Chapter 9
A Parameterized Method for Optimal Multi-Period Mean-Variance Portfolio Selection with Liability

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Abstract Big data is being generated by everything around us at all times. The massive amount and corresponding data of assets in the financial market naturally form a big data set. In this paper, we tackle the multi-period mean-variance portfolio of asset-liability management using the parameterized method addressed in Li et al. (SIAM J. Control Optim. 40:1540–1555, 2002) and the state variable transformation technique. By this simple yet efficient method, we derive the analytical optimal strategies and efficient frontiers accurately. A numerical example is presented to shed light on the results established in this work.

Keywords Multi-period portfolio • Mean-variance formulation • Asset-liability management

9.1 Introduction

Portfolio selection is concerned with finding the most desirable group of funds to hold. The mean-variance model proposed by Markowitz (1952) aims to seek a balance between the gain and the risk, which are expressed by expectation and

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variance of the investment return, respectively. In order to trace out the efficient frontier for this bi-objective optimization problem, one typically puts weights on the two criteria and transforms the problem into a single-objective optimization problem.

After Markowitz's vanward work in a single-period setting, the mean-variance portfolio selection framework was extended to multi-period setting by Li and Ng (2000) using an embedding technique. Zhou and Li (2000) considered a continuous-time mean-variance problem while Li et al. (2002) investigated the problem with no short setting. As any nonlinear term of expectation operator, the term $(\mathbb{E}[x_T])^2$ in the mean-variance case, induces nonseparability, the spirit of both the embedding scheme proposed by Li and Ng (2000) and Zhou and Li (2000) and the parameterized method developed by Li et al. (2002) is to embed $(\mathbb{E}[x_T])^2$ into an auxiliary function or to replace $\mathbb{E}[x_T]$ by an auxiliary variable in mean-variance models to deal with mean-variance problems in dynamic programming. Besides the above, Cui et al. (2014) presented another powerful tool named mean-field formulation to tackle the nonseparability of multi-period mean-variance portfolio selection problem and derived analytical optimal strategies and efficient frontiers. Yi et al. (2014) developed the mean-field formulation method to solve the multi-period mean-variance portfolio selection problem with an uncertain exit horizon.

Big data is being generated by everything around us at all times. The number of assets in the financial market and the corresponding data constitute a typical big data. Big data is also changing the way people investing. Insights from big data and extracting meaningful value from big data can enable all investors to make better profit. It is well known that the stability of financial institutions depends crucially on the matching of assets, and liabilities. Liability is being brought more and more into the limelight when investors establish their portfolios. The mean-variance framework of asset-liability management was first investigated by Sharp and Tint (1990) in a single-period setting. For the multi-period setting and by the embedding technique, Leippold et al. (2004) derived the closed form optimal policies and mean-variance frontiers under exogenous and endogenous liabilities using a geometric approach; Chiu and Li (2006) employed the stochastic optimal control theory to analytically solve the asset-liability management in a continuous time setting; Yi et al. (2008) considered the situation of uncertain investment horizon; Chen and Yang (2011) studied the case with regime switching; Zeng and Li (2011) investigated the model under benchmark and mean-variance criteria in a jump diffusion market; Li and Li (2012) took the risk control over bankruptcy into account; Yao et al. (2013) re-considered the uncertain time-horizon model of Yi et al. (2008) by adding an uncontrolled cash flow.

Most of the papers for multi-period mean-variance portfolio selection of asset-liability management mentioned above are based on the embedding technique. The embedding scheme is indeed an efficient way to deal with problems having the nonseparable property. However, it is prone to involve inefficient and complicated calculation during the derivation of the optimal strategies and efficient frontiers by embedding. Therefore, research is naturally required on developing a simple yet accurate method. In this paper, we study asset-liability management under a multi-period mean-variance portfolio selection framework using the parameterized method addressed in Li et al. (2002). We first deduce the case when the returns of assets and liability are correlated. Then we reduce it to the uncorrelated setting. One prominent feature of the dynamic mean-variance formulations is that the optimal portfolio policy is always linear with respect to the current wealth and liability. According to this feature, we derive the analytical optimal policies and efficient frontiers. The analytical form of the Lagrange multiplier is also given in expression of the expectation of the final surplus.

The rest of the paper is organized as follows. In Sect. 9.2, we present the mean-variance formulation of the multi-period portfolio selection model for asset-liability management. The optimal strategies and efficient frontiers are derived in Sect. 9.3. Section 9.4 provides some numerical examples to illustrate the results developed in this paper. Section 9.5 concludes this paper.

9.2 Mean-Variance Formulation

Assume that an investor joining the market at the beginning of period 0 with an initial wealth x_0 and initial liability l_0, plans to invest his/her wealth within a time horizon T. He/she can reallocate his/her portfolio at the beginning of each of the following $T-1$ consecutive periods. The capital market consists of one risk-free asset, n risky assets and one liability. At time period t, the given deterministic return of the risk-free asset, the random returns of the n risky assets, and the random return of the liability are denoted by s_t (>1), vector $\mathbf{e}_t = [e_t^1, \cdots, e_t^n]'$ and q_t, respectively. The random vector $\mathbf{e}_t = [e_t^1, \cdots, e_t^n]'$ and the random variable q_t are defined over the probability space $(\Omega, \mathcal{F}, P)$ and are supposed to be statistically independent at different time periods.

Suppose that M and N are symmetric matrices with the same order. We denote $M \succ N$ ($M \succcurlyeq N$) if and only if $M - N$ is positive definite (semidefinite). We assume that the only information known about $\mathbf{e}_t$ and q_t are their first two unconditional moments, $\mathbb{E}[\mathbf{e}_t] = \left(\mathbb{E}[e_t^1], \cdots, \mathbb{E}[e_t^n]\right)'$, $\mathbb{E}[q_t]$ and $(n+1) \times (n+1)$ positive definite covariance

$$\mathrm{Cov}\left(\begin{pmatrix}\mathbf{e}_t \\ q_t\end{pmatrix}\right) = \mathbb{E}\left[\begin{pmatrix}\mathbf{e}_t \\ q_t\end{pmatrix}(\mathbf{e}_t'\ q_t)\right] - \mathbb{E}\left[\begin{pmatrix}\mathbf{e}_t \\ q_t\end{pmatrix}\right]\mathbb{E}\left[(\mathbf{e}_t'\ q_t)\right].$$

From the above assumptions, we have

$$\begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{e}_t'] & s_t\mathbb{E}[q_t] \\ s_t\mathbb{E}[\mathbf{e}_t] & \mathbb{E}[\mathbf{e}_t\mathbf{e}_t'] & \mathbb{E}[\mathbf{e}_t q_t] \\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{e}_t'] & \mathbb{E}[q_t^2] \end{pmatrix} \succ 0.$$

We further define the excess return vector of risky assets $\mathbf{P}_t = (P_t^1, \cdots, P_t^n)'$ as $(e_t^1 - s_t, \cdots, e_t^n - s_t)'$. The following is then true for $t = 0, 1, \cdots, T-1$:

$$\begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{P}_t'] & s_t\mathbb{E}[q_t] \\ s_t\mathbb{E}[\mathbf{P}_t] & \mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] & \mathbb{E}[\mathbf{P}_t q_t] \\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{P}_t'] & \mathbb{E}[q_t^2] \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0}' & 0 \\ -\mathbf{1} & I & \mathbf{0} \\ 0 & \mathbf{0}' & 1 \end{pmatrix} \begin{pmatrix} s_t^2 & s_t\mathbb{E}[\mathbf{e}_t'] & s_t\mathbb{E}[q_t] \\ s_t\mathbb{E}[\mathbf{e}_t] & \mathbb{E}[\mathbf{e}_t\mathbf{e}_t'] & \mathbb{E}[\mathbf{e}_t q_t] \\ s_t\mathbb{E}[q_t] & \mathbb{E}[q_t\mathbf{e}_t'] & \mathbb{E}[q_t^2] \end{pmatrix} \begin{pmatrix} 1 & -\mathbf{1}' & 0 \\ \mathbf{0} & I & \mathbf{0} \\ 0 & \mathbf{0}' & 1 \end{pmatrix} \succ 0,$$

where $\mathbf{1}$ and $\mathbf{0}$ are the n-dimensional all-one and all-zero vectors, respectively, and I is the $n \times n$ identity matrix, which further implies, for $t = 0, 1, \cdots, T-1$,

$$\begin{pmatrix} \mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] & \mathbb{E}[\mathbf{P}_t q_t] \\ \mathbb{E}[q_t\mathbf{P}_t'] & \mathbb{E}[q_t^2] \end{pmatrix} \succ 0,$$

and $s_t^2(1 - B_t) > 0$, where $B_t \stackrel{\Delta}{=} \mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]$. This implies that $0 < B_t < 1$ for $t = 0, 1, \cdots, T-1$.

Let x_t and l_t be the wealth and liability of the investor at the beginning of period t, respectively, then $x_t - l_t$ is the net wealth. At period t, if π_t^i, $i = 1, 2, \cdots, n$ is the amount invested in the ith risky asset, then $x_t - \sum_{i=1}^n \pi_t^i$ is the amount invested in the risk-free asset. We assume in this paper that the liability is exogenous, which means it is uncontrollable and cannot be affected by the investor's strategies. Denote the information set at the beginning of period t, $t = 1, 2, \cdots, T-1$, as $\mathcal{F}_t = \sigma(\mathbf{P}_0, \mathbf{P}_1, \cdots, \mathbf{P}_{t-1}, q_0, q_1, \cdots, q_{t-1})$ and the trivial σ-algebra over Ω as $\mathcal{F}_0$. Therefore, $\mathbb{E}[\cdot|\mathcal{F}_0]$ is just the unconditional expectation $\mathbb{E}[\cdot]$. We confine all admissible investment strategies to be the $\mathcal{F}_t$-adapted Markov controls, i.e., $\pi_t = (\pi_t^1, \ \pi_t^2, \ \cdots, \ \pi_t^n)' \in \mathcal{F}_t$. Then, $\mathbf{P}_t$ and π_t are independent, $\{x_t, l_t\}$ is an adapted Markovian process and $\mathcal{F}_t = \sigma(x_t, l_t)$.

The multi-period mean-variance model of asset-liability management is to seek the best strategy, $\pi_t^* = [(\pi_t^1)^*, (\pi_t^2)^*, \cdots, (\pi_t^n)^*]'$, $t = 0, 1, \cdots, T-1$, which is the solution of the following dynamic stochastic optimization problem,

$$\begin{cases} \min \operatorname{Var}(x_T - l_T) \equiv \mathbb{E}[(x_T - l_T - d)^2], \\ \text{s.t. } \mathbb{E}[x_T - l_T] = d, \\ \qquad x_{t+1} = s_t\left(x_t - \sum_{i=1}^n \pi_t^i\right) + \sum_{i=1}^n e_t^i \pi_t^i \\ \qquad\qquad = s_t x_t + \mathbf{P}_t'\pi_t, \\ \qquad l_{t+1} = q_t l_t, \qquad t = 0, 1, \cdots, T-1. \end{cases} \tag{9.1}$$

Introducing a Lagrange multiplier $2\omega > 0$ yields

$$\begin{cases} \min \mathbb{E}[(x_T - l_T - d)^2] - 2\omega(\mathbb{E}[x_T - l_T] - d), \\ \text{s.t. } \{x_t, l_t, \pi_t\} \text{ satisfies the dynamic system of problem (9.1)}, \end{cases} \tag{9.2}$$

which is equivalent to the following problem,

$$\begin{cases} \min \mathbb{E}[(x_T - l_T - d - \omega)^2], \\ \text{ s.t. } \{x_t, l_t, \pi_t\} \text{ satisfies the dynamic system of problem (9.1)}, \end{cases} \tag{9.3}$$

in the sense that the two problems have the same optimal strategy. It can be rewritten as

$$\begin{cases} \min \mathbb{E}[(x_T - \gamma - l_T)^2], \\ \text{ s.t. } \{x_t, l_t, \pi_t\} \text{ satisfies the dynamic system of problem (9.1)}, \end{cases} \tag{9.4}$$

where $\gamma = d + \omega$. Set

$$y_t := x_t - \gamma \prod_{k=t}^{T-1} s_k^{-1}, \tag{9.5}$$

and denote $\prod_{k=T}^{T-1} s_k^{-1} := 1$. Then the dynamic system of problem (9.1) turns to

$$\begin{cases} y_{t+1} = s_t y_t + \mathbf{P}_t' \pi_t, \\ l_{t+1} = q_t l_t, \qquad\qquad t = 0, 1, \cdots, T-1, \end{cases} \tag{9.6}$$

where $y_0 = x_0 - \gamma \prod_{k=0}^{T-1} s_k^{-1}$. Problem (9.4) can be reformulated as

$$\begin{cases} \min \mathbb{E}[(y_T - l_T)^2], \\ \text{ s.t. } \{y_t, l_t, \pi_t\} \text{ satisfies Eq. (9.6)}, \end{cases} \tag{9.7}$$

and it is the ‘same’ as the following problem:

$$\begin{cases} \min \mathbb{E}[y_T^2 - 2 l_T y_T], \\ \text{ s.t. } \{y_t, l_t, \pi_t\} \text{ satisfies Eq. (9.6)}, \end{cases} \tag{9.8}$$

The ‘same’ here means that they have the same optimal strategy. By studying problem (9.8), we can obtain the optimal strategy of the original problem (9.1).

9.3 The Optimal Strategies

9.3.1 The Optimal Strategy with Correlation of Assets and Liability

In this subsection, assume that the returns of assets and liability are correlated at every period, i.e., $\mathbf{P}_t$ and q_t are dependent on each other at period $t = 0, 1, \cdots, T-1$. Before we derive the optimal strategy, we denote

$$\widehat{B}_t \triangleq \mathbb{E}[q_t \mathbf{P}_t'] \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t],$$

$$\widetilde{B}_t \triangleq \mathbb{E}[q_t \mathbf{P}_t'] \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[q_t \mathbf{P}_t],$$

for $t = 0, 1, 2, \cdots, T-1$.

Theorem 1 *Assume that the returns of assets and liability are correlated at every period. Then the optimal strategy of problem (9.1) is given by*

$$\pi_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[\mathbf{P}_t] s_t \left(x_t - \gamma^* \prod_{k=t}^{T-1} s_k^{-1} \right) + \left(\prod_{k=t+1}^{T-1} \frac{\mathbb{E}[q_k] - \widehat{B}_k}{(1 - B_k) s_k} \right) \mathbb{E}^{-1}[\mathbf{P}_t \mathbf{P}_t'] \mathbb{E}[q_t \mathbf{P}_t] l_t, \tag{9.9}$$

where

$$\gamma^* = \frac{x_0 \prod_{k=0}^{T-1} (1 - B_k) s_k - d - l_0 \prod_{k=0}^{T-1} \left(\mathbb{E}[q_k] - \widehat{B}_k \right)}{\prod_{k=0}^{T-1} (1 - B_k) - 1}. \tag{9.10}$$

Proof We prove it by making use of the dynamic programming approach. For the information set $\mathcal{F}_t$, the cost-to-go functional of problem (9.8) at period t is

$$J_t(y_t, l_t) = \min_{\pi_t} \mathbb{E}\left[J_{t+1}(y_{t+1}, l_{t+1}) \middle| \mathcal{F}_t \right],$$

where the terminal condition $J_T(y_T, l_T) = y_T^2 - 2 l_T y_T$. □

We start from the last stage $T-1$. While $t = T-1$, we have

$$\begin{aligned}
&\mathbb{E}\left[J_T(y_T, l_T) \middle| \mathcal{F}_{T-1} \right] \\
&= \mathbb{E}\left[y_T^2 - 2 l_T y_T \middle| \mathcal{F}_{T-1} \right] \\
&= s_{T-1}^2 y_{T-1}^2 + 2 s_{T-1} y_{T-1} \mathbb{E}[\mathbf{P}_{T-1}'] \pi_{T-1} + \pi_{T-1}' \mathbb{E}[\mathbf{P}_{T-1} \mathbf{P}_{T-1}'] \pi_{T-1} \\
&\quad - 2 \mathbb{E}[q_{T-1}] s_{T-1} l_{T-1} y_{T-1} - 2 \mathbb{E}[q_{T-1} \mathbf{P}_{T-1}'] l_{T-1} \pi_{T-1}.
\end{aligned}$$

Minimizing it with respect to π_{T-1} yields the optimal decision at period $T-1$ as follows:

$$\pi^*_{T-1} = -\mathbb{E}^{-1}[\mathbf{P}_{T-1}\mathbf{P}'_{T-1}]\mathbb{E}[\mathbf{P}_{T-1}]s_{T-1}y_{T-1} + \mathbb{E}^{-1}[\mathbf{P}_{T-1}\mathbf{P}'_{T-1}]\mathbb{E}[q_{T-1}\mathbf{P}_{T-1}]l_{T-1}.$$

Substituting π^*_{T-1} to $\mathbb{E}\big[J_T(y_T, l_T)\big|\mathcal{F}_{T-1}\big]$, we obtain

$$\begin{aligned} J_{T-1}(y_{T-1}, l_{T-1}) &= \min_{\pi_{T-1}} \mathbb{E}\big[J_T(y_T, l_T)\big|\mathcal{F}_{T-1}\big] \\ &= (1-B_{T-1})s^2_{T-1}y^2_{T-1} - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)s_{T-1}l_{T-1}y_{T-1} - \widetilde{B}_{T-1}l^2_{T-1}. \end{aligned}$$

In order to derive the cost-to-go functional and the optimal decision at period t clearly, we patiently repeat the procedure at time $T-2$. While $t = T-2$, we have

$$\begin{aligned} &\mathbb{E}\big[J_{T-1}(y_{T-1}, l_{T-1})\big|\mathcal{F}_{T-2}\big] \\ &= \mathbb{E}\big[(1-B_{T-1})s^2_{T-1}y^2_{T-1} - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)s_{T-1}l_{T-1}y_{T-1} - \widetilde{B}_{T-1}l^2_{T-1}\big|\mathcal{F}_{T-2}\big] \\ &= (1-B_{T-1})s^2_{T-1}\Big(s^2_{T-2}y^2_{T-2} + 2s_{T-2}y_{T-2}\mathbb{E}[\mathbf{P}'_{T-2}]\pi_{T-2} + \pi'_{T-2}\mathbb{E}[\mathbf{P}_{T-2}\mathbf{P}'_{T-2}]\pi_{T-2}\Big) \\ &\quad - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)\mathbb{E}\big[q_{T-2}]s_{T-1}s_{T-2}l_{T-2}y_{T-2} \\ &\quad - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)\mathbb{E}[q_{T-2}\mathbf{P}'_{T-2}]s_{T-1}l_{T-2}\pi_{T-2} \\ &\quad - \widetilde{B}_{T-1}\mathbb{E}[q^2_{T-2}]l^2_{T-2}. \end{aligned}$$

We derive the following optimal decision at period $T-2$ by minimizing the above functional with respect to π_{T-2}

$$\begin{aligned} \pi^*_{T-2} = &-\mathbb{E}^{-1}[\mathbf{P}_{T-2}\mathbf{P}'_{T-2}]\mathbb{E}[\mathbf{P}_{T-2}]s_{T-2}y_{T-2} \\ &+ \frac{\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}}{(1-B_{T-1})s_{T-1}}\mathbb{E}^{-1}[\mathbf{P}_{T-2}\mathbf{P}'_{T-2}]\mathbb{E}[q_{T-2}\mathbf{P}_{T-2}]l_{T-2}. \end{aligned}$$

Then the cost-to-go functional at period $T-2$ is

$$\begin{aligned} J_{T-2}(y_{T-2}, l_{T-2}) &= \min_{\pi_{T-2}} \mathbb{E}\big[J_{T-1}(y_{T-1}, l_{T-1})\big|\mathcal{F}_{T-2}\big] \\ &= (1-B_{T-1})(1-B_{T-2})s^2_{T-1}s^2_{T-2}y^2_{T-2} \\ &\quad - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)\big(\mathbb{E}\big[q_{T-2}] - \widehat{B}_{T-2}\big)s_{T-1}s_{T-2}l_{T-2}y_{T-2} \\ &\quad - \left(\frac{\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)^2}{1-B_{T-1}}\widetilde{B}_{T-2} + \widetilde{B}_{T-1}\mathbb{E}[q^2_{T-2}]\right)l^2_{T-2}. \end{aligned}$$

While $t = T-3$, we can similarly get

$$
\begin{aligned}
&\mathbb{E}\big[J_{T-2}(y_{T-2}, l_{T-2})\big|\mathcal{F}_{T-3}\big] \\
&= \mathbb{E}\big[(1-B_{T-1})(1-B_{T-2})s_{T-1}^2 s_{T-2}^2 y_{T-2}^2 \\
&\quad - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)\big(\mathbb{E}\big[q_{T-2}] - \widehat{B}_{T-2}\big)s_{T-1}s_{T-2}l_{T-2}y_{T-2} \\
&\quad - \left(\frac{\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)^2}{1-B_{T-1}}\widetilde{B}_{T-2} + \widetilde{B}_{T-1}\mathbb{E}[q_{T-2}^2]\right)l_{T-2}^2\bigg|\mathcal{F}_{T-3}\bigg] \\
&= (1-B_{T-1})(1-B_{T-2})s_{T-1}^2 s_{T-2}^2 \\
&\quad \times \left(s_{T-3}^2 y_{T-3}^2 + 2s_{T-3}y_{T-3}\mathbb{E}[\mathbf{P}'_{T-3}]\pi_{T-3} + \pi'_{T-3}\mathbb{E}[\mathbf{P}_{T-3}\mathbf{P}'_{T-3}]\pi_{T-3}\right) \\
&\quad - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)\big(\mathbb{E}\big[q_{T-2}] - \widehat{B}_{T-2}\big)\mathbb{E}[q_{T-3}]s_{T-1}s_{T-2}s_{T-3}l_{T-3}y_{T-3} \\
&\quad - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)\big(\mathbb{E}\big[q_{T-2}] - \widehat{B}_{T-2}\big)\mathbb{E}[q_{T-3}\mathbf{P}'_{T-3}]s_{T-1}s_{T-2}l_{T-3}\pi_{T-3} \\
&\quad - \left(\frac{\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)^2}{1-B_{T-1}}\widetilde{B}_{T-2} + \widetilde{B}_{T-1}\mathbb{E}[q_{T-2}^2]\right)\mathbb{E}[q_{T-3}^2]l_{T-3}^2.
\end{aligned}
$$

Thus the optimal decision at period $T-3$ is

$$
\begin{aligned}
\pi_{T-3}^* = &- \mathbb{E}^{-1}[\mathbf{P}_{T-3}\mathbf{P}'_{T-3}]\mathbb{E}[\mathbf{P}_{T-3}]s_{T-3}y_{T-3} \\
&+ \frac{\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}}{(1-B_{T-1})s_{T-1}}\frac{\mathbb{E}[q_{T-2}] - \widehat{B}_{T-2}}{(1-B_{T-2})s_{T-2}}\mathbb{E}^{-1}[\mathbf{P}_{T-3}\mathbf{P}'_{T-3}]\mathbb{E}[q_{T-3}\mathbf{P}_{T-3}]l_{T-3},
\end{aligned}
$$

and the cost-to-go functional at period $T-3$ is

$$
\begin{aligned}
&J_{T-3}(y_{T-3}, l_{T-3}) = \min_{\pi_{T-3}}\mathbb{E}\big[J_{T-2}(y_{T-2}, l_{T-2})\big|\mathcal{F}_{T-3}\big] \\
&= (1-B_{T-1})(1-B_{T-2})(1-B_{T-3})s_{T-1}^2 s_{T-2}^2 s_{T-3}^2 y_{T-3}^2 \\
&\quad - 2\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)\big(\mathbb{E}\big[q_{T-2}] - \widehat{B}_{T-2}\big)\big(\mathbb{E}[q_{T-3}] - \widehat{B}_{T-3}\big)s_{T-1}s_{T-2}s_{T-3}l_{T-3}y_{T-3} \\
&\quad - \bigg[\frac{\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)^2}{1-B_{T-1}}\frac{\big(\mathbb{E}[q_{T-2}] - \widehat{B}_{T-2}\big)^2}{1-B_{T-2}}\widetilde{B}_{T-3} \\
&\quad + \left(\frac{\big(\mathbb{E}[q_{T-1}] - \widehat{B}_{T-1}\big)^2}{1-B_{T-1}}\widetilde{B}_{T-2} + \widetilde{B}_{T-1}\mathbb{E}[q_{T-2}^2]\right)\mathbb{E}[q_{T-3}^2]\bigg]l_{T-3}^2.
\end{aligned}
$$

Inspired by the above three stages, we conjecture that the cost-to-go functional at period t can be expressed in the following form:

$$\begin{aligned} J_t(y_t, l_t) = &\Big(\prod_{k=t}^{T-1}(1-B_k)s_k^2\Big)y_t^2 - 2\Big(\prod_{k=t}^{T-1}(\mathbb{E}[q_k]-\widehat{B}_k)s_k\Big)l_t y_t \\ &-\sum_{j=t}^{T-1}\Big(\prod_{k=j+1}^{T-1}\frac{(\mathbb{E}[q_k]-\widehat{B}_k)^2}{1-B_k}\Big)\widetilde{B}_j\Big(\prod_{m=t}^{j-1}\mathbb{E}[q_m^2]\Big)l_t^2. \end{aligned} \tag{9.11}$$

Next, we prove it in mathematical induction. Assume that the cost-to-go functional (9.11) holds at period $t+1$. Then we shall prove that it still holds at time t. For the given information set $\mathcal{F}_t$, we have

$$\begin{aligned} &\mathbb{E}\big[J_{t+1}(y_{t+1}, l_{t+1})\big|\mathcal{F}_t\big] \\ &= \mathbb{E}\Big[\Big(\prod_{k=t+1}^{T-1}(1-B_k)s_k^2\Big)y_{t+1}^2 - 2\Big(\prod_{k=t+1}^{T-1}(\mathbb{E}[q_k]-\widehat{B}_k)s_k\Big)l_{t+1}y_{t+1} \\ &\quad -\sum_{j=t+1}^{T-1}\Big(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\widetilde{B}_j\Big(\prod_{m=t+1}^{j-1}\mathbb{E}[q_m^2]\Big)l_{t+1}^2\Big|\mathcal{F}_t\Big] \\ &= \Big(\prod_{k=t+1}^{T-1}(1-B_k)s_k^2\Big)\big(s_t^2y_t^2 + 2s_ty_t\mathbb{E}[\mathbf{P}_t']\pi_t + \pi_t'\mathbb{E}[\mathbf{P}_t\mathbf{P}_t']\pi_t\big) \\ &\quad -2\Big(\prod_{k=t+1}^{T-1}\big(\mathbb{E}[q_k]-\widehat{B}_k\big)s_k\Big)\big(\mathbb{E}[q_t]s_tl_ty_t + \mathbb{E}[q_t\mathbf{P}_t']l_t\pi_t\big) \\ &\quad -\sum_{j=t+1}^{T-1}\Big(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\widetilde{B}_j\Big(\prod_{m=t+1}^{j-1}\mathbb{E}[q_m^2]\Big)\mathbb{E}[q_t^2]l_t^2. \end{aligned}$$

Minimizing the above functional with respect to π_t, we get the optimal strategy decision at time t as follows:

$$\pi_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_ty_t + \Big(\prod_{k=t+1}^{T-1}\frac{\mathbb{E}[q_k]-\widehat{B}_k}{(1-B_k)s_k}\Big)\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[q_t\mathbf{P}_t]l_t.$$

Substituting it to $\mathbb{E}\big[J_{t+1}(y_{t+1}, l_{t+1})\big|\mathcal{F}_t\big]$ yields

$$\begin{aligned} J_t(y_t, l_t) &= \min_{\pi_t}\mathbb{E}\big[J_{t+1}(y_{t+1}, l_{t+1})\big|\mathcal{F}_t\big] \\ &= \Big(\prod_{k=t+1}^{T-1}(1-B_k)s_k^2\Big)s_t^2y_t^2 - 2\Big(\prod_{k=t+1}^{T-1}\big(\mathbb{E}[q_k]-\widehat{B}_k\big)s_k\Big)\mathbb{E}[q_t]s_tl_ty_t \end{aligned}$$

$$
\begin{aligned}
&-\Bigg(\prod_{k=t+1}^{T-1}(1-B_k)s_k^2\Bigg)\mathbb{E}[\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t^2y_t^2\\
&+2\Bigg(\prod_{k=t+1}^{T-1}\big(\mathbb{E}[q_k]-\widehat{B}_k\big)s_k\Bigg)\mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_tl_ty_t\\
&-\Bigg(\prod_{k=t+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Bigg)\mathbb{E}[q_t\mathbf{P}_t']\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[q_t\mathbf{P}_t]l_t^2\\
&-\sum_{j=t+1}^{T-1}\Bigg(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Bigg)\widetilde{B}_j\Bigg(\prod_{m=t+1}^{j-1}\mathbb{E}[q_m^2]\Bigg)\mathbb{E}[q_t^2]l_t^2\\
&\quad=\Bigg(\prod_{k=t}^{T-1}(1-B_k)s_k^2\Bigg)y_t^2-2\Bigg(\prod_{k=t}^{T-1}\big(\mathbb{E}[q_k]-\widehat{B}_k\big)s_k\Bigg)l_ty_t\\
&\qquad-\sum_{j=t}^{T-1}\Bigg(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Bigg)\widetilde{B}_j\Bigg(\prod_{m=t}^{j-1}\mathbb{E}[q_m^2]\Bigg)l_t^2,
\end{aligned}
$$

which proves (9.11).

To derive the expression (9.10) of γ, we first consider the value of the optimal objective function in (9.8). In fact,

$$
\begin{aligned}
\mathbb{E}\big[y_T^2-2l_Ty_T\big]&=\mathbb{E}\big[y_T^2-2l_Ty_T\big|\mathcal{F}_0\big]=J_0(y_0,l_0)\\
&=y_0^2\prod_{k=0}^{T-1}(1-B_k)s_k^2-2l_0y_0\prod_{k=0}^{T-1}\big(\mathbb{E}[q_k]-\widehat{B}_k\big)s_k\\
&\quad-l_0^2\sum_{j=0}^{T-1}\Bigg(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Bigg)\widetilde{B}_j\Bigg(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\Bigg).
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathrm{Var}(x_T-l_T)&=\mathbb{E}[(x_T-l_T-d)^2]\\
&=\mathbb{E}[(x_T-l_T-d)^2]-2\omega(\mathbb{E}[x_T-l_T]-d)+\omega^2-\omega^2\\
&=\mathbb{E}[(x_T-l_T-d)^2-2\omega(x_T-l_T-d)+\omega^2]-\omega^2\\
&=\mathbb{E}[(x_T-l_T-d-\omega)^2]-\omega^2\\
&=\mathbb{E}[(y_T-l_T)^2]-\omega^2\\
&=\mathbb{E}[y_T^2-2l_Ty_T]+\mathbb{E}[l_T^2]-\omega^2
\end{aligned}
$$

$$= y_0^2 \prod_{k=0}^{T-1}(1-B_k)s_k^2 - 2l_0 y_0 \prod_{k=0}^{T-1}\big(\mathbb{E}[q_k] - \widehat{B}_k\big)s_k$$
$$- l_0^2 \sum_{j=0}^{T-1}\Bigg(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Bigg)\widetilde{B}_j\Bigg(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\Bigg)$$
$$+ l_0^2 \prod_{k=0}^{T-1}\mathbb{E}[q_k^2] - \omega^2.$$

Since

$$y_0 = x_0 - \gamma \prod_{k=0}^{T-1} s_k^{-1} = x_0 - (d+\omega)\prod_{k=0}^{T-1} s_k^{-1},$$

we have

$$y_0^2 \prod_{k=0}^{T-1}(1-B_k)s_k^2 = \Bigg(x_0 - (d+\omega)\prod_{k=0}^{T-1}s_k^{-1}\Bigg)^2 \prod_{k=0}^{T-1}(1-B_k)s_k^2$$
$$= \Bigg(x_0 \prod_{k=0}^{T-1} s_k - (d+\omega)\Bigg)^2 \prod_{k=0}^{T-1}(1-B_k)$$

and

$$y_0 \prod_{k=0}^{T-1}(\mathbb{E}[q_k]-\widehat{B}_k)s_k = \Bigg(x_0 - (d+\omega)\prod_{k=0}^{T-1}s_k^{-1}\Bigg)\prod_{k=0}^{T-1}\big(\mathbb{E}[q_k]-\widehat{B}_k\big)s_k$$
$$= \Bigg(x_0 \prod_{k=0}^{T-1} s_k - (d+\omega)\Bigg)\prod_{k=0}^{T-1}\big(\mathbb{E}[q_k]-\widehat{B}_k\big).$$

Hence,

$$\begin{aligned}
&\mathrm{Var}(x_T - l_T)\\
&= \Bigg(x_0\prod_{k=0}^{T-1}s_k - (d+\omega)\Bigg)^2\prod_{k=0}^{T-1}(1-B_k) - 2l_0\Bigg(x_0\prod_{k=0}^{T-1}s_k-(d+\omega)\Bigg)\prod_{k=0}^{T-1}\big(\mathbb{E}[q_k]-\widehat{B}_k\big)\\
&\quad - l_0^2\sum_{j=0}^{T-1}\Bigg(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Bigg)\widetilde{B}_j\Bigg(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\Bigg) + l_0^2\prod_{k=0}^{T-1}\mathbb{E}[q_k^2] - \omega^2\\
&= \Bigg[\prod_{k=0}^{T-1}(1-B_k) - 1\Bigg]\Bigg(\omega - \frac{\big(x_0\prod_{k=0}^{T-1}s_k - d\big)\prod_{k=0}^{T-1}(1-B_k) - l_0\prod_{k=0}^{T-1}\big(\mathbb{E}[q_k]-\widehat{B}_k\big)}{\prod_{k=0}^{T-1}(1-B_k)-1}\Bigg)^2\\
&\quad + \frac{\prod_{k=0}^{T-1}(1-B_k)}{1-\prod_{k=0}^{T-1}(1-B_k)}\Bigg(d - x_0\prod_{k=0}^{T-1}s_k + l_0\prod_{k=0}^{T-1}\frac{\mathbb{E}[q_k]-\widehat{B}_k}{1-B_k}\Bigg)^2 + l_0^2 C_0,
\end{aligned} \tag{9.12}$$

where

$$C_0 = -\prod_{k=0}^{T-1} \frac{\left(\mathbb{E}[q_k] - \widehat{B}_k\right)^2}{1 - B_k} - \sum_{j=0}^{T-1} \left(\prod_{k=j+1}^{T-1} \frac{(\mathbb{E}[q_k] - \widehat{B}_k)^2}{1 - B_k} \right) \widetilde{B}_j \left(\prod_{m=0}^{j-1} \mathbb{E}[q_m^2] \right) + \prod_{k=0}^{T-1} \mathbb{E}[q_k^2]. \tag{9.13}$$

Since $0 < B_t < 1$ for $t = 0, 1, \cdots, T-1$,

$$0 < \prod_{k=0}^{T-1} (1 - B_k) < 1.$$

This implies that the variance term $\mathrm{Var}(x_T - l_T)$ in (9.12) is concave in ω. To obtain the minimum variance $\mathrm{Var}(x_T - l_T)$ and the optimal strategy for the original portfolio selection problem (9.1), one needs to maximize the value in (9.12) over $\omega \in \mathbb{R}$ according to the Lagrange duality theorem in Luenberger (1968). Taking the first order derivative for (9.12) with respect to ω yields

$$\omega^* = \frac{\left(x_0 \prod_{k=0}^{T-1} s_k - d \right) \prod_{k=0}^{T-1} (1 - B_k) - l_0 \prod_{k=0}^{T-1} \left(\mathbb{E}[q_k] - \widehat{B}_k \right)}{\prod_{k=0}^{T-1} (1 - B_k) - 1}.$$

A simple calculation of $\gamma^* = d + \omega^*$ implies the desired result (9.10). □

9.3.2 *Efficient Frontier*

For any matrix M, we denote by M^+ the Moore-Penrose pseudoinverse of M satisfying

$$MM^+M = M, M^+MM^+ = M^+, (MM^+)' = MM^+, (M^+M)' = M^+M.$$

It can be proved that M^+ is unique for any matrix M and if the inverse M^{-1} of M exists, then $M^+ = M^{-1}$.

Let M be a square matrix partitioned as

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}. \tag{9.14}$$

Then we have

Lemma 1 *If M_{22} is invertible, then* $|M| = |M_{22}|\left|M_{11} - M_{12}M_{22}^{-1}M_{21}\right|$.

Suppose that the square matrix M is symmetrical and partitioned as (9.14), where M_{11} and M_{22} are also symmetrical square matrices, then the following two lemmas hold.

Lemma 2 *The matrix* $M \succcurlyeq 0$ *is equivalent to* $M_{22} \succcurlyeq 0, M_{22}M_{22}^{+}M_{21} = M_{21}$ *and* $M_{11} - M_{12}M_{22}^{+}M_{21} \succcurlyeq 0$, *where* $M_{21} = M_{12}'$.

Lemma 3 *If* $M \succcurlyeq N \succcurlyeq 0$, *then* $|M| \geq |N|$.

The proof of Lemmas 1 and 3 can be found in Zhang (2011). And the proof of Lemma 2 can be found in Albert (1969).

Before we analyze the efficient frontier, we prove the following important result.

Lemma 4 *If* $\mathbb{E}\left[\begin{pmatrix}\mathbf{P}_k \\ q_k\end{pmatrix}\begin{pmatrix}\mathbf{P}_k' & q_k\end{pmatrix}\right]$ *is positive definite for* $k = 0, 1, \cdots, T-1$, *then*

$$C_0 \geq 0. \tag{9.15}$$

Proof Let $L_k = \begin{pmatrix}\mathbf{P}_k \\ 1\end{pmatrix}$ and $Q_k = \begin{pmatrix}\mathbf{P}_k \\ q_k\end{pmatrix}$, then

$$\begin{pmatrix}\mathbb{E}[\mathbf{P}_k\mathbf{P}_k'] & \mathbb{E}[\mathbf{P}_k] \\ \mathbb{E}[\mathbf{P}_k'] & 1\end{pmatrix} = \mathbb{E}\left[\begin{pmatrix}\mathbf{P}_k \\ 1\end{pmatrix}\begin{pmatrix}\mathbf{P}_k' & 1\end{pmatrix}\right] = \mathbb{E}[L_kL_k'], \tag{9.16}$$

$$\begin{pmatrix}\mathbb{E}[\mathbf{P}_k\mathbf{P}_k'] & \mathbb{E}[q_k\mathbf{P}_k] \\ \mathbb{E}[q_k\mathbf{P}_k'] & \mathbb{E}[q_k^2]\end{pmatrix} = \mathbb{E}\left[\begin{pmatrix}\mathbf{P}_k \\ q_k\end{pmatrix}\begin{pmatrix}\mathbf{P}_k' & q_k\end{pmatrix}\right] = \mathbb{E}[Q_kQ_k'], \tag{9.17}$$

$$\begin{pmatrix}\mathbb{E}[\mathbf{P}_k\mathbf{P}_k'] & \mathbb{E}[\mathbf{P}_k] \\ \mathbb{E}[q_k\mathbf{P}_k'] & \mathbb{E}[q_k]\end{pmatrix} = \mathbb{E}\left[\begin{pmatrix}\mathbf{P}_k \\ q_k\end{pmatrix}\begin{pmatrix}\mathbf{P}_k' & 1\end{pmatrix}\right] = \mathbb{E}[Q_kL_k']. \tag{9.18}$$

Taking determinant on both sides for (9.16)–(9.18) and according to Lemma 1, we get

$$\begin{vmatrix}\mathbb{E}[\mathbf{P}_k\mathbf{P}_k'] & \mathbb{E}[\mathbf{P}_k] \\ \mathbb{E}[\mathbf{P}_k'] & 1\end{vmatrix} = \left(1 - \mathbb{E}[\mathbf{P}_k']\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}_k']\mathbb{E}[\mathbf{P}_k]\right)\left|\mathbb{E}[\mathbf{P}_k\mathbf{P}_k']\right| = \left|\mathbb{E}[L_kL_k']\right|, \tag{9.19}$$

$$\begin{vmatrix}\mathbb{E}[\mathbf{P}_k\mathbf{P}_k'] & \mathbb{E}[q_k\mathbf{P}_k] \\ \mathbb{E}[q_k\mathbf{P}_k'] & \mathbb{E}[q_k^2]\end{vmatrix} = \left(\mathbb{E}[q_k^2] - \mathbb{E}[q_k\mathbf{P}_k']\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}_k']\mathbb{E}[q_k\mathbf{P}_k]\right)\left|\mathbb{E}[\mathbf{P}_k\mathbf{P}_k']\right| = \left|\mathbb{E}[Q_kQ_k']\right|, \tag{9.20}$$

$$\begin{vmatrix}\mathbb{E}[\mathbf{P}_k\mathbf{P}_k'] & \mathbb{E}[\mathbf{P}_k] \\ \mathbb{E}[q_k\mathbf{P}_k'] & \mathbb{E}[q_k]\end{vmatrix} = \left(\mathbb{E}[q_k] - \mathbb{E}[q_k\mathbf{P}_k']\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}_k']\mathbb{E}[\mathbf{P}_k]\right)\left|\mathbb{E}[\mathbf{P}_k\mathbf{P}_k']\right| = \left|\mathbb{E}[Q_kL_k']\right|. \tag{9.21}$$

By the assumption of $\mathbb{E}[Q_kQ_k'] \succ 0$, the inverse $\mathbb{E}^{-1}[Q_kQ_k']$ of $\mathbb{E}[Q_kQ_k']$ exists. Then $\mathbb{E}^{+}[Q_kQ_k'] = \mathbb{E}^{-1}[Q_kQ_k']$. Since

$$\mathbb{E}\left[\begin{pmatrix} L_k \\ Q_k \end{pmatrix} \begin{pmatrix} L_k' & Q_k' \end{pmatrix}\right] = \begin{pmatrix} \mathbb{E}[L_kL_k'] & \mathbb{E}[L_kQ_k'] \\ \mathbb{E}[Q_kL_k'] & \mathbb{E}[Q_kQ_k'] \end{pmatrix} \succcurlyeq 0, \tag{9.22}$$

it follows from Lemma 2 that

$$\mathbb{E}[L_kL_k'] - \mathbb{E}[L_kQ_k']\mathbb{E}^{-1}[Q_kQ_k']\mathbb{E}[Q_kL_k'] \succcurlyeq 0.$$

Obviously,

$$\mathbb{E}[L_kQ_k']\mathbb{E}[Q_kQ_k']^{-1}\mathbb{E}[Q_kL_k'] = \mathbb{E}[L_kQ_k']\mathbb{E}^{-1}[Q_kQ_k']\left(\mathbb{E}[L_kQ_k']\right)' \succcurlyeq 0.$$

Consequently,

$$\mathbb{E}[L_kL_k'] \succcurlyeq \mathbb{E}[L_kQ_k']\mathbb{E}^{-1}[Q_kQ_k']\mathbb{E}[Q_kL_k']. \tag{9.23}$$

Then according to (9.23) and Lemma 3, it follows that

$$\left|\mathbb{E}[L_kL_k']\right| \geq \left|\mathbb{E}[L_kQ_k']\mathbb{E}^{-1}[Q_kQ_k']\mathbb{E}[Q_kL_k']\right| = \left|\mathbb{E}[L_kQ_k']\right| \left|\mathbb{E}^{-1}[Q_kQ_k']\right| \left|\mathbb{E}[Q_kL_k']\right|. \tag{9.24}$$

Notice that $\left|\mathbb{E}[Q_kL_k']\right| = \left|\mathbb{E}[L_kQ_k']\right|$ and $\left|\mathbb{E}^{-1}[Q_kQ_k']\right| = \left|\mathbb{E}[Q_kQ_k']\right|^{-1}$, then (9.24) implies

$$\left|\mathbb{E}[Q_kL_k']\right|^2 \leq \left|\mathbb{E}[Q_kQ_k']\right| \left|\mathbb{E}[L_kL_k']\right|. \tag{9.25}$$

By (9.19)–(9.21) and (9.25), we obtain

$$\begin{aligned}&\left(1 - \mathbb{E}[\mathbf{P}_k']\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}_k']\mathbb{E}[\mathbf{P}_k]\right)\left(\mathbb{E}[q_k^2] - \mathbb{E}[q_k\mathbf{P}_k']\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}_k']\mathbb{E}[q_k\mathbf{P}_k]\right)\\ &\geq \left(\mathbb{E}[q_k] - \mathbb{E}[q_k\mathbf{P}_k']\mathbb{E}^{-1}[\mathbf{P}_k\mathbf{P}_k']\mathbb{E}[\mathbf{P}_k]\right)^2.\end{aligned}$$

Namely,

$$\left(\mathbb{E}[q_k] - \widehat{B}_k\right)^2 \leq \left(\mathbb{E}[q_k^2] - \widetilde{B}_k\right)(1 - B_k).$$

Then

$$\widetilde{B}_k \leq \mathbb{E}[q_k^2] - \frac{\left(\mathbb{E}[q_k] - \widehat{B}_k\right)^2}{1 - B_k}.$$

Therefore,

$$
\begin{aligned}
&\sum_{j=0}^{T-1}\Big(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\widetilde{B}_j\Big(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\Big)\\
&\leq \sum_{j=0}^{T-1}\Big(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\Big(\mathbb{E}[q_j^2]-\frac{(\mathbb{E}[q_j]-\widehat{B}_j)^2}{1-B_j}\Big)\Big(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\Big)\\
&= \sum_{j=0}^{T-1}\Big(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\mathbb{E}[q_j^2]\Big(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\Big)\\
&\quad -\sum_{j=0}^{T-1}\Big(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\frac{(\mathbb{E}[q_j]-\widehat{B}_j)^2}{1-B_j}\Big(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\Big)\\
&= \sum_{j=0}^{T-1}\Big(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\Big(\prod_{m=0}^{j}\mathbb{E}[q_m^2]\Big)-\sum_{j=0}^{T-1}\Big(\prod_{k=j}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\Big(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\Big)\\
&= \Big(\prod_{k=T}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\Big(\prod_{m=0}^{T-1}\mathbb{E}[q_m^2]\Big)-\Big(\prod_{k=0}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\Big(\prod_{m=0}^{-1}\mathbb{E}[q_m^2]\Big)\\
&= \Big(\prod_{m=0}^{T-1}\mathbb{E}[q_m^2]\Big)-\Big(\prod_{k=0}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\\
&= \Big(\prod_{k=0}^{T-1}\mathbb{E}[q_k^2]\Big)-\Big(\prod_{k=0}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big).
\end{aligned}
$$

As a result, it follows from the above inequality that

$$
\begin{aligned}
C_0 = &-\prod_{k=0}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}-\sum_{j=0}^{T-1}\Big(\prod_{k=j+1}^{T-1}\frac{\big(\mathbb{E}[q_k]-\widehat{B}_k\big)^2}{1-B_k}\Big)\widetilde{B}_j\Big(\prod_{m=0}^{j-1}\mathbb{E}[q_m^2]\Big)\\
&+\prod_{k=0}^{T-1}\mathbb{E}[q_k^2]\geq 0.
\end{aligned}
$$

This completes the proof of Lemma 4. □

It follows from Eq. (9.12) with ω^* that we have the following minimum variance theorem.

Theorem 2 *Assume that the returns of assets and liability are correlated at every period. Then the efficient frontier is given by*

$$
Var(x_T-l_T)=\frac{\prod_{k=0}^{T-1}(1-B_k)}{1-\prod_{k=0}^{T-1}(1-B_k)}\Big(d-x_0\prod_{k=0}^{T-1}s_k+l_0\prod_{k=0}^{T-1}\frac{\mathbb{E}[q_k]-\widehat{B}_k}{1-B_k}\Big)^2+l_0^2C_0.
$$

Setting the expected terminal surplus $d = x_0 \prod_{k=0}^{T-1} s_k - l_0 \prod_{k=0}^{T-1} \mathbb{E}[q_k]$, we obtain the global minimum variance as

$$\mathrm{Var}_{\min}(x_T - l_T) := C_0 {l_0}^2. \tag{9.26}$$

By Lemma 4, it follows that the global minimum variance $\mathrm{Var}_{\min}(x_T - l_T) \geq 0$.

9.3.3 *The Optimal Strategy with Uncorrelation of Assets and Liability*

Assume that the returns of asset and liability are uncorrelated at every period. Then

$$\widehat{B}_t = \mathbb{E}[q_t]B_t \quad \text{and} \quad \widetilde{B}_t = (\mathbb{E}[q_t])^2 B_t.$$

Hence, we have the following results:

$$\begin{aligned}
\prod_{k=t}^{T-1} \frac{\mathbb{E}[q_k] - \widehat{B}_k}{(1-B_k)s_k} &= \prod_{k=t}^{T-1} \mathbb{E}[q_k] s_k^{-1}, \\
\prod_{k=t}^{T-1} \big(\mathbb{E}[q_k] - \widehat{B}_k\big) &= \prod_{k=t}^{T-1} \mathbb{E}[q_k]\big(1 - B_k\big), \\
\prod_{k=t}^{T-1} \frac{\mathbb{E}[q_k] - \widehat{B}_k}{1-B_k} &= \prod_{k=t}^{T-1} \mathbb{E}[q_k], \\
\prod_{k=t}^{T-1} \frac{\big(\mathbb{E}[q_k] - \widehat{B}_k\big)^2}{1-B_k} &= \prod_{k=t}^{T-1} \big(\mathbb{E}[q_k]\big)^2 (1 - B_k)
\end{aligned}$$

and

$$\begin{aligned}
C_0 = &- \prod_{k=0}^{T-1} \big(\mathbb{E}[q_k]\big)^2 \big(1 - B_k\big) - l_0^2 \sum_{j=0}^{T-1} \left(\prod_{k=j+1}^{T-1} \big(\mathbb{E}[q_k]\big)^2 \big(1 - B_k\big) \right) \big(\mathbb{E}[q_j]\big)^2 \\
&\times B_j \left(\prod_{m=0}^{j-1} \mathbb{E}[q_m^2] \right) + \prod_{k=0}^{T-1} \mathbb{E}[q_k^2].
\end{aligned}$$

Therefore, we have the following two theorems.

Theorem 3 *Assume that the returns of assets and liability are uncorrelated at every period. Then the optimal strategy of problem (9.1) is given by*

$$\pi_t^* = -\mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t]s_t\left(x_t - \gamma^* \prod_{k=t}^{T-1} s_k^{-1} - l_t\prod_{k=t}^{T-1}\mathbb{E}[q_k]s_k^{-1}\right), \tag{9.27}$$

where

$$\gamma^* = \frac{x_0\prod_{k=0}^{T-1}(1-B_k)s_k - d - l_0\prod_{k=0}^{T-1}\mathbb{E}[q_k]\big(1-B_k\big)}{\prod_{k=0}^{T-1}(1-B_k) - 1}. \tag{9.28}$$

Theorem 4 *Assume that the returns of assets and liability are uncorrelated at every period. Then the efficient frontier is given by*

$$Var(x_T - l_T) = \frac{\prod_{k=0}^{T-1}(1-B_k)}{1-\prod_{k=0}^{T-1}(1-B_k)}\left(d - x_0\prod_{k=0}^{T-1} s_k + l_0\prod_{k=0}^{T-1}\mathbb{E}[q_k]\right)^2 + l_0^2C_0.$$

9.4 Numerical Examples

We consider an example of constructing a pension fund consisting of S&P 500 (SP), the index of Emerging Market (EM), Small Stock (MS) of the US market, and a bank account. Based on the data provided in Elton et al. (2007), Table 9.1 presents the expected values, variances, and correlation coefficients of the annual return rates of these indices.

Table 9.1 Data for the asset allocation example

	SP	EM (%)	MS (%)	Liability (%)
Expected return	14	16	17	10
Standard deviation	18.5	30	24	20
Correlation coefficient				
SP	1	0.64	0.79	ρ_1
EM	0.64	1	0.75	ρ_2
MS	0.79	0.75	1	ρ_3
Liability	ρ_1	ρ_2	ρ_3	1

Thus, for any time t, we have

$$\mathbb{E}[\mathbf{P}_t] = \begin{pmatrix} 0.09 \\ 0.11 \\ 0.12 \end{pmatrix}, \quad \mathrm{Cov}(\mathbf{P}_t) = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 \\ 0.0355 & 0.0900 & 0.0540 \\ 0.0351 & 0.0540 & 0.0576 \end{pmatrix},$$

$$\mathbb{E}[\mathbf{P}_t\mathbf{P}_t'] = \begin{pmatrix} 0.0423 & 0.0454 & 0.0459 \\ 0.0454 & 0.1021 & 0.0672 \\ 0.0459 & 0.0672 & 0.0720 \end{pmatrix}.$$

We consider five time periods and an annual risk free rate 5% ($s_t = 1.05$). Assume that the investor has an initial wealth $x_0 = 3$ and an initial liability $l_0 = 1$. Furthermore, for $t = 0, 1, 2, 3, 4$, the correlation of assets and the liability is $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3)$, where

$$\rho_i = \frac{\mathrm{Cov}(q_t, P_t^i)}{\sqrt{\mathrm{Var}(q_t)}\sqrt{\mathrm{Var}(P_t^i)}}$$

is the correlation coefficient of the ith asset and the liability. This means

$$\mathbb{E}[q_t P_t^i] = \mathbb{E}[q_t]\mathbb{E}[P_t^i] + \rho_i \sqrt{\mathrm{Var}(q_t)}\sqrt{\mathrm{Var}(P_t^i)}.$$

9.4.1 Correlation

In this subsection, assume that the returns of the assets and liability are correlated with $\boldsymbol{\rho} = (\rho_1, \rho_2, \rho_3) = (-0.25, 0.5, 0.25)$. Hence,

$$\mathrm{Cov}\left(\begin{pmatrix} \mathbf{P}_t \\ q_t \end{pmatrix}\right) = \begin{pmatrix} \mathrm{Cov}(\mathbf{P}_t) & \mathrm{Cov}(q_t, \mathbf{P}_t) \\ \mathrm{Cov}(q_t, \mathbf{P}_t') & \mathrm{Var}(q_t) \end{pmatrix} = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 & -0.0092 \\ 0.0355 & 0.0900 & 0.0540 & 0.0300 \\ 0.0351 & 0.0540 & 0.0576 & 0.0120 \\ -0.0092 & 0.0300 & 0.0120 & 0.0400 \end{pmatrix} \succ 0.$$

Using the above formula of $\mathbb{E}[q_t P_t^i]$, we have $\mathbb{E}[q_t \mathbf{P}_t] = (0.0898, 0.1510, 0.1440)'$. We seek for the expected terminal target with $d = 3.5$. According to Theorem 1, we can derive $\gamma^* = 4.0470$ and the optimal strategy of problem (9.1) is specified as follows:

$$\pi_0^* = -1.05(x_0 - 3.1710)\mathbf{K}_1 + 1.2053\mathbf{K}_2 l_0,$$
$$\pi_1^* = -1.05(x_1 - 3.3295)\mathbf{K}_1 + 1.1503\mathbf{K}_2 l_1,$$
$$\pi_2^* = -1.05(x_2 - 3.4960)\mathbf{K}_1 + 1.0979\mathbf{K}_2 l_2,$$
$$\pi_3^* = -1.05(x_3 - 3.6708)\mathbf{K}_1 + 1.0478\mathbf{K}_2 l_3,$$
$$\pi_4^* = -1.05(x_4 - 3.8543)\mathbf{K}_1 + 1.0000\mathbf{K}_2 l_4,$$

where

$$\mathbf{K}_1 = \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[\mathbf{P}_t] = \begin{bmatrix} 1.0580 \\ -0.1207 \\ 1.1052 \end{bmatrix}, \quad \mathbf{K}_2 = \mathbb{E}^{-1}[\mathbf{P}_t\mathbf{P}_t']\mathbb{E}[q_t\mathbf{P}_t] = \begin{bmatrix} -0.2398 \\ 0.4374 \\ 1.7446 \end{bmatrix}.$$

The variance of the final optimal surplus is $\mathrm{Var}(x_5 - l_5) = 0.7289$.

9.4.2 Uncorrelation

In this subsection, assume that the returns of the assets and liability are uncorrelated. Hence,

$$\mathrm{Cov}\left(\begin{pmatrix} \mathbf{P}_t \\ q_t \end{pmatrix}\right) = \begin{pmatrix} \mathrm{Cov}(\mathbf{P}_t) & \mathrm{Cov}(q_t, \mathbf{P}_t) \\ \mathrm{Cov}(q_t, \mathbf{P}_t') & \mathrm{Var}(q_t) \end{pmatrix} = \begin{pmatrix} 0.0342 & 0.0355 & 0.0351 & 0 \\ 0.0355 & 0.0900 & 0.0540 & 0 \\ 0.0351 & 0.0540 & 0.0576 & 0 \\ 0 & 0 & 0 & 0.04 \end{pmatrix} \succ 0.$$

We still seek to attain the same expected terminal target with $d = 3.5$. According to Theorem 3, we can derive $\gamma^* = 4.0464$ and the optimal strategy of problem (9.1) is specified as follows:

$$\pi_0^* = -1.05(x_0 - 3.1705 + 1.1472 l_0)\mathbf{K}_1,$$
$$\pi_1^* = -1.05(x_1 - 3.3290 + 1.0950 l_1)\mathbf{K}_1,$$
$$\pi_2^* = -1.05(x_2 - 3.4955 + 1.0452 l_2)\mathbf{K}_1,$$
$$\pi_3^* = -1.05(x_3 - 3.6702 + 0.9977 l_3)\mathbf{K}_1,$$
$$\pi_4^* = -1.05(x_4 - 3.8538 + 0.9524 l_4)\mathbf{K}_1,$$

where $\mathbf{K}_1$ is the same as in Sect. 9.4.1, and the variance of the final optimal surplus is $\mathrm{Var}(x_5 - l_5) = 1.0043$.

9.5 Conclusion

Using the parameterized method, the state variable transformation technique, and the dynamic programming approach, we obtain in this paper the closed-form expressions for the optimal investment strategy and the efficient frontier of our multi-period mean-variance asset-liability management problem. Compared with previous studies in the literature, our method is simpler yet more efficient, and the result is more concise and powerful since we do not need to solve an auxiliary problem and investigating the relationship of the auxiliary problem and the original one. Our method is hence especially useful in the big data era. In the future, we will try to use the parameterized method to solve the portfolio selection problem when the returns are correlated in every period, with probability constraint, with uncertain exit time and with Markov jumps.

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