Chapter 4
Two Degree-of-Freedom Oscillator Coupled to a Non-ideal Source

In this chapter the two degree-of-freedom structure excited with a non-ideal source is considered. The model corresponds to real energy harvester system (Felix et al. 2009), centrifugal vibration machine (Dantas and Balthazar 2006), tuned liquid column damper mounted on a structural frame (Felix et al. 2005a), portal frame (Felix et al. 2013) and portal frame foundation type shear building (Felix et al. 2005b), rotor-structure system which moves in-plane (Quinn 1997), etc. These systems are usually modelled as two mass systems with visco-elastic connection and excited with non-ideal motor. The main attention is given to resonance capture (Balthazar et al. 2001) in the presence of a 1:1 (Zniber and Quinn 2006) and 1:2 (Tsuchida et al. 2005) frequency ratio. However, we suggest to model the aforementioned real systems as an one-mass system with two degrees-of-freedom in two orthogonal directions as the mass moves in-plane. Such model is given in the paper of Goncalves et al. (2016) and treated numerically and experimentally. The model consists of a concentrated mass which is supported by a set of linear springs and dampers positioned in two orthogonal direction, such that the mass can move horizontally and vertically in a plane. A non-ideal motor is attached to the mass such that the phenomena of resonance capture can occur. In the paper of Goncalves et al. (2016) it is concluded that the resonance can occur in both directions, in only one direction or can not occur. Limits between these cases are determined in Cveticanin et al. (2017). The system is described with the set of three coupled second order differential equations: two of them describing the vibrations of the structure in two directions and one, which gives the motion of the motor. An analytical procedure for solving the equations is developed and the constraints for resonance are given.

The chapter is divided into six sections. In Sect. 4.2, the motion of the system which contains a mass supported in two directions and excited with non-ideal energy source is modeled. Mathematical model of the system is solved analytically in Sect. 4.3. The steady-state solution and the stability conditions for solutions are determined. In Sect. 4.4, two special cases are considered: first, the case when the frequencies of the system in x and y directions are equal and then, the case when the frequency

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in y direction is two times higher than that in x direction. The resonant motions are investigated. The influence of vertical and horizontal stiffness on the regions of double resonance motion are considered. The obtained results are compared with numerical results in Sect. 4.5. The chapter ends with conclusions.

4.1 Model of the System

The system considered in this paper consists of a mass M supported by springs and viscous dampers in two orthogonal directions (x and y). The spring constants are defined with k_x and k_y, while the damping coefficients are c_x and c_y. The subscripts x and y indicate the displacement directions. To the mass M a motor is attached with unbalanced mass m at the distance d from the center of the motor shaft (Fig. 4.1).

The motor shaft has moment of inertia defined by J. The motor - structure system shown in Fig. 4.1 has three degrees-of-freedom defined with three generalized coordinates x, y and φ. where the first two define the motion in two orthogonal directions and the third is the angle position of the unbalance. Equations of motion of such system are in general

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}} - \frac{\partial T}{\partial x} + \frac{\partial U}{\partial x} + \frac{\partial \Phi}{\partial \dot{x}} = Q_x,$$
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}} - \frac{\partial T}{\partial y} + \frac{\partial U}{\partial y} + \frac{\partial \Phi}{\partial \dot{y}} = Q_y,$$

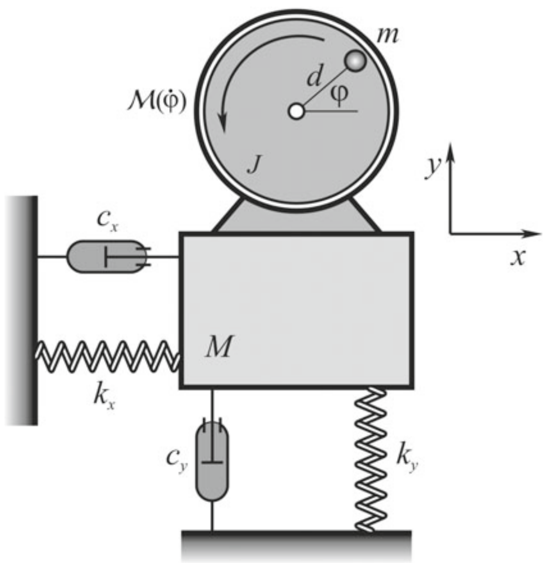

Fig. 4.1 Model of a two-degree-of-freedom oscillator coupled with a non-ideal unbalanced motor

$$\frac{d}{dt}\frac{\partial T}{\partial \dot\varphi} - \frac{\partial T}{\partial \varphi} + \frac{\partial U}{\partial \varphi} + \frac{\partial \Phi}{\partial \dot\varphi} = Q_\varphi, \tag{4.1}$$

where T and U are the kinetic and potential energy of the system, Φ is the dissipative function and Q_x, Q_y and Q_φ are generalized forces.

Motion of the system is excited with the motor torque $\mathcal{M}$ which depends on the angular velocity of the rotor $\dot\varphi$. For the DC motor the motor torque model is assumed to be a linear function of $\dot\varphi$, i.e.,

$$\mathcal{M}(\dot\varphi) = \mathcal{M}_0\left(1 - \frac{\dot\varphi}{\Omega_0}\right), \tag{4.2}$$

where $\mathcal{M}_0$ and Ω_0 are constant values. Then the generalized force due to motor torque is

$$Q_\varphi = \mathcal{M}(\dot\varphi). \tag{4.3}$$

For the model given in Fig. 4.1 the kinetic energy is

$$T = \frac{1}{2}M(\dot x^2 + \dot y^2) + \frac{1}{2}J\dot\varphi^2 + \frac{1}{2}m(\dot x_2^2 + \dot y_2^2), \tag{4.4}$$

where the position of unbalance mass m is

$$x_m = x + d\cos\varphi, \quad y_m = y + d\sin\varphi. \tag{4.5}$$

Substituting the time derivative of (4.5) into (4.4), we have

$$T = \frac{1}{2}(M+m)(\dot x^2 + \dot y^2) + \frac{1}{2}(J + md^2)\dot\varphi^2 + md\dot\varphi(\dot y\cos\varphi - \dot x\sin\varphi). \tag{4.6}$$

If the gravity potential energy is neglected, the potential energy of the system is

$$U = \frac{1}{2}k_x x^2 + \frac{1}{2}k_y y^2. \tag{4.7}$$

The dissipative function of the system is

$$\Phi = \frac{1}{2}c_x \dot x^2 + \frac{1}{2}c_y \dot y^2. \tag{4.8}$$

Substituting (4.3)–(4.6) into (4.1), we obtain the cart's equations in x and y direction

$$(M+m)\ddot x + k_x x + c_x \dot x = md(\dot\varphi^2\cos\varphi + \ddot\varphi\sin\varphi), \tag{4.9}$$

$$(M+m)\ddot y + k_y y + c_y \dot y = md(\dot\varphi^2\sin\varphi - \ddot\varphi\cos\varphi), \tag{4.10}$$

and the equation of motion of the unbalanced mass

$$(J + md^2)\ddot{\varphi} = md(\ddot{x}\sin\varphi - \ddot{y}\cos\varphi) + \mathcal{M}(\dot{\varphi}). \tag{4.11}$$

Using the parameter values

$$\omega_x = \sqrt{\frac{k_x}{M+m}}, \quad \omega_y = \sqrt{\frac{k_y}{M+m}}, \quad \eta = \frac{md}{J+md^2},$$
$$\zeta_x = \frac{c_x}{M+m}, \quad \zeta_y = \frac{c_y}{M+m}, \quad \mu = \frac{md}{M+m}, \quad \varepsilon = \frac{1}{J+md^2}, \tag{4.12}$$

Equations (4.9)–(4.11) are rewritten as

$$\begin{aligned} \ddot{x} + \omega_x^2 x + \zeta_x \dot{x} &= \mu(\dot{\varphi}^2\cos\varphi + \ddot{\varphi}\sin\varphi), \\ \ddot{y} + \omega_y^2 y + \zeta_y \dot{y} &= \mu(\dot{\varphi}^2\sin\varphi - \ddot{\varphi}\cos\varphi), \\ \ddot{\varphi} &= \eta(\ddot{x}\sin\varphi - \ddot{y}\cos\varphi) + \varepsilon\mathcal{M}(\dot{\varphi}). \end{aligned} \tag{4.13}$$

It has to be mention that the parameter ε is a small one, i.e., $\varepsilon << 1$ and the input motor torque is small. Our aim is to solve the Eq. (4.13).

4.2 Analytical Solution

Let us give the equations of motion (4.13) in terms of uncoupled accelerations

$$\begin{aligned} \ddot{x} = -\varepsilon\frac{\mu\mathcal{M}(\dot{\varphi})\sin\varphi}{\mu\eta - 1} &+ \frac{\mu\eta\sin(2\varphi)(\mu\dot{\varphi}^2\sin\varphi - F_y)}{2(\mu\eta - 1)} \\ &+ \frac{(\mu\eta\cos^2\varphi - 1)(\mu\dot{\varphi}^2\cos\varphi - F_x)}{\mu\eta - 1}, \end{aligned} \tag{4.14}$$

$$\begin{aligned} \ddot{y} = \varepsilon\frac{\mu\mathcal{M}(\dot{\varphi})\cos\varphi}{\mu\eta - 1} &+ \frac{\mu\eta\sin(2\varphi)(\mu\dot{\varphi}^2\cos\varphi - F_x)}{2(\mu\eta - 1)} \\ &+ \frac{(\mu\eta\sin^2\varphi - 1)(\mu\dot{\varphi}^2\sin\varphi - F_y)}{\mu\eta - 1}, \end{aligned} \tag{4.15}$$

$$\begin{aligned} \ddot{\varphi} = -\frac{\varepsilon\mathcal{M}(\dot{\varphi})}{\mu\eta - 1} &+ \frac{\eta\cos\varphi(\mu\dot{\varphi}^2\sin\varphi - F_y)}{\mu\eta - 1} \\ &- \frac{\eta\sin\varphi(\mu\dot{\varphi}^2\cos\varphi - F_x)}{\mu\eta - 1}, \end{aligned} \tag{4.16}$$

where

$$F_x = x\omega_x^2 + \zeta_x \dot{x}, \quad F_y = y\omega_y^2 + \zeta_y \dot{y}. \tag{4.17}$$

For the case when the parameters ζ_x, ζ_y, μ and η are small, i.e.,

$$\begin{aligned}&\zeta_x = \varepsilon\zeta_{x1}, \quad \zeta_y = \varepsilon\zeta_{y1}, \quad \mu = \varepsilon\mu_1,\\ &\eta = \varepsilon\eta_1,\end{aligned} \tag{4.18}$$

where $\varepsilon << 1$ is a small parameter, Eq. (4.13) transform into the form

$$\begin{aligned}(\varepsilon^2\mu_1\eta_1 - 1)\ddot{x} = &-\varepsilon^2\mu_1\mathcal{M}(\dot{\varphi})\sin\varphi\\ &+ \frac{\varepsilon^2\mu_1\eta_1\sin(2\varphi)(\varepsilon\mu_1\dot{\varphi}^2\sin\varphi - y\omega_y^2 - \varepsilon\zeta_{y1}\dot{y})}{2}\\ &+ (\varepsilon^2\mu_1\eta_1\cos^2\varphi - 1)(\varepsilon\mu_1\dot{\varphi}^2\cos\varphi - x\omega_x^2 - \varepsilon\zeta_{x1}\dot{x}),\end{aligned} \tag{4.19}$$

$$\begin{aligned}(\varepsilon^2\mu_1\eta_1 - 1)\ddot{y} = \varepsilon^2\mu_1\mathcal{M}(\dot{\varphi})\cos\varphi + \frac{\varepsilon^2\mu_1\eta_1\sin(2\varphi)(\mu\dot{\varphi}^2\cos\varphi - x\omega_x^2 - \varepsilon\zeta_{x1}\dot{x})}{2}\\ + (\varepsilon^2\mu_1\eta_1\sin^2\varphi - 1)(\varepsilon\mu_1\dot{\varphi}^2\sin\varphi - y\omega_y^2 - \varepsilon\zeta_{y1}\dot{y}),\end{aligned} \tag{4.20}$$

$$\begin{aligned}(\varepsilon^2\mu_1\eta_1 - 1)\ddot{\varphi} = &-\varepsilon\mathcal{M}(\dot{\varphi}) + \varepsilon\eta_1\cos\varphi(\varepsilon\mu_1\dot{\varphi}^2\sin\varphi - y\omega_y^2 - \varepsilon\zeta_{y1}\dot{y})\\ &- \varepsilon\eta_1\sin\varphi(\varepsilon\mu_1\dot{\varphi}^2\cos\varphi - x\omega_x^2 - \varepsilon\zeta_{x1}\dot{x}).\end{aligned} \tag{4.21}$$

Eliminating the terms with the small parameter ε of higher order than one, i.e., $O(\varepsilon^2)$, we have the following simplified equations

$$\ddot{x} + \omega_x^2 x = \varepsilon\mu_1\dot{\varphi}^2\cos\varphi - \varepsilon\zeta_{x1}\dot{x}, \tag{4.22}$$

$$\ddot{y} + \omega_y^2 y = \varepsilon\mu_1\dot{\varphi}^2\sin\varphi - \varepsilon\zeta_{y1}\dot{y}, \tag{4.23}$$

$$\ddot{\varphi} = \varepsilon\mathcal{M}(\dot{\varphi}) + \varepsilon\eta_1(y\omega_y^2\cos\varphi - x\omega_x^2\sin\varphi). \tag{4.24}$$

Using the notation

$$x_1 = x, \quad x_2 = \dot{x}, \quad y_1 = y, \quad y_2 = \dot{y}, \quad \Omega = \dot{\varphi}, \tag{4.25}$$

the Eqs. (4.22)–(4.24) are rewritten in the following system of first order differential equations

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -\omega_x^2 x_1 + \varepsilon\mu_1\Omega^2\cos\varphi - \varepsilon\zeta_{x1}x_2,\\ \dot{y}_1 &= y_2,\\ \dot{y}_2 &= -\omega_y^2 y_1 + \varepsilon\mu_1\Omega^2\sin\varphi - \varepsilon\zeta_{y1}y_2,\end{aligned}$$

$$\begin{aligned}\dot{\varphi} &= \Omega,\\ \dot{\Omega} &= \varepsilon\mathcal{M}(\Omega) + \varepsilon\eta_1(y_1\omega_y^2\cos\varphi - x_1\omega_x^2\sin\varphi),\end{aligned} \tag{4.26}$$

where according to (4.2) the motor torque is

$$\varepsilon\mathcal{M}(\Omega) = M_0\left(1 - \frac{\Omega}{\Omega_0}\right), \tag{4.27}$$

where M_0 and Ω_0 motor constants. Equation (4.26) represent a system of coupled differential equations whose solution is not easy to be obtained. For simplification, let us introduce the new variables

$$\begin{aligned}x_1 &= a_1\cos(\varphi+\psi_1),\\ x_2 &= -a_1\Omega\sin(\varphi+\psi_1),\\ y_1 &= a_2\cos(\varphi+\psi_2),\\ y_2 &= -a_2\Omega\sin(\varphi+\psi_2),\end{aligned} \tag{4.28}$$

where Ω, a_1, a_2, ψ_1 and ψ_2 are time dependent functions. Substituting (4.28) into (4.26) and using the relation $\dot{\varphi} = \Omega$ we obtain

$$\begin{aligned}0 &= \dot{a}_1\cos(\varphi+\psi_1) - a_1\dot{\psi}_1\sin(\varphi+\psi_1),\\ & -\dot{a}_1\Omega\sin(\varphi+\psi_1) - a_1\Omega\dot{\psi}_1\cos(\varphi+\psi_1)\\ -a_1\dot{\Omega}\sin(\varphi+\psi_1) &= -\omega_x^2 a_1\cos(\varphi+\psi_1) + a_1\Omega^2\cos(\varphi+\psi_1)\\ &\quad + \varepsilon\mu_1\Omega^2\cos\varphi + \varepsilon\zeta_{x1}a_1\Omega\sin(\varphi+\psi_1),\\ 0 &= \dot{a}_2\cos(\varphi+\psi_2) - a_2\dot{\psi}_2\sin(\varphi+\psi_2),\\ & -\dot{a}_2\Omega\sin(\varphi+\psi_2) - a_2\dot{\Omega}\sin(\varphi+\psi_2)\\ -a_2\Omega\dot{\psi}_2\cos(\varphi+\psi_2) &= -\omega_y^2 a_2\cos(\varphi+\psi_2) + a_2\Omega^2\cos(\varphi+\psi_2)\\ &\quad + \varepsilon\mu_1\Omega^2\sin\varphi + \varepsilon\zeta_{y1}a_2\Omega\sin(\varphi+\psi_2),\\ \dot{\Omega} &= \varepsilon\eta_1[a_2\cos(\varphi+\psi_2)\omega_y^2\cos\varphi - a_1\cos(\varphi+\psi_1)\omega_x^2\sin\varphi]\\ &\quad + \varepsilon\mathcal{M}(\Omega),\end{aligned} \tag{4.29}$$

and after some modification

$$\begin{aligned}\dot{a}_1 + a_1\frac{\dot{\Omega}}{\Omega}\sin^2(\varphi+\psi_1) &= \frac{\omega_x^2-\Omega^2}{2\Omega}a_1\sin 2(\varphi+\psi_1)\\ &\quad - \varepsilon\mu_1\Omega\cos\varphi\sin(\varphi+\psi_1)\\ &\quad - \varepsilon\zeta_{x1}a_1\sin^2(\varphi+\psi_1),\\ a_1\left(\dot{\psi}_1 + \frac{1}{2}\frac{\dot{\Omega}}{\Omega}\sin 2(\varphi+\psi_1)\right) &= \frac{\omega_x^2-\Omega^2}{\Omega}a_1\cos^2(\varphi+\psi_1)\\ &\quad - \varepsilon\mu_1\Omega\cos\varphi\cos(\varphi+\psi_1)\end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{1}{2}\varepsilon\zeta_{x1}a_1 \sin 2(\varphi + \psi_1), \\
\dot{a}_2 + a_2\frac{\dot{\Omega}}{\Omega}\sin^2(\varphi + \psi_2) &= \frac{\omega_y^2 - \Omega^2}{2\Omega}a_2 \sin 2(\varphi + \psi_2) \\
&\quad - \varepsilon\mu_1\Omega \sin\varphi \sin(\varphi + \psi_2) \\
&\quad - \varepsilon\zeta_{y1}a_2 \sin^2(\varphi + \psi_2), \\
a_2\left(\dot{\psi}_2 - \frac{\dot{\Omega}}{2\Omega}\sin 2(\varphi + \psi_2)\right) &= \frac{\omega_y^2 - \Omega^2}{\Omega}a_2 \cos^2(\varphi + \psi_2) \\
&\quad - \varepsilon\mu_1\Omega \sin\varphi \cos(\varphi + \psi_2) \\
&\quad - \frac{1}{2}\varepsilon\zeta_{y1}a_2 \sin 2(\varphi + \psi_2), \\
\dot{\Omega} &= \varepsilon\eta_1[a_2 \cos(\varphi + \psi_2)\omega_y^2 \cos\varphi \\
&\quad - a_1 \cos(\varphi + \psi_1)\omega_x^2 \sin\varphi] \\
&\quad + \varepsilon\mathcal{M}(\Omega).
\end{aligned} \tag{4.30}
$$

The Eq. (4.30) are the first order differential equations which correspond to second order equations (4.22)–(4.24). The system of equations (4.30) has to be solved for Ω, a_1, a_2, ψ_1 and ψ_2. As the Eq. (4.30) are coupled, to find the solution is not an easy task. It is at this moment where the simplification is done. Averaging the equations over the period 2π of the function φ, we obtain the averaged equations

$$
\begin{aligned}
\dot{a}_1 + a_1\frac{\dot{\Omega}}{2\Omega} &= -\frac{1}{2}\varepsilon\mu_1\Omega \sin\psi_1 - \frac{1}{2}\varepsilon\zeta_{x1}a_1, \\
a_1\dot{\psi}_1 &= \frac{\omega_x^2 - \Omega^2}{2\Omega}a_1 - \frac{1}{2}\varepsilon\mu_1\Omega \cos\psi_1, \\
\dot{a}_2 + a_2\frac{\dot{\Omega}}{2\Omega} &= -\frac{1}{2}\varepsilon\mu_1\Omega \cos\psi_2 - \frac{1}{2}\varepsilon\zeta_{y1}a_2, \\
a_2\dot{\psi}_2 &= \frac{\omega_y^2 - \Omega^2}{2\Omega}a_2 + \frac{1}{2}\varepsilon\mu_1\Omega \sin\psi_2, \\
\dot{\Omega} &= \varepsilon\mathcal{M}(\Omega) + \frac{1}{2}\varepsilon\eta_1\left(a_2\omega_y^2 \cos\psi_2 + a_1\omega_x^2 \sin\psi_1\right).
\end{aligned} \tag{4.31}
$$

4.2.1 Steady-State Motion

It is of interest to analyze the steady-state motion when $\dot{a}_1 = 0$, $\dot{a}_2 = 0$ and $\dot{\Omega} = 0$ and the corresponding Eq. (4.31) are

$$
\begin{aligned}
0 &= \mu\Omega_S \sin\psi_{1S} + \zeta_x a_{1S}, \\
0 &= \frac{\omega_x^2 - \Omega_S^2}{\Omega_S}a_{1S} - \mu\Omega_S \cos\psi_{1S},
\end{aligned} \tag{4.32}
$$

$$0 = \mu\Omega_S \cos\psi_{2S} + \zeta_y a_{2S},$$
$$0 = \frac{\omega_y^2 - \Omega_S^2}{\Omega_S} a_2 + \mu\Omega_S \sin\psi_{2S}, \tag{4.33}$$

$$0 = \varepsilon\mathcal{M}(\Omega) + \frac{1}{2}\eta\left(a_{2S}\omega_y^2 \cos\psi_{2S} + a_{1S}\omega_x^2 \sin\psi_{1S}\right). \tag{4.34}$$

Eliminating ψ_{1S} in Eq. (4.32) the steady-state amplitude a_{1S} as the function of Ω is obtained

$$a_{1S} = \frac{\mu\Omega_S}{\sqrt{(\zeta_x)^2 + (\frac{\omega_x^2 - \Omega_S^2}{\Omega_S})^2}}. \tag{4.35}$$

Using the same procedure and eliminating ψ_{2S} in Eq. (4.33) the steady-state amplitude a_{2S} as the function of Ω yields

$$a_{2S} = \frac{\mu\Omega_S}{\sqrt{(\zeta_y)^2 + (\frac{\omega_y^2 - \Omega_S^2}{\Omega_S})^2}}. \tag{4.36}$$

Dividing equations in (4.32) and (4.33) the phase angles in the both directions of motion are obtained

$$\tan\psi_{1S} = \frac{\zeta_x\Omega_S}{\Omega_S^2 - \omega_x^2}, \quad \tan\psi_{2S} = \frac{\omega_y^2 - \Omega_S^2}{\zeta_y\Omega_S}. \tag{4.37}$$

Comparing the amplitudes a_{1S} and a_{2S} we obtain the condition for which the motion is out of resonance and also when the resonance occurs in one or both directions of motion. For $\omega_x \neq \Omega_S$ and $\omega_y \neq \Omega_S$ the motion is out of resonance regime. The amplitude of vibration in x and y direction depends on the damping properties of the system and on the difference between the angular velocity of the motor and the frequency of the system ω_x and ω_y, respectively. However, it is of interest to analyze the motion when resonances appear.

Substituting (4.35)–(4.37) into (4.34), it is

$$0 = \varepsilon\mathcal{M}(\Omega) + \frac{\zeta_y\mu\Omega_S\eta\omega_y^2}{2(\zeta_y^2 + (\frac{\omega_y^2 - \Omega_S^2}{\Omega_S})^2} + \frac{\zeta_x\mu\Omega_S\eta\omega_x^2}{2(\zeta_x^2 + (\frac{\omega_x^2 - \Omega_S^2}{\Omega_S})^2}. \tag{4.38}$$

For the torque property (4.27) the relation (4.38) transforms into

$$\varepsilon M_0(1 - \frac{\Omega_S}{\Omega_0}) = -\frac{\eta\mu\Omega_S^3}{2}\left[\frac{\zeta_x\omega_x^2}{(\zeta_x\Omega_S)^2 + (\omega_x^2 - \Omega_S^2)^2} + \frac{\zeta_y\omega_y^2}{(\zeta_y\Omega_S)^2 + (\omega_y^2 - \Omega_S^2)^2}\right]. \tag{4.39}$$

The Eq. (4.39) gives the relation between the angular velocity Ω and parameter Ω_0. Let us rewrite (4.39) as

$$\Omega_0 = f(\Omega_S) = \frac{\Omega_S}{1 + \frac{\eta\mu\Omega_S^3}{2\varepsilon M_0}\left[\frac{\zeta_x\omega_x^2}{(\zeta_x\Omega_S)^2 + (\omega_x^2 - \Omega_S^2)^2} + \frac{\zeta_y\omega_y^2}{(\zeta_y\Omega_S)^2 + (\omega_y^2 - \Omega_S^2)^2}\right]}. \tag{4.40}$$

Calculating the first derivative $d\Omega_0/d\Omega_S$ and equating with zero, we obtain the values of Ω_S which give the extreme values of Ω_0, i.e., $\Omega_{0\,\min}(\Omega_S)$ and $\Omega_{0\,\max}(\Omega_S)$.

4.2.2 Stability Analysis

Using the results of steady-state motion (4.32)–(4.34) the perturbed amplitudes, phases and angular velocities are

$$\begin{aligned} a_1 &= a_{1S} + \xi_1, \quad a_2 = a_{2S} + \xi_2, \\ \psi_1 &= \psi_{1S} + \theta_1, \quad \psi_2 = \psi_{2S} + \theta_2, \\ \Omega &= \Omega_S + \varpi. \end{aligned} \tag{4.41}$$

where ξ_1, ξ_2, θ_1, θ_2 and ϖ are small perturbation functions. Substituting (4.41) into (4.31) and after linearization the system of coupled first order differential equations follows

$$\begin{aligned} 2\dot{\xi}_1\Omega_S + a_{1S}\dot{\varpi} &= -(\zeta_x a_{1S} + 2\mu\Omega_S \sin\psi_{1S})\varpi \\ &\quad - \zeta_x\Omega_S\xi_1 - (\mu\Omega_S^2\cos\psi_{1S})\theta_1, \\ 2a_{1S}\dot{\theta}_1\Omega_S &= -2\Omega_S(a_{1S} - \mu\cos\psi_{1S})\varpi \\ &\quad + (\omega_x^2 - \Omega_S^2)\xi_1 + (\mu\Omega_S^2\sin\psi_{1S})\theta_1, \\ 2\dot{\xi}_2\Omega_S + a_{2S}\dot{\varpi} &= -(\zeta_y a_{2S} + 2\mu\Omega_S\cos\psi_{2S})\varpi \\ &\quad - (\zeta_y\Omega_S)\xi_2 + (\mu\Omega_S^2\sin\psi_{2S})\theta_2, \\ 2a_{2S}\dot{\theta}_2\Omega_S &= -2\Omega_S(a_{2S} - \varepsilon\mu_1\sin\psi_{2S})\varpi \\ &\quad + (\omega_y^2 - \Omega_S^2)\xi_2 + (\mu\Omega_S^2\cos\psi_{2S})\theta_2, \\ 2\dot{\varpi} &= -\frac{2\varepsilon M_0}{\Omega_0}\varpi \\ &\quad - (\eta a_{2S}\omega_y^2\sin\psi_{2S})\theta_2 + (\eta\omega_y^2\cos\psi_{2S})\xi_2 \\ &\quad + (\eta\omega_x^2\sin\psi_{1S})\xi_1 + (\eta a_{1S}\omega_x^2\cos\psi_{1S})\theta_1. \end{aligned} \tag{4.42}$$

Assuming the solution of (4.42) in the form

$$\begin{aligned}\xi_1 &= A_1\exp(\lambda t), \quad \xi_2 = A_2\exp(\lambda t),\\ \theta_1 &= A_3\exp(\lambda t), \quad \theta_2 = A_4\exp(\lambda t),\\ \varpi &= A_5\exp(\lambda t),\end{aligned} \tag{4.43}$$

and substituting into (4.42), the system of linear algebraic equations is obtained

$$\begin{aligned}0 &= -(2\lambda\Omega_S + \zeta_x\Omega_S)A_1 - (\mu\Omega_S^2\cos\psi_{1S})A_3\\ &\quad -(a_{1S}\lambda + \zeta_x a_{1S} + 2\mu\Omega_S\sin\psi_{1S})A_5\\ 0 &= (\omega_x^2 - \Omega_S^2)A_1 + (\mu\Omega_S^2\sin\psi_{1S} - 2a_{1S}\lambda\Omega_S)A_3\\ &\quad -2\Omega_S(a_{1S} - \mu\cos\psi_{1S})A_5,\\ 0 &= -(2\Omega_S\lambda + \varepsilon\zeta_{y1}\Omega_S)A_2 + (\mu\Omega_S^2\sin\psi_{2S})A_4\\ &\quad -(a_{2S}\lambda + \zeta_y a_{2S} + 2\mu\Omega_S\cos\psi_{2S})A_5,\\ 0 &= (\omega_y^2 - \Omega_S^2)A_2 + (\mu\Omega_S^2\cos\psi_{2S} - 2a_{2S}\lambda\Omega_S)A_4\\ &\quad -2\Omega_S(a_{2S} - \mu\sin\psi_{2S})A_5\\ 0 &= (\eta\omega_x^2\sin\psi_{1S})A_1 + (\eta\omega_y^2\cos\psi_2)A_2\\ &\quad +(\eta a_1\omega_x^2\cos\psi_{1S})A_3 - (\eta a_2\omega_y^2\sin\psi_{2S})A_4\\ &\quad -(2\lambda + \frac{2\varepsilon M_0}{\Omega_0})A_5.\end{aligned} \tag{4.44}$$

The system has the nontrivial solution if the determinant is zero. The determinant of the system is a fifth order algebraic equation. Solving the equation and applying the Routh-Hurwitz criteria, the stability of the solutions is determined.

4.3 Special Cases

Two special cases are considered: one, when the resonance frequencies in both orthogonal directions are equal, and the second, when the resonance frequency in one direction is defined by half of the resonance frequency in the other direction.

4.3.1 Resonance Frequencies in Orthogonal Directions Are Equal

For the special case when the frequencies in both direction are equal, i.e., $\omega_x = \omega_y = \omega$, the steady state amplitudes of vibration are

$$a_{1S} = \frac{\mu\Omega_S}{\sqrt{\zeta_x^2 + (\frac{\omega^2-\Omega_S^2}{\Omega_S})^2}}, \quad a_{2S} = \frac{\mu\Omega_S}{\sqrt{\zeta_y^2 + (\frac{\omega^2-\Omega_S^2}{\Omega_S})^2}},$$

while the corresponding phases are

$$\tan\psi_{1S} = \frac{\zeta_x\Omega_S}{\Omega_S^2 - \omega^2}, \quad \tan\psi_{2S} = \frac{\omega^2 - \Omega_S^2}{\zeta_y\Omega_S}.$$

For the resonant case when

$$\Omega_S = \omega - \Delta, \tag{4.45}$$

and the detuning function is $\Delta = \varepsilon\sigma$, the equations transform into

$$a_{1S} = \frac{\mu\Omega_S}{\sqrt{(\zeta_x^2 + 4(\varepsilon\sigma)^2}}, \quad a_{2S} = \frac{\mu\Omega_S}{\sqrt{\zeta_y^2 + 4(\varepsilon\sigma)^2}},$$
$$\tan\psi_{1S} = -\frac{\zeta_x}{2\varepsilon\sigma}, \quad \tan\psi_{2S} = \frac{2\varepsilon\sigma}{\zeta_y},$$

i.e.,

$$a_{1S} = \frac{\mu\Omega_S}{\sqrt{\zeta_x^2 + 4(\omega - \Omega_S)^2}}, \tag{4.46}$$

$$a_{2S} = \frac{\mu\Omega_S}{\sqrt{\zeta_y^2 + 4(\omega - \Omega_S)^2}}, \tag{4.47}$$

while the phase angles in the both directions of motion are

$$\tan\psi_{1S} = \frac{\zeta_x}{2(\omega - \Omega_S)}, \quad \tan\psi_{2S} = \frac{2(\omega - \Omega_S)}{\zeta_y}. \tag{4.48}$$

Substituting (4.46)–(4.48) into (4.34) it is

$$\varepsilon\mathcal{M}(\Omega) = -\frac{1}{2}\eta\mu\Omega_S^3 \left[\frac{\zeta_y}{\zeta_y^2 + 4(\varepsilon\sigma)^2} + \frac{\zeta_x}{\zeta_x^2 + 4(\varepsilon\sigma)^2}\right], \tag{4.49}$$

i.e.,

$$\varepsilon\mathcal{M}(\Omega_S) = -\frac{1}{2}\eta\mu\Omega_S^3 \left[\frac{\zeta_y}{\zeta_y^2 + 4(\omega - \Omega_S)^2} + \frac{\zeta_x}{\varepsilon\zeta_x^2 + 4(\omega - \Omega_S)^2}\right]. \tag{4.50}$$

The influence of the detuning parameter on the motor torque is evident. For the torque property (4.2) the relation (4.50) transforms into

$$\varepsilon M_0\left(1 - \frac{\Omega_S}{\Omega_0}\right) = -\frac{1}{2}\eta\mu\Omega_S^3 \left[\frac{\zeta_y}{\zeta_y^2 + 4(\omega - \Omega_S)^2} + \frac{\zeta_x}{\zeta_x^2 + 4(\omega - \Omega_S)^2}\right]. \tag{4.51}$$

If the damping properties in both direction are equal, i.e., for $\zeta_x = \zeta_y = \zeta_1$, the resonance occurs in both directions and the amplitude in both directions are the same

$$a_{1S} = a_{2S} = \frac{\mu\Omega_S}{\sqrt{\zeta_1^2 + 4(\varepsilon\sigma)^2}} = \frac{\mu\Omega_S}{\sqrt{\zeta_1^2 + 4(\omega - \Omega_S)^2}} = a_S. \tag{4.52}$$

For this assumption the angular velocity as the function of frequency Ω_0 is obtained as

$$\varepsilon M_0\left(1 - \frac{\Omega_S}{\Omega_0}\right) = -\frac{\eta\mu\omega^3\zeta_1}{{\zeta_1}^2 + 4(\omega - \Omega_S)^2}. \tag{4.53}$$

In Fig. 4.2 according to (4.53) the evolution of the angular velocity is plotted. The parameter values are: $M = 0.064$, $m = 0.0021$, $J = 10^{-7}$, $d = 0.005$ and

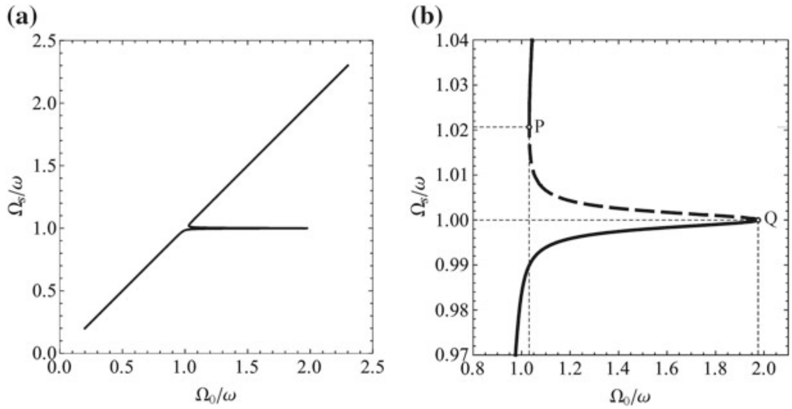

Fig. 4.2 Angular velocity as the function of the parameter Ω_0: **a** resonant case, **b** extrem angular velocities

$M_0 = 0.005$, the damping coefficient is $\zeta_1 = 0.006\omega$ where $\omega = 30\pi$. The motor is accelerated from rest to a fixed velocity by changing the parameter Ω_0 and the $(\Omega_0/\omega) - (\Omega_S/\omega)$ curve is shown. As it can be seen only one resonant case appears.

Let us rewrite the relation (4.53) into

$$\Omega_0 = \Omega_S \left[1 + \frac{\eta\mu\omega^3\zeta_1}{{\zeta_1}^2 + 4(\omega - \Omega_S)^2} \frac{1}{\varepsilon M_0} \right]^{-1}. \tag{4.54}$$

Equating the first time derivative $\frac{d\Omega_0}{d\Omega_S}$ with 0, the condition for existence of extreme values for Ω_0 is obtained

$$0 = 16(\omega - \Omega_S)^4 + 4\zeta_1 \left[2\zeta_1 - \frac{\eta\mu\omega^3}{\varepsilon M_0} \right] (\omega - \Omega_S)^2 - \frac{8\eta\mu\omega^4\zeta_1}{\varepsilon M_0}(\omega - \Omega_S)$$
$$+ \zeta_1^4 + \eta\mu\omega^3 \frac{1}{\varepsilon M_0} \zeta_1^3. \tag{4.55}$$

Solving the algebraic equation (4.55) for Ω_S two real values are obtained for which the extreme angular velocities exist (see Fig. 4.2b). For the assumed parameter values the extreme values are:

$$\left(\frac{\Omega_{0\,\min}}{\omega}, \frac{\Omega_S}{\omega} \right)_P = (1.03117,\ 1.0206400)\,,$$
$$\left(\frac{\Omega_{0\,\max}}{\omega}, \frac{\Omega_S}{\omega} \right)_Q = (1.97570,\ 1.0000046)\,. \tag{4.56}$$

From the Fig. 4.2b it is obvious that the number of solutions is one, two or three. Between P and Q three solutions exist. To examine the stability properties of the solutions the procedure suggested in previous section is applied. It is obvious that the stability of the solution depends on the torque characteristics M_0 and Ω_0, characteristics of the system μ, η and ς. In Fig. 4.2 it is shown that two solutions between P and Q are stable and one is unstable. The stable solutions are shown with the full-line, while the unstable solution is given with dotted-line.

Based on (4.52) and (4.53) the steady state amplitude is rewritten as

$$a_S^2 = \frac{\varepsilon M_0}{\eta\omega^3\zeta_1} \left(1 - \frac{\Omega_S}{\Omega_0} \right) \mu\Omega_S^2. \tag{4.57}$$

Using (4.53) and (4.57) the amplitude as the function of parameter Ω_0 is plotted in Fig. 4.3a, while in Fig. 4.3b the amplitude-frequency diagram (4.57) and the curves which depend on the motor torque for various values of parameter Ω_0 are shown. In Fig. 4.3b) the curves are obtained for three various values of Ω_0: 1. $\Omega_{0\,\min}/\omega = 1.03117$, 2. $\Omega_0/\omega = 1.5$,3. $\Omega_{0\,\max}/\omega = 1.97570$. motor properties for three values of angular velocity are presented.

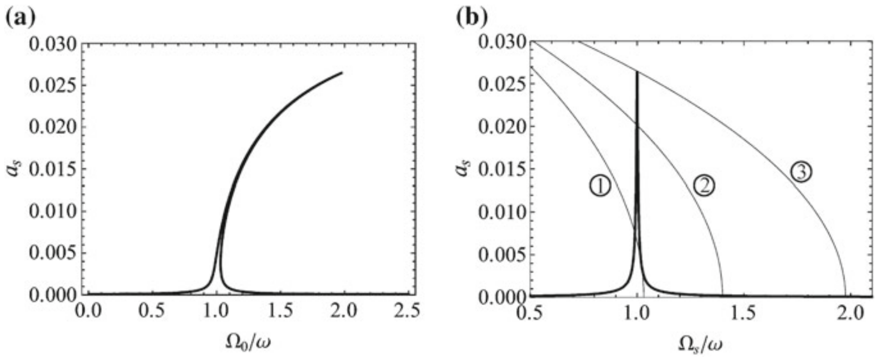

Fig. 4.3 **a** Amplitude as the function of parameter Ω_0; **b** Amplitude-frequency diagram and curves dependent on motor torque for: 1. $\Omega_{0\,\min}/\omega = 1.03117$, 2. $\Omega_0/\omega = 1.5$,3. $\Omega_{0\,\max}/\omega = 1.97570$

The curves 1 and 3 are boundary ones which satisfy the extreme conditions (4.56). The intersection of these curves and of the amplitude-frequency diagram gives the steady-state solutions. For boundary conditions (1) and (3) two steady-state solutions exist, while inside this interval there are three solutions (see intersection of (2) and the amplitude-frequency curve). In the regions outside these boundary ones, only one steady state solution exists.

4.3.2 Resonance Frequency in One Direction Is Half of the Resonance frequency in Other Direction

If the resonance frequency ω_x is defined by a half of the resonance frequency in the y direction, i.e., for $\omega_x = \omega$ it is $\omega_y = 2\omega_x = 2\omega$, the two resonance frequencies are separated and two resonance features occur (Fig. 4.4).

For that case we obtain the steady state amplitudes (4.35) and (4.36) as

$$a_{1S} = \frac{\mu\Omega_S^2}{\sqrt{(\zeta_1\Omega_S)^2 + (\omega^2 - \Omega_S^2)^2}},$$
$$a_{2S} = \frac{\mu\Omega_S^2}{\sqrt{(\zeta_y\Omega_S)^2 + (4\omega^2 - \Omega_S^2)^2}}, \tag{4.58}$$

while the steady state phases (4.37) are

$$\tan\psi_{1S} = \frac{\zeta_x\Omega_S}{\Omega_S^2 - \omega^2}, \quad \tan\psi_{2S} = \frac{4\omega^2 - \Omega_S^2}{\zeta_y\Omega_S}. \tag{4.59}$$

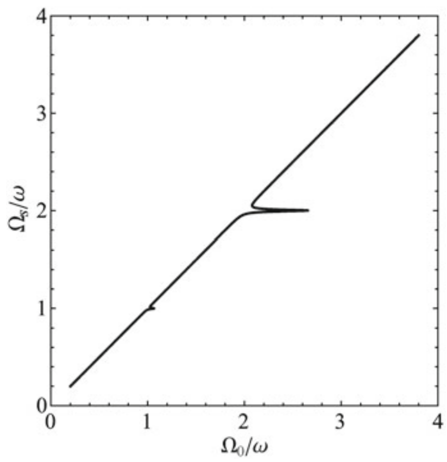

Fig. 4.4 Frequency as a function of the motor torque parameter Ω_0

The corresponding angular velocity - frequency relation (4.38) is

$$0 = \varepsilon \mathcal{M}(\Omega_S) + \frac{2\eta\zeta_y \mu \Omega_S^3 \omega^2}{(\zeta_y \Omega_S)^2 + (4\omega^2 - \Omega_S^2)^2} + \frac{1}{2}\frac{\eta\zeta_x \mu \Omega_S^3 \omega^2}{(\zeta_x \Omega_S)^2 + (\omega^2 - \Omega_S^2)^2}. \tag{4.60}$$

i.e.,

$$0 = \varepsilon M_0 \left(1 - \frac{\Omega_S}{\Omega_0}\right) + \frac{2\eta\zeta_y \mu \Omega_S^3 \omega^2}{(\zeta_y \Omega_S)^2 + (4\omega^2 - \Omega_S^2)^2}. \tag{4.61}$$

For numerical calculation the following numerical values are applied: $M = 0.064$, $m = 0.0021$, $J = 10^{-7}$, $d = 0.005$ and $M_0 = 0.005$, damping coefficients $\zeta_x = 0.012\omega$ and $\zeta_y = 0.024\omega$ where $\omega = 30\pi$.

Due to Fig. 4.4 it is evident that two resonances appear. In Fig. 4.5a, b the first and the second resonances with extreme values are plotted.

The extreme values for the first resonance are

$$\left(\frac{\Omega_{0\,\min}}{\omega}, \frac{\Omega_S}{\omega}\right)_P = (1.02397,\ 1.01507),$$
$$\left(\frac{\Omega_{0\,\max}}{\omega}, \frac{\Omega_S}{\omega}\right)_Q = (1.06596,\ 1.00029), \tag{4.62}$$

and for the second

$$\left(\frac{\Omega_{0\,\min}}{\omega}, \frac{\Omega_S}{\omega}\right)_R = (2.07929,\ 2.05142),$$

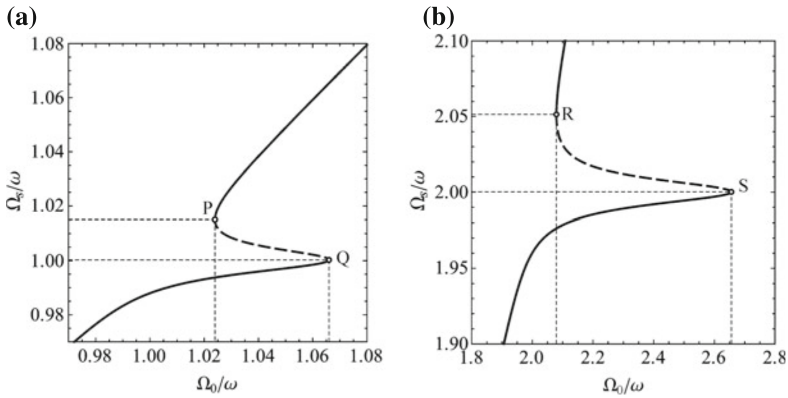

Fig. 4.5 **a** First resonance, **b** Second resonance

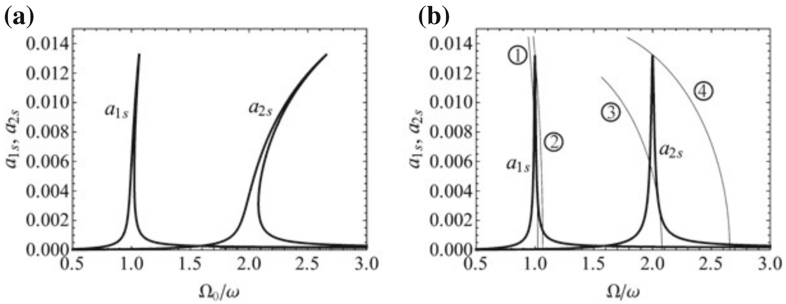

Fig. 4.6 **a** Amplitudes as functions of the parameter Ω_0; **b** Amplitude-frequency curves with motor characteristics for the extremal values of Ω_0: *1* $(\Omega_{0\,\min}/\omega)_P = 1.02397$, *2* $(\Omega_{0\,\max}/\omega)_Q = 1.06596$, *3* $(\Omega_{0\,\min}/\omega)_R = 2.07929$, *4* $(\Omega_{0\,\max}/\omega)_S = 2.65594$

$$\left(\frac{\Omega_{0\,\max}}{\omega}, \frac{\Omega_S}{\omega}\right)_S = (2.65594,\ 2.00014)\,. \tag{4.63}$$

The second resonant is more significant. In Fig. 4.6a the amplitudes of vibration as functions of the parameter Ω_0 of the motor torque and in Fig. 4.6b the amplitude-frequency curve with motor characteristics for the extreme values of Ω_0 are plotted.

For the both resonant regimes the Sommerfeld effect occurs. The amplitude solution between P and Q (Fig. 4.7a) and S and R (Fig. 4.7b) is unstable and the jump phenomena occurs. From Fig. 4.7 it is evident that there are three steady state solutions in the interval of curves (1) and (2) for the first resonance and in the interval (3) and (4) for the second resonance. The solutions between P and Q and also R and S are unstable.

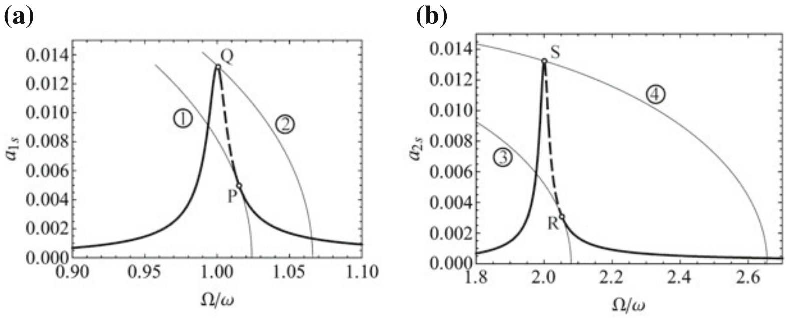

Fig. 4.7 Amplitude-frequency diagram with motor characteristics for extrem angular velocities for: **a** first resonance, **b** second resonance

4.4 Numerical Simulation

Let us rewrite the equations of motion (4.14)–(4.16) into six first order differential equations

$$\dot{x} = x_1, \quad \dot{y} = y_1, \quad \dot{\varphi} = \Omega, \tag{4.64}$$

$$\dot{x}_1 = -\frac{\varepsilon\mu\mathcal{M}(\varphi_1)\sin\varphi}{\mu\eta - 1} + \frac{\mu\eta\sin(2\varphi)(\mu\varphi_1^2\sin\varphi - y\omega_y^2 - \zeta_y y_1)}{2(\mu\eta - 1)} + \frac{(\mu\eta\cos^2\varphi - 1)(\mu\Omega^2\cos\varphi - x\omega_x^2 - \zeta_x x_1)}{\mu\eta - 1}, \tag{4.65}$$

$$\dot{y}_1 = \frac{\varepsilon\mathcal{M}(\Omega)\mu\cos\varphi}{\mu\eta - 1} + \frac{\mu\eta\sin(2\varphi)(\mu\Omega^2\cos\varphi - x\omega_x^2 - \zeta_x x_1)}{2(\mu\eta - 1)} + \frac{(\mu\eta\sin^2\varphi - 1)(\mu\Omega^2\sin\varphi - y\omega_y^2 - \zeta_y y_1)}{\mu\eta - 1}, \tag{4.66}$$

$$\dot{\Omega} = -\frac{\varepsilon\mathcal{M}(\Omega)}{\mu\eta - 1} + \frac{\eta\cos\varphi(\mu\Omega^2\sin\varphi - y\omega_y^2 - \zeta_y y_1)}{\mu\eta - 1} - \frac{\eta\sin\varphi(\mu\Omega^2\cos\varphi - x\omega_x^2 - \zeta_x x_1)}{\mu\eta - 1}. \tag{4.67}$$

with motor torque function (4.27). Applying the fourth order Runge–Kutta procedure the equations are solved numerically.

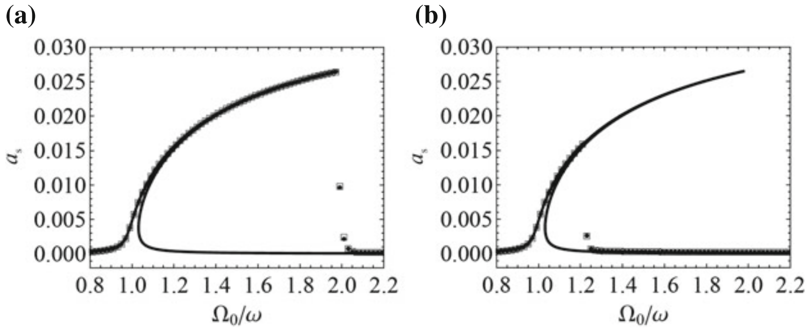

Fig. 4.8 Amplitude as the function of motor torque parameter Ω_0 for: **a** speeding up, **b** slowing down (analytical solution - *full line*, numerical solution in x direction - *squares*, numerical solution in y direction - *circles*)

Numerical solution is obtained for $\omega_x = \omega_y = \omega = 30\pi$ and parameter values $M = 0.064$, $m = 0.0021$, $J = 10^{-7}$, $d = 0.005$, $M_0 = 0.005$, $\zeta = 0.006\omega$ and plotted in Fig. 4.8.

The procedure to obtain the result shown in Fig. 4.8a is performed by slowly increasing the parameter Ω_0 of the DC motor and in Fig. 4.8b by slowly decreasing the parameter Ω_0 applied to the motor. The numerically obtained results are given with squares in x direction and with circles in y direction, while the analytical solution with the full line. The numerical solution is compared with analytical one and shows good agreement.

Let us consider the case when $\omega_x = \omega$ and $\omega_y = 2\omega$ and the parameters of the system are $M = 0.064$, $m = 0.0021$, $J = 10^{-7}$, $d = 0.005$, $M_0 = 0.005$, $\zeta_x = 0.012\omega$ and $\zeta_y = 0.024\omega$ where $\omega = 30\pi$. Using the suggested procedure the solution for slow increase of the parameter Ω_0 is plotted in Fig. 4.9. Two resonances occur: one in

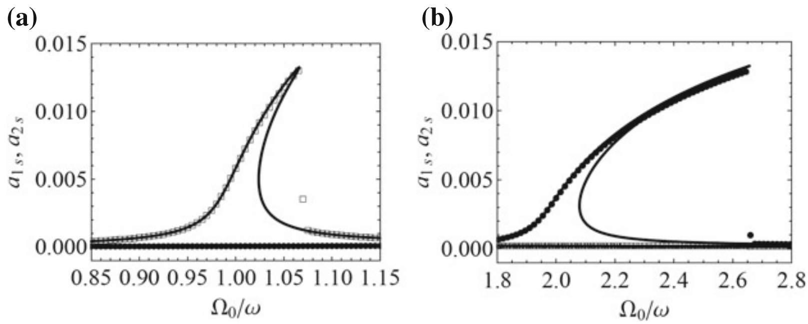

Fig. 4.9 Amplitude as the function of motor torque parameter Ω_0 for slow increase of Ω_0: **a** first resonance, **b** second resonance. Analytical solution - *full line*, numerical solution in x direction - *squares*, numerical solution in y direction - *circles*

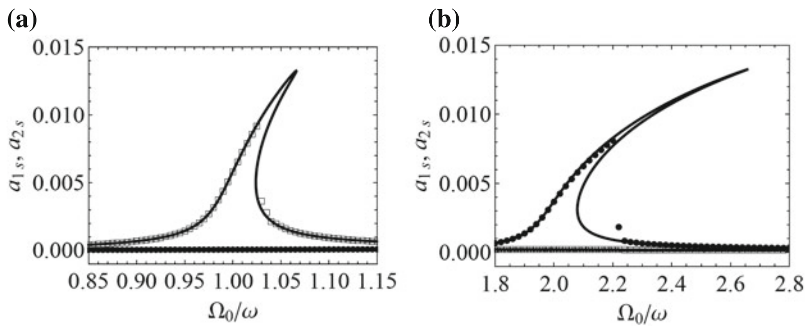

Fig. 4.10 Amplitude as the function of motor torque parameter Ω_0 for slow decrease of Ω_0: **a** first resonance, **b** second resonance. Analytical solution - *full line*, numerical solution in x direction - *squares*, numerical solution in y direction - *circles*

x direction (Fig. 4.9a) and the other in y direction (Fig. 4.9b). The same procedure is applied for obtaining solutions during decreasing of the parameter Ω_0. We obtain the first resonance in x direction (Fig. 4.10a) and the second in y direction (Fig. 4.10b). In the diagrams the jumps are observed when the system escapes resonance. Numerical results given with squares in x direction and circles in y direction are compared with analytical solution shown with full line. The numerical results are in good agreement with the analytical results.

4.5 Conclusions

In this chapter we considered a discrete parameter spring-mass-damper which moves in two orthogonal directions which is attached to a non-ideal rotating machine. Based on the analysis the following is concluded:

1. The system behaves different according to the values of resonance frequencies in the two orthogonal directions. Depending on the values of these frequencies the resonance can occur in both directions, only in one direction or can not occur. In the paper the limits between these cases are determined.
2. For the case when the frequencies of system in x and y direction are equal only one resonance regime appears. If the damping in both directions are also equal, the amplitudes of vibration in x and y direction are the same.
3. If the frequency of the system in y direction is two times higher than in x direction two resonances occur: one, in x direction and other, in y direction.
4. In the mechanical two degrees-of-freedom system with non-ideal excitation the Sommerfeld effect occurs. There is the jump in amplitude when the resonance regime is escaped. For the case when the frequency of the system in x and y directions are equal the Sommerfeld effect appears once time. If the frequency in y direction is two times higher than that in x direction the jump phenomena

occurs for two times for two different values of torque parameter. The procedure for analytical calculation of angular velocities and the corresponding amplitudes for which the jump occurs is suggested and the values are calculated for certain numerical data. The analytical procedure predicts the appearance of Sommerfeld effect for other relations between frequencies in two orthogonal directions, too.
5. Analytically obtained solutions are in good agreement with numerically obtained ones.
6. The analytical solutions are compared with experimental results given in Goncalves et al. (2016) and show good agreement.
7. The results published in this paper would be of special interest for engineers and technicians in prediction of resonances and their elimination.

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