Chapter 19
Positive Consensus Problem: The Case of Complete Communication

Maria Elena Valcher and Irene Zorzan

Abstract In this chapter the positive consensus problem for homogeneous multi-agent systems is investigated, by assuming that agents are described by positive single-input and continuous-time systems, and that each agent communicates with all the other agents. Under certain conditions on the Laplacian of the communication graph, that arise only when the graph is complete, some of the main necessary conditions for the problem solvability derived in [17–19] do not hold, and this makes the problem solution more complex. In this chapter we investigate this specific problem, by providing either necessary or sufficient conditions for its solvability and by analysing some special cases.

Keywords Multi agent system · Continuous time positive system · Consensus · Complete communication graph

19.1 Introduction

Research on multi-agent systems and consensus problems has been flourishing in the last decades [2, 7, 9, 11, 13, 14, 16], strongly stimulated by the large number of different applications areas where practical problems that can be formalized as consensus problems among autonomous agents/units arise. Just to mention the most popular ones, flocking and swarming in animal groups, dynamics of opinion forming, coordination in sensor networks, clock synchronization, distributed tasks among mobile robots/vehicles. These apparently different set-ups share some common features: in each of them there is a group of individuals/units (the agents), whose behavior can be regarded as homogeneous. Each agent performs tasks and updates

M.E. Valcher (✉) · I. Zorzan
Dip. di Ingegneria dell'Informazione, Univ. di Padova, via Gradenigo 6/B, 35131 Padova, Italy
e-mail: meme@dei.unipd.it

I. Zorzan
e-mail: irene.zorzan@studenti.unipd.it

F. Cacace et al. (eds.), *Positive Systems*, Lecture Notes in Control and Information Sciences 471, DOI 10.1007/978-3-319-54211-9_19

a vector of describing parameters (its state) based on the information received from neighbouring agents, with the final goal of agreeing on a common value for such a vector.

In a number of contexts, the information vector that the agents update (based on communication exchange with their neighbours), aiming to achieve consensus, is the value of variables that are intrinsically nonnegative. For instance, wireless sensor networks in greenhouses [1] exchange information regarding physical parameters as temperature, humidity, and CO_2 concentration, and the sensors must converge to some common values for these parameters, based on which ventilation/heating systems will be activated, shading or artificial lights will be controlled, CO_2 will be injected, and so on.

Another interesting problem, that is formalized as a consensus problem with positivity constraint, is the emissions control for a fleet of Plugin Hybrid Vehicles [8]. Each vehicle has a parallel power-train configuration that allows for any arbitrary combination of the power generated by the combustion engine and the electric motor. Moreover, the vehicles can communicate. Under these assumptions, an algorithm is proposed to regulate in a cooperative way the CO_2 emissions, so that no vehicle has a higher emission level than the others.

In a series of recent papers [17–19] we have investigated the consensus problem for homogeneous multi-agent systems, whose agents are modelled as continuous-time, single-input, positive state-space models. We assumed that interactions among agents are cooperative and the communication graph regulating the agents' mutual interactions is weighted, undirected and connected but not complete, namely not every agent directly exchanges information with all the other agents. As the agents' states are intrinsically nonnegative, a natural requirement to introduce, in addition to consensus, is the positivity of the overall controlled multi-agent system and hence that the state feedback law adopted to achieve consensus constrains all the state trajectories to remain in the positive orthant. A rather complete characterization of the problem solvability has been provided, and special cases, arising under special conditions either on the agents' description or on the communication graph, have been discussed.

The simple assumption that the communication graph is connected but not complete allowed to exclude the rather peculiar situation when the maximum weighted degree of an agent, namely the largest of the diagonal entries of the Laplacian associated with the communication graph, is smaller than all the positive eigenvalues of the Laplacian. By ruling out this case, we were able to derive some powerful necessary conditions for the solvability of the positive consensus problem that provided the backbone of the analysis carried on in [17–19]. This chapter addresses the critical case, namely the situation when the communication among the agents is described by a complete graph and all the positive eigenvalues of its Laplacian are greater than its diagonal entries. As we will see, the necessary conditions derived in this context are weaker, and conditions that in the previous investigation turned out to be necessary and sufficient for the problem solvability under the current assumptions are only sufficient.

In detail, Sect. 19.2 provides some background material. In Sect. 19.3 the positive consensus problem is formalized. A set of necessary or sufficient conditions for the problem solvability is provided in Sect. 19.4. The case when the input to state matrix involved in the agents' description is monomial is investigated in Sect. 19.5. Finally, in Sect. 19.6, we address the case of two-dimensional agents.

19.2 Background Material

Given a positive integer N, we let $[1, N]$ denote the set $\{1, 2, \ldots, N\}$. $\mathbf{e}_i$ is the ith *canonical vector* (whose size is always clear from the context). The (i, j)th entry of a matrix A will be denoted either by a_{ij} or by $[A]_{ij}$, the ith entry of a vector $\mathbf{v}$ by v_i or $[\mathbf{v}]_i$.A vector $\mathbf{v} = v_i \mathbf{e}_i$, $v_i > 0$, is called ith *monomial vector*. $\mathbf{1}_N$ is the N-dimensional vector whose entries are all unitary. The *Kronecker product* of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is the matrix

$$C = [A \otimes B] := \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1n}B \\ a_{21}B & a_{22}B & \ldots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \ldots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{pm\times qn}.$$

Given a matrix $A \in \mathbb{R}^{n\times n}$, we denote by $\sigma(A)$ its *spectrum*. A is *Hurwitz* if all its eigenvalues lie in the open left complex halfplane, i.e. $\lambda \in \sigma(A)$ implies $\Re(\lambda) < 0$. $\mathbb{R}_+$ is the set of nonnegative real numbers. A matrix (in particular, a vector) A_+ with entries in $\mathbb{R}_+$ is a *nonnegative matrix* ($A_+ \geq 0$); if $A_+ \geq 0$ and at least one entry is positive, A_+ is a *positive matrix* ($A_+ > 0$), while if all its entries are positive it is a *strictly positive matrix* ($A_+ \gg 0$). A matrix $A \in \mathbb{R}^{n\times n}$ is a *Metzler matrix* if its off-diagonal entries are nonnegative.

Given $A \in \mathbb{R}^{n\times n}$, we define the *spectral abscissa* of A as

$$\lambda_{\max}(A) := \max\{\Re(\lambda), \lambda \in \sigma(A)\}. \tag{19.1}$$

For a Metzler matrix, the spectral abscissa is always an eigenvalue (namely the eigenvalue with maximal real part is always real) and it is called *Frobenius eigenvalue*. Also, Metzler matrices exhibit a monotonicity property [15]: if A and $\bar{A} \in \mathbb{R}^{n\times n}$ are Metzler matrices and $A \leq \bar{A}$, then $\lambda_{\max}(A) \leq \lambda_{\max}(\bar{A})$.

An *undirected, weighted graph* is a triple [10, 12] $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$, where $\mathscr{V} = \{1, \ldots, N\}$ is the set of vertices, $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ is the set of arcs. $(i, j) \in \mathscr{E}$ if and only if $(j, i) \in \mathscr{E}$. Finally, $\mathscr{A} \in \mathbb{R}_+^{N\times N}$ is the (positive and symmetric) *adjacency matrix* of the weighted graph $\mathscr{G}$. We assume that the graph $\mathscr{G}$ has no self-loops, i.e. $[\mathscr{A}]_{ii} = 0$ for every index $i \in [1, N]$. If $\mathscr{A}$ is irreducible, the graph is *connected*. If $[\mathscr{A}]_{ij} > 0$ for every $i, j \in \mathscr{V}, i \neq j$, the graph $\mathscr{G}$ is *complete*. If $[\mathscr{A}]_{ij} > 0$ implies $[\mathscr{A}]_{ij} = 1$ the graph is called *unweighted*. We define the *Laplacian matrix*

$\mathscr{L} \in \mathbb{R}^{N\times N}$ associated with the graph $\mathscr{G}$ as $\mathscr{L} := \mathscr{C} - \mathscr{A}$, where $\mathscr{C} \in \mathbb{R}_+^{N\times N}$ is a diagonal matrix with $[\mathscr{C}]_{ii} := \sum_{j=1}^N [\mathscr{A}]_{ij}$, $\forall\, i \in [1, N]$. Accordingly, the Laplacian matrix $\mathscr{L} = \mathscr{L}^\top$ takes the following form:

$$\mathscr{L} = \begin{bmatrix} \ell_{11} & \ell_{12} & \dots & \ell_{1N} \\ \ell_{12} & \ell_{22} & \dots & \ell_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \ell_{1N} & \ell_{2N} & \dots & \ell_{NN} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^N [\mathscr{A}]_{1j} & -[\mathscr{A}]_{12} & \dots & -[\mathscr{A}]_{1N} \\ -[\mathscr{A}]_{12} & \sum_{j=1}^N [\mathscr{A}]_{2j} & \dots & -[\mathscr{A}]_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ -[\mathscr{A}]_{1N} & -[\mathscr{A}]_{2N} & \dots & \sum_{j=1}^N [\mathscr{A}]_{Nj} \end{bmatrix}.$$

If $\mathscr{G}$ is connected then $\ell_{ii} > 0$ for every $i \in [1, N]$, and hence $\ell^* := \max_{i\in[1,N]} \ell_{ii} > 0$. Notice that all rows of $\mathscr{L}$ sum up to 0, and hence $\mathbf{1}_N$ is always a right eigenvector of $\mathscr{L}$ corresponding to the null eigenvalue [3].

Lemma 19.1 *[3, 13, 20] If the undirected, weighted graph $\mathscr{G}$ is connected, then $\mathscr{L}$ is a symmetric positive semidefinite matrix, and 0 is a simple eigenvalue of $\mathscr{L}$.*

Therefore, if we denote by $\{\lambda_1, \lambda_2, \dots, \lambda_N\}$ the spectrum $\sigma(\mathscr{L})$, then $\lambda_i \in \mathbb{R}_+$ for every $i \in [1, N]$, and we can always assume that the λ_i's are sorted in non-decreasing order, namely

$$0 = \lambda_1 \le \lambda_2 \le \dots \le \lambda_N. \tag{19.2}$$

It is well-known that if the eigenvalues of $\mathscr{L}$ are sorted as in (19.2), then [4, 5] $\ell^* \le \lambda_N$. In addition, if $\mathscr{L}$ is irreducible, then $\ell^* < \lambda_N$ (see Theorem 3 in [4]).

Lemma 19.2 *(1) Let $\mathscr{G}$ be an undirected, weighted graph with N vertices. If $\ell^* < \lambda_2$, then [12] $\mathscr{G}$ is complete.*

(2) If $\mathscr{G}$ is the undirected, unweighted graph with N vertices, then [3, 5, 10] $\ell^* < \lambda_2$ if and only if $\mathscr{G}$ is complete. Moreover, in this case $\ell^* = N - 1$ and $\lambda_2 = \dots = \lambda_N = N$.

Notice that, differently from the unweighted case, completeness of a weighted graph $\mathscr{G}$ does not imply $\ell^* < \lambda_2$. Consider, e.g., the weighted Laplacian matrix

$$\mathscr{L} = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 2 & -1 \\ -2 & -1 & 3 \end{bmatrix},$$

and notice that $\lambda_2 = 3$ and hence $\lambda_2 = \ell^* = 3$ even if $\mathscr{G}$ is complete. In the following the complete, undirected and unweighted graph will be denoted by $\mathscr{G}_N$. Clearly, its Laplacian can be expressed as $\mathscr{L} = N I_N - \mathbf{1}_N \mathbf{1}_N^\top$ and its eigenvalues are $\lambda_2 = \dots = \lambda_N = N$, while $\ell^* = N - 1$.

19.3 Problem Statement

We consider a homogeneous multi-agent system consisting of N identical agents whose dynamics is described by the continuous-time positive single-input system:

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t), \qquad t \in \mathbb{R}_+,$$

where $\mathbf{x}_i \in \mathbb{R}^n$ and $u_i \in \mathbb{R}$ are the state vector and the (scalar) input, respectively, of the ith agent. $A = \left[a_{ij}\right] \in \mathbb{R}^{n\times n}$ is a non-Hurwitz Metzler matrix, and $B = \left[b_i\right] \in \mathbb{R}^n_+$ is a positive vector. The mutual interactions among agents are described by a (connected, undirected, weighted) *communication graph* $\mathscr{G} = (\mathscr{V}, \mathscr{E}, \mathscr{A})$, where $\mathscr{V} = \{1, \ldots, N\}$ and $\mathscr{A} = \mathscr{A}^\top \in \mathbb{R}^{N\times N}_+$. Note that we assume that the mutual interactions are cooperative and hence $\mathscr{A}$ is a nonnegative matrix. Differently from what we did in [17–19], we assume that the graph $\mathscr{G}$ is complete, namely each agent communicates with all the other agents, and that $\ell^* < \lambda_2$. As we will see, this apparently more restrictive situation makes the problem solution more difficult. In this scenario, $\mathscr{A} \in \mathbb{R}^{N\times N}_+$ is irreducible (in fact, primitive if $N > 2$), and if we sort the eigenvalues of $\mathscr{L}$ as in (19.2), then

$$0 = \lambda_1 < \ell^* < \lambda_2 \leq \cdots \leq \lambda_N.$$

Let $K \in \mathbb{R}^{1\times n}$ be a state-feedback matrix (to be designed) and assume that each ith agent adopts the following DeGroot type control law [20]:

$$u_i(t) = K \sum_{j=1}^{N} [\mathscr{A}]_{ij}[\mathbf{x}_j(t) - \mathbf{x}_i(t)].$$

Define $\mathbf{x}(t) \in \mathbb{R}^{Nn}$ and $\mathbf{u}(t) \in \mathbb{R}^N$ as

$$\mathbf{x}(t) := \left[\mathbf{x}_1^\top(t) \ \ldots \ \mathbf{x}_N^\top(t)\right]^\top \qquad \mathbf{u}(t) := \left[u_1(t) \ \ldots \ u_N(t)\right]^\top$$

respectively. The state-space description of the overall system becomes:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= (I_N \otimes A)\mathbf{x}(t) + (I_N \otimes B)\mathbf{u}(t)\\ \mathbf{u}(t) &= -(\mathscr{L} \otimes K)\mathbf{x}(t)\end{aligned}$$

and the resulting autonomous closed-loop system is described by

$$\dot{\mathbf{x}}(t) = [(I_N \otimes A) - (I_N \otimes B)(\mathscr{L} \otimes K)]\mathbf{x}(t). \tag{19.3}$$

The *positive consensus problem* is naturally posed as follows: determine, if possible, a state-feedback matrix $K = \left[k_i\right] \in \mathbb{R}^{1\times n}$ such that the (closed-loop multi-agent) system (19.3) satisfies the following conditions:

(I) *positivity*: $\mathbb{A} := (I_N \otimes A) - (I_N \otimes B)(\mathscr{L} \otimes K)$ is a Metzler matrix;
(II) *consensus*: meaning that

$$\lim_{t\to+\infty} \mathbf{x}_i(t) - \mathbf{x}_j(t) = 0, \quad \forall i, j \in [1, N].$$

As well-known in the literature [2, 20], a necessary and sufficient condition for the homogeneous agents to reach consensus is that all matrices $A - \lambda_i BK$, $i \in [2, N]$, are Hurwitz. A necessary condition for this to happen is that the pair (A, B) is stabilizable, a steady assumption from now onward.

As far as condition (I) is concerned, once we explicitly write the expression of the overall state matrix $\mathbb{A}$:

$$\mathbb{A} = \begin{bmatrix} A - \ell_{11}BK & -\ell_{12}BK & \dots & -\ell_{1N}BK \\ -\ell_{12}BK & A - \ell_{22}BK & \dots & -\ell_{2N}BK \\ \vdots & \vdots & \ddots & \vdots \\ -\ell_{1N}BK & -\ell_{2N}BK & \dots & A - \ell_{NN}BK \end{bmatrix}$$

it is easy to see [17–19] that $\mathbb{A}$ is Metzler if and only if (a) the off-diagonal blocks $-\ell_{ij}BK$, $i, j \in [1, N], i \neq j$, are non-negative; and (b) the diagonal blocks $A - \ell_{ii}BK$, $i \in [1, N]$, are Metzler. So, keeping in mind the assumptions on A and B, once we define the vector $K^* = [k_i^*] \in \mathbb{R}_+^{1\times n}$ as:

$$k_i^* := \begin{cases} \min_{\substack{j=1,\dots,n \\ j\neq i}} \dfrac{a_{ji}}{b_j} \dfrac{1}{\ell^*}, & \text{if } \exists\, j \neq i \text{ s.t. } b_j \neq 0; \\ +\infty, & \text{otherwise,} \end{cases}$$

it is immediate to prove that condition (I) holds if and only if $0 \leq K \leq K^*$. Note that in the special case when B is a monomial vector, say $B = b_i \mathbf{e}_i$, for some $i \in [1, n]$ and $b_i > 0$, the ith entry of K^* is infinite. In all the other cases (namely if B has at least two non-zero entries) K^* is always finite.

To summarize, the positive consensus problem can be equivalently posed as follows:

Positive consensus problem: determine, if possible, $K \in \mathbb{R}_+^{1\times n}, 0 \leq K \leq K^*$, such that all matrices $A - \lambda_i BK$, $i \in [2, N]$, are Hurwitz.

19.4 Necessary and/or Sufficient Conditions

A major consequence of the apparently more restrictive assumption that all the agents communicate with each other and $\ell^* < \lambda_i$, $i \in [2, N]$, is that one of the main necessary conditions for the positive consensus problem solvability we exploited in the previous analysis, namely the fact that the matrix $A - \lambda_2 BK^*$ is Metzler and Hurwitz, does not hold anymore. As ℓ^* is smaller than λ_2, by the way K^* is defined the matrix $A - \lambda_2 BK^*$ (and hence all matrices $A - \lambda_i BK^*$, $i \in [2, N]$) is not Metzler, and the case may occur that $A - \lambda_2 BK$ is Hurwitz even if $A - \lambda_2 BK^*$ is not.

Some necessary conditions for the problem solvability, however, can be determined, as they are independent of the relationship between ℓ^* and λ_2.

Proposition 19.1 *Assume that A is an* $n \times n$ *Metzler non-Hurwitz matrix,* $B \in \mathbb{R}^n_+$ *is a positive vector and* $0 < \ell^* < \lambda_i$, $i \in [2, N]$. *If the positive consensus problem is solvable, then*

(i) $\lambda_{\max}(A)$ *is a simple nonnegative eigenvalue;*
(ii) $K^*B > \mathrm{tr}(A)/\lambda_2$.

Proof (i) The fact that $\lambda_{\max}(A)$ is a simple eigenvalue follows from Proposition 1 and Remark 1 in [18], since those results are independent of the relationship between ℓ^* and λ_2. The fact that it is real and nonnegative follows from the assumption that A is a Metzler non-Hurwitz matrix.
(ii) As the trace of a matrix equals the sum of its eigenvalues, a necessary condition for the matrices $A - \lambda_i BK$, $i \in [2, N]$, to be Hurwitz is that their traces are negative, i.e., $\mathrm{tr}(A - \lambda_i BK) = \mathrm{tr}(A) - \lambda_i KB < 0, \forall i \in [2, N]$. However, since both B and K are positive vectors, if there exists a matrix K such that $0 \leq K \leq K^*$ and $A - \lambda_i BK$ is Hurwitz, then $K^*B \geq KB > \frac{\mathrm{tr}(A)}{\lambda_i}, \forall\, i \in [2, N]$. Finally, note that if $\mathrm{tr}(A) < 0$ the previous condition is trivial. If $\mathrm{tr}(A) \geq 0$ then

$$\frac{\mathrm{tr}(A)}{\lambda_2} \geq \frac{\mathrm{tr}(A)}{\lambda_i}$$

for every $i \in [2, N]$. So, in both cases, condition $K^*B > \frac{\mathrm{tr}(A)}{\lambda_i}$ holds for every $i \in [2, N]$ if and only if $K^*B > \frac{\mathrm{tr}(A)}{\lambda_2}$.

Conditions (i) and (ii) of the above proposition are not sufficient, not even when dealing with $N = 2$ agents described by a two-dimensional ($n = 2$) model, as the following elementary example shows.

Example 19.1 Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} 3 & 1 \\ 1 & -1 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u_i(t)$$

A is a Metzler and non-Hurwitz matrix and the pair (A, B) is stabilizable. The matrix A has a simple positive eigenvalue and a negative one. Assume that there are $N = 2$ agents and assume that the interconnection topology is described by the complete, undirected and unweighted graph $\mathscr{G}_2$, namely

$$\mathscr{L} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Then (see Lemma 19.2) $0 = \lambda_1 < \ell^* = 1 < \lambda_2 = 2$. The matrix K^* is easily proved to be $K^* = \begin{bmatrix} 1 & 1 \end{bmatrix}$, and hence condition $2 = K^* B > \mathrm{tr}(A)/\lambda_2 = 1$ holds. Yet, for every $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, with $0 \leq k_i \leq 1, i \in [1, 2]$, $A - \lambda_2 BK$ is not Hurwitz. So, the positive consensus problem is not solvable. ♣

Example 19.2 Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 6 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u_i(t)$$

Notice that A is a Metzler and non-Hurwitz matrix and that the pair (A, B) is stabilizable. Consider $N = 3$ agents and assume that the interconnection topology is described by the complete, undirected and unweighted graph $\mathscr{G}_3$. In this case (see Lemma 19.2) $\ell^* = 2$ and the eigenvalues of $\mathscr{L}$ are $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = 3$. The matrix K^* is easily proved to be $K^* = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$. As $K^* B = 1 < \frac{4}{3} = \frac{\mathrm{tr}(A)}{\lambda_2}$, we conclude that the positive consensus problem is not solvable. ♣

In order to investigate the problem solvability, let us define the set of solutions of the positive consensus problem as

$$\mathscr{K}^H := \{K : 0 \leq K \leq K^*, \ A - \lambda_i BK \text{ Hurwitz}, \ i \in [2, N]\}.$$

A sufficient condition for the solvability of the positive consensus problem is represented by the case when there is a matrix K, satisfying the given bounds, that makes all matrices $A - \lambda_i BK, i \in [2, N]$, Metzler and Hurwitz. To investigate this situation, we define

$$\mathscr{K}^{MH} := \{K \in \mathscr{K}^H : A - \lambda_i BK \text{ Metzler}, \ i \in [2, N]\}.$$

The following result provides, in the case when $\ell^* < \lambda_2$, an analysis that parallels the one carried on in Sect. 6 of [18]. In the case we are currently investigating the matrix $A - \lambda_2 BK^*$ is no longer Metzler and Hurwitz. However, $A - \ell^* BK^*$ is necessarily Metzler and hence we can ensure that all matrices taking the form $K = \alpha K^*$, with $\alpha \in [0, 1]$, make $A - \ell^* BK$ Metzler. So, we focus on this class of state feedback matrices to determine whether some of them belong to $\mathscr{K}^{MH}$.

Proposition 19.2 *Assume that A is an $n \times n$ Metzler non-Hurwitz matrix, $B \in \mathbb{R}^n_+$ is a positive vector and $0 < \ell^* < \lambda_i$, $i \in [2, N]$. The following conditions are equivalent:*

(i) $\mathscr{K}^{MH} \neq \emptyset$;
(ii) $\frac{\ell^*}{\lambda_N} K^* \in \mathscr{K}^{MH}$;
(iii) *the set* $\{\alpha \in (0, 1] \colon A - \alpha \ell^* BK^* \textit{ is Hurwitz}\}$ *is not empty and*

$$\tilde{\alpha} := \inf\{\alpha \in (0, 1] \colon A - \alpha \ell^* BK^* \textit{ is Hurwitz}\} \tag{19.4}$$

satisfies $\tilde{\alpha} < \frac{\lambda_2}{\lambda_N}$.

Proof (i) $\Rightarrow$ (ii) Suppose that $\mathcal{K}^{MH} \neq \emptyset$ and let $K \in \mathcal{K}^{MH}$. As $K \in \mathcal{K}^{MH}$ then $A - \lambda_N BK$ is Metzler (and Hurwitz) and this implies that $\lambda_N K \leq \ell^* K^*$, namely $K \leq \frac{\ell^*}{\lambda_N} K^*$. On the other hand, the Metzler matrix $A - \ell^* BK^* \leq A - \lambda_N BK$, being upper bounded by a Metzler and Hurwitz matrix, is Hurwitz in turn. Therefore, for every $k \in [2, N]$, $A - \lambda_k BK \geq A - \lambda_k \frac{\ell^*}{\lambda_N} BK^* \geq A - \ell^* BK^*$. Since $A - \ell^* BK^*$ is Metzler, then $A - \lambda_k \frac{\ell^*}{\lambda_N} BK^*$ is Metzler, too, and being upper-bounded by a Metzler Hurwitz matrix, it is Hurwitz, in turn. This proves that $A - \lambda_k \frac{\ell^*}{\lambda_N} BK^*$ is Metzler and Hurwitz for every $k \in [2, N]$, namely $\frac{\ell^*}{\lambda_N} K^* \in \mathcal{K}^{MH}$.
(ii) $\Rightarrow$ (iii) If $\frac{\ell^*}{\lambda_N} K^* \in \mathcal{K}^{MH}$, then $A - \lambda_2 \frac{\ell^*}{\lambda_N} BK^*$ is Metzler and Hurwitz, and hence $\frac{\lambda_2}{\lambda_N} \in \{\alpha \in (0, 1] \colon A - \alpha \ell^* BK^* \text{ is Hurwitz}\}$. This also implies that $\tilde{\alpha} < \frac{\lambda_2}{\lambda_N}$.
(iii) $\Rightarrow$ (i) Observe, first, that if $\{\alpha \in (0, 1] \colon A - \alpha \ell^* BK^* \text{ is Hurwitz}\}$ is not empty and $\tilde{\alpha}$ is the infimum value of the set, then for every $\alpha \in (\tilde{\alpha}, 1]$ the matrix $A - \alpha \ell^* BK^*$ satisfies $A - \ell^* BK^* \leq A - \alpha \ell^* BK^* < A - \tilde{\alpha} \ell^* BK^*$ and hence it is Metzler Hurwitz. Set $K = \frac{\ell^*}{\lambda_N} K^*$. By assumption, $\tilde{\alpha} < \frac{\lambda_2}{\lambda_N}$, and hence $A - \lambda_2 BK$ is Hurwitz. On the other hand, $A - \lambda_N BK = A - \ell^* BK^*$ is Metzler. This implies that $A - \lambda_2 BK \geq A - \lambda_3 BK \geq \cdots \geq A - \lambda_N BK$ are all Metzler matrices, and since the largest one is Hurwitz, by the monotonicity property of the spectral abscissa we can claim that they are all Hurwitz. So, $K \in \mathcal{K}^{MH}$.

Remark 19.1 It is easy to see that since $A - \ell^* BK^*$ is Metzler, then the set $\{\alpha \in (0, 1] \colon A - \alpha \ell^* BK^* \text{ is Hurwitz}\}$ coincides with the set $\{\alpha \in (0, 1] \colon A - \alpha \ell^* BK^* \text{ is Metzler and Hurwitz}\}$. Moreover, if the set is not empty then the Metzler matrix $A - \ell^* BK^*$ satisfies $A - \ell^* BK^* \leq A - \tilde{\alpha} \ell^* BK^*$ and hence it is necessarily Hurwitz. So, Proposition 19.2 above, essentially states that the set $\mathcal{K}^{MH}$ is not empty, namely there exists a state feedback matrix K, satisfying the usual bounding conditions, that makes all matrices $A - \lambda_i BK$, $i \in [2, N]$, Metzler and Hurwitz, if and only if such a solution can be found in the set of matrices $\{\alpha K^* : \alpha \in (0, 1]\}$. Note that not only the set $\{\alpha \in (0, 1] \colon A - \alpha \ell^* BK^* \text{ is Hurwitz}\}$ must be not empty, and hence the parameter $\tilde{\alpha}$ well defined, but the interval $(\tilde{\alpha}, 1]$ must be sufficiently "large" to include the interval $\left[\frac{\lambda_2}{\lambda_N}, 1\right]$. Only in this way we can determine a matrix of the form $K = \alpha K^*$ that makes $A - \lambda_i BK$ Metzler and Hurwitz for every $\lambda \in [\lambda_2, \lambda_N]$.

19.5 *B* Is a Monomial Vector

We consider now the case when B is a monomial vector. Without loss of generality we assume that $B = \mathbf{e}_1$, since we can always reduce ourselves to this case by resorting to a permutation and a rescaling that do not influence the problem solvability, only the value of the specific solution.

Proposition 19.3 *Assume that* $B = \mathbf{e}_1$ *and denote by* A_{22} *the principal submatrix obtained from* A *by deleting its first row and column.*

(i) If the positive consensus problem is solvable then every eigenvalue of A_{22} with nonnegative real part has geometric multiplicity equal to 1*;*
(ii) If A_{22} is Hurwitz, then the positive consensus problem is solvable.

Proof (i) Assume that the positive consensus problem is solvable and suppose by contradiction that there exists $\mu \in \sigma(A_{22})$ with $\Re\{\mu\} \geq 0$ and geometric multiplicity $d > 1$. Partition the matrix A as:

$$A = \begin{bmatrix} a_{11} & \mathbf{r}^\top \\ \mathbf{c} & A_{22} \end{bmatrix},$$

where $a_{11} \in \mathbb{R}$, $\mathbf{r}, \mathbf{c} \in \mathbb{R}_+^{n-1}$ are nonnegative vectors, and $A_{22} \in \mathbb{R}^{(n-1)\times(n-1)}$ is a Metzler matrix. Partition the feedback matrix $K \in \mathbb{R}_+^{1\times n}$, $0 \leq K \leq K^*$, in a consistent way, namely as $K = \begin{bmatrix} k_1 & \mathbf{k}_2 \end{bmatrix}$, where $\mathbf{k}_2 \in \mathbb{R}_+^{1\times(n-1)}$. Now, notice that for every $i \in [2, N]$ the characteristic polynomial of $A - \lambda_i BK$ can be written as

$$\begin{aligned}
\det(sI_n - A + \lambda_i BK) &= \det(sI_n - A) + \lambda_i K \operatorname{adj}(sI_n - A)B \\
&= \det(sI_{n-1} - A_{22}) \left[s - a_{11} - \mathbf{r}^\top (sI_{n-1} - A_{22})^{-1}\mathbf{c}\right] \\
&\quad + \lambda_i \begin{bmatrix} k_1 & \mathbf{k}_2 \end{bmatrix} \begin{bmatrix} \det(sI_{n-1} - A_{22}) \\ \operatorname{adj}(sI_{n-1} - A_{22})\mathbf{c} \end{bmatrix} \\
&= (s - a_{11} + \lambda_i k_1) \det(sI_{n-1} - A_{22}) \\
&\quad + \left(\lambda_i \mathbf{k}_2 - \mathbf{r}^\top\right) \operatorname{adj}(sI_{n-1} - A_{22})\mathbf{c}.
\end{aligned}$$

If $\mu \in \sigma(A_{22})$, then $\det(\mu I_{n-1} - A_{22}) = 0$ and, since the geometric multiplicity of μ as an eigenvalue of A_{22} is $d > 1$, it also holds that $\operatorname{adj}(\mu I_{n-1} - A_{22}) = 0$, and hence $\det(\mu I_n - A + \lambda_i BK) = 0$ for every $K \in \mathbb{R}_+^{1\times n}$, which contradicts the assumption of the solvability of the positive consensus problem.
(ii) It is the same as the proof of the sufficiency part of Proposition 7 in [18].

Differently from the case $\lambda_2 \leq \ell^*$, the Hurwitz condition on the submatrix A_{22} is sufficient for the problem solvability, but it is not necessary, as shown in Example 19.3 below.

Example 19.3 Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u_i(t)$$

Notice that A is a Metzler and non-Hurwitz matrix and that the pair (A, B) is stabilizable. Consider $N = 3$ agents and the same adjacency matrix as in Example 19.2, so that $\ell^* = 2$ and $\lambda_2 = \lambda_3 = 3$. $B = \mathbf{e}_3$ and the matrix A_{11}, obtained from A by deleting the third row and the third column, is non-Hurwitz, however this does not preclude the problem solvability. If we consider

$$A - \ell^* BK = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -2k_1 & 2-2k_2 & -1-2k_3 \end{bmatrix}$$

we notice that $K^* = \begin{bmatrix} 0 & 1 & +\infty \end{bmatrix}$. It is easy to verify that the positive consensus problem is solvable since for $K = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \in \mathbb{R}^{1\times 3}$, with $0 \le K \le K^*$, we get

$$A - \lambda_2 BK = A - \lambda_3 BK = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

which is Hurwitz. ♣

19.6 Second-Order Agents

We investigate now the case when the agents are modelled by a second-order (positive) linear system, i.e.

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{x}_i(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u_i(t), \tag{19.5}$$

with a_{12}, a_{21}, b_1 and b_2 nonnegative real numbers. Recalling that any matrix $M \in \mathbb{R}^{2\times 2}$ is Hurwitz if and only if $\mathrm{tr(M)} < 0$ and $\det(M) > 0$, after elementary manipulations it can be seen that for every $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^2$ and $K \in \mathbb{R}^{1\times 2}$, the matrix $M := A - \lambda BK$ is Hurwitz if and only if

$$\begin{cases} \lambda KB > \mathrm{tr(A)}; \\ \lambda K \mathrm{adj(A)B} < \det(A). \end{cases} \tag{19.6}$$

This simple observation leads to the following Lemma.

Lemma 19.3 *[18] Given $A \in \mathbb{R}^{2\times 2}$ and $B \in \mathbb{R}^2$ and $K \in \mathbb{R}^{1\times 2}$, for every choice of the $N-1$ positive real numbers $0 < \lambda_2 \le \lambda_3 \le \cdots \le \lambda_N$, the following facts are equivalent:*

(i) $A - \lambda BK$ is Hurwitz for every $\lambda \in [\lambda_2, \lambda_N]$;
(ii) $A - \lambda_i BK$ is Hurwitz for every $i \in [2, N]$;
(iii) $A - \lambda_i BK$ is Hurwitz for $i = 2, N$.

As a straightforward consequence of Lemma 19.3 and of the fact that $KB \ge 0$ (and hence $\lambda_N KB \ge \lambda_2 KB$), it follows that for two-dimensional agents the set of feedback matrices that solve the positive consensus problem is the set of matrices $K \in \mathbb{R}^{1\times 2}$ that satisfy the following LMIs:

$$K^* \ge K \quad \ge 0;$$

$$\lambda_2 KB > \text{tr(A)};$$
$$\det(A) > \lambda_i K \text{adj(A)B}, \qquad \text{i} = 2, \text{N}.$$

This ensures that the set of solutions is necessarily convex.

When the agents are described by second-order state-space models the case of B monomial can be completely solved. To this aim recall that from Proposition 19.3 part (ii) it follows that condition $a_{22} < 0$ ensures the solvability of the positive consensus problem, but as we have shown this is not a necessary condition. So, in the following we assume $a_{22} \geq 0$, $\ell^* < \lambda_2$, and investigate under which additional conditions on the matrix A and on the interconnection topology the positive consensus problem is solvable.

Proposition 19.4 *Assume that* $B = \mathbf{e}_1$, $A_{22} = a_{22} \geq 0$ *and* $\ell^* < \lambda_2$. *Then, the positive consensus problem for second-order agents is solvable if and only if* $a_{21} > 0$ *and the following condition holds:*

$$\max\left\{0, \frac{\text{tr}(A)a_{22}}{\lambda_2}\right\} < \frac{a_{12}a_{21}}{\ell^*} + \min\left\{\frac{\det(A)}{\lambda_2}, \frac{\det(A)}{\lambda_N}\right\}. \tag{19.7}$$

When so, there is always a solution of the form $K = \left[\max\left\{0, \frac{\text{tr}(A)a_{22}}{\lambda_2}\right\} + \epsilon \;\; \frac{a_{12}}{\ell^*}\right]$, *with* $\epsilon > 0$ *and arbitrarily small.*

Proof Note first that as $B = \mathbf{e}_1$ and $a_{22} \geq 0$, if the positive consensus problem is solvable, then a_{21} must be positive, otherwise a_{22} would be an eigenvalue of every matrix $A - \lambda_i BK$, $i \in [2, N]$. Conversely, it is easy to see that condition (19.7) implies $a_{21} > 0$. So, in the following we will assume $a_{21} > 0$. Set $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. Then $KB = k_1$, $K^* = \begin{bmatrix} +\infty & \frac{a_{12}}{\ell^*} \end{bmatrix}$, and the previous LMIs become

$$k_1 \geq 0, \quad k_1 > \frac{\text{tr}(A)}{\lambda_2}, \quad \frac{a_{12}}{\ell^*} \geq k_2 \geq 0, \tag{19.8}$$

$$\begin{bmatrix} k_1 & k_2 \end{bmatrix}\begin{bmatrix} a_{22} \\ -a_{21} \end{bmatrix} < \min\left\{\frac{\det(A)}{\lambda_2}, \frac{\det(A)}{\lambda_N}\right\}. \tag{19.9}$$

It is clear that, as $a_{21} > 0$, inequality (19.9) holds if and only if it holds for $k_2 = k_2^* = \frac{a_{12}}{\ell^*}$. So, inequality (19.9) becomes

$$k_1 a_{22} < \frac{a_{12}a_{21}}{\ell^*} + \min\left\{\frac{\det(A)}{\lambda_2}, \frac{\det(A)}{\lambda_N}\right\}. \tag{19.10}$$

If $\text{tr}(A) < 0$ then the only constraint on k_1 is the nonnegativity and condition (19.10) holds if and only if it holds for $k_1 = 0$. And if this is the case it also holds for $k_1 = \epsilon$, with $\epsilon > 0$ and arbitrarily small. On the other hand, if $\text{tr}(A) \geq 0$, then the problem is solvable if and only if it is solvable by assuming $k_1 = \frac{\text{tr}(A)}{\lambda_2} + \epsilon$, with $\epsilon > 0$ arbitrarily small, and this happens if and only if

$$\frac{\mathrm{tr}(A)}{\lambda_2}a_{22} < \frac{a_{12}a_{21}}{\ell^*} + \min\left\{\frac{\det(A)}{\lambda_2}, \frac{\det(A)}{\lambda_N}\right\}.$$

When the N agents are described by a second-order state-space model, $B = \mathbf{e}_1$, $A_{22} = a_{22} > 0$, and the communication among them is described by $\mathscr{G}_N$, Proposition 19.4 allows us to draw the following conclusion concerning the number of agents.

Corollary 19.1 *Assume that* $B = \mathbf{e}_1$, $A_{22} = a_{22} > 0$ *and the communication graph is described by the complete undirected and unweighted graph* $\mathscr{G}_N$ *(and hence* $\ell^* < \lambda_2$*). Then, there exists* $\bar{N}$ *such that for every* $N \geq \bar{N}$ *positive consensus cannot be reached.*

Proof The Laplacian of $\mathscr{G}_N$ has $\ell^* = N - 1$ and eigenvalues $\lambda_2 = \cdots = \lambda_N = N$. So, condition (19.7) becomes

$$\max\left\{0, \frac{\mathrm{tr}(A)a_{22}}{N}\right\} < \frac{a_{12}a_{21}}{N-1} + \frac{\det(A)}{N},$$

and it implies $a_{22}^2 < \frac{1}{N-1}a_{12}a_{21}$. Clearly, the term on the right goes to 0 as N tends to $+\infty$, while $a_{22}^2 > 0$. So, there exists $\bar{N}$ such for every $N \geq \bar{N}$ the previous inequality and hence condition (19.7) do not hold, i.e. positive consensus cannot be reached.

Example 19.4 Consider the positive single-input agent

$$\dot{\mathbf{x}}_i(t) = A\mathbf{x}_i(t) + Bu_i(t) = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}\mathbf{x}_i(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u_i(t)$$

Assume that the communication among the agents is described by $\mathscr{G}_N$: it follows from (19.7) that for every $N \geq \bar{N} = 4$ the positive consensus problem is not solvable.

References

1. Chaudhary, D.D., Nayse, S.P., Waghmare, L.M.: Application of wireless sensor networks for greenhouse parameter control in precision agriculture. Int. J. Wirel. Mobile Netw. (IJWMN) **3**(1) (2011)
2. Fax, J.A., Murray, R.M.: Information flow and cooperative control of vehicle formations. IEEE Trans. Autom. Control **49**(9), 1465–1476 (2004)
3. Fiedler, M.: Algebraic connectivity of graphs. Czechoslovak Math. J. **23**, 298–305 (1973)
4. Fu, E., Markham, T.: On the eigenvalues and diagonal entries of a hermitian matrix. Linear Algebra Appl. **179**, 7–10 (1993)
5. Goldberg, F.: Bounding the gap between extremal Laplacian eigenvalues of graphs. Linear Algebra Appl. **416**, 68–74 (2006)
6. Hinrichsen, D., Plischke, E.: Robust stability and transient behaviour of positive linear systems. Vietnam J. Math. **35**, 429–462 (2007)
7. Jadbabaie, A., Lin, J., Morse, A.S.: Coordination of groups of mobile autonomous agents using nearest neighbor rules. IEEE Trans. Autom. Control **48**(6), 988–1001 (2003)

8. Knorn, F., Corless, M.J., Shorten, R.N.: A result on implicit consensus with application to emissions control. In: Proceedings of the 2011 CDC-ECC, pp. 1299–1304, Orlando, FL (2011)
9. Lin, J., Morse, A.S., Anderson, B.D.O.: The multi-agent rendezvous problem. In: Proceedings of the 42nd IEEE Conference on Decision and Control, pp. 1508–1513, Maui, Hawaii (2003)
10. Mohar, B.: The Laplacian spectrum of graphs. Graph Theory Comb. Appl. **2**, 871–898 (1991)
11. Olfati-Saber, R., Fax, J.A., Murray, R.M.: Consensus and cooperation in networked multi-agent systems. Proc. IEEE **95**(1), 215–233 (2007)
12. Pejic, S.: Algebraic graph theory in the analysis of frequency assignment problems. PhD thesis, London School of Economics and Political Science (2008)
13. Ren, W., Beard, R.W.: Consensus seeking in multiagent systems under dynamically changing interaction topologies. IEEE Trans. Autom. Control **50**(5), 655–661 (2005)
14. Smith, H.L.: Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems, vol. 41. AMS, Mathematical Surveys and Monographs, Providence, RI (1995)
15. Son, N.K., Hinrichsen, D.: Robust stability of positive continuous time systems. Numer. Funct. Anal. Optimiz. **17**(5–6), 649–659 (1996)
16. Tsitsiklis, J.N.: Problems in Decentralized Decision Making and Computation. PhD thesis, Department of EECS, MIT (1984)
17. Valcher, M.E., Zorzan, I.: New results on the solution of the positive consensus problem. In: Proceedings of the 55th IEEE Conf. on Decision and Control, pp. 5251–5256, Las Vegas, Nevada, December 12–14 (2016)
18. Valcher, M.E., Zorzan, I.: On the consensus of homogeneous multi-agent systems with positivity constraints (2017, under review)
19. Valcher, M.E., Zorzan, I.: On the consensus problem with positivity constraints. In: Proceedings of the 2016 American Control Conference, pp. 2846–2851, Boston, MA (2016)
20. Wieland, P., Kim, J.-S., Allgover, F.: On topology and dynamics of consensus among linear high-order agents. Int. J. Syst. Sci. **42**(10), 1831–1842 (2011)