Chapter 12
Polyhedral Invariance for Convolution Systems over the Callier-Desoer Class

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Abstract BIBO stability is a central concept for convolution systems, introduced in control theory by Callier, Desoer and Vidyasagar, in the seventies. It means that a bounded input leads to a bounded output, and is characterized by the fact that the kernel of the system is integrable. We generalize this result in this chapter, giving conditions for the output of a convolution system to evolve in a given polyhedron, for any input evolving in another given convex polyhedron. The conditions are formulated in terms of integrals deduced from the kernel of the considered system and the given polyhedra. The condition is exact. It permits to construct exact inner and outer polyhedral approximations of the reachable set of a linear system. The result is compared to various known results, and illustrated on the example of a system with two delays.

Keywords Convolution systems · Callier-Desoer class · Invariance · Reachable set · Polyhedra · Approximations

12.1 Introduction

The evaluation of the reachable space of a dynamical system is important for the verification of properties [4], planification of trajectories and design of control laws to achieve closed-loop specifications [7]. Exact formulae can not always be determined, so that various methods have been developed to compute approximations of the reachable set. The case of linear finite dimensional systems has been deeply investigated [19, 24]. The basic approach consists in reformulating the problem in terms of optimal control, which can be extended to the case of nonlinear systems [11] and hybrid systems [4, 9]. The effect of uncertainties or disturbances can also be handled using similar ideas and interval analysis [15].

The case of distributed systems has also been addressed. Systems with state delays are considered in [8], where a bounding ellipsoid of the reachable state is derived

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using Linear Matrix Inequalities. This idea gave rise to many generalizations, to distributed delays and variable delays, see e.g. [2] and the references therein. The question is generalized in [18] to that of the determination of invariant sets, for a class of discrete systems with delays.

A different approach was recently introduced. The question is formulated in [16] in an input-output setting. This is the basis of the present work. It concerns a large class of convolution systems, that includes localized or distributed time delay systems, ordinary or neutral time-delay systems, fractional systems and many other distributed systems. The basic idea is to observe that the input-output gain of a convolution system is bounded by the L_1 norm of its kernel. This can be reinterpreted in terms of reachability: the output of a system with input in the unit ball is included into the ball which radius is the L_1 norm of the kernel. When the underlying topology is the infinite norm, this observation comes down to a polytopic bound of the reachable set of a constrained system. The aim of this communication is to develop this idea, and to provide basic tools for the determination of polytopic approximations of the reachable set for a large class of convolution systems. For a multivariable convolution system, which input is constrained in a given polyhedron, we formulate conditions for the output of the system to evolve in another given polyhedron. The conditions are formulated in terms of integrals deduced from the kernel of the considered system and the given polyhedra. The conditions are necessary and sufficient, which shows that the bounds are in some sense exact.

The article is organized as follows. In Sect. 12.2, we recall the basic concepts that are used, in particular the definition of the Wiener algebra, and of a polytope. We identify bounds for the output of a given constrained system over the Wiener algebra in Sect. 12.3. These bounds are used to design overapproximations and underapproximations of the reachable set of the system at a given time horizon. In Sect. 12.4, the result is discussed, and illustrated on examples. Section 12.5 is a short conclusion.

12.2 Background Concepts

12.2.1 Convolution Kernels

An input-output linear system given in the form of a convolution,

$$y = h \star u \ , \tag{12.1}$$

is BIBO-stable if its kernel h belongs to the class $\mathscr{A}$ of generalized functions of the form

$$h(t) = h_a(t) + \sum_{i \in \mathbb{N}} h_i \delta(t - t_i) \ , \tag{12.2}$$

where h_a is in L_1, $h_i \in \mathbb{R}$, $t_i \in \mathbb{R}_+$, $t_i < t_{i+1}$ for $i \geq 0$, and $\sum_{i\in\mathbb{N}} |h_i| < \infty$. The set $\mathscr{A}$ endowed with the convolution product forms a Banach commutative algebra for the norm

$$||h||_{\mathscr{A}} = \int_0^{+\infty} |h_a(t)|dt + \sum_{i\in\mathbb{N}} |h_i| \ . \tag{12.3}$$

This norm was shown to be the induced norm when h is seen as an operator over L_∞. We indeed have

$$\sup_{u\neq 0} \frac{||h \star u||_\infty}{||u||_\infty} = ||h||_{\mathscr{A}} \ , \tag{12.4}$$

for every h in $\mathscr{A}$. Here, as usually, $||\cdot||_\infty$ denotes the sup-norm on L_∞, say $||u||_\infty = \text{ess}\sup_{t\geq 0} |u(t)|$, $||y||_\infty = \text{ess}\sup_{t\geq 0} |y(t)|$. This shows that every bounded input leads to a bounded output, and that $||h||_{\mathscr{A}}$ gives an exact bound on the output $y(t)$.

The set $\mathscr{A}$ is sometimes called Wiener algebra (see, e.g. [20]). Many properties of the set $\mathscr{A}$ are exposed in [10], and its use in control theory was gradually introduced by various authors, among them Desoer [1, 5, 6], Callier [1, 5] and Vidyasagar [6]. The set of fractions of elements of $\mathscr{A}(\sigma) = \mathrm{e}^{-\sigma t}\mathscr{A}$ is called the Callier-Desoer class and is a key concept to describe robust stabilization methods for a large class of distributed systems. The matter continues to generate interesting results, see for instance Quadrat [20], or Lakkonen [13] for a recent survey.

The transfer of a system of the form (12.1) is the Laplace transform $\hat{h}(s)$ of the kernel $h(t)$. For instance, the class $\mathscr{A}$ includes:

- the class of linear finite dimensional systems with rational transfer, e.g.

$$\hat{h}(s) = (sI - A)^{-1} \ , \ h(t) = \mathrm{e}^{At} \ ,$$

- the class of time-delay systems, e.g.

$$\hat{h}(s) = \frac{\mathrm{e}^{-\theta s}}{1 + sT} \ , \ h(t) = \begin{cases} 0 & , \ \text{for } t < \theta \ , \\ \mathrm{e}^{t-\theta} & , \ \text{for } t \geq \theta \ , \end{cases} \ ,$$

  that are important models in many applications,

- the class of systems with distributed delays, e.g.

$$\hat{h}(s) = \frac{1 - \mathrm{e}^{\theta a}\mathrm{e}^{-\theta s}}{s - a} \ , \ h(t) = \begin{cases} \mathrm{e}^{at} & , \ \text{for } t \leq \theta \ , \\ 0 & , \ \text{for } t > \theta \ , \end{cases} \ ,$$

  that are important for the stabilization of time-delay systems,

- BIBO stable diffusive systems, e.g.

$$\hat{h}(s) = \frac{1 - \mathrm{e}^{-\alpha\sqrt{s}}}{\sqrt{s}} \;, \; h(t) = 1 - \mathrm{erfc}\left(\frac{\alpha}{2\sqrt{t}}\right) \;.$$

This short list is not exhaustive. The class also includes many other linear distributed systems, and covers many application fields [3, 22].

The system (12.2) is said to be regular if $h(t) = h_a(t)$, or equivalently if the singular part is absent, say $h_i = 0$ for $i \in \mathbb{N}$. Notice that the class of regular systems is also very large, for instance the four examples of transfer functions mentioned above belong to this family.

Finally notice that in the present work, we basically consider systems with kernels of the form (12.2) that are well defined, in the sense that the kernel $h(t)$ is integrable over every finite interval $[0, t]$. This includes the Callier-Desoer class, which justifies the use of this expression in the title of the chapter. In Sect. 12.3.3, we shall assume that the kernel of the system is defined over $\mathscr{A}$.

12.2.2 Reachable Sets

We now consider a multivariable convolution system, defined by a kernel H, say

$$y = H \star u \;, \tag{12.5}$$

where $u(t) \in \mathscr{U} \subset \mathbb{R}^m$, for $t \geq 0$. Recall that the convolution product $\star$ is defined as

$$y_i(t) = \int_0^t \sum_j H_{ij}(t-\tau) u_j(\tau) d\tau \;. \tag{12.6}$$

We consider a system with entries of the form (12.2), so that $H_{ij}(t) = h_{a_{ij}}(t) + \sum_{k\in\mathbb{N}} h_{kij}(t - t_k)$, for $i = 1$ to p and $j = 1$ to m. We hence have, for $i = 1$ to p:

$$y_i(t) = \sum_j \left(\int_0^t h_{a_{ij}}(t-\tau) u_j(\tau) d\tau + \sum_{k | t_k \leq t} h_{kij} u_j(t - t_k) \right) \;.$$

We are interested into the characterization of the range of system (12.5). The basic concept is that of reachable set.

Definition 12.1 System (12.5) and a subset $\mathscr{U}$ of $\mathbb{R}^m$ being given, we say that an input function u is admissible, if $u(t) \in \mathscr{U}$, for $t \geq 0$. The reachable set $\mathscr{R}(\mathscr{U})$ is then defined as the set of vectors $x \in \mathbb{R}^p$ for which there exists an admissible control u such that the output $y(t)$ defined by (12.5) satisfies $y(t) = x$ for some $t \geq 0$. We also define the set $\mathscr{R}(\mathscr{U}, t)$ of vectors x that are reachable at t, so that $x = y(t)$ for some admissible input u, and the set $\mathscr{R}_t(\mathscr{U})$ of the vectors x reachable within t, so that $x = y(\tau)$, for some instant τ satisfying $0 \leq \tau \leq t$.

These definitions are taken from [24], a seminal paper on the computation of reachable sets for systems without memory. We remark that

$$\mathscr{R}_t(\mathscr{U}) = \bigcup_{\tau\in[0,t]} \mathscr{R}(\mathscr{U}, \tau) ,$$

and

$$\mathscr{R}(\mathscr{U}) = \bigcup_{t>0} \mathscr{R}(\mathscr{U}, t) = \bigcup_{t>0} \mathscr{R}_t(\mathscr{U}) .$$

One can see that $\mathscr{R}(\mathscr{U}, t)$ is convex, if $\mathscr{U}$ is convex. In Sect. 12.3, we shall in particular study the case where $\mathscr{U}$ is given in the form of a polytope $\mathscr{C}(M)$. The sets $\mathscr{R}_t(\mathscr{U})$ and $\mathscr{R}(\mathscr{U})$ are not convex, in general. Let us discuss these aspects.

The sets $\mathscr{R}_t(\mathscr{U})$ and $\mathscr{R}(\mathscr{U})$ are not connected, in general. This is due to the singular part of the kernels of the form (12.2), that may cause discontinuity of the solution $y(t)$. Consider for instance the kernel $h(t) = \delta(t-\theta)$, where θ is any positive number, and $\mathscr{U} = \{1\}$. We have in this example $\mathscr{R}(\mathscr{U}) = \{0, 1\}$, that is not connected. One can find conditions under which the sets are connected, or convex.

Proposition 12.1 *System (12.5) being given, together with a subset $\mathscr{U}$ of $\mathbb{R}^m$, and a real number $t \geq 0$, the following claims are true.*

(i) The set $\mathscr{R}(\mathscr{U}, t)$ is convex if $\mathscr{U}$ is convex.
(ii) The sets $\mathscr{R}_t(\mathscr{U})$ and $\mathscr{R}(\mathscr{U})$ are connected if $\mathscr{U}$ is convex and the kernel of the system is regular.
(iii) The sets $\mathscr{R}_t(\mathscr{U})$ are growing with t if $0 \in \mathscr{U}$.
(iv) The sets $\mathscr{R}_t(\mathscr{U})$ and $\mathscr{R}(\mathscr{U})$ are convex if $\mathscr{U}$ is convex, and $0 \in \mathscr{U}$.

Proof Notice first that if $\mathscr{U}$ is convex, and y and y' are reached using the admissible input trajectories $u(t)$ and $u'(t)$, respectively, then $\alpha u(t) + (1-\alpha)u'(t)$ is admissible too, and permits to reach $\alpha y + (1-\alpha)y'$. This shows that $\mathscr{R}(\mathscr{U}, t)$ is convex if $\mathscr{U}$ is convex. Further, the trajectories $y(t)$ of the system are continuous when the kernel is regular. Consider now two points y and y' in $\mathscr{R}(U)$. There exist admissible inputs u and u', and two instants $t, t' \geq 0$ such that $y = (H \star u)(t)$ and $y' = (H \star u')(t')$. We can assume, without any limitation, that $t' < t$. Defining $y'' = (H \star u)(t')$, one can see that there is a path from y' to y'' in $\mathscr{R}(\mathscr{U}, t')$, since this set is convex. There is also a path from y'' to y in $\mathscr{R}_t(\mathscr{U})$, since $y(\tau)$ is continuous, and takes its values into $\mathscr{R}_t(\mathscr{U})$, by definition of this set. Therefore, since $\mathscr{R}(\mathscr{U}, t')$ is a subset of $\mathscr{R}_t(\mathscr{U})$, one deduces that there exists in the latter set a path from y' to y, which shows the second assertion of the proposition. The third assertion is obtained remarking that if $y \in \mathscr{R}(\mathscr{U}, t)$ and $0 \in \mathscr{U}$, then there exists an admissible function u, and an instant t, such that $y = (H \star u)(t)$. One can see that $y = (H \star u')(t')$, taking $u'(\tau) = 0$, for $\tau \in [0, t'-t[$, and $u'(\tau) = u(t+\tau-t')$, for $\tau \geq t'-t$. This shows that $y \in \mathscr{R}(\mathscr{U}, t')$, for every t' greater than t, and establishes the third assertion. The last assertion is a consequence of (i) and (iii). □

As a consequence to this remark, the hypotheses that $\mathscr{U}$ is convex and $0 \in \mathscr{U}$ are often formulated in the literature, even in the case of localized systems. Of course, these assumptions are limitative. The identification of more accurate conditions might be useful in certain applications with discontinuous behaviors.

12.2.3 Elements of Convex Analysis

We now recall the definition of a polytope and some basic concepts of convex analysis. These concepts are taken from [21] (see in particular Sects. 6 and 13), and will be useful to analyse the reachability of constrained convolution systems.

A convex set $\mathscr{C} \subset \mathbb{R}^n$ is such that, for every $x, y \in \mathscr{C}$, and every $\lambda \in [0, 1]$, the vector $z = \lambda x + (1 - \lambda) y$ lies in $\mathscr{C}$. The support function of $\mathscr{C}$ is $f_{\mathscr{C}} : \mathbb{R}^n \to \mathbb{R}^n$, defined by

$$f_{\mathscr{C}}(v) = \sup_{x \in \mathscr{C}} v^T x \ ,$$

for $v \in \mathbb{R}^n$. Notice that $f_{\mathscr{C}}(v)$ takes only finite values if $\mathscr{C}$ is bounded. The ball of radius ε centered on $x \in \mathbb{R}^n$ is denoted $B(x, \varepsilon)$, as usually. A convex set $\mathbb{C}$ is open if there exists $\varepsilon > 0$ such that the ball $B(x, \varepsilon)$ is included into $\mathbb{C}$. It is closed if its complement is open. The least closed set containing $\mathbb{C}$ is called its closure, denoted $\overline{\mathscr{C}}$. The greatest open set included into $\mathscr{C}$ is called the interior of $\mathscr{C}$.

The concept of relative interior, that we now recall, is specific to the convex sets. The affine hull of a convex set $\mathscr{C}$ is denoted aff $\mathscr{C}$ and is defined as the set

$$\operatorname{aff} \mathscr{C} = \{z \in \mathbb{R}^n \mid \exists x, y \in \mathscr{C}, \alpha \in \mathbb{R}, z = x + \alpha(y - x)\} \ .$$

An affine set can also be written as $\operatorname{aff} \mathscr{C} = x + \operatorname{lin} \mathscr{C}$, for any element $x \in \mathscr{C}$, where $\operatorname{lin}\mathscr{C}$ is the vector space generated by the differences $y - x$, with $y \in \mathscr{C}$. The relative interior of $\mathscr{C}$, denoted $\operatorname{ri} \mathscr{C}$, is the interior of $\mathscr{C}$ when it is considered as a subset of aff $\mathscr{C}$, say

$$\operatorname{ri}\mathscr{C} = \{x \in \mathbb{R}^n \mid \exists \varepsilon > 0, B(x, \varepsilon) \cap \operatorname{aff} \mathscr{C} \subset \mathscr{C}\} \ .$$

One says that $\mathscr{C}$ is relatively open if it equals its relative interior. If $\mathscr{C}$ is reduced to a unique point, then $\operatorname{lin} \mathscr{C} = 0$ and $\operatorname{ri}\mathscr{C} = \overline{\mathscr{C}} = \mathscr{C}$. In general, the three sets $\mathscr{C}$, $\operatorname{ri}\mathscr{C}$, and $\overline{\mathscr{C}}$ are different, and we have the following.

Theorem 12.1 *Two convex sets $\mathscr{C}_1$ and $\mathscr{C}_2$ being given, the following claims are equivalent.*

(i) $f_{\mathscr{C}_1}(v) = f_{\mathscr{C}_2}(v)$, *for every vector* $v \in \mathbb{R}^n$,
(ii) $\overline{\mathscr{C}_1} = \overline{\mathscr{C}_2}$,
(iii) $\operatorname{ri} \mathscr{C}_1 = \operatorname{ri} \mathscr{C}_2$.

For a convex set $\mathscr{C}$, and a vector $v \in \mathbb{R}^n$, we have the inclusion $\{v^T x \mid x \in \mathscr{C}\} \subset [-f_{\mathscr{C}}(-v), f_{\mathscr{C}}(v)]$, because $\inf_{x \in \mathscr{C}}\{v^T x\} = -\sup_{x \in \mathscr{C}}\{-v^T x\}$. The bounds are exact, and they belong or not to the set, depending on its topological property. The following theorem precises this aspect (this is Theorem 13.1 of [21]).

Theorem 12.2 *A nonempty convex set $\mathscr{C}$ being given, the following claims hold true.*

(i) $\mathscr{C}$ *is closed if and only if* $\forall v \in \mathbb{R}^n$, $\{v^T x \mid x \in \mathscr{C}\} = [-f_{\mathscr{C}}(-v), f_{\mathscr{C}}(v)]$.
(ii) $\mathscr{C}$ *is open if and only if* $\forall v \in \mathbb{R}^n$, $\{v^T x \mid x \in \mathscr{C}\} =]-f_{\mathscr{C}}(-v), f_{\mathscr{C}}(v)[$.
(iii) $\mathscr{C}$ *is relatively open if and only if* $\{v^T x \mid x \in \mathscr{C}\} =]-f_{\mathscr{C}}(-v), f_{\mathscr{C}}(v)[$, $\forall v \in \mathbb{R}^n$ *such that* $-f_{\mathscr{C}}(-v) < f_{\mathscr{C}}(v)$.

We finally recall basic facts and definitions concerning polytopes.

Definition 12.2 A matrix $M \in \mathbb{R}^{m \times n}$ being given, the convex polytope of $\mathbb{R}^m$ generated by the columns of M is the set denoted $\mathscr{C}(M)$, and defined by

$$\mathscr{C}(M) = \left\{ x \in \mathbb{R}^m \mid \exists v \in \mathbb{R}^n, v \geq 0, \sum_{i=1}^{n} v_i = 1, x = Mv \right\} .$$

The relatively open polytope generated by M is defined by

$$\mathscr{C}_{ro}(M) = \left\{ x \in \mathbb{R}^m \mid \exists v \in \mathbb{R}^n, v_i > 0, \sum_{i=1}^{n} v_i = 1, x = Mv \right\} .$$

In other words, introducing the notation $\Gamma = \{v \in \mathbb{R}^n, v \geq 0, \sum_{i=1}^{n} v_i = 1\}$, we have $\mathscr{C}(M) = M\Gamma$, and $\mathscr{C}_{ro}(M) = M \operatorname{ri} \Gamma = \operatorname{ri} M\Gamma$. The following result is then clear (see Theorems 6.6 and 6.9 from [21]).

Proposition 12.2 *For every matrix M, we have the equality*

$$\operatorname{ri} \mathscr{C}(M) = \mathscr{C}_{ro}(M) .$$

Definition 12.3 A matrix $P \in \mathbb{R}^{q \times p}$ and a vector $\pi \in \mathbb{R}^q$ being given, the polyhedron denoted $\mathscr{P}(P, \pi)$ is the set defined as

$$\mathscr{P}(P, \pi) = \left\{ z \in \mathbb{R}^p \mid Pz \leq \pi \right\} .$$

The relatively open polyhedron $\mathscr{P}_{ro}(P, \pi)$ is defined by

$$\mathscr{P}_{ro}(P, \pi) = \left\{ z \in \mathscr{P}(P, \pi) \mid \sum_{j=1}^{p} P_{ij} z_j < \pi_i, \text{ for } i \in J(P, \pi) \right\} ,$$

with $J(P, \pi) = \{i \mid \exists z \in \mathscr{P}(P, \pi), \sum_{j=1}^{p} P_{ij} z_j < \pi_i\}$.

In other words, some of the constraints corresponding to the rows of the matrix P and the vector π actually define the affine hull of $\mathscr{P}(P, \pi)$. The other ones, corresponding to the set $J(P,\pi)$, define a subset of aff $\mathscr{P}(P,\pi)$, with a nonempty interior that equals ri $\mathscr{P}(P,\pi)$. We hence have the following result.

Proposition 12.3 *A matrix $P \in \mathbb{R}^{p\times m}$ and a vector $\pi \in \mathbb{R}^p$ being given, the following equality holds true*

$$\text{ri}\, \mathscr{P}(P,\pi) = \mathscr{P}_{ro}(P,\pi)\ .$$

We are now ready to study the reachable set of convolution systems.

12.3 Polyhedral Bounds of the Reachable Set

12.3.1 Elementary Bounds

A basic question consists in determining the range of the output $y(t)$ of system (12.5). We are precisely interested in verifying whether or not the output $y(t)$ belongs to a given polyhedron, provided that the input $u(t)$ evolves in another given polyhedron. The following elementary remark will be useful in the sequel.

Lemma 12.1 *Let be given a vector $x \in \mathbb{R}^n$ and a vector v in the convex set $\Gamma = \{v \in \mathbb{R}^n, v \geq 0, \sum_{i=1}^n v_i = 1\}$ defined as in Definition 12.2. Then, we have*

$$\max_{v\in\Gamma} \left\{ \sum_{j=1}^n x_j v_j \right\} = \max_j\ x_j\ .$$

Proof Since $x_j \leq \max_j\ x_j$, it is clear that $\sum_{j=1}^n x_j v_j \leq (\max_j\ x_j)(\sum_{j=1}^n v_j)$. By definition of Γ, it appears that $\sum_{j=1}^n x_j v_j \leq \max_j\ x_j$, so that $\max_{v\in\Gamma} \left\{\sum_{j=1}^n x_j v_j\right\}$ $\leq \max_j\ x_j$. This is an exact bound, which follows considering the vector v defined by $v_k = 1$ and $v_j = 0$, for $j \neq k$, with $k = \arg\max_j x_j$. This ends the proof. □

We are now able to formulate the basic result on polyhedral bounds of system (12.5).

Theorem 12.3 *System (12.5) being given, together with matrices $M \in \mathbb{R}^{m\times n}$, $P \in \mathbb{R}^{q\times p}$, a vector $\pi \in \mathbb{R}^q$, and $t > 0$, then $y(t)$ belongs to $\mathscr{P}(p,\pi)$ for every input satisfying $u(\tau) \in \mathscr{C}(M)$, for $\tau \geq 0$, if and only if the following condition holds true for $i = 1$ to q:*

$$\int_0^t \max_j \left\{(PH(\tau)M)_{ij}\right\} d\tau \leq \pi_i\ .$$

Proof To begin with the proof, we proceed by equivalences:

$$\begin{aligned} y(t) \in \mathscr{P}(P,\pi) &\iff Py(t) \le \pi &&, \text{ by definition of } \mathscr{P}(P,\pi), \\ &\iff \textstyle\int_0^t PH(\tau)u(t-\tau)d\tau \le \pi &&, \text{ by definition of the system}, \\ &\iff \textstyle\int_0^t PH(\tau)Mv(t-\tau)d\tau \le \pi &&, \text{ by definition of } \mathscr{C}(M), \\ &\iff \textstyle\int_0^t \max_j \left\{(PH(\tau)M)_{ij}\right\} d\tau \le \pi_i &&, \text{ by Lemma 12.1.} \end{aligned}$$

Assuming that H is a matrix with inputs that are integrable over $[0, t]$, we observe that the integrals in these equivalences are well defined. They are indeed bounded by the product $p \cdot m \cdot \max_k\{|P_{ik}|\} \cdot B \cdot \max_l |\{M_{lj}|\}$, if B is a bound of the integrals of the entries of H, for instance $B = \max_{k,l} ||H_{kl}||_{\mathscr{A}}$, if H is a matrix over $\mathscr{A}$. In these statements, the vector $v(t-\tau)$ lies in Γ, by hypothesis, which permits to apply Lemma 12.1. The fact that this lemma gives exact bounds is essential to obtain the last equivalence, from which the theorem is deduced. □

A preliminary version of this result was obtained in [16]. We first remark that upper bounds and lower bounds of the behavior of the given system can be derived from Theorem 12.3. For this purpose, one defines

$$\lambda_i(t) = \int_0^t \min_j \left\{(H(\tau)M)_{ij}\right\} d\tau \ , \tag{12.7}$$

and

$$\mu_i(t) = \int_0^t \max_j \left\{(H(\tau)M)_{ij}\right\} d\tau \ . \tag{12.8}$$

Corollary 12.1 *The matrix M and the system (12.5) being given as in Theorem 12.3, and $\lambda_i(t)$, $\mu_i(t)$ being defined as in (12.7), (12.8), we have*

$$\lambda_i(t) \le y_i(t) \le \mu_i(t) \ ,$$

for $i = 1$ to p. In addition, the bounds are reached, so that the range of $y_i(t)$, when the input satisfies $u(\tau) \in \mathscr{C}(M)$, for $\tau \ge 0$, is exactly the closed interval $[\lambda_i(t), \mu_i(t)]$.

Proof The upper bound of Corollary 12.1 is obtained taking $P = I_p$ in Theorem 12.3. The lower bound is obtained with $P = -I_p$, since $\min_j\{x_j\} = -\max_j\{-x_j\}$, and $-\max_{v\in\Gamma}\{-x_j v_j\} = \min_{v\in\Gamma} x_j v_j$, with Γ defined as in Lemma 12.1.

To complete the proof, we remark that the upper bound $\mu_i(t)$ is indeed reached using the control defined by $u_k(\tau) = M_{kj(t-\tau)}$, for $k = 1$ to m and $\tau \in [0, t]$, with

$$j(\tau) = \arg\max_j \left\{(H(\tau)M)_{ij}\right\} \ .$$

Similarly, the lower bound is reached using the control that maximizes $-y_i(t)$, that is defined in terms of an argument of $\max_j \left\{-(H(\tau)M)_{ij}\right\}$. □

We can finally remark the following fact, that will be useful in Sect. 12.3.3.

Corollary 12.2 *Under the conditions of Corollary 12.1, the range of* $y_i(t)$ *when* $u(\tau) \in \mathscr{C}_{ro}(M)$ *equals the open interval* $]\lambda_i(t), \mu_i(t)[$, *if* $\lambda_i(t) < \mu_i(t)$, *and is reduced to* $\{\mu_i(t)\}$, *if* $\lambda_i(t) = \mu_i(t)$.

Proof If the equality $\lambda_i(t) = \mu_i(t)$ is satisfied, one can see that $\min_j \left\{(H(\tau)M)_{ij}\right\} = \max_j \left\{(H(\tau)M)_{ij}\right\}$ almost everywhere in the interval $[0, t]$, and therefore the kernels $(H(\tau)M)_{ij}$, for $j = 1$ to n, are equal almost everywhere in this interval. In this case, $y(t)$ takes a unique value, say $\int_0^t (H(\tau)M)_{i1} d\tau$. If $\lambda_i(t) < \mu_i(t)$, then the different kernels $(H(\tau)M)_{ij}$, for $j = 1$ to n, are not equal on a subset of $[0, t]$ having a nonzero measure. Taking an instant t from this set, we observe that $\min_j \left\{(H(\tau)M)_{ij}\right\} < H(\tau)u(t-\tau) < \max_j \left\{(H(\tau)M)_{ij}\right\}$ holds true, for every input $u(t-\tau) \in \mathscr{C}_{ro}(M)$, from which one deduces that $\lambda_i(t) < y_i(t) < \mu_i(t)$. The conclusion is obtained remarking that the bounds can be approached with an arbitrary precision. To this aim, define $K = \int_0^t \sum_k \left(\max_j \left\{(H(\tau)M)_{ij}\right\} - (H(\tau)M)_{ik}\right) d\tau$. We can see that K is positive, and has a finite value if the kernel is integrable over $[0, t]$. Taking $u(t-\tau) = M\nu(t-\tau)$, with $\nu_j(t-\tau) = 1 - (n-1)\varepsilon/K$, and $\nu_k = \varepsilon/K$, for $k \neq j(t-\tau)$, we obtain $y_i(t) = \mu_i(t) - \varepsilon$. The lower bound $\lambda_i(t)$ is approached in the same way, using an argument $j(\tau)$ of $\max_j \left\{-(H(\tau)M)_{ij}\right\}$ and defining now $K = \int_0^t \sum_k \left((H(\tau)M)_{ik} - \min_j \left\{(H(\tau)M)_{ij}\right\}\right)$. One checks that the input defined by $u(t-\tau) = M\nu(t-\tau)$, with $\nu_j(t-\tau) = 1 - (n-1)\varepsilon/K$, and $\nu_k = \varepsilon/K$, for $k \neq j(t-\tau)$ leads to an output verifying $y_i(t) = \lambda_i(t) + \varepsilon$, which ends the proof. □

12.3.2 Polyhedral Approximations of the Reachable Set

The previous results can be interpreted in terms of reachability.

Remark that the difference between the left and right members of the condition of Theorem 12.3 is the distance between the reachable set and the plan $\{y \in \mathbb{R}^p \mid \sum_j P_{ij} y_j = \pi_i\}$. The left member of the condition, say

$$\rho_i(t) = \int_0^t \max_j \left\{(PH(\tau)M)_{ij}\right\} d\tau \ , \tag{12.9}$$

is therefore so that the plan $\{y \in \mathbb{R}^p \mid \sum_j P_{ij} y_j = \rho_i(t)\}$ is tangent to the reachable space at t, say $\mathscr{R}(\mathscr{C}(M), t)$, of the system constrained by $\mathscr{U} = \mathscr{C}(M)$. If the matrix P is given, the polyhedron $\mathscr{P}(P, \rho(t))$ is the least polyhedron whose faces are oriented according to P, and that contains the reachable set. One can also compute a point of the intersection between the face and the reachable set. We first define the integers

$$j_k(\tau) = \arg \max_j \left\{(PH(\tau)M)_{kj}\right\} \ ,$$

for $k = 1$ to q, and the output vectors

$$\nu_i(k,t) = \int_0^t (H(\tau)M)_{i j_k(\tau)} d\tau \ ,$$

for $k = 1$ to q, and $i = 1$ to p. Then, N is defined as the matrix which columns are the vectors $\nu(k,t)$, say

$$N_{ij} = \nu_i(j,t) \ ,$$

for $i = 1$ to p, $j = 1$ to q. The following definitions are inspired by [24].

Definition 12.4 A compact convex set $\mathscr{R}$ being given, we say that a polyhedron is an exact outer approximation of $\mathscr{R}$ if its faces are tangent to $\mathscr{R}$, and that it is an exact inner approximation of $\mathscr{R}$, if its vertices are on the boundary of $\mathscr{R}$.

Theorem 12.4 *The system (12.1) being given, together with an integer q and two matrices $P \in \mathbb{R}^{q\times p}$ and $M \in \mathbb{R}^{m\times q}$, and taking N and ρ defined as above, the convex polytope $\mathscr{C}(N)$ is an exact inner approximation, and the polyhedron $\mathscr{P}(P, \rho(t))$ is an exact outer approximation, of $\mathscr{R}(\mathscr{C}(M), t)$.*

Proof For $k = 1$ to q, the control defined by $u^{(k)}(t-\tau) = Mv^{(k)}(\tau)$, with $v_j^{(k)}(\tau) = 1$, if $j = j_k(\tau)$, and $v_j^{(k)}(\tau) = 0$, if $j \neq j_k(\tau)$, satisfies $(Py)_k(t) = \rho_k(t)$. This shows that the faces of $\mathscr{P}(P, \rho(t))$ are tangent to $\mathscr{R}(\mathscr{C}(M), t)$, and the vertices of $\mathscr{C}(N)$ are on the boundary of $\mathscr{R}(\mathscr{C}(M), t)$, which ends the proof. □

In other words, we have the chain of inclusions $\mathscr{C}(N) \subset \mathscr{R}(\mathscr{C}(M), t) \subset \mathscr{P}(P, \rho(t))$, and the distance between the three sets is null:

$$\inf\{d(x, y) \mid x \in \mathscr{C}(N), y \in \mathscr{R}(\mathscr{C}(M), t)\} = 0 \ ,$$

and

$$\inf\{d(y, z) \mid y \in \mathscr{R}(\mathscr{C}(M), t), z \in \mathscr{P}(P, \rho(t))\} = 0.$$

The precision of the approximation can be defined as the Hausdorff distance between the upper and lower approximations, defined, since $\mathscr{C}(N)) \subset \mathscr{P}(P, \rho(t))$, as:

$$\max\{d(z, \mathscr{C}(N)) \mid z \in \mathscr{P}(P, \rho(t))\} \ .$$

This distance is decreasing when rows are added to the matrix P. This permits to reach an arbitrary precision choosing a matrix P that corresponds to plans in many different directions. In practice, the number of rows is rapidly growing with the dimension of the system. For this reason, one may prefer rough approximations in high dimension. Anyway, this formulation is well fitted for numerical computations. The integrals can be easily approximated using Matlab or Scilab, for instance, provided that the kernel $H(t)$ is explicitly known, or can be numerically computed. We shall give a simple example in Sect. 12.4.

We complete this study with remarks concerning the topological structure and the approximation of $\mathscr{R}_t(\mathscr{U})$ and $\mathscr{R}(\mathscr{U})$.

12.3.3 Additional Comments on the Structure of the Reachable Set

We first complete the previous results in terms of the reachable set at a given instant.

Proposition 12.4 *The matrix $M \in \mathbb{R}^{m\times n}$, an instant $t \geq 0$ and the system (12.5) being given, then the following claims are true.*

(i) The set $\mathscr{R}(\mathscr{C}(M), t)$ is closed.

(ii) The set $\mathscr{R}(\mathscr{C}_{ro}(M), t)$ is relatively open.

(iii) We have the equalities $\mathscr{R}(\mathscr{C}_{ro}(M), t) = \mathrm{ri}\,\mathscr{R}(\mathscr{C}(M), t)$.

Proof The proof uses Theorem 12.2 (that is Theorem 13.1 of [21]), and a variant of Theorem 12.3 and Corollary 12.2. According to claim (i) of Proposition 12.1, the set $\mathscr{R}(\mathscr{C}(M), t)$ is convex. We then remark that the support function of $\mathscr{R}(\mathscr{C}(M), t)$ is defined, in any direction $v \in \mathbb{R}^p$, by $f_{\mathscr{R}(\mathscr{C}(M),t)}(v) = \int_0^t \max_j\{(v^T H(\tau)M)_j\}d\tau$. Applying Corollary 12.2, one obtains that $v^T\mathscr{R}(\mathscr{C}(M), t)$ is either reduced to a single element, if $f_{\mathscr{R}(\mathscr{C}(M),t)}(v) = -f_{\mathscr{R}(\mathscr{C}(M),t)}(-v)$, or equal to the open interval $]-f_{\mathscr{R}(\mathscr{C}(M),t)}(-v), f_{\mathscr{R}(\mathscr{C}(M),t)}(v)[$, if $-f_{\mathscr{R}(\mathscr{C}(M),t)}(-v) < f_{\mathscr{R}(\mathscr{C}(M),t)}(v)$. The second claim is therefore deduced from claim (iii) of Theorem 12.2. In a similar way, one can see that for every $v \in \mathbb{R}^p$, the set $v^T\mathscr{R}(\mathscr{C}(M), t)$ is a closed interval. The claim (i) is then deduced from claim (i) of Theorem 12.2. From Corollaries 12.1 and 12.2, we conclude that actually $\mathscr{R}(\mathscr{C}(M), t)$ is the closure of $\mathscr{R}(\mathscr{C}_{ro}(M), t)$. We further obtain from Theorem 12.2 that the open interior of both sets are equal, and the conclusion follows since $\mathscr{R}(\mathscr{C}_{ro}(M), t)$ is equal to its relative interior, from claim (i). □

In other words, the set $\mathscr{R}(\mathscr{C}(M), t)$ is closed if the kernel $H(t)$ is integrable over $[0, t]$, because the limits are reached in the inequalities presented in Sect. 12.3.1, and its relative interior coincides with the set of points that are reachable using inputs in the relative interior of the polyhedron $\mathscr{U} = \mathscr{C}(M)$. When t tends to the infinity, the upper bound found for $y(t)$ when the system is subject to a bounded input $u(t)$ converges to a bounded limit (assuming that system (12.5) is over $\mathscr{A}$), but this limit may be reachable, or not, depending on $H(t)$, and the choosen direction v. As a consequence, $\mathscr{R}(\mathscr{C}(M))$ is not closed, in general. In the same way, when the kernels include delayed diracs, the function $\mu_i(t)$ may be discontinuous, so that the set of points that are reachable within a finite time, $\mathscr{R}_t(\mathscr{C}(M))$, is not always closed.

Consider for instance the system $y = h * u$, with $h(\tau) = f_a(\tau) - \delta(1-\tau)$, with $f_a(\tau) = 1$, for $\tau \in [0, 1]$, and $f_a(\tau) = 0$, for $\tau > 0$. We have

$$y(t) = \begin{cases} \int_0^t f_a(\tau)u(t-\tau)d\tau & , \text{ for } t < 1, \\ \int_0^t f_a(\tau)u(t-\tau)d\tau - u(t-1) & , \text{ for } t \geq 1. \end{cases} \tag{12.10}$$

One can verify that taking $u(\tau) = 1$ on this example, we obtain $y(t) = t$, for $t \in [0, 1[$, and $y(t) = 0$, for $t \geq 1$. The point $y = 1$ is not reachable within $t = 1$, if $\mathscr{U} = \{1\}$. We have in this case $M = (1)$, $\mathscr{C}(M) = \{1\}$, and $\mathscr{R}_t(\mathscr{C}(M)) = [0, t]$, for $t \in [0, 1[$, and $\mathscr{R}_t(\mathscr{C}(M)) = [0, 1[$, for $t \geq 1$.

A singular kernel may also cause that $\mathscr{R}_t(\mathscr{C}(M))$ and $\mathscr{R}(\mathscr{C}(M))$ are not connected set. The consequences of these remarks are different in terms of outer or inner approximations.

Remark 12.1 We can adapt Theorem 12.3 and Corollary 12.1 to have the constraint $y(\tau) \in \mathscr{P}(P, \pi)$ satisfied within a finite time interval, say $[0, t]$, or respectively for $t \geq 0$. For this purpose, one now defines

$$\rho_i(t) = \sup_{0 \leq \theta \leq t} \int_0^\theta \max_j \left\{(PH(\tau)M)_{ij}\right\} d\tau \ , \tag{12.11}$$

or, respectively,

$$\rho_i = \sup_{t \geq 0} \int_0^t \max_j \left\{(PH(\tau)M)_{ij}\right\} d\tau \ . \tag{12.12}$$

We then obtain the following bounds within t:

$$y_i(\theta) \leq \rho_i(t) \ ,$$

for $\theta \in [0, t]$, or, respectively

$$y_i(t) \leq \rho_i$$

for $t \geq 0$.

Going on in this direction, we remark that the polyhedra $\mathscr{P}(P, \rho(t))$, or $\mathscr{P}(P, \rho)$, respectively, are outer approximations of $\mathscr{R}_t(\mathscr{C}(M))$ and $\mathscr{R}(\mathscr{C}(M))$, respectively. As introduced in Proposition 12.1, additional hypotheses can be introduced to be able to calculate inner approximations of the reachable sets.

Remark 12.2 The integral that appears in (12.12) is an increasing function of the time t, when its integrand is non-negative. This is always the case when 0 lies in $\mathscr{C}(M)$, or when the kernel $H(t)$ and the matrix M are non-negative. In this case, the bound (12.12) is equal to

$$\rho_i = \int_0^\infty \max_j \left\{(PH(\tau)M)_{ij}\right\} d\tau \ ,$$

that is well-defined if $H(t)$ is defined over $\mathscr{A}$.

Under the same hypothesis, that $0 \in \mathscr{C}(M)$, we observe that $\rho_i(t)$ is actually given by (12.9), and $\mathscr{R}_t(\mathscr{C}(M)) = \mathscr{R}(\mathscr{C}(M), t)$. In this case, the procedure presented in

Sect. 12.3.2 can be used to calculate a matrix N that corresponds to a lower approximation of the reachable set $\mathscr{R}_t(\mathscr{C}(M))$. We can also adapt this procedure to the case of an indefinite integral. For $k = 1$ to q, we define N as in Sect. 12.3.2, with $t = \infty$. According to Definition 12.4, we have obtained an exact approximation of the closure of the reachable set. We may remark that $\mathscr{C}(N)$ is not included into $\mathscr{R}(\mathscr{C}(M))$, in general, but we have the inclusion $\mathscr{C}_{ro}(N) \subset \mathscr{R}(\mathscr{C}_{ro}(M)) \subset \mathscr{P}_{ro}(P, \rho)$. In this sense, the matrices P, N, and the vector ρ also define exact approximations of the relatively open reachable set.

Remark 12.3 In many applications, one wants to compute approximations of the tube $(\mathscr{R}(\mathscr{C}(M), t), t) \subset \mathbb{R}^p \times \mathbb{R}_+$. As suggested in claim (ii) of Proposition 12.1, this tube is well defined if the kernel of system (12.4) is regular. The tube is then approximated using polyhedral approximations of $\mathscr{R}(\mathscr{C}(M), t_i)$ at successive instants t_i.

12.4 Remarks and Examples

12.4.1 Positive Kernels

The classical characterization of the positivity of a system in terms of the positivity of its kernel can also be seen as a consequence of Theorem 12.3.

Definition 12.5 The system (12.1) is said to be non-negative if every non-negative input $u(t)$ leads to a non-negative output $y(t)$. The multivariable system (12.5) is non-negative if its entries are all non-negative.

Corollary 12.3 *The system (12.1) is non-negative if and only if its kernel (12.2) is non-negative almost everywhere. The system (12.5) is non-negative if and only if all the entries of its kernel $H(t)$ are non-negative almost everywhere.*

Proof By definition, the system (12.1) is non-negative if $\mathscr{R}(\mathscr{C}(M)) \subset \mathscr{P}(P, \pi)$, with $M = (0, 1)$, $P = (-1)$, and $\pi = (0)$. Applying Theorem 12.3, we conclude that $\int_0^t \max\{0, -h(\tau)\}d\tau \leq 0$, for $t \geq 0$, from which we deduce that $h(\tau)$ takes non-negative values almost everywhere. □

If the system (12.2) is positive, and $u(t)$ lies in $[\alpha, \beta]$, we have the following inequalities, for $t \geq 0$

$$\alpha \int_0^t h(\tau)d\tau \leq y(t) \leq \beta \int_0^t h(\tau)d\tau \ .$$

In addition, these bounds are exact, in the sense that they are reached. If in addition the kernel $h(t)$ is an element of $\mathscr{A}$, then we have $y(t) \in [\alpha\|h\|_{\mathscr{A}}, \beta\|h\|_{\mathscr{A}}]$. The limits of this interval may be reached or not, but they are exact in the sense of the discussion of Sect. 12.3.3, for instance we have $\mathscr{R}(\mathscr{C}_{ro}(M)) =]\alpha\|h\|_{\mathscr{A}}, \beta\|h\|_{\mathscr{A}}[$.

This first result can be generalized to kernels that are not necessarily positive. Every measure h in $\mathscr{A}$ can be uniquely decomposed into a difference $h = h^+ - h^-$, where h^+ and h^- are two positive measures in $\mathscr{A}$ with disjoint supports. Then if $\alpha \leq u(t) \leq \beta$, the range of $y(t)$ is given by

$$\alpha \int_0^t h^+(\tau)d\tau - \beta \int_0^t h^-(\tau)d\tau \leq y(t) \leq \beta \int_0^t h^+(\tau)d\tau - \alpha \int_0^t h^-(\tau)d\tau \ ,$$

for any positive t, that can be rewritten as

$$\int_0^t \min\{\alpha h(\tau), \beta h(\tau)\}\, d\tau \leq y(t) \leq \int_0^t \max\{\alpha h(\tau), \beta h(\tau)\}\, d\tau \ ,$$

that in turns appears to be a consequence of Theorem 12.3. The latter formulation is well fitted for numerical computations, since it avoids the computation of h^+ and h^-. Indeed the infinite integral can be easily approximated using Matlab or Scilab, provided that $h(t)$ is explicitly known, or can be numerically computed. We also remark that this formula gives the way to calculate a control law $u^{\max}$ that maximizes the output. This control law is given by

$$u^{\max}(t-\tau) = \begin{cases} \alpha \ , \ \text{if } \max\{\alpha h(\tau), \beta h(\tau)\} = \alpha h(\tau) \ , \\ \beta \ , \ \text{else} \ , \end{cases} \tag{12.13}$$

for any positive t. In the same way, the control given by

$$u^{\min}(t-\tau) = \begin{cases} \alpha \ , \ \text{if } \min\{\alpha h(\tau), \beta h(\tau)\} = \alpha h(\tau) \ , \\ \beta \ , \ \text{else} \ , \end{cases}$$

permits to reach the lower value of the output. When t goes to infinity, we obtain the results that follow. They are well-known and often used (or rediscovered) in the literature.
(i) If $u(t) \in]-u_{\max}, +u_{\max}[$, then $y(t) \in]-y_{\max}, +y_{\max}[$, with $y_{\max} = \|h\| u_{\max}$.
(ii) If $h(t)$ is positive, and $u(t) \in [0, +u_{\max}[$, then $y(t) \in [0, y_{\max}[$.
(iii) If $h(t) = h_+(t) - h_-(t)$ with h_+ and h_- positive, and $u(t) \in]-u_{\min}, +u_{\max}[$, then $y(t) \in]-y_{\min}, +y_{\max}[$, with $y_{\min} = \|h_+\| u_{\min} + \|h_-\| u_{\max}$, and $y_{\max} = \|h_+\| u_{\max} + \|h_-\| u_{\min}$.

12.4.2 Constrained Control and $\mathscr{D}$-Invariance

We give here a simple example of the explicit computation of the bounds of input-output systems. It illustrates that these techniques may be useful to design control laws for constrained systems.

We consider the following model, which was introduced by Simon some years ago [23]. The inventory level $y(t)$ of a simple logistic system follows the law

$$\dot{y}(t) = u(t-\theta) - d(t) \ ,$$

where $u(t)$ is the production rate order and $d(t)$is the instantaneous demand. We assume that for $t < \theta$, we have $\dot{y}(t) = \phi(t) - d(t)$, where $\phi(t)$ corresponds to some initial condition. We choose the control law in the form

$$u(t) = K(y_c - z(t)) \ ,$$

with

$$z(t) = \begin{cases} y(t) + \int_{t-\theta}^{t} u(\tau)d\tau & \text{for } t \geq \theta \, , \\ y(t) + \int_{t}^{\theta} \phi(\tau)d\tau + \int_{0}^{t} u(\tau)d\tau & \text{for } t < \theta \, . \end{cases}$$

One can show that the solution is written

$$\hat{y}(s) = \frac{1 + K\frac{1-\mathrm{e}^{-s\theta}}{s}}{s+K}\left(y_0 + \hat{\phi}(s) - \hat{d}(s)\right) + \frac{K\mathrm{e}^{-s\theta}}{s+K}.\left(\frac{y_c}{s} + \hat{\phi}\right) \ .$$

We therefore introduce the notations

$$\hat{h}_1(s) = \frac{\left(1 + K\frac{1-\mathrm{e}^{-s\theta}}{s}\right)}{s+K} \quad \hat{h}_2(s) = \frac{K\mathrm{e}^{-s\theta}}{s+K}$$

that are the Laplace transform of the kernels

$$h_1(t) = \begin{cases} 1 & , \text{ for } t \in [0, \theta[\ , \\ \mathrm{e}^{-K(t-\theta)} & , \text{ for } t \geq \theta \ , \end{cases} \qquad h_2(t) = \begin{cases} 0 & , \text{ for } t \in [0, \theta[\ , \\ \mathrm{e}^{-K(t-\theta)} & , \text{ for } t \geq \theta \ , \end{cases}$$

and we notice that $\|h_1\|_{\mathscr{A}} = \theta + 1/K$, and $\|h_2\|_{\mathscr{A}} = 1$. Assuming that the range of the external demand $d(t)$ is $[0, d_{\max}]$, we deduce the bounds

$$-d_{\max}\|h_1\|_{\mathscr{A}} + y_c\,\|h_2\|_{\mathscr{A}} \leq y(t) \leq y_c\,\|h_2\|_{\mathscr{A}}$$

that lead to explicit bounds on $y(t)$

$$y_c - d_{\max}\left(\theta + \frac{1}{K}\right) \leq y(t) \leq y_c \, ,$$

for $t \geq \theta$, and on the admissible initial conditions

$$y_0 + wip_0 - \theta \ \leq y(t) \leq y_0 + wip_0 \ .$$

over the initial period $t \in [0, \theta[$, with $wip_0 = \int_0^\theta \phi(\tau)d\tau$. From these bounds, one can easily deduce conditions to meet the constraints on the production and inventory capacity, that are given as $u(t) \in [0, u_{\max}]$ and $y(t) \in [0, y_{\max}]$, for every demand in the range $d(t) \in [0, d_{\max}]$. The admissible values of the control parameters are:

$$y_c \in [\theta d_{\max}, y_{\max}] \ , \ \ K \geq \frac{d_{\max}}{y_c} - \theta \ ,$$

and the admissible values of the sizing parameters are:

$$u_{\max} \geq d_{\max} \ , \ \ \theta d_{\max} < y_{\max} \ .$$

These results were obtained using other methods in [17]. The same model can also be used to study the congestion control in communication networks, and similar results have been expounded in [12].

12.4.3 Example of Approximation of the Reachable Set

Let us consider the following time delay system

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} x(t-1) + \begin{bmatrix} 0 & 1 \\ 0.5 & 0 \end{bmatrix} x(t-\pi) + \begin{bmatrix} -0.5 \\ 1 \end{bmatrix} u(t) \, ,$$

where the initial state of the system is taken as $x(t) = 0$ for $t \in [-\pi, 0]$, and $u(t)$ verifies $u(t) \in \mathscr{U} = \{0 \leq u(t) \leq 1\}$, for $t \geq 0$. Formally, this system can be rewritten in the form

$$x(t) = (H \star u)(t) \ ,$$

where $H \in \mathscr{A}^{2\times 1}$. The first step of the design is to numerically compute the kernels $H_{11}(t)$ and $H_{21}(t)$ using the solver dde23 of MATLAB. The result is plotted in Fig. 12.1. The second step of the design consists in the computation of the outer and inner approximations of the reachable set of the system. For this purpose, we consider the matrix P that is obtained by the concatenation of row vectors of the form $(\cos \frac{2k\pi}{K}, \sin \frac{2k\pi}{K})$, for $k = 1$ to K, and apply the procedure indicated in Sect. 12.3.2 to compute the vector ν and the matrix N, so that the outer and inner approximations of the reachable set are respectively $\mathscr{P}(P, \nu)$ and $\mathscr{C}(N)$. The polyhedra obtained for $K = 5$ are shown in Fig. 12.2. We also represent on the figure the reachable set, that was finely approximated using $K = 360$.

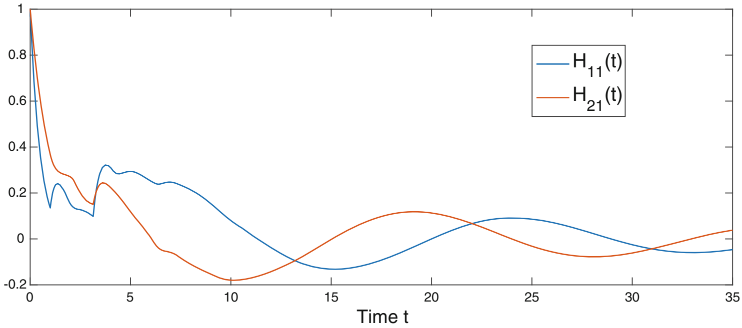

Fig. 12.1 Graphs of the kernels $H_{11}(t)$ and $H_{21}(t)$

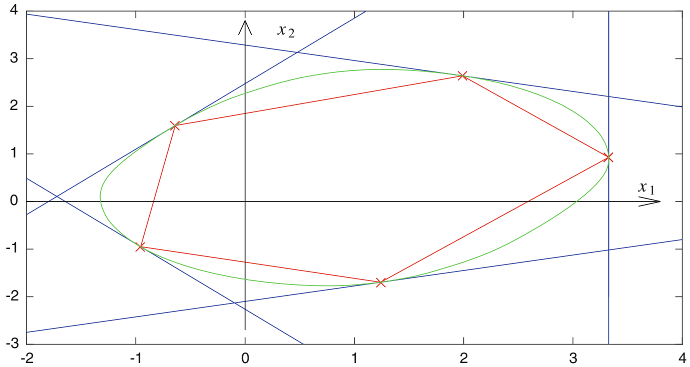

Fig. 12.2 Approximation of the reachable set

12.5 Conclusions

We have characterized bounds for a class of input-output systems defined by a convolution. They are derived from the concept of BIBO stability, and are given in terms of integrals that are easy to compute numerically. A method for the approximation of reachable sets of convolution systems was obtained from these bounds. We shortly commented the topological structure of the reachable set and the case of positive systems. The method was illustrated on a simple regulation problem of inventory level in a logistic system, and on an academic example of system with two non-commensurable delays.

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