Chapter 9
Analysis of the Positivity and Stability of Fractional Discrete-Time Nonlinear Systems

Tadeusz Kaczorek

Abstract The positivity and asymptotic stability of the discrete-time nonlinear systems are addressed. Necessary and sufficient conditions for the positivity and sufficient conditions for the asymptotic stability of the nonlinear systems are established. The proposed stability tests are based on an extension of the Lyapunov method to the positive nonlinear systems. The effectiveness of the tests are demonstrated on examples.

Keywords Positivity · Stability · Fractional · Nonlinear · System

9.1 Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial condition state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive system theory is given in the monographs [8, 11] and in the papers [10, 12, 18, 21]. Models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

The Lyapunov, Bohl and Perron exponents and stability of time-varying discrete-time linear systems have been investigated in [1–7]. The positive standard and descriptor systems and their stability have been analyzed in [10–12, 18, 21]. The positive linear systems with different fractional orders have been addressed in [12, 13] and the descriptor discrete-time linear systems in [9, 10]. Descriptor positive discrete-time and continuous-time nonlinear systems have been analyzed in [14, 19, 20] and the positivity and linearization of nonlinear discrete-time systems by state-feedbacks in [18]. The minimum energy control of positive linear systems has been addressed in [15–17]. The stability and robust stabilization of discrete-time switched systems have been analyzed in [23, 24].

T. Kaczorek (✉)
Faculty of Electrical Engineering, Bialystok University of Technology, Bialystok, Poland
e-mail: kaczorek@isep.pw.edu.pl

F. Cacace et al. (eds.), *Positive Systems*, Lecture Notes in Control and Information Sciences 471, DOI 10.1007/978-3-319-54211-9_9

In this chapter the positivity and asymptotic stability of the fractional discrete-time nonlinear systems will be investigated.

The chapter is organized as follows. In Sect. 9.2 the definitions and theorems concerning the positivity and stability of positive discrete-time and continuous-time linear systems are recalled. Necessary and sufficient conditions for the positivity of the fractional discrete-time nonlinear systems are established in Sect. 9.3. The asymptotic stability of the positive nonlinear systems is addressed in Sect. 9.4, where the sufficient conditions for the stability are proposed. Concluding remarks are given in Sect. 9.5.

The following notation will be used: $\mathbb{R}$—the set of real numbers, $\mathbb{R}^{n\times m}$—the set of $n \times m$ real matrices, $\mathbb{R}_+^{n\times m}$—the set of $n \times m$ matrices with nonnegative entries and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$, $\mathbb{Z}_+$—the set of nonnegative integers, $\mathbb{M}_n$—the set of $n \times n$ Metzler matrices (with nonnegative off-diagonal entries), $\mathbb{I}_n$—the $n \times n$ identity matrix.

9.2 Positive Discrete-Time and Continuous-Time Linear Systems and Their Stability

Consider the discrete-time linear system

$$x_{i+1} = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+ = \{0, 1, \ldots\}, \tag{9.1a}$$

$$y_i = Cx_i + Du_i, \tag{9.1b}$$

where $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$.

Definition 9.1 [8, 11] The discrete-time linear system (9.1) is called (internally) positive if $x_i \in \mathbb{R}_+^n$, $y_i \in \mathbb{R}_+^p$, $i \in \mathbb{Z}_+$ for any initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$.

Theorem 9.1 [8, 11] *The discrete-time linear system (9.1) is positive if and only if*

$$A \in \mathbb{R}_+^{n\times n}, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}.$$

Definition 9.2 [8, 11] The positive discrete-time linear system (9.1) is called asymptotically stable if

$$\lim_{i\to\infty} x_i = 0 \quad \text{for any} \quad x_0 \in \mathbb{R}_+^n.$$

Theorem 9.2 *The positive discrete-time linear system (9.1) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

1. *all coefficients of the polynomial*

$$p_n(z) = \det[\mathbb{I}_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$$

are positive, i.e. $a_i > 0$ *for* $i = 0, 1, \ldots, n-1$.

2. *all principal minors of the matrix* $\overline{A} = \mathbb{I}_n - A = [\overline{a}_{ij}]$ *are positive, i.e.*

$$M_1 = \left|\overline{a}_{11}\right| > 0, \quad M_2 = \begin{vmatrix} \overline{a}_{11} & \overline{a}_{12} \\ \overline{a}_{21} & \overline{a}_{22} \end{vmatrix} > 0, \ \ldots, \ M_n = \det \overline{A} > 0.$$

Proof The proof is given in [11].

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu, \tag{9.2a}$$

$$y = Cx + Du, \tag{9.2b}$$

where $x = x(t) \in \mathbb{R}^n$, $u = u(t) \in \mathbb{R}^m$, $y = y(t) \in \mathbb{R}^p$ are the state, input and output vectors and $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$.

Definition 9.3 [8, 11] The continuous-time linear system (9.2) is called (internally) positive if $x \in \mathbb{R}_+^n$, $y \in \mathbb{R}_+^p$, $t \geq 0$ for any initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u \in \mathbb{R}_+^m$, $t \geq 0$.

Theorem 9.3 [8, 11] *The continuous-time linear system (9.2) is positive if and only if*

$$A \in \mathbb{M}_n, \quad B \in \mathbb{R}_+^{n \times m}, \quad C \in \mathbb{R}_+^{p \times n}, \quad D \in \mathbb{R}_+^{p \times m},$$

Definition 9.4 [8, 11] The positive continuous-time linear system (9.2) is called asymptotically stable if

$$\lim_{t \to \infty} x(t) = 0 \quad \text{for all} \quad x_0 \in \mathbb{R}_+^n.$$

Theorem 9.4 *The positive continuous-time linear system (9.2) is asymptotically stable if and only if one of the following equivalent conditions is satisfied:*

1. *all coefficients of the polynomial*

$$p_n(s) = \det[\mathbb{I}_n s - A] = s^n + \hat{a}_{n-1}s^{n-1} + \cdots + \hat{a}_1 s + \hat{a}_0$$

are positive, i.e. $\hat{a}_k > 0$ *for* $k = 0, 1, \ldots, n-1$.

2. *all principal minors of the matrix* $\hat{A} = -A = [\hat{a}_{ij}]$ *are positive, i.e.*

$$\hat{M}_1 = \left|\hat{a}_{11}\right| > 0, \quad \hat{M}_2 = \begin{vmatrix} \hat{a}_{11} & \hat{a}_{12} \\ \hat{a}_{21} & \hat{a}_{22} \end{vmatrix} > 0, \ \ldots, \ \hat{M}_n = \det \hat{A} > 0.$$

Proof The proof is given in [11].

Theorem 9.5 *The matrix* $A \in \mathbb{M}_n$ *satisfies the condition*

$$-A^{-1} \in \mathbb{R}_{+}^{n\times n}$$

if and only if the positive system (9.2) is asymptotically stable.

Proof The proof is given in [11].

9.3 Positivity of the Fractional Nonlinear Systems

Consider the fractional discrete-time nonlinear system

$$\Delta^{\alpha} x_i = Ax_i + f(x_{i-1}, u_i), \tag{9.3a}$$

$$y_i = g(x_i, u_i) \tag{9.3b}$$

and $0 < \alpha \leq 1$, $i \in \mathbb{Z}_{+} = \{0, 1, \ldots\}$, where

$$\Delta^{\alpha} x_i = \sum_{j=0}^{i} a_j^{\alpha} x_{i-j} \tag{9.4a}$$

with

$$a_j^{\alpha} = (-1)^j \binom{\alpha}{j}, \quad \binom{\alpha}{j} = \begin{cases} 1 & \text{for } k = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } k = 1, 2, 3, \ldots \end{cases} \tag{9.4b}$$

is the α-order difference of x_i, $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $y_i \in \mathbb{R}^p$ are the state, input and output vectors, $A \in \mathbb{R}^{n\times n}$ and $f(x_{i-1}, u_i) \in \mathbb{R}^n$, $g(x_i, u_i) \in \mathbb{R}^p$ are vector functions continuous in x_i and u_i.

Note that the fractional difference (9.4a) is defined in the point "i" not as usually in the point "$i + 1$" [13, 22].

Substituting (9.4a) into (9.3a) we obtain

$$\sum_{j=0}^{i} a_j^{\alpha} x_{i-j} = Ax_i + f(x_{i-1}, u_i)$$

and

$$x_i = \sum_{j=1}^{i} A_1 c_j^{\alpha} x_{i-j} + f_1(x_{i-1}, u_i), \quad i \in \mathbb{Z}_{+}, \tag{9.5}$$

where

$$c_j^\alpha = -a_j^\alpha, \quad j = 1, \ldots, i; \quad A_1 = [\mathbb{I}_n - A]^{-1} \in \mathbb{R}^{n \times n},$$
$$f_1(x_{i-1}, u_i) = A_1 f(x_{i-1}, u_i).$$

Assuming $x_i = 0$, $i = 1, 2, \ldots$ from (9.5) for $i = 0$ we have

$$x_0 = f_1(0, u_0). \tag{9.6}$$

Therefore, the initial condition x_0 is related with u_0 by (9.6).

Lemma 9.1 *The matrix*

$$A_1 = [\mathbb{I}_n - A]^{-1} \in \mathbb{R}_+^{n \times n} \tag{9.7}$$

if and only if the positive linear system

$$x_{i+1} = Ax_i, \quad A \in \mathbb{R}_+^{n \times n} \tag{9.8}$$

is asymptotically stable.

Proof By Theorem 9.2 the positive discrete-time linear system (9.8) is asymptotically stable if and only if the matrix $A - \mathbb{I}_n \in \mathbb{M}_n$ is asymptotically stable (is Hurwitz) and by Theorem 9.5 the condition (9.7) is satisfied if the system (9.8) is asymptotically stable. □

Theorem 9.6 *The solution x_i of the Eq. (9.5) for given initial condition $x_0 \in \mathbb{R}^n$ and input $u_i \in \mathbb{R}^m$, $i \in \mathbb{Z}_+$ has the form*

$$x_i = \Phi_i x_0 + \sum_{j=1}^{i} \Phi_{i-j} f_1(x_{j-1}, u_j), \tag{9.9a}$$

where

$$\Phi_j = \sum_{k=1}^{j} c_k^\alpha A_1 \Phi_{j-k}, \quad j = 1, 2, \ldots, i; \quad \Phi_0 = \mathbb{I}_n. \tag{9.9b}$$

Proof The proof can be accomplished by induction or by checking that (9.9) satisfies the Eq. (9.5). □

In particular case for linear system

$$x_i = \sum_{j=1}^{i} A_1 c_j^\alpha x_{i-j} + B_1 u_i, \quad i \in \mathbb{Z}_+, \quad B_1 \in \mathbb{R}^{n \times m} \tag{9.10a}$$

the solution x_i has the form

$$x_i = \Phi_i x_0 + \sum_{j=1}^{i} \Phi_{i-j} B_1 u_j \tag{9.10b}$$

and the matrix Φ_j is given by (9.9b).

Remark 9.1 The solution x_i of the Eq. (9.5) can be computed using the formulae (9.9) iteratively for $i = 1, 2, \ldots$ and substituting x_{j-1} given by (9.9a) into the vector function $f_1(x_{j-1}, u_j)$ for $i = 1, 2, \ldots$.

Definition 9.5 The discrete-time nonlinear system (9.3) is called (internally) positive if $x_i \in \mathbb{R}_+^n$, $y_i \in \mathbb{R}_+^p$, $i \in \mathbb{Z}_+$ for any initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$.

Theorem 9.7 *The discrete-time nonlinear system (9.3) is positive if and only if* $0 < \alpha \leq 1$, *the matrix* $A \in \mathbb{R}_+^{n\times n}$ *is asymptotically stable and*

$$f(x_{i-1}, u_i) \in \mathbb{R}_+^n \quad for \quad x_i \in \mathbb{R}_+^n \quad and \quad u_i \in \mathbb{R}_+^m, \ i \in \mathbb{Z}_+, \tag{9.11}$$

$$g(x_i, u_i) \in \mathbb{R}_+^p \quad for \quad x_i \in \mathbb{R}_+^n \quad and \quad u_i \in \mathbb{R}_+^m, \ i \in \mathbb{Z}_+. \tag{9.12}$$

Proof Sufficiency. By Lemma 9.1 if $A \in \mathbb{R}_+^{n\times n}$ is asymptotically stable then $A_1 \in \mathbb{R}_+^{n\times n}$. It is well-known [13] that if $0 < \alpha \leq 1$ then $c_j^\alpha > 0$ for $j = 1, 2, \ldots$. Therefore, from (9.9b) we have $\Phi_j \in \mathbb{R}_+^{n\times n}$ for $j = 0, 1, 2, \ldots$ and from (9.9a) $x_i \in \mathbb{R}_+^n$ for $i = 1, 2, \ldots$, since by assumption (9.11) $f_1(x_{i-1}, u_i) = A_1 f(x_{i-1}, u_i) \in \mathbb{R}_+^n$ for $x_i \in \mathbb{R}_+^n$ and $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$. If (9.12) holds then from (9.3b) we have $y_i \in \mathbb{R}_+^p$ for $i \in \mathbb{Z}_+$.

Necessity. If $f(x_{i-1}, u_i) = 0$ then $x_i \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$ only if $A_1 \in \mathbb{R}_+^{n\times n}$ and by Lemma 9.1 implies the asymptotic stability of the matrix $A \in \mathbb{R}_+^{n\times n}$. Note that $x_i \in \mathbb{R}_+^n$ for $i \in \mathbb{Z}_+$ implies the condition (9.11). Similarly, $y_i \in \mathbb{R}_+^p$ for $i \in \mathbb{Z}_+$ implies the condition (9.12). □

9.4 Stability of the Positive Nonlinear Systems

Consider the fractional nonlinear system for zero inputs ($u_i = 0$ and $f(x_{i-1}, 0) = \bar{f}_2(x_{i-1})$ in the form

$$\Delta^\alpha x_i = A x_i + \bar{f}_2(x_{i-1}), \quad i \in \mathbb{Z}_+, \quad 0 < \alpha \leq 1 \tag{9.13}$$

or

$$x_i = \sum_{j=1}^{i} A_1 c_j^\alpha x_{i-j} + f_2(x_{i-1}), \quad i \in \mathbb{Z}_+, \quad 0 < \alpha \leq 1, \tag{9.14a}$$

where

$$f_2(x_{i-1}) = A_1 \bar{f}_2(x_{i-1}), \quad i \in \mathbb{Z}_+ \tag{9.14b}$$

and A_1 is defined by (9.7).

Definition 9.6 The fractional positive nonlinear system (9.13) is called asymptotically stable in the region $D \in \mathbb{R}_+^n$ if $x_i \in \mathbb{R}_+^n$, $i \in \mathbb{Z}_+$ and

$$\lim_{i\to\infty} x_i = 0 \quad \text{for} \quad x_0 \in D \in \mathbb{R}_+^n.$$

To test the asymptotic stability of the system the Lyapunov method will be used. As a candidate of the Lyapunov function we choose

$$V(x_i) = c^T x_i > 0 \quad \text{for} \quad x_i \in \mathbb{R}_+^n, \quad i \in \mathbb{Z}_+, \tag{9.15}$$

where $c \in \mathbb{R}_+^n$ is a vector with strictly positive components $c_i > 0$ for $i = 1, \ldots, n$.
Using (9.14) and (9.15) we obtain

$$\begin{aligned}\Delta V(x_i) &= V(x_{i+1}) - V(x_i) = c^T x_{i+1} - c^T x_i \\ &= c^T \left[\sum_{j=1}^{i+1} A_1 c_j^\alpha x_{i-j+1} + f_2(x_i) - \left(\sum_{j=1}^{i} A_1 c_j^\alpha x_{i-j} + f_2(x_{i-1})\right)\right] \\ &= c^T \left[\sum_{j=1}^{i} A_1 c_j^\alpha (x_{i-j+1} - x_{i-j}) + A_1 c_{i+1}^\alpha x_0 + f_2(x_i) - f_2(x_{i-1})\right] < 0\end{aligned}$$

and

$$\sum_{j=1}^{i} A_1 c_j^\alpha (x_{i-j+1} - x_{i-j}) + A_1 c_{i+1}^\alpha x_0 + f_2(x_i) - f_2(x_{i-1}) < 0, \quad x_i \in D \in \mathbb{R}_+^n \tag{9.16}$$

$i \in \mathbb{Z}_+$, since $c \in \mathbb{R}_+^n$ is strictly positive.
Therefore, the following theorem has been proved.

Theorem 9.8 *The positive discrete-time nonlinear system (9.13) is asymptotically stable in the region $D \in \mathbb{R}_+^n$ if the condition (9.16) is satisfied.*

Example 9.1 Consider the discrete-time nonlinear system (9.13) with

$$x_i = \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}, \quad A = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.4 \end{bmatrix}, \quad f_2(x_i) = \begin{bmatrix} x_{1,i} x_{2,i} \\ x_{2,i}^2 \end{bmatrix}.$$

In this case

$$A_1 = [\mathbb{I}_2 - A]^{-1} = \begin{bmatrix} 0.7 & -0.1 \\ -0.2 & 0.6 \end{bmatrix}^{-1} = \frac{1}{0.4}\begin{bmatrix} 0.6 & 0.1 \\ 0.2 & 0.7 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 6 & 1 \\ 2 & 7 \end{bmatrix} \in \mathbb{R}_+^{2\times 2}.$$

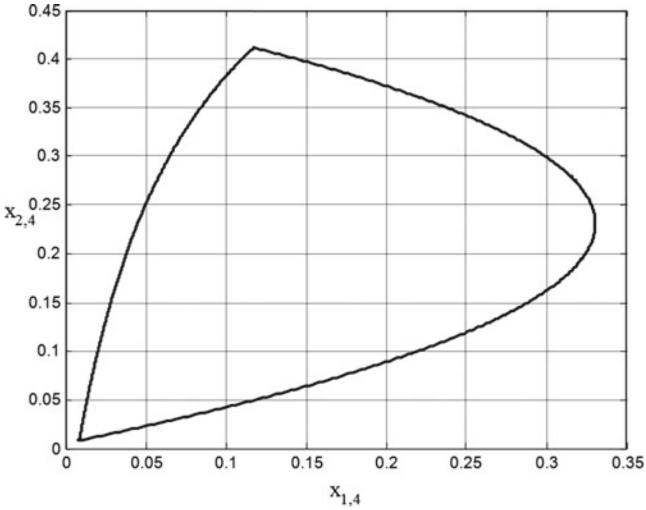

Fig. 9.1 Stability region (inside the *curved line*)

The nonlinear system is positive, since the matrix $A \in \mathbb{R}_+^{2\times 2}$ is asymptotically stable and $f_2(x_i) \in \mathbb{R}_+^2$ for all $x_i \in \mathbb{R}_+^2$, $i \in \mathbb{Z}_+$.

The region $D \in \mathbb{R}_+^2$ is defined by

$$D := \{x_{1,i}, x_{2,i}\} = \sum_{j=1}^{i} A_1 c_j^\alpha x_{i-j+1} + A_1 c_{i+1}^\alpha x_0 - x_i + f_2(x_i)$$
$$= \begin{bmatrix} 1.5\left(\sum_{j=1}^{i} c_j^\alpha x_{1,i-j+1} + c_{i+1}^\alpha x_{10}\right) + 0.25\left(\sum_{j=1}^{i} c_j^\alpha x_{2,i-j+1} + c_{i+1}^\alpha x_{20}\right) - x_{1,i} + x_{1,i}x_{2,i} \\ 0.5\left(\sum_{j=1}^{i} c_j^\alpha x_{1,i-j+1} + c_{i+1}^\alpha x_{10}\right) + 1.75\left(\sum_{j=1}^{i} c_j^\alpha x_{2,i-j+1} + c_{i+1}^\alpha x_{20}\right) - x_{2,i} + x_{2,i}^2 \end{bmatrix} \tag{9.17}$$

Let us assume

$$x_{10} = 0.1, \quad x_{20} = 0.2, \quad \alpha = 0.5, \quad i = 4. \tag{9.18}$$

The region defined by (9.17) with (9.18) is shown in Fig. 9.1.

9.5 Concluding Remarks

The positivity and asymptotic stability of the discrete-time nonlinear systems have been addressed. Necessary and sufficient conditions for the positivity of the discrete-time nonlinear systems have been established (Theorem 9.7). Using the Lyapunov direct method the sufficient conditions for asymptotic stability of the discrete-time nonlinear systems have been proposed (Theorem 9.8). The effectiveness of the conditions has been demonstrated on Example 9.1. The considerations can be extended to fractional continuous-time nonlinear systems.

An open problem is an extension of the conditions to the descriptor fractional discrete-time and continuous-time nonlinear systems.

Acknowledgements This work was supported by National Science Centre in Poland under work No. 2014/13/B/ST7/03467.

References

1. Czornik, A.: Perturbation Theory for Lyapunov Exponents of Discrete Linear Systems, vol. 17. AGH University of Science and Technology Press, Cracow (2012)
2. Czornik, A., Klamka, J., Niezabitowski, M.: On the set of Perron exponents of discrete linear systems. IFAC Proc. **47(4)**, 11740–11742 (2014)
3. Czornik, A., Newrat, A., Niezabitowski, M.: On the Lyapunov exponents of a class of the second-order discrete time linear systems with bounded perturbations. Int. J. Dyn. Syst. **4**(28), 473–483 (2013)
4. Czornik, A., Newrat, A., Niezabitowski, M., Szyda A.: On the Lyapunov and Bohl exponent of time-varying discrete linear system. In: Mediterranean Conference on Control and Automation (MED), pp. 194–197, Barcelona (2012)
5. Czornik, A., Niezabitowski, M.: Lyapunov exponents for system with unbounded coefficients. Int. J. Dyn. Syst. **2**(28), 140–153 (2013)
6. Czornik, A., Niezabitowski, M.: On the spectrum of discrete time-varying linear systems. Nonlinear Anal. Hybrid Syst. **9**, 27–41 (2013)
7. Czornik, A., Niezabitowski, M.: On the stability of Lyapunov exponents of discrete linear system. In: Proceedings of European Control Conference, pp. 2210–2213, Zurich (2013)
8. Farina, L., Rinaldi, S.: Positive Linear Systems: Theory and Applications. Wiley, New York (2000)
9. Kaczorek, T.: Positive singular discrete time linear systems. Bull. Pol. Acad. Sci. Tech. Sci. **45**(4), 619–631 (1997)
10. Kaczorek, T.: Positive descriptor discrete-time linear systems. Probl. Nonlinear Anal. Eng. Syst. **1**(7), 38–54 (1998)
11. Kaczorek, T.: Positive 1D and 2D Systems. Springer, London (2001)
12. Kaczorek, T.: Positive linear systems consisting of n subsystems with different fractional orders. IEEE Trans. Circ. Syst. **58**(6), 1203–1210 (2011)
13. Kaczorek, T.: Selected Problems of Fractional Systems Theory. Springer, Berlin (2012)
14. Kaczorek, T.: Descriptor positive discrete-time and continuous-time nonlinear systems. Proc. SPIE **9290** (2014). doi:10.1117/12.2074558
15. Kaczorek, T.: Minimum energy control of descriptor positive discrete-time linear systems. COMPEL **33**(3), 976–988 (2014)
16. Kaczorek, T.: Minimum energy control of fractional descriptor positive discrete-time linear systems. Int. J. Appl. Math. Comput. Sci. **24**(4), 735–743 (2014)
17. Kaczorek, T.: Necessary and sufficient conditions for minimum energy control of positive discrete-time linear systems with bounded inputs. Bull. Pol. Acad. Sci. Tech. Sci. **62**(1), 85–89 (2014)
18. Kaczorek, T.: Positivity and linearization of a class of nonlinear discrete-time systems by state feedbacks. Logistyka **6**, 5078–5083 (2014)

19. Kaczorek, T.: Analysis of the positivity and stability of discrete-time and continuous-time nonlinear systems. Comput. Prob. Electr. Eng. **5**(1), 11–16 (2015)
20. Kaczorek, T.: Descriptor standard and positive discrete-time nonlinear systems. Arch. Control Sci. **25**(2), 227–235 (2015)
21. Kaczorek, T.: Positivity and stability of discrete-time nonlinear systems. In: Proceedings of IEEE 2nd International Conference on Cybernetics (CYBERCONF) (2015)
22. Ostalczyk, P.: Discrete Fractional Calculus. Applications in Control and Image Processing. World Scientific (2015)
23. Zhang, J., Han, Z., Wu, H., Huang, J.: Robust stabilization of discrete-time positive switched systems with uncertainties and average dwell time switching. Circ. Syst. Signal Process. **33**(1), 71–95 (2014)
24. Zhang, H., Xie, D., Zhang, H., Wang, G.: Stability analysis for discrete-time switched systems with unstable subsystems by a mode-dependent average dwell time approach. ISA Trans. **53**(4), 1081–1086 (2014)