Measurement-Induced Nonlocality and Quantum Correlations Under Local Operations

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Abstract One significant feature of quantum theory is the existence of non-local quantum correlations which have no classical counterpart. There are various measures quantifying quantum correlations from different view points. Here, we present some recent developments about the quantum correlation measures known as measurement-induced nonlocality, in the sense that quantum measurement may destroy the quantum correlations for quantum states resulting in measures of non-locality. Quantum correlations remain invariant under local unitary operations, they may decrease under general local operations, however, sometimes they can also show increasing for some local operations. We will review the properties of quantum correlations under local operations.

1 Introduction

Quantum mechanics is non-local. There exist non-local quantum correlations which have no classical counterpart. The study of quantum correlations can trace back to the well-known debate about whether quantum mechanics is complete, known as the

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Einstein-Podolsky-Rosen paradox [1]. It was proposed that there may exist hidden variables for quantum theory being complete. Later, various Bell-type inequalities were proposed which are derived based on the local hidden variable theory, see, e.g., Ref. [2] for an overview. It was found that the violation of Bell inequalities implies quantum entanglement in a system, while the opposite case is not always true [3, 4]. Entanglement also plays a critical role in many protocols of quantum information processing. Great progress has been made in studying quantum entanglement, which is one kind of quantum correlations showing non-locality of quantum mechanics. Entanglement is also believed to be the key resource for the advantages of quantum computation and protocols of quantum information processing. Very recently, it is realized that entanglement is not the only quantum correlation which has no classical counterpart. Other type of quantum correlations, such as the quantum discord and measurement-induced nonlocality, may also be responsible for the speedups in some quantum algorithms while entanglement may be vanishing or negligible, see review [5].

In general, most of the quantum correlations for pure quantum states may coincide, and sometimes may demonstrate similar behaviors for mixed states. However, there are also subtle differences for those quantum correlations and their physical interpretations are also different. All these indicate that the properties of quantum correlations, or nonlocality, of a system are intricate, and the characterizations of them from different aspects are in demanding. Here we will review some recent results of measurement-induced nonlocality and quantum correlations under local operations.

We would like to point out that measurement-induced nonlocality is one type of quantum correlations. However, we remark that nonlocality, for example in form of non-local correlation, which is non-classical from one side, may not always be possessed by quantum states from other side, such as the PR box [6, 7]. Here the measurement-induced nonlocality means the quantum correlation possessed by quantum states which is naturally quantum mechanical. For quantum correlations, we mean that some non-classical correlations possessed by quantum states. One may realize that there also exist classical correlation for quantum states. We remark that quantum correlation, as valuable resource, cannot be cloned (broadcasted) because of no-cloning theorem, in contrast with the classical correlation [8]. We know that entanglement cannot be created by local quantum operations even assisted by classical communications. However, some other quantum correlations may increase by local operations. Here, we will review results of quantum correlations under local operations.

Before we proceed, let us first introduce some notations. Quantum states are presented as the density operators ρ in the Hilbert space $\mathcal{H}_d$, where d denotes the dimension. A qubit is a two-dimension quantum system. Let $\vec{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ denote the Pauli basis and $\sigma_0 = I$ be the single-qubit identity operator. Any single-qubit state ρ^A can be written as $\rho^A = \frac{1}{2}\sum_{\mu=0}^{3} a_\mu \sigma_\mu^A$, where $a_\mu = \mathrm{tr}(\rho^A \sigma_\mu^A)$ and $\boldsymbol{a} \equiv (a_1, a_2, a_3)^{\mathrm{T}}$ is called the Bloch vector of the state ρ^A. We label $\tilde{\boldsymbol{a}} \equiv (a_0, a_1, a_2, a_3)^{\mathrm{T}}$ for later convenience. Similarly, a two-qubit state ρ^{AB} can be expanded in the Pauli basis as

$\rho^{AB} = \frac{1}{4}\sum_{\mu\nu}\Theta_{\mu\nu}\sigma_\mu^A \otimes \sigma_\nu^B$, where the coefficient matrix $\Theta_{\mu\nu} = \text{tr}(\rho^{AB}\sigma_\mu^A \otimes \sigma_\nu^B)$ can be written in the block form $\Theta = \begin{pmatrix} 1 & \boldsymbol{b}^{\text{T}} \\ \boldsymbol{a} & T \end{pmatrix}$.

Here, $\boldsymbol{a}$ and $\boldsymbol{b}$ are the Bloch vectors of the reduced density matrices ρ^A and ρ^B respectively, and the 3×3 matrix T represents the correlations.

The von Neumann entropy of a quantum state is denoted as $S(\rho) := -\text{tr}(\rho \log_2 \rho)$. The relative entropy of two quantum states ρ and σ is $S(\rho||\sigma) := -\text{tr}(\rho \log_2 \sigma) - S(\rho)$. When we consider the bipartite quantum state ρ^{AB}, the conditional entropy is $S_{A|B}(\rho^{AB}) := S(\rho^{AB}) - S(\rho^B)$.

A quantum channel is a trace-preserving completely positive (TPCP) linear map $\Lambda : \mathcal{D}(\mathcal{H}_d) \to \mathcal{D}(\mathcal{H}_{d'})$. Here $\mathcal{D}(\mathcal{H}_d)$ denotes the operator space defined on the Hilbert space $\mathcal{H}_d$. In the following context, we take $d = d'$. Any quantum channel can be presented as the Kraus decomposition $\Lambda(\cdot) = \sum_j E_j(\cdot)E_j^\dagger$, where E_j are called Kraus operators.

2 What are Quantum Correlations

A bipartite state ρ^{AB} is called quantum-classical (QC) if there exist a positive operator-valued measure (POVM) on B which does not disturb the whole state. The term "classical" is used to stress the nondisturbing property of classical measurements. Mathematically, a QC state can be written as

$$\rho^{\text{QC}} = \sum_i p_i \rho_i^A \otimes |\phi_i\rangle_B \langle\phi_i|, \tag{1}$$

where $\{|\phi_i\rangle_B\}$ consist of an orthogonal basis for the Hilbert space of subsystem B, and ρ_i^A are density operators of A. The set of quantum-classical states are denoted as $\mathcal{QC}$. An equivalent expression for the QC states is

$$\rho^{\text{QC}} = \sum_i p_i \zeta_i^A \otimes \xi_i^B, \tag{2}$$

where ζ_i^A are linearly independent, and ξ_i^B commute with each other.

A state is said to have nonzero quantum correlation on B if and only if it does not belong to $\mathcal{QC}$. Like entanglement, the amount of quantum correlation can be measured in various ways [5]. The measures of quantum correlation Q we discuss here satisfy the following three conditions:

(C1) $Q(\rho) = 0$ if and only if $\rho \in \mathcal{QC}$;
(C2) $Q(U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger) = Q(\rho)$, where U_A and U_B are arbitrary unitary operators on A and B;
(C3) $Q(\Lambda_A \otimes \mathbb{1}_B(\rho)) \le Q(\rho)$.

Notice that the measures of quantum correlation are asymmetric for A and B, here and hereafter, we discuss only the quantum correlation defined on B. In the following, we list some quantum correlation measures which satisfy (C1–C3).

Quantum discord is defined as the minimum part of the mutual information shared between A and B that cannot be obtained by the measurement on B [9]:

$$\delta_{A|B}(\rho) := \min_{\{M_i^B\}} S_{A|B}\left(\sum_i I_A \otimes M_i^B \rho I_A \otimes M_i^{B\dagger}\right) - S_{A|B}(\rho), \tag{3}$$

where $\{M_i^B\}$ is a POVM on B. We point out here that the calculation of quantum discord is a hard task in general. Even for the two-qubit states, the analytical solutions of it exist only for certain special states [10].

Distance-based measure of quantum correlation is the minimum distance between the state ρ and the set of QC states [11]

$$Q_D(\rho) := \min_{\sigma \in \mathcal{QC}} D(\rho, \sigma), \tag{4}$$

where the distance D does not increase under any quantum operation, such that Q_D satisfies (C3). When the relative entropy $S(\rho||\sigma) := \mathrm{tr}[\rho(\log_2 \rho - \log_2 \sigma)]$ is employed as the distance measure, we obtain the **one-way quantum deficit**

$$\Delta_{A|B} := \min_{\{\Pi_i^B\}} S\left(\sum_i I_A \otimes \Pi_i^B \rho I_A \otimes \Pi_i^{B\dagger}\right) - S(\rho), \tag{5}$$

where $\{\Pi_i^B\}$ is a projective measurement on B. $\Delta_{A|B}$ equals to the minimal distance between ρ and ρ^{QC}, and its operational connection with quantum entanglement has also been established [12, 13].

Measurement-induced disturbance is defined as the minimum disturbance caused by local projective measurements that do not change the reduced state $\rho^B \equiv \mathrm{tr}_A \rho$ [14]

$$Q_M(\rho) := \min_{\{\mathrm{E}_i^B\}} D\left(\rho, \sum_i I_A \otimes \mathrm{E}_i^B \rho I_A \otimes \mathrm{E}_i^{B\dagger}\right), \tag{6}$$

where $\{\mathrm{E}_i^B\}$ is a projective measurement on B which satisfies $\sum_i \mathrm{E}_i^B \rho^B \mathrm{E}_i^{B\dagger} = \rho^B$. This measure of quantum correlation is favored for its easy of calculation, and its generalization to continuous-variable systems has also been established [15].

3 Measurement-Induced Nonlocality

In the following, we recall the recently proposed measure of nonlocality which was termed as measurement-induced nonlocality (MIN) [16], as well as various forms of its extension [17–21]. They were all defined from the measurement perspective, and were motivated by those of the discord-like correlation measures [5]. We shall focus mainly on the bipartite systems described by the density operator ρ in the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. But the related concepts and ideas can in fact be generalized to multipartite systems straightforwardly.

Motivated by the idea that the distance from a given state ρ to the set of states without the desired property is a measure of that property (e.g., the distance to the closest separable state is a measure of entanglement, and to the closest classical state is a measure of discord) [5], the MIN can be defined as the maximal distance that the considered state ρ to the set $\mathcal{L}$ of local quantum states, namely

$$N(\rho) = \max_{\delta \in \mathcal{L}} D(\rho, \delta), \tag{7}$$

where D denotes an arbitrary distance measure that does not increase under the action of TPCP map, while the maximum is taken over the full set of $\mathcal{L}$, which contains those of the quantum states δ obtained by the locally invariant measurements Π^A, that is, $\delta = \sum_k \Pi_k^A \rho \Pi_k^A$ for all $\Pi^A = \{\Pi_k^A\}$ satisfying $\sum_k \Pi_k^A \rho^A \Pi_k^A = \rho^A$, with $p_k = \mathrm{tr}(\Pi_k^A \rho \Pi_k^A)$, and $\rho^A = \mathrm{tr}_B \rho$ being the reduced state of ρ.

This definition of nonlocality measure was motivated by the consideration that a local state should not be disturbed by arbitrary locally invariant measurement Π^A on party A (or Π^B on B), while a nonlocal state may be disturbed by Π^A, and the maximal disturbance can be used to quantify the nonlocal property of it.

By adopting different distance measures D, one can define different MIN measures which possess distinct novel characteristics, and have been shown to play crucial role in many fields of quantum technology.

3.1 MIN Quantified by the Hilbert–Schmidt Norm

The notion of MIN was first introduced by Luo and Fu by using the Hilbert–Schmidt norm [16]. For a bipartite state ρ shared by two parties A and B, it was defined as

$$N_2(\rho) = \max_{\Pi^A} ||\rho - \Pi^A(\rho)||_2^2, \tag{8}$$

where Π^A denotes the locally invariant von Neumann measurements, and $||X||_2 = [\mathrm{tr}(X^\dagger X)]^{1/2}$ is the Hilbert–Schmidt norm.

Physically, $N_2(\rho)$ can be considered as the maximal global disturbance induced by the locally invariant measurements Π^A, or the maximal square Hilbert–Schmidt distance that the postmeasurement state $\Pi^A(\rho)$ departs from the premeasurement state ρ. From an applicative point of view, it is also hoped to be useful in the related field of quantum state steering, remote state control, superdense coding, and cryptography [16].

The MIN measure $N_2(\rho)$ has the following basic properties: (i) $N_2(\rho)$ is nonnegative, and equals zero for any product state $\rho = \rho^A \otimes \rho^B$. (ii) $N_2(\rho)$ is locally unitary invariant, namely, $N_2((U^A \otimes U^B)\rho(U^A \otimes U^B)^\dagger) = N_2(\rho)$ for any unitary operators U^A and U^B. (iii) If the reduced state ρ^A is nondegenerate with the spectral decomposition $\rho^A = \sum_k \lambda_k |k\rangle\langle k|$, then the optimal $\tilde{\Pi}^A$ for obtaining $N_2(\rho)$ is given by $\tilde{\Pi}^A(\rho) = \sum_k |k\rangle\langle k|\rho|k\rangle\langle k|$.

For the $(m \times n)$-dimensional bipartite states represented as

$$\rho = \sum_{ij} r_{ij} X_i \otimes Y_j, \tag{9}$$

with $\{X_i : i = 0, 1, \ldots, m^2 - 1\}$ ($X_0 = I_m/\sqrt{m}$) is the orthonormal operator base for subsystem A that satisfy $\mathrm{tr}(X_i^\dagger X_{i'}) = \delta_{ii'}$ (and likewise for Y_j), the MIN measure $N_2(\rho)$ has been shown to be upper bounded by

$$N_2(\rho) \leq \sum_{i=1}^{m^2-m} \lambda_i, \tag{10}$$

where λ_i ($i = 1, 2, \ldots, m^2 - 1$) denote the eigenvalues of RR^T in nonincreasing order, $R = (r_{ij})$ with $i, j \geq 1$ is a real matrix, and the superscript T denotes transpose of vectors or matrices.

The MIN measure $N_2(\rho)$ is favored for its ease of calculation for a wide range of quantum states. First, for any bipartite pure state $|\psi\rangle$ with the Schmidt decomposition $|\psi\rangle = \sum_k \lambda_k |\phi_k^A\rangle \otimes |\phi_k^B\rangle$, one has

$$N_2(|\psi\rangle\langle\psi|) = 1 - \sum_k \lambda_k^2, \tag{11}$$

and for the $(2 \times n)$-dimensional states represented as Eq. (9), one has

$$N_2(\rho) = \begin{cases} ||R||_2^2 - \dfrac{1}{||\vec{x}||_2^2} \vec{x}^T R R^T \vec{x} & \text{if } \vec{x} \neq 0, \\ ||R||_2^2 - \lambda_{\min}(RR^T) & \text{if } \vec{x} = 0. \end{cases} \tag{12}$$

where $\lambda_{\min}(RR^T)$ is the smallest eigenvalue of RR^T, and $\vec{x} = (r_{10}, r_{20}, r_{30})^T$.

Moreover, for certain higher dimensional states with symmetry, $N_2(\rho)$ can also be calculated analytically [22], e.g., for the $(d \times d)$-dimensional Werner state ρ_W and isotropic state ρ_I of the following form

$$\begin{aligned}\rho_W &= \frac{d-x}{d^3-d} I_{d^2} + \frac{dx-1}{d^3-d} \sum_{ij} |ij\rangle\langle ji|, \quad x \in [-1,1], \\ \rho_I &= \frac{1-x}{d^2-1} I_{d^2} + \frac{d^2x-1}{d^3-d} \sum_{ij} |ii\rangle\langle jj|, \quad x \in [0,1],\end{aligned} \tag{13}$$

one has

$$N_2(\rho_W) = \frac{(dx-1)^2}{d(d+1)(d^2-1)}, \quad N_2(\rho_I) = \frac{(d^2x-1)^2}{d(d+1)(d^2-1)}. \tag{14}$$

The analytical solutions of $N_2(\rho)$ or its bound for certain bound entangled states [23] and other special states with degenerate ρ^A [24] have also been reported in the literature.

3.2 MIN Quantified by the Trace Norm

Although the MIN measure $N_2(\rho)$ is favored for its convenience of calculation, it is problematic as it can increase or decrease under trivial local reversible operations on the unmeasured subsystem B of ρ. For example, consider a map $\mathcal{E}_B$ which gives rise to $\mathcal{E}_B(\rho) = \rho \otimes \rho^C$, (i.e., it introduces a local ancilla to B), then by making use of the multiplicativity of the Hilbert–Schmidt norm under tensor products, we obtain $N_2(\rho^{A:BC}) = N_2(\rho)\mathrm{tr}(\rho^C)^2$. As the purity of a state is no larger than one, this equality means that the MIN is decreased by simply introducing an uncorrelated local ancillary sytem.

To avoid the aforementioned problem, another geometric measure of MIN based on the trace norm was introduced. It is given by [17]

$$N_1(\rho) = \max_{\Pi^A} ||\rho - \Pi^A(\rho)||_1, \tag{15}$$

where $||X||_1 = \mathrm{tr}\sqrt{X^\dagger X}$ is the trace norm, and Π^A denotes still the locally non-disturbing von Neumann measurements.

The new MIN measure can be interpreted as the maximal trace distance that the premeasurement state ρ departs from those of the postmeasurement states $\Pi^A(\rho)$ caused by the locally invariant measurements. In particular, it is nonincreasing under the action of any TPCP map $\mathcal{E}_B$ on the unmeasured party B [17], namely,

$$N_1(\rho) \geq N_1(\mathcal{E}_B(\rho)). \tag{16}$$

The proof is as follows. Let $\bar{\Pi}^A$ the optimal measurement for obtaining $N_1(\rho)$, and $\tilde{\Pi}^A$ be the optimal measurement for obtaining $N_1(\mathcal{E}_B(\rho))$, then as $\mathcal{E}_B$ and $\tilde{\Pi}^A$ commute, we obtain $\tilde{\Pi}^A(\mathcal{E}_B(\rho)) = \mathcal{E}_B(\tilde{\Pi}^A(\rho))$, and therefore

$$
\begin{aligned}
N_1(\rho) &= ||\rho - \bar{\Pi}^A(\rho)||_1 \\
&\geq ||\rho - \tilde{\Pi}^A(\rho)||_1 \\
&\geq ||\mathcal{E}_B(\rho) - \mathcal{E}_B[\tilde{\Pi}^A(\rho)]||_1 \\
&= N_1[\mathcal{E}_B(\rho)],
\end{aligned} \tag{17}
$$

where the first inequality comes from the fact that $\tilde{\Pi}^A \neq \bar{\Pi}^A$ in general, and the second inequality is due to the contractivity of the trace norm under TPCP map. Therefore, $N_1(\rho)$ circumvents successfully the problem incurred for $N_2(\rho)$.

For certain quantum states, analytical solutions of $N_1(\rho)$ can be obtained, e.g., for the $(2 \times n)$-dimensional pure state $|\psi\rangle$ with the Schmidt decomposition $|\psi\rangle = \sum_{k=1}^{2} \lambda_k |\phi_k^A\rangle \otimes |\phi_k^B\rangle$, the trace norm MIN is given by

$$
N_1(|\psi\rangle\langle\psi|) = 2\sqrt{\lambda_1 \lambda_2}, \tag{18}
$$

while for two-qubit state ρ of the form of Eq. (9) (i.e., $m = 2$) with the addition $r_{ij} = 0$ for $i \neq j$, we have

$$
N_1(\rho) = \begin{cases} \dfrac{\sqrt{\chi_+} + \sqrt{\chi_-}}{||\vec{x}||_1} & \text{if } \vec{x} \neq 0, \\ 2\max\{|r_{11}|, |r_{22}|, |r_{33}|\} & \text{if } \vec{x} = 0, \end{cases} \tag{19}
$$

where $\chi_\pm = \alpha \pm 4\sqrt{\beta}|\vec{x}|$, with $\alpha = |\vec{r}|^2|\vec{x}|^2 - |\vec{r} \cdot \vec{x}|^2$, $\vec{r} = (r_{11}, r_{22}, r_{33})$, $\beta = \sum_{\langle ijk \rangle} x_i^2 r_{jj}^2 r_{kk}^2$, and the summation runs over all the cyclic permutations of $\{1, 2, 3\}$.

Moreover, for the Werner state ρ_W and isotropic state ρ_I of Eq. (13), solutions of the the trace norm MIN are given, respectively, by

$$
N_1(\rho_W) = \frac{|dx - 1|}{d + 1}, \quad N_1(\rho_I) = \frac{2|d^2x - 1|}{d(d + 1)}, \tag{20}
$$

which show qualitatively the same x-dependence as those of the MIN measure $N_2(\rho_W)$ and $N_2(\rho_I)$ with finite d. That is to say, for the symmetric states ρ_W and ρ_I, both the MIN measures N_1 and N_2 give the same descriptions of nonlocality.

3.3 MIN Quantified by the Bures Distance

The Bures distance $d_B(\rho, \chi) = [2(1 - F^{1/2}(\rho, \chi)]^{1/2}$ between two states ρ and χ, which is joint convexity, and is monotonous under the action of any TPCP map [25], can also be used to give a well-defined measure of MIN [17]. Without loss of generality, we define it as

$$
N_B(\rho) = \max_{\Pi^A} \{1 - \sqrt{F(\rho, \Pi^A(\rho))}\}, \tag{21}
$$

where Π^A is the locally invariant measurements on party A, and $F(\rho, \sigma)$ is the Uhlmann fidelity that is defined as

$$F(\rho, \sigma) = [\mathrm{tr}(\sqrt{\rho}\sigma\sqrt{\rho})^{1/2}]^2. \tag{22}$$

For states ρ with nondegenerate ρ^A, $N_B(\rho)$ can be obtained directly, as the optimal $\Pi^A = \{\Pi_i^A\}$ are induced by the spectral resolutions of $\rho^A = \sum_i p_i^A \Pi_i^A$. If ρ^A is degenerate, the calculation of $N_B(\rho)$ is difficult. But for the $(2 \times n)$-dimensional states ρ, the minimum Uhlmann fidelity $F_{\min}(\rho, \Pi^A(\rho)) = \min_{\Pi^A} F(\rho, \Pi^A(\rho))$ can be calculated via

$$F_{\min}(\rho, \Pi^A(\rho)) = \frac{1}{2} \min_{\|\vec{u}\|=1} \left(1 - \mathrm{tr}\Lambda + 2\sum_{k=1}^{n} \lambda_k(\Lambda)\right), \tag{23}$$

with $\vec{u} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ being a unit vector in $\mathbb{R}^3$, n the dimension of subsystem B, and $\lambda_k(\Lambda)$ eigenvalues of Λ arranged in non-increasing order. By denoting $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ the vector of the usual Pauli operators, and I_n the $n \times n$ identity matrix, we have

$$\Lambda = \sqrt{\rho}(\vec{u} \cdot \vec{\sigma} \otimes I_n)\sqrt{\rho}. \tag{24}$$

For the special case of the two-qubit Bell-diagonal states

$$\rho_{\mathrm{Bell}} = \frac{1}{4}\left(I_4 + \sum_{i=1}^{3} c_i \sigma_i \otimes \sigma_i\right), \tag{25}$$

as $\sqrt{\rho_{\mathrm{Bell}}}$ can be derived explicitly, $F_{\min}(\rho, \Pi^A(\rho))$ takes the form

$$F_{\min}(\rho, \Pi^A(\rho)) = \frac{1}{2}\left(1 + \min_{\{\theta,\phi\}} \sqrt{b_3^2 + [b_1^2 - b_3^2 + (b_2^2 - b_1^2)\sin^2\phi]\sin^2\theta}\right), \tag{26}$$

where $b_i = 8(t_0^2 + t_i^2) - 1$ $(i = 1, 2, 3)$, and by writing $c_{\mathrm{sum}} = c_1 + c_2 + c_3$, we have

$$\begin{aligned} t_0 &= \frac{1}{8}\sqrt{1 - c_{\mathrm{sum}}} + \frac{1}{8}\sum_{k=1}^{3}\sqrt{1 + c_{\mathrm{sum}} - 2c_k}, \\ t_i &= -\frac{1}{8}\sqrt{1 - c_{\mathrm{sum}}} + \frac{1}{8}\sum_{k=1}^{3}\sqrt{1 + c_{\mathrm{sum}} - 2c_k} - \frac{1}{4}\sqrt{1 + c_{\mathrm{sum}} - 2c_i}. \end{aligned} \tag{27}$$

From Eq. (26) one can see that $F_{\min}(\rho, \Pi^A(\rho)) = (1 + |b_1|)/2$ if $|b_1| \leq \min\{|b_2|, |b_3|\}$, $F_{\min}(\rho, \Pi^A(\rho)) = (1 + |b_2|)/2$ if $|b_2| \leqslant \min\{|b_1|, |b_3|\}$, and $F_{\min}(\rho, \Pi^A(\rho)) = (1 + |b_3|)/2$ otherwise.

3.4 MIN Quantified by the von Neumann Entropy

Apart from the geometric measures, the MIN can also be quantified from the entropic perspective. In this respect, if we accept that the quantum mutual information (QMI) is a good measure of total correlations in ρ, and the entropic measure of MIN, in the spirit of its original definition [16], can be defined as the maximal discrepancy between the QMI of the pre- and post-measurement states as [18]

$$N_E(\rho) = I(\rho) - \min_{\Pi^A} I[\Pi^A(\rho)], \tag{28}$$

where $I(\rho) = S(\rho^A) + S(\rho^B) - S(\rho)$ is the QMI, and Π^A denotes still the locally measurements which do not disturb the reduced state ρ^A.

This measure of MIN quantifies in fact, the maximal loss of total correlations under locally non-disturbing measurements on party A. Moreover, as both ρ and $\Pi^A(\rho)$ have the same reduced states ρ^A and ρ^B, $N_E(\rho)$ defined above is equivalent to

$$N_E(\rho) = \max_{\Pi^A} S[\Pi^A(\rho)] - S(\rho), \tag{29}$$

which indicates that $N_E(\rho)$ quantifies in fact the maximal increment of von Neumann entropy induced by the locally invariant measurements. Moreover, as the entropy of a state measures how much uncertainty there is in it, $N_E(\rho)$ can also be interpreted as the maximal increment of our uncertainty about that system induced by the locally invariant measurements.

The entropic measure of MIN possesses the same basic properties (i), (ii), and (iii) as that of the Hilbert–Schmidt norm MIN. Furthermore, it is monotonous under the action of any TPCP map $\mathcal{E}_B$ on the unmeasured party B, i.e., $N_E(\mathcal{E}_B(\rho)) \leq N_E(\rho)$ [18], which shows that it is a well-defined measure of MIN. Moreover, $N_E(\rho)$ vanishes for the classical-quantum state $\rho_{CQ} = \sum_i p_i |i\rangle\langle i| \otimes \rho_i^B$ with nondegenerate reduced ρ_{CQ}^A, or ρ_{CQ} with degenerate ρ_{CQ}^A and $\rho_i^B = \rho_j^B$ for all i and j.

We point out here that $N_E(\rho)$ is also equivalent to the MIN measure defined based on the relative entropy, namely, $N_E(\rho) = N_{RE}(\rho)$, with

$$N_{RE}(\rho) = \max_{\Pi^A} S(\rho||\Pi^A(\rho)), \tag{30}$$

where $\Pi^A(\rho) = \sum_i \Pi_i^A \rho \Pi_i^A$, and $\{\Pi_i^A\}$ is the set of von Neumann measurements which do not disturb ρ^A locally. In fact, the relative entropy between two states can also be considered as a measure of their distance, although technically it does has a geometric interpretation.

For the state ρ with nondegenerate ρ^A, the optimal measurement operators $\tilde{\Pi}_k^A = |k\rangle\langle k|$ for obtaining $N_E(\rho)$ are induced by the spectral resolutions of $\rho^A = \sum_k \lambda_k |k\rangle\langle k|$. For general cases, $N_E(\rho)$ can be obtained numerically. It is lower bounded by $-S(A|B)$ and upper bounded by $\min\{I(\rho), S(\rho^A)\}$, with $S(A|B) = S(\rho) - S(\rho^B)$. As an example, we list here the analytical solution of $N_E(\rho)$ for the two-qubit Bell-diagonal state ρ_{Bell} of Eq. (25), which is given by

$$N_E(\rho_{\text{Bell}}) = H\left(\frac{1+c_{\min}}{2}\right) + \frac{1-c_{\text{sum}}}{4}\log_2\frac{1-c_{\text{sum}}}{4} + \sum_{k=1}^{3}\frac{1+c_{\text{sum}}-2c_k}{4}\log_2\frac{1+c_{\text{sum}}-2c_k}{4} + 1, \tag{31}$$

with $c_{\min} = \min\{|c_1|, |c_2|, |c_3|\}$, and $H(x) = -(1+x)\log_2(1+x) - (1-x)\log_2(1-x)$ is the binary Shannon entropy.

The quantitative relation between $N_E(\rho)$ and $N_2(\rho)$ has also been established, which is given by [19]

$$N_E(\rho) \geq \frac{1}{2\ln 2}N_2^2(\rho), \tag{32}$$

that is to say, the entropic MIN $N_E(\rho)$ is always greater than or equal to the square of the geometric MIN $N_2(\rho)$ divided by $2\ln 2$ for any state ρ. As the calculation of $N_2(\rho)$ is somewhat easy, the above inequality can serve as a lower bound of $N_E(\rho)$.

3.5 MIN Quantified by the Wigner–Yanase Skew Information

The Wigner–Yanase skew information is given by $\mathcal{I}(\rho, K) = -\text{tr}\{[\rho^{1/2}, K]^2\}/2$, with K being an observable [26]. $\mathcal{I}(\rho, K)$ is upper bounded by the variance of K, i.e., $\mathcal{I}(\rho, K) \leq \langle K^2\rangle_\rho - \langle K\rangle_\rho^2$, and vanishes iff the state and the observable commute. It has also been employed to quantify local quantum uncertainty and coherence.

The MIN based on Wigner–Yanase skew information is defined as [20]

$$N_{SI}(\rho) = \max_{\{\tilde{K}_i^A\}}\sum_{i=1}^{m}\mathcal{I}(\rho, \tilde{K}_i^A \otimes I_{d_B}), \tag{33}$$

where the local observables $\tilde{K}^A = \{\tilde{K}_i^A\}$ are restricted to rank-one projectors (i.e., $\tilde{K}_i^A = |i^A\rangle\langle i^A|$) which do not disturb the local state ρ^A, and I_{d_B} is the identity operator for subsystem B, with $d_B = \dim \mathcal{H}_B$. This MIN measure has been shown to be invariant under locally unitary operations, to be contractive under any TPCP map $\mathcal{E}_B$ on party B, and vanishes for the product states $\rho = \rho^A \otimes \rho^B$ and the classical-quantum states ρ_{CQ} with nondegenerate ρ^A.

If we decompose the bipartite state ρ as follows

$$\sqrt{\rho} = \sum_{ij}\gamma_{ij}X_i \otimes Y_j, \tag{34}$$

then it can be shown that $N_{SI}(\rho)$ is upper bounded by [20]

$$N_{SI}(\rho) \le 1 - \sum_{i=1}^{m-1} \mu_i, \tag{35}$$

with μ_i $(i = 1, 2, \ldots, m^2)$ being the eigenvalues of $\Gamma\Gamma^T$ listed in decreasing order (counting multiplicity), and $\Gamma = (\gamma_{ij})$ is the $(m^2 \times n^2)$-dimensional correlation matrix.

For the pure states $|\psi\rangle$, $N_{SI}(|\psi\rangle\langle\psi|) = N_2(|\psi\rangle\langle\psi|)$, while for the $(2 \times n)$-dimensional states ρ, one has

$$N_{SI}(\rho) = \begin{cases} 1 - \mu_1 & \text{if } \vec{u} = 0, \\ 1 - \frac{1}{2}\mathrm{tr}\left(\begin{pmatrix} 1 & \vec{u}_0 \\ 1 & -\vec{u}_0 \end{pmatrix} \Gamma\Gamma^T \begin{pmatrix} 1 & \vec{u}_0 \\ 1 & -\vec{u}_0 \end{pmatrix}^T \right) & \text{if } \vec{u} \ne 0. \end{cases} \tag{36}$$

where $\vec{u} = (u_1, u_2, u_3)$ with $u_i = \mathrm{tr}(\rho^A \sigma_i)/\sqrt{2}$, and $\vec{u}_0 = \vec{u}/|\vec{u}|$.

Similarly, for the Werner state ρ_W and the isotropic state ρ_I, the skew information MIN are given, respectively, by

$$\begin{aligned} N_{SI}(\rho_W) &= \frac{1}{2}\left(\frac{d-x}{d+1} - \sqrt{\frac{d-1}{d+1}(1-x^2)} \right), \\ N_{SI}(\rho_I) &= \frac{1}{d}\left(\sqrt{(d-1)x} - \sqrt{\frac{1-x}{d+1}} \right)^2. \end{aligned} \tag{37}$$

The above measure of MIN is somewhat different from that of the MIN-like nonlocality measure defined as [27]

$$\mathcal{U}_{SI}(\rho) = \max_{K^A} \mathcal{I}(\rho, K^A \otimes I_{d_B}), \tag{38}$$

which was motivated by the notion of local quantum uncertainty [28], and was termed as uncertainty-induced nonlocality (UIN), with K^A being the Hermitian observable on A with non-degenerate spectrum and commuting with ρ^A.

The UIN $\mathcal{U}_{SI}(\rho)$ is invariant under locally unitary operation $U_A \otimes U_B$, and is contractive under TPCP map $\mathcal{E}_B$ on subsystem B. It also equals to the maximal squared Hellinger distance between ρ and $K^A \rho K^A$, namely, $\mathcal{U}_{SI}(\rho) = \max_{K^A} D_H^2 (\rho, K^A \rho K^A)$, with $D_H^2(\rho, \chi) = \mathrm{tr}\{(\rho^{1/2} - \chi^{1/2})^2\}/2$.

For the $(2 \times n)$-dimensional state of Eq. (9), the UIN is obtained explicitly as

$$\mathcal{U}_{SI}(\rho) = \begin{cases} 1 - \lambda_{\min}(W) & \text{if } \vec{x} = 0, \\ 1 - \frac{1}{|\vec{x}|^2}\vec{x}^T W \vec{x} & \text{if } \vec{x} \ne 0, \end{cases} \tag{39}$$

where $\vec{x} = (r_{10}, r_{20}, r_{30})^T$, and $\lambda_{\min}(W)$ is the smallest eigenvalue of the 3×3 matrix W, the elements of which is given by

$$W_{ij} = \text{tr}\{\rho^{1/2}(\sigma_i \otimes I_{d_B})\rho^{1/2}(\sigma_j \otimes I_{d_B})\}, \tag{40}$$

and from Eq. (39) one can also obtain that for the pure $(2 \times n)$-dimensional state $|\psi\rangle$, $\mathcal{U}_{SI}(|\psi\rangle\langle\psi|)$ reduces to the linear entropy of entanglement $2[1 - \text{tr}(\rho^A)^2]$.

3.6 Generalization of the MIN Measure to Multipartite States

The MIN measures presented above are all defined based on the one-sided locally invariant measurements Π^A on party A. They characterize in fact only partial information about the nonlocal properties of a state ρ. This is because a local state with respect to the subsystem A may be nonlocal with respect to the subsystem B.

The MIN measures can be extended to the cases with two-sided locally invariant measurements. Without loss of generality, we define it as

$$\tilde{N}(\rho) = \max_{\tilde{\delta} \in \mathcal{L}} D(\rho, \tilde{\delta}), \tag{41}$$

with $\tilde{\delta}$ being states that are obtained by the full set of locally invariant measurements $\Pi^A \otimes \Pi^B$, that is to say, $(\Pi^A \otimes \Pi^B)\tilde{\delta}(\Pi^A \otimes \Pi^B) = \tilde{\delta}$, and the measurement operators satisfy the equality $\sum_k \Pi_k^A \rho^A \Pi_k^A = \rho^A$ and $\sum_k \Pi_k^B \rho^B \Pi_k^B = \rho^B$ for arbitrary ρ. This definition of MIN reveals the genuine nonlocal characteristic of a bipartite state, and $\tilde{N}(\rho) = 0$ implies locality with respect to both the subsystems of A and B.

An example of the MIN over two-sided measurements is as follows [21]

$$\tilde{N}_2(\rho) = \max_{\Pi^A \otimes \Pi^B} ||\rho - \Pi^A \otimes \Pi^B(\rho)||_2^2, \tag{42}$$

which is locally unitary invariant, and vanishes for the product states $\rho = \rho^A \otimes \rho^B$.

For pure state $|\psi\rangle$, solution of $\tilde{N}_2(|\psi\rangle\langle\psi|)$ is completely the same as $N_2(|\psi\rangle\langle\psi|)$. For the special case that both ρ^A and ρ^B are nondegenerate, the optimal measurement operators are uniquely determined by the eigenvectors of ρ^A and ρ^B, while for more general case, it can be calculated using the numerical method.

Other measures of MIN presented in the above sections can be redefined in a similar way, namely, by replacing the original one-sided locally invariant measurements Π^A with the two-sided locally invariant measurements $\Pi^A \otimes \Pi^B$.

In fact, the MIN measure $\tilde{N}_2(\rho)$ can also be extended to the more general case of N-partite state ρ. The definition can be written in the same form of Eq. (41), with however the set $\mathcal{L}$ of local quantum states being obtained by performing all possible locally invariant measurements $\Pi^{A_1} \otimes \Pi^{A_2} \otimes \cdots \otimes \Pi^{A_N}$, with $\sum_k \Pi_k^{A_i} \rho^{A_i} \Pi_k^{A_i} =$

ρ^{A_i} for $i = \{1, 2, \ldots, N\}$, and ρ^{A_i} the reduced state of the subsystem A_i. But now the evaluation of their analytical expression becomes a hard work.

Here we summarize briefly the MIN. The notion of MIN is a recently introduced measure of nonlocality which is defined from a measurement perspective, and provides a better division between the local and nonlocal features of a system. In general, the MIN can be defined as distance between the quantum states before and after the measurement is performed.

Next, we consider the behaviors of quantum correlations under local operations.

4 Quantum Correlations Increased by Local Operations

The states that can be prepared by local operations and classical communications (LOCC) are called the separable states. The set of bipartite separable states can be written as $\mathcal{S} := \left\{\rho|\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B\right\}$. By definition, $\mathcal{QC}$ is a strict subset of the separable states $\mathcal{S}$. There exist quantum correlated states which are separable and hence can be prepared via LOCC. Actually, the local operations (LO) alone can turn a QC state to a quantum correlated one. For example, consider a channel Λ with Kraus decompositions $K_1 = |0\rangle\langle 0|$ and $K_2 = |+\rangle\langle 1|$. When Λ is applied to B of the two-qubit QC state $\rho = \frac{1}{2}|00\rangle_{AB}\langle 00| + \frac{1}{2}|11\rangle_{AB}\langle 11|$, the output state $\rho_{out} = \frac{1}{2}|00\rangle_{AB}\langle 00| + \frac{1}{2}|1+\rangle_{AB}\langle 1+| \notin \mathcal{QC}$ has nonvanishing quantum correlations.

Then questions naturally arises:

(a) What kind of local operations have the ability to create quantum correlations?
(b) What is the power of a given local operation to create quantum correlations?
(c) What kind of states whose quantum correlations are more likely to be increased

In this section, we will give answers to these three questions.

4.1 *Condition for Local Creation of Quantum Correlations*

The main purpose of this subsection is to characterize the whole set of quantum operations satisfying

$$\mathbb{1}_A \otimes \Lambda_B(\rho) \in \mathcal{QC}, \ \forall \rho \in \mathcal{QC}. \tag{43}$$

Before solving the problem, let us first introduce a class of quantum channels, which we call the commutativity-preserving channels.

Definition 1 (*Commutativity-preserving channel*) A commutativity-preserving channel Λ^{CP} is the channel that can preserve the commutativity of any input density operators; i.e.

$$[\Lambda^{\text{CP}}(\xi), \Lambda^{\text{CP}}(\xi')] = 0 \tag{44}$$

holds for any density operators ξ and ξ' satisfying $[\xi, \xi'] = 0$.

When a commutativity-preserving channel acts on B of a QC state, the output state $\rho_{out} = \sum_i p_i \rho_i^A \otimes \Lambda^{\mathrm{CP}}(\phi_i^B)$ is still a QC state. This is because Bob's states $\Lambda^{\mathrm{CP}}(\phi_i^B)$ commute with each other and a projective measurement on their common eigenbasis does not change the state ρ_{out}.

Conversely, if Bob's channel Λ satisfies Eq. (43), it must be a commutativity-preserving channel. To see this, let us write the input state in the form of Eq. (2), and the output state $\rho_{out} = \sum_i p_i \zeta_i^A \otimes \Lambda(\xi_i^B)$ is still quantum-classical only when $[\Lambda(\xi_i^B), \Lambda(\xi_j^B)] = 0,\ \forall\, i, j$. As in Eq. (43), we have considered the whole set of QC states as input state, the channel Λ must preserves the commutativity of any two commutable states. Hence we arrive at the following theorem.

Theorem 1 *A local quantum channel Λ acting on a subsystem of a multipartite system can create quantum correlation if and only if it is not a commutativity-preserving channel.*

This theorem characterize the set of quantum channels which does not create quantum correlation in any QC states. The rest of this subsection will be devoted to provide the explicit form of the commutativity-preserving channels. We will see that when B is a qubit, the quantum correlations can never be created by a unital channel, but when B is a higher-dimension system, even unital channels can create quantum correlation.

Let us first consider the qubit case. In the Bloch presentation, any qubit state $\rho = \frac{I+\boldsymbol{r}\cdot\boldsymbol{\sigma}}{2}$ corresponds to a three-dimension real vector $\boldsymbol{r}$, where $\boldsymbol{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ are Pauli matrices and the Bloch vector $\boldsymbol{r}$ lives inside or on the surface of a unit ball, which is called the Bloch ball. The Bloch vectors of two commutative states are of the same or opposite orientation, so the necessary and sufficient condition for the commutativity-preserving channels is that they map radial segments onto radial segments in the Bloch ball. The unital channels are defined as those preserve the identity $\Lambda^u(I) = I$, i.e., the origin of the Bloch ball, and thus satisfies the above condition. Another set of channels which are apparently commutativity-preserving are the semiclassical channels, which map all input states onto states diagonal on the same basis. It can be strictly proved that a commutativity-preserving qubit channel is either a unital channel or a semiclassical channel. This leads to the following theorem.

Theorem 2 *A local quantum channel acting on a single qubit can create quantum correlations in a multiqubit system if and only if it is neither semiclassical nor unital.*

Now we turn to multipartite systems of higher-dimension. Apparently, the semiclassical channels do not have the ability to create quantum correlations. Here we propose another set of quantum channels, which we call the isotropic channels, that never create quantum correlations.

Definition 2 (*isotropic channel*) An isotropic channel is a channel $\Lambda : \mathcal{D}(\mathcal{H}_d) \to \mathcal{D}(\mathcal{H}_d)$ of the form

$$\Lambda^{\mathrm{iso}}(\rho) = p\Gamma(\rho) + (1-p)\frac{I}{d}, \tag{45}$$

where Γ is any linear channel that preserves the eigenvalues of ρ, and the parameter p is chosen to make sure that Λ is a completely positive channel.

According to Ref. [29], Γ is either a unitary operation or unitarily equivalent to transpose. Direct calculations lead to $-1/(d-1) \leq p \leq 1$ when Γ is a unitary operation, and $-1/(d-1) \leq p \leq 1/(d+1)$ when Γ is unitarily equivalent to transpose.
Because the unitary operations and the transpose preserve the commutativity and the identity commutes with any state, the isotropic channels are commutativity preserving for arbitrary d. For $d = 3$, we have strictly proved that a commutativity channel is either semiclassical or isotropic.

Theorem 3 *A local quantum channel acting on a single qutrit of a multipartite system can create quantum correlations if and only if it is neither semiclassical nor isotropic.*

Since isotropic channels are a strict subset of unital channels, there exist unital channels that are able to locally create quantum correlations. Here we give an example to look more closely at why a unital channel can create quantum correlation in multipartite states of higher dimensions. Let us consider the unital channel $\Lambda(\cdot) = \sum_i E^{(i)}(\cdot)E^{(i)\dagger}$ with

$$E^{(0)} = |2\rangle\langle 2|, E^{(1)} = |0\rangle\langle 0| + |1\rangle\langle 1|. \tag{46}$$

It is *not* a commutativity-preserving channel, because when we choose the orthogonal pure state $|\psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle)$ and $|\phi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |2\rangle)$ as input states, the output states do not commute. Two higher-dimension orthogonal states may become nonorthogonal when projected to subspaces. This is just the reason for creating quantum correlation using a local unital channel. Isotropic channels act on all of the states equivalently, so they are likely the only subset of unital channels which belongs to the set of commutativity-preserving channels. This observation leads to the following conjecture.

Conjecture *A local quantum channel acting on a single qubit with $d > 3$ can create quantum correlations in a multipartite system if and only if it is neither semiclassical nor isotropic.*

4.2 Quantum Correlating Power of Local Operations

So far, we have discussed the problem of *whether* a quantum channel can create quantum correlations. The problem of *how much* quantum correlations can be created by a quantum channel is the theme of this subsection. The quantum-correlating power of quantum channel is defined as the maximum amount of quantum correlations that can be created when the channel is applied locally to a subsystem of a multipartite system [31]. The formal definition is given as follows.

Definition 3 (*Quantum correlating power, QCP*) The quantum correlating power of a quantum channel Λ is defined as

$$\mathcal{Q}(\Lambda) := \max_{\rho \in \mathcal{QC}} Q(\mathbb{1} \otimes \Lambda(\rho)). \tag{47}$$

The QCP is an intrinsic attribute of a quantum channel, which quantifies the channel's ability to create quantum correlations. In the definition of QCP, the maximization is taken over the set of all quantum-classical states. The input states that correspond to the maximization are called the optimal input states, which are proved to be in the set of classical-classical (CC) states

$$\mathcal{CC} = \left\{ \rho | \rho = \sum_i p_i |\psi_i\rangle_A \langle\psi_i| \otimes |\phi_i\rangle_B \langle\phi_i| \right\}. \tag{48}$$

where $\{|\psi_i\rangle_A\}$ and $\{|\phi_i\rangle_B\}$ are orthogonal basis of $\mathcal{H}_A$ and $\mathcal{H}_B$ respectively. The proof can be easily sketched. For any output state ρ' that corresponds to a general QC input state, we can find a CC state, whose corresponding output state ρ can be transformed to ρ' by a local channel on A, i.e., $\rho' = \Lambda_A \otimes \mathbb{1}\rho$. From the condition (C3), we have $Q(\rho) \geq Q(\rho')$. Hence the definition of QCP can be optimized to

$$\mathcal{Q}(\Lambda) := \max_{\rho \in \mathcal{CC}} Q(\mathbb{1} \otimes \Lambda(\rho)). \tag{49}$$

A channel with larger amount of QCP is more quantum, in the sense of the ability to create quantum correlation. Hence it is of interest to find out the quantum channels with the most QCP. It can be proved that, the local single-qubit channel which maximum QCP can be found in the set of measuring-preparing channels

$$\mathcal{MP} = \left\{ \Lambda | \Lambda(\cdot) = \sum_{i=1}^{1} |\alpha_i\rangle\langle\phi_i| \cdot |\phi_i\rangle\langle\alpha_i| \right\}, \tag{50}$$

where $|\alpha_0\rangle$ and $|\alpha_1\rangle$ are two nonorthogonal pure states.

When two channels used paralleled, the QCP of the composed channel is no less than the sum of the QCPs of the two channels. We call this property the superadditivity of QCP [32]. We here give an example of phase-damping (PD) channel to show exactly how this property works. The Kraus operators of PD channel are $E_0^{\mathrm{PD}} = |0\rangle\langle 0| + \sqrt{1-p}|1\rangle\langle 1|$ and $E_1^{\mathrm{PD}} = \sqrt{p}|1\rangle\langle 1|$. Here we consider the nontrivial case where $0 < p < 1$. Clearly, PD channel is a unital channel and thus has vanishing QCP.

Now consider a four-qubit initial state shared between Alice and Bob

$$\rho_{AA'BB'} = \frac{1}{4} \sum_{i,j} |ij\rangle_{AA'} \langle ij| \otimes |\psi_{ij}\rangle_{BB'} \langle\psi_{ij}|, \tag{51}$$

where $|\psi_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\psi_{11}\rangle = \frac{1}{\sqrt{2}}(|0+\rangle + |1-\rangle)$, $|\psi_{01}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, and $|\psi_{10}\rangle = \frac{1}{\sqrt{2}}(|0-\rangle - |1+\rangle)$. Here qubits AA' belong to Alice and BB' belong to Bob. Since $|\psi_{ij}\rangle$ are orthogonal to each other, the quantum correlation on Bob is zero. Then qubits B and B' each transmits through a PD channel, and the output state becomes $\rho'_{AA'BB'} = \mathbb{1}_{AA'} \otimes \Lambda_B^{\mathrm{PD}} \otimes \Lambda_{B'}^{\mathrm{PD}}(\rho_{AA'BB'})$. Because $[\Lambda^{\mathrm{PD}} \otimes \Lambda^{\mathrm{PD}}(\psi_{00}), \Lambda^{\mathrm{PD}} \otimes \Lambda^{\mathrm{PD}}(\psi_{11})] = \frac{1}{8}\tilde{i}p\sqrt{1-p}(\sigma^y \otimes \sigma^z + \sigma_z \otimes \sigma^y) \neq 0$, the output state $\rho'_{AA'BB'}$ is not a QC state. Therefore, the quantum correlation on Bob's qubits BB' is created by the channel $\Lambda_B^{\mathrm{PD}} \otimes \Lambda_{B'}^{\mathrm{PD}}$.

The super-activation of QCP is a collective effect. For both the input state $\rho_{AA'BB'}$ and the output state $\rho'_{AA'BB'}$, any two-qubit marginal is a completely mixed state. In other words, no correlation exists between any two qubits of the four-qubit state $\rho'_{AA'BB'}$. Therefore, we suppose that the effect of super-additivity of QCP is due to the genuine quantum correlation.

4.3 States Whose Quantum Correlations Can Be Increased

Our aim is now to characterize the quantum states whose quantum correlations can be increased locally. This is a less studied subject than the condition on quantum channels to locally create quantum correlations. Obviously, the quantum correlation of all the QC state can be increased locally, by the quantum channels which are not commutativity preserving. However, it is not obvious whether the quantum correlation of a discordant state can be increased locally.

Before study the problem, we first introduce the quantum steering ellipsoids (QSE), which provides a natural geometric presentation of two-qubit states. The quantum steering ellipsoid of a two-qubit state ρ_{AB} is the whole set of Bloch vectors that the qubit A can be collapsed to by a positive-operator valued measurement (POVM) on qubit B. When the Bloch vector $\boldsymbol{b}$ of ρ_B satisfies $b = 1$, ρ_B is a pure state which is not correlated to A; hence the QSE at A reduces to a single point $\boldsymbol{a}$.

Now we consider the case with $b \in [0, 1)$. Suppose the qubit B is projected to a pure state $\rho_{\boldsymbol{x}}$ with Bloch vector $\boldsymbol{x}$. The state of A is steered to $\rho_A^S = \mathrm{tr}_B[\rho_{AB}(I \otimes \rho_{\boldsymbol{x}})]/\mathrm{tr}[\rho_{AB}(I \otimes \rho_{\boldsymbol{x}})]$, whose Bloch vector is $\boldsymbol{a}^S = \frac{\boldsymbol{a}+T\boldsymbol{x}}{1+\boldsymbol{b}\cdot\boldsymbol{x}}$. Let $\boldsymbol{x}$ varies through the Bloch ball, the set of corresponding $\boldsymbol{a}^S$ forms an ellipsoid

$$\mathcal{E}_A = \left\{ \frac{\boldsymbol{a} + T\boldsymbol{x}}{1 + \boldsymbol{b} \cdot \boldsymbol{x}} | x \leq 1 \right\}. \tag{52}$$

To obtain $\mathcal{E}_B$, one only need to make the substitution $\boldsymbol{a} \to \boldsymbol{b}, \boldsymbol{b} \to \boldsymbol{a}, T \to T^{\mathrm{T}}$. It is worth mentioning that the QSE $\mathcal{E}_A$ and $\mathcal{E}_B$ of state ρ_{AB} have the same dimension, which equals to $\mathrm{rank}(\Theta) - 1$.

The state ρ_{AB} is a QC state if and only if $\mathcal{E}_B$ is a radial line segment. Local channels on qubit B can create B-side quantum discord from the above quantum-classical state. The output discordant state can be written as

$$\rho \equiv \mathbb{I}_A \otimes \Lambda_B(\rho_{AB}) = p_0\rho_0^A \otimes \rho_0^B + p_1\rho_1^A \otimes \rho_1^B. \tag{53}$$

Here $\rho_i^B \equiv \Lambda_B(|\phi_i\rangle_B\langle\phi_i|)$ $(i = 0, 1)$ do not commute with each other [30] and thus are linearly independent. The following statement builds the connection between locally created discordant states and the states with needle-shape QSE. *A B-side discordant two-qubit state can be created from a classical state by a trace-preserving local channel on B if and only if its QSE at qubit B* $\mathcal{E}_B$ *is a non-radial line segment* [33]. It means that all of the quantum states with pancake-shape or obese-shape QSE, even though not entangled, can not be prepared by local operations.

Next we focus on the Bell diagonal states and study the relation between the effect of locally increased quantum discord and the shape and position of QSE. For a Bell diagonal two-qubit state, the density matrix can be written as

$$\tilde{\rho} = \frac{1}{4}\left(\sigma_0 \otimes \sigma_0 + \sum_{i=1}^{3} c_i\sigma_i \otimes \sigma_i\right). \tag{54}$$

For such a state, both $\mathcal{E}_A$ and $\mathcal{E}_B$ are unit spheres shrunk by c_1, c_2 and c_3 in the x, y and z direction, respectively. After the action of an amplitude damping channel on B, the QSE at B becomes

$$\mathcal{E}_B^{\mathrm{AD}} = \left\{ \begin{pmatrix} 0 \\ 0 \\ p \end{pmatrix} + \begin{pmatrix} \sqrt{1-p}c_1x_1 \\ \sqrt{1-p}c_2x_2 \\ (1-p)c_3x_3 \end{pmatrix} \middle| x \leq 1 \right\}. \tag{55}$$

The effect of Λ_B^{AD} on $\mathcal{E}_B$ is to translate it by p in the z direction and meanwhile shrink the ellipsoid on three directions. Notice that when $p > \frac{c_3}{1+c_3}$, the ellipsoid does not contain the origin point any more.

Increase of quantum discord by local amplitude damping channel on B can occur for all of the three cases when the initial QSE is a needle, a pancake and an obese. For the last two cases, the local increase of discord occurs when $|c_1| \gg |c_2|, |c_3|$, which means that the shape of the plate or the ball is like a baguette perpendicular to the z axis. It is worth noticing that, the local quantum operation can increase the quantum discord of an entangled state.

5 Summary

A local quantum channel acting on a subsystem of a multipartite system can create quantum correlation if and only if it is not a commutativity-preserving channel. A qubit channel is commutativity-preserving if and only if it is unital or semiclassical. For the high-dimension case, some unital channels have the ability to create quantum correlations. In order to characterise the maximum quantum correlation that a quantum channel can create, the quantum-correlating power is defined. It is an intrinsic property of quantum channels. The superactivity of QCP is proved. Concerning the two-qubit states whose quantum correlation can be increased locally, it is observed that the quantum correlation of those states possesses baguette-like quantum steering ellipsoids.

We also presented that nonlocality of a quantum state can be described from different aspects. We provided a short review about the recently introduced quantification of MIN by considering the distances for states before and after the measurement is taken. The MIN in terms of distance can be defined based on the Hilbert–Schmidt norm, the trace norm, the Bures distance, the von Neumann entropy, and the Wigner–Yanase skew information. The basic formulae for their respective definitions, the analytical solutions of them for certain special states, and a comparison of their similarities and differences, are given in detail. We have also provided a outlook for its further development such as its generalization to multipartite systems.

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References

1. A. Einstein, B. Podolsky, N. Rosen, Can quantum-mechanical description of physical reality be considered complete? Phys. Rev. **47**, 777–780 (1935)
2. M. Genovese, Research on hidden variable theories: a review of recent progresses. Phys. Rep. **413**, 319–396 (2005)
3. R.F. Werner, Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. Phys. Rev. A **40**, 4277–4281 (1989)
4. L. Masanes, Y.C. Liang, A.C. Doherty, All bipartite entangled states display some hidden nonlocality. Phys. Rev. Lett. **100**, 090403 (2008)
5. K. Modi, A. Brodutch, H. Cable, Z. Paterek, V. Vedral, The classical-quantum boundary for correlations: discord and related measures. Rev. Mod. Phys. **84**, 1655–1707 (2012)
6. S. Popescu, R. Rohrlich, Nonlocality as an axiom. Found. Phys. **24**, 379 (1994)
7. S. Popescu, Nonlocality beyond quantum mechanics. Nat. Phys. **10**, 264 (2014)
8. H. Fan, Y.N. Wang, L. Jing, J.D. Yue, H.D. Shi, Y.L. Zhang, L.Z. Mu, Quantum cloning machines and the applications. Phys. Rep. **544**, 241–322 (2014)
9. H. Ollivier, W.H. Zurek, Quantum discord: a measure of the quantumness of correlations. Phys. Rev. Lett. **88**, 017901 (2002)
10. D. Girolami, G. Adesso, Quantum discord for general two-qubit states: analytical progress. Phys. Rev. A **83**, 052108 (2011)
11. K. Modi, T. Paterek, W. Son, V. Vedral, M. Williamson, Unified view of quantum and classical correlations. Phys. Rev. Lett. **104**, 080501 (2010)
12. M. Piani, S. Gharibian, G. Adesso, J. Calsamiglia, P. Horodecki, A. Winter, All nonclassical correlations can be activated into distillable entanglement. Phys. Rev. Lett. **106**, 220403 (2011)
13. M.F. Cornelio, M.C. de Oliveira, F.F. Fanchini, Entanglement irreversibility from quantum discord and quantum deficit. Phys. Rev. Lett. **107**, 020502 (2011)
14. S. Luo, Using measurement-induced disturbance to characterize correlations as classical or quantum. Phys. Rev. A **77**, 022301 (2008)
15. L. Mista Jr., R. Tatham, D. Girolami, N. Korolkova, G. Adesso, Measurement-induced disturbances and nonclassical correlations of Gaussian states. Phys. Rev. A **83**, 042325 (2011)

16. S. Luo, S. Fu, Measurement-induced nonlocality. Phys. Rev. Lett. **106**, 120401 (2011)
17. M.L. Hu, H. Fan, Measurement-induced nonlocality based on the trace norm. New J. Phys. **17**, 033004 (2015)
18. M.L. Hu, H. Fan, Dynamics of entropic measurement-induced nonlocality in structured reservoirs. Ann. Phys. **327**, 2343–2353 (2012)
19. Z. Xi, X. Wang, Y. Li, Measurement-induced nonlocality based on the relative entropy. Phys. Rev. A **85**, 042325 (2012)
20. L. Li, Q.W. Wang, S.Q. Shen, M. Li, Measurement-induced nonlocality based on Wigner-Yanase skew information. Europhys. Lett. **114**, 10007 (2016)
21. Y. Guo, Measurement-induced nonlocality over two-sided projective measurements. Int. J. Mod. Phys. B **27**, 1350067 (2013)
22. S. Luo, S. Fu, Global effects of quantum states induced by locally invariant measurements. Europhys. Lett. **92**, 20004 (2010)
23. S. Rana, P. Parashar, Geometric discord and measurement-induced nonlocality for well known bound entangled states. Quantum Inf. Process. **12**, 2523–2534 (2013)
24. S.Y. Mirafzali, I. Sargolzahi, A. Ahanj, K. Javidan, M. Sarbishaei, Measurement-induced nonlocality for an arbitrary bipartite state. Quantum Inf. Comput. **13**, 479–489 (2013)
25. R. Jozsa, Fidelity for mixed quantum states. J. Modern Opt. **41**, 2315–2323 (1994)
26. E.P. Wigner, M.M. Yanase, Information contents of distributions. Proc. Natl. Acad. Sci. U. S. A. **49**, 910–918 (1963)
27. S. Wu, J. Zhang, C. Yu, H. Song, Uncertainty-induced quantum nonlocality. Phys. Lett. A **378**, 344–347 (2014)
28. D. Girolami, T. Tufarelli, G. Adesso, Characterizing nonclassical correlations via local quantum uncertainty. Phys. Rev. Lett. **110**, 240402 (2013)
29. M. Marcus, B.N. Moyls, Linear transformations on algebras of matrices. Can. J. Math. **11**, 61 (1959)
30. X.Y. Hu, H. Fan, D.L. Zhou, W.M. Liu, Necessary and sufficient conditions for local creation of quantum correlation. Phys. Rev. A **85**, 032102 (2012)
31. X. Hu, H. Fan, D.L. Zhou, W.M. Liu, Quantum correlating power of local quantum channels. Phys. Rev. A **87**, 032340 (2013)
32. X. Hu, H. Fan, D.L. Zhou, W.M. Liu, Superadditivity of quantum-correlating power. Phys. Rev. A **88**, 012315 (2013)
33. X. Hu, H. Fan, Effect of local channels on quantum steering ellipsoids. Phys. Rev. A **91**, 022301 (2015)