The Stability of the Isoperimetric Inequality

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1 Introduction

These lecture notes contain the material that I presented in two summer courses in 2013, one at the Carnegie Mellon University and the other one in a CIME school at Cetraro. The aim of both courses was to give a quick but comprehensive introduction to some recent results on the stability of the isoperimetric inequality.

The starting point is the De Giorgi's proof of the *isoperimetric inequality*. Many other proofs of this inequality are now available. Some of them are classical, like the one based on the *Brunn-Minkowski inequality*, see for instance [15, Theorem 8.1.1], or the one based on the Alexandrov rigidity theorem [2]. More recent proofs are the one based on mass transportation due to Gromov, see Sect. 6, and the PDE proof due to Cabré [16]. Among all these proofs the one by De Giorgi still stands as the most intuitive from a geometric point of view and at the same time the most general one since his isoperimetric inequality (17) applies to any measurable set of finite measure.

In order to explain this proof a few basic properties of sets of finite perimeter are required. They are presented in Sect. 2, while Sect. 3 contains a slightly modified version of the original proof of De Giorgi.

The remaining part of these notes are devoted to the *stability of the isoperimetric inequality*. In fact, once we know that for a given volume balls are the unique area minimizers the next natural question is to understand what happens when a set E has the same volume of a ball B and a slightly bigger surface area. Precisely, one would like to show that in this case E must be close in a proper sense to a translation of B.

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Already a few years after the Hurwitz proof [49] of the isoperimetric inequality in the plane, this problem was studied by Bernstein [8] and later on by Bonnesen [11] for planar convex sets. The case of convex sets in any dimension was settled much later by Fuglede in [39]. Section 4 contains the complete proof of the Fuglede's Theorem 26.

The stability of the isoperimetric inequality for general sets of finite perimeter is a different story, see the discussion at the beginning of Sect. 5. The first result in this direction was proved by Hall [47] in 1992 with a not optimal estimate of the distance between E and the closest ball, while the estimate with the sharp exponent was obtained by Maggi, Pratelli and myself in [44], see Theorem 34. Section 5 contains a fairly detailed discussion of this result, whose proof is based on a suitable symmetrization argument aimed to reduce from a general set of finite perimeter to an axially symmetric bounded set with a center of symmetry.

Other proofs and generalizations of the quantitative isoperimetric inequality (35) were later on obtained by Figalli, Maggi and Pratelli in [34] and by Cicalese and Leonardi in [23], see also [42] and [1]. These alternative proofs are presented in Sect. 6.

The aforementioned papers were the starting point for an intensive study of the stability of other geometric and functional inequalities such as other inequalities of isoperimetric type [3, 5, 6, 9, 10, 22, 24, 25, 31, 41, 46, 56, 58], the *Sobolev inequality* [21, 35, 36, 43], the *Brunn-Minkowski inequality* [33], the *Faber-Krahn inequality* [12, 45] and several others [7, 13, 18, 20, 32, 52]. We shall not discuss here these further developments. The interested reader may have a look at a the survey paper [40] which contains a detailed account of all the recent results, updated to Spring 2015.

Finally, I would like to thank Ryan Murray who typed the notes of the course I gave in Pittsburgh, Matteo Rinaldi who added some extra material from some hand written notes of mine and Laura Bufford and Andrea Fusco for all the pictures.

2 A Quick Review of Sets of Finite Perimeter

We start by reviewing the definition and the main properties of sets of finite perimeter which are the objects for which the isoperimetric inequality will be proved in the next section. A good reference for the results stated here are the books [4, 29, 51] and the original papers of De Giorgi collected in [28]. Note, however that the definition below is equivalent, but different from the one originally proposed by De Giorgi.

In the following we denote by $B_r(x)$ the ball with radius $r > 0$ and center x and we use the following simplified notation

$$B_r := B_r(0), \quad B(x) := B_1(x) \quad B := B_1(0)\,.$$

The measure of the unit ball B will be denoted by ω_n. As a starting point we consider the classical divergence theorem stating that if E is a smooth bounded open set in $\mathbb{R}^n$, and φ is a smooth vector field in $\mathbb{R}^n$ with compact support, then

$$\int_E \operatorname{div}\varphi \, dx = \int_{\partial E} \varphi \cdot \nu \, d\mathcal{H}^{n-1} . \tag{1}$$

Here, if k is a nonnegative integer, by $\mathcal{H}^k$ we denote the k-dimensional *Hausdorff measure* in $\mathbb{R}^n$. Observe that from the previous formula, by taking the supremum over all vector fields $\varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n)$, with $||\varphi||_\infty \le 1$, we get

$$\mathcal{H}^{n-1}(\partial E) = \sup\left\{ \int_E \operatorname{div}\varphi \, dx : \; \varphi \in C^1_c(\mathbb{R}^n;\mathbb{R}^n), ||\varphi||_\infty \le 1 \right\} . \tag{2}$$

Since the first integral in (1) makes sense for any measurable set, equality (2) suggests how to extend the notion of boundary measure to any measurable set $E \subset \mathbb{R}^n$.

Definition 1 Let Ω be an open set in $\mathbb{R}^n$. The *perimeter of E in* Ω is defined as

$$P(E;\Omega) := \sup\left\{ \int_E \operatorname{div}\varphi \, dx : \; \varphi \in C^1_c(\Omega;\mathbb{R}^n), ||\varphi||_\infty \le 1 \right\} .$$

An important feature of this definition is that the perimeter is not affected by modifications on sets of measure zero. Thus the two sets shown in Fig. 1 have the same perimeter. Note also that $P(E;\Omega) = P(\mathbb{R}^n \setminus E;\Omega)$.

Observe that if $P(E;\Omega) < \infty$, then the map

$$\varphi \in C^1_c(\Omega;\mathbb{R}^n) \mapsto \int_E \operatorname{div}\varphi \, dx$$

is linear and continuous with respect to the uniform convergence on $C^1_c(\Omega;\mathbb{R}^n)$. Therefore Riesz's theorem yields that there exists a vector valued Radon measure

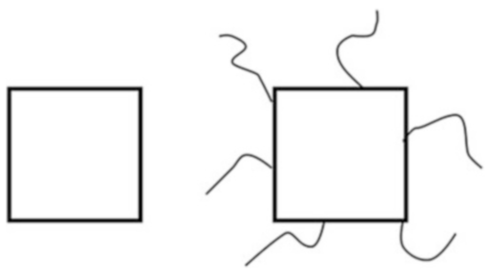


Fig. 1 Two sets with the same perimeter

$\mu = (\mu_1, \ldots, \mu_n)$ in Ω such that

$$\int_\Omega \chi_E \mathrm{div}\varphi \, dx = \int_E \mathrm{div}\varphi \, dx = \int_\Omega \varphi \cdot d\mu = \sum_{i=1}^n \int_\Omega \varphi_i \, d\mu_i$$

for all $\varphi \in C^1_c(\Omega, \mathbb{R}^n)$. Thus $\mu = -D\chi_E$, where $D\chi_E$ is the distributional derivative of χ_E and the above formula can be rewritten as

$$\int_E \mathrm{div}\varphi \, dx = -\int_\Omega \varphi \cdot dD\chi_E. \tag{3}$$

In conclusion, E has finite perimeter in Ω if and only if $D\chi_E$ is a Radon measure with values in $\mathbb{R}^n$ and finite total variation. In fact, from Definition 1 we immediately get that

$$P(E; \Omega) = |D\chi_E|(\Omega).$$

If $\Omega = \mathbb{R}^n$ we simply write $P(E)$ in place of $P(E; \mathbb{R}^n)$ and if $P(E) < \infty$ we say that E is a *set of finite perimeter*. If $P(E; \Omega) < \infty$ for every bounded open set, then we say that E has *locally finite perimeter*. The following properties are immediate consequences of Definition 1. For any measurable set E

$$P(\lambda E) = \lambda^{n-1} P(E) \qquad \text{for all } \lambda > 0\,; \tag{4}$$

moreover, for any open set Ω,

$$P(E; \mathring{E} \cap \Omega) = P(E; \Omega \setminus \overline{E}) = 0$$

Therefore the measure $D\chi_E$ is concentrated on $\partial E \cap \Omega$ and (3) can be rewritten as

$$\int_E \mathrm{div}\varphi \, dx = -\int_{\partial E \cap \Omega} \varphi \cdot D\chi_E, \qquad \text{for all } \varphi \in C^1_c(\Omega; (\mathbb{R}^n). \tag{5}$$

Observe also that from Besicovitch derivation theorem [4, Theorem 2.22] we have that for $|D\chi_E|$–a.e. $x \in \mathrm{supp}|D\chi_E|$ there exists the derivative of $D\chi_E$ with respect to its total variation $|D\chi_E|$ and that it is a vector of length 1. For such points we have

$$\frac{D\chi_E}{|D\chi_E|}(x) = \lim_{r \to 0} \frac{D\chi_E(B_r(x))}{|D\chi_E|(B_r(x))} =: -\nu^E(x) \quad \text{and} \quad |\nu^E(x)| = 1. \tag{6}$$

Definition 2 We shall denote by $\partial^* E$ the set of all points in $\mathrm{supp}|D\chi_E|$ where (6) holds. The set $\partial^* E$ is called the *reduced boundary* of E, while the vector $\nu^E(x)$ is the *generalized exterior normal at* x.

From (6) it follows that the measure $D\chi_E$ can be represented by integrating $-\nu^E$ with respect to $|D\chi_E|$, i.e.,

$$D\chi_E = -\nu^E |D\chi_E|.$$

Thus (5) can be rewritten as

$$\int_E \text{div}\varphi dx = \int_{\partial^* E \cap \Omega} \varphi \cdot \nu^E d|D\chi_E|, \qquad \forall\, \varphi \in C_c^1(\Omega, \mathbb{R}^n). \tag{7}$$

Since $\partial^* E \subset \text{supp}|D\chi_E| \subset \partial E$, the reduced boundary of E is a subset of the topological boundary. Moreover, as a consequence of De Giorgi structure Theorem 6, if E has finite perimeter, then $\mathcal{H}^{n-1}(\partial^* E) = P(E) < \infty$. Next example shows that in general $\partial^* E$ can be much smaller than ∂E.

Example 3 Let us take a sequence $\{q_i\}$ dense in $\mathbb{R}^n$ and set $E := \bigcup_{i=1}^{\infty} B_{2^{-i}}(q_i)$. Observe that $|\partial E| = \infty$. Nevertheless E is a set of finite perimeter. To see this take $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$, $||\varphi||_\infty \leq 1$, and note that

$$\begin{aligned} \int_E \text{div}\varphi\, dx &= \lim_{N\to\infty} \int_{\bigcup_{i=1}^N B_{2^{-i}}(q_i)} \text{div}\varphi\, dx = \lim_{N\to\infty} \int_{\partial(\bigcup_{i=1}^N B_{2^{-i}}(q_i))} \varphi \cdot \nu\, d\mathcal{H}^{n-1} \\ &\leq \lim_{N\to\infty} \mathcal{H}^{n-1}\Big(\partial\Big(\bigcup_{i=1}^N B_{2^{-i}}(q_i)\Big)\Big) \leq \lim_{N\to\infty} \sum_{i=1}^N \mathcal{H}^{n-1}\big(\partial B_{2^{-i}}(q_i)\big) \\ &= n\omega_n \sum_{i=1}^{\infty} 2^{-i(n-1)} < \infty\,. \end{aligned}$$

In dimension 1, sets of finite perimeter are easily characterized (see [4, Proposition 3.52]).

Theorem 4 *Let $E \subset \mathbb{R}$ be a measurable set. Then E has finite perimeter in $\mathbb{R}$ if and only if there exist $-\infty \leq a_1 < b_1 < a_2 < b_2 < \ldots < b_n \leq +\infty$ such that*

$$E = \bigcup_{i=1}^{n} (a_i, b_i)$$

up to a set of zero Lebesgue measure. Moreover, if $\Omega \subset \mathbb{R}$ is an open set,

$$P(E; \Omega) = \#(\{a_i, b_i \in \Omega\}).$$

Remark 5 Thus, if for instance $E = (0, 1) \cup (1, 2)$, then $P(E) = 2$ and $\partial^* E = \{0, 2\}$. In fact, as we already observed, the measure $D\chi_E$ does not change if we modify E by a set of measure zero and thus E and $(0, 2)$ have the same reduced boundary.

The characterization of sets of finite perimeter in $\mathbb{R}^n$ is more complicate and is contained in the next theorem due to De Giorgi. For a proof see [29, Sect. 5.7.3] or [4, Theorem 3.59].

Theorem 6 (De Giorgi) *Let $E \subset \mathbb{R}^n$ be a measurable set of finite perimeter. Then the following hold:*

(i) $\partial^* E$ *is* $(n-1)$*-countably rectifiable, i.e.,* $\partial^* E = \bigcup_{i=1}^{\infty} K_i \cup N_0$, *where* $\mathcal{H}^{n-1}(N_0) = 0$ *and* K_i *are compact subsets of* C^1 *manifolds* M_i *of dimension* $n-1$;

(ii) $|D\chi_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$;

(iii) *for* $\mathcal{H}^{n-1}$*-a.e.* $x \in K_i$, *the generalized exterior normal* $\nu^E(x)$ *is orthogonal to the tangent plane* $T_x M_i$ *to the manifold* M_i *at* x;

(iv) *for all* $x \in \partial^* E$, $\dfrac{|E \cap B_r(x)|}{B_r(x)} \to \dfrac{1}{2}$ *as* $r \to 0$;

(v) *for all* $x \in \partial^* E$, $\lim_{r \to 0} \dfrac{\mathcal{H}^{n-1}(\partial^* E \cap B_r(x))}{\omega_{n-1} r^{n-1}} = 1.$

As a consequence of the equality (ii) above we have that (7) can we rewritten as

$$\int_E \operatorname{div}\varphi \, dx = \int_{\partial E} \varphi \cdot \nu^E d\mathcal{H}^{n-1}, \qquad \forall\, \varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n).$$

Example 7 Let Q be a square in $\mathbb{R}^2$. The reduced boundary is given by $\partial^* Q = \partial Q \setminus \bigcup_{i=1}^{4} \{v_i\}$, where v_i are the vertices of Q. In fact, for any sufficiently small ball $B_r(v_i)$ we have that $|Q \cap B_r(v_i)|/|B_r| = \frac{1}{4}$. Therefore from the property (iv) in Theorem 6 it follows that the v_i do not belong to the reduced boundary $\partial^* Q$, see Fig. 2.

Property (v) tells us that if $x \in \partial^* E$ then the reduced boundary $\partial^* E$ looks flatter and flatter at small scales. Observe in fact that if we rescale $\partial^* E$ around x, we have, see Fig. 3,

$$\mathcal{H}^{n-1}\left(\frac{\partial^* E - x}{r} \cap B\right) = \frac{\mathcal{H}^{n-1}(\partial^* E \cap B_r(x))}{r^{n-1}} \to \omega_{n-1}.$$

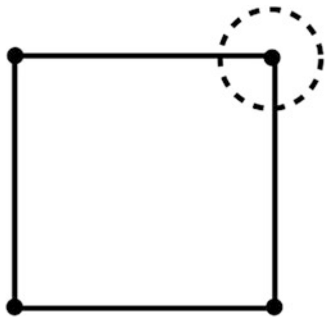


Fig. 2 The density of the vertices is 1/4

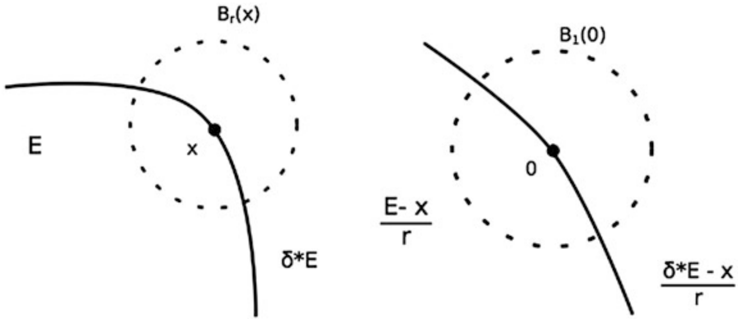


Fig. 3 Rescaling around x

Definition 8 Given a measurable set E and $x \in \mathbb{R}^n$, the *density of* E *at* x, $D(x; E)$, is defined as

$$D(x; E) := \lim_{r \to 0} \frac{|E \cap B_r(x)|}{\omega_n r^n}.$$

If $0 \leq a \leq 1$ we denote by $E^{(a)}$ the set of all points where the density of E is equal to a.

Observe that from the above definition it follows immediately that

$$x \in E^{(1)} \text{ if and only if } \lim_{r \to 0} \frac{|E \cap Q_r(x)|}{2^n r^n} = 1, \tag{8}$$

where $Q_r(x)$ is the cube with center at x with edge length equal to $2r$ and faces parallel to the coordinate planes. A similar characterization holds also for the points in $E^{(0)}$.

Using densities, part (iv) of De Giorgi's Theorem 6 can be written as $\partial^* E \subset E^{(1/2)}$. We recall also that if E is a measurable set in $\mathbb{R}^n$ its *measure theoretic boundary* $\partial^M E$ is defined by setting

$$\partial^M E := \mathbb{R}^n \setminus \left(E^{(0)} \cup E^{(1)}\right). \tag{9}$$

The next result gives a precise description of what is going on with sets of finite perimeter. For the proof see for instance [4, Theorem 3.61].

Theorem 9 (Federer) *Let E be a set of finite perimeter in $\mathbb{R}^n$. Then*

$$\partial^* E \subset E^{(1/2)} \subset \partial^M E \qquad \textit{and} \qquad \mathcal{H}^{n-1}\left(\partial^M E \setminus \partial^* E\right) = 0.$$

Note that if E is a set of finite perimeter in Ω Theorems 6 and 9 hold in local form.

Example 10 Let $U \subset \mathbb{R}^{n-1}$ be a bounded open set and $\Omega = U \times \mathbb{R}$. Let $f : U \to \mathbb{R}$ be a Lipschitz function. Let us denote by $\mathcal{S}_f := \{(x,t) \in \Omega : t < f(x)\}$ the *subgraph* of f. Then it may be easily checked that $\mathcal{S}_f$ has finite perimeter in Ω and that $\partial^* \mathcal{S}_f$ coincides with $\Gamma_f := \{(x, f(x)) : x \in U\}$ up to a set of zero $\mathcal{H}^{n-1}$ measure. Moreover, the generalized normal $\nu^{\mathcal{S}_f}(x)$ coincides $\mathcal{H}^{n-1}$-a.e. on Γ_f with the usual exterior normal $\dfrac{(-\nabla f, 1)}{\sqrt{1 + |\nabla f|^2}}$.

Example 11 Let $f : \mathbb{R} \to \mathbb{R}$ be the function $f(x) = x^2 \sin \frac{1}{x}$ and let $E := \mathcal{S}_f$ be the subgraph of f. Using the fact that $f'(0) = 0$ we get easily that

$$\frac{|E \cap B_r|}{|B_r|} \to \frac{1}{2}.$$

However $(0, 0) \notin \partial^* E$ since it can be checked that

$$\limsup_{r \to 0} \frac{\mathcal{H}^1(\partial^* E \cap B_r)}{2r} > 1 .$$

Thus property (v) stated in Theorem 6 does not hold.

Approximating sets of finite perimeter with nicer sets is very useful to deduce various properties from the corresponding ones of smooth sets. To this aim we introduce the following notion of convergence.

Definition 12 Given a sequence of measurable sets E_j and a measurable set E, we say that $E_j \to E$ *in measure* in Ω if $\chi_{E_j} \to \chi_E$ in $L^1(\Omega)$, i.e., $|(E_j \Delta E) \cap \Omega| \to 0$, as $j \to \infty$.

An important property of the perimeters is the lower semicontinuity with respect to the convergence in measure. This is a straightforward consequence of Definition 1. Precisely, if E_j is a sequence of measurable sets converging in measure in Ω to E, then

$$P(E; \Omega) \leq \liminf_{j \to \infty} P(E_j; \Omega).$$

For the proof of the next approximation result see for instance [4, Theorem 3.42].

Theorem 13 *Let E be a set of finite perimeter. Then there exists a sequence of smooth, bounded open sets E_j such that $E_j \to E$ in measure in $\mathbb{R}^n$ and $P(E_j) \to P(E)$.*

In view of this theorem and of the lower semicontinuity of the perimeter we have that E is a set of finite perimeter in $\mathbb{R}^n$ if and only if there exists a sequence of smooth open sets $E_j \subset \mathbb{R}^n$, such that

$$E_j \to E \text{ in measure in } \mathbb{R}^n \quad \text{and} \quad \sup_j P(E_j) < \infty.$$

Note also that in Theorem 13 one may replace the smooth sets E_j with polyhedra, i.e., bounded open sets obtained as the intersection of finitely many half–spaces. A local version of Theorem 13 is also true (see [4, Remark 3.43]). As a consequence of Theorem 13 observe that if E and F are sets of finite perimeter, the same is true for $E \cup F$, $E \cap F$ and $E \setminus F$ and that

$$P(E \cap F) + P(E \cup F) \leq P(E) + P(F)\,.$$

Simple examples show that the above inequality may be strict. In general the precise expression of the reduced boundaries of $E \cap F$ or $E \cup F$ in terms of the reduced boundaries of E and F is a little involved. The next statement provides the precise picture. For a proof see for instance [34, (2.8), (2.9) and Lemma 2.2].

Proposition 14 *Let $E, F \subset \mathbb{R}^n$ be sets of finite perimeter. Then, up to a set of zero $\mathcal{H}^{n-1}$ measure*

$$\partial^*(E \cap F) = \{y \in \partial^* E \cap \partial^* F :\ \nu^E(y) = \nu^F(y)\} \cup [\partial^* E \cap F^{(1)}] \cup [\partial^* F \cap E^{(1)}]$$

and for $\mathcal{H}^{n-1}$-a.e. $x \in \partial^*(E \cap F)$

$$\nu^{E\cap F}(x) = \begin{cases} \nu^E(x) = \nu^F(x) & \text{if } x \in \{y \in \partial^* E \cap \partial^* F :\ \nu^E(y) = \nu^F(y)\}, \\ \nu^E(x) & \text{if } x \in \partial^* E \cap F^{(1)}, \\ \nu^F(x) & \text{if } x \in \partial^* F \cap E^{(1)}\,. \end{cases}$$

Moreover, if $|E \cap F| = 0$, then, up to a set of zero $\mathcal{H}^{n-1}$-measure, $\partial^*(E \cup F) = \partial^* E \Delta \partial^* F$ and

$$\nu^{E\cup F}(x) = \begin{cases} \nu^E(x) & \text{if } x \in \partial^* E \setminus \partial^* F, \\ \nu^F(x) & \text{if } x \in \partial^* F \setminus \partial^* E\,. \end{cases}$$

The next result, is just the Rellich-Kondrachov compactness theorem stated in the framework of sets of finite perimeter (see [4, Theorem 3.39]).

Theorem 15 *Given a bounded open set $\Omega \subset \mathbb{R}^n$ and a sequence of measurable sets E_j such that $\sup_j P(E_j; \Omega) < \infty$, there exists a set E of finite perimeter in Ω such that, up to a subsequence, $E_j \to E$ in measure in Ω.*

The theory of sets of finite perimeter can be viewed as a special part of the theory of *functions of bounded variation*. Recall that if Ω is an open set a function $u \in L^1(\Omega)$ is said to be of bounded variation if the distributional gradient Du is a vector-valued measure in Ω with finite total variation. Observe that by definition of distributional gradient

$$\int_\Omega u \,\mathrm{div}\varphi \, dx = -\int_\Omega \varphi \, dDu,$$

for all C^1 vector fields φ with compact support in Ω. From this formula it follows immediately that the total variation $|Du|(\Omega)$ of Du in Ω is given by

$$|Du|(\Omega) = \sup\left\{\int_\Omega u\,\mathrm{div}\varphi\,dx : \varphi \in C^1_c(\Omega;\mathbb{R}^n), ||\varphi||_\infty \le 1\right\}.$$

We shall denote by $BV(\Omega)$ the space of all functions of bounded variation in Ω.

We conclude by recalling the *coarea formula* for sets of finite perimeter. For our purposes it will be enough to consider only C^1 maps, though these formulas may easily generalized to Lipschitz and even less regular maps, see [4, Chap. 2] and [30, Sect. 3.2]. Thus, let $f : \mathbb{R}^n \to \mathbb{R}^k$ be a C^1 map, $1 \le k \le n-1$, and E a set of finite perimeter. By Definition 2 at every point x of the reduced boundary $\partial^* E$ we have a generalized exterior normal $\nu^E(x)$, hence a generalized tangent plane, that we denote by $T_x\partial^* E$. Therefore, we can consider the *tangential differential* of f at x, that is the map $df(x) : T_x\partial^* E \to \mathbb{R}^k$ given by

$$df(x)(\tau) = \nabla f(x)(\tau), \qquad \text{for all } \tau \in T_x\partial^* E. \tag{10}$$

Furthermore, we define the *coarea factor at x* as

$$\mathbf{C}_k df(x) = \sqrt{\det(df(x)\circ(df(x))^T)},$$

where $(df(x))^T$ is the transpose of the matrix $df(x)$. It can be shown that $\mathbf{C}_k df(x)$ is the square root of the sum of the squares of the k–order minors of the matrix representing $df(x)$ with respect to a base in $T_x\partial^* E$ and a base in $\mathbb{R}^k$ (see [4, (2.71)]).

Theorem 16 (Coarea Formula for Sets of Finite Perimeter) *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter and $f : \mathbb{R}^n \to \mathbb{R}^k$ a C^1 map, $1 \le k \le n-1$. If $g : \mathbb{R}^n \to [0,+\infty]$ is a Borel function, then*

$$\int_{\partial^* E} g(x)\mathbf{C}_k df(x)\,d\mathcal{H}^{n-1}(x) = \int_{\mathbb{R}^k} dz \int_{f^{-1}(z)\cap\partial^* E} g(x)\,d\mathcal{H}^{n-1-k}(x).$$

Observe that if $\pi : \mathbb{R}^n \to \mathbb{R}^k$ is the projection over the first k components, i.e. $\pi(x,y) = x$ for all $(x,y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, then $\mathbf{C}_k d\pi(x,y) = |\nu^E_y(x,y)|$, for all $(x,y) \in \partial^* E$, where $\nu^E = (\nu^E_x, \nu^E_y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$. To prove this consider an orthonormal base $\{\tau_1,\dots,\tau_{n-1}\}$ for $T_{(x,y)}\partial^* E$, such that the frame $\{\tau_1,\dots,\tau_{n-1},\nu^E(x,y)\}$ is positively oriented. Then the matrix representing $d\pi(x,y)$ with respect to the given orthonormal base of $T_{(x,y)}\partial^* E$ and the standard base $\{e_1,\dots,e_k\}$ in $\mathbb{R}^k$ has coefficients $e_i \cdot \tau_\ell$, for $i = 1,\dots,k$, $\ell = 1,\dots,n-1$. Therefore, the matrix representing $\det(d\pi(x)\circ(d\pi(x))^T)$ has coefficients

$$a_{ij} := \sum_{\ell=1}^{n-1}(e_i\cdot\tau_\ell)(e_j\cdot\tau_\ell) = \delta_{ij} - \nu^E_i\nu^E_j \qquad \text{for } i,j = 1,\dots,k.$$

Recall that if $a, b \in \mathbb{R}^k$ and I denotes the identity matrix, then

$$\det(I + a \otimes b) = 1 + a \cdot b\,. \tag{11}$$

Thus $\mathbf{C}_k d\pi(x,y) = \sqrt{\det(a_{ij})} = \sqrt{1 - |\nu_x^E|^2} = |\nu_y^E|$ and the coarea formula reduces to

$$\int_{\partial^* E} g(x,y)|\nu_y^E(x,y)|d\mathcal{H}^{n-1}(x,y) = \int_{\mathbb{R}^k} dx \int_{(\partial^* E)_x} g(x,y) d\mathcal{H}^{n-1-k}(y), \tag{12}$$

where

$$(\partial^* E)_x = \{y \in \mathbb{R}^{n-k} : (x,y) \in \partial^* E\},$$

see Fig. 4. If we apply (12) to the particular case of the projection over the first $n-1$ components, recalling that $\mathcal{H}^0$ is the counting measure, we have that for every Borel function $g : \mathbb{R}^n \to [0, +\infty]$.

$$\int_{\partial^* E} g(x,y)|\nu_y^E(x,y)|d\mathcal{H}^{n-1}(x,y) = \int_{\mathbb{R}^{n-1}} \Big(\sum_{y \in (\partial^* E)_x} g(x,y) \Big) dx\,. \tag{13}$$

From this formula we deduce that the vertical part of the reduced boundary $\partial^* E$ "is not seen from below".

To be precise, let us define the *vertical part* of the reduced boundary by setting $V := \{(x,y) \in \partial^* E : \nu_y^E(x,y) = 0\}$. If we apply (13) with $g = \chi_V$ we get $\int_{\partial^* E} \chi_V(x,y)|\nu_y^E(x,y)|d\mathcal{H}^{n-1}(x,y) = 0$. Therefore, the right hand side of (13) is

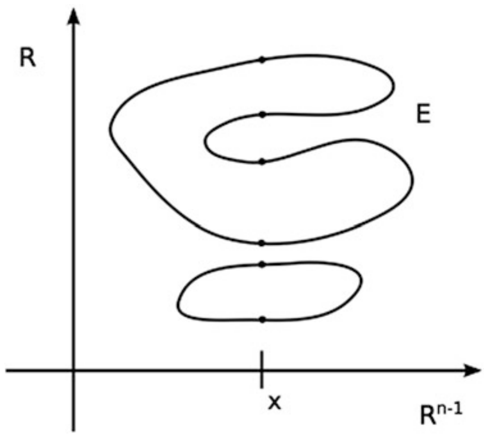
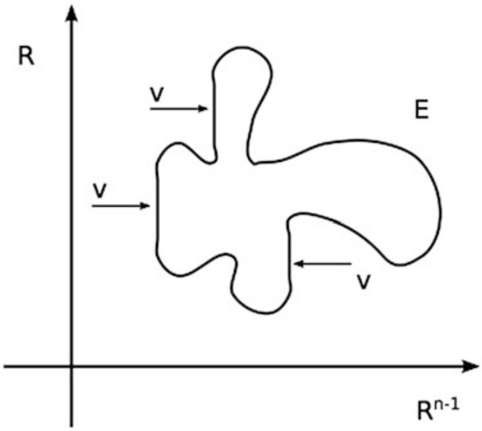


Fig. 4 Section of $\partial^* E$

Fig. 5 The set V



also zero, i.e.,

$$\int_{\mathbb{R}^{n-1}} \#\left(\{y \in \mathbb{R} : (x, y) \in V\}\right) dx = 0.$$

This implies that for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^{n-1}$, the section V_x is empty, see Fig. 5.

3 De Giorgi's Proof of the Isoperimetric Inequality

In the framework of sets of finite perimeter the isoperimetric inequality takes the following very general form.

Theorem 17 *Let $E \subset \mathbb{R}^n$ be a measurable set with $|E| = |B_r|$. Then*

$$P(B_r) \leq P(E) \tag{14}$$

with the equality holding if and only if E is a ball.

De Giorgi's proof follows an idea that Steiner had one century before [59]. Actually, the proof of the isoperimetric property of the ball was the original motivation for Steiner to introduce the symmetrization that nowadays bears his name.

Definition 18 Let $E \subset \mathbb{R}^n$ be a measurable set. For $x \in \mathbb{R}^{n-1}$ set $E_x := \{y \in \mathbb{R} : (x, y) \in E\}$ and $\ell(x) := \mathcal{H}^1(E_x)$. Then the *Steiner symmetrization of E* with respect to the hyperplane $\{x_n = 0\}$ is given by $E^s = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : -\ell(x)/2 < y < \ell(x)/2\}$.

The previous definition can be extended in an obvious way to any hyperplane π_ν passing through the origin and orthogonal to a unit vector ν. The resulting Steiner

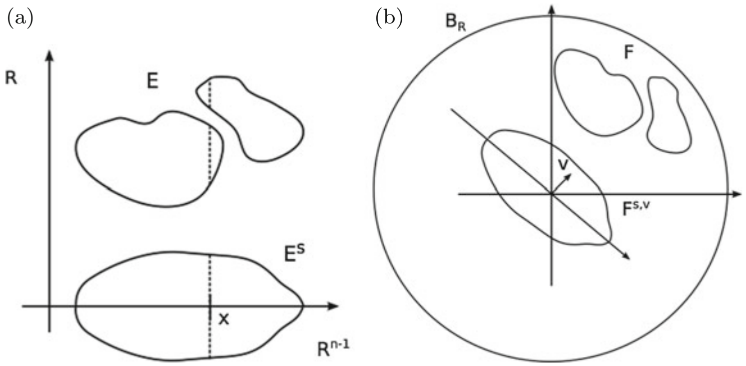


Fig. 6 Steiner symmetral of a measurable set. (**a**) Symmetrization wrt the plane $\{x_n = 0\}$. (**b**) Symmetrization wrt a plane with normal $\nu \neq e_n$

symmetrization of E with respect to π_ν will be denoted by $E^{s,\nu}$, see Fig. 6. The symmetrization of E with respect to $\{x_n = 0\}$ will be denoted by E^s.

From Fubini's theorem we have immediately that $|E| = |E^{s,\nu}|$, while it is not too difficult to show, see for instance [29, Lemma 2, Sect. 2.2], that $\mathrm{diam}(E^{s,\nu}) \leq \mathrm{diam}(E)$. If E is a measurable set, then ℓ is a measurable function. Instead, if E is a set of finite perimeter it can be proved that ℓ is a function of bounded variation in $\mathbb{R}^{n-1}$ and even a Sobolev function if $\partial^* E$ has no vertical part. However, for the proof of the isoperimetric inequality the relevant fact is that the Steiner symmetrization of a set keeps the volume and decreases the perimeter.

Theorem 19 *Let E be a set of finite perimeter with $|E| < \infty$. Then the following properties hold:*

(i) $\ell \in BV(\mathbb{R}^{n-1})$*;*
(ii) $\ell \in W^{1,1}(\mathbb{R}^{n-1})$ *if and only if* $\mathcal{H}^{n-1}(\{z \in \partial^* E : \nu_n^E(z) = 0\}) = 0$*;*
(iii) $P(E^s) \leq P(E)$*;*
(iv) *if* $P(E^s) = P(E)$ *then for* $\mathcal{H}^{n-1}$*-a.e.* $x \in \mathbb{R}^{n-1}$, E_x *coincides up to a set of zero* $\mathcal{H}^1$ *measure with a line segment.*

Inequality (iii) is classical and is proved for smooth sets in the beautiful book of Pólya–Szegö [57]. Property (iv) appears in a weaker form in De Giorgi's original paper on the isoperimetric property of balls [27]. The above statement of (iv) as well as (i) and (ii) are proved in [19, Theorem 1.1, Lemma 3.1, Proposition 1.2].

Note that if $P(E) = P(E^s)$, then E and E^s are not necessarily equal up to a translation, as shown in Fig. 7. In both pictures $P(E) = P(E^s)$, conclusion (iv) of the theorem holds but $E \neq E^s$ up to a translation. However, it is possible to characterize the cases when the equality $P(E) = P(E^s)$ implies that E and E^s coincide up to a translations, see [19] and [17], where a deeper analysis is carried on. We now turn to the proof of the isoperimetric inequality via Steiner symmetrization.

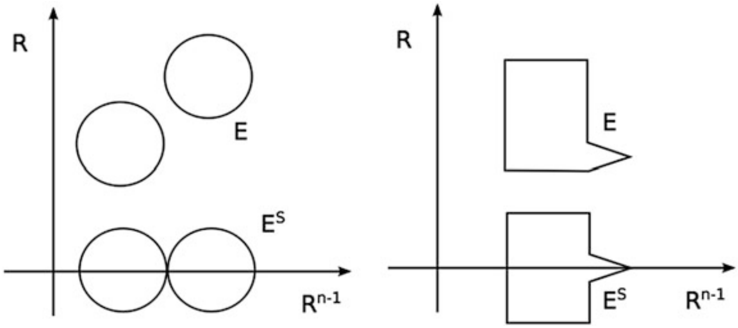


Fig. 7 In general, symmetrals are not translated of the original sets

Proof From the rescaling property (4) it follows that in order to prove (14) it is enough to show that $P(E) \geq P(B)$ for all sets E such that $|E| = |B| = \omega_n$, with the equality holding if and only if E is a ball.

Step 1. We first fix B_R, with $R > 1$ and consider the minimum problem

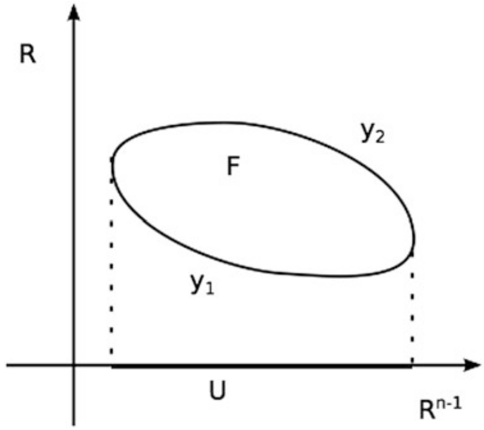
$$\inf\{P(E) : \ E \subset B_R, \ |E| = \omega_n\}.$$

Observe that the above infimum is always attained. In fact, let $E_j \subset B_R$, with $|E_j| = \omega_n$, be a minimizing sequence, i.e., $\lim_j P(E_j) = \inf\{P(E) : E \subset B_R, \ |E| = \omega_n\}$. By the compactness Theorem 15 we may assume, up to a not relabelled subsequence, that E_j converge in measure to some set $F \subset B_R$ with $|F| = \omega_n$. By the lower semicontinuity of the perimeter we have $P(F) \leq \liminf P(E_j)$ and thus F is a minimizer.

We claim that F coincides, up to a set of measure zero, with a convex set. To prove this, fix $\nu \in \mathbb{S}^{n-1}$ and consider the Steiner symmetrization $F^{s,\nu}$ of F with respect to the hyperplane π_ν passing through the origin and orthogonal to ν. Observe that $|F^{s,\nu}| = |F| = \omega_n$ and that $F^{s,\nu} \subset B_R$. Moreover, from part (iii) of Theorem 19 we have that $P(F^{s,\nu}) \leq P(F)$ and thus, by the minimality of F, we may conclude that $P(F^{s,\nu}) = P(F)$. Thus, recalling the property (iv) stated in Theorem 19, we have that for $\mathcal{H}^{n-1}$-a.e. $x \in \pi_\nu$, the section $\{t \in \mathbb{R} : x + t\nu \in F\}$ coincides up to a set of $\mathcal{H}^1$ measure zero with an open interval. By the arbitrariness of ν this property clearly holds for all directions $\nu \in \mathbb{S}^{n-1}$. Notice that if we knew that each section $\{t \in \mathbb{R} : x + t\nu \in F\}$ is an open interval for any $\nu \in \mathbb{S}^{n-1}$ and any $x \in \pi_\nu$, then we could conclude at once that F is a convex set. Although this may be not true, Lemma 20 guarantees that there exists a set equivalent to F up to a set of zero Lebesgue measure which has this property. This set is precisely $F^{(1)}$, the set of all points where F has density 1. Hence, $F^{(1)}$ is an open convex set.

To simplify the notation let us set $F = F^{(1)}$. Our goal now is to show that F is a ball. Denote by U the projection of F on $\mathbb{R}^{n-1}$. Then there exist two functions $y_1, y_2 : U \to \mathbb{R}$, y_1 convex and y_2 concave, such that $F = \{(x, y) : \ x \in U, \ y_1(x) <$

Fig. 8 Projection of F



$y < y_2(x)\}$, see Fig. 8. Moreover, $F^s = \{(x, y) : x \in U,\ (y_1 - y_2)(x)/2 < y < (y_2 - y_1)(x)/2\}$. We have:

$$P(F) = \int_U \sqrt{1 + |\nabla y_1|^2} + \int_U \sqrt{1 + |\nabla y_2|^2}, \qquad P(F^s) = 2\int_U \sqrt{1 + |\nabla (y_2 - y_1)/2|^2}.$$

Since F is a minimizer, $P(F) = P(F^s)$ and thus by the strict convexity of the function $t \mapsto \sqrt{1+t^2}$ we get that $\nabla y_2 = -\nabla y_1$, hence $y_2 = -y_1 + c$, thus proving that $F = F^s$ up to a translation. Repeating this argument for all the Steiner symmetrizations $F^{s,\nu}$, with $\nu \in \mathbb{S}^{n-1}$, we finally conclude that F must be a ball. This proves the isoperimetric inequality for a bounded set E.

Step 2. Let us now consider the case of an unbounded set E with $|E| = \omega_n$. From Theorem 13 we get a sequence of smooth bounded sets E_j such that E_j converge in measure to E in $\mathbb{R}^n$ and $P(E_j) \to P(E)$ as $j \to \infty$. From Step 1 we then have that $P(E_j) \geq P(B_{r_j})$ where $|E_j| = |B_{r_j}|$. From this inequality and using the fact that $|E_j| \to |E|$, letting $j \to \infty$, we have that $P(E) \geq P(B)$. Finally, if $P(E) = P(B)$ we may repeat the same argument used in Step 1 to conclude first that $E^{(1)}$ is an open convex set and then that it is a ball. □

Let us now give the proof of the technical lemma used before. Note that this lemma was not explicitly stated in the original paper [27]. For the proof below I thank Giovanni Alberti with whom I discussed the issue a few years ago. To this aim, given a measurable set E, we denote by $\pi(E)^+$ the *essential projection* of E over the first $n-1$ coordinates plane, that is

$$\pi(E)^+ := \{x \in \mathbb{R}^{n-1} : \ \mathcal{H}^1(E_x) > 0\}\,.$$

Lemma 20 *Let E be a measurable set in $\mathbb{R}^n$ such that for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{R}^{n-1}$ the section $E_x = \{y \in \mathbb{R} : (x, y) \in E\}$ is equivalent to a segment up to a set of zero $\mathcal{H}^1$*

measure. Then, denoting by F the set of points of density 1 *with respect to E,* F_x *is a segment for every* $x \in \mathbb{R}^{n-1}$.

Proof Let $z_1 = (x, y_1)$, $z_2 = (x, y_2)$ be two points in F_x with $y_1 < y_2$. Let us fix $\overline{y} \in (y_1, y_2)$. We claim that $\overline{z} = (x, \overline{y}) \in F_x$. Since E has density 1 at x_1 and x_2 the same is true also for F. Therefore, given $\varepsilon > 0$, there exists r_ε such that, if $0 < r < r_\varepsilon$, then, see (8),

$$\frac{|F \cap Q_r(z_i)|}{2^n r^n} > 1 - \varepsilon \qquad \text{for } i = 1, 2\,.$$

By Fubini's theorem we have that

$$2^n r^n (1-\varepsilon) < |F \cap Q_r(z_i)| = \int_{\pi(F\cap Q_r(z_i))^+} \mathcal{H}^1((F \cap Q_r(z_i))_x)\, dx \le 2r\mathcal{H}^{n-1}(\pi(F\cap Q_r(z_i))^+)$$

and thus

$$\mathcal{H}^{n-1}(\pi(F \cap Q_r(z_i))^+) > 2^{n-1} r^{n-1}(1-\varepsilon) \qquad \text{for } i = 1, 2\,. \tag{15}$$

Since the essential projections of $F \cap Q_r(z_1)$ and $F \cap Q_r(z_2)$ are both contained in the same $(n-1)$-dimensional cube of edge length $2r$, from (15) we get that

$$\mathcal{H}^{n-1}\big(\pi(F \cap Q_r(z_1))^+ \cap \pi(F \cap Q_r(x_2))^+\big) > 2^{n-1} r^{n-1}(1-2\varepsilon)\,. \tag{16}$$

Now, recall that by assumption for $\mathcal{H}^{n-1}$-a.e. $x \in \pi(F \cap Q_r(z_1))^+ \cap \pi(F \cap Q_r(z_2))^+$ the set F_x is equivalent to a segment such that $\mathcal{H}^1(F_x \cap Q_r(z_i)) > 0$ for $i = 1, 2$. Therefore, if we take $r_\varepsilon < \dfrac{1}{2}\min\{\overline{y} - y_1, y_2 - \overline{y}\}$, we get that

$$\mathcal{H}^1(F_x \cap Q_r(\overline{z})) = 2r\,.$$

This inequality, together with (16) implies that

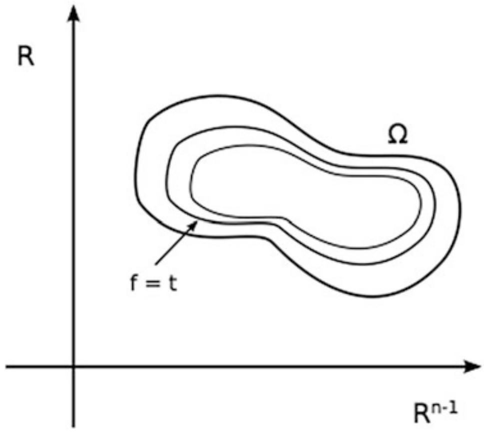
$$|F \cap Q_r(\overline{x})| > 2^n r^n (1-2\varepsilon), \qquad \text{for all } r < r_\varepsilon\,.$$

Therefore, letting first $r \to 0$ and then $\varepsilon \to 0$, we immediately get that F has density 1 at $\overline{z}$ and thus $\overline{z} \in F$. Hence the result follows. □

An equivalent way of stating the isoperimetric inequality can be obtained noting that if $|E| = |B_r|$ for some $r > 0$, then $|E| = \omega_n r^n$ and $P(B_r) = n\omega_n r^{n-1}$. Therefore (14) becomes

$$P(E) \ge n\omega_n^{1/n} |E|^{1-1/n}. \tag{17}$$

Fig. 9 The level sets of f



Observe that since $P(E) = |D\chi_E|(\mathbb{R}^n)$ and $\|\chi_E\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)} = |E|^{1-1/n}$ this inequality can be viewed as a particular case of the Sobolev inequality for $W^{1,1}(\mathbb{R}^n)$ or $BV(\mathbb{R}^n)$. To understand better this connection we need to introduce an important formula, first proved by Fleming and Rishel in [37]. As shown in the picture below, it is a sort of curvilinear version of the familiar Fubini theorem, Fig. 9.

Theorem 21 (Coarea Formula for Lipschitz Function) *Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{R}$ a Lipschitz function. Then $\{f > t\}$ is a set of finite perimeter for $\mathcal{H}^1$-a.e. $t \in \mathbb{R}$. Moreover, if $g : \Omega \to [0, +\infty]$ is a Borel function,*

$$\int_\Omega g(x)|\nabla f|\, dx = \int_{\mathbb{R}} dt \int_{\partial^*\{f>t\}} g(x)\, d\mathcal{H}^{n-1}(x). \tag{18}$$

The next result shows that the isoperimetric inequality (17) is equivalent to the Sobolev inequality (with the same constant).

Theorem 22 *The following statements are equivalent:*

(1) *for all measurable set E with finite measure $P(E) \geq C_0|E|^{\frac{n-1}{n}}$;*
(2) *for all $f \in W^{1,1}(\mathbb{R}^n)$ we have that $\|\nabla f\|_{L^1(\mathbb{R}^n)} \geq C_0\|f\|_{L^{\frac{n}{n-1}}(\mathbb{R}^n)}$.*

Proof To show that the Sobolev inequality (2) implies the isoperimetric inequality (1), we use mollifiers. For $\varepsilon > 0$ set $f_\varepsilon := \rho_\varepsilon * \chi_E$, where $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$ is a standard mollifier. Note that $f_\varepsilon \in W^{1,1}(\mathbb{R}^n)$ and that $f_\varepsilon \to \chi_E$ a.e. in $\mathbb{R}^n$. Then, fix $\varphi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ with $||\varphi||_\infty \leq 1$. Using the definition of f_ε, performing a change of variable and recalling Definition 1, we easily get

$$\begin{aligned}-\int_{\mathbb{R}^n} \nabla f_\varepsilon \cdot \varphi\, dx &= \int_{\mathbb{R}^n} f_\varepsilon \operatorname{div}\varphi\, dx = \int_{\mathbb{R}^n} dx \int_{\mathbb{R}^n} \rho_\varepsilon(z)\chi_E(x-z) \operatorname{div}\varphi(x)dz \\ &= \int_{\mathbb{R}^n} \rho_\varepsilon(z)dz \int_{\mathbb{R}^n} \chi_E(y) \operatorname{div}\varphi(y+z)dy \leq P(E) \int_{\mathbb{R}^n} \rho_\varepsilon(z)dz = P(E).\end{aligned}$$

Taking the supremum over all such φ, from (2) we get

$$P(E) \geq \int_{\mathbb{R}^n} |\nabla f_\varepsilon| \geq C_0 \|f_\varepsilon\|_{\frac{n}{n-1}}.$$

Hence (1) follows, letting $\varepsilon \to 0$ and recalling that $f_\varepsilon(x) \to \chi_E(x)$ for a.e. $x \in \mathbb{R}^n$.

To prove that the isoperimetric inequality implies the Sobolev inequality we are going to use the coarea formula (18). Note that by density it is enough to prove (2) for a function $f \in C_c^1(\mathbb{R}^n)$. Moreover, splitting f in its positive and negative part, we may always assume without loss of generality that $f \geq 0$. Then, for any $t \geq 0$ we truncate f from below by setting $f_t := \min\{f, t\}$. We set also $\phi(t) := \|f_t\|_{\frac{n}{n-1}}$. Note that ϕ is an increasing function and that for $h > 0$

$$\phi(t+h) - \phi(t) \leq \|f_{t+h} - f_t\|_{\frac{n}{n-1}} \leq h|\{f > t\}|^{1-1/n}$$

Thus ϕ is Lipschitz and $\phi'(t) \leq |\{f > t\}|^{1-1/n}$ for $\mathcal{H}^1$-a.e. $t \in \mathbb{R}$. Furthermore, using the isoperimetric inequality (14), we have

$$\begin{aligned}
\|f\|_{\frac{n}{n-1}} &= \lim_{t\to+\infty} \phi(t) = \int_0^{+\infty} \phi'(s)ds \leq \int_0^{+\infty} |\{f > s\}|^{1-1/n} ds \\
&\leq C_0^{-1} \int_0^{+\infty} P(\{f > s\})ds = C_0^{-1} \int_{-\infty}^{+\infty} ds \int_{\partial^*\{f>s\}} d\mathcal{H}^{n-1} = C_0^{-1} \int_{\mathbb{R}^n} |\nabla f|,
\end{aligned}$$

where the last equality follows from (18) with $g \equiv 1$. □

4 Stability of the Isoperimetric Inequality: Convex and Nearly Spherical Sets

After having proved the isoperimetric inequality we now turn to the next issue, namely the stability of this inequality. In other words, if E is a set such that $|E| = |B_r|$ and $P(E) = P(B_r) + \delta$ for some small δ, can we say that E is somehow close to a ball? And how can we measure the distance from a ball in terms of δ?

The first results in this direction were proven for planar convex sets by Bernstein [8] in 1905 and Bonnesen [11] in 1924. As we shall see in this section, it took some time before the problem was completely solved for convex sets in any dimension.

Theorem 23 (Bonnesen) *Given a convex set $E \subset \mathbb{R}^2$, with $|E| = |B|$, there exist two concentric disks $B_{r_1}(x_0) \subset E \subset B_{r_2}(x_0)$ such that*

$$(r_2 - r_1)^2 \leq \frac{P^2(E) - P^2(B)}{4\pi}. \tag{19}$$

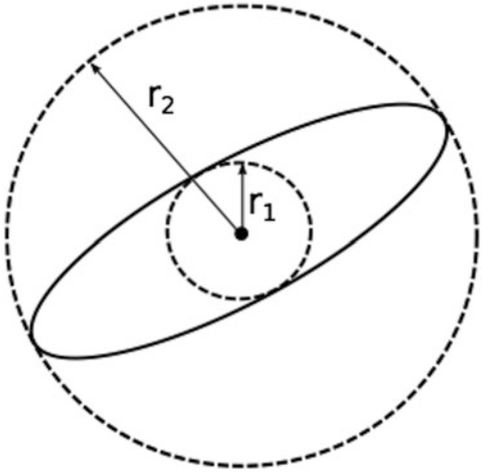


Fig. 10 Bonnesen's theorem

Figure 10 illustrates the statement of the theorem. A remarkable feature of inequality (19) is that the constant appearing on the right hand side is optimal. However, we cannot expect to prove also in higher dimension such a precise inequality. Thus, it may be useful to restate it in a weaker form that we may hope to extend to the general n-dimensional case. To this aim, observe that from (19) it follows that if $P(E) - P(B) \leq 1$ there exists $x_0 \in \mathbb{R}^2$ such that

$$d_H^2(E, B(x_0)) \leq C(P(E) - P(B))$$

for some positive constant C. Here and in the following we denote by

$$d_H(E, F) := \inf\{\varepsilon > 0 : \ E \subset F + B_\varepsilon, \ F \subset E + B_\varepsilon\}$$

the *Hausdorff distance* between any two sets $E, F \subset \mathbb{R}^n$.

Remark 24 Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Set $\mathcal{C}(\overline{\Omega}) := \{K \subset \overline{\Omega} : K \text{ compact}\}$. Then the set $\mathcal{C}(\overline{\Omega})$, endowed with the Hausdorff distance is a compact metric space, see for instance [4, Theorem 6.1]. Moreover the convergence of K_j to K in the metric space $(\mathcal{C}(\overline{\Omega}), d_H)$ is equivalent to the two following conditions

(i) for all $x \in K$ there exist $x_j \in K_j$ such that $x_j \to x$;
(ii) if $x_j \in K_j$, then any limit point of the sequence $\{x_j\}$ belongs to K.

The convergence defined by (i) and (ii) is also known as *convergence in the sense of Kuratowski.*

Throughout all this section we shall only deal with sets E of the same volume as B. This is not a restriction at all since all the statements that we shall prove under this assumption also apply to sets of any measure, up to a suitable rescaling.

Definition 25 Let $E \subset \mathbb{R}^n$ be a convex set with $|E| = |B|$. We define the *isoperimetric deficit* and the *asymmetry index* of E by setting

$$\mathcal{D}(E) := P(E) - P(B), \quad \mathcal{A}(E) := \min_{x \in R^n} d_H(E, B(x)),$$

respectively.

The extension of Bonnesen result Theorem 23 to high dimension was obtained by Fuglede in 1989, see [39].

Theorem 26 (Fuglede) *Let $n \geq 2$. There exist δ, C, depending only on n, such that if E is convex, $|E| = |B|$, and $\mathcal{D}(E) \leq \delta$, then:*

$$\mathcal{A}(E) \leq \begin{cases} C\sqrt{\mathcal{D}(E)}, & n = 2 \\ C\sqrt{\mathcal{D}(E)\log\left(\dfrac{1}{\mathcal{D}(E)}\right)}, & n = 3 \\ C(\mathcal{D}(E))^{\frac{2}{n+1}}, & n \geq 4. \end{cases} \tag{20}$$

As we already observed, for $n = 2$ the above estimate is just a weaker version of the more precise inequality (19). As shown in [39, Sect. 3] also when $n \geq 3$ the estimates above are optimal. In fact if $n \geq 4$ one cannot replace the power $\frac{2}{n+1}$ by a bigger one and if $n = 3$ one cannot remove the logarithm of $1/\mathcal{D}(E)$ from the right hand side of (20).

Fuglede's theorem is based on the following result for nearly spherical sets, that is sets which are very close to the unit ball, see Fig. 11. It turns out that for such sets one may estimate very precisely the distance from the ball by writing up the Taylor expansion of the perimeter. As we shall see, the next result will be also useful to prove the stability of the isoperimetric inequality for general sets of finite perimeter.

Theorem 27 *Let $u : \mathbb{S}^{n-1} \to (-1, 1)$ be a Lipschitz function and let*

$$E := \{tz(1 + u(z)) : z \in \mathbb{S}^{n-1}, 0 \leq t < 1\}. \tag{21}$$

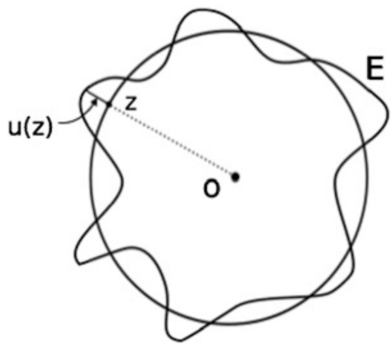


Fig. 11 A nearly spherical set

There exists $\varepsilon(n) > 0$ *such that if* $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < \varepsilon$, $|E| = |B|$ *and the barycenter of E is the origin, then*

$$\mathcal{D}(E) \geq \frac{1}{4}\|\nabla_\tau u\|^2_{L^2(S^{n-1})} \geq \frac{1}{8\omega_n}|E\Delta B|^2. \tag{22}$$

Note that in (22) we have denoted by $\nabla_\tau u$ the tangential gradient of u on $\mathbb{S}^{n-1}$. In the sequel we shall refer to a set $E \subset \mathbb{R}^n$ satisfying (21) as to *a nearly spherical set*. In order to prove Theorem 27 we need the formulas stated in the next lemma.

Lemma 28 *Let E be as in* (21)*, with* $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} < 1$. *Then*

$$P(E) = \int_{\mathbb{S}^{n-1}} \sqrt{(1+u)^{2(n-1)} + (1+u)^{2(n-2)}|\nabla_\tau u|^2}\, d\mathcal{H}^{n-1}\,. \tag{23}$$

Moreover,

$$|E| = \frac{1}{n}\int_{\mathbb{S}^{n-1}} (1+u(z))^n\, d\mathcal{H}^{n-1}, \qquad \int_E x\, dx = \frac{1}{n+1}\int_{\mathbb{S}^{n-1}} z(1+u(z))^{n+1}\, d\mathcal{H}^{n-1}\,. \tag{24}$$

Proof We start by proving (24). To this aim we extend u to $\mathbb{R}^n\setminus\{0\}$ by setting $u(x) := u(x/|x|)$ for all $x \neq 0$. In this way we have that $E = \Phi(B)$, where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is the map $\Phi(x) := x(1+u(x))$, $x \in B$. Note that $D\Phi(x) = (1+u(x))I + x \otimes Du$ and that since u is homogeneous of degree zero, then $x \cdot Du(x) = 0$ for all $x \neq 0$. Thus, recalling (11) we conclude that the Jacobian $J\Phi$ of Φ is given by $(1+u(x))^n$. Therefore

$$|E| = \int_B J\Phi\, dx = \int_B (1+u(x))^n\, dx = \int_0^1 r^{n-1}\, dr \int_{\mathbb{S}^{n-1}} (1+u(x))^n\, d\mathcal{H}^{n-1}\,.$$

Hence the first equality in (24) follows. The second one is obtained similarly.

Since E is a bounded open set with Lipschitz boundary, $P(E) = \mathcal{H}^{n-1}(\partial E)$, see Example 10 or [4, Proposition 3.62]. Then, recalling that $\partial E = \Phi(\mathbb{S}^{n-1})$, from the *area formula*, see for instance [4, Theorem 2.92], we have

$$P(E) = \mathcal{H}^{n-1}(\partial E) = \int_{\mathbb{S}^{n-1}} J_{n-1}\Phi\, d\mathcal{H}^{n-1}, \tag{25}$$

where the $(n-1)$*-dimensional Jacobian* $J_{n-1}\Phi$ of the map Φ is given by

$$J_{n-1}\Phi = \sqrt{\det\big((d\Phi(x))^T \circ d\Phi(x)\big)}\,.$$

Here the linear map $d\Phi(z) : T_z\mathbb{S}^{n-1} \mapsto \mathbb{R}^n$ is the tangential differential of Φ defined in (10) and $(d\Phi(z))^T$ is its adjoint. Note that for any $\tau \in T_z\mathbb{S}^{n-1}$ we have $d\Phi(z)(\tau) = \tau(1+u(z)) + zD_\tau u(z)$, where $D_\tau u(z) = \nabla u(z) \cdot \tau$. Therefore the coefficients of the matrix $d\Phi(z)$ relative to an orthonormal base $\{\tau_1,\dots,\tau_{n-1}\}$ of $T_z\mathbb{S}^{n-1}$ and to the standard base $\{e_1,\dots,e_n\}$ are given by $\tau_i\cdot e_h(1+u(z))+z_hD_{\tau_i}u(z)$, for $i=1,\dots,n-1$, $h=1,\dots,n$. Thus, for all $i,j\in\{1,\dots,n-1\}$, the coefficients a_{ij} of the matrix $(d\Phi(z))^T\circ d\Phi(z)$ are given by

$$a_{ij}=\sum_{h=1}^{n}\big(\tau_i\cdot e_h(1+u)+z_hD_{\tau_i}u\big)\big(\tau_j\cdot e_h(1+u)+z_hD_{\tau_j}u\big)=\delta_{ij}(1+u)^2+D_{\tau_i}uD_{\tau_j}u,$$

where in the last equality we have used the fact that $\tau_i\cdot\tau_j=\delta_{ij}$ and $\tau_i\cdot z=0$ for all $i,j=1,\dots,n-1$. Hence, recalling (11) we have

$$J_{n-1}\Phi=\sqrt{\det(a_{ij})}=\sqrt{(1+u)^{2(n-1)}+(1+u)^{2(n-2)}|\nabla_\tau u|^2}$$

and thus (23) follows immediately from (25). □

We are now in position to give the proof of Theorem 27. The proof below follows closely the one given in [40] which has the advantage of avoiding some heavy computations of the original proof by Fuglede.

Proof of Theorem 27 **Step 1.** From (23) we have

$$\begin{aligned}P(E)-P(B)&=\int_{\mathbb{S}^{n-1}}\left[(1+u)^{n-1}\sqrt{1+\frac{|\nabla_\tau u|^2}{(1+u)^2}}-1\right]d\mathcal{H}^{n-1}\\&=\int_{\mathbb{S}^{n-1}}\big[(1+u)^{n-1}-1\big]\,d\mathcal{H}^{n-1}\\&\quad+\int_{\mathbb{S}^{n-1}}(1+u)^{n-1}\left[\sqrt{1+\frac{|\nabla_\tau u|^2}{(1+u)^2}}-1\right]d\mathcal{H}^{n-1}.\end{aligned}$$

From the Taylor expansion of the square root it follows that for $t>0$ sufficiently small $\sqrt{1+t}\geq 1+\frac{t}{2}-\frac{t^2}{7}$. Hence, if ε is small, from the assumption $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})}<\varepsilon$ we get

$$\begin{aligned}P(E)-P(B)&\geq\int_{\mathbb{S}^{n-1}}\big[(1+u)^{n-1}-1\big]\,d\mathcal{H}^{n-1}\\&\quad+\int_{\mathbb{S}^{n-1}}(1+u)^{n-1}\left[\frac{1}{2}\frac{|\nabla_\tau u|^2}{(1+u)^2}-\frac{1}{7}\frac{|\nabla_\tau u|^4}{(1+u)^4}\right]d\mathcal{H}^{n-1}\\&\geq\int_{\mathbb{S}^{n-1}}\big[(1+u)^{n-1}-1\big]\,d\mathcal{H}^{n-1}+\left(\frac{1}{2}-C\varepsilon\right)\int_{\mathbb{S}^{n-1}}|\nabla_\tau u|^2d\mathcal{H}^{n-1},\end{aligned}\tag{26}$$

where C is a constant depending only on n. From the first equality in (24) it follows that the assumption $|E| = |B|$ is equivalent to

$$\int_{\mathbb{S}^{n-1}} \big[(1+u)^n - 1\big] d\mathcal{H}^{n-1} = 0, \tag{27}$$

that is

$$\int_{\mathbb{S}^{n-1}} \Big(nu + \sum_{h=2}^{n} \binom{n}{h} u^h\Big) d\mathcal{H}^{n-1} = 0. \tag{28}$$

From this identity, recalling again that $\|u\|_{L^\infty(\mathbb{S}^{n-1})} < \varepsilon$, we have

$$\int_{\mathbb{S}^{n-1}} u\, d\mathcal{H}^{n-1} \geq -\frac{n-1}{2} \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} - C\varepsilon \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1}.$$

Therefore, using this last inequality and the smallness assumption, we may estimate

$$\begin{aligned}
\int_{\mathbb{S}^{n-1}} [(1+u)^{n-1} - 1] d\mathcal{H}^{n-1} &= (n-1) \int_{\mathbb{S}^{n-1}} u\, d\mathcal{H}^{n-1} + \sum_{h=2}^{n-1} \binom{n-1}{h} \int_{\mathbb{S}^{n-1}} u^h d\mathcal{H}^{n-1} \\
&\geq (n-1) \int_{\mathbb{S}^{n-1}} u\, d\mathcal{H}^{n-1} + \frac{(n-1)(n-2)}{2} \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} \\
&\quad - C\varepsilon \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} \\
&\geq -\frac{n-1}{2} \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1} - C\varepsilon \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1}.
\end{aligned}$$

In conclusion, recalling (26), we have proved that if $\|u\|_{W^{1,\infty}(\mathbb{S}^{n-1})} \leq \varepsilon$, then

$$P(E) - P(B) \geq \Big(\frac{1}{2} - C\varepsilon\Big) \int_{\mathbb{S}^{n-1}} |\nabla_\tau u|^2 d\mathcal{H}^{n-1} - \Big(\frac{n-1}{2} + C\varepsilon\Big) \int_{\mathbb{S}^{n-1}} u^2 d\mathcal{H}^{n-1}, \tag{29}$$

for some constant C depending only on the dimension n.

Step 2. Now, for any integer $k \geq 0$, let us denote by $y_{k,i}$, $i = 1, \ldots, G(n,k)$, the spherical harmonics of order k, i.e., the restriction to $\mathbb{S}^{n-1}$ of the homogeneous harmonic polynomials of degree k, normalized so that $||y_{k,i}||_{L^2(\mathbb{S}^{n-1})} = 1$, for all k and for $i \in \{1, \ldots, G(n,k)\}$. Taking into account the normalization, we have that $y_0 = 1/\sqrt{n\omega_n}$ and $y_{1,i} = z_i/\sqrt{\omega_n}$, for $i = 1, \ldots, n$. Recall that the polynomials $y_{k,i}$ are eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^{n-1}$ and that for all k and i

$$-\Delta_{\mathbb{S}^{n-1}} y_{k,i} = k(k+n-2) y_{k,i}\,.$$

Therefore if we write

$$u = \sum_{k=0}^{\infty} \sum_{i=1}^{G(n,k)} a_{k,i} y_{k,i}, \quad \text{where} \quad a_{k,i} = \int_{\mathbb{S}^{n-1}} u y_{k,i} d\mathcal{H}^{n-1},$$

we have

$$||u||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=0}^{\infty} \sum_{i=1}^{G(n,k)} a^2_{k,i}, \quad ||\nabla_\tau u||^2_{L^2(\mathbb{S}^{n-1})} = \sum_{k=1}^{\infty} k(k+n-2) \sum_{i=1}^{G(n,k)} a^2_{k,i}\,. \tag{30}$$

Observe that from formula (28) we have

$$a_0 = \frac{1}{\sqrt{n\omega_n}} \int_{\mathbb{S}^{n-1}} u d\mathcal{H}^{n-1} = -\frac{1}{n\sqrt{n\omega_n}} \sum_{h=2}^{n} \binom{n}{h} \int_{\mathbb{S}^{n-1}} u^h d\mathcal{H}^{n-1},$$

hence

$$|a_0| \le C||u||_2^2 \le C\varepsilon ||u||_2\,.$$

From the assumption that the barycenter of E is at the origin and from the second equality in (24) we have

$$\int_{\mathbb{S}^{n-1}} z(1+u(z))^{n+1}\, d\mathcal{H}^{n-1} = 0\,.$$

Then, using the equality $\int_{\mathbb{S}^{n-1}} z = 0$ and arguing as before, we immediately get that for all $i = 1, \dots, n$,

$$|a_{1,i}| = \left| \frac{1}{\sqrt{\omega_n}} \int_{\mathbb{S}^{n-1}} u z_i \, d\mathcal{H}^{n-1} \right| \le C\varepsilon ||u||_2.$$

Therefore, from (30) we get

$$||u||_2^2 \le C\varepsilon^2 ||u||_2^2 + \sum_{k=2}^{\infty} \sum_{i=1}^{G(n,k)} |a_{k,i}|^2 \Longrightarrow ||u||_2^2 \le \frac{1}{1-C\varepsilon} \sum_{k=2}^{\infty} \sum_{i=1}^{G(n,k)} |a_{k,i}|^2.$$

But since for $k \ge 2$, $k(k+n-2) \ge 2n$, from (30) we have

$$||u||_2^2 \le \frac{1}{2n(1-C\varepsilon)} ||\nabla_\tau u||_2^2$$

and thus, recalling (29) and choosing ε sufficiently small, in dependence on n, we get

$$P(E)-P(B)\geq\Big(\frac{1}{2}-C\varepsilon\Big)\int_{\mathbb{S}^{n-1}}|\nabla_\tau u|^2 d\mathcal{H}^{n-1}-\Big(\frac{n-1}{2}+C\varepsilon\Big)\frac{1}{2n(1-C\varepsilon)}||\nabla_\tau u||_2^2$$
$$\geq\frac{1}{4}\int_{\mathbb{S}^{n-1}}|\nabla_\tau u|^2 d\mathcal{H}^{n-1}\geq\frac{n}{3}||u||^2_{L^2(\mathbb{S}^{n-1})}\geq\frac{1}{3\omega_n}||u||^2_{L^1(\mathbb{S}^{n-1})}. \tag{31}$$

This proves the first inequality in (22). To get the second inequality we observe that, choosing again ε sufficiently small

$$|E\Delta B|=\frac{1}{n}\int_{\mathbb{S}^{n-1}}|(1+u(x))^n-1|\,d\mathcal{H}^{n-1}\leq\frac{n+1}{n}\int_{\mathbb{S}^{n-1}}|u|\,d\mathcal{H}^{n-1}.$$

Therefore, from the last inequality of (31) we conclude that

$$P(E)-P(B)\geq\frac{1}{3\omega_n}||u||^2_{L^1(\mathbb{S}^{n-1})}\geq\frac{n^2}{3(n+1)^2\omega_n}|E\Delta B|^2\geq\frac{1}{8\omega_n}|E\Delta B|^2.$$

□

The theorem we have just proved allows us to estimate the distance in $W^{1,2}$ of a nearly spherical set E from the unit ball with the isoperimetric deficit. Now, an interpolation result will tell us that indeed we may also control the L^∞ distance, hence the Hausdorff distance, between E and B. For the proof see [39, Lemma 1.4].

Lemma 29 (Interpolation Lemma) *If $v\in W^{1,\infty}(\mathbb{S}^{n-1})$ and $\int_{\mathbb{S}^{n-1}}v=0$, then*

$$||v||^{n-1}_{L^\infty(\mathbb{S}^{n-1})}\leq\begin{cases}\pi||\nabla_\tau v||_2, & n=2\\ 4||\nabla_\tau v||_2^2\log\dfrac{8e||\nabla_\tau v||_\infty}{||\nabla_\tau v||_2^2}, & n=3\\ C||\nabla_\tau v||_2^2||\nabla_\tau v||_\infty^{n-3}, & n\geq 4,\end{cases}$$

where the constant C depends only on the dimension.

Combining Lemma 29 with Theorem 27 we immediately get the estimate of the L^∞ distance between a nearly spherical set E and the unit ball.

Theorem 30 *Under the assumptions of Theorem 27, there exist $\varepsilon, C>0$ depending only on n such that if $||u||_{W^{1,\infty}(\mathbb{S}^{n-1})}\leq\varepsilon$, then*

$$||u||^{n-1}_{L^\infty(\mathbb{S}^{n-1})}\leq\begin{cases}C\sqrt{\mathcal{D}(E)}, & n=2\\ C\mathcal{D}(E)\log\left(\dfrac{1}{\mathcal{D}(E)}\right), & n=3\\ C\mathcal{D}(E)||\nabla_\tau u||_\infty^{n-3}, & n\geq 4.\end{cases}$$

Proof Set $v := \dfrac{(1+u)^n - 1}{n}$. From the volume constraint $|E| = |B|$ we have, see (27),

$$\int_{\mathbb{S}^{n-1}} v \, d\mathcal{H}^{n-1} = \frac{1}{n} \int_{\mathbb{S}^{n-1}} \big[(1+u)^n - 1\big] d\mathcal{H}^{n-1} = 0.$$

Moreover, since

$$v = u + \frac{1}{n} \sum_{h=2}^{n} \binom{n}{h} u^h,$$

if $\varepsilon > 0$ is small enough we have

$$\frac{1}{2}|u| \leq |v| \leq 2|u|, \qquad \frac{1}{2}|\nabla_\tau u| \leq |\nabla_\tau v| \leq 2|\nabla_\tau u|.$$

Then the result follows immediately from Theorem 27 and the interpolation Lemma 29. □

Let us now consider the case of a convex set with small isoperimetric deficit and let us indicate the main steps in the proof of Fuglede's Theorem 26. The first step, see Lemma 32, is to show that a convex set with small isoperimetric deficit is close in the Hausdorff distance to a ball with the same volume. At this stage, however, we are not yet able to quantify how close is the set to the ball in terms of the isoperimetric deficit. Next, we observe that if a convex set is close in the Hausdorff sense to a ball of the same volume, then it is also close to the same ball in $W^{1,\infty}$, see Lemma 33. Then, the final step of the proof consists in combining these observations with the precise estimate provided by Theorem 30.

Let us start with a simple lemma relating the diameter diam(E) of a convex set E with its volume and perimeter. To this aim, let us recall that

$$P(E) \leq P(F) \qquad \text{if } E,\ F \text{ are convex and } E \subset F\,. \tag{32}$$

Lemma 31 *Let $E \subset \mathbb{R}^n$ be a bounded open convex set. Then*

$$\operatorname{diam}(E) \leq c(n) \frac{[P(E)]^{n-1}}{|E|^{n-2}}.$$

Proof First observe that if $n = 2$ we trivially have $\operatorname{diam}(E) \leq \frac{1}{2} P(E)$.

So let us assume $n \geq 3$. Let $x, y \in \partial E$ be such that $\operatorname{diam}(E) := d = |x - y|$. Then, rotate and translate E so to reduce to the situation shown in Fig. 12.

By Fubini's Theorem, $|E| = \int_0^d \mathcal{H}^{n-1}(E_t)dt$, where $E_t = E \cap \{x_n = t\}$. Observe that there exists $s \in (0, d)$ such that $\mathcal{H}^{n-1}(E_s) \geq |E|/d$. Note that we may always assume that $0 < s \leq d/2$ (otherwise we just rotate E upside down). Let C be the

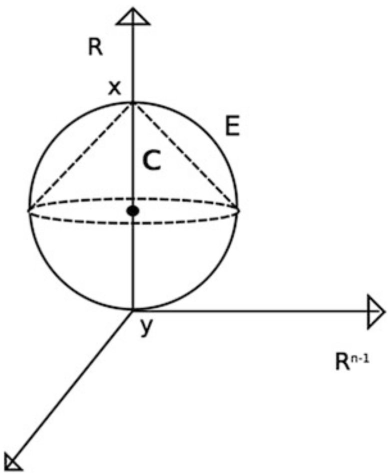


Fig. 12 The construction in the proof of Lemma 31

cone in Fig. 12 with base E_s and vertex x. Using the coarea formula (12) and (32) we may estimate

$$P(E) \geq P(C) \geq \mathcal{H}^{n-1}(\partial C \setminus E_s) = \int_s^d dt \int_{\partial C_t} \frac{1}{|\nu_t^C|}\, d\mathcal{H}^{n-2} \geq \int_s^d \mathcal{H}^{n-2}(\partial C_t)dt$$

$$= \int_s^d \left(\frac{d-t}{d-s}\right)^{n-2} \mathcal{H}^{n-2}(\partial E_s)dt = \frac{(d-s)\mathcal{H}^{n-2}(\partial E_s)}{n-1} \geq \frac{d}{2}\frac{\mathcal{H}^{n-2}(\partial E_s)}{n-1}.$$

From the isoperimetric inequality (17) we get

$$\mathcal{H}^{n-2}(\partial E_s) \geq (n-1)\omega_{n-1}^{1/(n-1)} \left[\mathcal{H}^{n-1}(E_s)\right]^{\frac{n-2}{n-1}} \geq (n-1)\omega_{n-1}^{1/(n-1)} \left(\frac{|E|}{d}\right)^{\frac{n-2}{n-1}}.$$

Thus,

$$P(E) \geq c(n) d \left(\frac{|E|}{d}\right)^{\frac{n-2}{n-1}},$$

whence the result follows. □

Let us now prove that a convex set with small isoperimetric deficit is close in the Hasudorff distance to a ball.

Lemma 32 *For all $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that if E is convex, $|E| = |B|$, the barycenter of E is the origin and $\mathcal{D}(E) < \delta_\varepsilon$, then there exists a function $u \in W^{1,\infty}(\mathbb{S}^{n-1})$, with $||u||_{L^\infty(\mathbb{S}^{n-1})} \leq \varepsilon$, and such that*

$$E := \{tz(1+u(z)) : z \in \mathbb{S}^{n-1},\ 0 \leq t < 1\}.$$

Proof We argue by contradiction. Assume that there exist $\varepsilon_0 > 0$ and a sequence of closed convex sets E_j such that $|E_j| = |B|$, the barycenter of E_j is the origin, $\mathcal{D}(E_j) \to 0$, but $||u_j||_{L^\infty(\mathbb{S}^{n-1})} \geq \varepsilon_0$, where u_j is the Lipschitz function representing E_j as in (21). From Lemma 31 it follows that the sets E_j are equibounded and so, recalling Remark 24, we may assume that they converge in the Hausdorff distance to a closed set E. Note that E is convex and that the sequence E_j converge to E also in measure. In particular $|E| = |B|$. Since $\mathcal{D}(E_j) \to 0$, we have that $P(E_j) \to P(B)$. Therefore, from the isoperimetric inequality and the lower semicontinuity of the perimeter we get that

$$P(B) \leq P(E) \leq \lim_{j\to\infty} P(E_j) = P(B)\,.$$

Thus E is a ball, actually the unit ball centered at the origin, since all the E_j have barycenter at the origin. This gives a contradiction, since the E_j are converging in the Hausdorff sense to the unit ball B, while $||u_j||_{L^\infty(\mathbb{S}^{n-1})} \geq \varepsilon_0$ for all j. □

The following lemma shows that the Hausdorff distance of a convex set from a ball controls indeed also its distance in $W^{1,\infty}$.

Lemma 33 *Let E is a convex set such that*

$$E := \{tz(1 + u(z)) : z \in \mathbb{S}^{n-1},\ 0 \leq t < 1\}\,.$$

for some Lipschitz function $u : \mathbb{S}^{n-1} \to (-1/2, 1/2)$. Then

$$\|\nabla_\tau u\|_{L^\infty} \leq 2\sqrt{\|u\|_{L^\infty}\frac{1 + \|u\|_{L^\infty}}{1 - \|u\|_{L^\infty}}}\,.$$

Proof Let us fix $P_z \in \partial E$ and let $z \in \mathbb{S}^{n-1}$ be such that $P_z = z(1 + u(z))$ and u is differentiable at z. Recall that the tangent plane $T_x\partial E$ is spanned by the vectors $(1 + u(z))\tau_i + z\nabla u(z) \cdot \tau_i$, where $\{\tau_1, \ldots, \tau_{n-1}\}$ is an orthonormal base for $T_z\mathbb{S}^{n-1}$. Therefore the exterior normal to E at P_z is given by

$$\nu^E(P_z) = \frac{z(1 + u(z)) - \nabla_\tau u(z)}{\sqrt{(1 + u(z))^2 + |\nabla_\tau u(z)|^2}}\,. \tag{33}$$

Since $z \cdot \nabla_\tau u(z) = 0$, we have

$$z \cdot \nu^E(P_z) = \frac{1 + u(z)}{\sqrt{(1 + u(z))^2 + |\nabla_\tau u(z)|^2}}\,.$$

Then, denoting by H the projection of the origin on the tangent plane to E at P_z, we have, see Fig. 13, $\frac{\overline{OH}}{\overline{OP_z}} = z \cdot \nu^E(P_z)$. Observe that

$$\overline{OP_z} \leq 1 + ||u||_\infty, \quad \overline{OH} \geq 1 - ||u||_\infty,$$

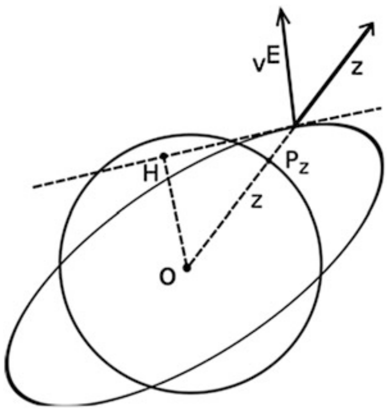


Fig. 13 The construction in the proof of Lemma 33

where the second inequality follows by the convexity of E. Thus,

$$\frac{1-||u||_\infty}{1+||u||_\infty} \le z \cdot \nu^E(P_z) = \frac{1+u(z)}{\sqrt{(1+u(z))^2+|\nabla_\tau u(z)|^2}},$$

from which we get

$$\frac{|\nabla_\tau u(z)|^2}{(1+u(z))^2} \le \left(\frac{1+||u||_\infty}{1-||u||_\infty}\right)^2 - 1 = \frac{4||u||_\infty}{(1-||u||_\infty)^2},$$

thus concluding

$$|\nabla_\tau u(z)|^2 \le 4||u||_\infty \left(\frac{1+||u||_\infty}{1-||u||_\infty}\right)^2,$$

whence the assertion follows. □

Let us conclude this section by giving the

Proof of Theorem 26 Let as assume $n \ge 4$, since otherwise the proof is similar and even easier.

From Lemmas 32 and 33 it follows that if E is a convex set with $|E| = |B|$ and $\mathcal{D}(E)$ is sufficiently small, then, up to a translation, E is a nearly spherical set as in (21) with barycenter at the origin, satisfying $\|u\|_{W^{1,\infty}} < \varepsilon$, where $\varepsilon > 0$ is the one provided by Theorem 30. Therefore, using this theorem and Lemma 33 again, we get

$$||u||_\infty^{n-1} \le c\mathcal{D}(E)||\nabla_\tau u||_\infty^{n-3} \le c||u||_\infty^{\frac{n-3}{2}}\mathcal{D}(E).$$

hence, $||u||_\infty^{\frac{n+1}{2}} \le c\mathcal{D}(E)$. Thus we may conclude that

$$\mathcal{A}(E) \le d_H(E,B) = ||u||_\infty \le c\,[\mathcal{D}(E)]^{\frac{2}{n+1}} . \qquad \square$$

5 Stability of the Isoperimetric Inequality: Proof by Symmetrization

We now discuss the quantitative isoperimetric inequality for general sets of finite perimeter. In this case it is clear that we cannot use the Hausdorff distance to measure the distance of a set E from a ball, since a set with the same volume of the ball B and a slightly larger perimeter may have small far away pieces or tiny long tentacles. Taking into account these examples it is then reasonable to introduce the so called *Fraenkel asymmetry* which is defined, for any measurable set E of finite measure, as

$$\alpha(E) := \inf_{x\in\mathbb{R}^n} \left\{ \frac{|E\Delta B_r(x)|}{r^n} : |E| = |B_r| \right\} .$$

Note that the above infimum is always attained. In the following we shall refer to a minimizer of the right hand side as to an *optimal ball* for E. Clearly, optimal balls do not need to be unique. Observe also that, since $|E\Delta B_r(x)|$ is exactly the L^1 distance between χ_E and $\chi_{B_r(x)}$, $\alpha(E)$ can be regarded as the normalized L^1 distance of E from its optimal ball. It is convenient to normalize also the isoperimetric deficit by setting

$$D(E) := \frac{P(E) - P(B_r)}{r^{n-1}},$$

where $|B_r| = |E|$.

In 1992 Hall [47], using some previous results proved in collaboration with by Hayman and Weitsman [48], showed that there exists a constant $c(n)$ such that for all measurable sets of finite measure

$$\alpha(E)^4 \le c(n)D(E) . \tag{34}$$

Note that the power on the left hand side of (34) is independent of the dimension of the ambient space. Note also that an inequality of this kind becomes critical only when the set E is a small perturbation of the ball. As an example consider for any $n \ge 2$ the ellipsoid

$$E_\varepsilon = \left\{ \frac{x_1^2}{1+\varepsilon} + x_2^2(1+\varepsilon) + x_3^2 + \ldots + x_n^2 \le 1 \right\} ,$$

with $\varepsilon > 0$. Then $\alpha(E_\varepsilon) = |E_\varepsilon \Delta B|$, see [6, Lemma 5.9] and it is not difficult to show that

$$\frac{D(E_\varepsilon)}{\alpha^2(E_\varepsilon)} \to \gamma > 0, \quad \text{as } \varepsilon \to 0^+.$$

This example led Hall to conjecture in [47] that inequality (34) should hold in any dimension with the (optimal) exponent 2. This was proved by Maggi, Pratelli and the author in [44]. The precise statement goes as follows.

Theorem 34 (Quantitative Isoperimetric Inequality) *There exists a constant $\kappa(n)$ such that for any measurable set E of finite measure*

$$\alpha(E)^2 \leq \kappa(n) D(E). \tag{35}$$

In this section we are going to discuss the proof of this result originally given in [44], which relies mostly on symmetrization arguments.

Note that inequality (35) can be rewritten in the following equivalent way: if $|E| = |B_r|$, then

$$P(E) \geq P(B_r)\Big(1 + \frac{1}{n\omega_n \kappa(n)} \alpha(E)^2\Big).$$

Thus the asymmetry index $\alpha(E)$ estimates from below the second order term in the Taylor expression of $P(E)$ in terms of $P(B_r)$.

Before going into the proof of (35) let us make some preliminary remarks.

First, observe that since both $\alpha(E)$ and $D(E)$ are scale invariant, to prove Theorem 34 we may always assume $|E| = |B|$. Note also that if one proves (35) for a set with small isoperimetric gap, i.e., $D(E) \leq \delta_0$, then the general case follows. As a matter of fact, if $D(E) > \delta_0$ and $|E| = |B_r|$, then

$$\alpha(E) \leq \frac{|E \Delta B_r|}{r^n} \leq 2\omega_n \leq \frac{2\omega_n}{\sqrt{\delta_0}} \sqrt{D(E)}.$$

The strategy of the proof consists in reducing the general case to more and more special classes of sets. Precisely, in the first step one reduces to sets contained in a sufficiently large square, see Lemma 35. Then one wants to reduce to bounded *n-symmetric* sets, i.e., sets which are symmetric with respect to n orthogonal hyperplanes, Theorem 40. These sets have the nice property that the ball centered at their center of symmetry is "almost optimal" in the sense stated in Lemma 38. The last reduction consists in passing from n-symmetric to axially symmetric sets whose profile is obtained by rotating a one-dimensional graph. Note that the proof of the quantitative isoperimetric inequality (35) for axially symmetric sets was already contained in Hall's paper [47, Theorem 2]. Different proofs are given in [44, Sect. 4] and in [50, Sect. 7]. The approach to stability issues via symmetrization has been used also used to deal with the Sobolev inequality, the isoperimetric inequality in

Gauss space and with other relevant geometric and functional inequalities, see for instance [6, 7, 13, 21, 22, 32, 33, 35, 43, 45, 46], and also [41, 52, 58].

The first reduction step is provided by the next result, see [44, Lemma 5.1]).

Lemma 35 *There exist positive constants L, C, δ depending only on n such that if $|E| = |B|$ and $D(E) \le \delta$ one can find a set $F \subset [-L, L]^n$, with $|F| = |B|$, such that*

$$\alpha(E) \le \alpha(F) + CD(F) \text{ and } D(F) \le CD(E).$$

We will not give the detailed proof of this lemma, which consists in cutting the far away parts of E and rescaling the remaining part of the set. The main ingredients of the proof are the isoperimetric inequality and the strict concavity of the function $t^{\frac{n-1}{n}}$ for $t > 0$, which allows to estimate in a quantitative way the asymmetry created by splitting a set in two parts. To understand how this estimate works observe that for all $\lambda \in (0, 1)$

$$\lambda^{\frac{n-1}{n}} + (1-\lambda)^{\frac{n-1}{n}} - 1 \ge c(n) \min\{\lambda, 1-\lambda\}. \tag{36}$$

Let $E = B_r(x) \cup B_\rho(y)$ the union of two disjoint balls such that $|E| = |B|$ and $r \ge \rho$. Then

$$r^n + \rho^n = 1$$

and from (36) we may estimate the isoperimetric deficit of E by

$$D(E) = P(B_r(x)) + P(B_\rho(y)) - P(B) \ge c(n) \min\{r^n, \rho^n\} \ge c(n)\rho^n \,.$$

Hence the estimate on the Fraenkel asymmetry of E immediately follows:

$$\frac{1}{2}\,\alpha(E) \le |B(x) \setminus B_r(x)| = \omega_n(1 - r^n) = \omega_n \rho^n \le \omega_n c(n) D(E)\,.$$

It is clear how to use Lemma 35. Indeed if the quantitative isoperimetric inequality (35) holds for a bounded set, then, given any set E with $|E| = |B|$ and $D(E) \le 1$, denoting by F the set provided by Lemma 35, we have

$$\alpha(E) \le \alpha(F) + CD(F) \le \sqrt{\kappa(n)D(F)} + CD(F) \le C' \sqrt{D(E)}\,.$$

for a constant C' depending only on n.

Thus, from now one we may assume without loss of generality that the set E has volume ω_n, that $E \subset [-L, L]^n$, for some given $L > 0$, and that $D(E) \le \delta$ for some conveniently small δ.

The advantage of working with bounded sets is that in this case the compactness theorem for sets of finite perimeter Theorem 13 implies that $\alpha(E)$ depends continuously on $D(E)$.

Lemma 36 *Let $L > 0$. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $E \subset [-L, L]^n$, $|E| = |B|$, and $D(E) \leq \delta$ then $\alpha(E) \leq \varepsilon$.*

Proof The proof is by contradiction. Assume that there exist $\varepsilon > 0$ and a sequence of sets $E_j \subset [-L, L]^n$, with $|E_j| = |B|$, $D(E_j) \to 0$ and $\alpha(E_j) \geq \varepsilon > 0$ for all $j \in \mathbb{N}$. Since the sets E_j are equibounded, by Theorem 15 we may assume that up to a not relabeled subsequence the E_j converge in measure to some set E_∞ of finite perimeter. Thus $|E_\infty| = |B|$, and by the lower semicontinuity of the perimeters $P(E_\infty) \leq P(B)$, so E_∞ is a ball. However the convergence in measure of E_j to E_∞ immediately implies that $|E_j \Delta E_\infty| \to 0$, against the assumption $\alpha(E_j) \geq \varepsilon$. The contradiction concludes the proof. □

Next step in the proof of the quantitative isoperimetric inequality is to reduce to the simpler case of an n-symmetric set.

Definition 37 We say that $E \subset \mathbb{R}^n$ is *n-symmetric* if, up to a translation and a rotation, E is symmetric about each coordinate plane.

Note that even if E is n-symmetric it is not true in general that the optimal ball is the one centered at the center of symmetry of E, as shown in Fig. 14. However, the next lemma shows that for n-symmetric sets this ball is optimal "up to a constant".

Lemma 38 *Let E be n-symmetric with centre of symmetry at the origin, $|E| = |B|$. Then*

$$\alpha(E) \leq |E \Delta B| \leq 3\alpha(E)$$

Proof Let $B(x_0)$ be an optimal ball for E, i.e. $\alpha(E) = |E \Delta B(x_0)|$. Then by the triangular inequality we have

$$|E \Delta B| \leq |E \Delta B(x_0)| + |B(x_0) \Delta B|.$$

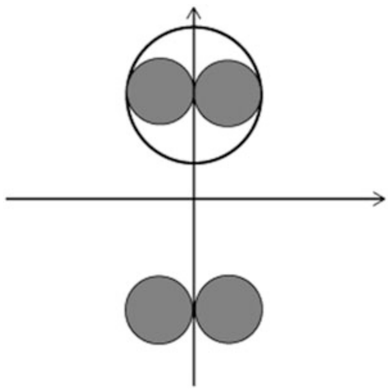


Fig. 14 An optimal ball not centered at the origin

Note that since E is n-symmetric $B(-x_0)$ is optimal as well, i.e., $\alpha(E) = |E\Delta B(-x_0)|$. Therefore from the inequality above we have

$$\begin{aligned}\alpha(E) &\le |E\Delta B| \le \alpha(E) + |B(x_0)\Delta B| \le \alpha(E) + |B(x_0)\Delta B(-x_0)| \\ &\le \alpha(E) + |E\Delta B(x_0)| + |E\Delta B(-x_0)| = 3\alpha(E).\end{aligned}$$

□

The next step is to reduce the proof of the quantitative isoperimetric inequality (35) to n-symmetric bounded sets. But before discussing how this can be done let us first introduce a few definitions.

Given a direction $\nu \in \mathbb{S}^{n-1}$ and a measurable set E of finite measure, let us consider the affine hyperplane π_ν orthogonal to ν splitting E into two parts of equal measure. We denote by E' the part of E contained in the open half space H_ν^+ with inner normal ν and by E'' the part of E contained in the open half space H_ν^- with inner normal $-\nu$. Then, we set $E_\nu^+ := E' \cup r_\nu(E')$, where r_ν is the reflection about the hyperplane π_ν and $E_\nu^- := E'' \cup r_\nu(E'')$. See Fig. 15 where, to simplify the notation, we dropped the subscript ν. We claim that

$$P(E_\nu^+) + P(E_\nu^-) \le 2P(E) \tag{37}$$

with the inequality being possibly strict. To see this observe that from the definition of density we easily have that

$$\left[E^{(0)} \cup E^{(1)}\right] \cap \pi_\nu \subset \left[(E_\nu^+)^{(0)} \cup (E_\nu^+)^{(1)}\right] \cap \left[(E_\nu^-)^{(0)} \cup (E_\nu^-)^{(1)}\right] \cap \pi_\nu\,.$$

Therefore, from the definition of measure theoretic boundary given in (9) we deduce in particular that $\partial^M E_\nu^\pm \cap \pi_\nu \subseteq \partial^M E \cap \pi_\nu$ and thus, recalling Theorem 9, $\mathcal{H}^{n-1}(\partial^* E_\nu^\pm \cap \pi_\nu) \le \mathcal{H}^{n-1}(\partial^* E \cap \pi_\nu)$. Hence, (37) follows, since

$$\begin{aligned}P(E_\nu^+) + P(E_\nu^-) &= 2P(E \cap H_\nu^+) + \mathcal{H}^{n-1}(\partial^* E_\nu^+ \cap \pi_\nu) + 2P(E \cap H_\nu^-) \\ &\quad + \mathcal{H}^{n-1}(\partial^* E_\nu^- \cap \pi_\nu) \\ &\le 2P(E \cap H_\nu^+) + 2P(E \cap H_\nu^-) + 2\mathcal{H}^{n-1}(\partial^* E \cap \pi_\nu) = 2P(E)\,.\end{aligned}$$

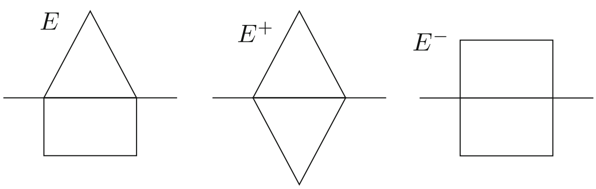


Fig. 15 The sets E^+ and E^- are obtained by reflecting the upper and lower half of E with respect to the horizontal plane

Observe that inequality (37) implies that

$$D(E_\nu^+) + D(E_\nu^-) \leq 2D(E)\,. \tag{38}$$

Therefore if we could prove that for some positive constant C_0 depending only on n

$$\alpha(E) \leq C_0[\alpha(E_\nu^+) + \alpha(E_\nu^-)], \tag{39}$$

we would conclude that, setting F either equal to E_ν^+ or E_ν^-, then

$$\alpha(E) \leq 2C_0\alpha(F), \qquad D(F) \leq 2D(E)\,.$$

Then by applying this argument to all coordinate directions we would find a n-symmetric set G with the same volume of E such that

$$\alpha(E) \leq 2^n C_0^n \alpha(G), \qquad D(G) \leq 2^n D(E)$$

and from these inequalities we would conclude that in order to prove (35) for E it is enough to prove it for the n-symmetric set G.

Inequality (39) is not true in general as we can see looking at the set E in Fig. 16. In fact, by reflecting the upper and lower halves of E with respect to the horizontal plane we get that $E^\pm$ are both balls, hence $\alpha(E^\pm) = 0$. However, if we symmetrize the same set with respect to the vertical direction the asymmetry index may even increase, as one can see in Fig. 17.

The following lemma shows that the phenomenon illustrated by this example is a general fact. Indeed, if for some ν the asymmetry of E_ν^+ and E_ν^- is much lower than the one of E, then given any other orthogonal direction ν', at least one of the two sets $E_{\nu'}^\pm$ has a larger asymmetry than E, up to a multiplicative constant depending only on the dimension.

Lemma 39 *There exist δ, C, depending only on n, such that if $E \subset [-L, L]^n$, $|E| = |B|$ and $D(E) \leq \delta$, given any two orthogonal direction ν_1, ν_2 and the four sets*

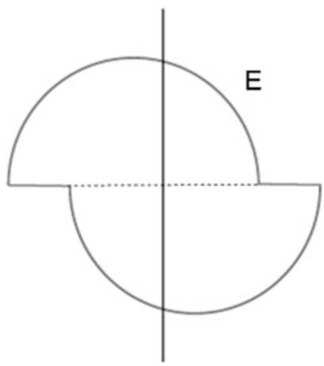


Fig. 16 A set for which (39) is not true

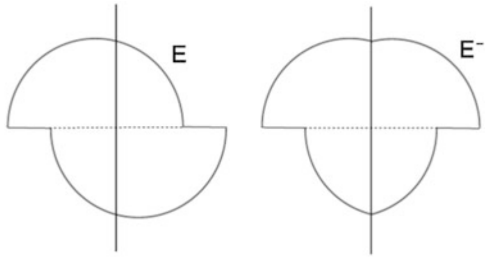


Fig. 17 A different symmetrization may give a bigger asymmetry

$E^+_{\nu_1}, E^-_{\nu_1}, E^+_{\nu_2}, E^-_{\nu_2}$, *we have that* $D(E^{\pm}_{\nu_i}) \leq 2D(E)$, *for* $i = 1, 2$. *Moreover, at least one of them, call it* F, *satisfies the estimate*

$$\alpha(E) \leq C\alpha(F).$$

We are not giving the proof of this lemma, for which we refer to (see [44, Lemma 2.5]). Instead we show how to use it in order to reduce the proof of (35) to n-symmetric sets.

Theorem 40 *There exist* δ_1 *and* C_1 *depending only on* n *such that if* $E \subset [-L, L]^n$, $|E| = |B|$, $\delta(E) \leq \delta_1$, *then there exists an* n*-symmetric set* F *such that* $F \subset [-2L, 2L]^n$, $|F| = |B|$ *and*

$$\alpha(E) \leq C_1\alpha(F), \quad D(F) \leq 2^n D(E). \tag{40}$$

Proof Take $\delta_1 = 2^{-(n-1)}\delta$, where δ is the constant of Lemma 39. By applying the lemma $n-1$ times to different pairs of orthogonal directions we find a set $\widetilde{E} \subset [-L, L]^n$ with $n-1$ symmetries, $|\widetilde{E}| = |B|$ and such that

$$\alpha(E) \leq C^{n-1}\alpha(\widetilde{E}), \quad D(\widetilde{E}) \leq 2^{n-1}D(E),$$

where C is the constant given by Lemma 39. Without loss of generality we may assume that $\widetilde{E}$ is symmetric with respect to the first $n-1$ directions $e_1, \ldots, e_{n-1}$. Let us consider a hyperplane π_{e_n} orthogonal to e_n and dividing $\widetilde{E}$ into two parts of equal measure, $\widetilde{E}', \widetilde{E}''$, and the corresponding sets $\widetilde{E}^{\pm}_{e_n}$. From (38) we have that

$$D(\widetilde{E}^{\pm}_{e_n}) \leq 2D(\widetilde{E}) \leq 2^n D(E).$$

To control the asymmetry of $\widetilde{E}^{\pm}_{e_n}$ observe that since $\widetilde{E}$ is symmetric with respect to the first $n-1$ directions, the sets $\widetilde{E}^{\pm}_{e_n}$ are both n-symmetric. Moreover, by suitably translating $\widetilde{E}$ if necessary, we may also assume that they are both symmetric around the origin. Thus we may apply Lemma 38 to estimate

$$\alpha(\widetilde{E}) \leq |\widetilde{E}\Delta B| = \frac{1}{2}\big[|\widetilde{E}^+_{e_n}\Delta B| + |\widetilde{E}^-_{e_n}\Delta B|\big] \leq \frac{3}{2}\big[\alpha(\widetilde{E}^+_{e_n}) + \alpha(\widetilde{E}^-_{e_n})\big].$$

Thus, at least one of the sets $\widetilde{E}^{\pm}_{e_n}$ has asymmetry index greater than $\dfrac{1}{3}\alpha(\widetilde{E})$. Therefore, denoting by F this set, we have

$$D(F) \leq 2D(\widetilde{E}) \leq 2^n D(E)$$

and

$$\alpha(E) \leq C^{n-1}\alpha(\widetilde{E}) \leq 3C^{n-1}\alpha(F). \qquad \square$$

Having proved Theorem 40, from now on we may assume that E is an n-symmetric set such that $E \subset [-L, L]^n$ for some L depending only on n, $|E| = |B|$. We now want to pass from n-symmetric sets to *axially symmetric* sets, i.e., sets E having an axis of symmetry such that every non-empty cross-section of E perpendicular to this axis is a $(n-1)$-dimensional ball.

In order to perform this further simplification, let us recall the definition of *Schwartz symmetrization* of a set E (Fig. 18). To this aim, given a measurable set E, for all $t \in \mathbb{R}$ we set

$$E_t = \{x \in \mathbb{R}^{n-1} : (x, t) \in E\}.$$

A result due to Vol'pert states that if E is a set of finite perimeter then E_t is a set of finite perimeter in $\mathbb{R}^{n-1}$ for a.e. $t \in \mathbb{R}$. For a proof of this important property see for instance [6, Theorem 2.4].

Definition 41 Given a measurable set $E \subset \mathbb{R}^n$, its *Schwartz symmetrization* is defined as

$$E^* = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} : t \in \mathbb{R},\ |x| < r_E(t)\},$$

where $\omega_{n-1} r_E^{n-1}(t) = \mathcal{H}^{n-1}(E_t)$.

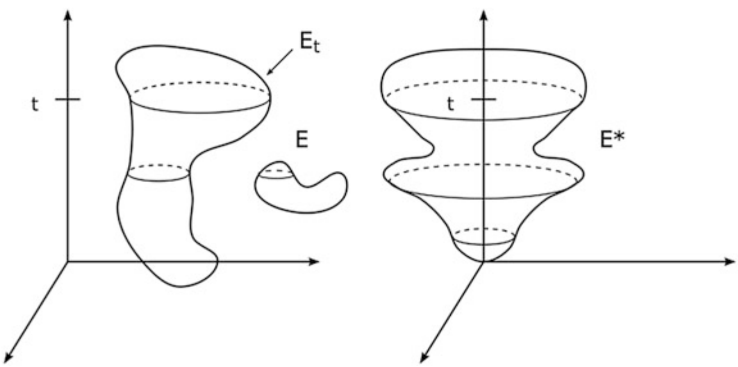


Fig. 18 The Schwartz symmetrization of the set E

Note that $|E^*| = |E|$. Moreover, as for Steiner symmetrization, also Schwartz symmetrization decreases the perimeter. The next result (see [44, Lemma 3.3]) provides a useful formula for the perimeter of an axially symmetric set whose boundary has no horizontal flat parts. To this aim, given a measurable set $E \subset \mathbb{R}^n$, for $\mathcal{H}^1$-a.e. $t \in \mathbb{R}$ we set

$$v_E(t) := \mathcal{H}^{n-1}(E_t), \qquad p_E(t) := P_{n-1}(E_t),$$

where $P_{n-1}(\cdot)$ denotes the perimeter of a subset of $\mathbb{R}^{n-1}$. Observe that this definition makes sense since for $\mathcal{H}^1$-a.e. $t \in \mathbb{R}$ the slice E_t is a set of finite perimeter in $\mathbb{R}^{n-1}$. Note also that from Definition 41 we have $v_E(t) = v_{E^*}(t)$ for all t. Moreover, the isoperimetric inequality in $\mathbb{R}^{n-1}$ yields that $p_{E^*}(t) \le p_E(t)$, since $(E^*)_t$ is a ball with the same measure of E_t.

Theorem 42 *Let $E \subset \mathbb{R}^n$ be a set of finite perimeter and let E^* be its Steiner symmetrization. Then*

$$P(E^*) \le P(E)\,. \tag{41}$$

Moreover, if

$$\mathcal{H}^{n-1}(\partial^* E \cap \{\nu_t^E = \pm 1\}) = 0, \tag{42}$$

then v_E belongs to $W^{1,1}(\mathbb{R})$ and the following formulas hold:

$$P(E) \ge \int_{\mathbb{R}} \sqrt{v_E'^2 + p_E^2}\, dt, \qquad P(E^*) = \int_{\mathbb{R}} \sqrt{v_E'^2 + p_{E^*}^2}\, dt\,.$$

The next step in the proof of the quantitative isoperimetric inequality is given by the following theorem, which states that we may eventually reduce to the case of axially symmetric sets.

Theorem 43 *Let $E \subset [-L, L]^n$ be an n-symmetric set satisfying (42) such that $|E| = |B|$ and $D(E) \le 1$. If $n = 2$ or if $n \ge 3$ and the quantitative isoperimetric inequality (35) holds true in $\mathbb{R}^{n-1}$, there exists a constant C depending only on n such that*

$$\alpha(E) \le \alpha(E^*) + C\sqrt{D(E)}, \quad \textit{and} \quad D(E^*) \le D(E). \tag{43}$$

We shall give the proof of this theorem at the end of this section. First, we show how to conclude the proof of the quantitative isoperimetric inequality (35) by combining this result with a final estimate for axially symmetric sets. This estimate is provided by the next theorem, which is a particular case of a more general one proved by Hall in [47, Theorem 2] for general axially symmetric sets. As we already mentioned, two different proofs of Theorem 44 below are given in [44, Sect. 4] and in [50, Sect. 7].

Theorem 44 *Let $E \subset [-L, L]^n$ be an axially and n-symmetric set with center of symmetry at the origin, such that $|E| = |B|$. Then*

$$|E \Delta B(x_0)| \leq C' \sqrt{D(E)}, \tag{44}$$

for some constant C' depending only on the dimension n.

The two previous theorems immediately yield the proof of (35).

Proof of Theorem 34 We argue by induction on the dimension n assuming that either $n = 2$ or $n \geq 3$ and the isoperimetric inequality (35) holds in $\mathbb{R}^{n-1}$.

As we already observed, in order to prove (35) it is enough to consider a set $E \subset [-L, L]^n$, such that $|E| = |B|$ and that $D(E) \leq \delta$ for some conveniently small $\delta \in (0, 1)$. Moreover, since the set of directions $\nu \in \mathbb{S}^{n-1}$ such that $\mathcal{H}^{n-1}(\partial^* E \cap \{\nu_t^E = \pm 1\}) > 0$ is at most countable, by rotating E if necessary we may always assume that (42) holds. Recall that Theorem 40 allows us to replace E by a n-symmetric set $F \subset [-2L, 2L]^n$ satisfying (40). And observe that from the proof of Theorem 40 and the statement of Lemma 39 it is clear that also F satisfies (42). Therefore, by replacing E with the n-symmetric set F if necessary, we may always reduce the proof of (35) to the case of a set E satisfying all the assumptions of Theorem 41.

Thus, recalling (43) and applying (44) to E^* we conclude, assuming without loss of generality that the center of symmetry of E^* is at the origin,

$$\begin{aligned} \alpha(E) &\leq \alpha(E^*) + C\sqrt{D(E)} \leq |E^* \Delta B| + C\sqrt{D(E)} \\ &\leq C' \sqrt{D(E^*)} + C\sqrt{D(E)} \leq C'' \sqrt{D(E)}\,. \end{aligned}$$

where the constant C'' depends only on the dimension n. □

We now turn to the proof of Theorem 43.

Proof of Theorem 43 Denoting by $B(x_0)$ an optimal ball for E^*, we have

$$\alpha(E) \leq |E \Delta B(x_0)| \leq |E^* \Delta B(x_0)| + |E \Delta E^*| = \alpha(E^*) + |E \Delta E^*|.$$

Hence, in order to prove the first inequality in (43) it is enough to show that

$$|E \Delta E^*| \leq c(n) \sqrt{D(E)}, \tag{45}$$

for some positive constant c depending only on n. The second inequality in (43) follows immediately from (41). To prove (45) we use again Theorem 42 to estimate

$$\begin{aligned}
D(E) = P(E) - P(B) \geq P(E) - P(E^*) &\geq \int_{\mathbb{R}} \sqrt{v_E'^2 + p_E^2} - \sqrt{v_E'^2 + p_{E^*}^2}\, dt \\
&= \int_{\mathbb{R}} \frac{p_E^2 - p_{E^*}^2}{\sqrt{v_E'^2 + p_E^2} + \sqrt{v_E'^2 + p_{E^*}^2}}\, dt \\
&\geq \left(\int_{\mathbb{R}} \sqrt{p_E^2 - p_{E^*}^2}\, dt \right)^2 \frac{1}{\int_{\mathbb{R}} \sqrt{v_E'^2 + p_E^2} + \sqrt{v_E'^2 + p_{E^*}^2}\, dt} \\
&\geq \left(\int_{\mathbb{R}} \sqrt{p_E^2 - p_{E^*}^2}\, dt \right)^2 \frac{1}{P(E) + P(E^*)},
\end{aligned}$$

where the inequality before the last one follows from Hölder's inequality. Since $D(E) \leq 1$, we have $P(E^*) \leq P(E) \leq P(B) + 1$. Therefore from the above estimate we get, recalling that $p_E \geq p_{E^*}$,

$$\begin{aligned}
\sqrt{D(E)} &\geq c_n \int_{\mathbb{R}} \sqrt{p_E^2 - p_{E^*}^2}\, dt \\
&= c_n \int_{\mathbb{R}} \sqrt{p_E + p_{E^*}} \sqrt{p_{E^*}} \sqrt{(p_E - p_{E^*})/p_{E^*}}\, dt \\
&\geq \sqrt{2} c_n \int_{\mathbb{R}} p_{E^*} \sqrt{(p_E - p_{E^*})/p_{E^*}}\, dt\,.
\end{aligned} \tag{46}$$

Now assume that $n \geq 3$ and observe that since $(E^*)_t$ is an $(n-1)$–dimensional ball of radius $r_E(t)$ with $\omega_{n-1} r_E^{n-1}(t) = \mathcal{H}^{n-1}(E_t)$, the ratio

$$\frac{p_E(t) - p_{E^*}(t)}{r_E^{n-2}(t)}$$

is precisely the isoperimetric gap of E_t in $\mathbb{R}^{n-1}$. Since by assumption, the quantitative isoperimetric inequality (35) holds true in $\mathbb{R}^{n-1}$, we have

$$\kappa(n-1) \sqrt{\frac{p_E(t) - p_{E^*}(t)}{r_E^{n-2}(t)}} \geq \alpha_{n-1}(E_t),$$

where $\alpha_{n-1}(E_t)$ is the $(n-1)$-dimensional Fraenkel asymmetry of E_t. But E_t is an $(n-1)$-symmetric set in $\mathbb{R}^{n-1}$ and $(E^*)_t$ is the ball centered at the center of symmetry of E_t. Therefore from Lemma 38 we get

$$\kappa(n-1) \sqrt{\frac{p_E(t) - p_{E^*}(t)}{r_E^{n-2}(t)}} \geq \alpha_{n-1}(E_t) \geq \frac{1}{3} \frac{\mathcal{H}^{n-1}(E_t \Delta (E^*)_t)}{r_E^{n-1}(t)}.$$

Inserting this inequality in (46) we then get

$$\sqrt{D(E)} \geq c \int_{\mathbb{R}} r_E^{n-2}(t) \sqrt{\frac{p_E(t) - p_{E^*}(t)}{r_E^{n-2}(t)}}\, dt \geq c \int_{\mathbb{R}} \frac{\mathcal{H}^{n-1}(E_t \Delta E_t^*)}{r_E(t)}\, dt$$

$$\geq \frac{c}{L} \int_{-L}^{L} \mathcal{H}^{n-1}(E_t \Delta E_t^*)\, dt = \frac{c}{L} |E_t \Delta E_t^*|,$$

where the inequality before the last one follows from the inclusion $E \subset [-L, L]^n$ and the last equality is just Fubini's theorem. This proves (45). Hence the assertion follows when $n \geq 3$.

If $n = 2$, since E is 2-symmetric, either E_t is a symmetric interval (and thus $E_t = E_t^*$) or E_t is the union of at least two essentially disjoint intervals and thus $p_E(t) \geq 4$, while $p_{E^*}(t) = 2$. Note also that since $E \subset [-L, L]^2$, then $\mathcal{H}^1(E_t \Delta E_t^*) \leq 2L$ for all $t \in \mathbb{R}$. Therefore, from (46) we easily get

$$\sqrt{D(E)} \geq \sqrt{2} c_2 \int_{\mathbb{R}} p_{E^*} \sqrt{(p_E - p_{E^*})/p_{E^*}}\, dt = 2c_2 \int_{\{t:\, E_t \neq E_t^*\}} \sqrt{p_E - p_{E^*}}\, dt$$

$$\geq 2c_2 \int_{\{t:\, E_t \neq E_t^*\}} \sqrt{2}\, dt \geq \frac{\sqrt{2} c_2}{L} \int_{\{t:\, E_t \neq E_t^*\}} \mathcal{H}^1(E_t \Delta E_t^*) dt = \frac{\sqrt{2} c_2}{L} |E \Delta E^*|,$$

thus concluding the proof also in this case. □

6 Alternative Proofs of the Quantitative Isoperimetric Inequality

In this section we discuss two different approaches to the quantitative isoperimetric inequality, the first one via the regularity theory of sets of finite perimeter and the second one via mass transportation. The latter approach will provide us with the extension of (35) to the anisotropic perimeter, a result that cannot be achieved via symmetrization techniques. In the final part of this section we shall give an account of a stronger version of (35) which is very much in the spirit of the estimate (22) proved for nearly spherical sets.

We start by presenting the approach to the quantitative isoperimetric inequalities introduced by Cicalese and Leonardi in [23] with some further simplifications due to Acerbi, Morini and the author, see [1]. In comparing this new proof with the one that we have seen in the previous section one can see two main differences. The proof by symmetrization is more elementary since it relies on some geometric ideas that do not require the use of deep previous results. But that proof is quite long. The approach of Cicalese and Leonardi to the quantitative isoperimetric inequality is based on the deep results of De Giorgi's regularity theory for area minimizing sets of finite perimeter, but it has the advantage of providing a quicker proof. Moreover this approach has proved to be useful in the study of the stability of other inequalities, see [1, 9, 10, 12, 25].

As we said before we need a regularity result on area minimizing sets of finite perimeter or, more generally, of area almost minimizers.

Definition 45 Let ω, r be positive numbers. A set E of finite perimeter is an (ω, r)-*area almost minimizer* if, for all balls $B_\varrho(x_0)$ with $\varrho < r$ and all measurable sets F such that $E\Delta F \subset\subset B_\varrho(x_0)$, we have

$$P(E) \leq P(F) + \omega\varrho^n.$$

So, an almost minimizer minimizes the perimeter with respect to local variations of the set up to a higher order volume term. De Giorgi's regularity theory, originally established only for minimizers, readily extends to almost minimizers, see [60, Sects. 1.9 and 1.10] and [51, Theorems 26.5 and 26.6].

Theorem 46 *If E is an (ω, r)-area almost minimizer, then $\partial^* E$ is a manifold of class $C^{1,1/2}$, $\partial E \setminus \partial^* E$ is relatively closed in ∂E and $H^s(\partial E \setminus \partial^* E) = 0$ for all $s > n - 8$.*

Moreover, if E_j is a sequence of equibounded (ω, r)-area almost minimizers converging in measure to an open set E of class C^2, then for j large each E_j is of class $C^{1,1/2}$ and the sequence E_j converges to E in $C^{1,\alpha}$ for all $0 < \alpha < 1/2$.

Next lemma is a simple consequence of the isoperimetric inequality.

Lemma 47 *If $\Lambda > n$, the unique solution up to translations of the problem*

$$\min \left\{P(F) + \Lambda\big||F| - |B|\big| : F \subset \mathbb{R}^n\right\} \tag{47}$$

is the unit ball.

Proof By the isoperimetric inequality it follows that in order to minimize the functional in (47), we may restrict to the balls B_r. Thus the above problem becomes

$$\min_{r>0} \{n\omega_n r^{n-1} + \Lambda\omega_n |r^n - 1|\},$$

which has a unique minimum for $r = 1$, if $\Lambda > n$. □

Lets us now describe how the new proof of the isoperimetric inequality works. The main idea in Cicalese and Leonardi approach was to reduce the proof of (35) to nearly spherical sets via a contradiction argument. They start by assuming that there exists a sequence of sets E_j with infinitesimal isoperimetric deficits for which the quantitative inequality does not hold. Then they replace it with a different sequence of sets F_j, still not satisfying the quantitative inequality and converging to B in C^1, thus contradicting Fuglede's Theorem 27 for nearly spherical sets. The sets F_j are constructed as the solutions of certain minimum problems and their convergence in C^1 to the unit ball is a consequence of Theorem 46.

In the following we shall set for any measurable set of finite measure

$$A(E) := \min_{x\in\mathbb{R}^n} \{|E\Delta B(x)|\}.$$

Clearly $A(E) = \alpha(E)$ if $|E| = |B|$.

Proof of (35) *by regularity* **Step 1.** Fix $R > 0$ so that the ball B_R contains the cube $[-L, L]^n$ given by Lemma 35. As we observed in the previous section, it is enough to prove (35) for a set $E \subset B_R$, with $|E| = |B|$ and with $D(E) \le \delta$ for some fixed $\delta > 0$. Thus, let us argue by contradiction assuming that there exists a sequence $E_j \subset B_R$, $|E_j| = |B|$, with $D(E_j) \to 0$ and

$$D(E_j) \le C_0\alpha(E_j)^2, \tag{48}$$

for some constant C_0 to be chosen later. Observe that Lemma 36 implies that $A(E_j) = \alpha(E_j) \to 0$. Let us now introduce a new sequence F_j, where for each j the set F_j is a minimizer of the following problem

$$\min \{P(F) + |A(F) - A(E_j)| + \Lambda ||F| - |B|| : F \subset B_R\},$$

where $\Lambda > n$ is a fixed constant. Note that the penalization term $\Lambda||F| - |B||$ forces the minimizers F_j to have almost the same volume of the unit ball, while the presence of $|A(F) - A(E_j)|$ has the effect that the asymmetry of F_j is very close to the one of E_j, hence converges to zero.

Since the perimeters of the F_j are equibounded, the compactness Theorem 15 implies that, up to a not relabeled subsequence, they converge in measure to some set F_∞. Moreover, the lower semicontinuity of the perimeter immediately yields that F_∞ is a minimizer of the problem: $\min \{P(E) + A(E) + \Lambda||E| - |B|| : E \subset B_R\}$. Therefore, for every set E of finite perimeter, from Lemma 47 we have

$$P(F_\infty) + A(F_\infty) + \Lambda||F_\infty| - |B|| \le P(B) \le P(E) + \Lambda||E| - |B||.$$

In particular, F_∞ is a minimizer of the problem in (47), hence Lemma 47 implies that F_∞ is a ball and thus that the F_j converge in measure to some ball $B_1(x_0)$.

We now want to show that the there exists $\omega > 0$ such that all sets are F_j are (ω, R)-area almost minimizers. This fact, thanks to Theorem 46, will imply the convergence to $B_1(x_0)$ in C^1. To prove the almost minimality of the F_j, let us fix a set F such that $F_j\Delta F \subset\subset B_r(x)$ for some ball $B_r(x)$ with radius $r < R$ and let us consider two cases.

First. let us assume that $F \subset B_R$. Then, by the minimality of F_j we get

$$\begin{aligned}
P(F_j) &\le P(F) + |A(F) - A(E_j)| - |A(F_j) - A(E_j)| + \Lambda\big[||F| - |B|| - ||F_j| - |B||\big] \\
&\le P(F) + |A(F) - A(F_j)| + \Lambda \,\big||F| - |F_j|\big| \\
&\le P(F) + (\Lambda + 1)|F\Delta F_j| \le P(F) + (\Lambda + 1)\omega_n r^n.
\end{aligned}$$

If instead $|F \setminus B_R| > 0$, we split F in two parts, one inside and the other one outside B_R. Hence,

$$P(F_j) - P(F) = [P(F_j) - P(F \cap B_R)] + [P(F \cap B_R) - P(F)].$$

Since $F \cap B_R \subset B_R$, as before we have

$$P(F_j) - P(F \cap B_R) \leq (\Lambda + 1)\omega_n r^n,$$

while

$$P(F \cap B_R) - P(F) = P(B_R) - P(F \cup B_R) \leq 0$$

by the isoperimetric inequality. Therefore we may conclude that the sets F_j are all $((\Lambda + 1)\omega_n, R)$–almost minimizers and that the sequence F_j converges to $B_1(x_0)$ in $C^{1,\alpha}$ for all $\alpha < 1/2$.

Step 2. By the minimality of the F_j, recalling (48) and using Lemma 47, we get

$$\begin{aligned} P(F_j) + \Lambda\big||F_j| - |B|\big| + |A(F_j) - A(E_j)| &\leq P(E_j) \\ &\leq P(B) + C_0 A(E_j)^2 \leq P(F_j) + \Lambda\big||F_j| - |B|\big| + C_0 A(E_j)^2. \end{aligned} \tag{49}$$

Therefore, $|A(F_j) - A(E_j)| \leq C_0 A(E_j)^2$. Since $A(E_j) \to 0$, we get that $A(F_j)/A(E_j) \to 1$.

To conclude the proof we need only to rescale the sets F_j to the same volume of the unit ball by setting $\widetilde{F}_j = \lambda_j F_j + x_j$, where $\lambda_j^n |F_j| = |B|$ and x_j is chosen so that $\widetilde{F}_j$ has the baricenter at the origin. Note that $\lambda_j \to 1$ since the F_j converge in C^1 to a unit ball. Observe also that, since $P(F_j) \to P(B)$ and $\Lambda > n$, for j large we have $P(F_j) < \Lambda |F_j|$. Thus for j large we have also

$$\big|P(\widetilde{F}_j) - P(F_j)\big| = P(F_j)|\lambda_j^{n-1} - 1| \leq P(F_j)|\lambda_j^n - 1| \leq \Lambda |\lambda_j^n - 1||F_j| = \Lambda \big||\widetilde{F}_j| - |F_j|\big|.$$

From this estimate, recalling (49) we get that

$$P(\widetilde{F}_j) \leq P(F_j) + \Lambda\big||\widetilde{F}_j| - |F_j|\big| = P(F_j) + \Lambda\big||F_j| - |B|\big| \leq P(B) + C_0 A(E_j)^2. \tag{50}$$

However, since $A(F_j)/A(E_j) \to 1$ as $j \to \infty$, we have $A(E_j)^2 < 2A(\widetilde{F}_j)^2$ for j large. Therefore, from (50) we obtain

$$P(\widetilde{F}_j) - P(B) < 2C_0 A(\widetilde{F}_j)^2,$$

which leads to contradiction to (22) if $C_0 < 1/(16\omega_n)$, since the $\widetilde{F}_j$ are converging in C^1 to B, This contradiction concludes the proof. □

In the remaining part of this section we will present two extensions of the isoperimetric inequality (35). The first one deals with the *anisotropic perimeter*. We recall that if $\gamma : \mathbb{R}^n \to [0, \infty)$ is a positively 1-homogeneous function such that $\gamma(x) > 0$ for all $x \neq 0$ the *anisotropic perimeter* associated with γ is defined for any set E of locally finite perimeter by setting

$$P_\gamma(E) := \int_{\partial^* E} \gamma(\nu^E(x)) d\mathcal{H}^{n-1}(x).$$

It is well known that the isoperimetric sets with respect to this perimeter are the homothetic and translated of the so called *Wulff shape set* associated to γ, see [38] and also [26] for two-dimensional case, which is given by

$$W_\gamma := \{x \in \mathbb{R}^n : \langle x, \nu\rangle - \gamma(\nu) < 0 \ \text{ for all } \nu \in \mathbb{S}^{n-1}\}.$$

Then, the anisotropic isoperimetric inequality states that

$$P_\gamma(E) \geq P_\gamma(W_\gamma)$$

for all sets of finite perimeter such that $|E| = |W_\gamma|$, with equality holding if and only if E is a translated of the Wulff shape set W_γ.

The quantitative version of the anisotropic isoperimetric inequality was proved by Figalli, Maggi and Pratelli in [34]. It states that there exists a constant C, depending only on n, such that for any set of finite perimeter E such that $|E| = r^n|W_\gamma|$

$$\alpha_\gamma(E)^2 \leq CD_\gamma(E), \tag{51}$$

where

$$\alpha_\gamma(E) := \min_{x\in\mathbb{R}^n}\left\{\frac{|E\Delta(x + rW_\gamma)|}{r^n}\right\}, \qquad D_\gamma(E) := \frac{P_\gamma(E) - P_\gamma(rW_\gamma)}{r^{n-1}}$$

denote the *anisotropic asymmetry index* and the *anisotropic isoperimetric deficit*, respectively.

Since the Wulff shape W_γ can be any bounded open convex set, it is clear that no symmetrization argument can be used to prove the anisotropic isoperimetric inequality or its quantitative counterpart (51). An extra difficulty is also due to the extreme rigidity of the anisotropic perimeter which is not invariant by rotation. Moreover, even the equality $P_\gamma(E) = P_\gamma(\mathbb{R}^n \setminus E)$ holds true only if γ is symmetric with respect to the origin. Observe also that since the Wulff shape set W_γ is in general a non smooth convex set, no strategy based on regularity may ever work. And in fact a completely different strategy was devised in [34] to prove inequality (51), based on optimal mass transportation and on the proof of the isoperimetric inequality given by Gromov in [55].

The idea of this proof, that we present in the simpler case of the standard perimeter, is to use a *transport map* from the set E to the an isoperimetric set of the same volume. Though the original proof of Gromov used the *Knothe map*, which has the advantage of being defined by an explicit construction, it is more convenient to use the so called *Brenier map* whose properties are stated in the following theorem, see [14], and also [53] and [54].

Theorem 48 *Let E be a set of finite perimeter with $|E| = |B|$. There exists a convex function $\varphi : \mathbb{R}^n \to \mathbb{R}$ such that if we set $T = \nabla\varphi$, then $T(x) \in B$ for a.e. $x \in \mathbb{R}^n$ and* $\det \nabla T(x) = 1$ *for a.e. $x \in E$.*

Let us now give the

Gromov's proof of the isoperimetric inequality Being the gradient of a convex function, T is a BV map, see [29, Sect. 6.3, Theorem 2]. However, in order to avoid unnecessary technical difficulties, let us assume that T is Lipschitz. For every $x \in E$ let us denote by $\lambda_i(x)$, $i = 1, \dots, n$, the eingenvalues of the symmetric matrix $\nabla T(x)$. Using the arithmetic–geometric mean inequality, we have

$$
\begin{aligned}
P(B) = n\omega_n = n\int_{B_1} dy = n\int_E (\det \nabla T)^{1/n} dx = n\int_E (\lambda_1 \dots \lambda_n)^{1/n} dx \\
\leq \int_E (\lambda_1 + \dots + \lambda_n) dx = \int_E \operatorname{div} T\, dx = \int_{\partial E} T \cdot \nu^E d\mathcal{H}^{n-1} \leq P(E).
\end{aligned}
$$

Moreover, since $\det \nabla T(x) = 1$, if $P(E) = P(B)$ we have that $\lambda_1(x) = \lambda_2(x) = \dots = \lambda_n(x) = 1$ for a.e. $x \in E$. Therefore, T is a translation and E is a ball. □

Beside being extremely simple, this argument gives some non trivial quantitative information. In fact, by subtracting the last and the first terms in the above chain of inequalities we get that

$$
\int_E \left[(\lambda_1 + \dots \lambda_n)/n - (\lambda_1 \dots \lambda_n)^{1/n}\right] \leq \frac{1}{n} D(E), \tag{52}
$$

$$
\int_{\partial E} (1 - T \cdot \nu^E) d\mathcal{H}^{n-1} \leq D(E). \tag{53}
$$

The first inequality (52) is telling us that if the isoperimetric deficit $D(E)$ is small the eigenvalues of $T(x)$ are almost equal, at least in an integral sense. From this inequality one can deduce, see [34, Corollary 2.4]), that there exists a constant c depending only on n such that

$$
\int_E |\nabla T - I| \leq c\sqrt{D(E)}\,. \tag{54}
$$

Let us assume, without loss of generality, that B is the optimal ball for E and let us observe, as proved in [34, Lemma 3.5], that

$$|E\Delta B| \le c(n)\int_{\partial^* E} \big|1-|x|\big|\,d\mathcal{H}^{n-1}.$$

Then, in order to prove (51) one should control the right hand side of the previous inequality with $\sqrt{D(E)}$. To this aim, using (53) we have

$$\int_{\partial^* E} \big|1-|x|\big|\,d\mathcal{H}^{n-1} \le \int_{\partial^* E} \big[\big|1-|T(x)|\big| + \big||T(x)|-|x|\big|\big]\,d\mathcal{H}^{n-1}$$
$$\le \int_{\partial^* E} \Big[\Big(1-T(x)\cdot \nu^E(x)\Big) + |T(x)-x|\Big]\,d\mathcal{H}^{n-1} \le D(E) + \int_{\partial^* E} |T(x)-x|\,d\mathcal{H}^{n-1}.$$

The difficult part of the proof of Figalli, Maggi and Pratelli consists in showing that if the isoperimetric deficit $D(E)$ is small one may always reduce to the case when a Poincaré type inequality for the boundary traces holds with a constant $c(n)$ depending only on n. If this is true, recalling (54), one gets

$$\int_{\partial^* E} |T(x)-x|\,d\mathcal{H}^{n-1} \le c(n)\int_E |\nabla T - I| \le c\sqrt{D(E)}\,.$$

Beside providing an alternative proof of the quantitative isoperimetric inequality in the wider framework of anisotropic perimeter, the paper by Figalli, Maggi and Pratelli contains several interesting results. In particular, Theorem 3.4 in [34] states that given any set of finite perimeter E with small deficit one may always extract from E a maximal set for which a trace inequality holds with a universal constant. This is a new and deep result that may have several applications. Moreover, the mass transportation approach used in [34] has been also successfully used to obtain the quantitative versions of other important inequalities (see [21, 33, 35]).

At the beginning of this section we observed how the proof of the isoperimetric inequality of Cicalese and Leonardi shows that one may always reduce to the case of a nearly spherical set and thus to Fuglede's Theorem 27. However, the two sets E and F in Fig. 19 have the same measure, the same asymmetry index, but $D(E) << 1$, while $D(F) >> 1$. Therefore the quantitative isoperimetric inequality (35) gives a sharp information on the set E while gives no information at all on the set F. The reason is that while the asymmetry index looks only at the distance in measure of a set from a ball, the isoperimetric gap encodes also an information on the oscillation of the boundary of the set.

This suggests that we should introduce a more precise index which takes into account also the oscillation of the normals to the boundary of the set E. To this aim, given a set of finite perimeter E and a ball $B_r(y)$ with the same volume of E, we are

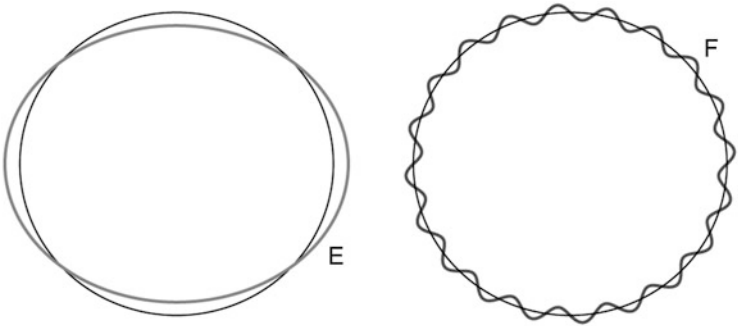


Fig. 19 E and F have the same measure and the same Fraenkel asymmetry

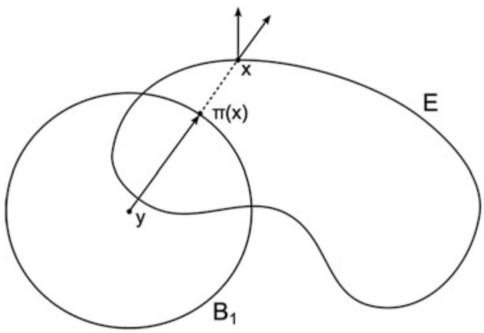


Fig. 20 The construction of the asymmetry index $\beta(E)$

going to measure the distance from E to the ball in the following way (see Fig. 20). For every point $x \in \partial^* E$ we take the projection $\pi_{y,r}(x)$ of x on the boundary of $\partial B_r(y)$ and consider the distance $|\nu^E(x) - \nu^{r,y}(\pi_{y,r}(x))|$ between the exterior normal to E at the point x and the exterior normal to $B_r(y)$ at the projection point $\pi_{y,r}(x)$. Then, we take the L^2 norm of this distance and minimize the resulting norm among all possible balls, thus getting

$$\beta(E) := \min_{y \in \mathbb{R}^n} \left\{ \left(\frac{1}{2r^{n-1}} \int_{\partial^* E} |\nu^E(x) - \nu^{r,y}(\pi_{y,r}(x))|^2 \, d\mathcal{H}^{n-1}(x) \right)^{1/2} \right\}.$$

We shall refer to $\beta(E)$ as to *the oscillation index* of E. Observe that Fuglede's Theorem 27 provides an estimate for both the asymmetry and the oscillation index. In fact, if E is a nearly spherical set satisfying (21) with a sufficiently small ε, recall, see (33), that for every point $x \in \partial^* E$ the exterior normal to E is given by

$$\nu^E(x) = \frac{z(1 + u(z)) - \nabla_\tau u(z)}{\sqrt{(1 + u(z))^2 + |\nabla_\tau u(z)|^2}},$$

where $z = x/|x|$ and thus $x = z(1 + u(z))$. Thus, from (22) we have

$$\begin{aligned}
\alpha(E)^2+\beta(E)^2 &\leq |E\Delta B|^2 + \frac{1}{2}\int_{\partial^* E}\Big|\nu^E(x)-\frac{x}{|x|}\Big|^2 d\mathcal{H}^{n-1} \\
&= |E\Delta B|^2 + \int_{\partial^* E}\Big(1-\nu^E(x)\cdot\frac{x}{|x|}\Big)d\mathcal{H}^{n-1} \\
&\leq c\int_{\mathbb{S}^{n-1}}|u|^2 d\mathcal{H}^{n-1} + c\int_{\mathbb{S}^{n-1}}\Big(1-\frac{1+u(z)}{\sqrt{(1+u)^2+|\nabla u|^2}}\Big)d\mathcal{H}^{n-1} \\
&= c\int_{\mathbb{S}^{n-1}}|u|^2 d\mathcal{H}^{n-1} + c\int_{\mathbb{S}^{n-1}}\frac{\sqrt{(1+u)^2+|\nabla u|^2}-(1+u)}{\sqrt{(1+u)^2+|\nabla u|^2}}\,d\mathcal{H}^{n-1} \\
&\leq c\int_{\mathbb{S}^{n-1}}|u|^2 d\mathcal{H}^{n-1} + c\int_{\mathbb{S}^{n-1}}|\nabla u|^2\,d\mathcal{H}^{n-1} \leq cD(E).
\end{aligned}$$

Next result, proved by Julin and the author in [42], is an improved version of the quantitative isoperimetric inequality.

Theorem 49 *There exists a constant $\gamma(n)$ such that for any set of finite perimeter E*

$$\beta(E)^2 \leq \gamma D(E).$$

Note that the inequality above is stronger than the quantitative isoperimetric inequality (35) since it can be shown (see [42, Proposition 1.2]) that there exists a constant $C(n)$ such that

$$\alpha(E) + \sqrt{D(E)} \leq C\beta(E).$$

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