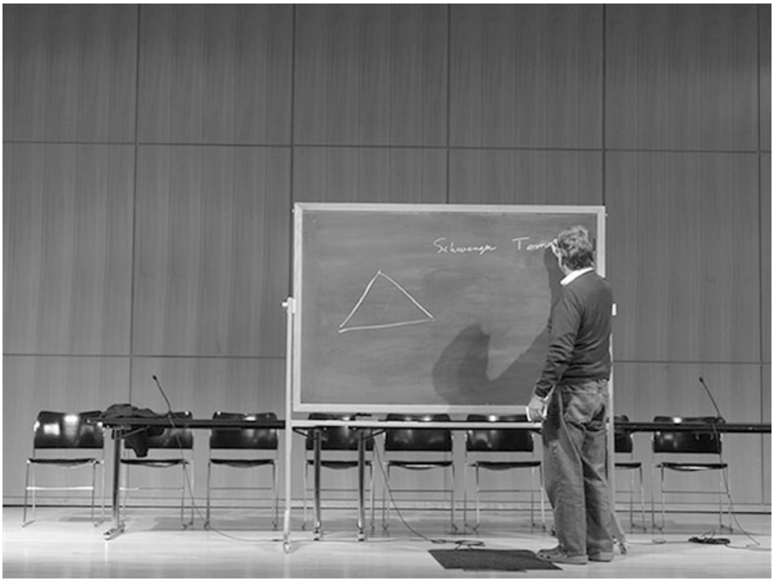

Dennis Sullivan
Photo by María Clara Cortés

Simplicity Is the Point

Dennis Sullivan

Editors' note: This text is an edited transcript of the author's conference talk.

I've been concentrating on mathematics for 52 years now, and I have a lot of opinions about it. I've tried to distill them down to a few things, which I want to share with you. That's the nature of this talk.

I've really liked the idea of simple things in mathematics, and I feel that it's incumbent upon a mathematical subject's participants to try to get it into the simplest form so that it is easy to communicate, easy to teach, easy to understand. Understanding is more important to me than proofs, although the way I come to understand things is often through just a few different proofs or proof forms, which you kind of move around in different settings. So, proof and understanding are intimately tied, but understanding is, for me, the primary goal, and simplicity plays a role in that. If you've found the simple organizing points of some discussion, then it's easy to understand. Now, it could be that you start from those points and develop fairly elaborate discussions from them while staying aware of what's essential.

I was just telling someone here at the conference—this is a digression—one way to find out what the important points are—I determined this when I was young and went to a lot of talks I didn't understand—is first of all you find out who the masters are and who they aren't. Now, since [Misha] Gromov is here, I should say that he is a master, but this doesn't work so well for him. For some masters, like Alain Connes, it works. You listen to them, you don't understand what they are saying, but you wait until they get excited about something. For example, Connes, he's very excited about the fact that L_∞ is a dual space. Some mathematicians here can try to

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figure out why that's important, but it's a very banal fact if you're a graduate student and you might miss it even though it's very important. The reason it doesn't work with Gromov is that he's excited at every moment in time, so you have to listen until he repeats something, then you know that's an important point. So, there are these simple points, and one searches for these simple points when trying to understand a field.

The two stories I'll tell are about finding these points, whether it's before you understand something or after when you say, "Ah, these are the two, three, or four main points that make this thing build up." Often there's another aspect to simplicity—this is banal, everything I'm going to say is rather pretty banal—if something's very simple, it's easy to use. You can tell it to people. They can learn it, and it's easy to use. And you can use it many times. For example—I'm getting used to making jokes about Gromov, I'm sorry, it's only because he's here—I looked at all of his work and I decided he just knows one thing: the triangle inequality. The triangle inequality says that if you have a triangle, the sum of the distances along two sides is at least as big as the distance along the third. A lot of his work is just using that key point *[To Mikhail Gromov in the audience]*. Would you disagree or not? *[Gromov replies, "It's a good point."]* Okay, this inequality is a very simple idea, but because it's simple, you can use it a million times. And Gromov used it like crazy. Anyway, on to the first story.

I'm telling this story because I just finished reading a book about Richard Feynman, but the point I want to make isn't unique to him. So, Feynman along with Julian Schwinger and Sin-Itiro Tomonaga shared the 1965 Nobel Prize in Physics. Let's say it's for Quantum Electro-Dynamics (QED) and understanding what's called "renormalization." This is part of a big story which is still ongoing, and it's not something that's understood. It seems to be related to mathematics however, so let's say that some part of it can be understood as a so-far-not-understood part of mathematics as well as being very important in physics. Here's a two-minute lecture on this entire theory: When you look down into water at an angle and see an object below the surface, you don't see the object where it's actually located. It turns out that the light rays that contribute to your sensory response to this object in the water haven't traveled in a straight line path. Anyone who's ever looked at their own foot in the water knows that. There's this principle, called the action principle, which is the first simple idea. The idea is that physical systems work to minimize some function, some value, of the state of the system. Feynman generalized this idea by considering every path, whether straight or not, that the light might follow and weighted each path with a certain efficiency. If a path is very costly, then light will not use that path very much. Summing over all paths produces the outcomes of any physical experiments, and mathematically, you write this sum as an integral

$$\int e^{\text{Action}}$$

where the action, this thing to be minimized, goes in the exponent.

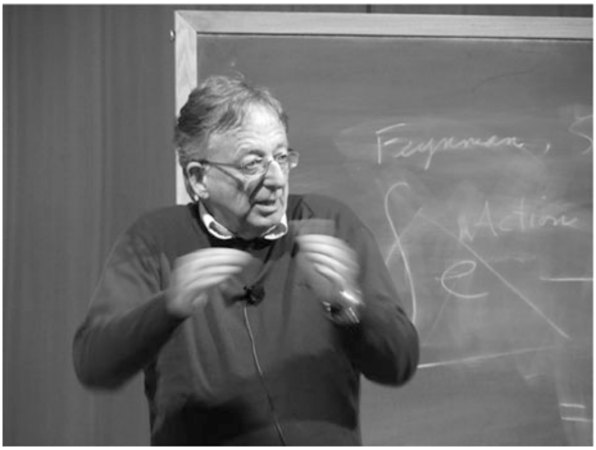

Fig. 1 "Once Feynman's idea emerged on the scene, the very fancy way of doing things that Schwinger had developed just disappeared." Photo by Wanda Siedlecka

One thing that makes QED so famous is that, in some sense, it's the most successful scientific algorithm there is. It could compute a certain measurable quantity to a large number of decimal places, say ten. That's sort of remarkable to have a theory that could fit with experiment to that many decimal places, so they got the Nobel Prize. Tomonaga in Japan, Schwinger at Harvard, and Feynman at Princeton, CalTech, and Cornell, independently all achieved a certain algorithm for QED. (Freeman Dyson, a mathematician at the Institute for Advanced Study in Princeton, also proved this independently.) And the point of the story is that—well, I don't know what happened in Japan, I mean Tomonaga completed his work around 1941, and then there was the war and I don't know what propagated from then. But, to compare Schwinger's version and Feynman's version, first of all, it's interesting to compare the two scientists as individuals. Schwinger was distinguished, from a well-to-do family, a limousine would take him to his lectures at Harvard. He was the youngest full professor at Harvard of all time. His lectures were beautiful. He had 200 PhD students. His formulas were elegant, complicated, awe-inspiring. Feynman now, Feynman was a smart Jewish kid from Brooklyn who talked like a World War II guy. He figured out how to do integrals in high school, and he liked to do integrals. He found out that if you put a parameter, it's usually called h, in front of the exponent then you could think of the integral as a function of the parameter:

$$\int e^{h\text{Action}}.$$

Now you can play around with this and differentiate it with respect to the parameter and get an equation and you work out integrals with parameters. So, like a high school student, he just kind of did the integral for physics. He actually worked out this integral for examples and found a big infinite series and in terms of this parameter,

$$\int e^{h\mathrm{Action}} = \lambda + \Delta h + O(h^2) + \cdots$$

Anyway, it's very simple to talk about this, I mean you have to have a little math, if you are a freshman in college you can understand this computation in form. But Feynman went further by making a graphical picture of this calculation in terms of so-called "Feynman diagrams" that imagine these terms as particles, photons, and things moving around and interacting. Feynman's idea, even though it involves fairly complicated ingredients, it's basically a simple idea.

Once Feynman's idea emerged on the scene, the very fancy way of doing things that Schwinger had developed just disappeared. Well, the fact that he was at Harvard and had 200 graduate students and gave excellent lectures, that kept it alive for a while. But Feynman, this guy had virtually no graduate students, maybe one or two, because of who he was; his personality was such that he had to do everything himself, he had to be the smartest guy in the room. He wasn't a good co-worker. But then he figured out something simple that would describe this idea and then, everything just switched. Schwinger and Tomonaga's techniques just got erased, I mean you don't hear about them anymore. But Feynman's idea, it has legs, as we say. An idea has legs if it just goes, and this idea just goes and goes.

Not everybody agrees with the principle that the goal of mathematicians is to reduce mathematical subjects to these simple essential points. I agree very much with what Gromov said earlier in the conference, that things may be simple only in appearance. When I look down and I immediately see my white shoes and another person's black shoes, that's super complicated actually. I remember having a big argument with [Shing-Tung] Yau. Many years ago, we were at a dinner, and he was talking about physics. I was saying, "you know, for me, this glass of water is a lot more complicated than a Riemann surface." So he started to argue with that. My reasoning was that I can go all the way back to Hugh Woodin's set theory and start from there, and I can build up the integers, the real numbers, Euclidean space, manifolds, differential structures, conformal structures, and I can define a Riemann surface. But, even deterministically speaking, I still can't say what a glass of water is. What is that water? Molecules moving around; looks like a fluid, while it's supposed to be made of atoms. Is there glass around? We are nowhere near understanding a glass of water. It's not simple, in fact, it's very complicated. The Riemann surface is abstract and it's simple. I can tell you what it is. I can take a smart high school student, and in a year, teach them everything that mathematicians know about the definition of a Riemann surface. That's one point about simplicity that agrees with Gromov's point. Actually, it also agrees with what Dusa [McDuff] said during the discussion this afternoon about definitions and proofs—actually, we want concepts and definitions that define and annunciate the discussion.

My second story is a personal story. It means a lot to me and it illustrates my abstract. I was an undergraduate at Rice University. In the first year we took Math 100, Physics 100, Chemistry 100—the big three. Those were hard courses, and you had to learn how to study, learn how to pass exams, learn the material. Then I went to graduate school. I kept the Rice method, I knew how to work, how to learn things. For my oral exams, I was reading a book by [John] Milnor, called *Topology from the Differentiable Viewpoint*. I applied my Rice method. I read and understood the whole book. I can tell you everything about this book, because it's all in my head, like a computer program: homotopy, cobordism, transversality, manifolds, mappings between spheres, all this sort of stuff. But the day before the exam, even though I knew the book backwards and forwards, its theorems and proofs, I decided to go back and look at it one more time. I went to the library, and I took the book out. While I was looking at it I saw this picture of a slinky, which I will try to explain.

Take a flat piece of paper, you can wrap it over the surface of a ball, tie it all together at the top, and you get a sphere. Okay? That's clear. You can also do that in 3-space; you can take a volume of space, imagine you are outside of it, and wrap it all up the same way to form the 3-dimensional sphere. And the problem is to study all ways of taking this wrapped up 3-dimensional space and pushing it down around this wrapped up 2-dimensional space. This slinky picture tells you basically all the ways you can do this. If everybody is ready for it, I can sort of prove something now. Imagine a slinky made of some perfectly elastic and strong material, like mithril. It's very long, and I extend it, twirl it around in 3-space into some kind of knot, and then bring its two ends back together. This fills up a part of the 3-dimensional space. Now I want to define a map from the three sphere down onto the two sphere. Here's what I am going to do with the long knotted slinky loop. I cut it in one place and make a little mark on each side. Then I let the slinky collapse on itself the way they do. And since it's mithril, when it comes together, this large knot comes to almost nothing, just a very thin cylinder. Then I push this coil down on to the 2-dimensional sphere, and I make sure the marks line up. Remember, the two marks came from the spot where I pulled the slinky loop apart. Everything inside the slinky tube goes along with it and gets pushed down its edge. Whereas I map all the points in three space outside of the slinky to the one point on the 2-sphere where that surface is tied together.

Fig. 2 "Imagine a slinky made of some perfectly elastic and strong material, like mithril. It's very long, and I extend it, twirl it around in 3 space into some kind of knot, and then bring its two ends back together." Photo by Wanda Siedlecka

Okay, that's a picture. It turns out that all the maps from the 3-sphere to the 2-sphere are essentially like that, except you might have several slinkies. But, from that picture, that one picture, you could operate on my brain and remove the memory of having read that entire book and understood it as a Rice undergraduate, and with just that picture (assuming I know the language of homotopy, manifolds, bordism, etc.), I can write out the whole book. I had this great feeling: that's what it means to understand a piece of mathematics! I see this one picture, and the whole theory evolves from that picture. I studied the whole book up and down, and then I made this redundant step, like supersaturation. Of course, this picture is what the proof says, but they don't say it like this, they go through it logically. But that's the one simple point; if you understand that picture, you can explain it. So that's the way I'd like to see a mathematical discussion, it might look very complicated, but there are central points like these.

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