Chapter 4
The Statistical Analysis of Experimental Results

We will now examine the way in which we can best use the results of experimental measurements to reach conclusions relating to the magnitude measured. Understanding the concepts and methods presented in this chapter possibly constitutes the main benefit the reader may derive from studying this book.

4.1 The Mean and the Dispersion of the Results of Measurements

Let us assume that we have measured a quantity N times, under exactly the same experimental conditions. We will use the results of the measurements in order to get an estimate of the value of the magnitude measured. Let the *real value* of the measured quantity be x_0, which is, of course, unknown to us. If the result of the i-th measurement is x_i, the *error* in this measurement is defined as

$$e_i \equiv x_i - x_0. \tag{4.1}$$

If a total of N measurements have been performed, the *mean* of their results, x_i $(i = 1, 2, \ldots, N)$, is defined as

$$\bar{x} \equiv \frac{1}{N}\sum_{i=1}^{N} x_i. \tag{4.2}$$

The difference of the mean from the true value is defined as the *error in the mean*,

$$e_\mu = e_{\bar{x}} \equiv \bar{x} - x_0. \tag{4.3}$$

C. Christodoulides and G. Christodoulides, *Analysis and Presentation of Experimental Results*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-53345-2_4

Combining these equations, we have

$$e_\mu = e_{\bar{x}} = \bar{x} - x_0 = \frac{1}{N}\sum_{i=1}^{N} x_i - x_0 = \frac{1}{N}\sum_{i=1}^{N}(x_0 + e_i) - x_0 = \frac{1}{N}\sum_{i=1}^{N} e_i. \quad (4.4)$$

The error in the mean,

$$e_\mu = e_{\bar{x}} = \frac{1}{N}(e_1 + e_2 + \ldots + e_i + \ldots + e_N), \quad (4.5)$$

is, therefore, equal to the mean of the errors of the results. Given that the errors e_i are both negative and positive, and that we usually assume that it is equally probable for an error to be negative or positive, the absolute value of e_μ will be smaller than the largest absolute value among the errors e_i. In general, therefore, the mean $\bar{x}$ will be nearer to x_0 than the worst result of the measurements. We actually expect the mean e_μ of the errors to decrease as we increase the number of measurements N. Thus, we accept that the mean $\bar{x}$ is the best estimate we have for x_0 after N measurements of the magnitude x. A rigorous proof of this statement, based on the theory of errors, will be given below.

Example 4.1

Let the unknown real length of a rod be 100 mm and that 10 measurements of the length of the rod gave the following results (in mm):

100.1 100.2 99.8 100.3 99.9 100.2 99.9 100.4 100.0 100.3.

We want to find $\bar{x}$ and e_μ.

We construct a table with columns showing the corresponding values of i, x_i and e_i, for $N = 10$ ($i = 1, 2, \ldots, 10$).

i	x_i (mm)	e_i (mm)
1	100.1	0.1
2	100.2	0.2
3	99.8	−0.2
4	100.3	0.3
5	99.9	−0.1
6	100.2	0.2
7	99.9	−0.1
8	100.4	0.4
9	100.0	0.0
10	100.3	0.3
Σ =	1001.1	1.1

The sum of the x_i's is: $\sum_{i=1}^{10} x_i = 1001.1$ mm.

The value of their mean is: $\overline{x} = \frac{1}{10}\sum_{i=1}^{10} x_i = 100.11 \approx 100.1\,\text{mm}$.
The error in the mean is equal to $e_\mu = \overline{x} - x_0 = 0.11 \approx 0.1\,\text{mm}$.
This error is not known to us.

Since the real value x_0 of the quantity being measured is unknown to us, the errors e_i and e_μ are also unknown. It is therefore impossible for us to examine the dispersion of the measurements x_i relative to the real value. We can, however, examine the dispersion of the measurements relative to the mean $\overline{x}$ of the measurements, which is known to us. We define the difference of the measurement x_i from the mean of all the measurements as

$$d_i \equiv x_i - \overline{x}, \tag{4.6}$$

which is known as the *deviation of measurement x_i from the mean* or as the *residual* of the measurement.

From the definitions $e_i = x_i - x_0$ and $d_i = x_i - \overline{x}$, it is $x_i = x_0 + e_i = \overline{x} + d_i$ and

$$e_i - d_i = \overline{x} - x_0. \tag{4.7}$$

Therefore, $e_1 + e_2 + \ldots + e_N = (x_1 - x_0) + (x_2 - x_0) + \ldots + (x_N - x_0) = N(\overline{x} - x_0)$

or

$$\sum_{i=1}^{N} e_i = N(\overline{x} - x_0) \tag{4.8}$$

and $d_1 + d_2 + \ldots + d_N = (x_1 - \overline{x}) + (x_2 - \overline{x}) + \ldots + (x_N - \overline{x}) = N\overline{x} - N\overline{x} = 0$

or

$$\sum_{i=1}^{N} d_i = 0. \tag{4.9}$$

Example 4.2

In the table of Example 4.1 we now also record the values of $|e_i|$, d_i and $|d_i|$.

i	x_i (mm)	e_i (mm)	$\lvert e_i \rvert$ (mm)	d_i (mm)	$\lvert d_i \rvert$ (mm)
1	100.1	0.1	0.1	−0.01	0.01
2	100.2	0.2	0.2	0.09	0.09
3	99.8	−0.2	0.2	−0.31	0.31
4	100.3	0.3	0.3	0.19	0.19
5	99.9	−0.1	0.1	−0.21	0.21
6	100.2	0.2	0.2	0.09	0.09

(continued)

(continued)

i	x_i (mm)	e_i (mm)	$\|e_i\|$ (mm)	d_i (mm)	$\|d_i\|$ (mm)
7	99.9	−0.1	0.1	−0.21	0.21
8	100.4	0.4	0.4	0.29	0.29
9	100.0	0.0	0.0	−0.11	0.11
10	100.3	0.3	0.3	0.19	0.19
Σ =	1001.1	1.1	1.9	0	1.70

We use the value $\bar{x} = 100.11$ mm in the estimation of the d_i and $|d_i|$.

As expected, we find that $\sum_{i=1}^{N} d_i = 0$.

Also, $\overline{|e_i|} = \frac{1}{N}\sum_{i=1}^{N}|e_i| = \frac{1.9}{10} = 0.19 \approx 0.2$ mm, $\sum_{i=1}^{N}|d_i| = 1.70$ mm and $\overline{|d|} = \frac{1}{N}\sum_{i=1}^{N}|d_i| = 0.17 \approx 0.2$ mm.

4.2 The Standard Deviations

4.2.1 The Standard Deviation of the Measurements

The dispersion of the results of the measurements about their mean, is described by the *standard deviation from the mean* of the measurements. The deviation $d_i = x_i - \bar{x}$ is measured from the mean of the measurements. Thus, the standard deviation from the mean $\bar{x}$ of a series of measurements consisting of N measurements x_i ($i = 1, 2, \ldots, N$), is defined as

$$s_x \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}. \qquad (4.10)$$

This is the standard deviation of a series of N measurements, which constitute a *sample* from the infinite measurements of the magnitude x which might be performed. These infinite possible measurements are the *parent population*, or simply the *population*, from which we have taken a sample consisting of N random values. We must bear in mind that $\bar{x}$ is the *sample mean* and s_x is the *sample standard deviation*. The quantity s_x is also referred to as the *standard deviation of a single measurement in the sample*. Our aim is to derive as much information as possible about the properties of the statistical distribution of the parent population. The degree to which we can achieve this goal by performing only N measurements, will be discussed below.

As already mentioned in Chap. 1, if the N measurements are distributed in K classes, where the r-th class contains n_r measurements that gave a result x_r, Eq. (4.10) may also be written in the forms

$$s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r (x_r - \bar{x})^2} \tag{4.11}$$

and

$$s_x = \sqrt{\sum_{r=1}^{K} f_r (x_r - \bar{x})^2}, \tag{4.12}$$

where $f_r = \dfrac{n_r}{N}$.

A relation often used is derived in the following manner:

$$\begin{aligned} s_x^2 &= \frac{1}{N}\sum_{r=1}^{K} n_r (x_r - \bar{x})^2 = \frac{1}{N}\sum_{r=1}^{K} (x_r^2 - 2\bar{x}x_r + \bar{x}^2) n_r \\ &= \frac{1}{N}\sum_{r=1}^{K} x_r^2 n_r - \frac{2}{N}\bar{x}\sum_{r=1}^{K} x_r n_r + \frac{\bar{x}^2}{N}\sum_{r=1}^{K} n_r = \frac{1}{N}\sum_{r=1}^{K} x_r^2 n_r - \bar{x}^2 \end{aligned} \tag{4.13}$$

and, therefore,

$$s_x^2 = \overline{x^2} - \bar{x}^2 \quad \text{or} \quad \overline{(\Delta x)^2} = \overline{x^2} - \bar{x}^2. \tag{4.14}$$

Example 4.3

Find the standard deviation from the mean of the values of Example 4.1.

We use the value of $\bar{x}$ = 100.11 in evaluating $x_i - \bar{x}$.

i	x_i (mm)	$x_i - \bar{x}$ (mm)	$(x_i - \bar{x})^2$ (mm^2)
1	100.1	−0.01	0.0001
2	100.2	0.09	0.0081
3	99.8	−0.31	0.0961
4	100.3	0.19	0.0361
5	99.9	−0.21	0.0441
6	100.2	0.09	0.0081
7	99.9	−0.21	0.0441
8	100.4	0.29	0.0841
9	100.0	−0.11	0.0121
10	100.3	0.19	0.0361
Σ =	1001.1	0.00	0.3690

We find that $\sum_{i=1}^{N} (x_i - \bar{x})^2 = 0.369$ mm^2.

Therefore, from Eq. (4.10), $s_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2} = \sqrt{\frac{0.369}{10}} = \sqrt{0.0369} = 0.192$ mm.

The standard deviation of the 10 values is $s_x = 0.19$ mm.

Example 4.4 [E]

For the data of Example 4.3, using Excel®, find the mean, $\bar{x}$, the sample standard deviation, s_x, and the mean absolute deviation, $\overline{|d|}$.

We enter the values of x_i in cells A1 to A10. We highlight cells A1 to A10. Then,

Data > Data Analysis > Descriptive Statistics > OK

In the dialog box that opens, we set **Input, Input Range > $A$1:$A$10**, **Grouped by > Columns** and tick the box for **Summary statistics**. Press **OK**. The program returns a table, from which we read:

[Mean] = 100.11, [Standard Deviation] = 0.202485.

It must be remembered that Excel returns as **Standard Deviation** not the value of the standard deviation of the sample, $s_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}$, but rather the best estimate for the standard deviation of the parent population, $\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$ (see Sect. 4.2.4). We may evaluate s_x using the relation $s_x = \sqrt{\frac{N-1}{N}}\,\hat{\sigma}$. The result is $s_x = \sqrt{\frac{9}{10}}\,0.202485 = 0.192094$ mm, as expected.

To calculate the mean absolute deviation, $\overline{|d|}$, we proceed as follows:

Set cell B1 **= ABS(A1-100.11)**. Fill Down cells B2 to B10. Column B now contains the values of $|x_i - \bar{x}|$. Highlight cells A1 to A10. Open the **Autosum** dialog box and press **Sum**. The result returned is 1.7. Dividing by N = 10, we have the result: $\overline{|d|} = 0.17$ mm.

Summarizing, $\bar{x} = 100.11$ mm and $s_x = 0.19$ mm and $\overline{|d|} = 0.17$ mm.

Example 4.5 [O]

For the data of Example 4.3, using Origin®, find the mean, $\bar{x}$, the sample standard deviation, s_x, and the mean absolute deviation, $\overline{|d|}$.

We enter the numbers in column A. We highlight column A. Then,

Statistics > Descriptive Statistics > Statistics on Columns > Open Dialog…

In the window that opens, we tick the following:

Input > Input Data > Range 1 > Data Range > A(X)
Quantities > Tick **Mean, Standard Deviation, Mean Absolute Deviation**

Open **Computation Control > Weight Method > Direct Weight**
Then **Variance Divisor of Moment > N**

The last setting puts the number N in the denominator of Eq. (4.10) (The choice DF would put $N - 1$ in the denominator).

Pressing **OK** we obtain the results:

[Mean] = 100.11, [Standard Deviation] = 0.19209, [Mean Absolute Deviation] = 0.17

Summarizing, $\bar{x} = 100.11$ mm and $s_x = 0.19$ mm and $\overline{|d|} = 0.17$ mm.

Example 4.6 [P]

For the data of Example 4.3, find the mean, $\bar{x}$, the sample standard deviation, s_x, and the mean absolute deviation, $\overline{|d|}$, using Python.

```
from __future__ import division
import numpy as np
import math

# Enter the values given as the components of the vector x:

x = np.array([100.1, 100.2, 99.8, 100.3, 99.9, 100.2, 99.9, 100.4, 100.0,
100.3])

# Evaluation of the parameters:

N = len(x)
mean_x = x.mean()
mean_abs_dev_mean = np.sum(np.abs(x-mean_x)) / N
std_dev_sample = x.std(ddof = 1) * math.sqrt((N-1)/N)

# Preparing the printout:

print ("Number of values N =", N)
print ("Mean =", mean_x)
print ("Standard deviation of the sample =", std_dev_sample)
print ("Mean absolute deviation from the mean =", mean_abs_dev_mean)
```

Running the program, returns:

```
Number of values N = 10
Mean = 100.11
Standard deviation of the sample = 0.192093727123
Mean absolute deviation from the mean = 0.17
```

Example 4.7 [R]

For the data of Example 4.3, find the mean, $\overline{x}$, the sample standard deviation, s_x, and the mean absolute deviation, $\overline{|d|}$, using R.

We first find the mean, $\overline{x}$, and the standard deviation, **s.d.**

```
> x <- c(100.1, 100.2, 99.8, 100.3, 99.9, 100.2, 99.9, 100.4, 100.0, 100.3)
> meanx = mean(x)
> meanx
[1] 100.11
> sd(x)
[1] 0.2024846
```

The mean was found to be $\overline{x} = 100.11$ mm.

It should be pointed out R returns as **sd** not the value of the standard deviation of the sample, $s_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2}$, but rather the best estimate for the standard deviation of the parent population, $\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}$ (see Sect. 4.2.4). We may evaluate s_x using the relation $s_x = \sqrt{\frac{N-1}{N}}\,\hat{\sigma}$. The result is $s_x = \sqrt{\frac{9}{10}}\,0.202485 = 0.192094$ mm or $s_x = 0.192$ mm, as expected.

The mean absolute deviation, $\overline{|d|}$, is found by

```
> sum(abs(x-meanx))/10
[1] 0.17
```

We have found that $\overline{x} = 100.11$ mm $s_x = 0.19$ mm, $\overline{|d|} = 0.17$ mm.

Example 4.8

A total of 33 measurements are classified into 10 classes (of 10 different values of the result) as seen in the table below. Find the mean and the standard deviation from the mean of the measurements.

r	x_r (mm)	n_r	$n_r x_r$ (mm)	$x_r - \overline{x}$ (mm)	$(x_r - \overline{x})^2$ (mm^2)	$n_r(x_r - \overline{x})^2$ (mm^2)
1	9.4	1	9.4	−0.4848	0.2350	0.2350
2	9.5	1	9.5	−0.3848	0.1481	0.1481
3	9.6	4	38.4	−0.2848	0.0811	0.3244
4	9.7	3	29.1	−0.1848	0.0342	0.1025
5	9.8	5	49.0	−0.0848	0.0072	0.0360
6	9.9	5	49.5	0.0152	0.0002	0.0012
7	10.0	6	60.0	0.1152	0.0133	0.0796

(continued)

(continued)

r	x_r (mm)	n_r	$n_r x_r$ (mm)	$x_r - \bar{x}$ (mm)	$(x_r - \bar{x})^2$ (mm^2)	$n_r(x_r - \bar{x})^2$ (mm^2)
8	10.1	4	40.4	0.2152	0.0463	0.1852
9	10.2	3	30.6	0.3152	0.0994	0.2981
10	10.3	1	10.3	0.4152	0.1724	0.1724
	$\Sigma =$	33	326.2			1.5824

We find the mean $\bar{x} = \frac{1}{33}\sum_{r=1}^{10} n_r x_r = \frac{326.2}{33} = 9.8848 \approx 9.88$ mm.

Since $\sum_{r=1}^{K} n_r (x_r - \bar{x})^2 = 1.5833$ mm^2, the standard deviation of the 33 values from the mean is $s_x = \sqrt{\frac{1.5833}{33}} = \sqrt{0.04795} = 0.2190 \approx 0.22$ mm.

Example 4.9 [E]

Solve Example 4.8 using Excel®.

We enter the values of x_r and n_r in columns A and B, respectively. We need to evaluate the weighted standard deviation of x, with n as weights. The weighted mean and weighted standard deviation are defined in Sect. 9.4, but, essentially, we use Eq. (1.9), $\bar{x} = \dfrac{1}{N}\sum_{r=1}^{K} n_r x_r$ for the mean and Eq. (1.19), $s_x = \sqrt{\dfrac{1}{N}\sum_{r=1}^{K} n_r (x_r - \bar{x})^2}$ for the standard deviation, with n_r as weights and N as the sum of the weights.

We will first evaluate the weighted mean. Highlight an empty cell, say E1. Left click on cell E1 and type:

=SUMPRODUCT(A1:A10;B1:B10)/SUM(B1:B10)

Pressing **ENTER** will return the number 9.8848 in cell E1. This is the required mean, $\bar{x} = 9.88$ mm.

We will give this number the name . To do this, we right click on cell E1. In the dialog box that opens, we select **Define Name…** and in the cell for **NameM** we write **M**.

We will now evaluate the weighted standard deviation. We first evaluate the terms $(x_r - \bar{x})^2$. We highlight cell C1 and type: **= (A1-M)^2**. Pressing **ENTER** returns the number `0.235078` in cell C1. To fill cells C1 to C10 with the values of $(x_r - \bar{x})^2$, we highlight cells C1-C10 and press

Fill > Down

To evaluate the standard deviation, we highlight an empty cell, say D13 and type

=SQRT(SUMPRODUCT(B1:B10;C1:C10)/SUM(B1:B10))

Pressing **ENTER** returns the number `0.21898`. We have found that the standard deviation of the sample is $s_x = 0.22$ mm, in agreement with the results of Example 4.8.

Example 4.10 [O]

Solve Example 4.8 using Origin®.

We enter x_r and n_r in columns A and B. We highlight columns A and B. Then,

Statistics > Descriptive Statistics > Statistics on Columns > Open Dialog…

In the window that opens, we tick the following:

Input > Input Data > Range 1 > Data Range > A(X) > Weighting Range > B(Y)
Quantities > Tick **Mean, Standard Deviation**

Open **Computation Control > Weight Method > Direct Weight**
The last choice ensures that the numbers n_r will act as weights.
Then **Variance Divisor of Moment > WS**
The last setting puts the quantity number $\sum_r n_r = N$ in the denominator of Eq. (4.11).
Pressing **OK** we obtain the results:
[Mean] = 9.88485, [Standard Deviation] = 0.21898
Summarizing, $\bar{x} = 9.88$ mm, $s_x = 0.22$ mm.

Example 4.11 [P]

Solve Example 4.8 using Python.

r	x_r (mm)	n_r
1	9.4	1
2	9.5	1
3	9.6	4
4	9.7	3
5	9.8	5
6	9.9	5
7	10.0	6
8	10.1	4
9	10.2	3
10	10.3	1

We need to calculate $s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r (x_r - \bar{x})^2}$. First we need to evaluate the mean $\bar{x} = \frac{1}{N}\sum_{r=1}^{K} n_r x_r$. The task is equivalent to calculating the weighted mean and the weighted standard deviation of the sample, for the measurements x_r, with corresponding weights n_r, where $N = \sum_{r=1}^{K} n_r$. The weighted mean and weighted standard deviation are defined in Sect. 9.4. The equations derived there are the same as those given above, with the weights w_r replacing n_r, and $\sum_{r=1}^{K} w_r$ replacing N.

We will use the weighted average function from the numpy package. We also import the math module in order to use the square root function. The measurement values are stored in the vector x and the corresponding weights in the vector w.

```
import math
import numpy as np
x = np.array([9.4, 9.5, 9.6, 9.7, 9.8, 9.9, 10, 10.1, 10.2, 10.3])
w = np.array([1,1, 4, 3, 5, 5, 6, 4, 3, 1])
wmean = np.average(x, weights=w)
variance = np.average((x-wmean) ** 2, weights=w)
s = math.sqrt(variance)

# Preparing the printout:
print ("Weighted mean =", wmean)
print ("Weighted standard deviation of the sample =", s)
```

Running the program returns:

```
Weighted mean = 9.88484848485
Weighted standard deviation of the sample = 0.21898002139563225
```

Example 4.12 [R]

Solve Example 4.8 using R.

r	x_r (mm)	n_r
1	9.4	1
2	9.5	1
3	9.6	4
4	9.7	3
5	9.8	5
6	9.9	5
7	10.0	6
8	10.1	4
9	10.2	3
10	10.3	1

We need to calculate $s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r(x_r - \overline{x})^2}$. First we need to evaluate the mean $\overline{x} = \frac{1}{N}\sum_{r=1}^{K} n_r x_r$. The task is equivalent to calculating the weighted mean and the weighted standard deviation of the sample, for the measurements x_r, with corresponding weights n_r, where $N = \sum_{r=1}^{K} n_r$. The weighted mean and weighted standard deviation are defined in Sect. 9.4. The equations derived there are the same as those given above, with the weights w_r replacing n_r, and $\sum_{r=1}^{K} w_r$ replacing N. We may, therefore use the function **weighted.mean(x, w, …)** available in R.

We first define the vectors of the x and w = n values and then find the weighted mean:

```
> x <- c(9.4, 9.5, 9.6, 9.7, 9.8, 9.9, 10, 10.1, 10.2, 10.3)
> w <- c(1, 1, 4, 3, 5, 5, 6, 4, 3, 1)
> wmean = weighted.mean(x, w)
> wmean
[1] 9.884848
```

We notice that the variance s_x^2 of the sample is simply the weighted mean of the quantity $(x_r - \bar{x})^2$. Therefore,

```
> variance = weighted.mean((x-wmean)^2, w)
> variance
[1] 0.04795225
> sqrt(variance)
[1] 0.21898
```

Summarizing, we have found that $\bar{x} = 9.88$ mm and $s_x = 0.22$ mm.

4.2.1.1 Use of a Working Mean in Order to Minimize Arithmetical Calculations

It is sometimes convenient to use a suitable working mean in evaluating the standard deviation, in order to minimize the work involved. If m is the working mean selected, then

$$\sum_{i=1}^{N} \frac{1}{N}(x_i - m) = \bar{x} - m. \tag{4.15}$$

Defining

$$\bar{x}_m = \sum_{i=1}^{N} \frac{1}{N}(x_i - m), \tag{4.16}$$

it follows that

$$\bar{x} = \bar{x}_m + m. \tag{4.17}$$

The standard deviation is found from

$$\begin{aligned} s_x^2 &= \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - m + m - \bar{x})^2 \\ &= \frac{1}{N}\sum_{i=1}^{N}\left[(x_i - m)^2 + 2(x_i - m)(m - \bar{x}) + (m - \bar{x})^2\right] \end{aligned} \tag{4.18}$$

which gives

$$\begin{aligned} s_x^2 &= \frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2 + 2(m - \bar{x})\frac{1}{N}\sum_{i=1}^{N}(x_i - m) + (m - \bar{x})^2 \\ &= \frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2 - 2(m - \bar{x})^2 + (m - \bar{x})^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2 - (m - \bar{x})^2 \end{aligned} \tag{4.19}$$

Defining

$$s_m^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2, \tag{4.20}$$

we have

$$s_x^2 = s_m^2 - (m - \bar{x})^2 \tag{4.21}$$

Summarizing:
If

$$\bar{x}_m = \sum_{i=1}^{N}\frac{1}{N}(x_i - m) \quad \text{and} \quad s_m = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2}, \tag{4.22}$$

it is

$$\bar{x} = m + \bar{x}_m \quad \text{and} \quad s_x = \sqrt{s_m^2 - (m - \bar{x})^2}. \tag{4.23}$$

Example 4.13

Five measurements of the speed of light, c, gave the following results (in m/s):

299 792 459.2 299 792 457.4 299 792 457.1 299 792 458.8 299 792 457.8.

Find the mean and the standard deviation of the measurements.

We use $m = 299\,792\,457$ m/s as a working average and evaluate $(c_i - m)$ and $(c_i - m)^2$ and their sums:

i	c_i (m/s)	$(c_i - m)$ (m/s)	$(c_i - m)^2$ (m^2/s^2)
1	299 792 459.2	2.2	4.84
2	299 792 457.4	0.4	0.16
3	299 792 457.1	0.1	0.01
4	299 792 458.8	1.8	3.24
5	299 792 457.8	0.8	0.64
	Σ =	5.3	8.89

We find $\overline{c}_m = \sum_{i=1}^{N} \frac{1}{N}(c_i - m) = \frac{5.3}{5} = 1.06$ m/s and $s_m = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(c_i - m)^2}$ $= \sqrt{\frac{8.89}{5}} = 1.33$ m/s.

Therefore, $\overline{c} = m + \overline{c}_m = 299\,792\,457 + 1.06 = 299\,792\,458.06$ m/s and $s_c = \sqrt{s_m^2 - (m - \overline{c})^2} = \sqrt{1.33^2 - 1.06^2} = \sqrt{0.645} = 0.80$ m/s.

The final results are: $\overline{c} = 299\,792\,458.1$ m/s and $s_c = 0.8$ m/s.

4.2.2 *The Standard Deviation of the Mean*

Assume that we perform M (for $k = 1, 2, \ldots, M$) identical series of measurements of the quantity x, each consisting of N (for $i = 1, 2, \ldots, N$) measurements, as seen in Table 4.1.

The value $<x>_i$ is the mean of the M values of a given i,

$$<x>_i = \frac{1}{M}\sum_{k=1}^{M} x_{k,i}. \tag{4.24}$$

We will return to these mean values later.

For each series of measurements ($k = 1, 2, \ldots, M$) we evaluate the mean of the measurements

Table 4.1 *M* series of measurements of the quantity *x*, consisting of *N* measurements each

$i =$	1	2	…	i	…	N	$\bar{x}_k,\ s_{xk}$
$k = 1$	$x_{1,1}$	$x_{1,2}$	…	$x_{1,i}$	…	$x_{1,N}$	$\bar{x}_1,\ s_{x,1}$
2	$x_{2,1}$	$x_{2,2}$	…	$x_{2,i}$	…	$x_{2,N}$	$\bar{x}_2,\ s_{x,2}$
…	…	…	…	…	…	…	…
k	$x_{k,1}$	$x_{k,2}$	…	$x_{k,i}$	…	$x_{k,N}$	$\bar{x}_k,\ s_{x,k}$
…	…	…	…	…	…	…	…
M	$x_{M,1}$	$x_{M,2}$	…	$x_{M,i}$	…	$x_{M,N}$	$\bar{x}_M,\ s_{x,M}$
	$<x>_1$	$<x>_2$	…	$<x>_i$	…	$<x>_N$	

$$\bar{x}_k = \frac{1}{N}\sum_{i=1}^{N} x_{k,i} \tag{4.25}$$

and their standard deviation

$$s_{x,k} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_{k,i} - \bar{x}_k)^2}. \tag{4.26}$$

The values of $\bar{x}_k$ and $s_{x,k}$ are given in the last column of the table. The M results $\bar{x}_k$ will, in general, differ from each other and we will thus have a distribution of the mean values. The $M \times N$ values of x in the table have a mean $\overline{X}$ and a standard deviation S, which are given by

$$\overline{X} = \frac{1}{MN}\sum_{k=1}^{M}\sum_{i=1}^{N} x_{k,i} \tag{4.27}$$

and

$$S = \sqrt{\frac{1}{MN}\sum_{k=1}^{M}\sum_{i=1}^{N}(x_{k,i} - \overline{X})^2}. \tag{4.28}$$

As the number of the series of measurements M tends to infinity, these values tend to the corresponding values of the total of the infinite measurements which it is possible to perform. These are the parent population from which each sample consisting of N measurements is taken. The mean value of x for the infinite population is denoted by μ and its standard deviation with σ (Greek letters are used, in general, for the parent population and Latin letters for the sample). Thus, we have

$$\lim_{M\to\infty} \overline{X} = \mu \quad \text{and} \quad \lim_{M\to\infty} S = \sigma. \tag{4.29}$$

The various values of $\overline{x}_k$ estimated from the M series of measurements have, themselves, a distribution about the real value x_0 of the quantity x, which is characterized by a standard deviation, known as the *standard deviation of the mean* and is denoted by $\sigma_{\overline{x}}$. Our aim is to find relations which will enable us to make estimates for σ and $\sigma_{\overline{x}}$, when we know s_x for one series of N measurements of x.

In the mathematical analysis that will follow, the sample means are symbolized as up to now, by a line over the symbol, e.g. $\overline{x}$. The mean values evaluated for the whole of the parent population will be symbolized as $<x>$, $<e>$, $<e^2>$ etc. These values are calculated for the $M \times N$ values of x, which result from the M series of N measurements each, as $M \to \infty$. Thus, for example, for each column of Table 4.1, the values $<x>_i$, as mean values of x which result from infinite measurements, will be equal to the real value x_0. The same is true for the totality of the table's values. For the same reasons, the standard deviation of the values of each column, or of the whole table, will tend to σ, while for the last column of the table, the mean of the values of $\overline{x}_k$ will tend to x_0, and the mean of their standard deviations $s_{x,k}$ to σ.

Example 4.14

$M = 8$ series of the quantity x, each consisting of $N = 6$ measurements, gave the results shown in the table that follows. Find the mean and the standard deviation for each series of measurements Then, find the mean of all the measurements and the standard deviation of the 8 mean values of the series of measurements.

	$x_{k,i}$						$\overline{x}_k$	$s_{x,k}$
$i =$	1	2	3	4	5	6		
$k = 1$	9.5	10.3	10.1	9.9	10.0	10.3	10.02	0.273
2	9.7	10.0	10.3	9.8	10.2	9.8	9.97	0.221
3	10.2	10.2	9.7	10.1	10.1	10.2	10.08	0.177
4	9.8	10.4	9.9	10.2	9.9	10.2	10.07	0.213
5	10.0	9.9	10.1	9.7	10.1	10.0	9.97	0.137
6	9.9	10.4	10.3	10.4	9.6	9.7	10.05	0.330
7	10.0	9.8	10.1	9.9	10.2	9.9	9.98	0.134
8	10.1	9.6	10.0	9.9	10.0	9.7	9.88	0.177
						Sums Σ =	80.02	1.662
						Mean =	10.00	0.208
						Standard deviation =	0.06	

The mean for each series of measurements is given in the column of $\overline{x}_k$.

The standard deviation for each series of measurements is given in the column of $s_{x,k}$. Their mean is 0.208.

Because the sum of the means $\overline{x}_k$ of the 8 series of measurements is 80.02, the mean of all the measurements is $\overline{X} = 80.02/8 = 10.00$.

The standard deviation of the 8 mean values $\overline{x}_k$ is 0.06.

The standard deviation of all the 48 values is $S = 0.23$.

4.2.3 The Relationship Between σ and $\sigma_{\bar{x}}$

If we have N measurements x_i ($i = 1, 2, \ldots, N$) of the quantity x, whose real value is x_0, the error in x_i is $e_i = x_i - x_0$ and the error in the mean $\bar{x}$ is $e_\mu = \bar{x} - x_0$. Since

$$e_\mu = \bar{x} - x_0 = \frac{1}{N}\sum_i e_i, \tag{4.30}$$

it will be

$$e_\mu^2 = \frac{1}{N^2}\left(\sum_i e_i\right)^2 = \frac{1}{N^2}\sum_i e_i^2 + \frac{1}{N^2}\sum_i \sum_{j, j\neq i} e_i e_j, \tag{4.31}$$

where the squares of e_i are summed in the first sum, while the products of different e_i are summed in the second sum. We will now assume that we have a large number M of series of N measurements each and we will take the (population) means of these two sums as $M \to \infty$.

The population mean of e_μ^2 is denoted by $< e_\mu^2 >$, while the population mean $\frac{1}{N}\sum_i e_i^2$ is $< e^2 >$, i.e. the mean of the square of the error. However, by definition, it is $< e_\mu^2 > \; = \sigma_\mu^2 = \sigma_{\bar{x}}^2$, where $\sigma_\mu = \sigma_{\bar{x}}$ is the standard deviation of the mean and $< e^2 > \; = \sigma^2$, where σ is the standard deviation of the population of the infinite measurements that may be performed.

The population mean of the sum $\sum_i \sum_{j, j\neq i} e_i e_j$ tends to zero, being the average of the products of a large number ($M \to \infty$) of mutually independent quantities, which are symmetrically distributed around zero. Equation (4.31) gives, therefore,

$$\sigma_\mu = \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}. \tag{4.32}$$

A different proof of this relation will be given in Example 6.5 of Chap. 6.

At present, we have no knowledge regarding σ, which describes the distribution about the real value x_0 of the infinite measurements x_i that can be made. In the next subsection we will find an estimate for this value, based on the known quantity s_x, the standard deviation of the N measurements we have performed. In this way, it will also be possible to have an estimate for the value of $\sigma_\mu = \sigma_{\bar{x}}$.

4.2.4 The Relationship Between s_x and σ and $\sigma_{\bar{x}}$

From the definition of the standard deviation of the measurements x_i

$$s_x^2 \equiv \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2 \tag{4.33}$$

and the fact that

$$x_i - \bar{x} = e_i - e_\mu, \tag{4.34}$$

we have the relation

$$s_x^2 = \frac{1}{N}\sum_{i=1}^{N}(e_i - e_\mu)^2 = \frac{1}{N}\sum_{i=1}^{N}e_i^2 - 2e_\mu\frac{1}{N}\sum_{i=1}^{N}e_i + e_\mu^2 = \frac{1}{N}\sum_{i=1}^{N}e_i^2 - e_\mu^2 \tag{4.35}$$

Evaluating the population means we have

$$< s_x^2 > \; = \sigma^2 - \sigma_\mu^2 \tag{4.36}$$

Combined with Eqs. (4.32) and (4.36) gives

$$\sigma^2 = \frac{N}{N-1} < s_x^2 > \quad \text{or} \quad \sigma = \sqrt{\frac{N}{N-1} < s_x^2 >} \tag{4.37}$$

and

$$\sigma_\mu^2 = \frac{1}{N-1} < s_x^2 > \quad \text{or} \quad \sigma_\mu = \sqrt{\frac{1}{N-1} < s_x^2 >}\,. \tag{4.38}$$

The quantity $< s_x^2 >$ is unknown to us, since, in theory, we need an infinite number of measurements for it to be determined with absolute accuracy. The best *estimate* that we have for it is s_x^2, which results from the N measurements we have made. Therefore, the best estimates we have at our disposal for σ and σ_μ are, respectively,

$$\hat{\sigma} = \sqrt{\frac{N}{N-1}}\, s_x \tag{4.39}$$

and

$$\hat{\sigma}_\mu = \frac{s_x}{\sqrt{N-1}}, \tag{4.40}$$

where the carets (hats) above σ and σ_μ state the fact that, strictly speaking, we do not have an equation but that the magnitude on the right is the best estimate for the magnitude on the left. The carets are usually omitted.

Using the fact that $s_x \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}$ and omitting the carets, we have

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2} \tag{4.41}$$

and

$$\sigma_{\bar{x}} = \sigma_\mu = \sqrt{\frac{1}{N(N-1)}\sum_{i=1}^{N}(x_i - \bar{x})^2} \tag{4.42}$$

for the standard deviation of the parent population of the infinite possible results of the measurements of the magnitude x that may be performed and for the standard deviation of the mean, respectively.

Example 4.15

Find the (best estimates of the) standard deviations of the means for the values of Examples 4.3 and 4.8.

Example 4.3: From $s_x = 0.192$ mm and $N = 10$, we find $\sigma_{\bar{x}} = \sigma_\mu \approx \hat{\sigma} = \frac{s_x}{\sqrt{N-1}}$ and, therefore,

$$\sigma_{\bar{x}} = \sigma_\mu \approx \hat{\sigma} = \frac{0.192}{3} = 0.062 \text{ mm.}$$

Example 4.8: From $s_x = 0.22$ mm and $N = 33$, it is $\sigma_{\bar{x}} = \sigma_\mu \approx \hat{\sigma} = \frac{0.22}{\sqrt{32}} = 0.039$ mm.

Example 4.16 [E]

Given the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, find their mean $\bar{x}$, sample standard deviation s_x, the best estimate for the standard deviation of the parent population, $\hat{\sigma}$, the standard deviation of the mean $\sigma_{\bar{x}}$ and mean absolute deviation $\overline{|d|}$.

We enter the values of x_i in cells A1 to A10. We highlight cells A1 to A10. Then,

Data > Data Analysis > Descriptive Statistics > OK

In the dialog box that opens, we set **Input, Input Range > $A$1:$A$10**, **Grouped by > Columns** and tick the box for **Summary statistics**. Press **OK**. The program returns a table, from which we read:

[Mean] = 5.500, [Standard Error] = 0.957427 [Standard Deviation] = 3.027650. This is also the best estimate of the standard deviation of the parent population.

By the term [Standard Error], Excel means the standard error of the mean or the standard deviation of the mean, $\sigma_{\bar{x}}$. Therefore, $\sigma_{\bar{x}} = 0.96$ mm.

Bearing in mind the comments made in Example 4.4, the [Standard Deviation] given by Excel is $\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$. From this, we find the standard deviation of the sample as $s_x = \sqrt{\frac{N-1}{N}}\,\hat{\sigma}$. The result is $s_x = \sqrt{\frac{9}{10}}\,3.027650 = 2.87228$ mm.

To calculate the mean absolute deviation, $\overline{|d|}$, we proceed as follows:

Set cell B1 **= ABS(A1-5.5)**. Fill Down cells B2 to B10. Column B now contains the values of $|x_i - \bar{x}|$. Highlight cells A1 to A10. Open the **Autosum** dialog box and press **Sum**. The result returned is 25.00. Dividing by $N = 10$, we have the result: $\overline{|d|} = 2.5$ mm.

Summarizing, $\bar{x} = 5.5$ mm, $s_x = 2.9$ mm, $\hat{\sigma} = 3.0$ mm, $\overline{|d|} = 2.5$ mm and $\sigma_{\bar{x}} = 0.96$ mm.

Example 4.17 [O]

Given the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, find their mean $\bar{x}$, sample standard deviation s_x, the best estimate for the standard deviation of the parent population, $\hat{\sigma}$, the standard deviation of the mean $\sigma_{\bar{x}}$ and mean absolute deviation $\overline{|d|}$.

We enter the numbers in column A. We select column A. Then,

Statistics > Descriptive Statistics > Statistics on Columns > Open Dialog…

In the window that opens, we tick the following:

Input > Input Data > Range 1 > Data Range > A(X)
Quantities > Tick **Mean, Standard Deviation, Mean Absolute Deviation**

Open **Computation Control > Weight Method > Direct Weight**
Then **Variance Divisor of Moment > N**

The last setting puts the number N in the denominator of Eq. (4.10) (The choice DF would put $N - 1$ in the denominator).

Pressing **OK** we obtain the results:

[Mean] = 5.5, [Standard Deviation] = 2.8723, [Mean Absolute Deviation] = 2.5

Summarizing, $\bar{x} = 5.5$ mm, $s_x = 2.9$ mm, $\hat{\sigma} = \sqrt{N/(N-1)}\, s_x = 3.0$ mm, $\overline{|d|} = 2.5$ mm and $\sigma_{\bar{x}} = s_x/\sqrt{N-1} = 0.96$ mm.

Example 4.18 [P]

Given the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, find their mean $\bar{x}$, sample standard deviation s_x, the best estimate for the standard deviation of the parent population, $\hat{\sigma}$, the standard deviation of the mean $\sigma_{\bar{x}}$ and mean absolute deviation $\overline{|d|}$.

```
from __future__ import division
import numpy as np
import math

# Enter the values given as the components of the vector x
x = np.array([1, 2, 3, 4, 5, 6, 7, 8, 9, 10])

# Evaluation:
N = len(x)
mean_x = x.mean()
std_dev_sample = x.std(ddof = 1) * math.sqrt((N-1)/N)
std_dev_popul = x.std(ddof = 1)
std_dev_mean = std_dev_sample * math.sqrt(1/(N-1))
mean_abs_dev_mean = np.sum(np.abs(x-mean_x)) / N

# Preparation for printout:
print ("Number of values N =", N)
print ("Mean =", mean_x)
print ("Standard deviation of the sample =", std_dev_sample)
print ("Standard deviation of the population =", std_dev_popul)
print ("Standard deviation of the mean =", std_dev_mean)
print ("Mean absolute deviation from the mean =", mean_abs_dev_mean)
```

Running the program returns the results:

```
Number of values N = 10
Mean = 5.5
Standard deviation of the sample = 2.87228132327
Standard deviation of the population = 3.0276503541
Standard deviation of the mean = 0.957427107756
Mean absolute deviation from the mean = 2.5
```

Example 4.19 [R]

Given the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, find their mean $\bar{x}$, sample standard deviation s_x, the best estimate for the standard deviation of the parent population, $\hat{\sigma}$, the standard deviation of the mean $\sigma_{\bar{x}}$ and mean absolute deviation $\overline{|d|}$.

The mean, $\bar{x}$, and mean absolute deviation $\overline{|d|}$ are:

```
> x <- c(1,2,3,4,5,6,7,8,9,10)
> mean(x)
[1] 5.5
> sum(abs(x - mean(x)))/10
[1] 2.5
```

R calculates the best estimate for the standard deviation of the parent population, $\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$ as **sd(x)**. From this, we find the standard deviation of the sample as $s_x = \sqrt{\frac{N-1}{N}}\,\hat{\sigma}$ and the standard deviation or error of the mean as $\sigma_{\bar{x}} = \frac{s_x}{\sqrt{N-1}}$:

```
> sd(x)
[1] 3.02765
> sx = sqrt(9/10)*sd(x)
> sx
[1] 2.872281
err = sx/sqrt(9)
> err
[1] 0.9574271
```

We have found that $\bar{x} = 5.5\,\text{mm}$, $\overline{|d|} = 2.5\,\text{mm}$, $\hat{\sigma} = 3.0\,\text{mm}$, $s_x = 2.9\,\text{mm}$, and $\sigma_{\bar{x}} = 0.96\,\text{mm}$.

4.3 The Standard Deviation of the Standard Deviation of the Mean

The value of the standard deviation $\sigma_{\bar{x}}$ of the mean $\bar{x}$ of a series of measurements was determined using the N measurements performed. If we perform another series of N measurements, what will the difference be between the two standard deviations? And finally, if we perform a large number M of series with N measurements each, what kind of dispersion will there be in the standard deviations $\sigma_{\bar{x},k}$ ($k = 1, 2, \ldots, M$) of the means of the M series of measurements? Having made only one series of measurements, the best estimate that we have for the mean of these standard deviations is $\sigma_{\bar{x}}$. The dispersion of the values $\sigma_{\bar{x},k}$ around the mean is expressed by a *standard deviation of the standard deviation of the mean* $\sigma_{\bar{x}}$. We will denote this by $\sigma(\sigma_{\bar{x}})$ and its *fractional* value by α, in which case it will be $\sigma(\sigma_{\bar{x}}) = \alpha\sigma_{\bar{x}}$.

It is proved that for N measurements, and under the same assumptions for the statistical behavior of the $\sigma_{\bar{x},k}$ that we accepted to hold for the measurements x_i and their errors, it is, to a good approximation,

$$\alpha(N) = \frac{1}{\sqrt{2(N-1)}}. \tag{4.43}$$

The values of α are given in the table that follows for different values of N.

N	5	6	8	10	15	20	30	50	100
α	0.354	0.316	0.267	0.236	0.189	0.162	0.131	0.101	0.071

The function $\alpha(N)$ is plotted in Fig. 4.1.

The value of α is useful as an estimate of the accuracy with which we know the standard deviation of the mean, $\sigma_{\bar{x}}$. We will see, for example, in the chapter for the presentation of numerical results, that, for the usual number of measurements, which is of the order of 10, the standard deviation is known with an uncertainty of about 24% and that it makes no sense to give the numerical value of $\sigma_{\bar{x}}$ with more than one or, at most, two significant figures. As a consequence it makes no sense to give the numerical value of the mean with greater accuracy.

Shown in Fig. 4.2 are the standard deviations s_x of the N measurements of the quantity x, and of their mean, $\sigma_{\bar{x}}$, as a function of N, as this increases from 2 to 500. These 500 values of 'measurements' were taken at random from a parent population similar to that of our 'thought experiment' of Chap. 3, which was chosen to have $\sigma = 1$. In order to cover the wide range of values of N without loss of detail in the behavior at low N, the scale of N in the figure was taken to be logarithmic. As N increases, s_x is seen to approach the value of $\sigma = 1$, and $\sigma_{\bar{x}}$ tends to the value $\sigma/\sqrt{N}$. Also drawn in the figure are the curves for $\sigma\left(1 \pm 1/\sqrt{2(N-1)}\right)$ and $\frac{\sigma}{\sqrt{N}}\left(1 \pm 1/\sqrt{2(N-1)}\right)$, between which s_x and $\sigma_{\bar{x}}$ are seen to lie for most values of N. The fluctuations in s_x and $\sigma_{\bar{x}}$ appear to be of the order of magnitude predicted by Eq. (4.43).

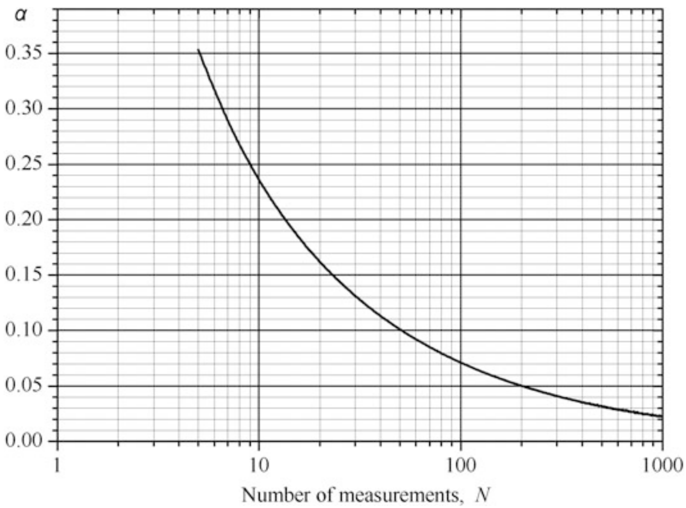

Fig. 4.1 The variation with the number N of the measurements of the fractional standard deviation, α, of the standard deviation of the mean. The scale of N is logarithmic

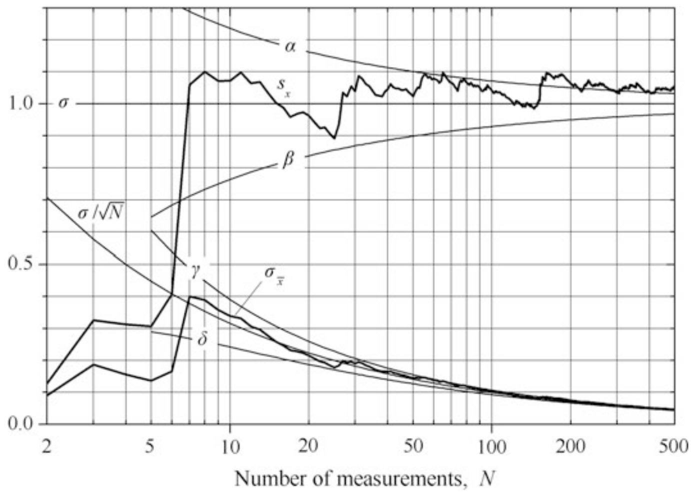

Fig. 4.2 The standard deviations s_x of the N measurements of the quantity x, and of their mean $\sigma_{\bar{x}}$, as a function of N, as this increases from 2 to 500 (the scale for N is logarithmic). Also drawn are the curves $\sigma/\sqrt{N}$, $\sigma\left(1 \pm 1/\sqrt{2(N-1)}\right)$ (α and β) and $\frac{\sigma}{\sqrt{N}}\left(1 \pm 1/\sqrt{2(N-1)}\right)$ (γ and δ)

The reader might be relieved to know that we have absolutely no use for the standard deviation of the standard deviation of the standard deviation.

4.4 Information Derived from the Measurement of $\bar{x}$ and s_x

From the values of $\bar{x}$ and s_x we have from the N measurements of x we made, we may extract some useful information regarding the distribution of the values of x in the parent population of all the possible results and the real value x_0 of x.

4.4.1 The Mean Value of the Results of the Measurements and Its Standard Deviation

We have already explained, qualitatively, why the mean $\bar{x}$ is the best estimate we have for the real value x_0, under the assumption that the parent population is symmetrical relative to x_0, i.e. that positive and negative errors are equally probable. A more rigorous proof will be given in Chap. 9, which deals with the Theory of Errors.

Using the definition

$$s_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2} \tag{4.44}$$

of the standard deviation of the results of the measurements, we will show that we can conclude that s_x tends to a constant value as the number of measurements increases. This value is the standard deviation of the parent population of the possible results of the measurements:

$$\lim_{N\to\infty} s_x = \sigma. \tag{4.45}$$

If we look at the successive values of s_x^2 as N increases, we have

$$\frac{(x_1-\bar{x}_1)^2}{1},\ \frac{(x_1-\bar{x}_2)^2+(x_2-\bar{x}_2)^2}{1+1},\ \frac{(x_1-\bar{x}_3)^2+(x_2-\bar{x}_3)^2+(x_3-\bar{x}_3)^2}{1+1+1},\ \ldots \tag{4.46}$$

for N equal to 1, 2, 3, …, respectively, where $\bar{x}_N$ is the mean after the first N measurements. We see that making another measurement, x_N, leads to the increase of the numerator by the quantity $(x_N - \bar{x}_N)^2$ and of the denominator by unity. As the mean value of the results gradually tends to a constant value, the same happens to the mean value of the quantities $(x_N - \bar{x}_N)^2$. This simultaneous proportional increase of numerator and denominator has an effect on the value of the fraction which becomes gradually smaller and the value of s_x tends to a limit. Since the sample becomes, with increasing N, more and more representative of the parent population, this limit must be σ.

Due to the fact that the standard deviation of the mean is, according to Eq. (4.32), equal to

$$\sigma_{\bar{x}} = \sigma_\mu = \frac{\sigma}{\sqrt{N}}, \tag{4.47}$$

it follows that, for large values of N, $\sigma_{\bar{x}}$ is inversely proportional to $\sqrt{N}$ and tends to zero. The deviation of the mean $\bar{x}$ from the real value x_0 tends to zero and $\bar{x}$ is increasingly a better estimate for the real value x_0. This procedure is seen in Fig. 4.3. Shown in this figure is the variation of the mean $\bar{x}_N$ for the 'measurements' of our thought experiment of Chap. 3, with the number of measurements performed, N, as this number increases, finally reaching the value of 10,000. The curves for $x_0 + \sigma/\sqrt{N}$ and $x_0 - \sigma/\sqrt{N}$ are also drawn in the figure. These values are known to us here, as the results of the measurements x_i were specially selected for the 'experiment', using random numbers, so that they have $x_0 = 100\,\text{mm}$ and $\sigma = 1\,\text{mm}$.

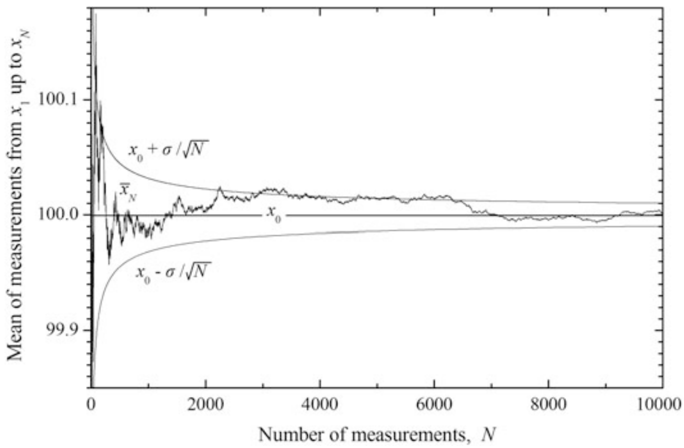

Fig. 4.3 The variation of the mean $\overline{x}_N$ for N measurements of the quantity x, as this number increases. The real value of x is $x_0 = 100$ mm and the standard deviation of the parent population of the possible measurements is $\sigma = 1$ mm. The curves for $x_0 \pm \sigma/\sqrt{N}$ were also drawn in the figure

The standard deviation of the mean, $\sigma_{\overline{x}}$, being the best estimate we have for the root of the mean square of the deviations from the real value, of the means of many series of measurements of x, gives an estimate of the expected difference of the determined value from the real. We will see below that it helps us make predictions for the statistical distribution of the means of many series of measurements of x. Concerning the determination of the real value of x, we may say that, most probably, it lies between the limits $\overline{x} \pm \sigma_{\overline{x}}$. We state this by writing

$$x = \overline{x} \pm \sigma_{\overline{x}}, \tag{4.48}$$

when giving the numerical values of $\overline{x}$ and $\sigma_{\overline{x}}$. For the first 1000 values of Fig. 4.3, for example, we find that it is $\overline{x} = 99.99$ mm and $\sigma_{\overline{x}} = 0.03$ mm, to an accuracy of two decimal digits. Thus, we write:

$$x = 99.99 \pm 0.03\,\text{mm}.$$

The presentation of numerical results will be examined in the next chapter.

The quantity

$$\delta x \equiv \sigma_{\overline{x}} = \sigma_{\mu} = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \overline{x})^2}{N(N-1)}} \tag{4.49}$$

is also called *standard error in the mean* or, simply, *error in the mean* $\overline{x}$.

Also used is the *relative or fractional standard deviation of the mean* or the *relative or fractional error in the mean*,

$$\frac{\delta x}{\bar{x}}, \tag{4.50}$$

which is also expressed as a percentage,

$$100\,\frac{\delta x}{\bar{x}}\,\%. \tag{4.51}$$

4.4.2 The Statistical Distribution of the Results of the Measurements

From the standard deviation s_x of the results of the measurements, we have an estimate $\hat{\sigma}$ for the standard deviation σ of the parent population [Eq. (4.39)]. If we knew the mathematical form of the probability density function $f(x)$ of the parent population of all the possible measurements that can be made, we would be able to make estimates for the parameters present in $f(x)$. For example, if the distribution had the form of the Laplace distribution,

$$f(x) = \frac{\alpha}{2}\mathrm{e}^{-\alpha|x-\mu|}, \tag{4.52}$$

we would have estimates for μ and α. In Example 1.6 we found that, for this normalized distribution, it is $\bar{x} = \mu$ and $\sigma = \sqrt{2}/\alpha$. Since the estimate we found for σ is

$$\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}, \tag{4.53}$$

we would find that it is

$$\alpha = \sqrt{2}/\hat{\sigma} \tag{4.54}$$

and, thus, have an estimate for $f(x)$. From this we would be able to derive quantitative conclusions regarding the distribution of the results of the measurements, such as, for example, the proportion of measurements expected to have values between certain limits, the probability for a result to exceed a certain value etc.

However, the function $f(x)$ is not known to us with certainty and the histogram of the measurements is usually too vague (due to the low number of measurements) to give us even an approximation for the form of $f(x)$. There are, however, well grounded reasons for us to believe that, under some very general conditions, the distribution of the results of the measurements is expressed by a probability density function which has the, so called, *Gaussian* form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-(x-\mu)^2/2\sigma^2} \equiv G_{\mu,\sigma}(x). \tag{4.55}$$

It is easily proved that μ is the mean value and σ is the standard deviation of the results of the parent population of the measurements, x. The distribution is also called the *normal distribution.*

Strictly speaking, the distribution is termed normal when it is stated in the form

$$G_{0,1}(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}, \tag{4.56}$$

i.e. when it has as mean the value of $\mu = 0$ and a standard deviation equal to $\sigma = 1$ (or, in other words, when the function has been normalized so that the deviations x are measured from the mean, in units of σ).

The conditions for this distribution to be valid are:

(a) the errors of the measurements are due to the superposition of a large number of small deviations from the real value and
(b) these deviations are equally probable to be positive or negative.

These conditions appear reasonable for the description of the behavior of the random errors of measurements, but it must not be taken for granted that the Gaussian distribution has general validity. In most cases, it is a satisfactory approximation of reality and it is used due to its mathematical simplicity. This will be discussed in more detail in Chap. 9.

The graph of the Gaussian distribution. is shown in Fig. 4.4 in a universal form. The quantity $\sigma f(x)$ has been plotted as a function of $(x-\mu)$, which is expressed in units of σ. Thus, the shape of the curve of the figure is independent of σ and μ.

The curve of the Gaussian function is symmetrical relative to the axis $x = \mu$. It has a maximum equal to $0.3989/\sigma$ at $x = \mu$, while for $x = \mu \pm \sigma$ it takes the value $0.2420/\sigma$. The points of the curve at $x = \mu \pm \sigma$ are points of inflection. For large values of $|x-\mu|$ the curve tends rapidly and asymptotically towards the x-axis.

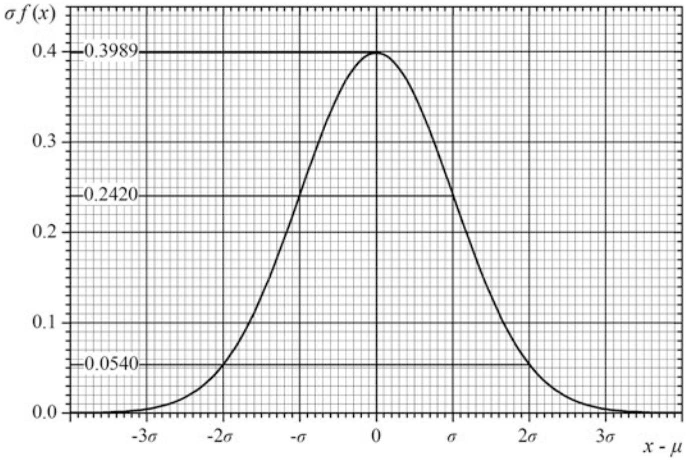

Fig. 4.4 The density function of the Gaussian distribution

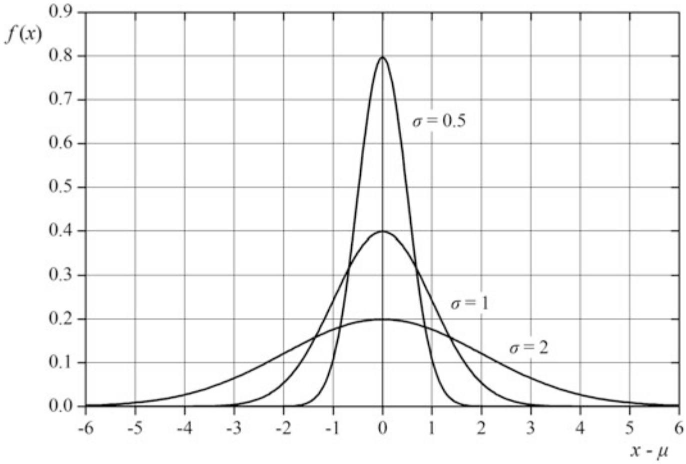

Fig. 4.5 Plots of the Gaussian function for σ equal to 0.5, 1 and 2

The significance of the parameter σ is seen in Fig. 4.5, where $f(x)$ was drawn as a function of $(x-\mu)$, for σ equal to 0.5, 1 and 2. It is immediately evident that large σ means a large dispersion of the values of x.

The total area between the curve and the x-axis is equal to unity:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}\,dx = 1. \tag{4.57}$$

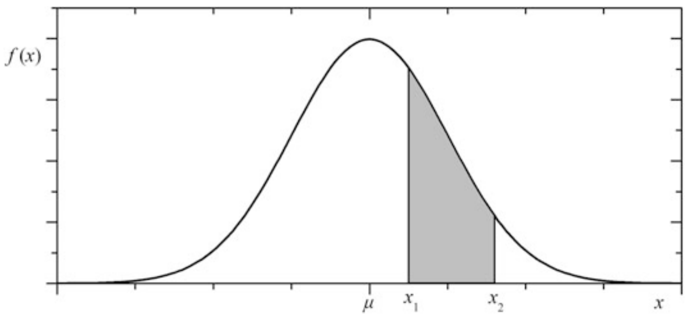

Fig. 4.6 The area between the Gaussian curve and the x-axis, in the range $[x_1, x_2]$

The function is, therefore, normalized. As a consequence, the area between the curve and the x-axis in the range $[x_1, x_2]$ gives the probability for a value of x to lie in the range $x_1 \leq x \leq x_2$ (see Fig. 4.6):

$$\Pr\{x_1 \leq x \leq x_2\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{-(x-\mu)^2/2\sigma^2} dx. \tag{4.58}$$

There are detailed tables of the Gaussian function and its integral. The function

$$\operatorname{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \tag{4.59}$$

which is called *error function* (thus erf from **er***ror* **f***unction*), is used in the evaluation of the integral of the probability density function of the normal distribution [1].

Also defined is the function [2]

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt. \tag{4.60}$$

It is

$$\Phi(x) = \frac{1}{2} \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right). \tag{4.61}$$

The functions $\operatorname{erf}(x)$ and $\Phi(x)$ are odd and, therefore, $\operatorname{erf}(-x) = -\operatorname{erf}(x)$ and $\Phi(-x) = -\Phi(x)$. Also, $\operatorname{erf}(0) = 0$, $\Phi(0) = 0$ and $\operatorname{erf}(\infty) = 1$, $\Phi(\infty) = \frac{1}{2}$.

Table 4.2 Values of the functions $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\,dt$

x	$G_{0,1}(x)$	$\Phi(x)$	x	$G_{0,1}(x)$	$\Phi(x)$
0	0.398 942	0	2.1	0.043 984	0.482 136
0.1	0.396 952	0.039 828	2.2	0.035 475	0.486 097
0.2	0.391 042	0.079 260	2.3	0.028 327	0.489 276
0.3	0.381 388	0.117 911	2.4	0.022 395	0.491 802
0.4	0.368 270	0.155 422	2.5	0.017 528	0.493 790
0.5	0.352 065	0.191 462	2.6	0.013 583	0.495 339
0.6	0.333 224	0.225 747	2.7	0.010 421	0.496 533
0.7	0.312 254	0.258 036	2.8	0.007 915	0.497 445
0.8	0.289 691	0.288 145	2.9	0.005 953	0.498 134
0.9	0.266 085	0.315 940	3.0	0.004 432	0.498 650
1.0	0.241 971	0.341 345	3.1	0.003 267	0.499 032
1.1	0.217 852	0.364 334	3.2	0.002 384	0.499 313
1.2	0.194 186	0.384 930	3.3	0.001 723	0.499 517
1.3	0.171 368	0.403 200	3.4	0.001 232	0.499 663
1.4	0.149 727	0.419 243	3.5	0.000 873	0.499 767
1.5	0.129 518	0.433 193	3.6	0.000 612	0.499 841
1.6	0.110 921	0.445 201	3.7	0.000 425	0.499 892
1.7	0.094 049	0.455 435	3.8	0.000 292	0.499 928
1.8	0.078 950	0.464 070	3.9	0.000 199	0.499 952
1.9	0.065 616	0.471 283	4.0	0.000 134	0.499 968
2.0	0.053 991	0.477 250	4.1	0.000 0893	0.499 979

Some values of the functions $G_{0,1}(x)$ and $\Phi(x)$ are given in Table 4.2.
The functions $G_{0,1}(x)$ and $\Phi(x)$ have been plotted in Fig. 4.7.
With the aid of the functions erf(x) and $\Phi(x)$ we find that, for $X \geq \mu$,

$$\Pr\{\mu \leq x \leq X\} = \frac{1}{2}\operatorname{erf}\left(\frac{X-\mu}{\sigma\sqrt{2}}\right) = \Phi\left(\frac{X-\mu}{\sigma}\right) \tag{4.62}$$

is the probability that a value of x lies in the region $\mu \leq x \leq X$. Defining as $\chi \equiv X - \mu$ the deviation of x from the mean μ, and making use of the symmetry of the Gaussian distribution with respect to the mean, we have

$$\Pr\{\mu - \chi \leq x \leq \mu + \chi\} = \operatorname{erf}\left(\frac{\chi}{\sigma\sqrt{2}}\right) = 2\,\Phi\left(\frac{\chi}{\sigma}\right) \tag{4.63}$$

as the probability for a value of x to differ from the mean by less than χ.

Measuring the deviation of x from the mean in multiples of the standard deviation, i.e. putting $\chi = v\sigma$, where v is a positive number, we have

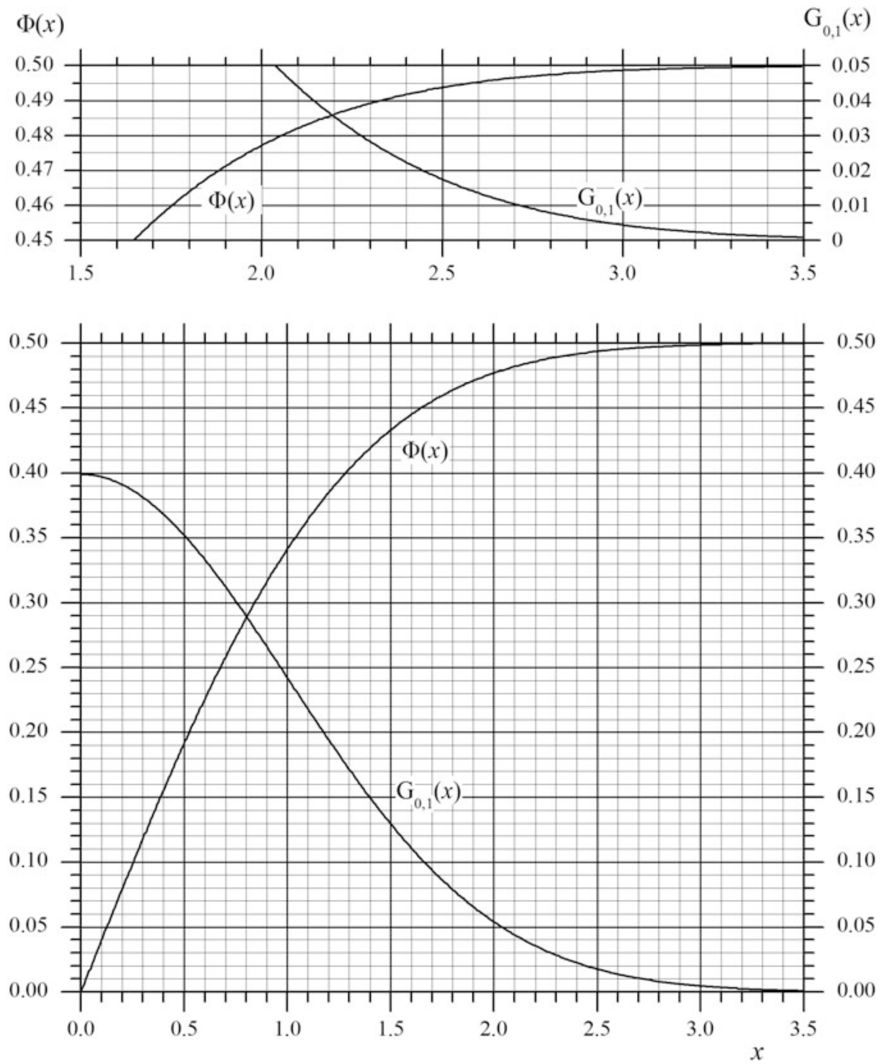

Fig. 4.7 The functions $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$

$$\Pr\{\mu - v\sigma \le x \le \mu + v\sigma\} = \operatorname{erf}\left(\frac{v}{\sqrt{2}}\right) = 2\,\Phi(v) \tag{4.64}$$

as the probability for a value of x to differ from the mean by less than v times the standard deviation.

The probability for a value of x to differ from the mean by more than v times the standard deviation is given by the relation

Table 4.3 The probabilities that a value of x lies in the range $\mu - \nu\sigma \leq x \leq \mu + \nu\sigma$ or outside it

ν	$\Pr\{\mu - \nu\sigma \leq x \leq \mu + \nu\sigma\}$	$\Pr\{x \leq \mu - \nu\sigma \text{ or } x \geq \mu + \nu\sigma\}$
0	0	1
0.001253	0.001	0.999
0.01253	0.01	0.99
0.1257	0.1	0.9
0.6745	0.5	0.5
1	0.68269	0.31731
1.282	0.8	0.2
1.665	0.9	0.1
1.96	0.95	0.05
2	0.95450	0.04550
2.33	0.98	0.02
2.58	0.99	0.01
3	0.99730	0.00270
3.29	0.999	0.001
∞	1	0

$$\Pr\{x \leq \mu - \nu\sigma \text{ or } x \geq \mu + \nu\sigma\} = 1 - \text{erf}\left(\frac{\nu}{\sqrt{2}}\right) \equiv \text{erfc}\left(\frac{\nu}{\sqrt{2}}\right) = 1 - 2\,\Phi(\nu), \qquad (4.65)$$

where erfc $(x) \equiv 1 - \text{erf}\ (x)$ is the *complementary error function*. The values of these probabilities for various values of ν are given in Table 4.3.

The table shows that:

Half the values of x are expected to differ from the mean by more than 0.6745 times the standard deviation. The value 0.6745 σ is called *probable error* in the results for x.

31.7% of the values, or about 1 in 3, are expected to differ from the mean by more than one standard deviation.

4.6% of the values, or 1 in 22, are expected to differ from the mean by more than 2σ.

0.27% of the values, or 1 in 370, are expected to differ from the mean by more than 3σ.

Other useful conclusions that follow from Eq. (4.65) are:

95% of the values lie in the region $\mu \pm 1.96\sigma$.

Only 1% of the values, or 1 in 100, are expected to differ from the mean by more than 2.58σ.

Only 0.1% of the values, or 1 in 1000, are expected to differ from the mean by more than 3.29σ.

In Fig. 4.8 the arrows show the regions, about 0, which contain 50, 68, 90, 95 and 99% of the values. Also shown are the regions of $x - \mu$ above which lie 1, 2.3, 5, 10, 16 and 25% of the values.

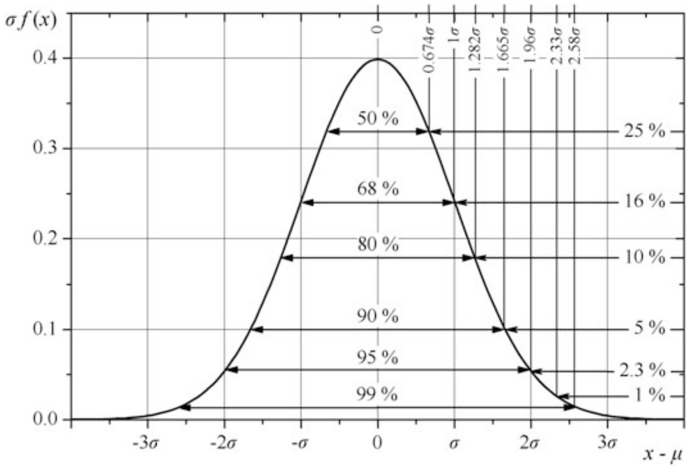

Fig. 4.8 The regions, about 0, which contain 50, 68, 90, 95 and 99% of the values, in the case of the Gaussian distribution. Also shown are the values of $x - \mu$ above which lie 1, 2.3, 5, 10, 16 and 25% of the values

These values are useful when we have to decide if a result that differs from the mean by a large difference should be accepted or rejected, since the difference is very improbable to be due to random errors. For example, since a measurement has a probability of 0.0027 to differ by more than 3σ from the mean, in 10 measurements, say, we expect $0.0027 \times 10 = 0.027$ measurements to differ by more than 3σ from the mean. If we actually have one such value, we might think that the probability of something like that happening as a result of random errors is too low and therefore we should exclude that particular value from the analysis of our measurements.

Having in mind the statistical estimates we mentioned above, we conventionally consider σ as an indicative value for the deviations of the measurements from the mean and, as a consequence, from the real value also. For this reason, σ is also called *standard deviation* (or *standard error* or simply *error*) *of a single measurement*. Having determined the value of s_x for a series of measurements of the quantity x and considering $\hat{\sigma}$, as this is derived from Eq. (4.39), as the best estimate we have for σ, we may say that the expected standard error of a single measurement of the quantity x that we may make, is equal to σ.

The statement 'the standard deviation of x is s_x' means that 68% of the results of the measurements of x are expected to lie in the region between $\bar{x} - s_x$ and $\bar{x} + s_x$.

Example 4.20 [E]

Given a Gaussian distribution with $\mu = 2$ and $\sigma = 1$, find the probability of a value between $x_1 = 3$ and $x_2 = 4$.

We follow the path:

Formulas > More Functions > Statistical

Select **NORM.DIST**. In the dialog box that opens, set

X = 4, Mean = 2, Standard dev = 1 and **Cumulative = TRUE**

Pressing **OK** returns the probability of x being smaller than 4, $\Pr\{x<4\} = 0.977250$.

Setting

X = 3, Mean = 2, Standard dev = 1 and **Cumulative = TRUE**

and pressing **OK** returns the probability of x being smaller than 3, $\Pr\{x<3\} = 0.841345$.

Taking the difference of the two probabilities, we have the probability of x having a value between $x_1 = 3$ and $x_2 = 4$ as being equal to $\Pr\{3<x<4\} = \Pr\{x<4\} - \Pr\{x<3\} = 0.977250 - 0.841345 = 0.135905$.

Example 4.21 [O]

Given a Gaussian distribution with $\mu = 2$ and $\sigma = 1$, find the probability of a value between $x_1 = 3$ and $x_2 = 4$.

The probability density distribution is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$.

Here, it is $\frac{1}{\sqrt{2\pi}\sigma} = 0.39894$ and $\frac{1}{2\sigma^2} = 0.5$, so $f(x) = 0.39894\, e^{-0.5(x-2)^2}$.

We will perform numerical integration of this function between $x_1 = 3$ and $x_2 = 4$.

We first fill the first 1000 cells of column A with values between 3.0005 and 3.9995, increasing in steps of $\delta x = 0.001$. This is done as follows:

We select column A by left-clicking on label A. Then, **Column > Set Column Values**.

Set col(A) equal to $3.0005 + (i-1) * 0.001$, for $i = 1$ to $i = 1000$.

We select column B by left-clicking on label B. Then, **Column > Set Column Values**.

Set col(B) equal to 0.39894 * exp(−0.5 * (col(A) − 2)^2), for $i = 1$ to $i = 1000$.

We add all the values in column B, using **Σ**. The result is 135.9043. Multiplying by $\delta x = 0.001$, we find the area under the curve between $x_1 = 3$ and $x_2 = 4$. This gives the value of 0.1359 as the probability of a value between $x_1 = 3$ and $x_2 = 4$.

It is not clear whether the accuracy with which the probability is given is justified. This can be checked by performing the numerical integration with a smaller δx, say $\delta x = 0.0005$. If the result is the same with 4 significant figures, then the result can be assumed to be accurate with the given significant figures.

Example 4.22 [P]

Given a Gaussian distribution with $\mu = 2$ and $\sigma = 1$, find the probability of a value between $x_1 = 3$ and $x_2 = 4$.

```
# http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.
html
from scipy.stats import norm

# Enter the values of the distribution's parameters:

mean = 2            # the mean
stdev = 1           # and standard deviation of the distribution

# Enter the values of the limits of x:
x1 = 3
x2 = 4

# Evaluation:
# The probability of a value of x between x1 and x2:
print ("The probability of a value of x between 3 and 4 is =", norm.cdf(x2,
mean, stdev) - norm.cdf(x1, mean, stdev))

# Result:
The probability of a value of x between 3 and 4 is = 0.135905121983
```

Example 4.23 [R]

Given a Gaussian distribution with $\mu = 2$ and $\sigma = 1$, find the probability of a value between $x_1 = 3$ and $x_2 = 4$.

The function **pnorm(*q*, μ, *σ*)** gives the probability that a value of x is smaller than q, $P\{x < q\}$. The probability of a value between $x_1 = 3$ and $x_2 = 4$ is $P\{x_1 < x < x_2\} = P\{x < x_2\} - P\{x < x_1\}$:

```
> P = pnorm(4,2,1)-pnorm(3,2,1)
> P
[1] 0.1359051
```

We have found that $\Pr\{3 < x < 4\} = 0.135905$.

Example 4.24

Using Table 4.2 verify the result of Examples 4.20–4.23.

It is given that $\mu = 2$, $\sigma = 1$, $x_1 = 3$ and $x_2 = 4$. From Table 4.2, for $x = x_1 - \mu = 1$ we find that $\Phi(1) = 0.341\ 345$ and for $x = x_2 - \mu = 2$ it is $\Phi(2) = 0.477\ 250$. The difference gives the probability of a value between 3 and 4 as 0.135905. The result of Examples 4.20 and 4.21 is 0.1359.

Example 4.25 [E]

Given the Gaussian probability distribution function $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}$, find the value x_0 of x for which there is a probability $\Pr\{x \le x_0\} = 0.9$ that it is $x \le x_0$.

We follow the path:

Formulas > More Functions > Statistical

And select the function **NORM.INV**. In the dialog box that opens, we set:
Probability = 0.9, Mean = 0 and Standard dev = 1
Pressing OK returns the required value as being $x_0 = 1.281552$.
If the standard deviation of the distribution is σ and not 1, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-x^2/2\sigma^2}$ we multiply x_0 by σ. If, for example, it is $\sigma = 2$, we multiply the value of x_0 by 2, obtaining $x_0' = 2.563103$. If also the mean is not 0 but it is μ, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-(x-\mu)^2/2\sigma^2}$, we add to x_0' the value of μ. If, say it is $\mu = 3$, we obtain $x_0'' = 5.563103$.

Example 4.26 [O]

Given the Gaussian probability distribution function $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}$, find the value x_0 of x for which there is a probability $\Pr\{x \le x_0\} = 0.9$ that it is $x \le x_0$.

We highlight any empty cell, in which we want the result to be written, by left-clicking on it. Then,

Column > Set column values > Functions > Distributions > INV > norminv (p)

Substituting p = 0.9 and pressing **OK**, we get for the required value $x_0 = 1.28155$.

If the standard deviation of the distribution is σ and not 1, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-x^2/2\sigma^2}$ we multiply x_0 by σ. If, for example, it is $\sigma = 2$, we multiply the value of x_0 by 2, obtaining $x_0' = 2.56310$. If also the mean is not 0 but it is μ, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-(x-\mu)^2/2\sigma^2}$, we add to x_0' the value of μ. If, say it is $\mu = 3$, we obtain $x_0'' = 5.56310$.

Example 4.27 [P]

Given the Gaussian probability distribution function $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}$, find the value x_0 of x for which there is a probability $\Pr\{x_0 \le x\} = 0.9$ that it is $x_0 \le x$.

```
# http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.
html
from scipy.stats import norm
# Enter the following of the distribution's parameters:

mean = 0              # the mean
stdev = 1             # and standard deviation of the distribution

# Enter the value of the cumulative probability, p:
p = 0.9

# Evaluation:
# The value of x at which the cumulative probability is equal to p is:
x0 = norm.ppf(p, mean, stdev)

# Result:
print ("The value of x for which the cumulative probability is p, is x0 =", x0)

The value of x for which the cumulative probability is p, is x0 =
1.28155156554
```

Example 4.28 [R]

Given the Gaussian probability distribution function $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-x^2/2}$, find the value x_0 of x for which there is a probability $\Pr\{x \le x_0\} = 0.9$ that it is $x \le x_0$.

The function **pnorm(*q*, μ, *σ*)** gives the value x_0 of x for which there is a probability $\Pr\{x_0 \le x\} = q$ that it is $x_0 \le x$. Here, $\mu = 1$, $\sigma = 1$ and $q = 0.9$. Therefore,

```
> qnorm(0.9, 0, 1)
[1] 1.281552
```

and $x_0 = 1.28155$.

If the standard deviation of the distribution is σ and not 1, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-x^2/2\sigma^2}$ we multiply x_0 by σ. If, for example, it is $\sigma = 2$, we multiply the value of x_0 by 2, obtaining $x_0' = 2.56310$. If also the mean is not 0 but it is μ, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}$, we add to x_0' the value of μ. If, say it is $\mu = 3$, we obtain $x_0'' = 5.56310$.

4.4.3 Statistical Estimates for the Mean

If we perform M series of measurements, each with N measurements, we will have M pairs of values for the mean $\bar{x}$ of the N measurements and its standard deviation, $\sigma_{\bar{x}}$. It is proved that:

If samples consisting of N values are taken from a parent population whose mean is μ and its standard deviation σ, and if N is large, the distribution of the means of the samples tends to a normal (Gaussian) distribution, independently of the form of the distribution of the parent population.

This is known as the *central limit theorem*. We will discuss this theorem in Chap. 9.

We have found that the relationship between the standard deviation of the mean, σ_μ, and the standard deviation of all the possible measurements of the parent population, σ, is

$$\sigma_\mu = \frac{\sigma}{\sqrt{N}}.$$

We therefore conclude, making use of the central limit theorem, that the mean values are normally distributed about the value of μ, with a standard deviation σ_μ. The distribution function of the means $\bar{x}$ is, therefore,

$$f(\bar{x}) = \frac{1}{\sqrt{2\pi}\,\sigma_\mu} \mathrm{e}^{-(\bar{x}-\mu)^2/2\sigma_\mu^2}. \tag{4.66}$$

Of course, μ and σ_μ are unknown. The best estimates we have for them, having performed only one series of N measurements, is the mean $\bar{x}$ and the standard deviation of the mean, $\sigma_{\bar{x}}$, of these N measurements. We know that for 'series' of measurements consisting of only one measurement each, σ_μ is equal to the standard deviation of the parent population, σ, while, according to the relation $\sigma_\mu = \sigma/\sqrt{N}$, as N increases, σ_μ tends to zero, as demonstrated in Fig. 4.9.

Figure 4.9 shows the distributions of the means, $\bar{x}$, of series of measurements consisting of 1, 5 and 25 measurements each. The standard deviation of the parent population of the measurements is σ. For $N = 1$ measurement, the means are the values x_i themselves and, therefore, their standard deviation will be that of the parent population, σ. For $N = 5$ measurements, the dispersion of the means about the real value μ is small and their standard deviation is $\sigma_\mu = \sigma/\sqrt{5} = 0.447\,\sigma$. Increasing the number of measurements to $N = 25$ has the result that the dispersion is further reduced and the standard deviation is reduced to $\sigma_\mu = \sigma/\sqrt{25} = 0.2\,\sigma$. The advantage achieved by increasing the number of measurements for a better determination of the unknown quantity μ is obvious. The larger the number of measurements used in the evaluation of the mean, the more probable it is that the mean is near the real value μ. The standard deviation of the mean, σ_μ, is a measure of the order of magnitude of the error present in the determination of μ using the N

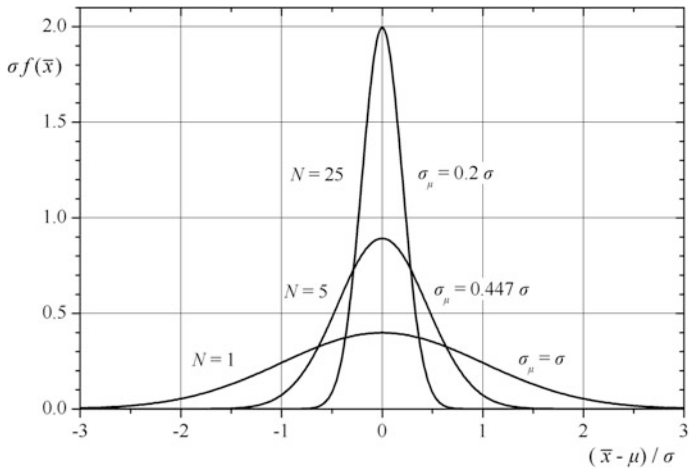

Fig. 4.9 Distributions of the means $\bar{x}$ derived from series of measurements of x consisting of $N = 1,\ 5$ and 25 measurements each. The real value of x is $x_0 = \mu$ and the standard deviation of the parent population of all the possible measurements x_i is equal to σ

measurements performed. The best estimate we have for the value of σ_μ is $\sigma_{\bar{x}}$. We can therefore consider the quantity

$$\delta x = \sigma_{\bar{x}} \tag{4.67}$$

as a measure of the error we have when we consider $\bar{x}$ to be an estimate of the real value $x_0 = \mu$ of the quantity measured.

It should finally be noted that, given that the distribution of the means is Gaussian, the statistical estimates valid for normally distributed variables, as these were presented in Sect. 4.4.2, are also true for the mean $\bar{x}$. The statement $x = \bar{x} \pm \sigma_{\bar{x}}$ means that there is a 68% probability that $\bar{x}$ differs from the real value x_0 of x by less than $\sigma_{\bar{x}}$.

4.4.4 Summary of the Method of Analysis of the Results

With reference to Fig. 4.10, we will now summarize the whole process of the statistical treatment of the results of the measurements of the quantity x.

(a) We suppose that the quantity x has a clearly defined real value x_0, which remains constant at least for the time needed for the measurements to be made. In our example, we will assume that it is $x_0 = 100$ mm exactly.
(b) The experimental procedure we will follow, the instruments we will use and the sources of noise determine the distribution of all the possible measurements that can be made, known as the parent population. This has a probability density $f(x)$. There are reasons for us to believe that this is Gaussian (normal). In our numerical example we take the standard deviation of the Gaussian to be $\sigma = 1$ mm.

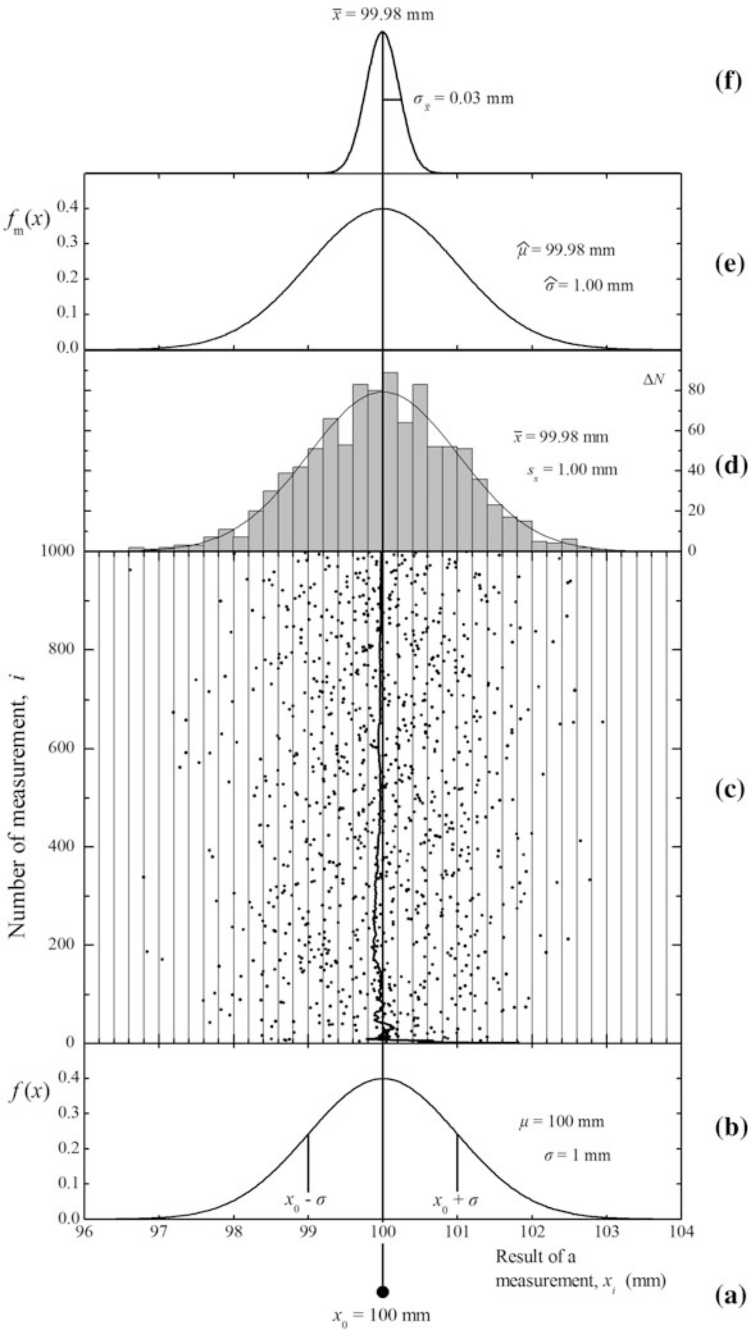

Fig. 4.10 Summary of the method of analysis of the results

(c) We perform, successively and under identical experimental conditions, a number N of measurements x_i of the quantity x. The results of the measurements have random deviations from the real value, whose statistical behavior is described by the function $f(x)$. The points in the figure show the results of these measurements in the order in which they were performed, while the thick line shows the variation of the mean $\bar{x}$ of the first N measurement, as the number of measurements increases from 1 to 1000. The asymptotic approach of $\bar{x}$ to x_0 is evident, with the random deviations being of the expected order of magnitude. We should point out that, almost always, the number of measurements performed is much smaller than used here. We use a large number of measurements, however, in order to have a sample large enough for its statistical properties to be clearly visible.

(d) The histogram of the measurements shows the grouping of the results around the real value and their dispersion according to the standard deviation of the parent population. We evaluate the mean $\bar{x}$ and the standard deviation, s_x, of the results. For our numerical result we find $\bar{x} = 99.98$ mm and $s_x = 1.00$ mm. The continuous curve shows the best Gaussian distribution that may be fitted to the histogram (using a method we will describe in a later chapter).

(e) The Gaussian distribution that results from the histogram of the measurements, $f(x)_{\mathrm{m}}$, has mean and standard deviation the estimates $\hat{\mu} = \bar{x} = 99.98$ mm and $\hat{\sigma} = \sqrt{\frac{N}{N-1}}\, s_x = 1.00$ mm we have for these quantities.

(f) The results of our measurements have a mean $\bar{x} = 99.98$ mm and a standard deviation of the mean $\sigma_{\bar{x}} = s_x/\sqrt{N-1} = 0.03$ mm. The final result is given, therefore, as

$$x = 99.98 \pm 0.03\,\text{mm}\;(1000\,\text{measurements}).$$

We note that, in this particular example, the error in the mean,

$$e_\mu \equiv \bar{x} - x_0 = 99.98 - 100.00 = -0.02\,\text{mm},$$

happens to be smaller than one standard deviation of the mean.

The result $x = 99.98 \pm 0.03$ mm defines a Gaussian curve with mean $\bar{x} = 99.98$ mm and standard deviation $\sigma_{\bar{x}} = 0.03$ mm, which gives the probability the real value x_0 of $\mathbf{x}$ to be in a certain range of values.

The reader may perhaps have reservations as to whether the example used, with its 1000 measurements, is realistic. Obviously, we used a large number of measurements in order to demonstrate their statistical behavior. The same analysis is used for smaller numbers of measurements, say 5–10, but in those cases the results must be considered to be less accurate.

Programs
Excel
Ch. 04. Excel—Mean and Standard Deviations
Origin
Ch. 04. Origin—Mean and Standard Deviations
Python
R
Ch. 04. R—Mean and Standard Deviations

Problems

4.1 **[E.O.P.R.]** Find the sample standard deviation of the values: 1.6, 1.4, 1.0, 2.4, 1.2, 2.0.

4.2 **[E.O.P.R.]** Find the mean, the standard deviation and the standard deviation of the mean of the measurements:

$$10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15 \quad 16 \quad 17 \quad 18 \quad 19.$$

4.3 Show that, for a number of measurements equal to 2, the standard deviation of their mean is equal to half the difference of the two measurements.

4.4 If an amount a, exactly, is added to the results x_i of a series of measurements, what will the change be in (a) the mean and (b) the standard deviation of the values?

4.5 If the results x_i of a series of measurements are multiplied by an exact factor of a, what will the change be in (a) the mean and (b) the standard deviation of the values?

4.6 **[E.O.P.R.]** Ten successive measurements of the period of a pendulum gave the following results:

i	1	2	3	4	5	6	7	8	9	10
T_i (s)	2.16	1.85	2.06	1.93	2.03	1.98	2.02	1.97	2.06	1.95

Calculate: (a) the mean value of T, (b) the standard deviation of the measurements and (c) the standard deviation of their mean.

4.7 Find the mean, the standard deviation and the standard deviation of the mean of the 30 measurements x_r of the table below, if their frequencies are n_r:

r	1	2	3	4	5	6	7	8	9	10
x_r	10	11	12	13	14	15	16	17	18	19
n_r	1	2	3	4	5	5	4	3	2	1

What will the values of these quantities be if (a) the values x_r are doubled and (b) the frequencies n_r are doubled?

4.8 A series of 51 measurements of x gave the following results x_r, with the frequencies n_r given:

r	1	2	3	4	5	6	7	8
x_r (cm)	125	126	127	128	129	130	131	132
n_r	2	3	9	16	11	7	2	1

Find the mean $\bar{x}$ and the standard deviation s_x of the measurements and the standard deviation $\sigma_{\bar{x}}$ of their mean.

4.9 **[E.O.P.R.]** For the values

$$x_i: \quad 10.3 \quad 10.1 \quad 10.5 \quad 10.4 \quad 10.7 \quad 10.4 \quad 10.2 \quad 10.5 \quad 10.3$$

find the mean value, $\bar{x}$, their standard deviation, s_x, the standard deviation of the mean, $\sigma_{\bar{x}}$, as well as the standard deviations of these two standard deviations.

What are the best estimates for the mean μ and the standard deviation σ of the parent population from which the sample was taken?

4.10 If the values of a magnitude **x**, as they are obtained by N measurements, have a standard deviation (of a single observation) $\sigma_x = 0.05$, for which values of N would the standard deviation of the mean be equal to (a) 0.03, (b) 0.02, (c) 0.005?

4.11 If we wish to have an estimate of the standard deviation of the parent population with an uncertainty of less than 5%, how many measurements must our series of measurements consist of?

4.12 **[E.O.P.R.]** If the random variable **x** is normally distributed with $\bar{x} = 10.0$ and $\sigma_x = 0.5$, what is the probability for the observation of a value (a) $x \leq 9$, (b) $9 \leq x \leq 11$, (c) $x \geq 11$, (d) $x \geq 11.5$?

4.13 From measurements performed on a large number of electric light bulbs produced by a certain factory, it was found that the durations of their lives had a mean of τ = 1200 h and a standard deviation of σ = 200 h.

What is the probability that a bulb from this factory will operate, before it fails, for less than (a) 800, (b) 1000, (c) 1200, (d) 1500 and (e) 1800 h?

(f) What is the probability that a bulb will fail after it has operated for a time between 800 and 1200 h?

4.14 For the measurement of the power consumed by electric light bulbs, a voltmeter was used which has a standard deviation of 0.20 V in its measurements and an ammeter with a standard deviation of 0.015 A in its measurements. What is the % standard deviation of a measurement of the power with these instruments in each of the following cases?

(a) A 500 W bulb operating at 115 V.
(b) A 60 W bulb operating at 115 V.
(c) A 60 W bulb operating at 32 V.
(d) A 60 W bulb operating at 8 V.

Note: The power consumed by a bulb is $P = IV$ where I is the current through the bulb and V is the potential difference across it.

4.15 What is the % accuracy in the determination of the density of a steel sphere which has a mass of 10 g and a density of about 7.85 g/cm^3, if the standard deviation in the measurement of its radius is 0.015 mm and in the measurement of its mass 0.05 mg?

4.16 **[E.O.P.R.]** A large number of measurements of the thermal conductivity of copper at the temperature of °C, have a Gaussian distribution with mean $k = 385$ W/(m·°C) and standard deviation $\sigma = 15$ W/(m·°C). What is the probability that a measurement lies between: (a) 370 and 400, (b) 355 and 415, and (c) 340 and 430 W/(m·°C)?

References

1. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (*with Formulas, Graphs and Mathematical Tables*), (Dover Publications, Inc., New York, 1965). Ch. 7
2. Op. cit., Ch. 26, for related functions