A New Delay-Independent Stability Test for LTI Systems with Single Delay

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Abstract A new method complying necessary and sufficient conditions to test delay-independent stability of the general linear time invariant (LTI) dynamics with single delay is presented. The method is based on investigating the location of zeros of an auxiliary characteristic polynomial obtained via Kronecker summation. The proposed approach enables to determine the exact regions of the unknown parameters, e.g., system and controller parameters, ensuring delay-independent stability.

1 Introduction

Stability and control design of time-delay systems are widely studied due to the effect of delay phenomena on system dynamics [1, 2]. The challenge in handling the characteristics of time-delayed systems arise from their nature of being infinite dimensional and yielding nonlinear eigenvalue problems. This paper deals with delay-independent stability (DIS), formally defined in [3] first, which indicates the stability of the system for all nonnegative values of delays whereas delay-dependent stability specifies the stability for a nonnegative subset of delays. Our main goal is to test DIS of a single delay LTI system and also to determine the region of system and/or controller parameters stabilizing the system independent of delay, in other words regardless of the delay value. This class of controllers is particularly important with the systems where the delayed feedback may cause critical and dangerous instabilities such as motion control, tank level control, high temperature furnace control. In these applications, controller has to drive the system to a stable operating point regardless of the delay where delay may occur due to a malfunction in the sensory system.

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The research on stability and stabilization of time-delay systems can be classified according to the approach for the solution of the problem roughly speaking. One of the main approaches is the time domain approach based on Lyapunov–Razumikhin and Lyapunov–Krasovskii functionals [4, 5]. Another one is the frequency domain approach including frequency sweeping and matrix-pencil techniques, e.g., [1, 6–8] considering both delay-independent and dependent stability. Frequency domain techniques are mainly motivated by the characteristic root crossings on the imaginary axis where the delay-free system is stable. Practical algebraic tools in frequency domain to test DIS are being studied in recent years. [9] concerns DIS of single delay systems and investigates the roots of an improved rational function. [10] is presenting a new DIS test for multiple delay systems based on resultant theory and discriminant polynomials associated with infinite dimensional characteristic polynomial. Both papers are utilizing the features of Rekasius Transformation [11] and are useful in real arithmetics, however, are not convenient for literal parameters. Furthermore there are some algebraic methods to determine controller parameters guaranteeing DIS. A conservative method based on resultant theory and Descartes rule of signs for determining the boundaries of delay-independent stabilizing parameters is presented in [12] for multiple time-delay systems. A nonconservative method for single delay case is presented in [13] also based on resultant theory. Then the method is extended to multiple delay case using Sturm Sequences [14], however, it is required to test a set of real values in unknown parameter space in each region to determine if the region is delay-independent stable or not in both methods. This paper can be considered as a continuous work of the previous studies [15, 16] presenting a conservative methodology for multiple delay case. Although this paper is for single delay case, the approach is convenient to be extended to multiple delay case.

In this study, we consider general time-delay systems of retarded type with single uncertain constant delay. A nonconservative method to determine analytical boundaries of delay-independent stabilizing unknown parameters is improved. The method is based on the results of [17] utilizing Kronecker Summation and obtained auxiliary characteristic polynomial (*ACP*) which is self-inversive to explore the imaginary axis crossing roots of the characteristic equation of the system. The infinite dimensional delayed system is represented in terms of a finite dimensional polynomial with interspersed zeros among the unit circle. The problem of determining the region of unknown parameters for DIS is transformed into assigning a certain number of the zeros of the derivative of *ACP* inside the unit circle. For this purpose, an efficient zero location test with respect to the unit circle [18, 19] is utilized. Since the test is complying to necessary and sufficient conditions, the proposed method yields the exact boundaries of region of unknown parameter space for DIS. The key novelty of this method is that the method is not conservative, is practically implementable for literal parameters, and does not require parameter sweeping. Also there are no restrictions on the order of the system and the number of the parametric uncertainties.

The paper is organized as follows: In Sect. 2 preliminary definitions and statements of the study are given. Section 3 presents the main results for delay-independent stability for LTI system with single delay. Section 4 contains example case studies. In the last section, conclusive remarks about the methodology are given.

2 Preliminaries

In the text, we use boldface notation for vector and matrix quantities. $\mathbb{R}$, $\mathbb{R}_+$ represent the set of real numbers and the set of nonnegative real numbers, respectively. Open left half, open right half, and the entire complex plane are represented as $\mathbb{C}_-$, $\mathbb{C}_+$, $\mathbb{C}$, respectively. Open unit disc, unit circle, and outside of unit circle are referred as $\mathbb{D}$, $\mathbb{T}$, and $\mathbb{S}$, respectively. Also, note that the "over-line" is used to represent being closed for the planes, e.g., $\overline{\mathbb{C}}_+$ for the closed right half complex plane.

2.1 Kronecker Sum and Delay-Independent Stability

We consider LTI single time-delay systems of retarded type. The general state space form is given as,

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{q})\mathbf{x}(t) + \mathbf{B}(\mathbf{q})\mathbf{x}(t-\tau) \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{A}(\mathbf{q}) \in \mathbb{R}^{n\times n}$, $\mathbf{B}(\mathbf{q}) \in \mathbb{R}^{n\times n}$, $\mathbf{q} \in \mathbb{R}^r$ is a vector of unknown parameters and τ is the time delay.

The characteristic equation of the system (1) is derived as

$$CE(s,\tau,\mathbf{q}) = \det\left[s\mathbf{I} - \mathbf{A}(\mathbf{q}) - \mathbf{B}(\mathbf{q})e^{-\tau s}\right] = a_0(s,\mathbf{q}) + \sum_{k=1}^{n} a_k(s,\mathbf{q})e^{-k\tau s} = 0 \tag{2}$$

where $a_0(s,\mathbf{q})$ is a polynomial of n^{th} degree in s and $a_k(s,\mathbf{q})$ are polynomials of degree lower than n. In characteristic quasipolynomial $CE(s,\mathbf{q},\tau)$, there is no transcendental term ($e^{-k\tau s}$) multiplied by s^n because the delay system is retarded type. There are infinite number of characteristic roots since (2) is a transcendental equation. It is obvious that all the roots of characteristic equation (2) must lie on $\mathbb{C}_-$ for asymptotic stability for certain values of τ and $\mathbf{q}$. In other words, if no roots of (2) exist in $\overline{\mathbb{C}}_+$, the system is stable.

The frequency domain approach focuses on the characteristic roots that cross the imaginary axis rather than determining the location of infinitely many roots of (2) to determine stability. It has to be noted that this approach is valid when the delay-free system ($\tau = 0$) is stable. In [17], the problem of examining imaginary axis crossing of infinitely many characteristic roots is transformed into determination of the root locations of the auxiliary characteristic equation of which represents the system at the stability switching points for multiple delay systems. The Extended Kronecker Summation method is used to convert the infinite imaginary axis crossing problem into finite unit circle crossing as in the following theorem. The theorem is given for single delay case in the interest of this study for simplicity.

Theorem 1 *Let the Auxiliary Characteristic Equation (ACE) of the system (1), with* $z = e^{-\tau s}$ *be as follows:*

$$ACE(z, \mathbf{q}) = \det \begin{bmatrix} \mathbf{A}(\mathbf{q}) \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}(\mathbf{q}) + \\ (\mathbf{B}(\mathbf{q}) \otimes \mathbf{I})\, z + (\mathbf{I} \otimes \mathbf{B}(\mathbf{q}))\, z^{-1} \end{bmatrix} = 0 \tag{3}$$

where $\otimes$ *denotes the Kronecker Product. Then, for the system (1) for a certain value set of* $\mathbf{q}$*, i.e.,* $\mathbf{q}^*$*, the following findings are equivalent:*

(i) A unitary complex number $z \in \mathbb{T}$ *satisfies ACE given in (3).*
(ii) There exists at least one pair of imaginary characteristic roots, $\pm\omega_c i$*, of (2).*
(iii) There exists a corresponding delay $\tau \in \mathbb{R}_+$ *which satisfies* $|z| = |e^{\pm\tau\omega_c i}| = 1$.

Proof Proof of the theorem is given in [17].

Theorem 1 states that one pair of imaginary characteristic roots, $\pm\omega_c i$, of (2) for a certain value of τ and $\mathbf{q}^*$ correspond to a unitary root of (3) necessarily and sufficiently.

DIS of a time-delay system represents that the delay system is stable regardless of the delay. For a system (1) ensuring DIS, none of the roots of the characteristic equation (2) must be on $\overline{\mathbb{C}}_+$ for all nonnegative values of the delay. Formal definition can be given as follows:

Definition 1 The system in (1) is said to be stable if

$$CE(s, \tau) \neq 0, \quad \forall s \in \overline{\mathbb{C}}_+ \tag{4}$$

where $CE(s, \tau)$ is given in (2). It is said that the system is delay-independent stable (DIS) if (4) holds for all $\tau \geq 0$.

Then the following theorem can be written with the perspective of Theorem 1 and Definition 1.

Theorem 2 *An LTI system (1) is delay-independent stable for a certain parameter set* $\mathbf{q}^*$ *if and only if the following conditions are satisfied simultaneously:*

(i) $\Re\left(\lambda_i\left[\mathbf{A}(\mathbf{q}^*) + \mathbf{B}(\mathbf{q}^*)\right]\right) < 0,\ i = 1, \ldots, n$
(ii) $Z = \{z \in \mathbb{T} \mid ACE(z) = 0\} = \emptyset.$

Proof In the first condition, the stability of the non-delayed system is guaranteed for the certain set of unknown parameters $\mathbf{q}^*$ by declaring the matrix $\mathbf{A}(\mathbf{q}^*) + \mathbf{B}(\mathbf{q}^*)$ is Hurwitz. In the second one, stating ACE has no roots on $\mathbb{T}$ assures that characteristic equation (2) has no $\pm i\omega_c$ roots on the imaginary axis for $\forall\tau \in \mathbb{R}_+$. Thus, the system is delay-independent stable and this condition is the result of root continuity property.

2.2 Bistritz Tabulation Method

In the literature, there are several methods to determine the location of zeros with respect to the unit circle for a given polynomial based on Schur–Chon matrices and Jury–Marden tables [20]. In this study, we present and utilize a different approach, Bistritz Tabulation, to determine the location of zeros with respect to the unit circle. The method is computationally efficient and easily implementable for unknown parameters. After giving the necessary definitions, the algorithm, and the theorems for the method are represented in this subsection. Then, the method is utilized on an auxiliary characteristic polynomial in the main results section.

Definition 2 Consider the polynomial with complex coefficients

$$P_n(z) = \sum_{i=0}^{n} d_i z^i \ . \tag{5}$$

Then the reciprocal of $P_n(z)$ is

$$P_n^{\#}(z) = \sum_{i=0}^{n} \overline{d}_{n-i} z^i = z^n \overline{P_n(1/\overline{z})} \tag{6}$$

where $\bar{(\cdot)}$ denotes the complex conjugate.

Definition 3 If $P_n(z) = P_n^{\#}(z)$ then the polynomial is called as *symmetric* or *self-inversive*.

Definition 4 $P_n(z)$ in (5) is called *normal* if $d_n \neq 0$. Otherwise it is called *abnormal*. In other words, being *normal* is the equivalence of formal degree (n) and the exact degree of the polynomial.

Definition 5 The *deficiency parameter*, λ_k, is the difference between the formal (i.e., expected) degree and the exact degree of a polynomial $P_k(z)$ where k denotes the degree of the polynomial. $P_k(z)$ is normal if $\lambda_k = 0$ and abnormal if $\lambda_k > 0$.

Bistritz Tabulation is a tabular method to determine the number of the zeros of a polynomial inside, on, and outside the unit circle ($\mathbb{D}, \mathbb{T}, \mathbb{S}$). It is presented in [18] for polynomials with real coefficients and then the study is extended for polynomials with complex coefficients in [21]. It is a Routh like tabulation method based on a three-term recursion of symmetric polynomials and the number of sign variations of these polynomials at $z = 1$. There are two kinds of singularities in the early mentioned papers. The method is improved to overcome one of the singularity types and a more compact form is given in [19].
For a polynomial $P_n(z)$ defined in (5), such that $P_n(1) \neq 0 \in \mathbb{R}$ and $d_n \neq 0$, the *regular* recursion algorithm [19] is as follows:

$$T_n(z) = \sum_{i=0}^{n} t_{ni} z^i = P_n(z) + P_n^{\#}(z) \tag{7}$$

$$T_{n-1}(z) = \sum_{i=0}^{n-1} t_{(n-1)i} z^i = \frac{P_n(z) - P_n^{\#}(z)}{z-1} \tag{8}$$

For $k = n-1, \dots, 0$

$$\delta_{k+1} = \begin{cases} \frac{t_{(k+1)0}}{t_{k\lambda_k}}, & \text{if } T_k(z) \not\equiv 0 \\ 0, & \text{if } t_{(k+1)0} = 0 \\ \text{not required}, & \text{if } t_{(k+1)0} \neq 0 \ \ \& \ \ T_k(z) \equiv 0 \end{cases}$$

$$T_{k-1}(z) = z^{-1}\left[\left(\delta_{k+1} z^{\lambda_k} + \overline{\delta}_{k+1} z^{\lambda_k+1}\right) T_k(z) - T_{k+1}(z)\right]. \tag{9}$$

The following theorem is for counting the number of zeros inside and outside the unit circle in regular (i.e., nonsingular) case.

Theorem 3 (Zero Location for Nonsingular Case) *Consider $P_n(z)$ with the assumptions $P_n(1) \neq 0 \in \mathbb{R}$ and $d_n \neq 0$. Assume that the procedure is regular. Then,*

(i) number of zeros in $\mathbb{D}$: $\alpha_n = n - \nu_n$,
(ii) number of zeros in $\mathbb{S}$: $\gamma_n = \nu_n$

where $\nu_n = Var\left\{\sigma_n, \sigma_{n-1}, \dots, \sigma_0\right\}$ such that $\sigma_k := T_k(1)$ and $Var\{\cdot\}$ denotes number of sign variations.

The only singularity situation which interrupts the regular recursion occurs if and only if a normal polynomial $T_\eta(z)$ ($\lambda_\eta = 0$) is followed by an identically zero polynomial, i.e., $T_{\eta-1}(z) \equiv 0$, in the given recursion algorithm. This occurs when a polynomial has unit circle and/or reciprocal zeros, i.e., conjugate pairs of zeros symmetrical to unit circle. Note that *ACE* in (3) has such roots. One can find the method to overcome singularities for proceeding the algorithm in [19]. We prefer not giving details since the singularity is handled for the polynomial of which the zeros are identical to the roots of *ACE* by means of a theorem. On the other hand, it is useful to present the following theorem to understand the effect of singularities on the zero distribution of the polynomial with respect to $\mathbb{T}$.

Theorem 4 (Zero Location for Singular Case) *Assume the algorithm in (7), (8), (9) is applied to a polynomial $P_n(z)$ in (5), such that $P_n(1) \neq 0 \in \mathbb{R}$ and $d_n \neq 0$, and a sequence of polynomials $\left\{T_k(z); k = n, \dots, 0\right\}$ is produced after handling the singularities. Let "η" denote the degree after which a singularity occurred for the first time (with $\eta = 0$ denoting a nonsingular case). Then,*

(i) number of zeros in $\mathbb{D}$: $\alpha_n = n - \nu_n$,

(ii) *number of zeros on* $\mathbb{T}$*:* $\beta_n = 2\nu_\eta - \eta$,
(iii) *number of zeros in* $\mathbb{S}$*:* $\gamma_n = n - \alpha_n - \beta_n$

where $\nu_n = Var\left\{\sigma_n, \sigma_{n-1}, \ldots, \sigma_0\right\}$ *and* $\nu_\eta = Var\left\{\sigma_\eta, \sigma_{\eta-1}, \ldots, \sigma_0\right\}$ *such that* $\sigma_k := T_k(1)$. *Also, there are* $\eta - \nu_\eta$ *pairs of reciprocal zeros.*

Proof The detailed proofs of Theorems 3 and 4 can be found in [19].

One can utilize Theorem 4 to show none of the roots of the *ACE* is on the unit circle for DIS as stated in Theorem 2. However, firstly the singularity occurred in the *ACE* has to be treated. We prefer to present a corollary for DIS in the next section which removes the necessity of handling the singularity in the first step of recursion algorithm in Bistritz Tabulation.

3 Main Results

The two conditions given in Theorem 2 are the framework of DIS. Briefly, the delay-free system must be Hurwitz stable and *ACE* must have no unitary roots. Firstly, we would like to focus on the distinctive property of (3) which is also indicated in [15] for multiple delay case. Note that this function has a special form:

$$ACE(z, \mathbf{q}) = p(\frac{1}{z}, \mathbf{q}) + n(\mathbf{q}) + p(z, \mathbf{q}) \tag{10}$$

where $p(z, \mathbf{q})$ is a polynomial in terms of z. One can easily check the above observation by calculating the given determinant in (3) for system matrices $\mathbf{A}(\mathbf{q})$ and $\mathbf{B}(\mathbf{q})$ of any dimension "n" with literal parameters $\mathbf{q}$. Then, it is obvious that the roots of the *ACE* are identical to the zeros of the following Auxiliary Characteristic Polynomial (*ACP*)

$$ACP(z, q) = z^l ACE(z, \mathbf{q}) \tag{11}$$

where l is the degree of $p(z, \mathbf{q})$. Due to the structure indicated in (10), the *ACP* is a self-inversive polynomial of even degree $m = 2l$, as in Definition 3 and its reciprocal coefficients are equal. The zeros of self-inversive polynomials lie either on the unit circle $\mathbb{T}$ and/or occur in complex pairs which are symmetrical (reciprocal) to $\mathbb{T}$. This causes a singularity in recursion algorithm since the symmetrical polynomial produced by Eq. (8) is identically zero. Instead of trying to overcome the singularity in the first step, we employ the remarkable relationship of the critical points of the self-inversive polynomial (zeros of its derivative wrt z) and the zeros of the polynomial itself. It is stated as in the theorem below.

Theorem 5 *[24] Let P be a self-inversive polynomial of degree p. Suppose that P has exactly* β *zeros on the unit circle* $\mathbb{T}$ *(multiplicity included) and exactly* μ *critical points in the closed unit disc* $\overline{\mathbb{D}}$ *(counted according to multiplicity). Then,*

$$\beta = 2(\mu + 1) - p. \tag{12}$$

Proof The proof of the theorem can be found in [24] in detail.

Theorem 5 has a crucial role for establishing the criterion for DIS. In Theorem 2 it is stated that system (1) is delay-independent stable if its *ACE* has no roots on $\mathbb{T}$, i.e., *ACP* has no unitary zeros. It is cumbersome to check whether zeros are unitary, especially when unknown parameters are present. Notice that the number of the unitary roots of *ACP* is related to the number of its critical points in $\overline{\mathbb{D}}$. We then present the following theorem.

Theorem 6 *Consider an LTI system with single time-delay given in (1) with a certain parameter set* $\mathbf{q}^*$. *Defining the derivative of its auxiliary characteristic polynomial as*

$$D(z) := \frac{dACP(z)}{dz} \tag{13}$$

of degree $m - 1$ *where the degree-m polynomial* $ACP(z)$ *is described in (3) and (11), the system (1) is delay-independent stable if and only if*

(i) $\Re\left(\lambda_i\left[\mathbf{A}(\mathbf{q}^*) + \mathbf{B}(\mathbf{q}^*)\right]\right) < 0,\ i = 1, \ldots, n$
(ii) $\nu = m/2$

where $\nu = Var\left\{\sigma_{m-1}, \sigma_{m-2}, \ldots, \sigma_0\right\}$ *such that* $\sigma_k = T_k(1)$ *are obtained by the recursion equations in (7), (8), (9) from* $D(z)$ *and* $Var\{\cdot\}$ *denotes the number of sign variations in a sequence.*

Proof For a Hurwitz stable delay-free system as declared in first condition above to be delay-independent stable, its *ACE* must have no roots on $\mathbb{T}$ (see Theorem 2). This also yields that self-inversive *ACP* must have no unitary zeros, i.e., $\beta = 0$ in Eq. (12). Considering Theorem 5, the number of zeros of $D(z)$ in $\overline{\mathbb{D}}$ must be $\mu = (m-2)/2$ to satisfy $\beta = 0$. From Theorems 3 and 4, the number of zeros of $D(z)$ in unit circle is $\mu = (m-1) - \nu$. Combining two equations for μ, we get $\nu = m/2$ indicating that the *ACP* and *ACE* have no roots on $\mathbb{T}$ yielding DIS.

Remark 1 The assumption $P_n(1) \neq 0 \in \mathbb{R}$, i.e., a zero of $P_n(z)$ at $z = 1$, to generate recursive polynomials given by (7), (8), (9), does not lead to a restriction in the theorem above for DIS. To have $D(1) = 0$, the *ACP* must have a zero at $z = 1$ with a multiplicity of at least two and since $z = e^{i\omega\tau}$, this can occur when there is an imaginary crossing $\pm i\omega$ for $\tau = 0$ or when there is an imaginary crossing at $\omega = 0$ for some τ. So, $D(1) = 0$ corrupts DIS and the assumption can be considered as a "weak" necessary condition.

In Theorem 6, necessary and sufficient conditions are given for DIS of LTI systems with single delay. Notice that it does not require any numerical calculation beyond simple algebraic operations, unlike other methods, e.g., [1, 22, 23] which include cumbersome numerical calculations such as frequency sweeping, finding spectral radius, or solving LMIs. As a result of Theorem 6, the following corollary can be written in case of the existence of unknown parameters.

Corollary 1 *Consider the system (1) and define the partial derivative of its auxiliary characteristic polynomial as*

$$D(z, \mathbf{q}) := \frac{\partial ACP(z, \mathbf{q})}{\partial z} \tag{14}$$

of degree $m-1$ *where the degree-m polynomial* $ACP(z, \mathbf{q})$ *is described by (3) and (11). Then, the exact parameter set stabilizing the system independent of delay is*

$$\boldsymbol{\Psi} = \left\{ \mathbf{q} \in \mathbb{R}^r | \ \Re\left(\lambda_i \left[\mathbf{A}(\mathbf{q}) + \mathbf{B}_1(\mathbf{q})\right]\right) < 0 \text{ and } \nu(\mathbf{q}) = m/2 \right\}, \ i = 1, \dots, n \tag{15}$$

where $\nu(\mathbf{q}) = \mathrm{Var}\{\sigma_{\mathrm{m}-1}(\mathbf{q}), \sigma_{\mathrm{m}-2}(\mathbf{q}), \dots, \sigma_0(\mathbf{q})\}$ *such that* $\sigma_l(\mathbf{q}) := T_l(1, \mathbf{q})$ *are obtained by the recursion equations in (7, 8, 9) from* $D(z, \mathbf{q})$ *and* $\mathrm{Var}\{\cdot\}$ *denotes the number of sign variations.*

The Corollary above can be used as a tool for delay-independent stabilizing controller design when $\mathbf{B}(\mathbf{q})$ is the matrix of state-feedback controller where $\mathbf{q}$ is the vector of gain parameters. The region of controller gain parameters which stabilize the system independent of delay is determined without conservativeness and parameter sweeping.

Remark 2 The *ACP* for the singe delay case is a polynomial with real coefficients so that $\delta_k = \overline{\delta}_k$ in (9) due to nonexistence of imaginary parts and the recursion equation (9) becomes

$$T_{k-1}(z) = z^{-1}\left[\delta_{k+1} z^{\lambda_k}(z+1)T_k(z) - T_{k+1}(z)\right] \tag{16}$$

allowing for effective computation in the recursion algorithm.

4 Case Studies

Example 1 Consider the time-delay system with single delay in [10] governed by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -20 & -13 & -4.1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 & 0.05 \\ 0.26 & 0 & 0 \\ 0 & 0.74 & 0 \end{bmatrix} \mathbf{x}(t-\tau). \tag{17}$$

The eigenvalues of the delay-free system ($\tau = 0$) are -2.167 and $-0.967 \pm i2.807$ which have negative real parts satisfying the first condition of Theorem 6. The auxiliary characteristic polynomial of the system (17) is

$$\begin{aligned}ACP(z) &= 8.903\times10^{-7} - 4.692\times10^{-5}z + 1.120\times10^{-3}z^2\\ &-1.936\times10^{-2}z^3 + 0.270\,z^4 - 2.777\,z^5 + 8.867\,z^6 + 103.7\,z^7\\ &-1031.\,z^8 - 1.769\times10^5 z^9 - 1031.\,z^{10} + 103.7\,z^{11} + 8.867\,z^{12}\\ &-2.777\,z^{13} + 0.270\,z^{14} - 1.936\times10^{-2}z^{15} + 1.120\times10^{-3}z^{16}\\ &-4.692\times10^{-5}z^{17} + 8.903\times10^{-7}z^{18}.\end{aligned} \tag{18}$$

The degree of the ACP in (18) is $m = 18$ and it is trivial to find its derivative (13), i.e., $D(z)$. After constructing the polynomials $T_k(z)$ by applying the recursion algorithm given in Eqs. (7), (8), (9) to $D(z)$, the sequence $\{\sigma_k\}$, such that $\sigma_k := T_k(1)$ for $k = 17, \dots, 0$, is obtained as follows:

$$\begin{aligned}\{\sigma_k\} = \{&1.572\times10^6, -1.479\times10^6, -1.672\times10^7, 1.513\times10^6,\\ &2.533\times10^6, -2.168\times10^5, -2.511\times10^6, 1.405\times10^6, 1.969\times10^7,\\ &-1.393\times10^6, -2.065\times10^6, 1.212\times10^5, 2.082\times10^6, -1.659\times10^6,\\ &-1.698\times10^7, 1.640\times10^6, 1.606\times10^6, -3.217\times10^6\}.\end{aligned} \tag{19}$$

The sign variation in the list (19) is $\nu = 9$ and it is exactly $m/2 = 18/2$ as stated condition (*ii*) of Theorem 6 indicating the system is delay-independent stable. The result coincides with the result of the original study [10]. Note that, a polynomial $D(\omega)$ of degree 18 also obtained and the nonexistence of real ω roots is checked for DIS in [10].

Example 2 Consider the system below from [15]:

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2.5 & -1.8 & -4 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -2\,\alpha & 0 & -1 \end{bmatrix}\mathbf{x}(t-\tau). \tag{20}$$

The characteristic equation of the system (20) is

$$CE(s,\alpha,\tau) = s^3 + 4s^2 + s^2e^{-s\tau} + 1.8s + 2.5 + 2\alpha e^{-s\tau} = 0. \tag{21}$$

The delay-free system ($\tau = 0$) is stable when $-1.25 < \alpha < 1.45$, see Theorem 6. The partial derivative (14) of the corresponding ACP of degree $m = 6$ obtained by (3) and (11) is

$$\begin{aligned}D(z,\alpha) &= z^5(-37.5 + 201.1\alpha - 297.6\alpha^2 - 48\alpha^3)\\ &+z^4(-331 + 1441.\alpha - 2284\alpha^2) + z^3(-651.8 - 192.6\alpha + 428.8\alpha^2 - 96\alpha^3)\\ &+z^2(-1122. - 2110.\alpha + 3403.\alpha^2) + z(-325.9 - 96.32\alpha + 214.4\alpha^2 - 48\alpha^3)\\ &-66.2 + 288.3\alpha - 456.8\alpha^2.\end{aligned} \tag{22}$$

We get the sequence of variance coefficients $\{\sigma_k(\alpha)\}$, such that $\sigma_k(\alpha) := T_k(1,\alpha)$ for $k = 5, \dots, 0$, applying the regular recursion algorithm in (7), (8), (9) to (22) as follows:

$$\sigma_5 = -6\,(13 - 4\alpha)^2\,(5 + 4\alpha)$$

$$\sigma_4 = 599 - \frac{4}{5}\alpha(-7619 + 4\alpha(3023 + 60\alpha))$$

$$\sigma_3 = \frac{-4(5 - 16\alpha)^2(5.319 \times 10^4 + 8\alpha(-2.99 \times 10^5 + 70\alpha(6227 + 600\alpha))}{25(-287 + 8\alpha(109 + \alpha(-199 + 60\alpha)))}$$

$$\sigma_2 = \frac{\begin{array}{c}-6(7.107 \times 10^7 + \ \alpha(6.789 \times 10^8 + 8\alpha(-1.347 \times 10^9 + 50\alpha(7.644 \times 10^7 \\ +4\alpha(-1.514 \times 10^7 + 40\alpha(1241 + 1950\alpha))))))\end{array}}{5(1.679 \times 10^5 + 10\alpha(-1.572 \times 10^5 + 40\alpha(6031 + 150\alpha)))}$$

$$\sigma_1 = \frac{\begin{array}{c}100(5 - 16\alpha)^4(4.251 \times 10^7 + \ 32\alpha(8.056 \times 10^6 + 5\alpha(-2.757 \times 10^6 \\ +200\alpha(1106 + 135\alpha))))\end{array}}{\begin{array}{c}3(1.614 \times 10^6 + \ 8\alpha(5.412 \times 10^8 + 5\alpha(-9.469 \times 10^8 + 20\alpha(1.232 \times 10^8 \\ +100\alpha(-9.941 \times 10^5 + 4\alpha(5013 + 1400\alpha))))))\end{array}}$$

$$\sigma_0 = -\frac{3}{25}(-5 + 4\alpha)(29 + 20\alpha)^2$$

Note that the delay-free stability condition is $-1.25 < \alpha < 1.45$ and it enforces $\sigma_5 < 0$ and $\sigma_0 > 0$. Thus, there may be ten possible sign variations in the sequence $\{\sigma_k\}$ which satisfy $\nu(\alpha) = m/2 = 3$ for DIS. The union of the regions for the unknown parameter satisfying $\nu = 3$ is found as $-0.134 < \alpha < 0.762$ by reducing the inequalities. The found range of α for DIS coincides with the numerical calculation where *ACE* has no unitary zeros as indicated in [15]. The DIS range for α found by the method complying sufficient condition in the mentioned paper is $0.135 < \alpha < 0.508$ and this result is highly conservative comparing to the proposed nonconservative result.

Example 3 A case for PD-controller design for a second order system is borrowed from [13] with the state space representation,

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 & 0 \\ -k_p & -k_d \end{bmatrix} \mathbf{x}(t - \tau) \tag{23}$$

where $\omega_n = 1$ and $\zeta = 0.95$. The characteristic equation of the system is

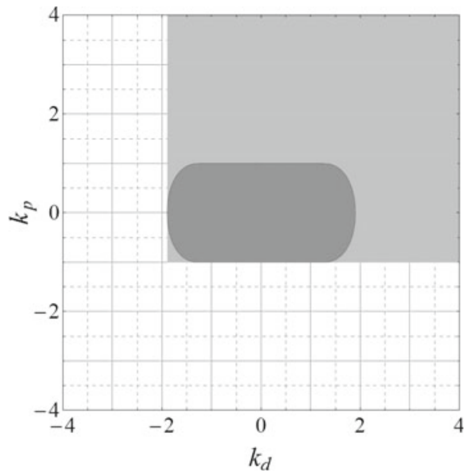
$$CE(s, k_p, k_d, \tau) = s^2 + 1.9s + k_d s e^{-s\tau} + k_p e^{-s\tau} + 1 = 0. \tag{24}$$

The delay-free system where $\tau = 0$ is Hurwitz stable for $k_p > -1$ and $k_d > -1.9$. The partial derivative (14) of the related *ACP* of degree $m = 4$ is

$$\begin{aligned} D(z, k_p, k_d) &= 7.6k_d + 7.22k_p + 2k_d^2 k_p + (28.88 + 4k_d^2 + 22.8k_d k_p - 4k_p^2)z \\ &+ \left(22.8k_d + 21.66k_p + 6k_d^2 k_p\right) z^2 + \left(4k_d^2 + 7.6k_d k_p + 4k_p^2\right) z^3. \end{aligned} \tag{25}$$

We obtain the elements of the sequence $\{\sigma_k(k_p, k_d)\}$, $k = 3, \ldots, 0$, such that $\sigma_k := T_k(1, k_p, k_d)$ as follows:

Fig. 1 Hurwitz stable region for the delay-free system (*light gray*) and delay-independent stabilizing region (*dark gray*) in PD-controller gain plane



$$\sigma_3 = -\frac{2}{25}(19 - 10k_d)^2(-1 + k_p)$$

$$\sigma_2 = \frac{4k_p^2\left(1.408 \times 10^4 + 200k_d^2\left(161 - 50k_d^2\right) - 4 \times 10^4 k_p^2\right)}{25\left(20(19 - 10k_d)k_d + (19 - 10k_d)^2 k_p - 200k_p^2\right)}$$

$$\sigma_1 = -\frac{722}{25} + 8k_d^2 + 16k_p^2$$

$$\sigma_0 = \frac{4}{25}(19 + 10k_d)^2(1 + k_p)$$

Note that $\nu(k_p, k_d) = 2$ must be satisfied for DIS as in Corollary 1. The stability condition for delay-free system enforces $\sigma_0 > 0$ and this also yields $\sigma_3 > 0$ since $\nu = 2$ is not possible for any combination of sign variations when $\sigma_3 < 0$. Consequently, there are three possible variations for $\{\sigma_2, \sigma_1\}$ as $\{+, -\}$, $\{-, +\}$, and $\{-, -\}$ which satisfies $\nu = 2$. The union of the regions of the PD-controller parameters satisfying $\nu = 2$, i.e., delay-independent stabilizing region of the parameters k_p and k_d, is shown in Fig. 1. The obtained region coincides with the results of the original paper [13].

5 Conclusion

A methodology preserving necessary and sufficient conditions, based on an auxiliary characteristic equation obtained via Kronecker summation is presented to test delay-independent stability of LTI systems with single delay. The method works for determining regions of DIS of systems with unknown parameters and for determining suitable state-feedback controllers ensuring DIS. The self-inversive property of the auxiliary characteristic polynomial, and the Bistritz Tabulation method are utilized for efficient computation. The method does not require either parameter or frequency sweeping. Also, checking the roots of an obtained polynomial and point-wise testing in candidate DIS regions of parameter space is not needed like in alternative methods. On the other hand, finding the parameter regions for DIS may computationally be cumbersome when the system order "n" and correspondingly the degree of the auxiliary characteristic polynomial get higher. Because the possible combinations of sign variations in the sequence, $\{\sigma_k\}$, increases. To remedy this problem the condition for the number of zeros on the unit circle should be simplified in future work. Also, we plan to extend the obtained results for single delay case to multiple time-delay systems. In multiple delay case, the auxiliary characteristic polynomial should turn into a multivariable polynomial. The Bistritz method will be utilized for this multivariable polynomial problem.

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