\}$, $\{60, 54, 69, 63\}$, $\{60, 54, 51\}$, $\{60, 54, 27\}$
54	$\{54, 45, 72\}$
72	$\{72, 45, 36\}$, $\{36, 69, 72\}$, $\{42, 72\}$, $\{9, 72\}$
69	$\{36, 69, 63\}$, $\{69, 33, 42\}$, $\{69, 6\}$, $\{69, 3\}$
42	$\{42, 33, 51\}$, $\{42, 33, 27\}$, $\{42, 6, 27\}$, $\{42, 6, 51\}$, $\{42, 63\}$
36	$\{36, 45, 63\}$, $\{36, 51, 45\}$, $\{36, 27, 45\}$
33	$\{45, 33, 51\}$, $\{45, 33, 27\}$
6	$\{9, 6, 15\}$, $\{9, 6, 27\}$, $\{9, 6, 51\}$, $\{9, 6, 24\}$
9	$\{9, 3, 15\}$, $\{9, 3, 24\}$, $\{9, 3, 27\}$, $\{9, 3, 51\}$, $\{9, 63\}$

4.3.1 Comparative Computations

In practice, the above table corresponds to only one aspect of information obtained from information systems. The scope of the anti chain based is intended to be beyond that including the inverse problem [21]. The empirical aspect is explained in this part.

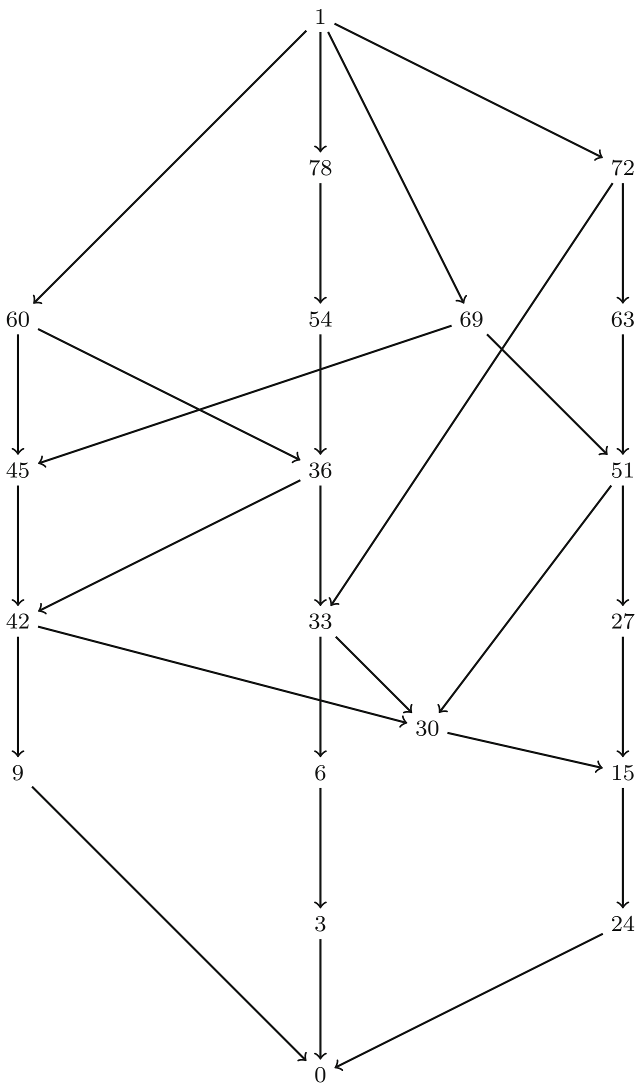

Fig. 1 Lattice of rough objects

Antichains are formed from $\wp(S)$ or subsets of it with some implicit temporal order (because of the order in which elements are accessed). If the elements of $\wp(S)$ are accessed in lexicographic order, and the sequence is decomposed by rough object discernibility alone, then it would have the following form ($\lceil, \rceil$ being group boundaries):

$$\{\lceil\{a\}\rceil, \lceil\{b\}\,\{c\}, \{e\}, \{f\}\rceil, \lceil\{a,b\}, \{a,c\}, \{a,e\}\rceil,$$
$$\lceil\{a,f\}, \{b,c\}, \{b,e\}\rceil,$$
$$\lceil\{b,f\}, \{c,e\}, \{c,f\}, \{e,f\}, \{a,b,c\}, \{a,b,e\}\}\rceil, \ldots\}$$

If these are refined by rough inclusion, then a decomposition into antichains would have the following form ($\lceil$, $\rceil$ now serve as determiners of antichain boundaries)

$$\{\lceil\{a\}\rceil, \lceil\{b\}\,\{c\}, \{e\}\rceil, \lceil\{f\}, \{a,b\}\rceil, \lceil\{a,c\}\rceil, \lceil\{a,e\}\rceil,$$
$$\lceil\{a,f\}\rceil, \lceil\{b,c\}, \{b,e\}\rceil,$$
$$\lceil\{b,f\}\rceil, \lceil\{c,e\}, \{c,f\}\rceil, \lceil\{e,f\}, \{a,b,c\}\rceil, \lceil\{a,b,e\}\}\rceil, \ldots\}$$

Implicit in all this is that the agent can perceive

- rough approximations,
- rough inclusion,
- rough equality and

have good intuitive algorithms for arriving at maximal antichains. In the brute force approach, the agent would need as much as $\frac{2^{\#(\wp(S))}!}{2}$ orders for obtaining all maximal antichains. The number of computations can be sharply reduced by the table of rough objects and known algorithms in the absence of intuitive algorithms.

A reading of the above sequence of antichains in terms of approximations (the compact notation introduced earlier is used) is

$$\{\lceil(a, abe)\rceil, \lceil(a, abe), (\emptyset, cef), (e, abec)\rceil, \lceil(\emptyset, cf), (a, abe)\rceil,$$
$$\lceil(a, S)\rceil, \lceil(a, abec)\rceil, \lceil(a, S)\rceil, \lceil(\emptyset, S), (e, abec)\rceil,$$
$$\lceil(\emptyset, S)\rceil, \lceil(ec, S), (cf, cef)\rceil, \lceil(e, S), (a, S)\rceil, \lceil(a, abec)\rceil, \ldots\}$$

Relative the order structure this reads as

$$\{\lceil 3\rceil, \lceil 3, 15, 9\rceil, \lceil 24, 3\rceil,$$
$$\lceil 33\rceil, \lceil 6\rceil, \lceil 33\rceil, \lceil 30, 9\rceil,$$
$$\lceil 30\rceil, \lceil 45, 27\rceil, \lceil 42, 33\rceil, \lceil 6\rceil, \ldots\}$$

4.4 Example: Micro-Fossils and Descriptively Remote Sets

This is a somewhat extended version of the example mentioned by the present author in [26]. In the case study on numeric visual data including micro-fossils with the help of nearness and remoteness granules in [34], the difference between granules and approximations is very fluid as the precision level of the former can be varied.

The data set consists of values of probe functions that extract pixel data from images of micro-fossils trapped inside other media like amethyst crystals.

The idea of remoteness granules is relative a fixed set of nearness granules formed from upper approximations—so the approach is about reasoning with sets of objects which in turn arise from tolerance relations on a set. In [34], antichains of rough objects are not used, but the computations can be extended to form maximal antichains at different levels of precision towards working out the best antichains from the point of view of classification.

Let X be an object space consisting of representation of some patterns and $\boldsymbol{\Phi} : X \longmapsto \mathbb{R}^n$ be a *probe function*, defined by

$$\boldsymbol{\Phi}(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_n(x)), \tag{23}$$

where $\phi_i(x)$ is intended as a measurement of the ith component in the feature space $\mathfrak{I}(\boldsymbol{\Phi})$. The concept of descriptive intersection of sets permits migration from classical ideas of proximity to ones based on descriptions. A subset $A \subseteq X$'s descriptive intersection with subset $B \subseteq X$ is defined by

$$A \cap_{\boldsymbol{\Phi}} B = \{x \in A \cup B : \boldsymbol{\Phi}(x) \in \boldsymbol{\Phi}(A) \,\&\, \boldsymbol{\Phi}(x) \in \boldsymbol{\Phi}(B)\} \tag{24}$$

A is then *descriptively near* B if and only if their descriptive intersection is nonempty. Peter's version of proximity $\pi_{\boldsymbol{\Phi}}$ is defined by

$$A\pi_{\boldsymbol{\Phi}}B \leftrightarrow \boldsymbol{\Phi}(A) \cap \boldsymbol{\Phi}(B) \neq \emptyset \tag{25}$$

In [8], weaker implications for defining *descriptive nearness* are considered :

$$A \cap_{\boldsymbol{\Phi}} B \neq \emptyset \rightarrow A\delta_{\boldsymbol{\Phi}}B. \tag{26}$$

Specifically, if δ is a proximity on R^n, then a descriptive proximity $\delta_{\boldsymbol{\Phi}}$ is definable via

$$A\delta_{\boldsymbol{\Phi}}B \leftrightarrow \boldsymbol{\Phi}(A)\delta\boldsymbol{\Phi}(B). \tag{27}$$

All these are again approachable from an anti-chain perspective.

4.5 Example: Beyond Data Tables

In this example subjective data is cast in terms of rough language for the purpose of understanding appropriate frameworks and solving context related problems.

Suppose agent X wants to complete a task and this task is likely to involve the use of a number of tools. X thinks tool-1 suffices for the task that a tool-2 is not suited for the purpose and that tool-3 is better suited than tool-1 for the same task. X also believes that tool-4 is as suitable as tool-1 for the task and that tool-5 provides more than what is necessary for the task. X thinks similarly about other tools but not much is known about the consistency of the information. X has a large repository of tools and limited knowledge about tools and their suitability for different purposes, and at the same time X might be knowing more about difficulty of tasks that in turn require better tools of different kinds.

Suppose also that similar heuristics are available about other similar tasks.

The reasoning of the agent in the situation can be recast in terms of lower, upper approximations and generalized equality and questions of interest include those relating to the agent's understanding of the features of tools, their appropriate usage contexts and whether the person thinks rationally.

To see this it should be noted that the key predicates in the context are as below:

- suffices for can be read as *includes potential lower approximation of* a right tool for the task.
- is not suited for can be read as *is neither a lower or upper approximation of* any of the right tools for the task.
- better suited than can be read as *potential rough inclusion*,
- is as suitable as can be read as *potential rough equality* and
- provides more than what is necessary for is for *upper approximation* of a right tool for the task.

If *table rationality* is the process of reasoning by information tables and approximations, then when does X's reasoning become table rational?

This problem fits in easily with the antichain perspective, but not the information table approach because the latter requires extra information about properties.

5 Quasi Equivalential Rough Sets and More

Entire semantics of various general rough set approaches can be recast in the antichain based perspective. For example, prototransitive rough sets [27] can be dealt with the same way. A finer characterization of the same will appear separately. Quasi Equivalential rough set were considered also as an example of the approach in [26]. As this can serve as an important example, the part is repeated below.

One of the most interesting type of granulation $\mathcal{G}$ in relational RST is one that satisfies

$$(\forall x, y)\,(\phi(x) = \phi(y) \leftrightarrow Rxy \,\&\, Ryx), \tag{28}$$

where $\phi(x)$ is the granule generated by $x \in \underline{S}$. This granular axiom says that if x is left-similar to y and y is left-similar to x, then the elements left similar to either of x and y must be the same. R is being read as *left-similarity* because it is directional and has effect similar to tolerances on neighborhood granules.

Reflexivity is not assumed as the present author's intention is to isolate the effect of the axiom alone.

For example, it is possible to find quasi equivalences that do not satisfy other properties from contexts relating to numeric measures. Let S be a set of variables such that Rxy if and only if $x \approx \kappa y \,\&\, y \approx \kappa' x \,\&\, \kappa, \kappa' \in (0.9, 1.1)$ for some interpretation of $\approx$.

Definition 15 By a *Quasi-Equivalential Approximation Space* will be meant a pair of the form $S = \langle \underline{S}, R \rangle$ with R being a quasi equivalence. For an arbitrary subset $A \in \wp(S)$, the following can be defined:

$$(\forall x \in \underline{S})\, [x] = \{y \,;\, y \in \underline{S} \,\&\, Ryx\}.$$

$$A^{l} = \bigcup\{[x]\,;\, [x] \subseteq A \,\&\, x \in \underline{S}\} \,\&\, A^{u} = \bigcup\{[x]\,;\, [x] \cap A \neq \emptyset \,\&\, x \in \underline{S}\}$$

$$A^{l_o} = \bigcup\{[x]\,;\, [x] \subseteq A \& x \in A\} \,\&\, A^{u_o} = \bigcup\{[x]\,;\, [x] \cap A \neq \emptyset \,\&\, x \in A\}$$

$$A^{L} = \{x\,;\, \emptyset \neq [x] \subseteq A \,\&\, x \in \underline{S}\} \,\&\, A^{U} = \{x\,;\, [x] \cap A \neq \emptyset \,\&\, x \in \underline{S}\}$$

$$A^{L_o} = \{x\,;\, [x] \subseteq A \,\&\, x \in A\} \,\&\, A^{U_o} = \{x\,;\, [x] \cap A \neq \emptyset \,\vee\, x \in A\}.$$

$$A^{L_1} = \{x\,;\, [x] \subseteq A \,\&\, x \in \underline{S}\} \,\&\, A^{U} = \{x\,;\, [x] \cap A \neq \emptyset \,\&\, x \in \underline{S}\}.$$

Note the requirement of non-emptiness of $[x]$ in the definition of A^{L}, but it is not necessary in that of A^{L_o}.

Theorem 10 *The following properties hold:*

1. *All of the approximations are distinct in general.*
2. $(\forall A \in \wp(S))\, A^{L_o} \subseteq A^{l_o} \subseteq A^{l} \subseteq A$ *and* $A^{L_o} \subseteq A^{L}$.
3. $(\forall A \in \wp(S))\, A^{l_o l} = A^{l_o} \,\&\, A^{l l_o} \subseteq A^{l_o} \,\&\, A^{l_o l_o} \subseteq A^{l_o}$
4. $(\forall A \in \wp(S))\, A^{u} = A^{ul} \subseteq A^{uu}$, *but it is possible that* $A \nsubseteq A^{u}$
5. *It is possible that* $A^{L} \nsubseteq A$ *and* $A \nsubseteq A^{U}$, *but* $(\forall A \in \wp(S))\, A^{L} \subseteq A^{U}$ *holds. In general* A^{L} *would not be comparable with* A^{l} *and similarly for* A^{U} *and* A^{u}.
6. $(\forall A \in \wp(S))\, A^{L_o L_o} \subseteq A^{L_o} \subseteq A \subseteq A^{U_o} \subseteq A^{U_o U_o}$. *Further* $A^{U} \subseteq A^{U_o}$.

Clearly the operators l, u are granular approximations, but the latter is controversial as an upper approximation operator. The point-wise approximations L, U are more problematic—not only do they fail to satisfy representability in terms of neighborhood granules, but the lower approximation fails inclusion.

Example 1 (General)

$$\text{Let } \underline{S} = \{a, b, c, e, f, k, h, q\} \tag{29}$$

and let R be a binary relation on it defined via

$$R = \{(a, a), (b, a), (c, a), (f, a), (k, k), (e, h), (f, c), (k, h) (b, b), (c, b), (f, b), (a, b), (c, e), (e, q)\}.$$

The neighborhood granules $\mathcal{G}$ are then

$$[a] = \{a, b, c, f\} = [b], [c] = \{f\}, [e] = \{c\},$$
$$[k] = \{k\}, [h] = \{k, e\}, [f] = \emptyset \,\&\, [q] = \{e\}.$$

So R is a quasi-equivalence relation.

If $A = \{a, k, q, f\}$, then

$$A^l = \{k, f\},\ A^u = \{a, b, c, f, k, e\},\ A^{uu} = \{a, b, c, f, k, e, h\}$$
$$A^{l_o} = \{k\},\ A^{u_o} = \{a, b, c, k, f\}.$$
$$A^L = \{k, f\},\ A^U = \{a, b, c, k, h, q\}.$$
$$A^{L_o} = \{q, k, f\},\ A^{U_o} = \{a, k, q, f, b, c, h\}.$$
$$A^{L_1} = \{k, c, f\},\ A^U = \{a, b, c, k, h, q\}.$$

$$\text{Note that } A^{L_1} \nsubseteq A \,\&\, A^{L_1} \nsubseteq A^U \,\&\, A \nsubseteq A^U. \tag{30}$$

6 Semantics of QE-Rough Sets

In this section a semantics of quasi-equivalential rough sets (QE-rough sets), using antichains generated from rough objects, is developed. Interestingly the properties of the approximation operators of QE-rough sets fall short of those of granular operator space. Denoting the set of maximal antichains of rough objects by $\mathfrak{S}$ and carrying over the operations $\ll, \varrho, \delta$, the following algebra can be defined.

Definition 16 A *maximal simply independent algebra* Q of quasi equivalential rough sets shall be an algebra of the form

$$Q = \langle \mathfrak{S}, \ll, \varrho, \delta \rangle \tag{31}$$

defined as in Sect. 4 with the approximation operators being l, u uniformly in all constructions and definitions.

Theorem 11 *Maximal simply independent algebras are well defined.*

Proof None of the steps in the definition of the maximal antichains, or the operations ϱ or δ are problematic because of the properties of the operators l, u. □

The above theorem suggests that it would be better to try and define more specific operations to improve the uniqueness aspect of the semantics or at least the properties of ϱ, δ. It is clearly easier to work with antichains as opposed to maximal antichains as more number of suitable operations are closed over the set of antichains as opposed to those over the set of maximal antichains.

Definition 17 Let $\mathfrak{K}$ be the set of antichains of rough objects of S then the following operations $\mathfrak{L}$, $\mathfrak{U}$ and extensions of others can be defined:

- Let $\alpha = \{\mathbb{A}_1, \mathbb{A}_2, \ldots, \mathbb{A}_n, \ldots\} \in \mathfrak{K}$ with $\mathbb{A}_i$ being rough objects; the lower and upper approximation of any subset in $\mathbb{A}_i$ will be denoted by $\mathbb{A}_i^l$ and $\mathbb{A}_i^u$ respectively.
- Define $\mathfrak{L}(\alpha) = \{\mathbb{A}_1^l, \mathbb{A}_2^l, \ldots, \mathbb{A}_r^l, \ldots\}$ with duplicates being dropped
- Define $\mathfrak{U}(\alpha) = \{\mathbb{A}_1^u, \mathbb{A}_2^u, \ldots, \mathbb{A}_r^u, \ldots\}$ with duplicates being dropped
- Define

$$\mu(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in \mathfrak{S} \\ \text{undefined}, & \text{else.} \end{cases} \tag{32}$$

- Partial operations ϱ^*, δ^* corresponding to ϱ, δ can also be defined as follows: Define

$$\varrho^*(\alpha, \beta) = \begin{cases} \varrho(\alpha, \beta) & \text{if } \alpha, \beta \in \mathfrak{S} \\ \text{undefined}, & \text{else.} \end{cases} \tag{33}$$

$$\delta^*(\alpha, \beta) = \begin{cases} \delta(\alpha, \beta) & \text{if } \alpha, \beta \in \mathfrak{S} \\ \text{undefined}, & \text{else.} \end{cases} \tag{34}$$

The resulting partial algebra $\mathfrak{K} = \langle \mathfrak{K}, \mu, \vee, \wedge, \varrho^*, \delta^*, \mathfrak{L}, \mathfrak{U}, 0 \rangle$ will be said to be a *simply independent QE algebra*.

Theorem 12 *Simply independent QE algebras are well defined and satisfy the following:*

- $\mathfrak{L}(\alpha) \vee \alpha = \alpha$.
- $\mathfrak{U}(\alpha) \vee \alpha = \mathfrak{U}(\alpha)$.

7 Ternary Deduction Terms

Since AC-algebras are distributive lattices with additional operations, a natural strategy should be to consider terms similar to Boolean algebras and p-Semilattices. For isolating deductive systems in the sense of Sect. 3, a strategy can be through complementation-like operations. This motivates the following definition:

Definition 18 In a AC-algebra $\mathfrak{S}$, if an antichain $\alpha = (X_1, X_2, \ldots, X_n)$, then some possible general complements on the schema

$$\alpha^c = \mathfrak{H}(X_1^c, X_2^c, \ldots, X_n^c)$$

are as follows:

$$X_i^* = \{w;\ (\forall a \in X_i)\, \neg \mathbf{P}aw \,\&\, \neg \mathbf{P}wa\} \tag{Class A}$$
$$X_i^\# = \{w;\ (\forall a \in X_i)\, \neg a^l = w^l \text{ or } \neg a^u = w^u\} \tag{Light}$$
$$X_i^\flat = \{w;\ (\forall a \in X_i)\, \neg a^l = w^l \text{ or } \neg a^{uu} = w^{uu}\} \tag{UU}$$

$\mathfrak{H}$ is intended to signify the maximal antichain containing the set if that is definable.

Note that under additional assumptions (similarity spaces), the light complementation is similar to the preclusivity operation in [3] for Quasi BZ-lattice or Heyting-Wajsburg semantics and variants.

The above operations on α are partial in general, but can be made total with the help of an additional order on α and the following procedure:

1. Let $\alpha = \{X_1, X_2, \ldots, X_n\}$ be a finite sequence,
2. Form α^c and split into longest ACs in sequence,
3. Form maximal ACs containing each AC in sequence
4. Join resulting maximal ACs.

Proposition 8 *Every general complement defined by the above procedure is well defined.*

Proof • Suppose $\{X_1^c, X_2^c\}, \{X_3^c, \ldots X_n^c\}$ form antichains, but $\{X_1^c, X_2^c, X_3^c\}$ is not an antichain.
• Then form the maximal antichains $\eta_1, \ldots, \eta_p$ containing either of the two antichains.
• The join of this finite set of maximal antichains is uniquely defined. By induction, it follows that the operations are well defined. □

7.1 Translations

As per the approach of Sect. 3, possible definitions of translations are as follows:

Definition 19 A *translation* in a AC-algebra $\mathfrak{S}$ is a $\sigma : \mathfrak{S} \longmapsto \mathfrak{S}$ that is defined in one of the following ways (for a fixed $a \in \mathfrak{S}$):

$$\sigma_\theta(x) = \theta(a, x) \; ; \theta \in \{\vee, \wedge, \varrho, \delta\}$$
$$\sigma_\mu(x) = \mu(x) \; ; \mu \in \{\Box, \Diamond\}$$
$$\sigma_t(x) = (x \oplus a) \oplus b \text{ for fixed a, b}$$
$$\sigma_{t+}(x) = (a \oplus b) \oplus x \text{ for fixed a, b}$$

Theorem 13

$$\sigma_\vee(0) = a = \sigma_\vee(a \, ; \sigma_\vee(1) = 1$$
$$\textit{Ran}(\sigma_\vee) \textit{ is the principal filter generated by a}$$
$$\textit{Ran}(\sigma_\wedge) \textit{ is the principal ideal generated by a}$$
$$x \lessdot w \longrightarrow \sigma_\vee(x) \lessdot \sigma_\vee(w) \,\&\, \sigma_\wedge(x) \lessdot \sigma_\wedge(w)$$

Proof • Let $\mathbb{F}(a)$ be the principal lattice filter generated by a.

- If $a \lessdot w$, then $a \vee w = w = \sigma_{\vee}(w)$. So $w \in Ran(\sigma_{\vee})$.
- $\sigma_{\vee}(x) \wedge \sigma_{\vee}(w) = (a \vee x) \wedge (a \vee w) = a \vee (x \wedge w) = \sigma_{\vee}(x \wedge w)$.
- So if $x, w \in Ran(\sigma_{\vee})$, then $x \wedge w, x \vee w \in Ran(\sigma_{\vee})$
- Similarly it is provable that $Ran(\sigma_{\wedge})$ is the principal ideal generated by a.

□

7.2 Ternary Terms and Deductive Systems

Possible ternary terms that can cohere with the assumptions of the semantics include the following $t(a, b, z) = a \wedge b \wedge z$, $t(a, b, z) = a \oplus b \oplus z$ ($\oplus$ being as indicated earlier) and $t(a, b, z) = \Box(a \wedge b) \wedge z$. These have admissible deductive systems associated. Further under some conditions on granularity, the distributive lattice structure associated with $\mathfrak{S}$ becomes pseudo complemented.

Theorem 14 *If $t(a, b, z) = a \wedge b \wedge z$, $\tau = \{t\}$, $z \in H$, $h(x) = x$ $\sigma(x) = x \wedge z$ and if H is a ternary τ-deduction system at z, then it suffices that H be an filter.*

Proof All of the following must hold:

- If $a \in H$, $t(z, a, z) = a \wedge z \in H$
- If $t(a, b, z) \in H$, then $t(\sigma(a), \sigma(b), z) = t(a, b, z) \in H$
- If $a, t(a, b, z) \in H$ then $t(a, b, z) = (a \wedge z) \wedge b \in H$. But H is a filter, so $b \in H$. □

Theorem 15 *If $t(a, b, z) = (a \vee (\Box b)) \wedge z$, $\tau = \{t\}$, $z \in H$, $h(x) = x$ $\sigma(x) = x \wedge z$ and if H is a ternary τ-deduction system at z, then it suffices that H be a principal LD-filter generated by z.*

Proof All of the following must hold:

- If $a \in H$, $t(z, a, z) = (z \vee (\Box a)) \wedge z \in H$ because $(z \vee (\Box a)) \in H$.
- If $t(a, b, z) \in H$, then $t(\sigma(a), \sigma(b), z) = t((a \wedge z), (b \wedge z), z) = ((a \wedge z) \vee \Box (b \wedge z)) \wedge z \in H$
- If $a, t(a, b, z) \in H$ then $t(a, b, z) = (a \vee \Box(b)) \wedge z = (a \wedge z) \vee (\Box(b) \wedge z) \in H$. But H is a LD-filter, so $a \vee \Box(b) \in H$. This implies $\Box(b) \in H$, which in turn yields $b \in H$.

□

In the above two theorems, the conditions on H can be weakened considerably. The converse questions are also of interest.

The existence of pseudo complements can also help in defining ternary terms that determine deductive systems (or subsets closed under consequence). In general, pseudo complementation $\circledast$ is a partial unary operation on $\mathfrak{S}$ that is defined by $x^{\circledast} = \max\{a\,;\ a \wedge x = 0\}$ (if the greatest element exists).

There is no one answer to the question of existence as it depends on the granularity assumptions of representation and stability of granules. The following result guarantees pseudo complementation (in the literature, there is no universal approach—it has always been the case that in some case they exist):

Theorem 16 *In the context of AC-algebras, if the granulation satisfies mereological atomicity and absolute crispness, then a pseudo complementation is definable.*

Proof Under the conditions on the granulation, it is possible to form the rough interpretation of each antichain. Moreover the granules can be moved in every case to construct the pseudo complement. The inductive steps in this proof have been omitted. □

8 Relation to Knowledge Interpretation

In Pawlak's concept of knowledge in classical RST [30, 32], if $\underline{S}$ is a set of attributes and P an indiscernibility relation on it, then sets of the form A^l and A^u represent clear and definite concepts (the semantic domain associated is the rough semantic domain). Extension of this to other kinds of RST have also been considered in [20, 22, 24, 27] by the present author. In [20], the concept of knowledge advanced by her is that of union of pairwise independent granules (in set context corresponding to empty intersection) correspond to clear concepts. This granular condition is desirable in other situations too, but may not correspond to the approximations of interest. In real life, clear concepts whose parts may not have clear relation between themselves are too common. If all of the granules are not definite objects, then analogous concepts of knowledge may be graded or typed based on the properties satisfied by them [24, 27]. *Then again the semantic domains in which these are being considered can be varied and so knowledge is relative*. Some examples of granular knowledge axioms are as follows:

1. Individual granules are atomic units of knowledge.
2. If collections of granules combine subject to a concept of mutual independence, then the result would be a concept of knowledge. The 'result' may be a single entity or a collection of granules depending on how one understands the concept of *fusion* in the underlying mereology. In set theoretic (ZF) setting the fusion operation reduces to set-theoretic union and so would result in a single entity.
3. Maximal collections of granules subject to a concept of mutual independence are admissible concepts of knowledge.
4. Parts common to subcollections of maximal collections are also knowledge.
5. All stable concepts of knowledge consistency should reduce to correspondences between granular components of knowledges. Two knowledges are *consistent* if and only if the granules generating the two have 'reasonable' correspondence.
6. Knowledge A is consistent with knowledge B if and only if the granules generating knowledge B are part of some of the granules generating A.

An antichain of rough objects is essentially a set of *some-sense mutually distinct rough concepts* relative that interpretation. Maximal antichains naturally correspond to represented knowledge that can be handled in a clear way in a context involving vagueness. The stress here should be on possible operations both within and over them. It is fairly clear that better the axioms satisfied by a concept of granular knowledge, better will be the nature of possible operations over sets of *some-sense mutually distinct rough concepts*.

From decision making perspectives, antichains of rough objects correspond to forming representative partitions of the whole and semantics relate to relation between different sets of representatives.

8.1 Knowledge Representation

In Sect. 4.2, the developed representation has the following features:

- Every object in a antichain is representable by a pair of objects (a, b) that are respectively of the form x^l and z^u.
- Some of these objects might be of the form (a, a) under the restriction $a = a^l = a^u$
- The above means that antichains can be written in terms of objects that are approximations of other objects or themselves.
- At another level, the concepts of rough objects mentioned in the background section suggest classification of the possible concepts of knowledge.
- The representation is perceivable in a rough semantic domain and this will be referred to as the *AC-representable rough domain* ACR.
- If *definable rough objects* are those rough objects representable in the form (a, b) with a, b being definite objects, then these together with definite objects may not correspond to maximal antichains in the classical semantic domain—the point is that some of the non crisp objects may fail to get represented under the constraints. The semantic domain associated with the definable rough objects with the representation and crisp objects will be referred to as the *strict rough domain* (SRD).

The above motivates the following definition sequence

Definition 20 All of the following constitute the basic knowledge structure in the context of AC-semantics:

- A *Proper Knowledge Sequence* in ACR corresponds to the representation of any of the maximal antichains.
- An *Abstract Proper Knowledge Sequence* in ACR corresponds to the representation of possible maximal antichains. These may be realized in particular models.
- A *Knowledge Sequence* in SRD corresponds to the relatively maximal antichains formed by sequences of definable rough objects and definite objects.
- Definable rough objects.
- Representation of rough and crisp objects.

More complex objects formed by antichains are also of interest. The important thing about the idea of knowledge sequences is the explicit admission of temporality and the relation to all of the information available in the context. This is considered next.

8.2 Perdurantism and Endurantism

The *endurantist* position is that objects persist over time by being *wholly present* at all times of their existence. Endurantism is also known as *Three dimensionalism*. The *perdurantist* position (or four dimensionalism),in contrast, is that objects persist over time by having *temporal parts* in addition to their spatial parts at all times of their existence. Some references are [16, 38, 42, 43]. In the context of rough sets, though temporality has been investigated, these concepts do not figure explicitly in earlier work in both the present author's work and the rough membership function based approach [35].

Classical extensional mereology CEM is seen by most endurantists and perdurantists as a reasonable framework for handling real objects with material existence. In the framework, such objects may be viewed as mereological aggregations including sum and fusion. But when it comes to concepts as in rough sets, such an approach need not suffice (see [21]). Given the assumptions of CEM, it can be seen that maximal antichains have temporal parts and by modification of temporal parts their identity changes. Again they can be seen as the same information with irrelevant temporal parts—but in doing this the semantic domain needs to be changed. This means that the positions of perdurantism and endurantism need to rely on choice of semantic domain for their validation. From the way the algebraic semantics has been constructed, it is clear that any two distinct maximal antichain have distinct temporal components. So the following meta theorem follows:

Theorem 17 *The semantic domain associated with the the AC-algebras is perdurantist (or four dimensional).*

But the argument does not stop here. It can be argued that two distinct maximal antichains

- are insufficient references to the same knowledge U (say).
- U is guaranteed to exist by generalized granular operator spaces.
- U is not affected by the *so-called temporal parts* and in the classical semantic domain maximal antichains correspond to distinct knowledge.

All this confirms the present author's position that the two positions happen because of choice of semantic domains—the main problem is then of constructing/describing consistent domains.

8.3 Patterns of Triads

Pierce's approach to semiotics does not provide a consistent perspective for ontology in general. This is often seen as a failure of the enterprise [1]. But attempts have been made to adopt aspects of the semiotics to get to new insights in the structure of knowledge and consequence. In this section, parts of how these might play with the concept of knowledge afforded by rough sets and the antichain based knowledge representation is examined. Admittedly the developed approach may differ substantially with Pierce's ideas and in particular on assumptions relating to existence of fixed ontologies.

The first thing to be noted is that the concept of knowledge in general rough sets must be viewed in a deconstructive perspective rather than in the perspective of aggregation. The deconstructive process in the triadic perspective may not always yield parts that correspond to exact knowledge and so a few redefinitions of the ontology are introduced. Multiple sources of knowledge can also be handled within the framework with scope for handling conflict resolution by algebraic strategies especially when the graphical take forever.

If *knowledge* is viewed as the result of a *process of knowing* an object or phenomena then the process in question may be about placing all of the parts or the whole in a logical space of reasons that refers to justification and the justified. This kind of knowledge will be referred to as *pre-rational knowledge* and if the understanding of *pre-rational* is something less than rational, then the machinery of general rough sets affords a specific instance of this definition of pre-rational knowledge. This definition may be seen as a variation of the definition of knowledge in [36] (passage 36) wherein the part-whole distinctions are missing. Unless quasi orders or preference orders is imposed on granulations or attributes or the mereological *is necessary part of* is specified between subsets of granules and objects, the concept of *rationality* in approximations is difficult to ensure scientific rationality.

In Peirce's view all forms of reality exist in Signs. A sign is a spatiotemporal reality that may be material or conceptual. These exist in relation to other signs and is determined by its Object, and determines Interpretant through the mediating role of representations. Somewhat controversially all this can be assumed to establish the nature of the sign as a triad of interactions or relations. Thus in [33] pp. 272–273 it is stated that

> A Sign, (or Representamen), is a First which stands in such a genuine triadic relation to a Second, called its Object, as to be capable of determining a Third, called its Interpretant, to assume the same triadic relation to its Object in which it stands itself to the same Object.

The semiotic process of Object-Representation; Representation-Representation; Representation-Interpretant is then a triad of three relations, correlated by the mediation of the Representation. Some authors [40] reject the singularity of *a triadic relation* because it contributes to setting up a kind of closed and isolated process, while others believe that it is but one of the possibilities.

Signs are definable through six relations (called semiosic predicates) and interact as functions that act as mediation between input and output. The triad may be read as Object-Mediation/Representation-Interpretant in Peirce's sense and being irreducible. One scheme described in [37, 40], consists of ten basic signs that can be seen as classes with further subdivisions. This approach requires further ontology for a reasonable ascription of thresholds for concepts of reasonable/rational approximations.

The triadic approach again refers some fragment of rationality because of the way in which sign-vehicles [1] refer parts of attributes possessed—for example all attributes associated with red traffic lights are not used by drivers for the intended interpretation. The main features of sign vehicles in the triadic approach are classified by the signified by virtue of qualities, existential facts, or conventions respectively into *qualisigns, sinsigns, and legisigns* respectively.

Explicit integration with rough theoretic ideas of knowledge is however hindered by the ontological commitments of the semiosis and is an interesting problem—Will the triadic approach yield good thresholds of rationality in approximations?

Omitting even the ten sign classification, knowledge can be associated with ontological types along the following simple lines—these can in turn be applied to knowledge and its parts.

- Simplified Firstness [f]: these directly refer possible qualities or primitives that interact or become combined in multiple ways to form real objects or entities.
- Simplified Secondness [s]: these directly refer real objects/entities.
- Simplified Thirdness [t]: these directly refer general principles, rules, laws, methodologies and categories.

The resulting triads may not be easily applicable for handling questions relating to rational approximations and concepts, but permits contextual reasoning to some degree of rigor. An example ontological assignment may be like the following [41]:

```
Entities-[f]
.[SingleEntities-[f]
...[Objects-[f]
...States-[s]
...Events-[t]]
.PartOfEntities-[s]
...[Members-[f]
...Parts-[s]
...FunctionalComponents-[t]]
.ComplexEntities-[t]
...[CollectiveStuff-[f]
...MixedStuff-[s]
...CompoundEntities-[t]]
```

In the ontology, it is claimed that the process of investigation of knowledge by way of thirdness is fractal in nature. In the present author's perspective it is obvious that the claim is provably false at sufficient depth and is not a mathematical one. How deep are fractals formed on three symbols?

The components of maximal antichains and antichains in general rough sets can also be perceived in the triadic systems of reasoning and in the above mentioned ontological scheme. For closely related considerations on ontology and parthood the reader is referred to [25].

9 Concluding Remarks

In this research, general approaches to semantics of rough sets using antichains of rough objects and maximal antichains have been developed further. Specifically the operations used in [26] have been improved. The semantics is shown to be valid for a very large class of general rough set theories. This has been possible mainly because the objects of study have been taken to be antichains of rough objects as opposed to plain rough objects.

The problem of finding deductive systems in the context of antichain based semantics for general rough sets has also been explored in considerable detail and key results have been proved by the present author. The lateral approach used by her is justified by the wide variety of possible concepts of rough consequence in the general setting.

In forthcoming papers including [28], the framework of granular operator spaces has been expanded with definable parthood relations and semantics has been considered through counting strategies. All this will be explored in greater detail in future work.

The concept of knowledge afforded by the antichain based approach is explored in much detail and contrasted with concepts of knowledge studied in earlier papers by the present author. New concepts of fullness of knowledge have also been isolated by her and it is shown (in a sense) that the knowledge afforded by antichains is four dimensional. Last but not least methods of integrating Peirce's triadic approach to semiotics is shown to be possible.

This research also motivates the following:

- Further study of specific rough sets from the perspective of antichains.
- Research into connections with the rough membership function based semantics of [6] and extensions by the present author in a forthcoming paper. This is justified by advances in concepts of so-called cut-sets in antichains.
- Research into computational aspects as the theory is well developed for antichains. The abstract example illustrates parts of this aspect in particular.
- Study of consequence and special ideals afforded by the semantics and
- Research into ontologies indicated by the triadic approach.

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