Chapter 16
Expected Present and Final Value of an Annuity when some Non-Central Moments of the Capitalization Factor are Unknown: Theory and an Application using R

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Abstract The aim of this chapter is the development of three approaches for obtaining the value of an n-payment annuity, with payments of 1 unit each, when the interest rate is random. To calculate the value of these annuities, we are going to assume that only some non-central moments of the capitalization factor are known. The first technique consists in using a tetraparametric function which depends on the arctangent function. The second expression is derived from the so-called quadratic discounting whereas the third approach is based on the approximation of the mathematical expectation of the ratio of two random variables by Mood et al. (1974). A comparison of these methodologies through an application, using the R statistical software, shows that all of them lead to different results.

Keywords Annuity · Random interest rate · Tetraparametric function · Discount factor · Mood et al. approximation

16.1 Introduction

This work aims to determine an approximate expression to obtain the present, or final, value of an annuity when the interest rate is random. In annuities assessment, fixing the interest rate has a great relevance because even small changes can result

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in major changes in the total annuity value. Thus, the determination of the interest rate value must be carried out as accurately as possible (Cruz Rambaud and Sánchez Pérez 2016; Cruz Rambaud et al. 2015).

Under the traditional approach, interest rates have been treated deterministically. In certainty contexts, the use of a single possible value for each period may be enough (Villalón et al. 2009). However, for those operations developed in uncertain environments, it is more reasonable the formulation of potential scenarios, which are subsequently reduced to one by statistical treatment (Cruz Rambaud and Valls Martínez 2002).

The determination of the interest rate value must be based on the current situation, as well as on its possible future evolution, of both the company and its environment. In this way, if prospects are unfavorable, interest rates must be higher, compared to more favorable situations, and hence the operation value is reduced as a consequence of the risk attached to it. However, in most cases, determining the interest rate of a financial operation is subject to the risk propensity/aversion of the agent to be responsible for the assessment (Suárez Suárez 2005). In this sense, the adopted interest rate would be affected by a degree of subjectivity that may over/undervalue the project.

We will consider the interest rate as a random variable which is represented as X. Therefore, the capitalization factor, $1 + i$ (Mira Navarro 2014), is also a random variable represented as U. Obviously, it is verified that $U = 1 + X$, so the relationship between means and standard deviations of both variables is:

$$\mu_U = 1 + \mu_X$$

and

$$\sigma_U = \sigma_X.$$

As a result, if X is defined in an interval $[a, b]$, U will be in the interval $[1 + a, 1 + b]$. Henceforth, when the mean and standard deviation are mentioned we will refer to the random variable U, unless otherwise specified.

In this case, the final value of an n-payment annuity, with payments of 1 unit each, made at the end of every year (annuity-immediate), valued at the rate $X = U - 1$, would be the following random variable:

$$s_{\overline{n}|U-1} = 1 + U + U^2 + \cdots + U^{n-1}. \tag{16.1}$$

Thus, its expected value is:

$$E(s_{\overline{n}|U-1}) = E(1) + E(U) + E(U^2) + \cdots + E(U^{n-1}) = 1 + \mu + \mu_2 + \cdots + \mu_{n-1}.$$

On the other hand, the final expected value of an n-payment annuity, with payments of 1 unit each, made at the beginning of every year (annuity-due), valued at the rate X, would be:

$$E(\ddot{s}_{\overline{n}|U-1}) = E(U) + E(U^2) + \cdots + E(U^n) = \mu + \mu_2 + \cdots + \mu_n, \tag{16.2}$$

where

$$\mu_r = E(U^r)$$

is the moment of order r with respect to the origin of the random variable U. In the case that the random variable is discrete, it adopts the following expression:

$$\mu_r = E(U^r) = \sum_{i=1}^{k} p_i u_i^r, \tag{16.3}$$

where p_i is the probability that the random variable takes the value u_i. In the continuous case, the expression of the moment of order r is:

$$\mu_r = E(U^r) = \int_{u_{\min}}^{u_{\max}} uf(u)\mathrm{d}u, \tag{16.4}$$

for all values of r, being $f(u)$ the density function of the random variable U. On the other hand, the mean of order r is defined as the r-th root of the moment of order r which, in the discrete case, adopts the following expression:

$$m_r = \left(\sum_{i=1}^{k} p_i u_i^r\right)^{1/r}, \tag{16.5}$$

whereas in the continuous case, the expression of the mean of order r is:

$$m_r = \left(\int_{u_{\min}}^{u_{\max}} u^r f(u)\mathrm{d}u\right)^{1/r}, \tag{16.6}$$

for all values of r.

Below, we are going to study the limit L of the mean of order r, when r tends to $+\infty$:

$$L := \lim_{r\to+\infty} m_r. \tag{16.7}$$

To do this, take into account that the sequence $\{m_r\}_{r=-\infty}^{+\infty}$ and, in general, the function $g(x) = m_x$, being $-\infty < x < +\infty$, is increasing since, according to the inequality of Lyapunov, for $1 < r < s$, it is verified that $[E(U^r)]^{1/r} \leq [E(U^s)]^{1/s}$. Moreover, as $u_{\max}$ (maximum value of the random variable U) is an upper bound of $g(x)$, we can deduce that m_r has a limit at infinity which will be denoted by L.

Obviously, $L \leq u_{\max}$. Let us suppose $L < u_{\max}$. In this case, there would be a u_0, such that $L < u_0 < u_{\max}$. Below, we decompose the integral which defines m_r in other two as follows:

$$m_r = \left(\int_{u_{\min}}^{u_0} u^r f(u) \mathrm{d}u + \int_{u_0}^{u_{\max}} u^r f(u) \mathrm{d}u \right)^{1/r} > \left(\int_{u_0}^{u_{\max}} u^r f(u) \mathrm{d}u \right)^{1/r} >$$

$$> \left(u_0^r \int_{u_0}^{u_{\max}} f(u) \mathrm{d}u \right)^{1/r} = u_0 k^{1/r},$$

where the density function's integral between u_0 and $u_{\max}$, $\int_{u_0}^{u_{\max}} f(u)\mathrm{d}u$, has been represented by k. Clearly, it is verified that $0 < k < 1$. So,

$$L = \lim_{r \to +\infty} m_r \geq u_0 \lim_{r \to +\infty} k^{1/r} = u_0 \cdot 1 = u_0,$$

in contradiction with the fact that $L < u_0$. Therefore, one has:

$$L = \lim_{r \to +\infty} m_r = u_{\max}. \tag{16.8}$$

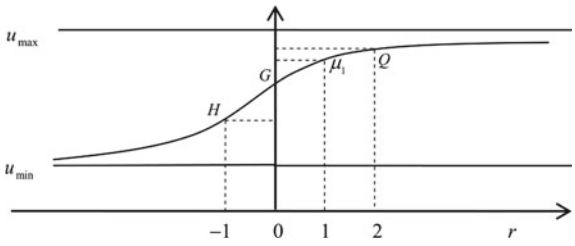
Analogously, it would be shown that $l := \lim_{r \to -\infty} m_r = u_{\min}$. Consequently, the function m_x, $-\infty < x < +\infty$, has a horizontal asymptote at $y = u_{\min}$ and another one at $y = u_{\max}$, so it changes its concavity (or its convexity), which means having, at least, an inflection point.

In this chapter we will analyze the mathematical expression of the present and final expected value of an n-payment annuity, with payments of 1 unit each, made at the end/beginning of every year (annuity-immediate and annuity-due), whose calculation entails a random interest rate. Specifically, in this work these expected values are analyzed when only some non-central moments of the capitalization factor are known. In Sect. 16.2, an approach on the basis of a tetraparametric function is studied. On the other hand, in Sect. 16.3, it is developed an approach by using the so-defined quadratic discounting. In Sect. 16.4, the expression to calculate the value of an annuity is by employing the approximate formula by Mood et al. (1974). In Sect. 16.5, we present a practical example using the R statistical software. Finally, Sect. 16.6 summarizes and concludes.

16.2 The Tetraparametric Function Approach

As a result of the reasoning shown in Sect. 16.1, the curve which represents the mean of order r is an increasing function of r which can be seen in Fig. 16.1. Thus, if Q denotes the quadratic mean, or mean of order 2, H is the harmonic mean, or mean

Fig. 16.1 Graphical representation of the mean of order r (Calot 1974)



of order -1, and G is the geometric mean or mean of order 0, we have the relations shown in Fig. 16.1.

In effect, according to the Lyapunov inequality (Fisz 1963), if the moments of order r of a random variable X exist for arbitrary values of r, it is verified the following inequality:

$$\mu_r^{1/r} \leq \mu_{r+1}^{1/(r+1)}; \tag{16.9}$$

thus, it is verified the following relationship between the corresponding means of order r and $r + 1$:

$$m_r \leq m_{r+1}. \tag{16.10}$$

Consequently, the curve representing the mean of order r can be fitted to the following tetraparametric function which exhibits its same shape:

$$g(r) = a \cdot \arctan(br + c) + d, \tag{16.11}$$

where, in a beginning, a and b are parameters greater than 0. Under these circumstances, we have:

$$\lim_{r \to +\infty} g(r) = a\frac{\pi}{2} + d = u_{\max}$$

and

$$\lim_{r \to -\infty} g(r) = -a\frac{\pi}{2} + d = u_{\min}.$$

By adding both equations, we deduce that $2d = u_{\min} + u_{\max}$, from where we can obtain the value of d:

$$d = \frac{u_{\min} + u_{\max}}{2}. \tag{16.12}$$

On the other hand, by subtracting both equations, we obtain that $\pi a = u_{\max} - u_{\min}$; thus, the parameter a is given by:

$$a = \frac{u_{\max} - u_{\min}}{\pi}. \tag{16.13}$$

Observe that, as expected, $a > 0$. As $g(1) = \mu_1$ and $g(2) = \mu_2^{1/2}$, one has:

$$a \cdot \arctan(b + c) + d = \mu_1 \tag{16.14}$$

and

$$a \cdot \arctan(2b + c) + d = \mu_2^{1/2}, \tag{16.15}$$

from where

$$b + c = \tan \frac{\mu_1 - d}{a} \tag{16.16}$$

and

$$2b + c = \tan \frac{\mu_2^{1/2} - d}{a}. \tag{16.17}$$

By subtracting the above two equations, we obtain:

$$b = \tan \frac{\mu_2^{1/2} - d}{a} - \tan \frac{\mu_1 - d}{a}, \tag{16.18}$$

which confirms that $b > 0$. Once determined a, b, c, and d, we can approximate the final expected value of an n-payment annuity, with payments of 1 unit each, made at the end/beginning of every year (annuity-immediate/annuity-due), valued at the rate X, by the following expressions:

$$E(s_{\overline{n}|U-1}) = 1 + m_1 + (m_2)^2 + \cdots + (m_{n-1})^{n-1} \approx 1 + \sum_{s=1}^{n-1} [a \cdot \arctan(bs + c) + d]^s \tag{16.19}$$

and

$$E(\ddot{s}_{\overline{n}|U-1}) = m_1 + (m_2)^2 + \cdots + (m_n)^n \approx \sum_{s=1}^{n} [a \cdot \arctan(bs + c) + d]^s. \tag{16.20}$$

On the other hand, the present expected value of an n-payment annuity, with payments of 1 unit each, made at the end/beginning of every year (annuity-immediate/annuity-due), valued at the rate X, respectively, would be as follows:

$$E(a_{\overline{n}|U-1}) = (m_{-1})^{-1} + (m_{-2})^{-2} + \cdots + (m_{-n})^{-n} \approx \sum_{s=1}^{n} [a \cdot \arctan(-bs + c) + d]^{-s} \tag{16.21}$$

and

$$E(\ddot{a}_{\overline{n}|U-1}) = 1 + (m_{-1})^{-1} + \cdots + (m_{-(n-1)})^{-(n-1)} \approx 1 + \sum_{s=1}^{n-1} [a \cdot \arctan(-bs + c) + d]^{-s}. \tag{16.22}$$

16.3 The Quadratic Discounting Approach

The variable which represents the expected present value of an n-payment annuity, with payments of 1 unit each, made at the end of every year (annuity-immediate), valued at the rate X, by using the exponential discounting is the following one:

$$a_{\overline{n}|X} = (1+X)^{-1} + (1+X)^{-2} + (1+X)^{-3} + \cdots + (1+X)^{-n} \tag{16.23}$$

or, equivalently, $a_{\overline{n}|X} = \frac{1-(1+X)^{-n}}{X}$. Below, we can simplify Eq. (16.23) by using the McLaurin formula. Indeed, by expanding the expression $(1+X)^{-t}$, we have that:

$$(1+X)^{-t} = 1 - tX + \frac{(-t)(-t-1)}{2!}X^2 - \cdots \tag{16.24}$$

In this sum, by removing all terms from the third one, we obtain that $(1+X)^{-t} \approx 1 - tX + \frac{t^2+t}{2}X^2$, which means that the present value may be approximately estimated by replacing the exponential by the quadratic discounting. Once made this simplification, the present value of an n-payment annuity, with payments of 1 unit each, made at the end of every year (annuity-immediate), valued at the rate X, can be calculated by using the following approximation:

$$a_{\overline{n}|X} = \left(1 - X\frac{1^2+1}{2}X^2\right) + \left(1 - 2X + \frac{2^2+2}{2}X^2\right) + \cdots + \left(1 - nX + \frac{n^2+n}{2}X^2\right). \tag{16.25}$$

By applying the formulas of the sum of n first natural numbers and that of their squares, we have:

$$a_{\overline{n}|X} = n - \frac{n(n+1)}{2}X + \frac{n(n+1)}{4}\left(1 + \frac{2n+1}{3}\right)X^2.$$

Therefore,

$$E(a_{\overline{n}|X}) = n - \frac{n(n+1)}{2}E(X) + \frac{n(n+1)}{4}\left(1 + \frac{2n+1}{3}\right)E(X^2). \tag{16.26}$$

In the same way, the formula for the final value of an n-payment annuity, with payments of 1 unit each, made at the end of every year (annuity-immediate), valued at the rate X, after applying the aforementioned discount factor approach, would be as follows:

$$s_{\overline{n}|X} = n + \frac{(n-1)n}{2}X + \frac{(n-1)n}{4}\left(1 + \frac{2n-1}{3}\right)X^2, \tag{16.27}$$

from which

$$E(s_{\overline{n}|X}) = n + \frac{(n-1)n}{2}E(X) + \frac{(n-1)n}{4}\left(1 + \frac{2n-1}{3}\right)E(X^2). \tag{16.28}$$

In the case of an n-payment annuity, with payments of 1 unit each, made at the beginning of every year (annuity-due), valued at the rate X, the expected present and final values are given by:

$$E(\ddot{a}_{\overline{n}|X}) = n - \frac{(n-1)n}{2}E(X) + \frac{(n-1)n}{4}\left(1 + \frac{2n-1}{3}\right)E(X^2)$$

and

$$E(\ddot{s}_{\overline{n}|X}) = n + \frac{n(n+1)}{2}E(X) + \frac{n(n+1)}{4}\left(1 + \frac{2n+1}{3}\right)E(X^2). \tag{16.29}$$

16.4 The Mood et al. Approach

By using the approximate formula of the mathematical expectation of the ratio of two random variables X and Y by Mood et al. (1974), introduced as well by Rice (2006):

$$E\left(\frac{X}{Y}\right) \approx \frac{E(X)}{E(Y)} - \frac{\operatorname{cov}(X,Y)}{[E(Y)]^2} + \frac{E(X)}{[E(Y)]^3}\operatorname{var}(Y), \tag{16.30}$$

we can obtain the expression of the final expected value of an n-payment annuity, with payments of 1 unit each, made at the end of every year (annuity-immediate), valued at the rate X, as follows:

$$E(s_{\overline{n}|U-1}) = E\left(\frac{U^n-1}{U-1}\right) \approx$$

$$\approx \frac{E(U^n-1)}{E(U-1)} - \frac{\operatorname{cov}(U-1, U^n-1)}{[E(U-1)]^2} + \frac{E(U^n-1)}{[E(U-1)]^3}\operatorname{var}(U-1).$$

Thus, taking into account that (Fisz 1963):

$$\mathrm{var}(U-1) = \mathrm{var}(U) = E(U^2) - [E(U)]^2 = \mu_2 - \mu^2$$

and

$$\mathrm{cov}(U-1, U^n - 1) = \mathrm{cov}(U, U^n) = E(U^{n+1}) - E(U^n)E(U) = \mu_{n+1} - \mu_n\mu,$$

we can write

$$E(s_{\overline{n}|U-1}) \approx \frac{\mu_n - 1}{\mu - 1} - \frac{\mu_{n+1} - \mu_n\mu}{(\mu-1)^2} + \frac{\mu_n - 1}{(\mu-1)^3}(\mu_2 - \mu^2). \tag{16.31}$$

Analogously, the final expected value of an n-payment annuity, with payments of 1 unit each, made at the beginning of every year (annuity-due), valued at the rate X, can be deduced. Indeed,

$$E(\ddot{s}_{\overline{n}|U-1}) = E\left(\frac{U^{n-1} - U}{U-1}\right) \approx$$

$$\approx \frac{E(U^{n-1} - U)}{E(U-1)} - \frac{\mathrm{cov}(U^{n+1} - U, U-1)}{[E(U-1)]^2} + \frac{E(U^{n+1} - U)}{[E(U-1)]^3}\mathrm{var}(U-1) =$$

$$= \frac{\mu_{n+1} - \mu}{\mu - 1} - \frac{\mu_{n+2} - \mu_{n+1}\mu - \mu_2 + \mu^2}{(\mu-1)^2} + \frac{\mu_{n+1} - \mu}{(\mu-1)^3}(\mu_2 - \mu^2). \tag{16.32}$$

The expected present value is:

$$E(a_{\overline{n}|U-1}) = E\left(\frac{U^n - 1}{U^{n+1} - U^n}\right) \approx \frac{E(U-1)}{E(U^{n-1} - U)} - \frac{\mathrm{cov}(U^{n+1} - U^n, U^n - 1)}{[E(U^{n+1} - U^n)]^2} +$$

$$+ \frac{E(U^n - 1)}{[E(U^{n+1} - U^n)]^3}\mathrm{var}(U^{n+1} - U^n) =$$

$$= \frac{\mu - 1}{\mu_{n+1} - \mu} - \frac{\mu_{2n+1} - \mu_{n+1}\mu_n - \mu_{2n} + \mu_n^2}{(\mu_{n+1} - \mu_n)^2} +$$

$$+ \frac{\mu_n - 1}{(\mu_{n+1} - \mu_n)^3}(\mu_{2n+2} - \mu_{n+1}^2 + \mu_{2n} - \mu_n^2 - 2(\mu_{2n+1} - \mu_{n+1}\mu_n)),$$

in the case of an n-payment annuity, with payments of 1 unit each, made at the end of every year (annuity-immediate), valued at the rate X, and

$$E(\ddot{a}_{\overline{n}|U-1}) = E\left(\frac{U^{n+1}-U}{U^{n+1}-U^n}\right) \approx \frac{E(U^{n+1}-U)}{E(U^{n+1}-U^n)} - \frac{\mathrm{cov}(U^{n+1}-U^n, U^{n+1}-U)}{[E(U^{n+1}-U^n)]^2} +$$

$$+\frac{E(U^{n+1}-U)}{[E(U^{n+1}-U^n)]^3}\,\mathrm{var}(U^{n+1}-U^n) =$$

$$= \frac{\mu_{n+1}-\mu}{\mu_{n+1}-\mu_n} - \frac{\mu_{2n+2}-\mu_{n+1}^2-2\mu_{n+2}+2\mu_{n+1}\mu+\mu_2-\mu^2}{(\mu_{n+1}-\mu_n)^2} +$$

$$+\frac{\mu_{n+1}-\mu}{(\mu_{n+1}-\mu_n)^3}(\mu_{2n+2}-\mu_{n+1}^2+\mu_{2n}-\mu_n^2-2(\mu_{2n+1}-\mu_{n+1}\mu_n)),$$

in the case of an n-payment annuity, with payments of 1 unit each, made at the beginning of every year (annuity-due), valued at the rate X.

16.5 A Numerical Example of the Expected Final Value of an Annuity by Developing R Functions

Next, we are going to calculate the expected final value of a 6-payment annuity, with payments of 1 unit each, made at the end/beginning of every year (annuity-immediate/ annuity-due) by employing the different expressions developed in this chapter. The present value calculation has been omitted since it can be carried out similarly. Following we present some R functions to perform the proposed approaches. Because we supposed that only some non-central moments of the capitalization factor are known, the three different approaches have been applied:

- The tetraparametric function approach.
- The quadratic discounting approach.
- The Mood et al. approach.

To calculate the mean and variance, we consider the historical data of Table 16.1, specifically the monthly updates of Euribor from January 2015 to April 2016.

Before applying our functions it is necessary to load the data and install two R packages with the following codes:

```
data=c(0.298,0.255,0.212,0.180,0.165,0.163,0.167,0.161,
       0.154,0.128,0.079,0.059,0.042,-0.008,-0.012,-0.002)

install.packages("moments")
library(moments)
install.packages("labstatR")
library(labstatR)
```

Table 16.1 Euribor (January 2015–April 2016). *Source* Bank of Spain

Period	% Euribor in 1 year
January 2015	0.298
February 2015	0.255
March 2015	0.212
April 2015	0.180
May 2015	0.165
June 2015	0.163
July 2015	0.167
August 2015	0.161
September 2015	0.154
October 2015	0.128
November 2015	0.079
December 2015	0.059
January 2016	0.042
February 2016	−0.008
March 2016	−0.012
April 2016	−0.002

Following a preliminary analysis, we obtain the following mean:

$$\mu = 1.12756398,$$

whereas the calculation of the standard deviation gives a value of

$$\sigma = 0.008138746.$$

The minimum and the maximum values of the random variable U are:

$$u_{\min} = a = 0.988$$

and

$$u_{\max} = b = 1.298,$$

respectively.

The Tetraparametric Function Approach

To approximate the final expected value of an n-payment annuity, with payments of 1 unit each, made at the end/beginning of every year (annuity-immediate/annuity-due), valued at the rate X, we formulated two functions to reproduce Eqs. (16.19) and (16.20). The first one computes the final expected value of an n-payment annuity, with payments of 1 unit each, made at the end of every year (annuity-immediate) with random interest rates using the arctangent method:

```
FV_post_artan=function(data, years){
    U=1+data
  u1=mean(U)
  var=sigma2(U)
  u2=sqrt(var+u1^2)
  u_max=1+max(data)
  u_min=1+min(data)
  d=(u_min+u_max)/2
  a=(u_max-u_min)/pi
  b=tan(x=((u2-d)/a))-tan(x=((u1-d)/a))
  c=tan(x=((u1-d)/a))-b
  appo=rep(NA, years)
  s=years-1
  for(i in 0:s) {appo[i+1]=(a*atan(x=(b*i+c))+d)^i}
  final_value=sum(appo)
  return(final_value)
}
```

The second function computes the final expected value of an n-payment annuity, with payments of 1 unit each, made at the beginning of every year (annuity-due) with random interest rates using the arctangent method:

```
FV_pre_artan=function(data, years){
  U=1+data
  u1=mean(U)
  var=sigma2(U)
  u2=sqrt(var+u1^2)
  u_min=1+min(data)
  u_max=1+max(data)
  d=(u_min+u_max)/2
  a=(u_max-u_min)/pi
  b=tan(x=((u2-d)/a))-tan(x=((u1-d)/a))
  c=tan(x=((u1-d)/a))-b
  appo=rep(NA, years)
  for(i in 1:years) {appo[i]=(a*atan(x=(b*i+c))+d)^i}
  final_value=sum(appo)
  return(final_value)
}
```

Using the above codes, we get the following results:

```
> FV_post_artan(data,6)
[1] 8.491768
> FV_pre_artan(data,6)
[1] 9.75462
```

The Quadratic Discounting Approach

To use Eqs. (16.28) and (16.29) for computing the final value of an n-payment annuity, with payments of 1 unit each, made at the end/beginning of every year (annuity-immediate/annuity-due), valued at the rate X, we generate the two different codes:

```
FV_post_quad=function(data,years){
  n=years
  u=mean(data)
  u2=mean(data^2)
  final_value=n+(n*(n-1)/2)*u+(n*(n-1)/4)*(1+(2*n-1)/3)*u2
  return(final_value)
}

FV_pre_quad=function(data,years){
  n=years
  u=mean(data)
  u2=mean(data^2)
  final_value=n+(n*(n+1)/2)*u+(n*(n+1)/4)*(1+(2*n-1)/3)*u2
  return(final_value)
}
```

These functions applied to our data give the following results:

```
> FV_post_quad(data,6)
[1] 8.76782
> FV_pre_quad(data,6)
[1] 9.874948
```

The Approach of Mood et al.

The last method developed in this chapter is the approach by Mood et al.; according to this perspective, we can approximate the final value of an n-payment annuity, with payments of 1 unit each, made at the end of every year (annuity-immediate), valued at the rate $X = U - 1$ Eq. (16.31), by using the following code:

```
FV_post_mood=function(data,years){
  n=years
  m=n+2
  momenti=rep(NA,m)
  U=1+data
  u=mean(U)
  for (i in 1:m) momenti[i]=moment(U,
  central = FALSE, absolute = FALSE, order =i)
  final_value=((momenti[n]-1)/(u-1))-
    ((momenti[n+1]-u*momenti[n])/((u-1)^2))+
    ((momenti[n]-1)/((u-1)^3))*
    (momenti[2]-u^2)
```

```
  return(final_value)
}
```

Analogously, the final value of an n-payment annuity, with payments of 1 unit each, made at the beginning of every year (annuity-due), valued at the rate $X = U - 1$ Eq. (16.32), may be computed by:

```
FV_pre_mood=function(data, years){
  n=years
  m=n+2
  momenti=rep(NA,m)
  U=1+data
  u=mean(U)
  for (i in 1:m) momenti[i]=moment(U,
  central = FALSE, absolute = FALSE, order =i)
  final_value=((momenti[n+1]-u)/(u-1))-
    ((momenti[n+2]-u*momenti[n+1]-momenti[2]+u^2)/
        ((u-1)^2))+
    ((momenti[n+1]-u)/((u-1)^3))*(momenti[2]-u^2)
  return(final_value)
}
```

Following our result (using data of Table 16.1):

```
> FV_post_mood(data,6)
[1] 9.072831
> FV_pre_mood(data,6)
[1] 10.59077
```

Our results show that the three similar give similar results. In summary, for the final value of an n-payment annuity, with payments of 1 unit each, made at the end of every year (annuity-immediate), valued at the rate $X = U - 1$, we obtain the following values:

- Tetraparametric function approach: 8.491.
- Quadratic discounting approach: 8.767.
- Mood et al. approach: 9.072.

Instead, for the final value of an n-payment annuity, with payments of 1 unit each, made at the beginning of every year (annuity-due), valued at the rate $X = U - 1$, we get:

- Tetraparametric function approach: 9.754.
- Quadratic discounting approach: 9.874.
- Mood et al. approach: 10.590.

We highlight that the final values of the Mood et al. approach are always the greatest. However, we underline that the Mood et al. approach is based on the moments; thus, the final value, computed with this last method, is strongly influenced by the distribution of the data. In this application, we assumed a normal distribution when calculating the moments of the distribution; therefore, it is reasonable to infer that if the data do not follow a normal distribution, we get very different results from those obtained with the first two methods.

To highlight this concept, we simulate interest rates following a normal distribution and repeat our test with the following code:

```
> data<-rnorm(n=365,m=0.31,sd=0.075)
> FV_post_artan(data,6)
[1] 13.18
> FV_pre_artan(data,6)
[1] 17.39
> FV_post_quad(data,6)
[1] 14.09
> FV_pre_quad(data,6)
[1] 17.33
> FV_post_mood(data,6)
[1] 13.20
> FV_pre_mood(data,6)
[1] 17.41
```

This simulation shows that, when we deal with interest rates following a normal distribution, the approaches give similar results. However, the more the interest rate are far from normality, the more the Mood et al. approach brings to results different from the tetraparametric function method.

16.6 Conclusion

In this chapter we have presented three methodologies to obtain the value of an annuity whose discount rate is a random variable. The first model is based on the curve representing the mean of order r as a tetraparametric function. On the other hand, the second model is based on the so-defined quadratic discounting, and the third one uses the approximate formula of the expected value of the ratio of two random variables. A comparison among these methodologies is presented with an R application. We considered an n-payment annuity, with payments of 1 unit each, made at the end/beginning of every year (annuity-immediate/annuity-due), valued at a random interest rate. The comparison among the developed methodologies shows that all lead to similar results.

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