Logical Analogies Between Intuitionistic Fuzzy Sets and Rough Sets

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1 An Introduction to Intuitionistic Fuzzy Sets

Intuitionistic sets were introduced by Atanasov [1] as an extension of fuzzy sets. However, it was observed already by Atanasov [2] that their logical properties differ in some aspects from those of fuzzy sets. In this note we prove that in the framework of intuitionistic fuzzy sets can be defined connective systems which satisfy logical rules analogous to the rules of the constructive logic with strong negation. Since the rough sets are models of the mentioned logic, we will compare their algebraic properties with those of intuitionistic sets.

The *intuitionistic fuzzy sets* are introduced as pairs $A = (\mu_T, \mu_F)$ of membership functions $\mu_T, \mu_F: U \longrightarrow [0, 1]$ defined on a fixed nonempty universe U such that $\mu_T(x) + \mu_F(x) \leq 1$, for all $x \in U$. Therefore, to any element $x \in U$ corresponds a logical value $(\mu_T(x), \mu_F(x))$, where $\mu_T(x)$ expresses the membership value of the element x in the set A and $\mu_F(x)$ the degree of the non-membership of x with respect to A. Of course, in this interpretation the value $1 - \mu_T(x) - \mu_F(x)$ denotes a measure of non-determinacy or of the incompleteness of our information. Thus the possible *logical values* form a lattice

$$\mathcal{L} = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \leq 1\}$$

where the partial order is defined as follows:

$$(x_1, x_2) \leq (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2. \tag{P}$$

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Remark 1 It is easy to see that the lattice operations in $\mathcal{L}$ have the form

$$(x_1, x_2) \vee (y_1, y_2) = (\max(x_1, y_1), \min(x_2, y_2)),$$
$$(x_1, x_2) \wedge (y_1, y_2) = (\min(x_1, y_1), \max(x_2, y_2)).$$

Clearly, the least element of $\mathcal{L}$ is $0_{\mathcal{L}} = (0, 1)$ and its greatest element is $1_{\mathcal{L}} = (1, 0)$. In Cornelis et al. [4], it is proved that $\mathcal{L}$ is a complete sublattice of $[0, 1] \times [0, 1]^d$, and this yields that $\mathcal{L}$ is completely distributive.

Example 1 A special area of knowledge management focuses on subjective ontology (epistemology), see Goertz and Mahoney [6]. A key property of subjective ontology is that the ontology is created with a learning process where some information may be incomplete or partial. Thus, the principle of excluded middle is not met in this domain. The subjective ontology is a conceptual model generated by cognitive processes. During the learning process, the aggregated input from the environment is used to build up the conceptual model.

Having an ontology fragment as shown in Fig. 1, every edge correspond to a statement, elementary proposition. The notation is based on the RDF model. A complex arc denotes the specialization (is a) relationship, while the simple arcs refer to the properties of the concepts. As also the properties are concepts, the specialization can be defined among the properties too.

It is assumed that at person level a proposition may be true, false or unknown. This approach is very common in data and information systems. For example, the SQL standard in Grant [7] uses a NULL value to denote the unknown value at a field level. For example, the truth value of a statement that a person eats apple can be either T, F or U.

```
P(eat,Mary, apple)   = T
P(eat,Tom, apple)    = F
P(eat,Peter, apple)  = U
```
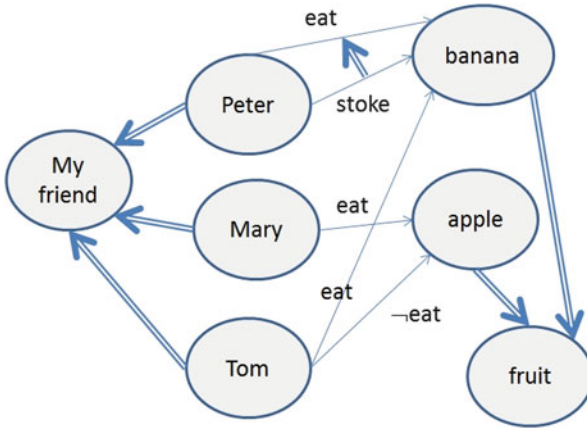

Fig. 1 Sample ontology fragment

If the engine aggregates the truth values of the different agents to construct the population level truth value, we get a membership like value.

```
m(eat,my_friend,fruit,T) = .500
m(eat,my_friend,fruit,F) = .167
m(eat,my_friend,fruit,U) = .333
```

On this way, we get use intuitionistic fuzzy representation of the relationship element *s* in the fuzzy ontology. This representation form allow a more efficient decision making in the ontology framework.

Application Areas A relevant application area is the *fuzzy clustering of a product palette*. The goal of the clustering is to determine the groups of similar objects. The generated groups can be used to build up the ontology of the products (see e.g. Liu [12]). This ontology enables an easier management of product categories and it helps us to discover hidden dependencies between the products. The similarity evaluation of the products can be based here on a survey or on a sentiment analysis of natural texts. We can assume that in the survey, the customers can leave some questions blank if they have no information on a given aspect of the products. In both cases, the missing data may occur frequently, this fact justifies the application of an intuitionistic fuzzy model. The intuitionistic model provides additional information to distinguish the missing value from the other not-missing values. Hence the clustering based on this model can provide a more sophisticated partitioning than the base evaluation methods.

Now let $\mathcal{F}(U)$ stand for the set of all membership functions $\mu \colon U \rightarrow [0, 1]$. Defining the operations $\vee, \wedge$ for any $f, g \in \mathcal{F}(U)$ as usually (see [13]):

$$(f \vee g)(x) := \max\{f(x), g(x)\} \text{ and } (f \wedge g)(x) := \min\{f(x), g(x)\}, \text{ for all } x \in U,$$

we obtain a completely distributive lattice $(\mathcal{F}(U), \vee, \wedge)$. The least element of $\mathcal{F}(U)$ is the constant 0 map on U, denoted by $\mathbf{0}$, and its greatest element is $\mathbf{1}$, the constant 1 map. Next, consider the set of intuitionistic fuzzy sets

$$\mathcal{I}(U) = \{(\mu_T, \mu_F) \in \mathcal{F}(U)^2 \mid \mu_T(x) + \mu_F(x) \leq 1, \textit{for all } x \in U\},$$

ordered as follows:

$$(\mu_T, \mu_F) \leq (\nu_T, \nu_F) \Leftrightarrow \mu_T(x) \leq \nu_T(x) \text{ and } \mu_F(x) \geq \nu_F(x). \text{ for all } x \in U.$$

$(\mathcal{I}(U), \leq)$ is a complete distributive lattice with least element $(\mathbf{0}, \mathbf{1})$ and greatest element $(\mathbf{1}, \mathbf{0})$ (see e.g. Novák et al. [13]). We define a unary operation $\sim$ on $\mathcal{L}$ and $\mathcal{I}(U)$ by setting

$\sim (x_1, x_2) = (x_2, x_1)$, for all $(x_1, x_2) \in \mathcal{L}$, and
$\sim (\mu_T, \mu_F) = (\mu_F, \mu_S)$, for all $(\mu_T, \mu_F) \in \mathcal{I}(U)$.

In Cornelis et al. [4] $\sim$ is called the *standard negator* on $\mathcal{L}$. Clearly, we have

$$\sim (\sim (x_1, x_2)) = (x_1, x_2), \text{ and}$$

$$(x_1, x_2) \leq (y_1, y_2) \Leftrightarrow \sim (x_1, x_2) \geq \sim (y_1, y_2).$$

for any $(x_1, x_2), (y_1, y_2) \in \mathcal{L}$. It is obvious that any $(\mu_T, \mu_F), (\nu_T, \nu_F) \in \mathcal{I}(U)$ satisfy the same rules. These properties mean that $(\mathcal{L}, \vee, \wedge, \sim, (\mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{0}))$ and $(\mathcal{I}(U), \vee, \wedge, \sim, (\mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{0}))$ are De Morgan algebras.

A *De Morgan algebra* $\mathbb{A} = (A, \vee, \wedge, \sim, 0, 1)$ is an algebra such that $(A, \vee, \wedge)$ is a bounded distributive lattice with a least element 0 and a greatest element 1, and $\sim$ is a unary operation that satisfies, for all $x, y \in A$,

$$\sim (\sim x) = x \text{ and } x \leq y \Leftrightarrow \sim x \geq \sim y.$$

This definition yields that $\sim$ is an isomorphism between the lattice A and its dual A^d. Therefore, $\sim$ satisfies the so-called De Morgan equations:

$$\sim (x \vee y) = \sim x \wedge \sim y, \sim (x \wedge y) = \sim x \vee \sim y. \tag{M}$$

Intuitionistic sets can be viewed as a common generalization of the fuzzy sets and of the rough sets. Indeed, fuzzy sets on U can be interpreted as *exact intuitionistic sets* (μ_T, μ_F) with $\mu_F = 1 - \mu_T$, i.e. they correspond to the family

$$Ex(U) = \{(\mu, 1 - \mu) \mid \mu \in \mathcal{F}(U)\}.$$

Conversely, any intuitionistic fuzzy set (μ_T, μ_F) can be viewed as a pair of "exact" fuzzy sets $(\mu_T, 1 - \mu_T)$ and $(1 - \mu_F, \mu_F)$, and obviously

$$(\mu_T, 1 - \mu_T) \leq (\mu_T, \mu_F) \leq (1 - \mu_F, \mu_F).$$

Thus the fuzzy set $(\mu_T, 1 - \mu_T)$ can be considered as a "lower approximation", and $(1 - \mu_F, \mu_F)$ as an "upper approximation" of the intuitionistic set (μ_T, μ_F). Hence, we can define an upper approximation operator $\overline{A}$ and a lower approximation operator $\underline{A}$ on the set $\mathcal{I}(U)$ as follows (see Atanasov [2]):

$\overline{A} : \mathcal{I}(U) \longrightarrow Ex(U), \overline{A}(\mu_T, \mu_F) := (1 - \mu_F, \mu_F),$
$\underline{A} : \mathcal{I}(U) \longrightarrow Ex(U), \underline{A}(\mu_T, \mu_F) := (\mu_T, 1 - \mu_T).$

2   T-norms and T-conorms Defined on the Lattice $\mathcal{L}$

In the literature, the lattice $\mathcal{L} = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \leq 1\}$ of logical values is usually equipped with some additional logical connectives different from $\vee$ and $\wedge$. For this we need to recall the notion of a t-norm and t-conorm defined on a lattice.

A *t-norm* on a bounded lattice L (with least element 0_L and greatest element 1_L) is an order-preserving, commutative, associative, binary operation $T:L^2 \to L$ satisfying $T(1_L, z) = z$, for all $z \in L$. Dually, a *t-conorm* $\mathcal{S}(x, y)$ on L is an order-preserving, commutative, associative binary operation $\mathcal{S}:L^2 \to L$, having 0_L as a neutral element. Following the terminology in Cornelis et al. [4], we say that a t-norm $\mathcal{T}$ (respectively a t-conorm $\mathcal{S}$) on $\mathcal{L}$ *is t-representable* if there exists a t-norm T and a t-conorm S on the lattice $([0, 1], \leq)$ such that, for any $x = (x_1, x_2) \in \mathcal{L}$ and any $y = (y_1, y_2) \in \mathcal{L}$ we have

$$\mathcal{T}(x, y) = (T(x_1, y_1), S(x_2, y_2)) \tag{1}$$

$$\mathcal{S}(x, y) = (S(x_1, y_1), T(x_2, y_2)). \tag{1'}$$

In [4, Theorem 2] is also proved that for any t-norm T and any t-conorm S on $[0, 1]$ satisfying

$$T(a, b) \leq 1 - S(1 - a, 1 - b), \text{ for all } a, b \in [0, 1], \tag{2}$$

the mappings $\mathcal{T}$ and $\mathcal{S}$ defined by the formulas (1) and (1') are a t-norm and a t-conorm on L, respectively. If the relation in (2) is satisfied with equality, i.e.

$$T(a, b) = 1 - S(1 - a, 1 - b), \text{ for all } a, b \in [0, 1], \tag{2'}$$

then the operators T and S are called *dual* (in the sense of L. Zadeh). Obviously, if T and S satisfies (2'), then S and T also satisfies (2').

Example 2 Let us consider the Łukasiewitz t-norm T_L with $T_L(x_1, x_2) = x_1 \circledast x_2 :=$ $\max(x_1 + x_2 - 1, 0)$, for all $x_1, x_2 \in [0, 1]$ and the Łukasiewitz t-conorm S_L with $S_L(x_1, x_2) = x_1 \oplus x_2 := \min(x_1 + x_2, 1)$, for all $x_1, x_2 \in [0, 1]$. It is easy to check that they satisfy relation (3'). Hence they are dual operators, and for any $x = (x_1, x_2) \in \mathcal{L}$ and any $y = (y_1, y_2) \in \mathcal{L}$ the operation

$(x_1, x_2) \circledast (y_1, y_2) = (x_1 \circledast y_1, x_2 \oplus y_2)$ and
$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \oplus y_1, x_2 \circledast y_2)$

will be a (representable) t-norm and t-conorm on $\mathcal{L}$, respectively. Clearly,

$$(x_1, x_2) \circledast (y_1, y_2) = (\max(x_1 + y_1 - 1, 0), \min(x_2 + y_2, 1)) \tag{3}$$

$$(x_1, x_2) \oplus (y_1, y_2) = (\min(x_1 + y_1, 1), \max(x_2 + y_2 - 1, 0)). \tag{3'}$$

3   Rough Sets and Their Representations

Rough sets were introduced by Pawlak [14] in order to provide a formal approach to deal with incomplete data. In rough set theory, any set of entities is characterized by a lower approximation and an upper approximation.

Approximations are then defined in terms of an indiscernibility space, that is, a relational structure (U, R) such that R is a binary relation on U. In original definition of Pawlak, R is an equivalence relation, but since inception, several generalizations of his construction had been proposed. Here we will consider $R \subseteq U \times U$ to be a quasiorder, i.e. a reflexive, transitive relation, since in this case the induced rough sets still form a completely distributive lattice (see Järvinen et al. [10]), similar to the case of an equivalence.

Let $R \subseteq U \times U$ be a quasiorder defined on the universe U. For any element $x \in U$ the set $R(x) = \{u \in U \mid (x, u) \in R\}$ is called the *relational neighbourhood* of x. Now, for any set $X \subseteq U$ its *lower approximation* and its *upper approximation* are defined as follows:

$$X_R := \{u \in U \mid R(u) \subseteq X\}, X^R := \{u \in U \mid R(u) \cap X \neq \emptyset\}.$$

The *rough set of X* is the pair (X_R, X^R), and the set of all rough sets is $RS = \{(X_R, X^R) \mid X \subseteq U\}$. In this approach, X_R can be viewed as the set of elements which *certainly* belong to X, and X^R is interpreted as the set of objects that *possibly* are in X, when elements are observed through the knowledge expressed by (U, R). The set RS can be ordered by the coordinatewise order:

$$(X_R, X^R) \leq (Y_R, Y^R) \Leftrightarrow X_R \subseteq Y_R \text{ and } X^R \subseteq Y^R.$$

It was proved in Järvinen et al. [10] that the ordered set $\mathbb{RS} = (RS, \leq)$ is a completely distributive De Morgan algebra, and latter it was shown that a particular Kleene algebra can be defined on it (see Järvinen and Radeleczki [8]). A *Kleene algebra* is a De Morgan algebra $\mathcal{K} = (K, \vee, \wedge, \sim, 0, 1)$ satisfying the (Kleene's) axiom:

$$x \wedge \sim x \leq y \vee \sim y, \text{ for all } x, y \in K. \tag{K}$$

A *Heyting algebra L* is a bounded lattice such that for all $a, b \in L$, there is a greatest element $x \in L$ satisfying $a \wedge x \leq b$. This element x is called the *relative pseudocomplement of a with respect to b*, and is denoted by $a \Rightarrow b$. It is known that any completely distributive complete lattice is a Heyting algebra $(L, \vee, \wedge, \Rightarrow, 0, 1)$ where the relative pseudocomplement is defined as follows:

$$x \Rightarrow y = \bigvee \{z \in L \mid x \wedge z \leq y\}.$$

A *Nelson algebra* is a Kleene algebra $\mathcal{N} = (N, \vee, \wedge, \sim, \mapsto, 0, 1)$ such that for any pair $a, b \in N$ the relative pseudocomplement $a \mapsto b$ of the element a with respect to $\sim a \vee b$ there exists (see Cignoli [3]), and for any $c \in N$ the equation:

$$(a \wedge b) \mapsto c = a \mapsto (b \mapsto c). \tag{N}$$

holds. In each Nelson algebra, the *weak negation* $\neg$ can be defined as

$$\neg a := a \to 0, \text{ for all } a \in N. \tag{4}$$

Since $\mathbb{RS}$ is a completely distributive lattice, a Heyting algebra $(RS, \vee, \wedge, \Rightarrow)$ can be defined on it. Moreover, in Järvinen and Radeleczki [8] it was proved that $\mathbb{RS}$ is a Nelson algebra.

Let $\mathcal{P}(U)_R = \{X_R, \mid X \subseteq U\}$ and $\mathcal{P}(U)^R = \{X^R \mid X \subseteq U\}$. In [10] is proved that $(\mathcal{P}(U)_R, \subseteq)$ and $(\mathcal{P}(U)^R, \subseteq)$ are Heyting algebras. Clearly, (X_R, X^R) is an element of $\mathcal{P}(U)_R \times \mathcal{P}(U)^R$. Let $U - A$ stand for the complement of a set $A \subseteq U$. It is known that $U - X^R$ is just the set $(U - X)_R \in \mathcal{P}(U)_R$. Since each rough set is uniquely determined by the approximation pair (X_R, X^R), one can represent the rough set of X as a pair $(X_R, U - X^R) = (X_R, (U - X)_R) \in \mathcal{P}(U)_R \times \mathcal{P}(U)_R = (\mathcal{P}(U)_R)^2$, too. Therefore, we construct the set

$$DRS = \{(X_R, U - X^R) \mid X \subseteq U\} \subseteq (\mathcal{P}(U)_R)^2, \tag{5}$$

which is ordered as follows:

$$(X_R, U - X^R) \leq (Y_R, U - Y^R) \Leftrightarrow X_R \subseteq Y_R \text{ and } U - X^R \supseteq U - Y^R.$$

Thus $(X_R, U - X^R) \leq (Y_R, U - Y^R) \Leftrightarrow \big(X_R \subseteq Y_R \text{ and } X^R \subseteq Y^R\big) \Leftrightarrow (X_R, X^R) \leq (Y_R, Y^R)$, and hence DRS and RS are order-isomorphic; therefore, they are isomorphic as Heyting algebras and as Nelson algebras, as well. Since X_R and $U - X^R$ are disjoint sets, the above representation is called *the disjoint representation* of the rough sets of R. The algebraic operations on DRS are defined for any $(A, B), (C, D) \in DRS$ as follows:

$(A, B) \vee (C, D) = (A \cup C, B \cap D),$
$(A, B) \wedge (C, D) = (A \cap C, B \cup D),$
$\sim (A, B) = (B, A)$

Because RS determines a Nelson algebra, for any pair $(X_R, X^R), (Y_R, Y^R) \in RS$ the Nelson implication $(X_R, X^R) \mapsto (Y_R, Y^R)$ is also defined. By isomorphism to this operation $\mapsto$ corresponds an operation $\rightarrowtail$ on DRS, expressed as

$$(A, B) \rightarrowtail (C, D) := (A \Rightarrow C, \ A \cap D),$$

where $\Rightarrow$ is the Heyting operation of the algebra $(\mathcal{P}(U)_R, \vee, \wedge, \Rightarrow)$.

Now, we are closing this section by proving that the disjoint representations of rough sets of R can be interpreted as intuitionistic fuzzy sets. We will do this by identifying the disjoint representation $(X_R, U - X^R)$ of a rough set $(X_R, X^R) \in RS$ with the pair $\left(\chi^{(X_R)}, \chi^{(U-X^R)}\right)$ formed by the characteristic function of the set X_R, respectively $U - X^R$. Observe that $\left(\chi^{(X_R)}, \chi^{(U-X^R)}\right)$ is an intuitionistic set, since

$X_R, U - X^R \subseteq U$ and $X_R \cap (U - X^R) \subseteq X^R \cap (U - X^R) = \emptyset$ imply that for any element $x \in U$, only the following cases are possible:

(1) $x \in X_R$. Then $x \notin U - X^R$, hence $\chi^{(X_R)}(x) + \chi^{(U-X^R)}(x) = 1 + 0 = 1$,
(2) $x \in U - X^R$. Then $x \notin X_R$, hence $\chi^{(X_R)}(x) + \chi^{(U-X^R)}(x) = 0 + 1 = 1$
(3) $x \notin X_R$ and $x \notin U - X^R$. Then $\chi^{(X_R)}(x) + \chi^{(U-X^R)}(x) = 0 + 0 = 0$.

Thus for any $x \in U$ we get $\chi^{(X_R)}(x) + \chi^{(U-X^R)}(x) \leq 1$.

4 Generalizing Nelson Logic via Intuitionistic Fuzzy Sets

Our aim is to generalize the notion of rough sets by using intuitionistic sets, in such a way to obtain algebraic structures similar to Nelson algebras, suitable to model similar logical rules. The significance of the Nelson algebras lies in the fact that they provide models for Constructive logic with strong negation (cf. Järvinen and Radeleczki [9]) introduced by D. Nelson. This logic is often called as *Nelson logic*. It is an extension of the intuitionistic propositional logic by *strong negation* $\sim$, as shown by Rasiowa [15]. This logic is axiomatized by extending intuitionistic logic with the formulas [where $p \leftrightarrow q = (p \rightarrow q) \wedge (q \rightarrow p)$]:

(NL1) $\sim p \rightarrow (p \rightarrow q)$;
(NL2) $\sim (p \rightarrow q) \leftrightarrow p \wedge \sim q$;
(NL3) $\sim (p \wedge q) \leftrightarrow \sim p \vee \sim q$;
(NL4) $\sim (p \vee q) \leftrightarrow \sim p \wedge \sim q$;
(NL5) $\sim \sim p \leftrightarrow p$;
(NL6) $\sim \neg p \leftrightarrow p$.

Since the rough sets defined by equivalence or quasiorder relations form Heyting algebras $(RS, \vee, \wedge, \Rightarrow)$ which are very particular objects in fuzzy setting (see e.g. Novák et al. [13]), first we need to consider a notion which represents a generalization of Heyting algebras and it is used in the framework of fuzzy sets.

Definition 1 A *bounded residuated lattice* is an algebra $\mathcal{A} = (A, \vee, \wedge, *, \rightarrow, 0, 1)$ of type (2,2,2,2,0,0) such that

(a) $(A, \vee, \wedge)$ is a lattice with least element 0 and greatest element 1,
(b) $(A, *)$ is a commutative semigroup, such that $1 * x = x * 1 = x$, for all $x \in L$.
(c) $\mathcal{A}$ satisfies the *adjointness* property, for all $x, y, z \in A$, that is

$$x * y \leq z \text{ if and only if } x \leq y \rightarrow z.$$

We note the operation $i(x, y) = x \rightarrow y$ is order reversing in the first variable, and order preserving in its second variable, moreover, for all $x, y, z \in A$ the rules

$x * (y \vee z) = x * y \vee x * z$;
$(x * y) \rightarrow z = x \rightarrow (y \rightarrow z)$
$x \leq y \Leftrightarrow x \rightarrow y = 1$

are always satisfied. We say that $\mathcal{A}$ satisfies the *double negation law* if

$$(x \to 0) \to 0 = x, \tag{DN}$$

for each $x \in A$. Denote $\rceil x := x \to 0$. Then (DN) means $\rceil (\rceil x) = x$, and the map $x \mapsto \rceil x$ is a De-Morgan operation on the lattice A (see e.g. Galatos et al. [5]).

Observe, that in fact any Heyting algebra is a particular residuated lattice with $* = \wedge$. It is also well-known that any continuous t-norm T on a complete lattice L induces a residuated structure on L as follows:

$$x \to_T y := \sup \{z \in L \mid T(x, z) \le y\}, \text{ for all } x, y \in L\}.$$

The implication $x \to y$ is called the *residuated implication induced by T*.

Example 3

(a) Consider the Lukasiewitz t-norm $x \circledast y = \max\{0, x + y - 1\}$, $x, y \in [0, 1]$. Then $x \to_L y = \min\{1, 1 - x + y\}$. Clearly, $\rceil x = x \to 0 = 1 - x$, for all $x \in [0, 1]$. Hence $\rceil (\rceil x) = x$, for all $x \in [0, 1]$. Thus $\mathcal{B} = ([0, 1], \max(,),$ $\min(,), \circledast, \to_L, 0, 1)$ is a bounded residuated lattice satisfying (DN). It is well-known that $\mathcal{B}$ satisfies the additional rules

$$x \circledast (x \to_L y) = \min(x, y) \text{ and } \max((x \to_L y), (y \to_L x)) = 1.$$

Residuated lattices satisfying these rules and (DN) are called *MV-lattices*.

(b) Consider the implication induced by $\mathcal{T}_L = \circledast$ defined on $\{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \le 1\}$ by (3). In Cornelis et al. [4] it is shown, that

$$(x_1, x_2) \Rightarrow_L (y_1, y_2) := (\min(1, 1 - x_1 + y_1, 1 - y_2 + x_2), \max(0, y_2 - x_2)), \tag{6}$$

for all $x, y \in [0, 1]$. It is easy to check that $(\mathcal{L}, \vee, \wedge, \circledast, \Rightarrow_L, (0, 1), (1, 0))$ does not satisfies the double negation rule, because $\rceil_L (x_1, x_2) = (x_1, x_2) \Rightarrow_L (0, 1) = (x_2, 1 - x_2)$, and hence it is not an MV-algebra.

Using the residuated implication $\to_L$ discussed in Example 3(a), we will define on the lattice $\mathcal{L}$ of logical values a *new binary operation* $\rightarrowtail_L$:

$$(x_1, x_2) \rightarrowtail_L (y_1, y_2) := (x_1 \to_L y_1, x_1 \circledast y_2), \text{ for all } (x_1, x_2), (y_1, y_2) \in \mathcal{L}. \tag{7}$$

Let L be a lattice with least element 0 and greatest element 1. An *implicator on L* is a mapping $i: L^2 \to L$ satisfying the conditions: $i(0, 0) = i(0, 1) = i(1, 1) = 1$ and $i(1, 0) = 0$. This notion is derived from fuzzy logic, where $L = [0, 1]$. An implicator i satisfies the *left neutrality principle*, if $i(1, x) = x$, for all $x \in L$, and i satisfies the *identity principle*, if $i(x, x) = x$, for all $x \in L$.

Proposition 1 *The lattice $\mathcal{L}$ is closed with respect to $\rightarrowtail_L$. The operation $\rightarrowtail_L$ is an implicator on $\mathcal{L}$ which satisfies the left neutrality and identity principle.*

Proof First, we note the following: Because $([0, 1], \max(,) \min(,), \circledast, \rightarrow_L, 0, 1)$ is a residuated lattice with $]x = 1 - x$, for any $(x_1, x_2) \in L$ we have

$$x_1 + x_2 \leq 1 \Leftrightarrow x_1 \leq 1 - x_2 =]x_2 \Leftrightarrow x_1 \circledast x_2 = 0.$$

Hence $(x_1, x_2) \in \mathcal{L}$ if and only if $x_1 \circledast x_2 = 0$.

Now take any $(x_1, x_2), (y_1, y_2) \in \mathcal{L}$. Then $x_1 \circledast x_2 = 0$ and $y_1 \circledast y_2 = 0$. Since by definition, $(x_1, x_2) \rightarrowtail_L (y_1, y_2) := (x_1 \rightarrow_L y_1, x_1 \circledast y_2)$, in order to prove $(x_1, x_2) \rightarrowtail_L (y_1, y_2) \in L$, it suffices to show that $(x_1 \circledast y_2) \circledast (x_1 \Rightarrow_L y_1) = 0$. Because $\circledast$ is commutative, associative and order preserving, we obtain:

$$(x_1 \circledast y_2) \circledast (x_1 \Rightarrow_L y_1) = y_2 \circledast x_1 \circledast (x_1 \Rightarrow_L y_1) \leq y_2 \circledast y_1 = y_1 \circledast y_2 = 0.$$

This proves that $(x_1, x_2) \rightarrowtail_L (y_1, y_2) \in \mathcal{L}$.

Next, recall that the least element and the greatest element in $\mathcal{L}$ is $(0, 1)$, respectively $(1, 0)$. Now is a routine to check that

$(0, 1) \rightarrowtail_L (0, 1) = (0, 1) \rightarrowtail_L (1, 0) = (1, 0) \rightarrowtail_L (1, 0) = (1, 0)$, and
$(1, 0) \rightarrowtail_L (0, 1) = (0, 1)$.

Thus $\rightarrowtail_L$ is an implicator on $\mathcal{L}$. We get also
$(1, 0) \rightarrowtail_L (x_1, x_2) = (1 \rightarrow_L x_1, 1 \circledast x_2) = (x_1, x_2)$, meaning that $\rightarrowtail_L$ satisfies the left neutrality. Similarly, $(x_1, x_2) \rightarrowtail_L (x_1, x_2) = (x_1 \rightarrow_L x_1, x_1 \circledast x_2) = (1, 0)$, hence the identity principle is also satisfied by $\rightarrowtail_L$. $\square$

Finally, we introduce the notion of a *quasi Kleene algebra* which is defined as a De Morgan algebra $(K, \vee, \wedge, \circledast, \oplus, \sim, 0, 1)$ extended with a strong conjunction $\circledast$ and a strong disjunction $\oplus$, and such that

$$\sim (x \circledast y) = (\sim x) \oplus (\sim y), \ \sim (x \oplus y) = (\sim x) \circledast (\sim y) \qquad (M^*)$$

$$x \otimes (\sim x) \leq y \oplus (\sim y), \text{ for all } x, y \in K. \qquad (K^*)$$

Of course, replacing $\otimes$ by $\wedge$ and $\oplus$ by $\vee$ we reobtain as a particular case the notion of a Kleene algebra.

Theorem 1 *The algebra $(\mathcal{L}, \vee, \wedge, \circledast, \oplus, \sim, (0, 1), (1, 0))$ is a quasi Kleene algebra, and the implicator $\rightarrowtail_L$ satisfies the identity:*

$$(a \circledast b) \rightarrowtail_L c = a \rightarrowtail_L (b \rightarrowtail_L c). \qquad (N^*)$$

Proof Since $(\mathcal{L}, \vee, \wedge, \sim, (0, 1),)(1, 0))$ is a De Morgan algebra, we have to check only (M^*) (K^*). Take any $x = (x_1, x_2) \in \mathcal{L}$ and $y = (y_1, y_2) \in \mathcal{L}$. By definition,
$\sim (x \circledast y) = \sim (x_1 \circledast y_1, x_2 \oplus y_2) = (x_2 \oplus y_2, x_1 \circledast y_1) = (x_2, x_1) \oplus (y_2, y_1) =$
$= (\sim x) \oplus (\sim y)$.

The equality $\sim (x \oplus y) = (\sim x) \circledast (\sim y)$ is proved dually. Thus (M^*) is satisfied.

We get also $(x_1, x_2) \circledast (\sim (x_1, x_2)) = (x_1, x_2) \circledast (x_2, x_1) = (x_1 * x_2, x_2 \oplus x_1) = (0, x_2 \oplus x_1)$, because $x_2 \leq 1 - x_1 \leq]x_1$ implies $x_1 * x_2 \leq x_1 *]x_1 = x_1 * (x_1 \rightarrow_L 0) = 0$ (by Definition 1(c)). Similarly, we obtain

$(y_1, y_2) \oplus (\sim (y_1, y_2)) = (y_1, y_2) \oplus (y_2, y_1) = (y_1 \oplus y_2, y_2 * y_1) = (y_1 \oplus y_2, 0)$.
Since $(0, x_2 \oplus x_1) \leq (y_1 \oplus y_2, 0)$, we deduce

$$x \otimes (\sim x) = (x_1, x_2) \circledast (\sim (x_1, x_2)) \leq (y_1, y_2) \oplus (\sim (y_1, y_2)) = y \oplus (\sim y),$$

proving that the algebra satisfies (K^*).

In order to prove (N^*), take any $a = (a_1, a_2)$, $b = (b_1, b_2)$, $c = (c_1, c_2)$ such that $a, b, c \in \mathcal{L}$. Then

$$(a \circledast b) \rightarrowtail_L c = (a_1 \circledast b_1, a_2 \oplus b_2) \rightarrowtail_L (c_1, c_2) = ((a_1 \circledast b_1) \rightarrow_L c_1, \ a_1 \circledast b_1 \circledast c_1).$$

As $(a_1 \circledast b_1) \rightarrow_L c_1 = a_1 \rightarrow_L (b_1 \rightarrow_L c_1)$ holds in $([0, 1], \max(,), \min(,), \circledast, \rightarrow_L, 0, 1)$, we get $(a \circledast b) \rightarrowtail_L c = (a_1 \rightarrow_L (b_1 \rightarrow_L c_1), a_1 \circledast b_1 \circledast c_1)$ in $\mathcal{L}$.

Since the right side of the equality in (N^*) yields

$$a \rightarrowtail_L (b \rightarrowtail_L c) = (a_1, a_2) \rightarrowtail_L ((b_1, b_2) \rightarrowtail_L (c_1, c_2)) =$$

$= (a_1, a_2) \rightarrowtail_L (b_1 \rightarrow_L c_1, b_1 \circledast c_1) = (a_1 \rightarrow_L (b_1 \rightarrow_L c_1), a_1 \circledast b_1 \circledast c_1)$,
we obtain that $(a \circledast b) \rightarrowtail_L c = a \rightarrowtail_L (b \rightarrowtail_L c)$. $\qquad\square$

Corollary 1 *The intuitionistic sets on U form a quasi Kleene algebra* $(\mathcal{I}(U), \vee \wedge, \circledast, \oplus, \sim, (\mathbf{0}, \mathbf{1}), (\mathbf{1}, \mathbf{0}))$.

Finally, we will prove that on $\mathcal{L}$ can be defined a propositional calculus that satisfies almost all additional logical rules (NL1)–(NL6) used to extend the intuitionistic logic into Nelson logic. In order show this, it suffices to prove that $\mathcal{L}$ satisfies some identities corresponding to the tautologies (NL1)–(NL6). More precisely, defining $x \leftrightarrow y = (x \rightarrowtail_L) \wedge (y \rightarrowtail_L x)$, for all $x, y \in \mathcal{L}$, we can prove

Theorem 2 *The algebra* $\mathbb{A} = (\mathcal{L}, \vee, \wedge, \circledast, \oplus, \sim, \rightarrowtail_L, (0, 1), (1, 0))$ *satisfies the identities:*

(NL1) $\sim x \rightarrowtail_L (x \rightarrowtail_L y) = (1, 0)$;
(NL2') $\sim (x \rightarrowtail_L y) \leftrightarrow (x \circledast \sim y) = (1, 0)$;
(NL3) $\sim (x \wedge y) \leftrightarrow (\sim x \vee \sim y) = (1, 0)$;
(NL3') $\sim (x \circledast y) \leftrightarrow (\sim x \oplus \sim y) = (1, 0)$;
(NL4) $\sim (x \vee y) \leftrightarrow (\sim x \wedge \sim y) = (1, 0)$
(NL4') $\sim (x \oplus y) \leftrightarrow (\sim x \circledast \sim y) = (1, 0)$;
(NL5) $\sim \sim x \leftrightarrow x = (1, 0)$;
(NL6) $\sim \neg x \leftrightarrow x = (1, 0)$.

Proof Since $\rightarrowtail_L$ satisfies the identity principle, we have $x \leftrightarrow x = (x \rightarrowtail_L x) \wedge (x \rightarrowtail_L x) = (1, 0) \wedge (1, 0) = (1, 0)$. Therefore, in the case when $A = B$ is an

identity in $\mathcal{L}$, we obtain $A \leftrightarrow B = (1, 0)$. Since $(\mathcal{L}, \vee, \wedge, \circledast, \oplus, \sim, (0, 1), (1, 0))$ is a quasi Kleene algebra, the identities (M) and (M^*) are satisfied. From here it follows that (NL3),(NL3'),(NL4), (NL4'), (NL5) hold for all $x, y \in L$.

(NL6). Here we recall that for any $x = (x_1, x_2) \in \mathcal{L}$, we have by definition

$$\neg x =: (x_1, x_2) \rightarrowtail_L (0, 1) = (x_1 \rightarrow_L 0, \ x_1 \circledast 1) = (1 - x_1, x_1), \text{ therefore}$$
$$\sim \neg x = (x_1, 1 - x_1).$$

Hence for all $x \in \mathcal{L}$ we obtain:

$$\sim \neg x \leftrightarrow x = ((x_1, 1 - x_1) \rightarrowtail_L (x_1, x_2)) \wedge ((x_1, x_2) \rightarrowtail_L (x_1, 1 - x_1)) =$$
$$= (x_1 \rightarrow_L x_1, x_1 \circledast x_2) \wedge (x_1 \rightarrow_L x_1, x_1 \circledast (1 - x_1)) = (1, 0) \wedge (1, 0) = (1, 0),$$

and this proves (NL6).

(NL1). As $\mathcal{L}$ satisfies (N^*), for any $x = (x_1, x_2) \in \mathcal{L}$, $y = (y_1, y_2) \in \mathcal{L}$ we get:

$$\sim x \rightarrowtail_L (x \rightarrowtail_L y) = (\sim x \circledast x) \rightarrowtail_L y = ((x_2, x_1) \circledast (x_1, x_2)) \rightarrowtail_L (y_1, y_2) =$$
$$= (x_2 \circledast x_1, x_1 \oplus x_2) \rightarrowtail_L (y_1, y_2) = (0, x_1 \oplus x_2) \rightarrowtail_L (y_1, y_2) = (0 \rightarrow_L y_1, 0) =$$
$$(1, 0), \text{ because } 0 \rightarrow_L y_1 = 1.$$

(NL2'). Take any $x, y \in \mathcal{L}$ with $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Then

$$\sim (x \rightarrowtail_L y) \rightarrowtail_L (x \circledast \sim y) = \sim (x_1 \rightarrow_L y_1, x_1 \circledast y_2) \rightarrowtail_L ((x_1, x_2) \circledast (y_2, y_1)) =$$
$$= (x_1 \circledast y_2, x_1 \rightarrow_L y_1) \rightarrowtail_L (x_1 \circledast y_2, x_2 \oplus y_1) =$$
$$= ((x_1 \circledast y_2) \rightarrow_L (x_1 \circledast y_2), x_1 \circledast x_2 \circledast y_2 \circledast y_1)) = (1, 0)$$

(since $\rightarrow_L$ satisfies the identity principle and $x_1 \circledast x_2 = y_2 \circledast y_1 = 0$).
Similarly, we obtain:

$$(x \circledast \sim y) \rightarrowtail_L \sim (x \rightarrowtail_L y) = (x_1 \circledast y_2, x_2 \oplus y_1) \rightarrowtail_L (x_1 \circledast y_2, x_1 \rightarrow_L y_1) =$$
$$= ((x_1 \circledast y_2) \rightarrow_L (x_1 \circledast y_2), x_1 \circledast y_2 \circledast (x_1 \rightarrow_L y_1)) =$$
$$(1, y_2 \circledast (x_1 \circledast (x_1 \rightarrow_L y_1))) = (1, 0), \text{ because}$$
$$x_1 \circledast (x_1 \rightarrow_L y_1) \leq y_1 \text{ implies } y_2 \circledast (x_1 \circledast (x_1 \rightarrow_L y_1)) \leq y_2 \circledast y_1 = 0.$$

Thus we deduce (NL2'): $\sim (x \rightarrowtail_L y) \leftrightarrow (x \circledast \sim y) = (1, 0) \wedge (1, 0) = (1, 0)$ $\square$

5   Conclusions

In the previous sections we have shown that there are several analogies between the intuitionistic fuzzy sets and rough sets.

First, we proved that the rough sets induced by a quasiorder relation R can be viewed as particular intuitionistic sets. Moreover, rough sets induced by R form particular Kleene algebras, and analogously, the intuitionistic sets on a universe U form quasi Kleene algebras.

Secondly, in a logical approach, both rough sets and both the intuitionistic sets are tools for handling the incomplete information, see e.g. Kovács and Radeleczki [11]. Rough sets induced by a quasiorder are fundamental models for Constructive

logic with strong negation (or Nelson logic) which is obtained by extending the intuitionistic propositional logic with some formulas involving the strong negation $\sim$ (see e.g. Järvinen and Radeleczki [9]). We note that the algebraic counterpart of the intuitionistic logic is formed by Heyting algebras. Similarly, for any continuous t-norm $\mathcal{T}$ defined on the lattice $\mathcal{L} = \{(x_1, x_2) \in [0, 1]^2 \mid x_1 + x_2 \leq 1\}$ of logical values a residuated lattice $(\mathcal{L}, \vee, \wedge, \mathcal{T}, \rightarrow_T, (0, 1), (1, 0))$ can be constructed; the residuated lattices constitute a natural generalization of Heyting algebras. It is known that the logics defined on several types of residuated lattices belong to the family of the so called *substructural logics* (see Galatos et al. [5], or Novák et al. [13]). Therefore, a connective system defined on $\mathcal{L}$ leads to a particular substructural logic. We proved that in the case of Łukasiewitz t-norm $\circledast$, this logic can be extended by adding to it almost the same rules involving the standard negation $\sim$ as in the case of Nelson logic. In fact, all the particular rules of this logic remain valid in the framework of $\mathcal{L}$, except (NL2), which is replaced by the rule

(NL2') $\sim (p \rightarrowtail_L q) \leftrightarrow (p \circledast \sim q)$.

Hence in the case of the intuitionistic sets and their logic, the standard negation $\sim (x_1, x_2) = (x_2, x_1)$ has the same role as the strong negation in Nelson logic.

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