Descriptor Fractional Continuous-Time Linear System and Its Solution – Comparison of Three Different Methods

Łukasz Sajewski(✉)

Faculty of Electrical Engineering, Bialystok University of Technology, Wiejska 45D, 15-351 Bialystok, Poland
l.sajewski@pb.edu.pl

Abstract. Descriptor fractional continuous-time linear systems are addressed. Three different methods for finding the solution to the state equation of the descriptor fractional linear system are considered. The methods are based on: Shuffle algorithm, Drazin inverse of the matrices and Weierstrass-Kronecker decomposition theorem. Effectiveness of the methods is demonstrated on simple numerical example.

Keywords: Descriptor · Fractional · Continuous-time · Linear · State space solution

1 Introduction

Descriptor (singular) linear systems have been considered in many papers and books [1–3, 6, 8, 12]. First definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [18, 19], another one was proposed in 20th century by Caputo [20] and next one in present times by Caputo-Fabrizio [16]. This idea has been used by engineers for modeling different processes [4, 5]. Mathematical fundamentals of fractional calculus are given in the monographs [17–20]. Solution of the state equations of descriptor fractional continuous-time linear systems have been given in [8, 12] and for discrete-time in [13, 14]. Application of the Drazin inverse method to analysis of descriptor fractional discrete-time and continuous-time linear systems have been given in [7, 9]. Solution of the state equation of descriptor fractional continuous-time linear systems with two different fractional orders has been introduced in [22]. Comparison of three different methods for finding the solution of descriptor fractional discrete-time linear system has been given in [21].

In this paper three different methods for finding the solution to descriptor fractional continuous-time linear systems will be considered and illustrated on single example.

The paper is organized as follows. In Sect. 2 the basic informations on the descriptor fractional continuous-time linear systems are recalled. Shuffle algorithm method is described in Sect. 3. Drazin inverse method is given in Sect. 4. Section 5 recalls Weierstrass-Kronecker decomposition method. In Sect. 6 single numerical example, illustrating three methods is presented. Concluding remarks are given in Sect. 7.

R. Szewczyk et al. (eds.), *Automation 2017*, Advances in Intelligent Systems and Computing 550, DOI 10.1007/978-3-319-54042-9_4

The following notation will be used: $\Re$ – the set of real numbers, $\Re^{n\times m}$ – the set of $n \times m$ real matrices, Z_+ – the set of nonnegative integers, I_n – the $n \times n$ identity matrix.

2 Preliminaries

Consider the descriptor fractional continuous-time linear system described by the state equation

$$E\,{}_0D_t^\alpha x(t) = Ax(t) + Bu(t), \tag{2.1}$$

where α is fractional order, $x(t) \in \Re^n$ is the state vector $u(t) \in \Re^m$ is the input vector, $E, A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ and $k-1<\alpha<k$, $k \in W = \{1, 2, \ldots\}$,

$${}_0D_t^\alpha f(t) = \frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(k-\alpha)}\int_0^\infty \frac{f^{(k)}(\tau)}{(t-\tau)^{\alpha+1-k}}d\tau,\; f^{(k)}(\tau) = \frac{d^k f(\tau)}{d\tau^k} \tag{2.2}$$

is the Caputo definition of $\alpha \in \Re$ order derivative of $x(t)$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ is the Euler gamma function.

Main property of the descriptor system is that

$$\det E = 0 \tag{2.3}$$

that is the matrix E is not full row (or column since E is square), rank $E = n_1 = n - q$ and matrix E contains only n_1 linearly independent rows (columns). Integer q often serves as index of singular matrices when it satisfy condition rank $E^q =$ rank E^{q+1} or as index of nilpotency of nilpotent matrices if it satisfy the condition $N^q = 0$ and $N^{q-1} \neq 0$.

Considering pencil (E, A) of the system (2.1) we can distinguish two types of descriptor systems:

- regular system where the pencil (E, A) is regular, i.e.

$$\det[Es^\alpha - A] \neq 0 \text{ for some } s \in \mathrm{C} \text{ (the field of complex numbers)} \tag{2.4}$$

- singular system where the pencil (E, A) is singular, i.e.

$$\det[Es^\alpha - A] = 0 \text{ for some } s \in \mathrm{C}. \tag{2.5}$$

Finding the solution to the Eq. (2.1) can be accomplished by the use of at least three different methods. That is: Shuffle algorithm method [10], Drazin inverse method [9], Weierstrass-Kronecker decomposition method [12]. Further in the paper, these method will be used to compute the solution of the descriptor fractional continuous-time linear system. MATLAB computational environment will be used to check the solution in according to number of historical elements of the system (length of practical realization).

3 Shuffle Algorithm Method

First method is based on row and column elementary operations [11] and use the Shuffle algorithm to determine the solution [10].

According to fact that det $E = 0$, by performing elementary row operations on the array

$$E \quad A \quad B \tag{3.1}$$

we always can rewrite (3.1) in the form

$$\begin{matrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{matrix} \tag{3.2}$$

which lead to following notation of state Eq. (2.1)

$$E_1 \frac{d^\alpha x(t)}{dt^\alpha} = A_1 x(t) + B_1 u(t), \tag{3.3a}$$

$$0 = A_2 x(t) + B_2 u(t), \tag{3.3b}$$

where E_1 has full row rank equal n_1. Now performing shuffle, that means α order differentiation of (3.3b) with respect to time, yields

$$-A_2 \frac{d^\alpha x(t)}{dt^\alpha} = B_2 \frac{d^\alpha u(t)}{dt^\alpha}. \tag{3.4}$$

The Eqs. (3.3a) and (3.4) formulate new state equation of the form

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} \frac{d^\alpha x(t)}{dt^\alpha} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \frac{d^\alpha u(t)}{dt^\alpha}. \tag{3.5}$$

In this case the system array (like (3.2)) has the form

$$\begin{matrix} E_1 & A_1 & B_1 & 0 \\ -A_2 & 0 & 0 & B_2 \end{matrix}. \tag{3.6}$$

If matrix

$$\begin{bmatrix} E_1 \\ -A_2 \end{bmatrix} \tag{3.7}$$

of the Eq. (3.5) is nonsingular, then we obtain standard differential equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \left(\begin{bmatrix} A_1 \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \frac{d^\alpha u(t)}{dt^\alpha} \right). \tag{3.8}$$

If the matrix (3.7) is singular, then we perform next shuffle, this time on matrix (3.6). Assuming regular pencil, after q steps we obtain a nonsingular matrix [15]

$$\begin{bmatrix} E_q \\ -A_{q+1} \end{bmatrix} \tag{3.9}$$

which lead to following differential equation

$$\frac{d^\alpha x(t)}{dt^\alpha} = \bar{A}x(t) + \bar{B}_0 u(t) + \bar{B}_1 \frac{d^\alpha u(t)}{dt^\alpha} + \ldots + \bar{B}_q \frac{d^{q\alpha} u(t)}{dt^{q\alpha}} = \bar{A}x(t) + \bar{B}\bar{u}(t), \tag{3.10}$$

where

$$\bar{A} = \begin{bmatrix} E_q \\ -A_{q+1} \end{bmatrix}^{-1} \begin{bmatrix} A_q \\ 0 \end{bmatrix}, \quad \bar{B}_0 = \begin{bmatrix} E_q \\ -A_{q+1} \end{bmatrix}^{-1} \begin{bmatrix} B_q \\ 0 \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} E_q \\ -A_{q+1} \end{bmatrix}^{-1} \begin{bmatrix} C_q \\ 0 \end{bmatrix}, \ldots, \quad \bar{B}_{q-1} = \begin{bmatrix} 0 \\ H_q \end{bmatrix},$$
$$\bar{B} = [\bar{B}_0 \quad \bar{B}_1 \quad \ldots \quad \bar{B}_q] \in \Re^{n \times \bar{n}}, \quad \bar{n} = (q+1)m, \quad \bar{u}(t) = \begin{bmatrix} u(t) \\ \frac{d^\alpha u(t)}{dt^\alpha} \\ \vdots \\ \frac{d^{q\alpha} u(t)}{dt^{q\alpha}} \end{bmatrix}. \tag{3.11}$$

In this process, we reduce the descriptor system to standard system with derivative of the inputs. To compute the solution $x(t)$ of (3.10), well-known formula [11] can be used

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)\bar{B}\bar{u}(\tau)d\tau, \tag{3.12a}$$

where x_0 is the admissible initial condition and the matrices Φ are determined by

$$\Phi_0(t) = \sum_{k=0}^{\infty} \frac{\bar{A}^k t^{k\alpha}}{\Gamma(k\alpha+1)}, \; \Phi(t) = \sum_{k=0}^{\infty} \frac{\bar{A}^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \tag{3.12b}$$

4 Drazin Inverse Method

Second method use Drazin inverses of the matrices $\bar{E}$ and $\bar{A}$ [9].

A matrix $\bar{E}^D$ is called the Drazin inverse of $\bar{E} \in \Re^{n \times n}$ if it satisfies the conditions [9]

$$\bar{E}\bar{E}^D = \bar{E}^D\bar{E}, \; \bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \; \bar{E}^D\bar{E}^{q+1} = \bar{E}^q, \tag{4.1}$$

where q is the index of $\bar{E}$. The Drazin inverse $\bar{E}^D$ of a square matrix $\bar{E}$ always exist and is unique [1]. If $\det \bar{E} \neq 0$ then $\bar{E}^D = \bar{E}^{-1}$.

Assuming that for some chosen $c \in$ C, $\det[Ec - A] \neq 0$ and premultiplying (2.1) by $[Ec - A]^{-1}$ we obtain

$$\bar{E}\,{}_0D_t^{\alpha}x(t) = \bar{A}x(t) + \bar{B}u(t), \tag{4.2a}$$

where

$$\bar{E} = [Ec - A]^{-1}E,\ \bar{A} = [Ec - A]^{-1}A,\ \bar{B} = [Ec - A]^{-1}B. \tag{4.2b}$$

Following [9], the solution to the Eq. (4.2a) with an admissible initial condition $x(0)$, is given by

$$x(t) = \Phi_0(t)\bar{E}\bar{E}^D v + \bar{E}^D \int_0^t \Phi(t-\tau)\bar{B}u(\tau)d\tau + (\bar{E}\bar{E}^D - I_n)\sum_{k=0}^{q-1}(\bar{E}\bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(t), \tag{4.3a}$$

where q is the index of $\bar{E}$ and

$$\Phi_0(t) = \sum_{k=0}^{\infty}\frac{(\bar{E}^D\bar{A})^k t^{k\alpha}}{\Gamma(k\alpha+1)},\ \Phi(t) = \sum_{k=0}^{\infty}\frac{(\bar{E}^D\bar{A})^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]},\ u^{(k\alpha)}(t) = {}_0D_t^{k\alpha}u(t) \tag{4.3b}$$

and the vector $v \in \Re^n$ is arbitrary.

From (4.3) for $t = 0$ we have the formula for admissible initial conditions

$$x(0) = x_0 = \bar{E}\bar{E}^D v + (\bar{E}\bar{E}^D - I_n)\sum_{k=0}^{q-1}(\bar{E}\bar{A}^D)^k\bar{A}^D\bar{B}u^{(k\alpha)}(0). \tag{4.4}$$

5 Weierstrass-Kronecker Decomposition Method

Third method use the following property of descriptor systems, upon which the solution to the state equation will be derived.

If (2.4) holds (descriptor system with regular pencil), then there exist nonsingular matrices $P, Q \in \Re^{n\times n}$ such that [11, 15]

$$PEQ = \mathrm{diag}(I_{n_1}, N),\ \ PAQ = \mathrm{diag}(A_1, I_{n_2}), \tag{5.1}$$

where $N \in \Re^{n_2\times n_2}$ is nilpotent matrix with the index μ, $A_1 \in \Re^{n_1\times n_1}$ and n_1 is equal to degree of the polynomial

$$\det[Es^{\alpha} - A] = a_{n_1}s^{\alpha n_1} + \ldots + a_1 s^{\alpha} + a_0,\ n_1 + n_2 = n. \tag{5.2}$$

Premultiplying the Eq. (2.1) by the matrix $P \in \Re^{n\times n}$ and introducing new state vector

$$\bar{x}(t) = Q^{-1}x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix},\ x_1(t) \in \Re^{n_1},\ x_2(t) \in \Re^{n_2}, \tag{5.3}$$

we obtain

$$PEQQ^{-1}\frac{d^\alpha}{dt^\alpha}x(t) = PEQ\frac{d^\alpha}{dt^\alpha}Q^{-1}x(t) = PAQQ^{-1}x(t) + PBu(t). \tag{5.4}$$

Applying (5.1) and (5.3) to (5.4) we have

$$\begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}\frac{d^\alpha}{dt^\alpha}\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}\begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}u(t), \tag{5.5}$$

where

$$\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = PB,\ B_1 \in \Re^{n_1 \times m},\ B_2 \in \Re^{n_2 \times m}. \tag{5.6}$$

From (5.5) we obtain

$$\frac{d^\alpha}{dt^\alpha}\bar{x}_1(t) = A_1\bar{x}_1(t) + B_1u(t) \tag{5.7}$$

and

$$N\frac{d^\alpha}{dt^\alpha}\bar{x}_2(t) = \bar{x}_2(t) + B_2u(t). \tag{5.8}$$

The solution $\bar{x}_1(t)$ to the Eq. (5.7) with admissible initial condition $\bar{x}_{10}$ is similar as in (3.12) and it is given by the formula

$$\bar{x}_1(t) = \Phi_0(t)\bar{x}_{10} + \int_0^t \Phi(t-\tau)B_1u(\tau)d\tau, \tag{5.9a}$$

where

$$\Phi_0(t) = \sum_{k=0}^{\infty}\frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha+1)},\ \Phi(t) = \sum_{k=0}^{\infty}\frac{A_1^k t^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}. \tag{5.9b}$$

The solution $\bar{x}_2(t)$ of the Eq. (5.8) with admissible initial condition $\bar{x}_{20}$ can be found by the use of straight and inverse Laplace transform and is given by [12]

$$\bar{x}_2(t) = -B_2u(t) - N\bar{x}_{20}\frac{t^{-\alpha}}{\Gamma(1-\alpha)} - \sum_{i=1}^{\mu-1}\left[N^iB_2\frac{d^{i\alpha}}{dt^{i\alpha}}u(t) + N^{i+1}\frac{d^{(i+1)\alpha-1}}{dt^{(i+1)\alpha-1}}\bar{x}_{20}\right]. \tag{5.10}$$

From (5.3), for known $\bar{x}_1(t)$ and $\bar{x}_2(t)$, we can find the desired solution of the Eq. (2.1).

6 Example

Find the solution $x(t)$ of the descriptor fractional continuous-time linear system (2.1) with the matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{6.1}$$

for $\alpha = 0.5$, constant input $u(t) = u = 1$ and initial condition $x(0) = [\,1 \quad 2 \quad -2\,]^T$ (T denotes the transpose).

In this case, $\det E = 0$ and the pencil of the system (2.1) with (6.1) is regular since

$$\det[Es^\alpha - A] = s^\alpha(s^\alpha - 1). \tag{6.2}$$

6.1 Case of Shuffle Algorithm Method

Following Sect. 3 we have

$$[\,E \quad A \quad B\,] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} E_1 & A_1 & B_1 \\ 0 & A_2 & B_2 \end{bmatrix} \tag{6.3}$$

and the Eqs. (3.3) have the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \frac{d^\alpha x(t)}{dt^\alpha} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t), \tag{6.4a}$$

$$0 = [-1 \quad 0 \quad -1\,]x(t) - u(t). \tag{6.4b}$$

α order differentiation of (6.4b) with respect to time yields

$$-[\,-1 \quad 0 \quad -1\,]\frac{d^\alpha x(t)}{dt^\alpha} = -\frac{d^\alpha u(t)}{dt^\alpha}. \tag{6.5}$$

Now, as in (3.5), we can write (6.4a) and (6.5) in the form

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \frac{d^\alpha x(t)}{dt^\alpha} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \frac{d^\alpha u(t)}{dt^\alpha}. \tag{6.6}$$

The state Eq. (6.6) can be written as

$$\frac{d^{\alpha}x(t)}{dt^{\alpha}} = \bar{A}x(t) + \bar{B}_0 u(t) + \bar{B}_1 \frac{d^{\alpha}u(t)}{dt^{\alpha}} = \bar{A}x(t) + [\bar{B}_0 \quad \bar{B}_1] \begin{bmatrix} u(t) \\ \frac{d^{\alpha}u(t)}{dt^{\alpha}} \end{bmatrix} = \bar{A}x(t) + \bar{B}\bar{u}(t), \quad (6.7)$$

where

$$\bar{A} = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \quad \bar{B}_0 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \bar{B}_1 = \begin{bmatrix} E_1 \\ -A_2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \quad (6.8)$$

and the solution of the descriptor fractional continuous-time linear system (2.1) with (6.1) can be computed by the use of the following formula

$$x(t) = \Phi_0(t)x_0 + \int_0^t \Phi(t-\tau)\bar{B}_0 u(\tau)d\tau = \sum_{k=0}^{\infty} \frac{\bar{A}^k t^{k\alpha}}{\Gamma(k\alpha+1)} x_0 + \sum_{k=0}^{\infty} \frac{\bar{A}^k t^{(k+1)\alpha} \bar{B}_0}{\Gamma[(k+1)\alpha](k+1)\alpha} u \quad (6.9)$$

since for constant input $\frac{d^{\alpha}u}{dt^{\alpha}} = 0$.

6.2 Case of Drazin Inverse Method

Following Sect. 4, using (4.2b) for $c = 2$ we have the matrices

$$\bar{E} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ -0.5 & 0 & 0 \end{bmatrix}, \bar{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}, \bar{B} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}. \quad (6.10)$$

Using rank factorization we obtain

$$\bar{E} = V_e W_e = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \\ -0.5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } \bar{A} = V_a W_a = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix} \quad (6.11)$$

and using formula $F^D = V[WFV]^{-1}W$ we compute

$$\bar{E}^D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix}, \bar{A}^D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & -1 \end{bmatrix}. \quad (6.12)$$

The desired solution for the descriptor fractional continuous-time linear system (2.1) with (6.1) has the form

$$\begin{aligned} x(t) &= \Phi_0(t)\bar{E}\bar{E}^D v + \bar{E}^D \int_0^t \Phi(t-\tau)\bar{B}u(\tau)d\tau + (\bar{E}\bar{E}^D - I_3)\bar{A}^D\bar{B}u(t) \\ &= \sum_{k=0}^{\infty} \frac{(\bar{E}^D\bar{A})^k t^{k\alpha}\bar{E}\bar{E}^D}{\Gamma(k\alpha+1)} v + \bar{E}^D \sum_{k=0}^{\infty} \frac{(\bar{E}^D\bar{A})^k t^{(k+1)\alpha}\bar{B}}{\Gamma[(k+1)\alpha](k+1)\alpha} u + (\bar{E}\bar{E}^D - I_3)\bar{A}^D\bar{B}u, \end{aligned} \tag{6.13}$$

since the index of $\bar{E}$ is equal 1 (μ = 1).

From (6.13) for t = 0 we have

$$x(0) = x_0 = \bar{E}\bar{E}^D v + (\bar{E}\bar{E}^D - I_3)\bar{A}^D\bar{B}u(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u_0, \tag{6.14}$$

hence, for given $u_0 = u = 1$, the initial condition $v = x_0 = [1 \quad 2 \quad -2]^T$ satisfy (6.14) and it is admissible.

6.3 Case of Weierstrass-Kronecker Decomposition Method

In this case the for (6.1) matrices P and Q have the form

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \tag{6.15}$$

and decomposition (5.1) is possible since

$$PEQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad PAQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad PB = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \tag{6.16}$$

$(n_1 = 2, n_2 = 1)$.

The Eqs. (5.7) and (5.8) have the form

$$\frac{d^\alpha}{dt^\alpha}\bar{x}_1(t) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bar{x}_1(t), \tag{6.17a}$$

$$0 = \bar{x}_2(t) + u(t), \tag{6.17b}$$

since $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B_1 = 0, \quad N = 0, \quad B_2 = 1.$

Taking under considerations (5.9) and (5.10), the solution of the Eq. (6.17a) has the form

$$\bar{x}_1(t) = \sum_{k=0}^{\infty} \frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha + 1)} \bar{x}_{10} \tag{6.18}$$

and of the Eq. (6.17b) has the form

$$\bar{x}_2(t) = -u(t). \tag{6.19}$$

The desired solution of the descriptor fractional continuous-time linear system (2.1) with (6.1) is given by

$$x(t) = Q \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} \sum_{k=0}^{\infty} \frac{A_1^k t^{k\alpha}}{\Gamma(k\alpha+1)} \bar{x}_{10} \\ -u(t) \end{bmatrix}, \tag{6.20}$$

where $\bar{x}_0 = Q^{-1} x_0 = \begin{bmatrix} \bar{x}_{10} \\ \bar{x}_{20} \end{bmatrix}$ and $\bar{x}_{10} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

6.4 Computational Results

Continuous-time systems give some numerical problems in computation of exact solution, since matrices Φ given by formula (3.12b), (4.4b) and (5.9b) assume infinite memory. In practical case infinite memory is substituted by finite number n called length of practical implementation and the matrices Φ takes the form

$$\Phi_0(t) = \sum_{k=0}^{n} \frac{A^k t^{k\alpha}}{\Gamma(k\alpha + 1)}, \; \Phi(t) = \sum_{k=0}^{n} \frac{A^k t^{(k+1)\alpha - 1}}{\Gamma[(k+1)\alpha]}. \tag{6.21}$$

This, approach allows us to compute desired solution, however new problem arise, how to find number n? In the paper, this problem, was solved by testing the solution of the system with matrices (6.1). Figure 1 shows solution for $\alpha = 0.5$ and $t = 1$ in the function of the number n. Figure 2 shows response of the descriptor system (Weierstrass-Kronecker decomposition method only) for constant $\alpha = 0.5$ and $t = 1$, $t = 10$, $t = 100$. As we can see, all three methods gives the same results, so they are equivalent. General conclusion is that, the further in time we go, the more historical elements is necessary, e.g. for $\alpha = 0.5$ and $t = 1$ we have $n \approx 10$, for $\alpha = 0.5$ and $t = 100$ we have $n \approx 300$. Length of practical implementation n also strongly depend on row α. The smaller the row α is, the more historical elements is necessary, e.g. for $t = 1$ and $\alpha = 0.9$ we have $n \approx 5$, for $t = 1$ and $\alpha = 0.1$ we have $n \approx 50$. Based on computational results, the following condition has been found

$$n \approx 5t/\alpha. \tag{6.22}$$

From practical point of view, Drazin inverse method is most suitable for practical implementation, since computation of Drazin inverse of the square matrices can by accomplished by singular value decomposition (SVD) (see Listing 1).

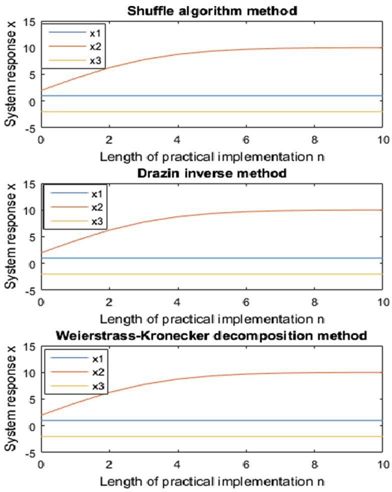

Fig. 1. Solution for $\alpha = 0.5$ and $t = 1$.

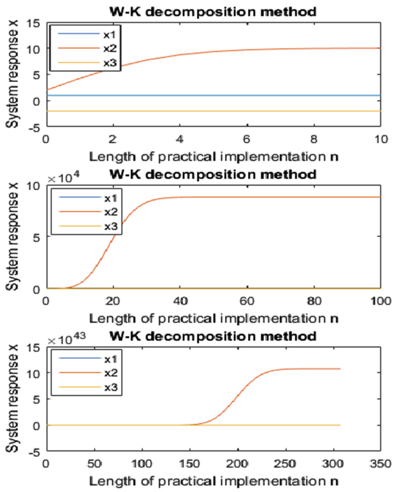

Fig. 2. Solution for $\alpha = 0.5$ and $t = 1$, $t = 10$, $t = 100$.

```
function [d_inv] = drazin_inv(A)
[u,s,d]=svd(A); n=rank(s); d=d';
V=u(:,1:n); W=s(1:n,1:n)*d(1:n,:);
d_inv=V*inv(W*A*V)*W;
```

7 Concluding Remarks

The descriptor fractional continuous-time linear systems have been recalled. Three different methods for finding the solution to the state equation of the descriptor fractional continuous-time linear system have been considered. Comparison of computational effort of the methods have been demonstrated on single numerical example.

In Drazin inverse method admissible initial conditions should be applied. In Shuffle algorithm method admissible initial conditions as well as derivative of the inputs should be known. The weak point of Weierstrass-Kronecker decomposition approach is computation of the P and Q matrices, where elementary row and column operations method is recommended. The same method uses Shuffle algorithm. In summary, the Drazin inverse method seems to be most suitable for numerical implementation. An open problem is extension of these considerations to the system with different fractional orders.

Acknowledgments. This work was supported by National Science Centre in Poland under work No. 2014/13/B/ST7/03467.

References

1. Campbell, S.L., Meyer, C.D., Rose, N.J.: Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients. SIAMJ. Appl. Math. **31** (3), 411–425 (1976)
2. Dai, L.: Singular Control Systems. LNCIS. Springer, Berlin (1989). doi:10.1007/BFb0002475
3. Dodig, M., Stosic, M.: Singular systems state feedbacks problems. Linear Algebra Appl. **431** (8), 1267–1292 (2009)
4. Dzieliński, A., Sierociuk, D., Sarwas, G.: Ultracapacitor parameters identification based on fractional order model. In: Proceedings of the ECC, Budapest (2009)
5. Ferreira, N.M.F., Machado, J.A.T.: Fractional-order hybrid control of robotic manipulators. In: Proceedings of the 11th International Conference on Advanced Robotics, ICAR, Coimbra, Portugal, pp. 393–398 (2003)
6. Guang-Ren, D.: Analysis and Design of Descriptor Linear Systems. Springer, New York (2010)
7. Kaczorek, T.: Application of Drazin inverse to analysis of descriptor fractional discrete-time linear systems with regular pencils. Int. J. Appl. Math. Comput. Sci. **23**(1), 29–33 (2013)
8. Kaczorek, T.: Descriptor fractional linear systems with regular pencils. Int. J. Appl. Math. Comput. Sci. **23**(2), 309–315 (2013)
9. Kaczorek, T.: Drazin inverse matrix method for fractional descriptor continuous-time linear systems. Bull. Pol. Acad. Sci. Tech. Sci. **62**(3), 409–412 (2014)
10. Kaczorek, T.: Reduction and decomposition of singular fractional discrete-time linear systems. Acta Mechanica Automatica **5**(4), 1–5 (2011)
11. Kaczorek, T.: Selected Problems in Fractional Systems Theory. Springer, Berlin (2011)
12. Kaczorek, T.: Singular fractional continuous-time and discrete-time linear systems. Acta Mechanica Automatica **7**(1), 26–33 (2013)

13. Kaczorek, T.: Singular fractional discrete-time linear systems. Control Cybern. **40**(3), 1–8 (2011)
14. Kaczorek, T.: Solution of the state equations of descriptor fractional discrete-time linear systems with regular pencils. Tech. Transp. Szyn. (10), 415–422 (2013)
15. Kaczorek, T.: Vectors and Matrices in Automation and Electrotechnics. WNT, Warszawa (1998)
16. Losada, J., Nieto, J.: Properties of a new fractional derivative without singular kernel. Progr. Fract. Differ. Appl. **1**(2), 87–92 (2015)
17. Miller, K.S., Ross, B.: An Introduction to the Fractional Calculus and Fractional Differential Equations. Wiley, New York (1993)
18. Nishimoto, K.: Fractional Calculus. Decartess Press, Koriama (1984)
19. Oldham, K.B., Spanier, J.: The Fractional Calculus. Academic Press, New York (1974)
20. Podlubny, I.: Fractional Differential Equations. Academic Press, San Diego (1999)
21. Sajewski, Ł.: Descriptor fractional discrete-time linear system and its solution—comparison of three different methods. In: Szewczyk, R., Zieliński, C., Kaliczyńska, M. (eds.) Challenges in Automation, Robotics and Measurement Techniques. AISC, vol. 440, pp. 37–50. Springer, Heidelberg (2016). doi:10.1007/978-3-319-29357-8_4
22. Sajewski, Ł.: Solution of the state equation of descriptor fractional continuous-time linear systems with two different fractional. In: Szewczyk, R., Zieliński, C., Kaliczyńska, M. (eds.) Progress in Automation, Robotics and Measuring Techniques. AISC, vol. 350, pp. 233–242. Springer, Heidelberg (2016). doi:10.1007/978-3-319-15796-2_24