Chapter 5
On Feedback Transformation and Integral Input-to-State Stability in Designing Robust Interval Observers for Control Systems

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Abstract The problem of designing interval observers is addressed for output feedback control of a class of nonlinear systems in this chapter. The framework of integral input-to-state stability is exploited to drive the estimated intervals and the state variables to the origin asymptotically when disturbances converge to zero. Moreover interval observers are tuned with feedback gain. A reduced-order interval observer is proposed, and the flexibility offered by gains in designing observer is related to the existence of reduced-order interval observers. Comparative simulations are given to illustrate the theoretical results.

Keywords Interval observers · Reduced-order observers · Nonlinear systems · Output feedback control · Guaranteed state estimation.

5.1 Introduction

Interval observers generate upper bounds and lower bounds of state variables of dynamical systems at each time instant based on given information about bounds of unknown disturbances and of unknown initial conditions [6]. The bounds give intervals where the state variables are sure to stay during transient periods in which classical observers do not provide any guarantee. The usefulness of interval estimates is evident for monitoring purposes when large disturbances or uncertainties are present [1]. A typical mechanism to allow the construction of such interval observers is to let the estimation errors be governed by positive systems. Some examples of extensive studies on design of interval observers have been reported in [4, 5, 8–14] (see also references therein).

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Recently, an interval observer was proposed in [3] for nonlinear control systems which are affine in unmeasured state variables, and it was investigated further in [7] to provide design guidelines for guaranteeing the length of estimated intervals to converge to zero for converging disturbances and guaranteeing (integral) input-to-state stability ((i)ISS) of the entire controlled system. The iISS approach developed in [7] has allowed one to deal with a larger class of nonlinearities than the original approach [3]. This chapter continues investigating the iISS framework and introduces a modification by incorporating feedback gain into the observer for control systems. The modification, in addition to state transformation of the error systems, offers flexibility in obtaining positive systems leading to tighter interval estimates and swifter convergence of the interval length and state variables of the plant to zero. This chapter also proposes a reduced-order interval observer aiming at swifter behavior of the estimates and the plant state with less control effort. It also discusses how the positivity of error systems allows the existence of a full-order observer to imply the existence of a reduced-order observer. Comparative simulations are given to illustrate these ideas.

In this chapter, the set of real numbers is denoted by $\mathbb{R}$. The set of non-negative real numbers is denoted by $\mathbb{R}_{\geq 0}$. The symbol $|\cdot|$ denotes Euclidean norm of vectors. Inequalities must be understood *component-wise*, i.e., for $a = [a_1, ..., a_n]^\top \in \mathbb{R}^n$ and $b = [b_1, ..., b_n]^\top \in \mathbb{R}^n$, $a \leq b$ if and only if, for all $i \in \{1, ..., n\}$, $a_i \leq b_i$. For a square matrix $Q \in \mathbb{R}^{n\times n}$, let $Q^+ \in \mathbb{R}^{n\times n}$ denote $Q^+ = \left(\max\{q_{i,j}, 0\}\right)_{i,j=1,1}^{n,n}$, where $Q = \left(q_{i,j}\right)_{i,j=1,1}^{n,n}$. Let $Q^- = Q^+ - Q$. This notation is limited to square matrices, and the superscripts $+$ and $-$ for other purposes are defined appropriately when they appear. A square matrix $Q \in \mathbb{R}^{n\times n}$ is said to be Metzler if each off-diagonal entry of this matrix is nonnegative. The symbol I denotes the identity matrix of appropriate dimension. For $\alpha, \beta : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, by $\alpha(s) \equiv \beta(s)$ we mean $\alpha(s) = \beta(s)$ for all $s \in \mathbb{R}_{\geq 0}$. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be positive definite and written as $\alpha \in \mathscr{P}$ if α is continuous and satisfies $\alpha(0) = 0$ and $\alpha(s) > 0$ for all $s \in (0, \infty)$. A function $\alpha \in \mathscr{P}$ is said to be of class $\mathscr{K}$ if α is strictly increasing. A class $\mathscr{K}$ function is said to be of class $\mathscr{K}_\infty$ if it is unbounded. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $\mathscr{KL}$ if, for each fixed $t \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, t)$ is of class $\mathscr{K}$ and, for each fixed $s > 0$, $\beta(s, \cdot)$ is strictly decreasing and $\lim_{t\to\infty} \beta(s, t) = 0$. Logical sum and logical product are denoted by $\vee$ and $\wedge$, respectively.

5.2 Setups and Objectives

Consider the system

$$\dot{x}(t) = A(y(t))x(t) + B(y(t))u(y(t), \hat{x}^+(t)) + \delta(t) \tag{5.1a}$$

$$y(t) = Cx(t) \tag{5.1b}$$

with time $t \in \mathbb{R}_{\geq 0}$, the state $x(t) \in \mathbb{R}^n$, the measurement output $y(t) \in \mathbb{R}^p$ and the initial condition $x(0) = x_0$, where the functions $A : \mathbb{R}^p \to \mathbb{R}^{n\times n}$ and $B : \mathbb{R}^p \to \mathbb{R}^{n\times q}$

are supposed to be locally Lipschitz, and $C \in \mathbb{R}^{p\times n}$ is a constant matrix. The term $u(y(t), \hat{x}^+(t)) \in \mathbb{R}^q$ is the control input indicating output feedback, and the function $u : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^q$ is supposed to be locally Lipschitz. The signal $\hat{x}^+(t) \in \mathbb{R}^n$ denotes an estimate of $x(t)$, which has yet to be defined. The disturbance vector $\delta : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ is supposed to be piecewise continuous. It is stressed that $x(t)$ is not measured. Instead, the output $y(t)$ is available as a measurement for all $t \in \mathbb{R}_{\geq 0}$. Assume that the vectors $x_0^-, x_0^+ \in \mathbb{R}^n$ and piecewise continuous functions $\delta^+, \delta^- : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$ satisfying

$$x_0^- \leq x_0 \leq x_0^+ \tag{5.2}$$

$$\delta^-(t) \leq \delta(t) \leq \delta^+(t), \quad \forall t \in \mathbb{R}_{\geq 0} \tag{5.3}$$

are known, while $x(0) = x_0$ and $\delta(t)$ are not known. The design problem to be addressed in this chapter is mainly to achieve two objectives simultaneously. One is to drive $x(t)$ to the origin asymptotically for an arbitrary initial condition satisfying (5.2) by output feedback control when $\delta(t)$ converges to zero. The other is to estimate an envelope $x^-(t), x^+(t) \in \mathbb{R}_{\geq 0}$ such that the framer property

$$x^-(t) \leq x(t) \leq x^+(t), \quad \forall t \in \mathbb{R}_{\geq 0} \tag{5.4}$$

holds in the presence of any piecewise continuous disturbance $\delta(t)$ satisfying (5.3). The former is for the purpose of control, and the latter is for monitoring. Other important features of the simultaneous control and monitoring problem are described mathematically in Sect. 5.4.

5.3 Observer Candidates

5.3.1 Full-Order Interval Observer

Divide the control input u into a direct output feedback term and the remainder as

$$u(y, \hat{x}^+) = K(y)y + u_a(y, \hat{x}^+). \tag{5.5}$$

The locally Lipschitz function function $K : \mathbb{R}^p \to \mathbb{R}^{q\times p}$ can be given arbitrarily since $K(y)y$ can be absorbed by the locally Lipschitz function $u_a : \mathbb{R}^p \times \mathbb{R}^n \to \mathbb{R}^q$. Define an observer candidate as

$$\dot{\hat{x}}^+ = (A(y) + B(y)K(y)C)\hat{x}^+ + B(y)u_a + H(y)[C\hat{x}^+ - y] + S[R^+\delta^+ - R^-\delta^-] \tag{5.6a}$$

$$\dot{\hat{x}}^- = (A(y) + B(y)K(y)C)\hat{x}^- + B(y)u_a + H(y)[C\hat{x}^- - y] + S[R^+\delta^- - R^-\delta^+] \tag{5.6b}$$

with the initial condition defined by

$$\hat{x}^+(0) = \hat{x}_0^+ := S[R^+ x_0^+ - R^- x_0^-] \tag{5.7a}$$

$$\hat{x}^-(0) = \hat{x}_0^- := S[R^+ x_0^- - R^- x_0^+] \tag{5.7b}$$

and the output equation

$$x^+ = S^+ R\hat{x}^+ - S^- R\hat{x}^-,\ x^- = S^+ R\hat{x}^- - S^- R\hat{x}^+, \tag{5.8}$$

where $S = R^{-1}$. The invertible matrix $R \in \mathbb{R}^{n\times n}$, the locally Lipschitz functions $H : \mathbb{R}^p \to \mathbb{R}^{n\times p}$ and $K : \mathbb{R}^p \to \mathbb{R}^{q\times p}$ are design parameters. The observer candidate (5.6) includes the one proposed in [3] as a special case given by $K = 0$. For $K = 0$, sufficient conditions for achieving (5.4) and the nominal convergence $(x(t), x^+(t), x^-(t) \to 0$ as $t \to \infty$ for $\delta(t) \equiv 0)$ are given in [3]. The convergence by the observer with $K = 0$ was made robust to allow $\delta(t) \not\equiv 0$ in [7]. Inspired by the result in [7], this chapter introduces the following two assumptions as guidelines for selecting K and H for (5.6).

Assumption 5.1 The matrix

$$\Gamma(y) = R[A(y) + B(y)K(y)C + H(y)C]R^{-1} \tag{5.9}$$

is Metzler for each fixed $y \in \mathbb{R}^p$.

Assumption 5.2 There exist a C^1 function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, continuous functions $\underline{\nu}$, $\overline{\nu} \in \mathcal{K}_\infty$, $\omega \in \mathcal{P}$ and $\eta^+, \eta^- \in \mathcal{K}$ such that $\underline{\nu}(|\xi|) \leq V(\xi) \leq \overline{\nu}(|\xi|)$ and

$$\begin{aligned}\frac{\partial V}{\partial \xi}(\xi)\left\{[A(y) + B(y)K(y)C + H(y)C]\xi + S[R^+\rho^+ + R^-\rho^-]\right\}\\ \leq -\omega(|\xi|) + \eta^+(|\rho^+|) + \eta^-(|\rho^-|)\end{aligned} \tag{5.10}$$

hold for all $\xi \in \mathbb{R}^n$, $y \in \mathbb{R}^p$, $\rho^+ \in \mathbb{R}^n$ and $\rho^- \in \mathbb{R}^n$.

The former assumption aims at securing the framer property (5.4). The latter assumption guarantees the convergence of $x^+(t) - x^-(t)$ to zero even in the presence of disturbance $\delta(t) \not\equiv 0$ by requiring the error systems of $\hat{x}^+ - x$ and $\hat{x}^- - x$ corresponding to (5.6a) and (5.6b) to be integral input-to-state stable (iISS) with respect to $\rho^+ := \delta^+ - \delta$ and $\rho^- := \delta - \delta^-$, respectively. Based on the idea of separating feedback design from the observer design, the following assumption is introduced as guidelines for selecting the feedback input u.

Assumption 5.3 There exist a positive definite radially unbounded C^1 function $U : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, continuous functions $\mu \in \mathcal{P}$ and $\gamma, \zeta \in \mathcal{K}$ such that

$$\frac{\partial U}{\partial x}(x)[A(Cx)x + B(Cx)u(Cx, x + d) + \delta] \leq -\mu(|x|) + \gamma(|d|) + \zeta(|\delta|) \tag{5.11}$$

holds for all $x \in \mathbb{R}^n$, $d \in \mathbb{R}^n$ and $\delta \in \mathbb{R}^n$.

This assumption requires the closed-loop system with the fictitious state feedback u using x instead of $\hat{x}^+$ to be iISS with respect to the estimation error $d = \hat{x}^+ - x$ and the disturbance δ.

5.3.2 Reduced-Order Interval Observer

Consider the following partition of the state vector x:

$$x = \begin{bmatrix} x_m \\ x_{\overline{m}} \end{bmatrix} \begin{matrix} \}\, p \text{ components} \\ \}\, n - p \text{ components.} \end{matrix} \tag{5.12}$$

Accordingly, A, B, δ and x_0 are partitioned as

$$A(y) = \begin{bmatrix} A_{m,m}(y) & A_{m,\overline{m}}(y) \\ A_{\overline{m},m}(y) & A_{\overline{m},\overline{m}}(y) \end{bmatrix}, \ B(y) = \begin{bmatrix} B_m(y) \\ B_{\overline{m}}(y) \end{bmatrix}, \ \delta = \begin{bmatrix} \delta_m \\ \delta_{\overline{m}} \end{bmatrix}, \ x_0 = \begin{bmatrix} x_{m,0} \\ x_{\overline{m},0} \end{bmatrix} \tag{5.13}$$

and it is assumed that

$$C = [I \quad 0] \in \mathbb{R}^{p \times n} \tag{5.14}$$

holds. Since the component vector $x_m(t) \in \mathbb{R}^p$ is measured, one needs to estimate the remainder $x_{\overline{m}}(t) \in \mathbb{R}^{n-p}$. Let $\hat{w}_{\overline{m}}(t)$ denote such an estimate which has yet to be defined. Then the output feedback control law based on the estimation can be represented by $u(y, \hat{w}_{\overline{m}})$ instead of $u(y, \hat{x}^+)$. For a constant matrix $G \in \mathbb{R}^{(n-p)\times p}$ to be chosen later, let $\hat{w}_{\overline{m}}$ be called an estimate of $x_{\overline{m}}$ by defining $\hat{w}_{\overline{m}} = \hat{x}^+_{\overline{m}} - Gy$ and generating $\hat{x}^+_{\overline{m}}(t)$ appropriately. Then we have

$$u(y, \hat{w}_{\overline{m}}) = u(y, \hat{x}^+_{\overline{m}} - Gy). \tag{5.15}$$

To construct a reduced-order observer, we replace (5.3) with

$$x^-_{\overline{m},0} \le x_{\overline{m},0} \le x^+_{\overline{m},0}, \tag{5.16}$$

$$\delta^-_m(t) \le G\delta_m(t) \le \delta^+_m(t), \quad \forall t \in \mathbb{R}_{\ge 0}, \tag{5.17}$$

$$\delta^-_{\overline{m}}(t) \le \delta_{\overline{m}}(t) \le \delta^+_{\overline{m}}(t), \quad \forall t \in \mathbb{R}_{\ge 0}, \tag{5.18}$$

where the vectors $x^-_{\overline{m},0}, x^+_{\overline{m},0} \in \mathbb{R}^{n-p}$ and piecewise continuous functions $\delta^+_m, \delta^-_m : \mathbb{R}_{\ge 0} \to \mathbb{R}^p$, $\delta^+_{\overline{m}}, \delta^-_{\overline{m}} : \mathbb{R}_{\ge 0} \to \mathbb{R}^{n-p}$ are assumed to be known and satisfy

$$G = 0 \ \Rightarrow \ \delta^-_m(t) \equiv \delta^+_m(t) \equiv 0. \tag{5.19}$$

The bounds δ^-_m and δ^+_m are meaningless unless (5.19) holds.

Define a reduced-order observer candidate as

$$\dot{\hat{x}}_{\overline{m}}^{+} = \left[A_{\overline{m},\overline{m}}(y) + GA_{m,\overline{m}}(y)\right]\hat{x}_{\overline{m}}^{+} + \left[A_{\overline{m},m}(y) - A_{\overline{m},\overline{m}}(y)G - GA_{m,\overline{m}}(y)G \ + GA_{m,m}(y)\right]y$$
$$+ \left[B_{\overline{m}}(y) + GB_m(y)\right]u + S_{\overline{m}}[R_{\overline{m}}^{+}\delta_{\overline{m}}^{+} - R_{\overline{m}}^{-}\delta_{\overline{m}}^{-}] + S_{\overline{m}}[R_{\overline{m}}^{+}\delta_{m}^{+} - R_{\overline{m}}^{-}\delta_{m}^{-}] \quad (5.20a)$$
$$\dot{\hat{x}}_{\overline{m}}^{-} = \left[A_{\overline{m},\overline{m}}(y) + GA_{m,\overline{m}}(y)\right]\hat{x}_{\overline{m}}^{-} + \left[A_{\overline{m},m}(y) - A_{\overline{m},\overline{m}}(y)G - GA_{m,\overline{m}}(y)G \ + GA_{m,m}(y)\right]y$$
$$+ \left[B_{\overline{m}}(y) + GB_m(y)\right]u + S_{\overline{m}}[R_{\overline{m}}^{+}\delta_{\overline{m}}^{-} - R_{\overline{m}}^{-}\delta_{\overline{m}}^{+}] + S_{\overline{m}}[R_{\overline{m}}^{+}\delta_{m}^{-} - R_{\overline{m}}^{-}\delta_{m}^{+}] \quad (5.20b)$$

with

$$\hat{x}_{\overline{m}}^{+}(0) = \hat{x}_{\overline{m},0}^{+} := S_{\overline{m}}[R_{\overline{m}}^{+}x_{\overline{m},0}^{+} - R_{\overline{m}}^{-}x_{\overline{m},0}^{-}] + Gy(0) \quad (5.21a)$$
$$\hat{x}_{\overline{m}}^{-}(0) = \hat{x}_{\overline{m},0}^{-} := S_{\overline{m}}[R_{\overline{m}}^{+}x_{\overline{m},0}^{-} - R_{\overline{m}}^{-}x_{\overline{m},0}^{+}] + Gy(0) \quad (5.21b)$$

and

$$x_{\overline{m}}^{+} = S_{\overline{m}}^{+}R_{\overline{m}}\hat{x}_{\overline{m}}^{+} - S_{\overline{m}}^{-}R_{\overline{m}}\hat{x}_{\overline{m}}^{-} - Gy \quad (5.22a)$$
$$x_{\overline{m}}^{-} = S_{\overline{m}}^{+}R_{\overline{m}}\hat{x}_{\overline{m}}^{-} - S_{\overline{m}}^{-}R_{\overline{m}}\hat{x}_{\overline{m}}^{+} - Gy \quad (5.22b)$$
$$x^{+} = \begin{bmatrix} y \\ x_{\overline{m}}^{+} \end{bmatrix}, \quad x^{-} = \begin{bmatrix} y \\ x_{\overline{m}}^{-} \end{bmatrix}. \quad (5.22c)$$

Here, $S_{\overline{m}} = R_{\overline{m}}^{-1}$. The invertible matrix $R_{\overline{m}} \in \mathbb{R}^{(n-p)\times(n-p)}$ is a design parameter. For the reduced-order observer, this chapter proposes the following assumptions as guidelines to select the gain G and the control input u.

Assumption 5.4 The matrix

$$\Gamma_{\overline{m}}(y) = R_{\overline{m}}\left[A_{\overline{m},\overline{m}}(y) + GA_{m,\overline{m}}(y)\right]R_{\overline{m}}^{-1} \quad (5.23)$$

is Metzler for each fixed $y \in \mathbb{R}^p$.

Assumption 5.5 There exist a C^1 function $V : \mathbb{R}^{n-p} \to \mathbb{R}_{\geq 0}$, continuous functions $\underline{\nu}, \overline{\nu} \in \mathscr{K}_\infty$, $\omega \in \mathscr{P}$ and $\eta^+, \eta^- \in \mathscr{K}$ such that $\underline{\nu}(|\xi|) \leq V(\xi) \leq \overline{\nu}(|\xi|)$ and

$$\frac{\partial V}{\partial \xi}(\xi)\left\{\left[A_{\overline{m},\overline{m}}(y) + GA_{m,\overline{m}}(y)\right]\xi \ + S_{\overline{m}}[R_{\overline{m}}^{+}\rho_{\overline{m}}^{+} + R_{\overline{m}}^{-}\rho_{\overline{m}}^{-}] + S_{\overline{m}}[R_{\overline{m}}^{+}\rho_{m}^{+} + R_{\overline{m}}^{-}\rho_{m}^{-}]\right\}$$
$$\leq -\omega(|\xi|) + \eta^{+}(|\rho^{+}|) + \eta^{-}(|\rho^{-}|) \quad (5.24)$$

hold for all $\xi \in \mathbb{R}^{n-p}$, $y \in \mathbb{R}^p$, $\rho^+ = [\rho_m^{+\top}, \rho_{\overline{m}}^{+\top}]^\top \in \mathbb{R}^{p+(n-p)}$ and $\rho^- = [\rho_m^{-\top}, \rho_{\overline{m}}^{-\top}]^\top \in \mathbb{R}^{p+(n-p)}$.

Assumption 5.6 There exist a positive definite radially unbounded C^1 function $U : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, continuous functions $\mu \in \mathscr{P}$ and $\gamma, \zeta \in \mathscr{K}$ such that

$$\frac{\partial U}{\partial x}(x)[A(x_m)x + B(x_m)u(x_m, x_{\overline{m}} + d_{\overline{m}}) + \delta] \leq -\mu(|x|) + \gamma(|d_{\overline{m}}|) + \zeta(|\delta|) \tag{5.25}$$

holds with (5.12) for all $x \in \mathbb{R}^n$, $d_{\overline{m}} \in \mathbb{R}^{n-p}$ and $\delta \in \mathbb{R}^n$.

5.4 Guarantees

Define the following vectors:

$$\begin{array}{l}\eta = \eta^+ + \eta^-,\\ \delta^\pm = \delta^+ - \delta^-,\end{array}\ X = \begin{bmatrix} x \\ \hat{x}^+ \\ \hat{x}^- \end{bmatrix},\ \varDelta = \begin{bmatrix} \delta \\ \delta^+ \\ \delta^- \end{bmatrix},\ \hat{z} = \begin{bmatrix} \hat{x}^+ - x \\ x^+ - x^- \end{bmatrix},\ \hat{\rho} = \begin{bmatrix} \delta^+ - \delta \\ \delta^- - \delta \end{bmatrix}.$$

Since the assumptions in Sect. 5.3 are imposed separately on the observer mechanism (5.6) and the feedback mechanism $u(\cdot, \cdot)$, the following two theorems provide conditions under which their coupling results in desired boundedness and convergence for control and monitoring.

Theorem 5.1 *Suppose that Assumptions 5.1, 5.2 and 5.3 are satisfied with $\mu \in \mathcal{K}$. Then in the case of $\delta(t) \equiv \delta^+(t) \equiv \delta^-(t) \equiv 0$, for any x_0 satisfying (5.2), the unique solution $X(t)$ to (5.1) and (5.6) satisfies (5.4) and $\lim_{t\to\infty} |x^+(t) - x^-(t)| = 0$, and moreover, $X = 0$ is globally asymptotically stable. If*

$$\omega \in \mathcal{K}_\infty \vee \left[\omega \in \mathcal{K} \wedge \left\{ \gamma \notin \mathcal{K}_\infty \ \vee \ \lim_{s\to\infty} \omega(s) > \sup_{t\in\mathbb{R}_{\geq 0}} \eta(\sqrt{2}|\delta^\pm(t)|) \right\} \right] \tag{5.26}$$

holds, there exist $\hat{\theta} \in \mathcal{KL}$, $\hat{\psi} \in \mathcal{K}$ and $\hat{\chi} \in \mathcal{K}_\infty$ such that

$$\hat{\chi}\left(|\hat{z}(t)|\right) \leq \hat{\theta}(|\hat{z}(0)|, t) + \int_0^t \hat{\psi}(|\hat{\rho}(\tau)|)d\tau,\ \forall t \in \mathbb{R}_{\geq 0}. \tag{5.27}$$

$$\int_0^\infty \hat{\psi}(|\hat{\rho}(\tau)|)d\tau < \infty \ \Rightarrow \ \lim_{t\to\infty} |\hat{z}(t)| = 0 \tag{5.28}$$

hold for any x_0 and δ satisfying (5.2) and (5.3), and moreover, the closed-loop system consisting of (5.1) and (5.6) is iISS with respect to the input $\varDelta$ and the state X. If

$$\mu \in \mathcal{K}_\infty \wedge \omega \in \mathcal{K}_\infty \tag{5.29}$$

holds, there exist $\hat{\theta} \in \mathcal{KL}$ and $\hat{\phi} \in \mathcal{K}$ such that

$$|\hat{z}(t)| \leq \hat{\theta}(|\hat{z}(0)|, t) + \hat{\phi}\left(\sup_{\tau\in[0,t]} |\hat{\rho}(\tau)|\right), \ \forall t \in \mathbb{R}_{\geq 0} \tag{5.30}$$

$$\lim_{t\to\infty} |\hat{\rho}(t)| = 0 \ \Rightarrow \ \lim_{t\to\infty} |\hat{z}(t)| = 0 \tag{5.31}$$

hold for any x_0 and δ satisfying (5.2) and (5.3), and moreover, the closed-loop system is ISS with respect to Δ and X.

Theorem 5.2 *The claims in Theorem 5.1 hold true even if $\mu \in \mathscr{K}$ and (5.26) are replaced by*

$$\int_0^1 \frac{\gamma \circ \underline{\nu}^{-1}(s)}{\omega \circ \overline{\nu}^{-1}(s)} ds < \infty \tag{5.32}$$

$$\omega \in \mathscr{K} \ \wedge \ \left\{\exists c > 0, \ \exists k \geq 1, \ \forall s \in \mathbb{R}_{\geq 0}, \ c\gamma \circ \underline{\nu}^{-1}(s) \leq [\omega \circ \overline{\nu}^{-1}(s)]^k\right\}, \tag{5.33}$$

respectively.

The proofs are omitted due to the space limitation. The above theorems can be verified by following the arguments developed in [7]. Modification of the arguments also proves that Theorems 5.1 and 5.2 hold true for the reduced-order observer candidate (5.20) by replacing Assumptions 5.1, 5.2 and 5.3 with Assumptions 5.4, 5.5 and 5.6, respectively, and redefining

$$X = \begin{bmatrix} x \\ \hat{x}^+_{\overline{m}} - Gy \\ \hat{x}^-_{\overline{m}} - Gy \end{bmatrix}, \ \hat{z} = \begin{bmatrix} \hat{x}^+_{\overline{m}} - x_{\overline{m}} - Gy \\ x^+_{\overline{m}} - x^-_{\overline{m}} \end{bmatrix}.$$

5.5 Difference Between Observers

5.5.1 Utility of H and K, and Difference

Property (5.11) is independent of the state transformation R and the gains $H(y)$ and $K(y)$. The state transformation R contributes to only (5.9), while the gain $H(y)$ contributes to (5.9) and (5.10) and has the same effect as $B(y)K(y)$. The observer (5.6) varies with the choice of $K(y)$ for a given and fixed u. Thus, $K(y)$ offers freedom to change the behavior of the interval estimates $x^+(t)$ and $x^-(t)$ within the aforementioned guarantees. This change in estimates influences the behavior of $x(t)$ of the plant. The standard Luenberger observer also admits $K(y)$ influencing the closed-loop response. However, the freedom is not much appreciated since the standard observer aims at only closed-loop stability and convergence and it is not built for monitoring. In contrast, interval observers provide estimates in the transient phase and the freedom of $K(y)$ matters. Notice that for a given and fixed feedback control law u, the choice of $H(y)$ does not influence u_a in the observer (5.6), while the choice of $K(y)$ does. This flexibility of $K(y)$ in addition to $H(y)$ can be utilized to construct a bundle of interval observers for generating a tighter estimate, as done for instance in [2].

5.5.2 *Benefits of Reduced-Order Design*

In the case of partial measurement (5.14), the reduced-order interval observer (5.20) lets the exact measurement x_m be used instead of estimating intervals for x_m. Since the reduced-order observer is free from dynamics estimating the measured part x_m, its closed loop can be expected to have relatively swifter response with less control effort than the control loop based on the full-order estimates.

To illustrate another advantage of the reduced-order observer, consider the simplest choice $G = 0$ in (5.20). Suppose that Assumption 5.1 is achieved with

$$R = \begin{bmatrix} R_m & 0 \\ 0 & R_{\overline{m}} \end{bmatrix}, \quad R_m \in \mathbb{R}^{p\times p}. \tag{5.34}$$

Then we have

$$R(BKC + HC)R^{-1} = [R(BK + H)R_m^{-1} \; 0] \tag{5.35}$$

Thus, to render $\Gamma(y)$ Metzler, the observer gain $H(y)$ and the feedback gain $K(y)$ modify the first p columns which correspond to the measurable part x_m of x, Therefore,

$$R[A(y) + B(y)K(y)C + H(y)C]R^{-1} \text{ is Metzler} \Rightarrow \; R_{\overline{m}}A_{\overline{mm}}(y)R_{\overline{m}}^{-1} \text{ is Metzler.} \tag{5.36}$$

holds for each fixed $y \in \mathbb{R}^p$ since every principal minor of a Metzler matrix is Metzler. The modification of A within the limited freedom of (5.35) is unnecessary if a reduced-order interval observer is constructed. The reduced-order design is concerned with only the part $R_{\overline{m}}A_{\overline{mm}}(y)R_{\overline{m}}^{-1}$ which can be influenced by neither $K(y)$ nor $H(y)$ of the full-order observer design. In this way, the reduced-order design allows us to get rid of the unnecessarily "Metzlerization" in the partial measurement case (5.14). In addition, the matrix G in the reduced-order design provides another degree of freedom to modify $R_{\overline{m}}A_{\overline{mm}}(y)R_{\overline{m}}^{-1}$ for the "Metzlerization".

Now, we pay attention to Assumption 5.2. The next proposition demonstrates that in many cases, attainability of (5.9) and (5.10) for the full-order interval observer (5.6) implies the existence of a reduced-order interval observer unless the state transformation R is fully exploited.

Proposition 5.1 *Suppose that (5.14) holds and $A_{\overline{mm}}(y)$ is independent of y. If Assumptions 5.1 and 5.2 are satisfied with a non-singular matrix $R \in \mathbb{R}^{n\times n}$ of the form (5.34) and a quadratic function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$. then Assumptions 5.4 and 5.5 hold with $G = 0$.*

Exploiting $G \neq 0$ can yield a better (larger) ω in (5.24). Furthermore, $A_{\overline{mm}}(y)$ is allowed to depend on y in Proposition 5.1 if the quadratic function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is chosen as a quadratic form of a block-diagonal matrix. In the partial measurement case (5.14), producing a Metzler matrix Γ within the freedom of (5.35) imposes severe constraints on the choice of H and K in obtaining a better (larger) ω in (5.10) for the full-order observer (5.6).

Finally, it should be stressed that the above discussions on benefits of the reduced-order observer are not precise when R is not block diagonal. The use of non-diagonal R is crucial for allowing $H(y)$ and $K(y)$ to offer more flexibility than the reduced-order design.

5.6 Comparative Simulations

To illustrate the design flexibility introduced in this chapter, we borrow the following plant from [7]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -x_1^2 - \frac{1}{2} \\ 0 & -2x_1^2 - \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \frac{x_2}{2} + u_1 + \delta_1 \\ -\frac{x_2}{2} + u_2 + \delta_2 \end{bmatrix} \tag{5.37a}$$

$$y = x_1. \tag{5.37b}$$

Fix the feedback control input as

$$u(y, \hat{x}_2) = \frac{1}{2} \begin{bmatrix} -4y^3 + \hat{x}_2 \\ -\hat{x}_2 \end{bmatrix}, \tag{5.38}$$

where $\hat{x}_2$ denotes an estimate of x_2. The full-order interval observer in Sect. 5.3.1 employs $\hat{x}_2 = \hat{x}_2^+$, while the reduced-order interval observer in Sect. 5.3.2 employs $\hat{x}_2 = \hat{w}_2^+$. Let $U(x) = x^\top x$. As verified in [7], (5.11) and (5.25) are satisfied with $\mu(s) = \frac{1}{4}\min\{s^4, s^2\}$, $\gamma(s) = \max\left\{\frac{3}{2}s^{\frac{4}{3}}, s^2\right\}$, $\zeta(s) = \max\left\{3s^{\frac{4}{3}}, 2s^2\right\}$. Let

$$H(y) = \begin{bmatrix} -2y^2 - 3/4 \\ -1/2 \end{bmatrix}. \tag{5.39}$$

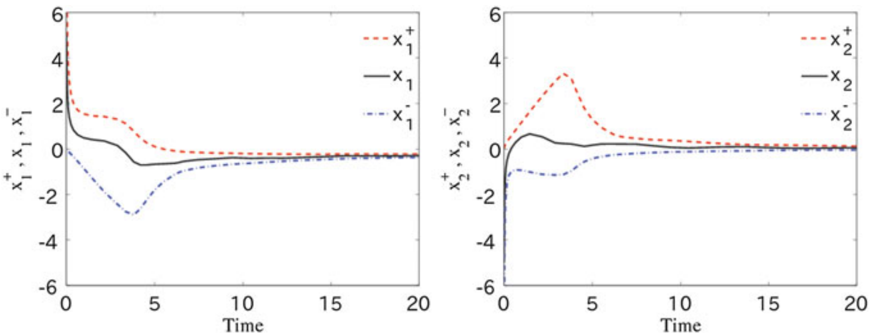
For the choice $K = 0$, (5.10) is satisfied for $V(\xi) = \xi^\top \xi$ with $\omega(s) = s^2/60$, $\eta^+(s) = 10s^2$ and $\eta^-(s) = 13s^2$. For

$$K(y) = \begin{bmatrix} -2y^2 - 1 \\ 0 \end{bmatrix} \tag{5.40}$$

property (5.10) is achieved by letting $\omega(s) = 2s^2/5$. The matrix $\Gamma(y)$ for both $K = 0$ and (5.40) is Metzler with (5.34) and $R_m = 1$, $R_{\overline{m}} = -1/2$. Thus, Assumptions 5.1, 5.2, 5.3 and (5.29) in Theorem 5.1 are satisfied. Since the diagonal matrix R and the quadratic function $V(\xi) = \xi^\top \xi$ led to the above two full-order observers (5.6) with $K = 0$, and (5.6) with (5.40), the discussion in Sect. 5.5.2 indicates that a reduced-order observer can be constructed. Define the reduced-order interval observer as (5.20). For any $G \geq 0$, Assumptions 5.4, 5.5, 5.6 and (5.29) are satisfied. For simulations, we use $x_0 = [5, -5]^\top$, $x_0^+ = [10, 0]^\top$, $x_0^- = [0, -10]^\top$ and

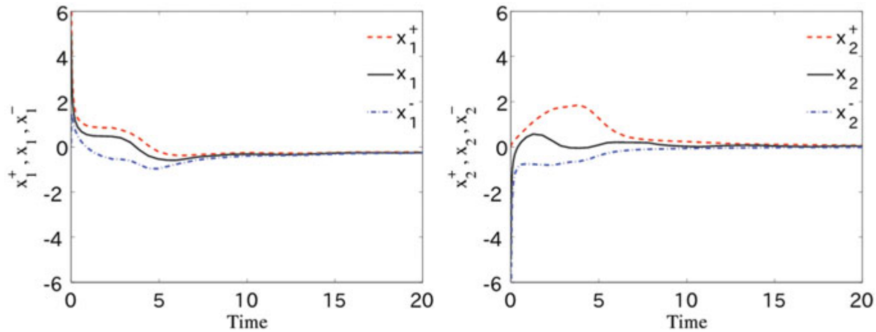
$$\delta(t) = \begin{bmatrix} \text{sgn}(\sin(t)) \min\left\{|\sin(t)|, 5/t^2\right\} \\ \text{sgn}(\cos(t)) \min\left\{|\cos(t)|, 5/t^2\right\} \end{bmatrix}. \tag{5.41}$$

Pick δ^+ by replacing $\sin(t)$ and $\cos(t)$ in (5.41) with 1. Use -1 instead for δ^-. The simulation results shown in Figs. 5.1, 5.2 and 5.3 verify that in all the three designs, the framer property (5.4) is achieved, and the estimated intervals and the plant state converge to the origin. Figures 5.1 and 5.2 show that the choice (5.40) in the observer (5.6) provides a tighter estimate than $K = 0$. Since the control law (5.38) uses the measured component $y = x_1$ instead of its estimate in the full-order designs, the behavior of x with the reduced-order observer (5.20) for $G = 0$ is almost identical with that of the full-order observers (The plots are omitted). For the reduced order observer (5.20) with $G = 2$, Fig. 5.3 not only verifies the achievement of the framer property and the convergence of the estimates and the plant state, but also shows that the change from $G = 0$ to $G = 2$ resulted in the slightly swifter convergence of the interval estimate and x to zero in Fig. 5.3.



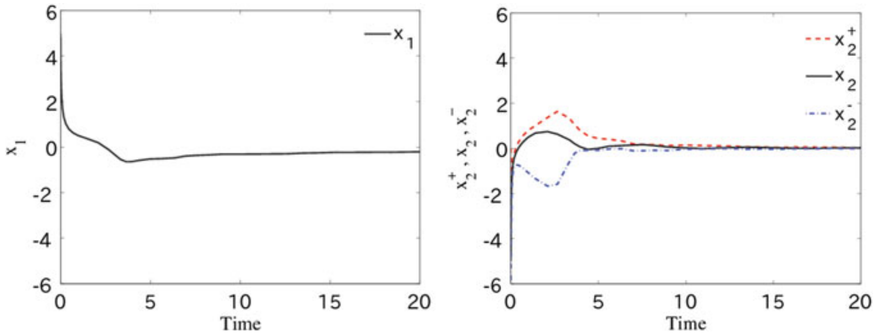
(a) $x_1^-(t), x_1(t), x_1^+(t)$ **(b)** $x_2^-(t), x_2(t), x_2^+(t)$

Fig. 5.1 Closed-loop response for (5.6) with $K = 0$ in the presence of (5.41)



(a) $x_1^-(t), x_1(t), x_1^+(t)$ **(b)** $x_2^-(t), x_2(t), x_2^+(t)$

Fig. 5.2 Closed-loop response for (5.6) with K as in (5.40) in the presence of (5.41)



(a) $x_1(t)$ (b) $x_2^-(t), x_2(t), x_2^+(t)$

Fig. 5.3 Closed-loop response for (5.20) with $G = 2$ in the presence of (5.41)

5.7 Conclusions

This chapter has presented an iISS approach to interval observer design for output feedback control of nonlinear systems to guarantee the convergence of the estimated interval length to zero in the presence of converging disturbances. A modification has been proposed by incorporating feedback gain into the interval observer presented in the preceding study [7]. The simple modification offers flexibility to obtain better transient behavior of estimated intervals without altering the observer gain and the control law. For possible improvement of performance for control and estimation, this chapter has also proposed a reduced-order interval observer to avoid estimating measured variables. As a unique consequence of the interval observer design based on Metzler matrices, it has been shown that the existence of a full-order observer implies the existence of a reduced-order observer unless state transformation is fully exploited.

References

1. Alcaraz-Gonzalez, V., Harmand, J., Rapaport, A., Steyer, J.P., Gonzalez-Alvarez, V., Pelayo-Ortiz, C.: Software sensors for highly uncertain WWTPs: a new approach based on interval observers. Water Res. **36**(10), 2515–2524 (2002)
2. Bernard, O., Gouzé, J.L.: Closed loop observers bundle for uncertain biotechnical models. J. Process Control **14**(7), 765–774 (2004)
3. Dinh, T.N., Mazenc, F., Niculescu, S.-I.: Interval observer composed of observers for nonlinear systems. In: Proceedings of the 13th European Control Conference, pp. 660–665 (2014)
4. Efimov, D., Perruquetti, W., Richard, J.P.: Interval estimation for uncertain systems with time-varying delays. Int. J. Control **86**(10), 1777–1787 (2013)
5. Efimov, D., Raissi, T., Chebotarev, S., Zolghadri, A.: Interval state observer for nonlinear time varying systems. Automatica **49**(1), 200–205 (2013)
6. Gouzé, J.L., Rapaport, A., Hadj-Sadok, M.Z.: Interval observers for uncertain biological systems. Ecol. Modell. **133**(1–2), 45–56 (2000)
7. Ito, H., Dinh, T.N.: Interval observers for nonlinear systems with appropriate output feedback. In: Proceedings of the 2nd SICE International Symposium on Control Systems, pp. 9–14 (2016)
8. Mazenc, F., Bernard, O.: Interval observers for linear time-invariant systems with disturbances. Automatica **47**(1), 140–147 (2011)
9. Mazenc, F., Dinh, T.N.: Construction of interval observers for continuous-time systems with discrete measurements. Automatica **50**(10), 2555–2560 (2014)
10. Mazenc, F., Dinh, T.N., Niculescu, S.-I.: Robust interval observers and stabilization design for discrete-time systems with input and output. Automatica **49**(11), 3490–3497 (2013)
11. Mazenc, F., Niculescu, S.-I., Bernard, O.: Exponentially stable interval observers for linear systems with delay. SIAM J. Control Optim. **50**(1), 286–305 (2012)
12. Polyakova, A., Efimov, D., Perruquetti, W.: Output stabilization of time-varying input delay systems using interval observation technique. Automatica **49**(11), 3402–3410 (2013)
13. Raissi, T., Efimov, D., Zolghadri, A.: Interval state estimation for a class of nonlinear systems. IEEE Trans. Autom. Control **57**(1), 260–265 (2012)
14. Raissi, T., Ramdani, N., Candau, Y.: Bounded error moving horizon state estimation for non-linear continuous time systems: application to a bioprocess system. J. Process Control **15**(5), 537–545 (2005)