Chapter 12
Graphs

12.1 Introduction

The graphical representation of experimental results may serve one or more of the following purposes:

1. To show the relationship between two quantities, bringing out characteristics which would not be obvious in a table of numerical values.
2. The curve of the graph may be used in the evaluation of the slope or the intercept with one of the axes, especially when the relationship between the two magnitudes is linear. Important physical quantities and natural constants are usually determined by this method.
3. To investigate the form of the relation connecting two variables (linear, exponential etc.) which then may be expressed in the form of a mathematical equation for greater accuracy.
4. For the verification or not of a theoretical relation between two magnitudes, by comparison of the theoretical curve with the experimental results.
5. To determine the calibration curve of an instrument or of a process or, in general, to enable finding the value of one of the variables corresponding to a certain value of the other. Even if the result is present in the table of results, reading the required value off the best curve drawn between the experimental points gives a better value, as it is based on more than one measurement.

Having used so many graphs in the previous chapters of the book, it is certainly unnecessary to try to put forward arguments in favor of using graphs in the presentation of information. The numerical values of a table are obviously useful, but they do not transmit the same amount of information as a graph does. Figure 12.1 demonstrates the truth of this statement.

In this chapter we will present the main characteristics of graphs and the criteria on the basis of which these are selected for the best presentation of the data. We will only examine cases in which the results of measurements we have at our

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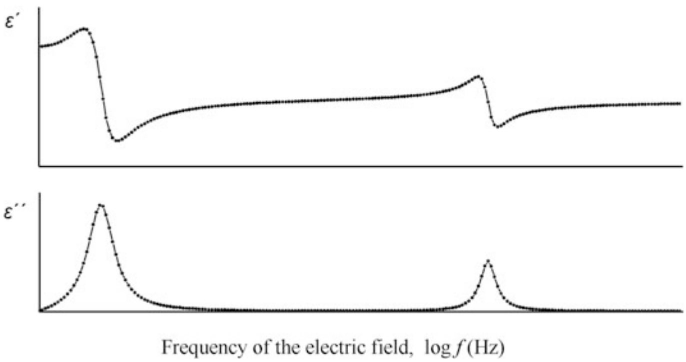


Fig. 12.1 Real (ε') and imaginary part (ε'') of the dielectric constant of a material, as functions of the frequency. The frequency scale is logarithmic and common to both quantities

disposal are N pairs of values $(x,\ y)$ of a dependent variable y as a function of an independent variable x. We wish to exhibit the relationship between the two magnitudes in the best way.

12.2 The Axes

The first things that are drawn in a graph are the axes. The choice of these will determine the kind of graph that will result, the range of values that will be covered as well as the kind of curve that will be obtained.

The magnitude considered to be the independent variable is usually recorded along the axis of the abscissae (x-axis), while the magnitude considered to be the dependent variable is recorded along the ordinate axis (y-axis). Although the distinction is not always possible, in general, if in an experiment we set the values of one of the variables (e.g. the potential difference across the ends of a resistance), then this variable is considered to be the independent variable and the result (e.g. the current through the resistance) is the dependent variable.

12.2.1 Linear Scales

In the simplest and most common graphs, we use linear scales for both x and y. Examples of axes with linear scales have been drawn in Fig. 12.2. Although the axes drawn are horizontal, ordinate axes may be drawn in the same manner.

The numerical values for a reasonable number of points are given on the axis, in such a way that intermediate values of the variable would be easy to find. This is done without overloading the axis with numbers which may cause confusion without

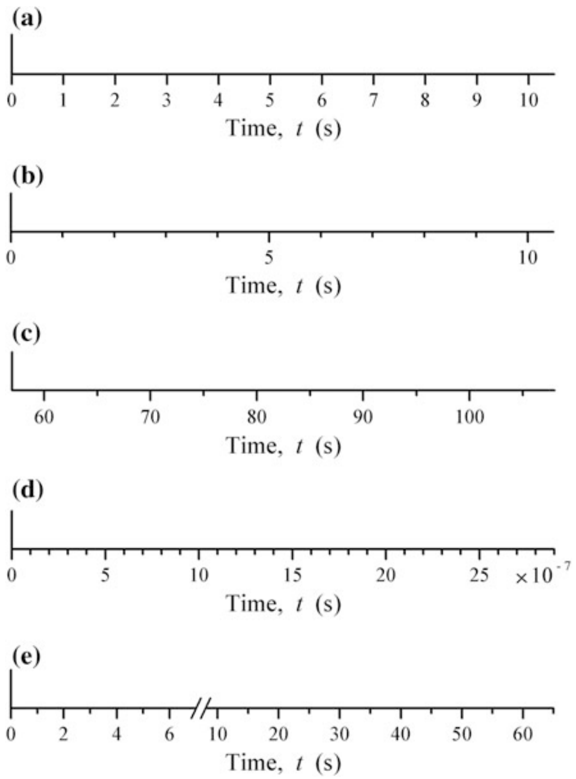


Fig. 12.2 Examples of axes with linear scales

giving any useful information. Having drawn the axis and the numerical scale corresponding to it, the name of the magnitude to which the axis corresponds is written near it and (usually in parentheses) the units used, e.g. time, $t\,$(s), height, $h\,$(m), or, simply, $\lambda I^2/m$ (S.I.).

Figure 12.2 shows some common kinds of linear axes:

(a) The simplest kind, with a number at every division.
(b) With numerical indications every 5 units and 4 subdivisions between them.
(c) The scale does not start at 0. In this way, the region of interest or the region in which experimental results exist may be shown in greater detail.
(d) The numerical indications have a common multiplying factor, e.g. $\times 10^{-7}$. The range covered by the axis in this case is between 0 and 29×10^{-7} s.
(e) Two different scales are used on the axis. This may be necessary in order to bring out the details in a region of values (here, between 0 and 6 s). The 'break' in the axis must be marked clearly.

For convenience, every subdivision of the graph paper used (usually with mm subdivisions) corresponds to 1, 2 or 5 units or to the corresponding multiples of a power of 10, depending on the range of values to be covered. This is done to

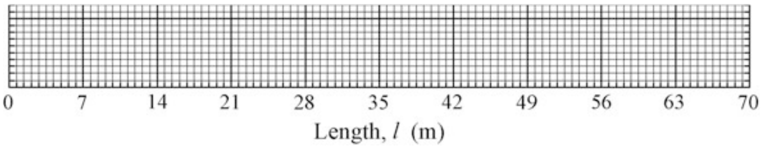


Fig. 12.3 The choice of a scale (1 cm = 7 units) which makes the reading of values off the graph difficult

facilitate marking the points on the graph, as well as in reading numerical values from it. A scale on which one cm corresponded to, say, 7 units, would be difficult to use (Fig. 12.3).

Subdivisions of the units in 1/2, 1/4, 1/8 etc. are actually used in scales in inches, as these are the subdivisions usually found on inch rulers. If years are used as units, they are usually subdivided in 12 months. Angles and times may be subdivided in 60 min and 60 s.

It is preferable to avoid using multi-digit numbers (e.g. 0.000 01, 0.000 02, or 10 000, 20 000 etc.). This is easily achieved by the suitable choice of units (e.g. mA or μA instead of A) or the use of a power of 10 as a common multiplying factor $\left(\times 10^3, \times 10^{-6}\,\text{etc.}\right)$. Some examples of linear scales are given in Figs. 12.4 and 12.5. Figure 12.5 shows the advantage of the suitable choice of the region of values in the scales, for the best presentation of the measurements.

12.2.2 Logarithmic Scales

Quite often, when a large range of values has to be covered without losing the details in the region of small values, we use as variable not the magnitude itself. e.g. x, but its logarithm $\log x$ (we usually use the common (decimal) logarithm $\log x$, while the use of the natural logarithm, $\ln x$, is rarer). For convenience in the

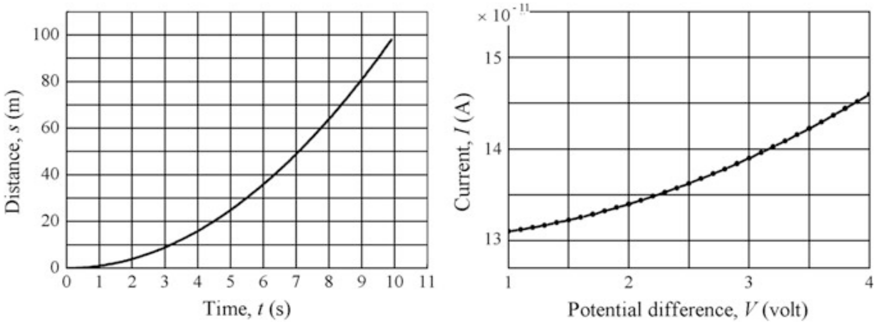


Fig. 12.4 Examples of use of various kinds of linear axes

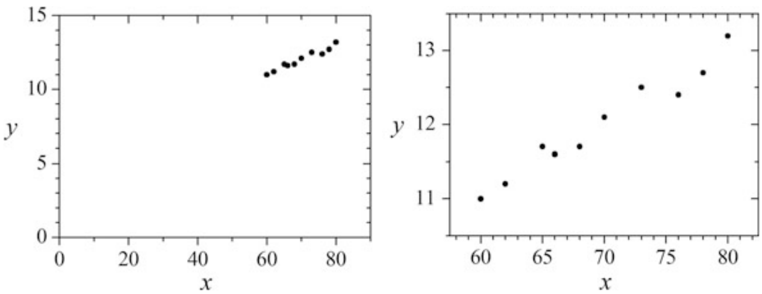
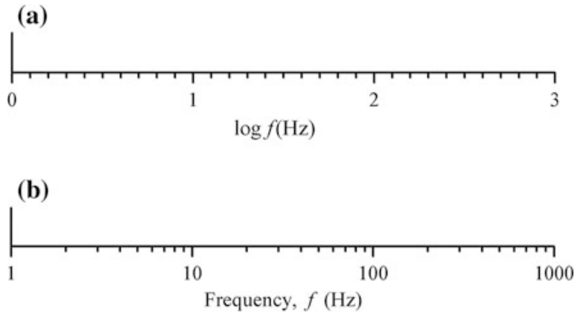


Fig. 12.5 The choice of the suitable range of values in the scales

Fig. 12.6 A logarithmic scale of three decades (or three cycles)



reading of values off the graph, the subdivisions of the scale as well as the numerical values marked on the axis are those of x and not those of $\log x$. This is demonstrated in Fig. 12.6a, where an axis was drawn with the logarithm of f, $\log f$, taking values from 1 to 3, while in Fig. 12.6b the same axis is drawn, on which now the subdivisions and values marked are those of f, which takes values between 1 and 1000, corresponding to the values of $\log f$ between 0 and 3. On the axis of Fig. 12.6b the subdivisions corresponding to the values of 2, 3, … 9, 20, 30, … 90, 200, 300, … 900 are also drawn. It should be noted that, although the logarithm is a dimensionless magnitude, the numerical value of the logarithm of a physical quantity depends on the units used in expressing this quantity. For this reason, it is necessary to state the units used, in parentheses, immediately after the symbol for the physical quantity, e.g. $\log f(\mathrm{Hz})$. Figure 12.7 shows four of the many kinds of semi-log (one linear and one logarithmic axis) and log-log (both axes logarithmic) graph paper available. Naturally, on the logarithmic graph paper available, there are given, whenever possible, more subdivisions than shown in our figures. For example, the interval between 1 and 2 is often subdivided into tenths, etc.

The choice of the scales to be used in any particular case depends mainly on the relationship expected to exist between the plotted variables. Thus, semi-log graph paper is used, apart from the case we have already mentioned in which we wish to

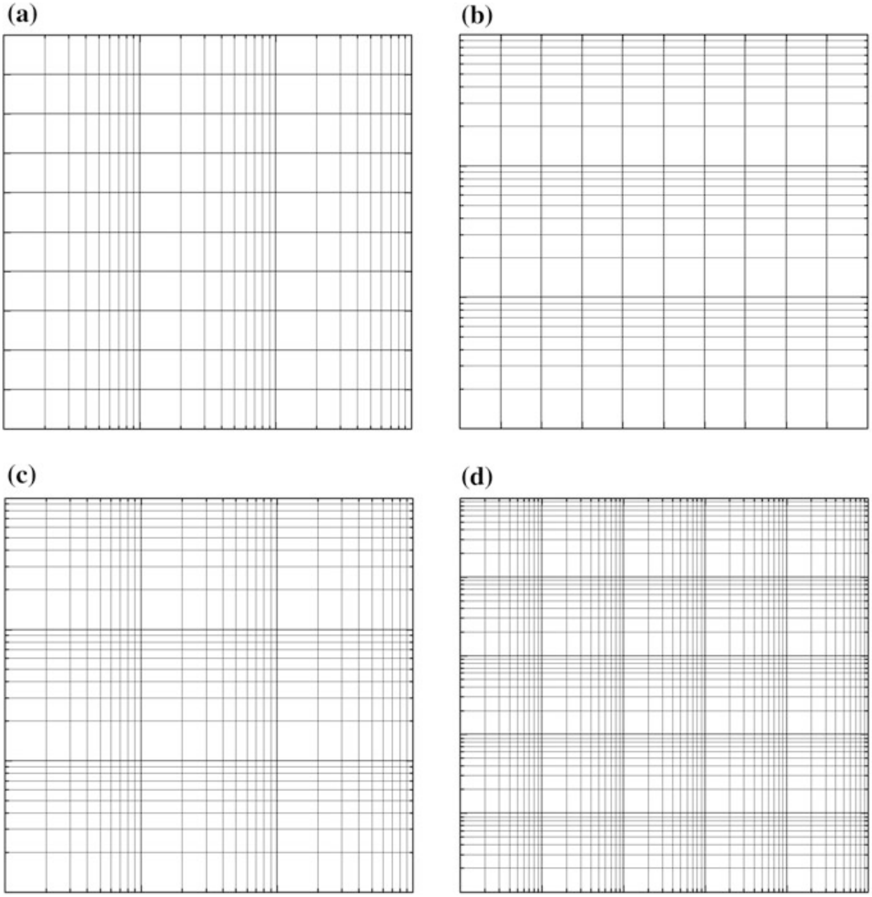


Fig. 12.7 **a**, **b** Three-decade semi-log graph paper, **c** log-log paper with 3 × 3 decades, **d** log-log paper with 5 × 5 decades

cover a large range of values, also when the relation between two variables is exponential, as, for example, when it is

$$y = A\,\mathrm{e}^{-\kappa x}. \tag{12.1}$$

Then, since it is

$$\ln y = \ln A - \kappa x, \tag{12.2}$$

plotting the natural logarithm of y as a function of x, will result in a linear graph. The same is true if the common logarithm of y is plotted as a function of x (since it is $\log y = \ln y / \ln 10$). A linear relation is desirable, as it is easier to draw a straight line between the points (x, $\log y$) than it is to draw the corresponding exponential curve.

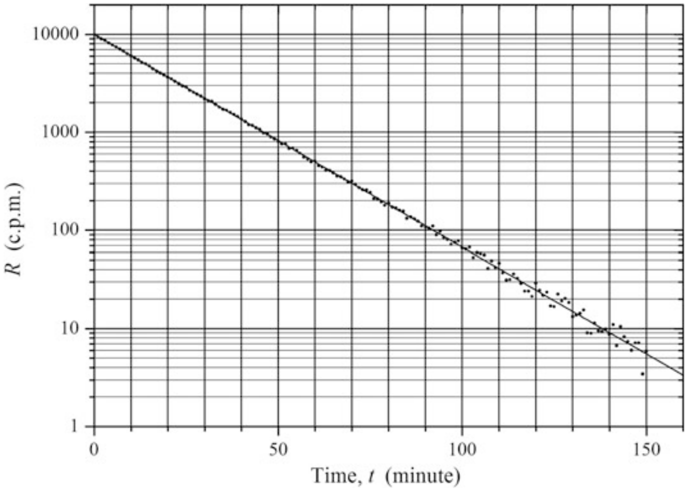


Fig. 12.8 The variation with time t of the activity R of a radioactive sample. The scale of R is logarithmic but the corresponding *values* of R are shown

Using $\log y$ as variable, we succeed in depicting the large values of y as well as the small ones. A classic example is the case of the decrease of the activity of a radioactive sample with time, presented in Fig. 8.2 which we reproduce here (Fig. 12.8).

The mathematical relation between the activity R and the time t is:

$$R(t) = R_0 \mathrm{e}^{-\lambda t}. \tag{12.3}$$

Plotting log R as a function of time, we have the straight line shown in the figure.

For relationships of the form $y = Ax^n$, the use of logarithmic scales for both variables leads to a linear relation:

$$\log y = \log A + n \ \log x. \tag{12.4}$$

For example, the Child-Langmuir law for the anode current I_a passing through a vacuum tube diode is found theoretically to be of the form

$$I_a = K V_a^{3/2}, \tag{12.5}$$

where V_a is the anode potential and K is a constant which depends on the geometry of the diode. Figure 12.9 shows the experimental results of I_a as a function of V_a, plotted using logarithmic scales. The linear relation

$$\log I_a = \log K + \frac{3}{2} \log V_a \tag{12.6}$$

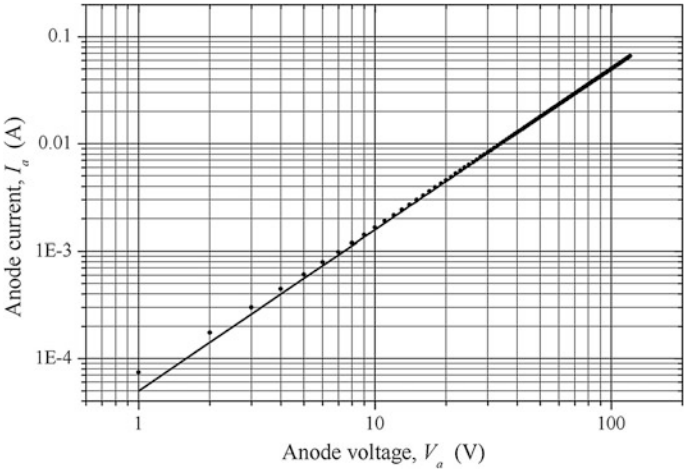


Fig. 12.9 Anode current as a function of anode potential for a vacuum tube diode

is seen to apply for large values of I_a and V_a, with the slope of the straight line actually being equal to 3/2. The deviation from linearity at low currents is due to the fact that current passes through the diode even when no potential difference is applied between cathode and anode. The relation is, in practice,

$$I_a = K(V_0 + V_a)^{3/2}, \tag{12.7}$$

with $V_0 = 0.3$ V for the diode of Fig. 12.9. If we plot I_a as a function of $V_a + 0.3$ V, using logarithmic scales, we will have the linear relation holding over all the range of values.

Another example is shown in Fig. 12.10, where the periods T of the planets were plotted as a function of the planets' distances from the Sun (the semi-major axes of their elliptic orbits), a, using logarithmic scales. The unit for the distance is the astronomic unit (1 ua = mean distance of Sun-Earth) and the unit for time is the year. As a result, the point for Earth is (1, 1). The resulting linear relation verifies Kepler's third law, $T^2 \propto a^3$ (the straight line has a slope of 3/2).

A variety of combinations of scales can be used to bring out a certain characteristic of a graph. Examples are shown in Fig. 12.11.

12.2.3 Polar Diagrams

It is often the case that the dependence of one magnitude on another is angular, $r(\theta)$. In these cases, it is useful to draw the relation using a polar diagram, such as that shown in Fig. 12.12. The independent variable is plotted as an angle on a circle

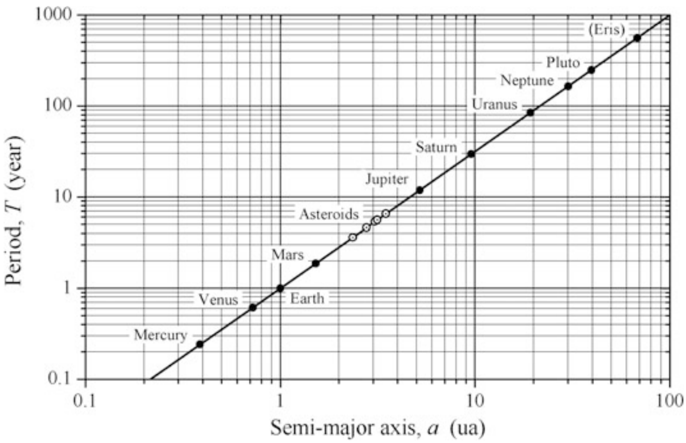


Fig. 12.10 Kepler's third law. The point for the newly discovered dwarf planet Eris should not be taken as verifying the law, as it is certain that the law was actually used in order to evaluate its period from the knowledge of its orbit

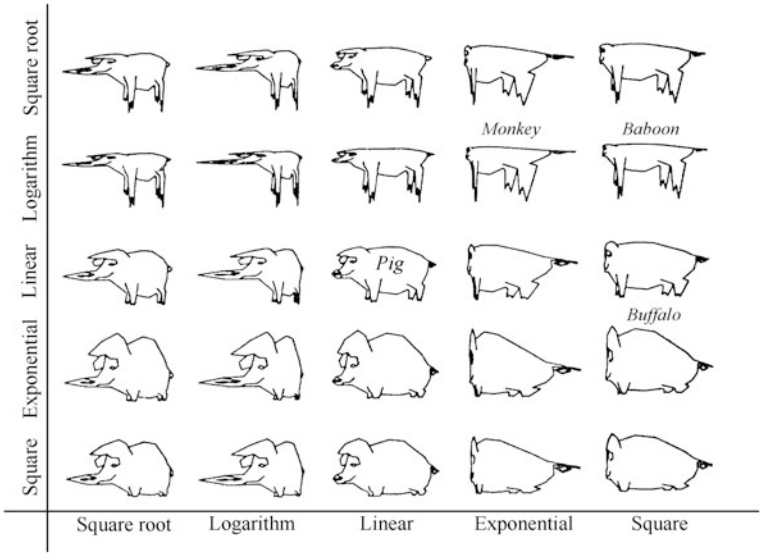


Fig. 12.11 The effect of different kinds of scales on a graph or mapping a pig (with acknowledgements to unknown author)

and the dependent variable is given by the corresponding distance from the center of the diagram.

An example of a polar plot is shown in Fig. 12.13, where the relative luminosity of an electric bulb has been drawn as a function of the direction, as this is given by the angle θ it forms with the axis of the bulb.

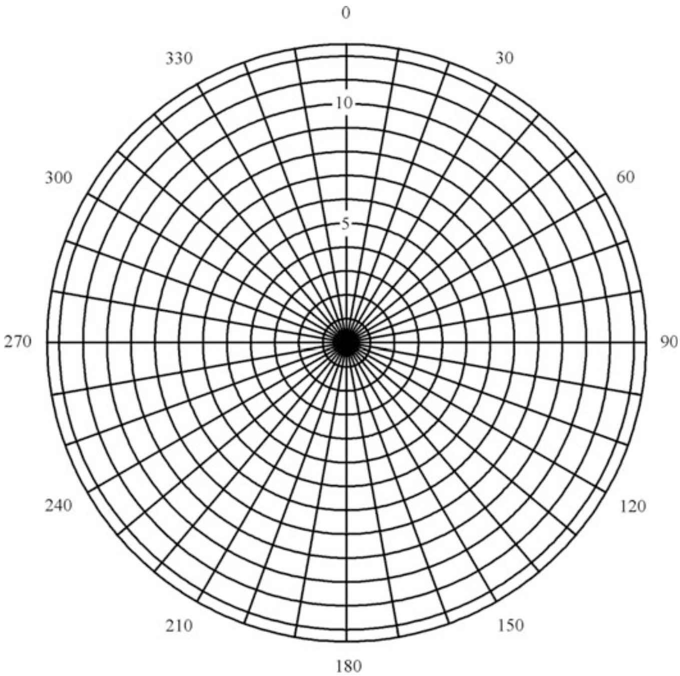


Fig. 12.12 The scales of a polar diagram

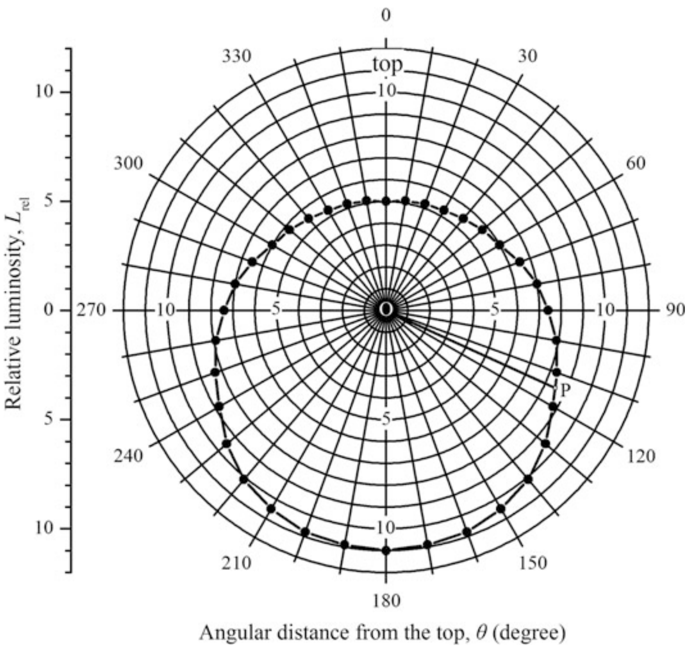


Fig. 12.13 A polar diagram of the relative luminosity of an electric bulb as a function of direction, as this is given by the angle θ it forms with the axis of the bulb. Point P has co-ordinates $(\theta = 115^{\text{o}},\ L_{\text{rel}} = 8.5)$

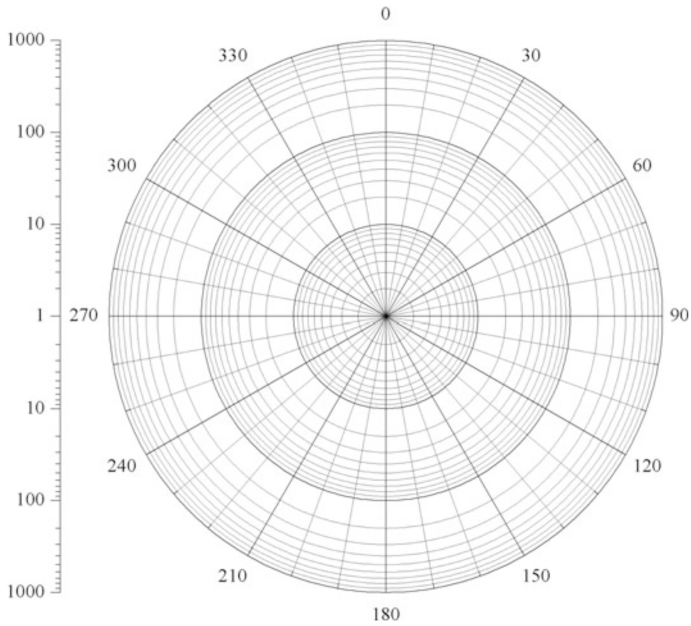


Fig. 12.14 Polar logarithmic graph paper

To give an example, point P shows that the relative luminosity of the bulb in a direction forming an angle of $\theta = 115^{\circ}$ with respect to the upward vertical is $L_{\mathrm{rel}} = 8.5$.

Polar diagrams which have a logarithmic radial scale are sometimes used, in order to cover a wide range of values without losing the details at low values. An example of graph paper used in such cases, with 3 cycles of logarithmic scale, is shown in Fig. 12.14.

12.2.4 Other Matters Relating to the Axes and the Scales of Graphs

It is sometimes desirable to give in a graph a second scale, which has some mathematical relationship with the main scale and which gives additional information. For example, let us examine the Arrhenius equation

$$\tau = \tau_0 \, \mathrm{e}^{E/kT}, \tag{12.8}$$

where τ_0, E and k are constants and T is the absolute temperature. Then, since it is

$$\ln \tau = \ln \tau_0 + \left(\frac{E}{k}\right) \frac{1}{T}, \tag{12.9}$$

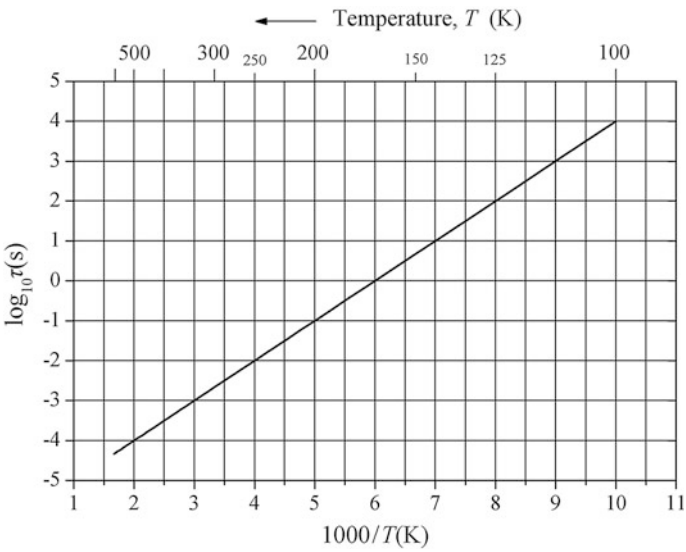


Fig. 12.15 The use of an additional auxiliary scale. Apart from the basic scale for $1/T$, the scale for the temperature values T is also given along the upper axis

if we plot log τ or ln τ as a function of $1/T$ we will have a straight line (Fig. 12.15). Apart from the main scale of $1/T$, it is useful to also have a scale for T, from which we can read directly the values of the absolute temperature. In Fig. 12.15 the scale of the temperature T is given along the top axis. This scale is not, of course, linear.

One more example of a graph with double axes which are mathematically related is shown in Fig. 12.16. Figure 12.17 shows an example where the second axis gives some specific values of the independent variable.

In some cases, when using linear scales, the values of the dependent variable may be too small in a certain region to be clearly visible. In such cases we re-draw the region of interest in a different scale. Next to this re-drawn section we give the factor by which the values had to be multiplied in order to reach their new level (e.g. $\times 10$, $\times 100$ etc.). Such an example is shown in Fig. 12.18, in which the second peak was re-drawn at a more sensitive scale in order to make visible details of the peak. As indicated in the figure, the values were re-drawn having been multiplied by 10. We must be careful here, as some researchers mark on the graph the factor by which the scale numbers must be multiplied in order to read a value off the graph.

If we wish to compare two different quantities, we may plot both in the same graph (Fig. 12.19). One axis (usually the x-axis) is common. In general, the other two axes are drawn, if they are different, on the left-hand-side and on the right-hand-side. As has been done in Fig. 12.19, the correspondence of curves and scales is shown by arrows. Alternatively, we might write next to the curves which one gives y_1 and which y_2.

More than one series of measurements may be presented in one graph, using different symbols for the points of each. In Fig. 12.20 the values of the specific heat

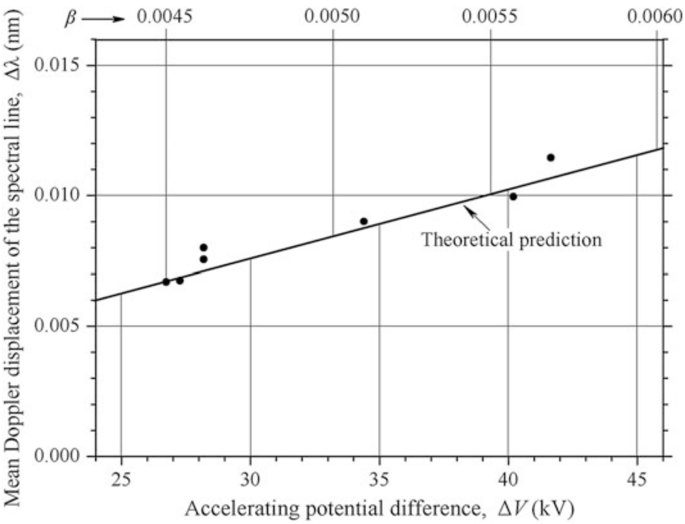


Fig. 12.16 The use of an additional auxiliary axis. β is the ratio of the particle's speed to the speed of light in vacuum, c

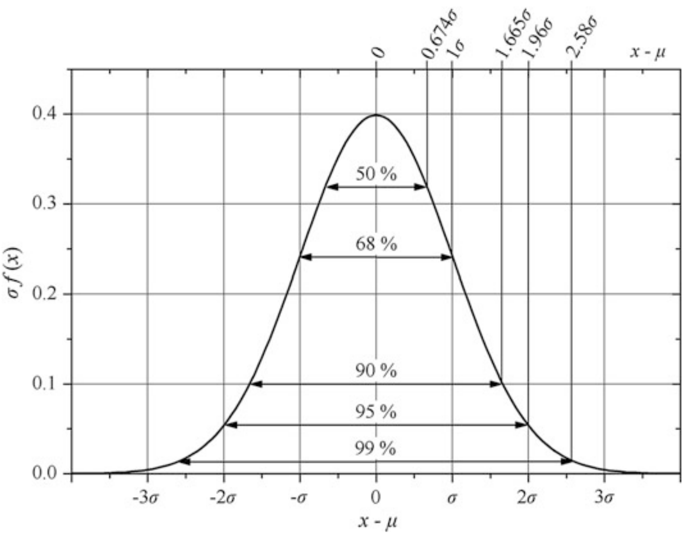


Fig. 12.17 The use of an additional auxiliary axis which gives some specific values of the independent variable

of four solids were drawn as functions of the reduced temperature T/θ_D, where θ_D is the Debye temperature of each solid. The symbol used for each solid is given in the text box on the graph. In this particular case when using as variable the reduced

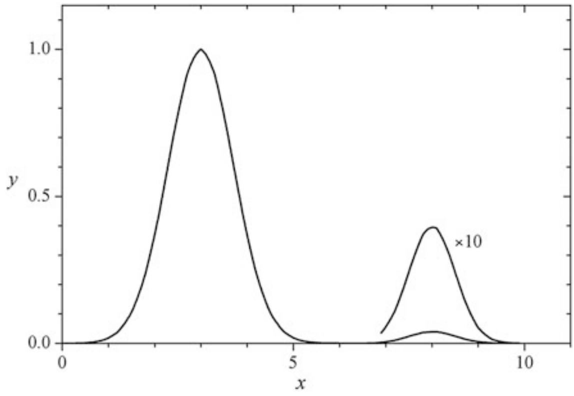


Fig. 12.18 Drawing a part of the graph in a different scale, to highlight possible details. The multiplying factor given should be the factor by which the values were multiplied in order to get the curve shown

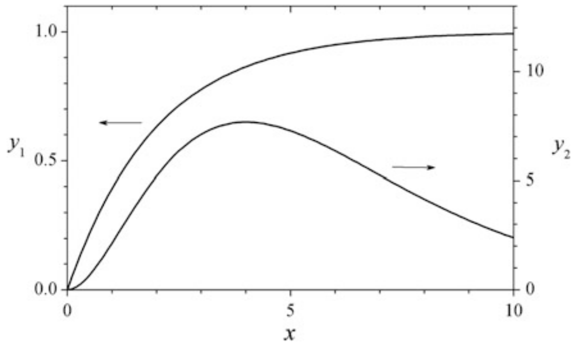


Fig. 12.19 Plotting, on the same graph, two different physical magnitudes, y_1 and y_2, having a common scale for the independent variable (x). The *arrows* point towards the scale to be used with each curve

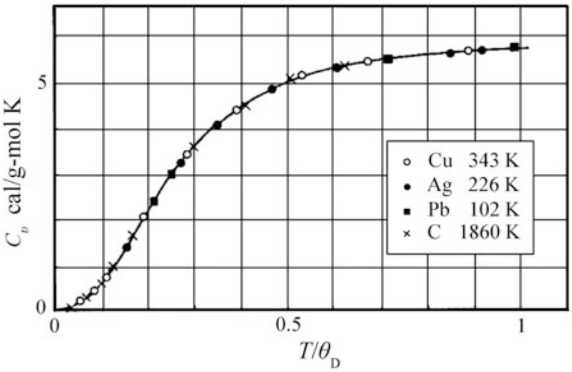


Fig. 12.20 The drawing of four series of measurements in one graph. The points give the results of measurements of the specific heat of four materials as a function of the absolute temperature T divided by the Debye temperature, θ_D, characteristic of each material. The distinction of the series is achieved by using different symbols for the points of each one of them. The values of the Debye temperature for each material are given in the box in the figure. The continuous curve shows the theoretical prediction of the Debye theory for the specific heats of solids. When C_υ is plotted as a function of the reduced temperature T/θ_D, a curve results which is universal for all solids

temperature T/θ_D for each solid, where T is the absolute temperature and θ_D a temperature characteristic of the material (the Debye temperature) the values for all the materials fall on a common, universal, theoretical curve, which is derived in Debye's theory of the specific heats of solids. This curve has been drawn in the figure. In most cases, however, in which this drawing technique is used, different curves are obtained for different materials or for different experimental conditions, if these vary from one series of measurements to another.

12.2.5 Legends of the Figure, Labels of the Axes and the Units of Physical Magnitudes

As a rule, the figure should convey as much information as possible by itself, without the reader having to resort to the text. The contents of the graph, together with its legend, should be enough to enable the reader to understand what the figure describes. The legend of a figure should be informative without being of excessive size, a rule which was probably not followed in Fig. 12.20. Of course, for the details, the text is indispensable.

Each axis of a graph should be clearly labeled with the magnitude it records, its symbol and its units. If the name is too long to be written along the axis, a suitable symbol may be used, which should be clearly explained in the legend. Examples are the following:

Increasing number of the measurement, i
Distance, $s\,(\mathrm{m})$
Speed, $\upsilon\,(\mathrm{m/s})$
Electric current, $I\,(\mu\mathrm{A})$
Temperature gradient, $\mathrm{d}T/\mathrm{d}x\,(\mathrm{K/m})$
Thermal conductivity, $\kappa\,(\mathrm{W/m{\cdot}K})$
Thermal conductivity/Electrical conductivity, $\kappa/\sigma\,(\mathrm{W{\cdot}\Omega/K})$
Differential cross-section, $\mathrm{d}\sigma/\mathrm{d}\,\Omega\,(\mathrm{mbarn/sterad})$
or just symbols: $E/k\,(\mathrm{K})$ $\quad 2Nh^2d/\kappa r_0\,(\mathrm{J{\cdot}K/(s\ m^4)})$ $\quad$ or $\quad 2Nh^2d/\kappa r_0\,(\mathrm{S.I.})$

There are different opinions regarding the way in which the units should be given. It is of course a rule that the units are written in upright letters (m, s, K, km, W, V,...), as opposed to the variables, which are symbolized by italics $(x, l, m, I, E, V, dx/dt, \ldots)$. There is also the point of view that fraction slashes (solidi) should not be used in stating the units. It is suggested that $\mathrm{m\,s^{-1}}$ should be written instead of m/s, $\mathrm{N\,m^2\,kg^{-2}}$ instead of $\mathrm{N\,m^2\,kg^2}$ etc. The reason for this is to avoid confusion when many fraction slashes appear without it being clear which quantity is divided each time. For example, by the expression N m/s/K/kg it is not clear that it is meant N m/(s/K/kg) or (N m/s)/(K/kg) or N (m/s/K)/kg or something else. The use of fraction slashes is accepted, however, if such uncertainties are avoided by the suitable use of parentheses, as, for example, in the case of W/(K/m). The problem also disappears if we take care for only one slash to appear, as, for example, in the case of W m/K. The S.I. system of units has already been presented in Chap. 5.

According to the Symbols Committee of the Royal Society [1], the symbols and their units should be written, in tables and graphs, as dimensionless numbers:

$$\text{Magnitude}/\text{Units}.$$

For example, a column of a table or the axis of a graph giving the temperature in degrees Kelvin must be labeled as T/K. Thus, the number 400 in a table or a graph means $T/\text{K} = 400$, or $T = 400$ K. This symbolism, however, presents a problem when we are dealing with complex quantities. For example, if (God forbid!) the magnitude

$$(\text{Stefan-Boltzmann's constant})\times(\text{electric conductivity})$$

should arise in some theoretical work and we express the Stefan-Boltzmann constant in terms of universal constants, together with their units, we would have the expression

$$(\pi^2/60)k^4\hbar^{-3}c^2\sigma_\eta/\text{W m}^{-3}\ \text{K}^{-4}\,\Omega^{-1}$$

which might cause some confusion.

Different scientific journals have adopted different notations regarding the presentation of units. In this book, for tables and graphs, we prefer to write the physical magnitude and its units separately, with the units given in parentheses, as in the examples given above.

12.3 The Points

A point (x, y) is marked in a graph using a symbol, such as the ones shown in Fig. 12.21.

The symbol must be large enough to be clearly visible. The same symbol is used for all the points corresponding to the same series of measurements, unless we wish to separate a point from the others due to some reason. If there are more than one series of measurements and their points are not sufficiently separated in the graph, we use different symbols for the different series of measurements (see, for example, Fig. 12.20).

If the errors $\delta x = \sigma_{\bar{x}}$ and $\delta y = \sigma_{\bar{y}}$ in the values of x and y, respectively, are known, these are marked on the graph, as shown in Fig. 12.22. For the point on the left in Fig. 12.22, only the value of y has an error. If the error in y is $\delta y = \sigma_{\bar{y}}$, then we draw a vertical line which stretches from the point (x, y) up and down to the points $(x, y \pm \delta y)$. For the point on the right in the figure, for which there are errors both in x and in y, the same procedure is followed for the error in x. Drawing the errors in the graph is important as it affects the form of the curve we will draw in order to describe the mathematical relationship between x and y, which results from the experimental values. Two examples of graphs drawn with errors in the measurements (x, y) are shown in Fig. 12.23.

Fig. 12.21 Some of the symbols commonly used for denoting points in graphs

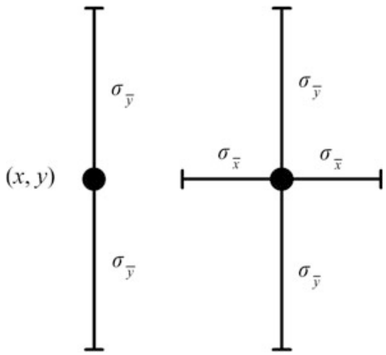


Fig. 12.22 Denoting the errors $\delta x = \sigma_{\bar{x}}$ and $\delta y = \sigma_{\bar{y}}$ in the values of x and y in a graph. In the case of the point on the *left*, there is an error only in y, while in the case of the point on the *right* there are errors in both x and y

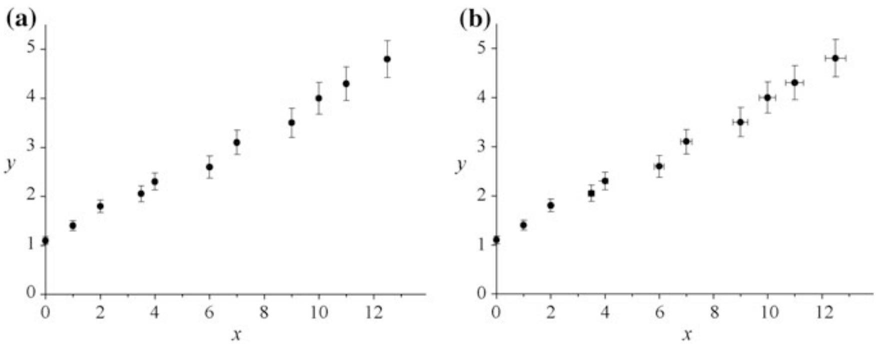


Fig. 12.23 Two graphs on which the errors in the values of x and y are indicated. **a** Errors exist only for the values of y, **b** errors exist for the values of both x and y

12.4 The Curve

Having chosen the axes and marked all the experimental points $(x_i,\ y_i)$ with their errors in the diagram, we wish to draw the best curve describing the relation $y(x)$. The tern '*best curve*' does not have a unique interpretation. Experience has taught us that mathematical simplicity is one criterion. This, however, cannot mean that a straight line is always preferable to a parabola and so on. The existence of a theoretical prediction for the particular experiment is usually the best way out of the difficulty. Even then, however, the possibility exists that the theoretical curve is not followed exactly by the experimental points due to errors. For example, a straight line may not pass through the origin, although theory predicts that it should do so. Summarizing, we may say that the main factors which determine our choice of the mathematical expression best describing a series of measurements are:

1. The relation 'suggested' by the positions of the points, taking into account their errors.
2. A theoretically predicted mathematical relation.
3. The simplicity of the mathematical relation.
4. Our experience from similar cases.

We will examine below some issues related to this procedure.

At the start, it should be stressed that the curve does not necessarily have to pass through all the experimental points (nor even through any of them). This is a frequent mistake. Two examples of this erroneous practice are shown in Fig. 12.24. Although the picture presented in the figure in not always impossible to be true, in the cases when something like this happens, it should be adequately documented. An example, from outside the physical sciences, is the daily variation of a stock exchange index. In this case, values between the points have no meaning and the connection of adjacent points with straight lines is justified, in order that the succession of values should be clear. In Physics or Chemistry, the recording of the values of a particular property of the elements as a function of their atomic number, for example, would justify a graph such as that of Fig. 12.24a. The straight lines help us follow the succession of points. In general, if the independent variable is quantized, a plot such as the one in Fig. 12.24a is usually justified.

In general, the curve adjusted to a series of measurements must be as smooth as possible. This can be seen as another application of Occam's razor. Deviation from a straight line or a smooth curve is justified only if there are an adequate number of reliable experimental points in the region of the deviation, taking into account the magnitude of the errors. Whenever possible, the curve should pass between the limits of the errors (see, for example, Fig. 12.25), always remembering that deviations from the real value by one or even more standard deviations are not rare. On the other hand, we should bear in mind that a point differing by more than about two standard deviations from the curve should be investigated, in order to decide whether the point is acceptable or it should be rejected.

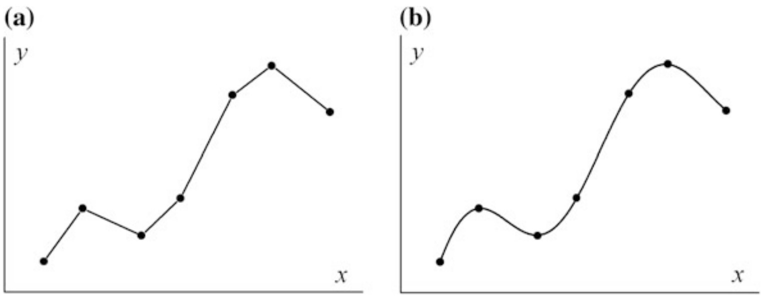


Fig. 12.24 Two examples of the wrong plotting of the curve between the experimental points, when the independent variable is continuous

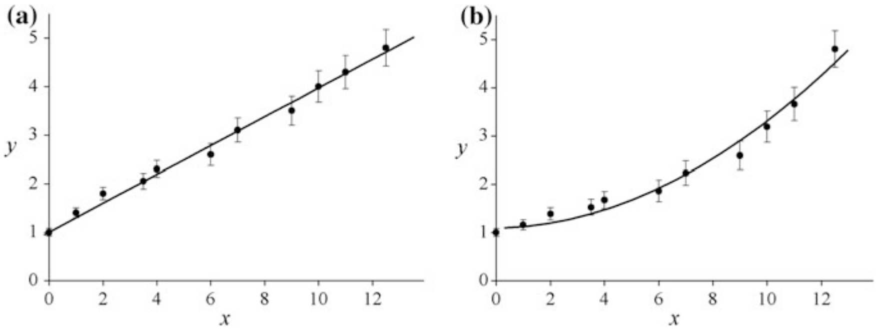


Fig. 12.25 The adjustment of a curve to a number of experimental points. **a** We see that, taking into account the errors in the values of y, a straight line adequately describes the relation $y(x)$. **b** In this example, a second degree curve is necessary for the description of $y(x)$

In Fig. 12.25a, a straight line expresses the relation $y(x)$ to a satisfactory degree. In Fig. 12.25b, however, given the systematic behavior of the points as well as the magnitudes of the errors in y, a straight line is not satisfactory and a curve of the second degree (parabola) is required.

Theory suggests that the best curve through the experimental points should cross, on average, only 2 out of 5 error bars. To demonstrate this, we apply the results of Example 9.5 to the points of graph in Fig. 12.23b. We draw ellipses with semi-axes equal to $\sigma_{\bar{x}}$ and $\sigma_{\bar{y}}$, and centers at the corresponding points $(\bar{x},\ \bar{y})$ (Fig. 9.26a). The probability of the real point $(x,\ y_0)$ lying within the corresponding ellipse is 39%. With this in mind, we expect the straight line drawn between the 11 experimental points (Fig. 9.26b) to pass through 4 of the ellipses. In fact, it cuts 8. This is evidence that the straight line fit is a good one. We also found that 87% of the ellipses with semi-axes equal to $2\sigma_{\bar{x}}$ and $2\sigma_{\bar{y}}$ should be intersected by the straight line. Here, all $2\sigma_{\bar{x}} - 2\sigma_{\bar{y}}$ ellipses are crossed by the line (Fig. 12.26).

In programs used in personal computers nowadays, there are large libraries of mathematical functions which may be fitted to the experimental results. Apart from

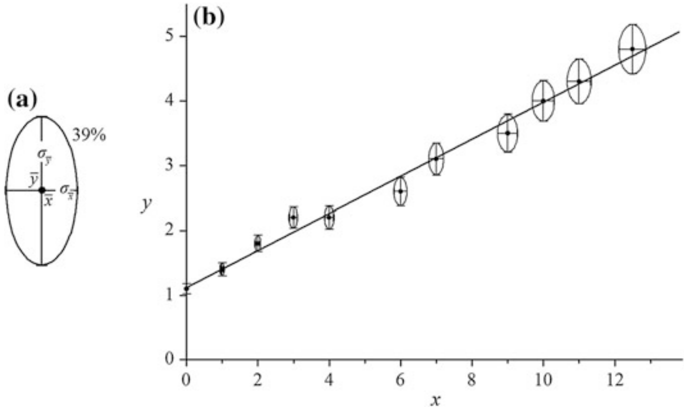


Fig. 12.26 **a** The ellipse with center an experimental point and semi-axes equal to the uncertainties $\delta x = \sigma_{\bar{x}}$ and $\delta y = \sigma_{\bar{y}}$ in $\bar{x}$ and $\bar{y}$ respectively. The real point corresponding to x and y has a 39% probability to lie within this ellipse. **b** The straight line fitted to the experimental points is expected to cut 39% or approximately 4 of the 11 ellipses such as that of **a** corresponding to the experimental points. Instead it cuts 8, a fact that must suggest that the linear fit is a very good one

the common mathematical functions such as polynomials of various degrees, exponential functions, trigonometric functions etc. and their combinations, there are specialized functions which are used in specific branches of science, such as Statistics, optical, dielectric and gamma-ray spectroscopy etc. The fast and expedient fitting of the best curve to the experimental points is, therefore, a matter of experience, mainly in the use of the suitable programs. Of course, the thing that cannot be done efficiently by any other method is the preliminary examination of the point 'by eye' in order to check whether there is a problem with some of the points, which has to be resolved before a curve is fitted.

Example 12.1 [E]

Graph plotting with Excel®

i	t_i (s)	z_i (m)	δt_i (s)	δz_i (m)
1	0	0	0.1	0
2	5	12	0.2	0.5
3	10	55	0.3	4
4	15	100	0.25	8
5	20	200	0.3	10
6	25	305	0.5	15
7	30	380	0.5	15
8	35	430	0.6	20
9	40	485	0.7	25
10	45	490	0.7	30

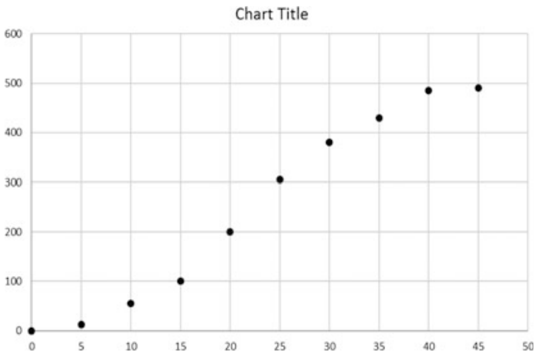
A table is given, containing the pairs of experimental results t_i and z_i of a relation $z(t)$ between the position of a particle and time, together with their respective errors, δt_i and δz_i.

Create a scatter plot of $z(t)$. Format the graph. Find the best polynomial curve $z(t)$ that fits the experimental points.

Differentiate the curve found, $z(t)$, to obtain the velocity of the particle, $\upsilon(t)$. Differentiate this curve, $\upsilon(t)$, to obtain the acceleration of the particle, $a(t)$. Plot the curves $\upsilon(t)$ and $a(t)$ in the graph of $z(t)$.

We enter the data t_i, $z(t)$, δt_i and δz_i in columns A, B, C and D, respectively. Highlight columns A and B. Open the **Insert** window and from **Charts** select **Insert Scatter (X, Y) or Bubble Chart**. A graph now appears, which is shown in the figure below. We will format this graph.

Pressing the ⊞ key that appears when we click on the top right hand side of the graph's box opens the **Chart Elements** dialog box. We choose



Error Bars > More Options > Format Error Bars > Error Bar Options

In **Vertical Bar Options**, we select **Direction: Both**, **End Style: Cap**, **Error Amount: Custom**. Then,

Specify Values > Positive Error Values and **Negative Error Values** type **= Sheet1!$D$4:$D$13**

Press **OK**. The vertical error bars appear in the figure at each point.

Ticking on a horizontal error bar opens a dialog box in which we select **Direction: Both**, **End Style: Cap**, **Error Amount: Custom**. Then,

Specify Values > Positive Error Values and **Negative Error Values** type **= Sheet1!$C$4:$C$13**

Press **OK**. The horizontal error bars also appear in the figure at each point.

We will fit a polynomial curve to the points.

Pressing the ⊞ key opens the **Chart Elements** dialog box. We choose

Error Bars > More Options > Trent Line Options > Polynomial: Order 4 and **Display Equation on chart**

The equation of the curve fitted is:

$$z(t) = 3.3741 - 4.355\,t + 1.0543\,t^2 - 0.0177\,t^3 + 4 \times 10^{-5}\,t^4\ (\text{in m for } t \text{ in s}).$$

Double-clicking on the curve opens the **Format Trendline**, **Trendline Options** window. For **Line**, we choose: **Solid Line**, **Color: Black**, **Width: 1 pt**, **Dash Line: solid line**.

We will now plot the curves for $\upsilon(t)$ and $a(t)$ in the graph.

Differentiating with respect to time, we have the speed

$$\upsilon(t) = -4.355\ + 2.1086\,t - 0.0531\,t^2 + 1.6 \times 10^{-4}\,t^3\ (\text{in m/s for } t \text{ in s}).$$

Differentiating with respect to time, we have the acceleration

$$a(t) = 2.1086 - 0.1062\,t + 4.8 \times 10^{-4}\,t^2\ \left(\text{in m/s}^2 \text{for } t \text{ in s}\right).$$

We label an empty column, say E(Y) as **v (m/s)**. In cell E4 we type

(−4.355 + 2.1086 * A1 − 0.0531 * A1^2 + 0.00016 * A1^3) * 10

and press **ENTER**. The factor of 10 is used since we are going to plot $z(t)$, $\upsilon(t)$ and $a(t)$ in the same graph, using a common axis, and we want to do this using comparable numbers, for easier readability. We **Fill Down** from E4 to E13.These cells now contain the values of $10\upsilon(t)$.

We label an empty column, say F(Y) as **a (m/s^2)**. In cell F4 we type

(2.1086 − 0.1062 * A1 + 0.00048 * A1^2) * 100

and press **ENTER**. The factor 100 serves the same purpose as the factor 10 for the velocity, as explained above. We **Fill Down** from F4 to F13. These cells now contain the values of $100a(t)$.

We highlight cells A4 to A13 and E4 to E13. In **Insert**, **Charts** we chose the option smooth line plot without points. This produces a graph of $\upsilon(t)$.

We highlight cells A4 to A13 and F4 to F13. In **Insert**, **Charts** we chose the option smooth line plot without points. This produces a graph of $a(t)$.

We **Cut** the graph $\upsilon(t)$ and **Paste** it on the graph $a(t)$. We then **Cut** the graph $z(t)$ and **Paste** it on the graph of $a(t)$ and $\upsilon(t)$.

We format the graph containing $z(t)$, $\upsilon(t)$ and $a(t)$ by changing the colors to black etc., as described below. In order to write something on the plot, we open **Insert** and insert a **Text Box**. We write the text in the box and then move it to the appropriate position.

We write z, 10υ and $100a$ near the corresponding curves, in order to identify them.

We click and select the numbers on the X-axis opening **Chart Tools**. In **Format** we open Text Fill and select **black**. We do the same with the Y-scale.

We click anywhere in the area of the numbers of the X-axis. This opens **Chart Tools**. **Open** the **Format** window. In the top left corner of the screen select **Horizontal (Value) Axis**. This opens the **Format Axis** window for the horizontal axis. In **Line** select **Solid Line**, **Color** black, **Width** 0.75 pt. In the same window, click the icon showing a histogram. Open **Axis Options** and select **Bounds**, **Minimum** 0 and **Maximum** 50. For **Units**, we select **Major** 5 and **Minor** 1. Open **Tick Marks** and select **Major Type**, **Outside** and **Minor Type**, **Outside**.

Repeat the same procedure for the **Vertical (Value) Axis**. In this case select **Bounds**, **Minimum** −200 and **Maximum** 600. For **Units**, we select **Major** 100 and **Minor** 20.

We click anywhere in the plot area. This opens **Chart Tools**. **Open** the **Format** window. In the top left corner of the screen select **Horizontal (Value) Axis Major Gridlines**. This opens the **Format Axis** window for the horizontal major gridlines. Select **Solid Line**, **Color** black and **Width** 0.75 pt. We now open the **Horizontal (Value) Axis Minor Gridlines**. This opens the **Format Axis** window for the horizontal minor gridlines. Select **Solid Line**, **Color** gray and **Width** 0.5 pt.

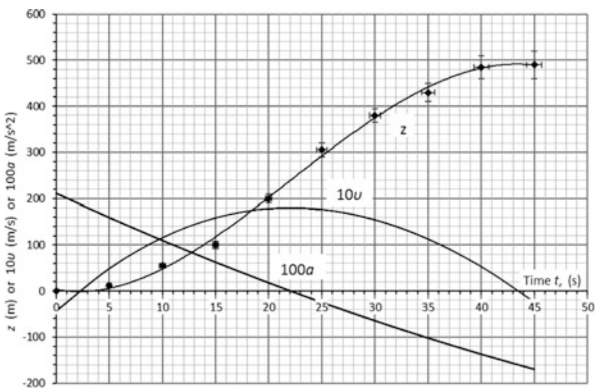
We repeat the same procedure for the vertical gridlines.

To white out the background of lettering so that they are easy to read, we double-click on the area of the X-axis numbers. This opens the **Format Axis** window. Clicking on the first icon, we select **Fill**, **Pattern Fill**, **Foreground** white and **Background** white.

To white out the background of text in the plot area, we right-click on the text. In the window that opens we select **Format Object**. Clicking on the first icon, we select **Fill**, **Pattern Fill**, **Foreground** white and **Background** white.

To remove the border line around the plot, we right-click on the plot area and open the **Format Chart Area** window. We click on the first icon in **Chart Options**. We select **No Line**.

The final result is shown in the figure below.



Example 12.2 [O]

Graph plotting with Origin®

A table is given, containing the pairs of experimental results t_i and z_i of a relation $z(t)$ between the position of a particle and time, together with their respective errors, δt_i and δz_i.

i	t_i (s)	z_i (m)	δt_i (s)	δz_i (m)
1	0	0	0.1	0
2	5	12	0.2	0.5
3	10	55	0.3	4
4	15	100	0.25	8
5	20	200	0.3	10
6	25	305	0.5	15
7	30	380	0.5	15
8	35	430	0.6	20
9	40	485	0.7	25
10	45	490	0.7	30

Create a scatter plot of $z(t)$. Format the graph. Find the best polynomial curve $z(t)$ that fits the experimental points.

Differentiate the curve found, $z(t)$, to obtain the velocity of the particle, $\upsilon(t)$. Differentiate this curve, $\upsilon(t)$, to obtain the acceleration of the particle, $a(t)$. Plot the curves $\upsilon(t)$ and $a(t)$ in the graph of $z(t)$.

We enter the data t_i, $z(t)$, δt_i and δz_i in columns A(X), B(Y), C(Y) and D(Y), respectively. Right-click on C(Y). Then

Set As > X Error

The label of the column now becomes C(xEr±). This indicates that the column contains the errors in the values of X, i.e. in t.

Right-click on D(Y). Then

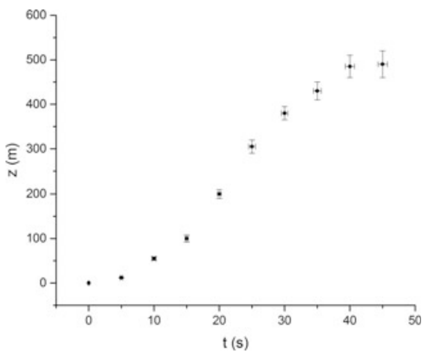
Set As > Y Error

The label of the column now becomes D(yEr±). This indicates that the column contains the errors in the values of Y, i.e. in z.

Highlight columns A, B, C and D. Then

Plot > Symbol > Scatter

The plot shown in the figure below appears.



We will format this graph.

We will fit a polynomial curve to the points. While in the plot environment, we follow the path

Analysis > Fitting > Nonlinear Curve Fit > Open Dialog…

In the window that opens, we select

Settings > Function Selection > Category set to **Polynomial > Function** set to **Poly4**

This will fit a polynomial of the fourth degree to the experimental points. Press **Fit**. The fitted curve appears on the graph. The equation of the curve fitted is:

$$z(t) = 3.34 - 4.312\,t + 1.0437\,t^2 - 0.01756\,t^3 + 3.946 \times 10^{-5}\,t^4 \ (\text{in m for } t \text{ in s}).$$

Differentiating with respect to time, we have the speed

$$\upsilon(t) = -4.312\ + 2.0874\,t - 0.05268\,t^2 + 1.3784 \times 10^{-4}\,t^3\,(\text{in m/s for } t \text{ in s}).$$

Differentiating with respect to time, we have the acceleration

$$a(t) = 2.0874 - 0.10536\,t + 4.135 \times 10^{-4}\,t^2\,\left(\text{in m/s}^2 \text{for } t \text{ in s}\right).$$

We will now plot the curves for $\upsilon(t)$ and $a(t)$ in the graph.

In the data sheet (Book1) we highlight an empty column, say E(Y), which we label as **v (m/s)**. Then

Column > Set Column Values

and in the dialog box that opens we type

−4.312 + 2.0874 * col(A) − 0.05268 * (col(A))^2 + 0.00013784 * (col(A))^3

Pressing **OK** fills column E with the values of $\upsilon(t)$.

In the data sheet (Book1) we highlight an empty column, say F(Y), which we label as **a (m/s)**. Then

Column > Set Column Values

and in the dialog box that opens we type

2.0874 − 0.10536 * col(A) + 0.0004135 * (col(A))^2

Pressing **OK** fills column F with the values of $a(t)$.

We will plot these values in the graph of $z(t)$. Before we do so, however, we want to bring the values to be plotted in the same range as those of $z(t)$. To achieve this, we fill column G with the values of $10\upsilon(t)$ and column H with the values of $100a(t)$.

Returning to the graph environment (**Window, Graph1**), we right-click on the number at the top left hand side of the page and then we click on **Layer Contents…** . In the window that opens, we highlight the line corresponding to column G by clicking on any point of the line. Then, pressing the arrow $\longrightarrow$, we include column G in the list on the right, which shows the columns plotted in the graph. We repeat for column H. Then press **Plot Setup…** . The graph appears, now containing the scatter plots of $10\upsilon(t)$ and $100a(t)$.

We will change the plots of $10\upsilon(t)$ and $100a(t)$ from scatter to line plots. To do this we double-click on one of the points, opening the **Plot Details—Plot Properties** and in the **Plot Type** select **Line**. We do this for both the plots $10\upsilon(t)$ and $100a(t)$.

We double-click on each of the three curves in turn and change the line width to 1.5 and the color to black.

To change the ranges of the scales, we double-click on one of the axes. In the window that opens we set the **Horizontal** scale **From** 0 **To** 50 and the **Vertical** scale **From** −200 **To** 600.

In the same window, we open **Tick Labels**. For the **Left** and **Bottom** axes we tick the **Show** box.

In the same window, we open **Grids**. For **Vertical**, **Major Grid Lines**, tick the **Show** box and select **Color** to be **Black**, **Style** to be **Solid** and **Thickness** to be 0.5. For **Minor Grid Lines**, we tick the **Show** box and select **Color** to be **Black**, **Style** to be **Solid** and **Thickness** to be 0.3. In Additional Lines tick Y = 0. For **Horizontal**, **Major Grid Lines**, we tick the **Show** box and select **Color** to be **Black**, **Style** to be **Solid** and **Thickness** to be 0.5. For **Minor Grid Lines**, we tick the **Show** box and select **Color** to be **Black**, **Style** to be **Solid** and **Thickness** to be 0.3. In **Additional Lines** tick Y = 0.

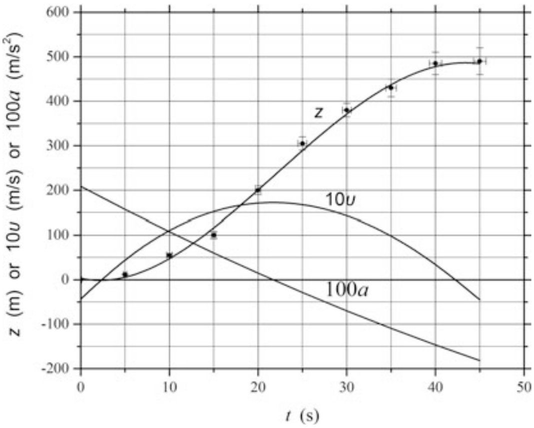
In the same window, we open **Lines and Ticks**.

For **Bottom**, we tick the **Show Line and Ticks** box and select: for **Line** we tick the **Show** box, **Color** to be **Black**, **Thickness 1.5** and **Axis Position Bottom**. For **Major Ticks** we select **Out**. For **Minor Ticks** we select **Out**. For **Top** we tick the **Show Line and Ticks** box and the **Use Same Options for Bottom and Top** box. For **Left** we use same settings as **Bottom**. For **Right** we tick the **Show Line and Ticks** box and the **Use Same Options for Left and Right** box. Press **OK**.

We double click on each of the three lines in turn and set for each **Line**: **Connect** to be **Straight**, **Style** to be **Solid**, **Width** to be **1** and **Color** to be **Black**.

We double click on the X label and write ***t* (s)**. We double click on the Y axis and write ***z* (m) or 10*υ* (m/s) or 100*a* (m/s^2)** . Identify the three curves by writing ***z***, **10*υ*** and **100*a*** near the corresponding curve.

The final result is shown in the figure below.



Example 12.3 [P]

Graph plotting with Python

A table is given, containing the pairs of experimental results t_i and z_i of a relation $z(t)$ between the position of a particle and time, together with their respective errors, δt_i and δz_i.

Create a scatter plot of $z(t)$. Format the graph. Find the best polynomial curve $z(t)$ that fits the experimental points.

Differentiate the curve found, $z(t)$, to obtain the velocity of the particle, $\upsilon(t)$. Differentiate this curve, $\upsilon(t)$, to obtain the acceleration of the particle, $a(t)$. Plot the curves $\upsilon(t)$ and $a(t)$ in the graph of $z(t)$.

i	t_i (s)	z_i (m)	δt_i (s)	δz_i (m)
1	0	0	0.1	0
2	5	12	0.2	0.5
3	10	55	0.3	4
4	15	100	0.25	8
5	20	200	0.3	10
6	25	305	0.5	15
7	30	380	0.5	15
8	35	430	0.6	20
9	40	485	0.7	25
10	45	490	0.7	30

As usual, we will import the numpy and matplotlib modules, and then entering the experimental data into vectors t and z, and the corresponding error values into vectors errt and errz.

```
import numpy as np
import matplotlib.pyplot as plt
t = np.array([0, 5, 10, 15, 20, 25, 30, 35, 40, 45])
z = np.array([0, 12, 55, 100, 200, 305, 380, 430, 485, 490])
errt = np.array([0.1, 0.2, 0.3, 0.25, 0.3, 0.5, 0.5, 0.6, 0.7, 0.7])
errz = np.array([0, 0.5, 4, 8, 10, 15, 15, 20, 25, 30])
```

We use the errorbar function of matplotlib to produce a scatter plot with error bars:

```
plt.errorbar(t, z, xerr=errt, yerr=errz, fmt='o', color='b')
plt.xlim(0, 50)
plt.ylim(-200, 600)
plt.xlabel("t (s)")
plt.ylabel("z (m) or 10υ (m/s) or 100a (m/s^2)"
plt.grid(True)
```

The fmt = 'o' option indicates that the experimental points will be drawn as small circles on the scatter plot. Other available markers include 's' for a square, '*' for a star, 'D' for a diamond, and '1', '2', '3', '4' for a triangle (down-, up-, left- or right-oriented respectively). A complete list of markers can be found on http://matplotlib.org/api/markers_api.html. These markers can also be used with the plot and scatter commands. The color parameter sets the colour of the points: 'b' is shorthand for blue, 'r' for red, 'g' for green, 'k' for black etc. A complete list of colours available for use in matplotlib graphs can be found on http://matplotlib.org/api/colors_api.html.

We then use the least-squares method to fit a fourth degree polynomial to the experimental data, as follows:

```
fit = np.polyfit(t, z, 4)
p = np.poly1d(fit)
```

In order to draw this polynomial as a curve, we create a series of 200 points between min(t) and max(t) using the linspace command from numpy, and then use the plot command. The third parameter ('-') to the plot command indicates that the points should be linked (to form a smooth curve).

```
xp = np.linspace(min(t), max(t), 200)
plt.plot(xp, p(xp), '-', color="red")
```

The numpy polynomial object supports the deriv function that calculates derivatives of the polynomial. We store the velocity (first derivative) and acceleration (second derivative) polynomials in objects v and a as follows:

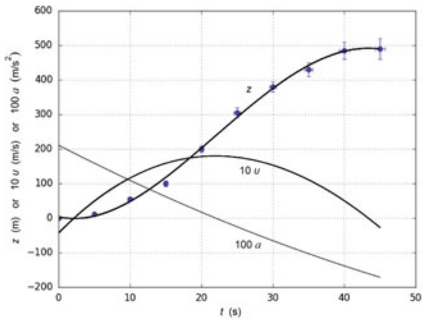
```
v = p.deriv(1)
a = p.deriv(2)
```

Using the same method as above, we can plot $10v$ and $100a$ on the graph, using the following commands:

```
plt.plot(xp, 10*v(xp), '-', color="blue")
plt.plot(xp, 100*a(xp), '-', color="black")
```

Our graph is ready: to see it on the screen, or export it as an image file, we issue the show() command:

```
plt.show()
```



Example 12.4 [R]

Graph plotting with R

A table is given, containing the pairs of experimental results t_i and z_i of a relation $z(t)$ between the position of a particle and time, together with their respective errors, δt_i and δz_i.

i	t_i (s)	z_i (m)	δt_i (s)	δz_i (m)
1	0	0	0.1	0
2	5	12	0.2	0.5
3	10	55	0.3	4
4	15	100	0.25	8
5	20	200	0.3	10
6	25	305	0.5	15
7	30	380	0.5	15
8	35	430	0.6	20
9	40	485	0.7	25
10	45	490	0.7	30

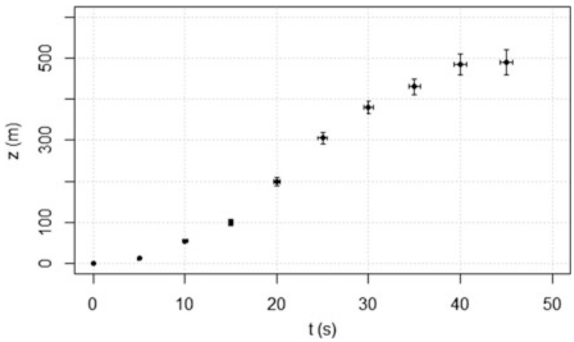
Create a scatter plot of $z(t)$. Format the graph. Find the best polynomial curve $z(t)$ that fits the experimental points.

Differentiate the curve found, $z(t)$, to obtain the velocity of the particle, $\upsilon(t)$. Differentiate this curve, $\upsilon(t)$, to obtain the acceleration of the particle, $a(t)$. Plot the curves $\upsilon(t)$ and $a(t)$ in the graph of $z(t)$.

We first create the scatter plot of the experimental points. We enter the vectors for t, z and their errors:

```
# t and z vectors
t <- c(0, 5, 10, 15, 20, 25, 30, 35, 40, 45)
z <- c(0, 12, 55, 100, 200, 305, 380, 430, 485, 490)
```

```
# errors in t and z
errt <- c(0.1, 0.2, 0.3, 0.25, 0.3, 0.5, 0.5, 0.6, 0.7, 0.7)
errz <- c(0, 0.5, 4, 8, 10, 15, 15, 20, 25, 30)

#scatter plot of data with x and y axes labels, lengths and grid
plot(t, z, pch=20, xlab="t (s)", ylab="z (m)", xlim=c(0, 50),
ylim=c(0, 600), grid())

# add t error bars
arrows(t-errt, z, t+errt, z, length=0.02, angle=90, code=3)

# add z error bars
arrows(t, z-errz, t, z+errz, length=0.02, angle=90, code=3)
# ENTER returns the scatter plot
```



We fit to the points a least-squares polynomial of the fourth degree:

```
# Least-squares curve fit
> nls(z~a0+a1*t+a2*t^2+a3*t^3+a4*t^4)
Nonlinear regression model
  model: z~a0+a1*t+a2*t^2+a3*t^3+a4*t^4
   data: parent.frame()
       a0        a1        a2        a3        a4
 3.374e+00 -4.355e+00  1.054e+00 -1.774e-02  3.986e-05
 residual sum-of-squares: 751.5
Number of iterations to convergence: 1
Achieved convergence tolerance: 5.72e-07
```

The equation of the curve fitted is:

$$z(t) = 3.374 - 4.355\,t + 1.054\,t^2 - 0.01774\,t^3 + 3.986 \times 10^{-5}\,t^4 \text{ (in m for } t \text{ in s).}$$

Differentiating with respect to time, we have the speed

$$\upsilon(t) = -4.355\ + 2.108\,t - 0.05322\,t^2 + 1.5944 \times 10^{-4}\,t^3\ (\text{in m/s for } t \text{ in s}).$$

Differentiating with respect to time, we have the acceleration

$$a(t) = 2.108 - 0.10644\,t + 4.7832 \times 10^{-4}\,t^2\ \left(\text{in m/s}^2 \text{for } t \text{ in s}\right).$$

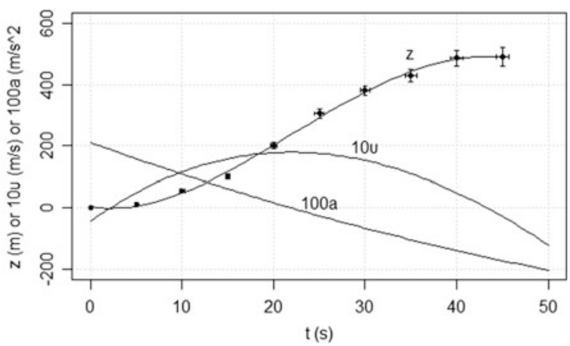
We will now plot the curves for $\upsilon(t)$ and $a(t)$ in the graph. We first re-plot the curve $z(t)$ with the z-axis taking values between −200 and 600. To enable easy reading, we will plot 10υ and $100a$ on the graph. We also change the Y-axis label to 'z (m) or 10υ (m/s) or $100a$ (m/s^2)'.

```
# Plot the curve z(t):
curve(3.374-4.355*x+1.054*x^2-0.01774*x^3
+3.986e-05*x^4, from=0, to=50, add=T)
# add the curves for 10υ and 100a
curve(-43.55+21.08*x-0.5322*x^2+0.001594*x^3, from=0, to=50, add=T)
curve(210.8-10.644*x+0.04783*x^2, from=0, to=50, add=T)
```

Label the curve for z and the two curves for 10υ and 100a:

```
# add labels to the curves
text(35, 500, "z", cex=1)
text(30, 200, "10υ", cex=1)
text(25, 20, "100a", cex=1)
```

The final graph is shown below



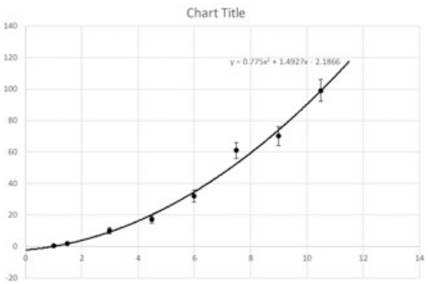
Example 12.5 [E]

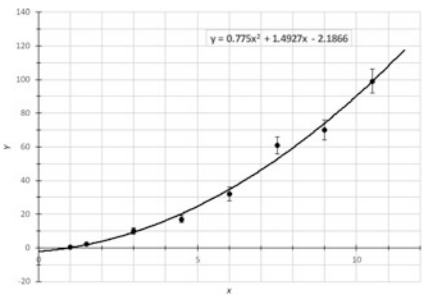
Eight results of measurements of the quantity $Y(x)$ are given in the table below. Using Excel©, create a graph showing the measurements and the best parabolic curve between them.

i	1	2	3	4	5	6	7	8
x	1.0	1.5	3.0	4.5	6.0	7.5	9.0	10.5
$Y \pm \delta Y$	0.5 ± 0.3	2.1 ± 0.5	10 ± 2	17 ± 2	32 ± 4	61 ± 5	70 ± 6	99 ± 7

We enter x, Y and δY in columns A, B and C, respectively. To plot $Y(x)$, we highlight cells A1–A8 and B1–B8 and open the **Insert** window. We choose scatter plot. We left-click at the top right corner of the plot and open ⊞. We open **Error Bars**, **More Options**, **Format Error Bars**. Double-click at a point near and to the left or the right of a point of the plot, in order to open the **Horizontal Errors** window. In **Format Error Bars > Error Bar Options** we open . We select **Line > No Line**. We double-click at a point near and above or below a point of the plot, we open the **Vertical Errors** window. In **Format Error Bars > Error Bar Options** we open . We select **Line > Solid Line**, **Color** black and **Width** 0.75 pt. We open **Vertical Error Bar**. We select **Direction Both**, **End Style Cap**, **Error Amount Custom**. Clicking **Specify Value** will open the window **Custom Error Bars**. In both **Positive Error Value** and **Negative Error Value** type **= Sheet1!$C$1:$C$8**. We press **OK**.

We will now fit the best parabola to the experimental points. We press ⊞, **Trendline**, **More Options**. In , **Line**, we select **Solid Line**, **Color** black, **Width** 1.5 pt, **Dash Type** continuous line. We open and select **Polynomial Order** 2. Also, select **Forecast Forward** 1 period and **Backward** 1 period. Finally, click the box Display Equation on Chart. Left-click on the straight line present in the plot and delete it. The graph shown on the left below is produced.





We will format this graph.

We delete the **Chart Title** text box. We click anywhere in the area of the numbers of the X-axis. This opens **Chart Tools**. **Open** the **Format** window. In the top left corner of the screen select **Horizontal (Value) Axis**. This opens the **Format**

Axis window for the horizontal axis. In **Line** select **Solid Line**, **Color** black, **Width** 0.75 pt, **Dash Type** continuous. In the same window, click the icon . Open **Axis Options** and select **Bounds**, **Minimum** 0 and **Maximum** 12. For **Units**, we select **Major** 5 and **Minor** 1. Open **Tick Marks** and select **Major Type**, **Outside** and **Minor Type**, **Outside**.

Repeat the same procedure for the **Vertical (Value) Axis**. In this case select **Bounds**, **Minimum** –20 and **Maximum** 140. For **Units**, we select **Major** 20 and **Minor** 10.

We click anywhere in the plot area. This opens **Chart Tools**. **Open** the **Format** window. In the top left corner of the screen select **Horizontal (Value) Axis Major Gridlines**. This opens the **Format Axis** window for the horizontal major gridlines. Select , **Solid Line**, **Color** gray and **Width** 0.5 pt. We now open the **Horizontal (Value) Axis Minor Gridlines**. This opens the **Format Axis** window for the horizontal minor gridlines. Select **Solid Line**, **Color** gray and **Width** 0.5 pt.

We repeat the same procedure for the vertical gridlines.

To insert axis titles we open , **Axis Titles**. For the X-Axis title we type *x*. For the Y-Axis title we type ***Y***. In **Format Axis Title > Text Options > Text Fill** select Solid fill color black. Repeat for the Y-Axis, typing ***y***.

We left-click on the equation of the parabola and change the lettering to **Bold** and **Size** 11 pts.

The final result is that shown in the right-hand-side figure above.

Example 12.6 [O]

Eight results of measurements of the quantity $Y(x)$ are given in the table below. Using Origin©, create a graph showing the measurements and the best parabolic curve between them.

i	1	2	3	4	5	6	7	8
x	1.0	1.5	3.0	4.5	6.0	7.5	9.0	10.5
$Y \pm \delta Y$	0.5 ± 0.3	2.1 ± 0.5	10 ± 2	17 ± 2	32 ± 4	61 ± 5	70 ± 6	99 ± 7

We enter the data x, Y and δY in columns A, B and C respectively. Highlight column C. Right-click on C. Then

Set As > Y Error

Highlight columns A, B and C. Then

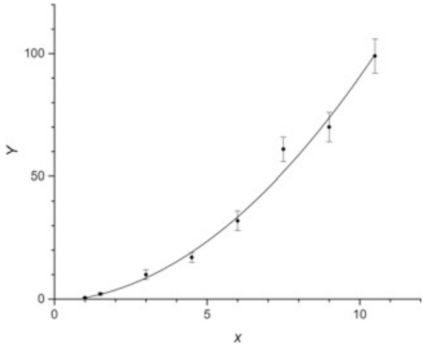
Plot > Symbol > Scatter

A plot is produced. We will fit a parabola to the points.

Analysis > Fitting > Polynomial Fit > Open Dialog…

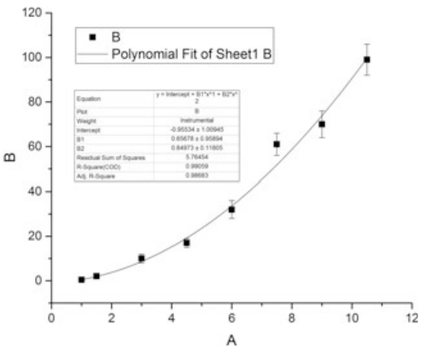
Select **Polynomial** of **Order 2**. On pressing **OK** the graph shown in the figure on the left below is produced. It is seen that the parabola $Y = (-0.95534 \pm 1.00945) + (0.65678 \pm 0.95894)x + (0.84973 \pm 0.11805)x^2$

We will improve the appearance of the graph.



1. We delete the two text boxes given in the figure.
2. We change the thickness of the line by double-clicking on it and changing **Width** from 0.5 to 1. We change the color of the line from red to black.
3. We double-click on a point and change the shape and size of the points from square and 9 pts to circular and 5 points.
4. We change the labels of the axes by double-clicking on them and writing x in place of A and Y in place of B. We use the default font of Arial 22 pts.
5. We change the **Horizontal** x axis by double-clicking on it. We set the **Scale** from 0 to 12, **Major Ticks Value** 5 and **Minor Ticks Count** 4.
6. We change the **Vertical** Y axis by double-clicking on it. We set the **Scale** from 0 to 120, **Major Ticks Value** 50 and **Minor Ticks Count** 4.
7. We may, if we wish, write the equation of the best fit parabola on the graph.

The result is shown in the figure on the right above.



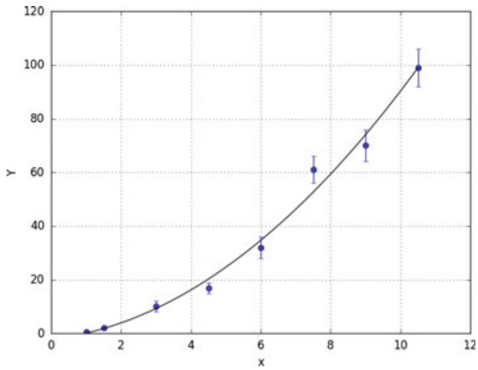
Example 12.7 [P]

Eight results of measurements of the quantity $Y(x)$ are given in the table below. Using Python, create a graph showing the measurements and the best parabolic curve between them.

i	1	2	3	4	5	6	7	8
x	1.0	1.5	3.0	4.5	6.0	7.5	9.0	10.5
$Y \pm \delta Y$	0.5 ± 0.3	2.1 ± 0.5	10 ± 2	17 ± 2	32 ± 4	61 ± 5	70 ± 6	99 ± 7

```
import numpy as np
import matplotlib.pyplot as plt

# Enter the values of x and Y:
x = np.array([1, 1.5, 3, 4.5, 6, 7.5, 9, 10.5])
Y = np.array([0.5, 2.1, 10, 17, 32, 61, 70, 99])

# Enter the errors in Y:
errx = np.array([0, 0, 0, 0, 0, 0, 0, 0])
errY = np.array([0.3, 0.5, 2, 2, 4, 5, 6, 7])

# Production of the scatter plot of data with linear axes and grid:
plt.errorbar(x, Y, xerr=errx, yerr=errY, fmt='o', color='b')
plt.xlim(0, 12)          # set the x-axis range of values
plt.ylim(0, 120)          # set the Y-axis range of values
plt.xlabel("x")        # set the x-axis label
plt.ylabel("Y")        # set the Y-axis label
plt.grid(True)

# Least-squares curve fit (parabola)
fit = np.polyfit(x, Y, 2)
p = np.poly1d(fit)

xp = np.linspace(min(x), max(x), 200)

# Plot result:
plt.plot(xp, p(xp), '-', color="black")

plt.show()
```

The plot produced is shown here.



Example 12.8 [R]

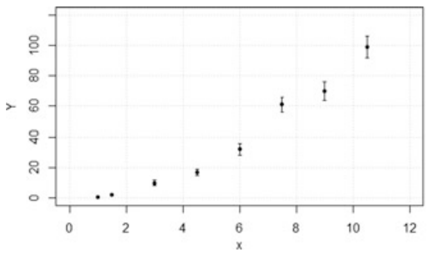
Eight results of measurements of the quantity $Y(x)$ are given in the table below. Using R, create a graph showing the measurements and the best parabolic curve between them.

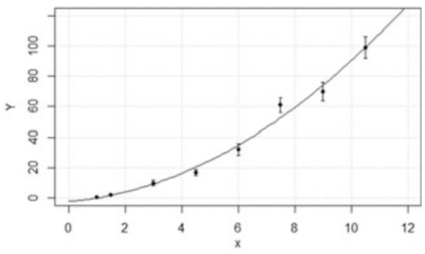
i	1	2	3	4	5	6	7	8
x	1.0	1.5	3.0	4.5	6.0	7.5	9.0	10.5
$Y \pm \delta Y$	0.5 ± 0.3	2.1 ± 0.5	10 ± 2	17 ± 2	32 ± 4	61 ± 5	70 ± 6	99 ± 7

We first create the scatter plot of the experimental points. We enter the vectors for x, Y and δY:

```
# x, Y and errY vectors:
x <- c(1.0, 1.5, 3.0, 4.5, 6.0, 7.5, 9.0, 10.5)
Y <- c(0.5, 2.1, 10, 17, 32, 61, 70, 99)
errY <- c(0.3, 0.5, 2, 2, 4, 5, 6, 7)

#scatter plot of data with x and Y axes labels, lengths and grid:
plot(x, Y, pch=20, xlab="x", ylab="Y", xlim=c(0, 12), ylim=c(0, 120), grid
())

# add Y error bars
arrows(x, Y-errY, x, Y+errY, length=0.02, angle=90, code=3)
# ENTER returns the scatter plot
```

The scatter plot is shown below on the left.





We fit to the points a least-squares parabola:

```
# Least-squares curve fit
> nls(Y~a0+a1*x+a2*x^2)
```

We add the least-squares curve to the graph:

```
# plot least-squares curve
curve(-2.187+1.493*x+0.775*x^2, from=0, to=12, add=T)
```

The final graph is shown above on the right.

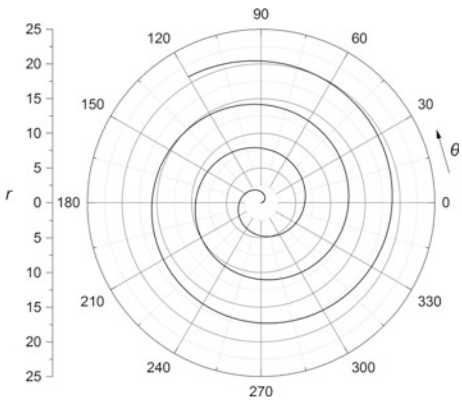
Example 12.9 [O]

Plot 3 turns of the Archimedean spiral, $r = \theta$, in polar coordinates.

We will give the angle θ in degrees but the program will evaluate r with θ in radians. For 3 turns, we therefore need values of θ between 0 and $3 \times 360^{\circ} = 1080^{\circ}$. We will use $0 \leq \theta \leq 1200^{\circ}$.

To plot the graph, we act as follows:

Values of column A. Highlight column A



Column > Set Column Values

and enter **(i − 1)** for i between 1 and 1201. Press **OK**.

Values of column B. Highlight column B

Column > Set Column Values

and enter **col(A) * 2 * pi/360** for i between 1 and 1201. Press **OK**.

Highlight both columns A and B. Then,

Plot > Specialized > Plot theta(X) r(Y)

The polar plot of $r = \theta$ appears.

We will improve the appearance of the graph.

We delete the text box containing B by clicking on it and pressing **Delete**.

We increase the thickness of the line in the graph by double-clicking on them and changing **Width** from 0.5 to 1.

On the r scale, we delete B. Using the **Text Tool T** we write r in italics and size 22.

For the θ scale, using the **Arrow Tool** we draw an arrow in the region between 0 and 30 degrees. If we wish, we may change the size, shape or color of the arrow by double-clicking on it and opening the appropriate window. Changing the font to Arial Greek we write θ next to the arrow.

We export the figure by using

File > Export Graphs > Open Dialog…

naming the file and selecting the directory in which we wish to save it.

The ease with which we can fit a straight line to the points for which a linear relation $y(x)$ holds, makes it desirable to reduce other mathematical expressions to linear. We have already seen that an exponential relation

$$R(t) = R_0\,\mathrm{e}^{-\lambda t} \tag{12.10}$$

may be reduced to a linear by the transformation

$$y \equiv \ln R, \quad a \equiv \ln R_o, \quad x \equiv t, \tag{12.11}$$

when we have

$$y = a - \lambda x \tag{12.12}$$

which is a linear relation. We may then draw, even by hand, a straight line between the points (x, y). The intercept of the line with the y-axis gives a and so also $R_0 = \mathrm{e}^a$, while the slope of the line gives $-\lambda$. [It should be noted that this presupposes the use of natural logarithms in the plot, as required by Eq. (12.11). The determination of λ from the slope of the straight line when common logarithms are used will be described below.]

Many common mathematical relations may be reduced to linear by a suitable change of variables. The more common of these cases are given in Table 12.1.

Table 12.1 Examples of variable transformations that reduce non-linear relations to linear

#	Non-linear relation	New variables		Resulting linear relation
		x	y	
1	$s = s_0 + \frac{1}{2}\gamma t^2$	t^2	s	$y = s_0 + \frac{1}{2}\gamma x$
2	$s = \upsilon_0 t + \frac{1}{2}\gamma t^2$	t	$\frac{s}{t}$	$y = \upsilon_0 + \frac{1}{2}\gamma x$
3	$V = -\frac{k}{r}$	$\frac{1}{r}$	V	$y = -kx$
4	$Y = \frac{1}{a + bX}$	X	$\frac{1}{Y}$	$y = a + bx$
5	$F = \frac{k}{r^2}$	$\frac{1}{r^2}$	F	$y = kx$
6	$\upsilon = k r^n$	$\log r$	$\log \upsilon$	$y = \log k + nx$
7	$N = N_0 e^{-\lambda t}$	t	$\ln N$	$y = \ln N_0 - \lambda x$
		t	$\log N$	$y = \log N_0 - (\lambda \log e)\, x$
8	$Y = ab^{cX}$	X	$\log Y$	$y = \log a + (c \log b)x$
9	$Y = ab^{cX} + d$	X	$\log(Y - d)$	$y = \log a + (c \log b)x$
10	$Y = aX^b$	$\log X$	$\log Y$	$y = \log a + bx$
11	$Y = aX^b + d$	$\log X$	$\log(Y - d)$	$y = \log a + bx$
12	$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$	$\frac{1}{s'}$	$\frac{1}{s}$	$y = \frac{1}{f} - x$

Example 12.10 [E]

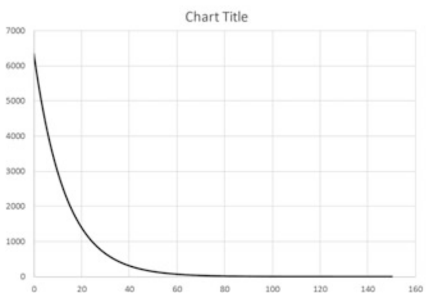
The activity of an isotope at $t = 0$ is $R_0 = 6350\,\text{cpm}$, ignoring errors. The mean lifetime of the isotope is $1/\lambda = 13.2$ min. Plot a graph of the expected activity $R(t)$ of the isotope up to $t = 150$ min. Use a logarithmic scale for R.

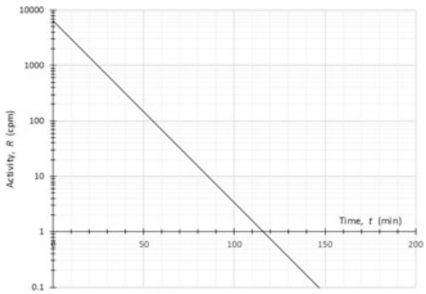
We will use the relation $R(t) = R_0 e^{-\lambda t}$. We will evaluate $R(t)$ for $0 \leq t \leq 150$ min in steps of 1 min. The values of t will be stored in column A, while the corresponding values of $R(t)$ will be stored in column B.

Evaluation of t values: Type 0 in cell A1. In cell A2 type = A1 + 1. After **ENTER**, we fill down from cell A2 to cell A151. Cells A1 to A151 now contain the numbers 0 to 150.

In cell B1 we type = 6350 * exp(−A1/13.2). After **ENTER**, we fill down from B1 to B151. Cells B1 to B151 now contain the values of $R(t)$ for $0 \leq t \leq 150$ min in steps of 1 min.

Highlight columns A and B and through **Insert** select the **Scatter** smooth line plot. Pressing **ENTER** results in the graph shown on the left, below.





We delete the text box for the **Chart Title**. We double-click on the Y-Axis and open the **Format Axis** window. In **Axis Options**, we click the **Logarithmic Base 10** box.

We format the plot in the way described in previous examples so that, finally, we have the graph shown on the right above.

Example 12.11 [O]

The activity of an isotope at $t = 0$ is $R_0 = 6350\,\text{cpm}$, ignoring errors. The mean lifetime of the isotope is $1/\lambda = 13.2$ min. Plot a graph of the expected activity $R(t)$ of the isotope up to $t = 150$ min. Use a logarithmic scale for R.

We will use the relation $R(t) = R_0 e^{-\lambda t}$. We will evaluate $R(t)$ for $0 \leq t \leq 150$ min in steps of 1 min. The values of t will be stored in column A, while the corresponding values of $R(t)$ will be stored in column B.

Evaluation of t values: Highlight column A. Then

Column > Set Column Values

In the window that opens we type **i − 1** for i from 1 to 151. We press **OK**. The values of t in min are entered in column A.

Evaluation of R values: Highlight column B. Then,

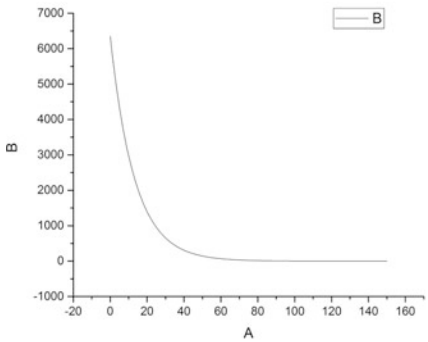
Column > Set Column Values

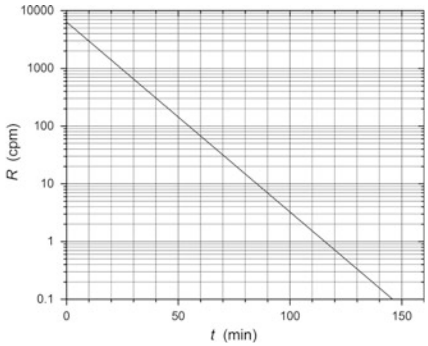
In the window that opens we type 6350 * exp(−col(A)/13.2) for i from 1 to 151. We press **OK**. The values of $R(t)$ in cpm are entered in column B.

We now plot $R(t)$. Highlight columns A and B. Then

Plot > Symbol > Scatter

The graph shown in the figure on the left below appears.





We will improve the appearance of the graph.

1. We delete the two text box given in the figure.
2. We change the thickness of the line by double-clicking on it and changing **Width** from 0.5 to 1.
3. We change the labels of the axes by double-clicking on them and writing ***t* (min)** in place of A and ***R* (c.p.m.)** in place of B. We use the default font of Arial 22 pts.
4. We change the **Horizontal** t axis by double-clicking on it. We set the **Scale** from 0 to 160, **Major Ticks Value** 50 and **Minor Ticks Count** 4.
5. We change the **Vertical** R axis by double-clicking on it. We set the **Scale** from 0.1 to 10 000, Type **Log10**. **Major Ticks** and **Minor Ticks** are set by the program to 1 and 8 respectively.
6. It is of great assistance in reading values off the graph to have the grid lines drawn. This is particularly true when one or both the axes are logarithmic. We will now draw the grid lines:

   i. We double click on the t axis, thus opening the **X Axis** window. Then, having opened **Horizontal, Grids**, set
   Major Grid Lines: Tick **Show**, **Color Black**, **Style Solid**, **Thickness 0.5**
   Minor Grid Lines: Tick **Show**, **Color Black**, **Style Solid**, **Thickness 0.3.**
   Then,
   Vertical, Grids, set the same as for **Horizontal, Grids**.

ii. We open the window for **Line and Ticks**.

Bottom: Tick **Show Line and Ticks**.

Line: Tick **Show**, **Color Black**, **Thickness 1.5**.
Major Ticks: Style Out.
Minor Ticks: Style Out.

Top: Tick **Show Line and Ticks**.

Line: Tick **Show**, **Color Black**, **Thickness 1.5**.
Major Ticks: Style In.
Minor Ticks: Style In.

Left: Same as Bottom.
Right: Same as Top.

Press **OK**.
The result is shown in the figure on the right above.

Example 12.12 [P]

The activity of an isotope at $t = 0$ is $R_0 = 6350$ cpm, ignoring errors. The mean lifetime of the isotope is $1/\lambda = 13.2$ min. Plot a graph of the expected activity $R(t)$ of the isotope up to $t = 150$ min. Use a logarithmic scale for R.

We will use the relation $R(t) = R_0 \mathrm{e}^{-\lambda t}$. We will evaluate $R(t)$ for $0 \le t \le 150$ min in steps of 1 min. We first plot $R(t)$ with linear scales:

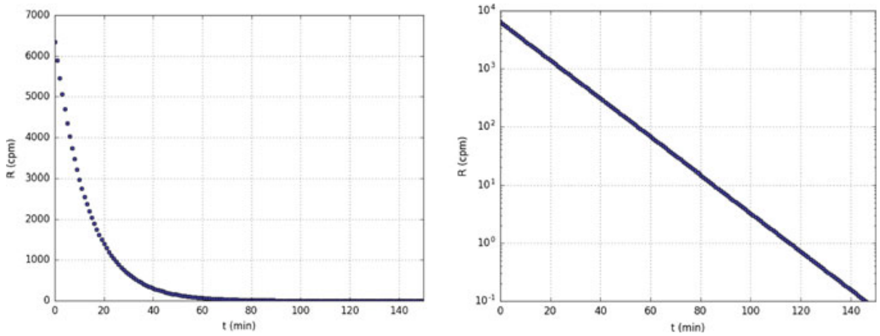
```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt
t = np.arange(0, 151)
R = 6350 * np.exp(-t/13.2)

plt.scatter(t, R)
plt.xlim(0, 150)
plt.ylim(0, 7000)
plt.xlabel("t (min)")
plt.ylabel("R (cpm)")
plt.grid(True)
plt.show()
```

We then plot R(t) with a logarithmic scale for R (the y axis):

```
plt.scatter(t, R)
plt.yscale('log')
plt.xlim(0, 150)
plt.ylim(0.1, 10000)
plt.xlabel("t (min)")
```

```
plt.ylabel("R (cpm)")
plt.grid(True)
plt.show()
```

The following figure shows the graph with linear scales (left) and the graph with logarithmic R scale (right).

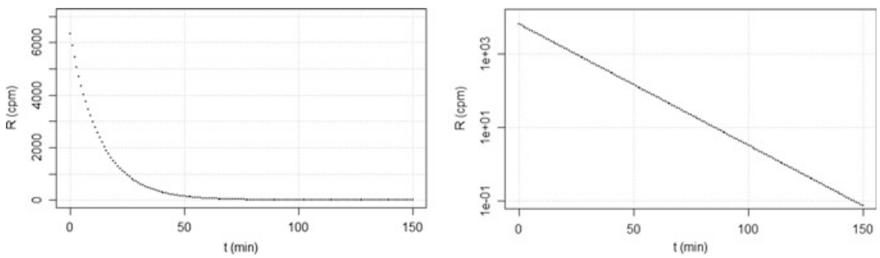


Example 12.13 [R]

The activity of an isotope at $t = 0$ is $R_0 = 6350\,\text{cpm}$, ignoring errors. The mean lifetime of the isotope is $1/\lambda = 13.2$ min. Plot a graph of the expected activity $R(t)$ of the isotope up to $t = 150$ min. Use a logarithmic scale for R.

We will use the relation $R(t) = R_0 e^{-\lambda t}$. We will evaluate $R(t)$ for $0 \le t \le 150$ min in steps of 1 min. We first plot $R(t)$ with linear scales:

```
> # scatter plot with linear scales:
> t <- seq(0, 150, by=1)
> R = 6350*exp(-t/13.2)
> plot(t, R, pch=20, cex=0.5, xlab="t (min)", ylab="R (cpm)",
  xlim=c(0, 150), ylim=c(0, 7000), grid())
```



This plot is shown above, on the left.

We now plot $R(t)$ using a logarithmic scale for R. So that we avoid infinities, we change the lower limit of R to 0.1 cpm.

```
# Scatter plot with logarithmic R scale:
plot(t, R, log="y", pch=20, cex=0.5, xlab="t (min)", ylab="R (cpm)",
xlim=c(0, 150), ylim=c(0.1, 10000), grid())
```

This plot is shown above, on the right.

Example 12.14

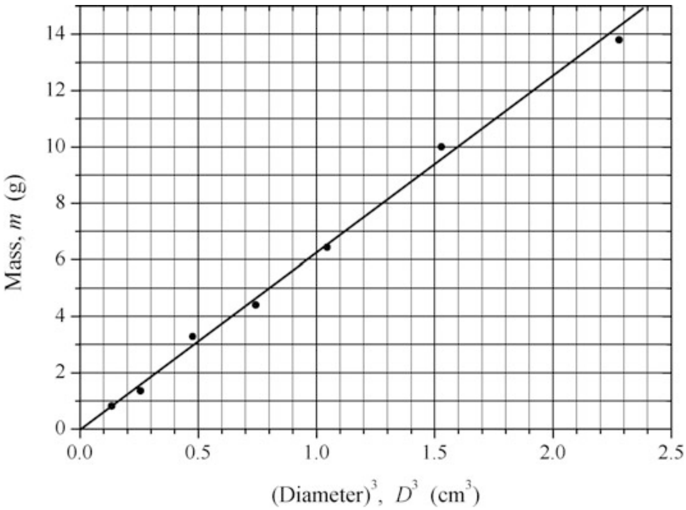
With the aim of determining their density, we measure the diameters of 7 metal spheres, as well as their masses. The results are given in columns 2 and 3 of the table below. Assuming that the 'spheres' are perfectly spherical, find their density using a graphical method.

The condition 'using a graphical method' is obviously imposed because we could determine the density of each sphere directly and then find the mean value and its standard deviation. However, for demonstration purposes, we will use the graphical method.

The relation between the mass m and the diameter D of a sphere is $m = \frac{\pi}{6}\rho D^3$, where ρ is its density. If we use the new variable $x = D^3$, we have the linear relation $m = \frac{\pi}{6}\rho x$. The straight line $m(x)$ passes through the origin and has a slope equal to $\lambda = \frac{\pi}{6}\rho$.

#	D (cm)	m (g)	D^3 (cm^3)
1	0.512	0.824	0.1342
2	0.635	1.362	0.2561
3	0.781	3.285	0.4764
4	0.906	4.401	0.7437
5	1.015	6.428	1.0457
6	1.152	10.853	1.5288
7	1.316	13.806	2.2791

The values of the variable D^3 are given in the table. The mass m is plotted as a function of D^3 in the figure that follows, where the linear relationship between the variables D^3 and m is seen.



The slope of the straight line is found to be approximately equal to $\lambda = 6.30\,\text{g/cm}^3$, from which we find that $\rho = \frac{6}{\pi}\lambda = 12.0\,\text{g/cm}^3$.

A simple method of obtaining an estimate for the error in the density will be presented below. It should be noted that for each point we could also have drawn the errors in both D^3 (as they are evaluated from the accuracy of measuring D) and in the masses m of the spheres.

12.5 The Slope of the Curve

The most accurate method for finding the best straight line or, in general, the best curve which corresponds to a series of experimental points, is the method of least squares which, at least in the case of the straight line, also gives the errors in the values of the two parameters which describe the line. The method has been presented in Chap. 11. Here, we will describe how we may find the slope of a straight line or of a curve at a point by a graphical method.

By the term 'best straight line' we mean a straight line that passes between the experimental points in such a way that, to the degree that this is possible, there are equal numbers of points above and below the line, both in the case of all the points and in as many smaller regions of values of the points as possible. Obviously this is not easy or possible, unless there are a great number of points with most of them not deviating by much more than the others from the general linear trend. Drawing this line requires more than the use of a transparent ruler. It is already becoming clear that this process depends on a number of subjective judgments. The same is true for non-linear curves, with the difference that in their case things are more difficult.

In general, large and sudden changes in the slope of the curve should be avoided, unless this is justified by the systematic behavior of an adequate number of points.

Having drawn what in our judgment is the best curve, we may find for a straight line its intercepts with the axes and its slope, and for a curve its slope at a certain point of interest. In Fig. 12.27a the method of finding the slope of a straight line is described. Having drawn the best straight line between the points, we choose two geometrical points on the line, A and B. These points *must not be* experimental points; otherwise the procedure would virtually cancel itself by giving weight to only two experimental points, ignoring all the others! For more accurate results, points A and B must at as great a distance from each other as possible. Also, for better reading of the co-ordinates of these points but also in order to simplify arithmetic, it is desirable that these points have abscissas (x) which correspond to whole numbers or integers or, in any case, to numbers for which there correspond lines on the graph paper used (for example, in Fig. 12.27a, the abscissas of points A and B are 2 and 12). The same comments apply in case we might choose to start by selecting two points with given values of the dependent variable (y). We read the co-ordinates of the points, A:(x_A, y_A) and B:(x_B, y_B). The slope of the straight line is

$$\lambda \equiv \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B - x_A}. \tag{12.13}$$

The need for as large as possible values of the differences $x_B - x_A$ and $y_B - y_A$ is obvious, since, otherwise, the reading errors in the co-ordinates will be significant fractions of the differences $x_B - x_A$ and $y_B - y_A$. It will be mentioned here that the units of the slope are given by the relation:

$$[\text{Units of slope}] = [\text{Units of } y]/[\text{Units of } x]. \tag{12.14}$$

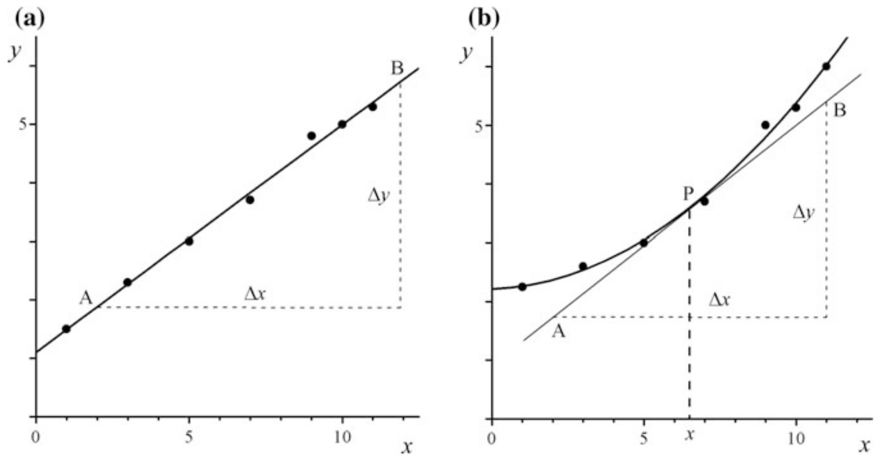


Fig. 12.27 The evaluation of the slope of **a** a straight line and **b** a curve at a point P

It must also be mentioned that the point of view that the slope is equal to $\tan\theta$, where θ is the angle formed by the straight line with the x-axis is wrong. This is true only in those cases that the scales of the variables x and y are such that it is

(units of x per unit of length of the x-axis) = (units of y per unit of length of the y-axis).

It must also be taken into account that the slope has dimensions while $\tan\theta$ is a pure (dimensionless) number.

In the general case of a curve, having drawn the curve, we also draw a tangent AB to the curve at the point P at which we wish to determine the slope of the curve, Fig. 12.27b. The required slope at P, $(\mathrm{d}y/\mathrm{d}x)_{\mathrm{P}}$, is equal to the slope of this tangent and is evaluated as described above.

12.5.1 A Graphical Method of Evaluating the Errors $\delta\alpha$ and $\delta\lambda$ in the Parameters α and λ of the Straight Line $y = \alpha + \lambda x$

A graphical method is described here for the determination of the errors $\delta\alpha$ and $\delta\lambda$ in the parameters α and λ of the straight line $y = \alpha + \lambda x$, which is simpler than the method of least squares. Although subjective judgments are made in applying the method, its results are satisfactory in most cases. It must be borne in mind, however, that the values of the errors found by this method have uncertainties of the order of 50% or even more.

Having drawn the best straight line for the given points and found the values of the parameters α and λ, we determine their errors by the following procedure (see Fig. 12.28):

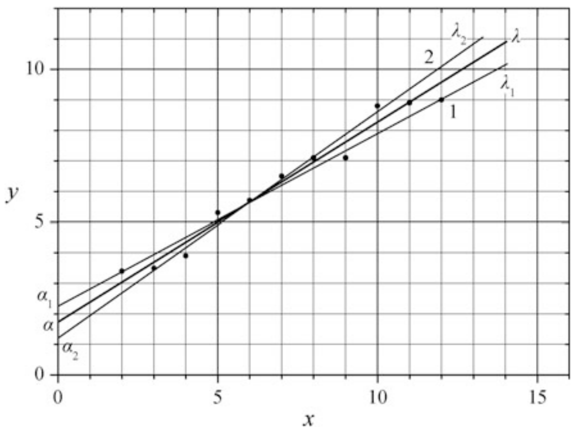


Fig. 12.28 The determination, by a graphical method, of the errors in the slope of a straight line and in the points of intersection of the axes by the line

1. We draw the best straight line, 1, passing between those points which lie *above* the main straight line on the left and *below* the main straight line on the right. The line we draw in this way is given by the equation $y = \alpha_1 + \lambda_1 x$ and from its slope and its intercept with the y-axis we determine the values of λ_1 and α_1, respectively.
2. We draw the best straight line, 2, passing between those points which lie *below* the main straight line on the left and *above* the main straight line on the right. The line we draw in this way is given by the equation $y = \alpha_2 + \lambda_2 x$ and from its slope and its intercept with the y-axis we determine the values of λ_2 and α_2, respectively.

Satisfactory estimates for the errors in α and λ are the values

$$\delta\alpha = \frac{1}{2}(\alpha_1 - \alpha_2) \quad \text{and} \quad \delta\lambda = \frac{1}{2}(\lambda_2 - \lambda_1). \tag{12.15}$$

Example 12.15

Find the values of the parameters α and λ and of their errors $\delta\alpha$ and $\delta\lambda$ for the straight line $y = \alpha + \lambda x$ of Fig. 12.28.

Referring to Fig. 12.28, we choose to find the ordinates of the three lines which correspond to the values $x = 0$ and $x = 13$. These are:

Main line:	$(x = 0,\ y = 1.75)$	$(x = 13,\ y = 10.25)$
Line 1:	$(x = 0,\ y = 2.25)$	$(x = 13,\ y = 9.55)$
Line 2:	$(x = 0,\ y = 1.20)$	$(x = 13,\ y = 10.85)$

We consider these magnitudes to be dimensionless, so that we do not have to give units.

The points of intersection of the y-axis are: $\alpha = 1.75$, $\alpha_1 = 2.25$, $\alpha_2 = 1.20$.

The slopes of the straight lines are:

$$\lambda = \frac{10.25 - 1.75}{13 - 0} = \frac{8.50}{13} = 0.654$$

$$\lambda_1 = \frac{9.55 - 2.25}{13 - 0} = \frac{7.30}{13} = 0.562 \quad \lambda_2 = \frac{10.85 - 1.20}{13 - 0} = \frac{9.65}{13} = 0.742.$$

The errors are

$$\delta\alpha = \frac{1}{2}(2.25 - 1.20) = 0.5 \quad \text{and} \quad \delta\lambda = \frac{1}{2}(0.742 - 0.562) = 0.09.$$

Finally,

$$\alpha = 1.8 \pm 0.5 \quad \text{and} \quad \lambda = 0.65 \pm 0.09.$$

Example 12.16 [E]

A converging lens is used for the formation of the image of a bright object. If the distance of the object from the center of the lens is s, then the distance of the image from the center of the lens is s' and the two distances are connected by the relation $\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$, where the length f is characteristic of the particular lens used and is called the *focal length* of the lens.

Given in the table that follows are pairs of values of s and s', as these were determined experimentally for a particular lens.

Column A	s (cm)	13	14	15	16	18	20	25
Column B	s' (cm)	43.0	35.0	30.0	27.0	22.5	20.0	16.7

(a) Mark in a graph the points $(s,\ s')$. Assume that the values of s are known with errors of ± 0.5 cm and that the values of s' have fractional errors equal to $\pm 5\%$ and draw the errors $\pm\delta s$ and $\pm\delta s'$ for every point of the graph. Investigate whether there are any points that you think should be neglected as the results of wrong measurements.

(b) Use the theoretical relation between s and s' for the determination of the focal length f of the lens as follows: From the theoretical relation it follows that $s' = f(1 + s'/s)$. Make a table in which you record the values of s and s', with their errors, δs and $\delta s'$, and the corresponding values of the variable $x = 1 + s'/s$ with their errors, $\delta\, x$.

Column A	Column B	Column C	Column D	Column E	Column F
s (cm)	s' (cm)	δs (cm)	$\delta s'$ (cm)	$x = 1 + \frac{s'}{s}$	δx

Plot the points $(x,\ s')$, with their errors $\pm\delta x$ and $\pm\delta s'$, and the best straight line that passes between these points. From the value of the slope of the straight line determine the focal length of the lens, f.

(a) We enter the values of $s, s', \delta s$ and $\delta s'$ in the columns A, B, C and D, respectively. We highlight columns A and B and, from **Insert**, we select the plot **Scatter** for them.

We delete the **Chart Title** text box. We double-click on a point and change the color of the points to black. We open ⊞, **Error Bars**, **More options**, to open the **Format Error Bars** window. In ▮▮▮, **Horizontal Error Bars**, we select **Direction: Both** and **End Style: Cup**. We also select **Error Amount: Custom** and tick the **Specify Value** box. In the window that opens, we type **= Sheet1!$C$1:$C$7** in both **Positive** and **Negative Error Value**. We repeat for the **Vertical Error Bars**. In the last step, we type **= Sheet1!$D$1:$D$7** in both **Positive** and **Negative Error Value**.

In ⊞, we select **Axis Titles**. We change the **X-Axis Title** to ***s* (cm)** and black color. We change the **Y-Axis Title** to ***s′* (cm)** and black color.

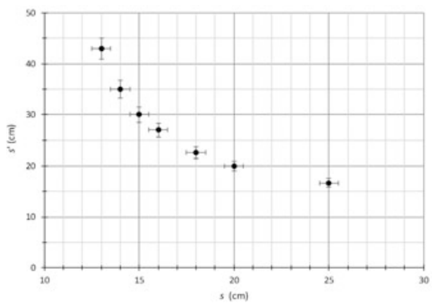
We double-click on the X-Axis and open the **Format Axis** window. In , **Axis Options**, we select **Bounds Minimum** 10, **Maximum** 30, **Units Major** 5, **Minor** 1. In **Tick Marks**, we select **Cross** for both **Major** and **Minor Type**.

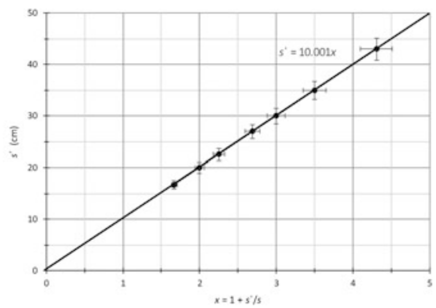
For the Y-Axis, in , **Axis Options**, we select **Bounds Minimum** 0, **Maximum** 50, **Units Major** 10, **Minor** 5. In **Tick Marks**, we select **Cross** for both **Major** and **Minor Type**.

In **Format Axis > Text Options > Text Fill**, we select color black for both axes.

In , **Grid Lines**, we also tick **Primary Minor Horizontal** and **Primary Minor Vertical** grid lines.

We finally obtain the graph shown on the left below.





We see that the points lie on a smooth curve and that there do not appear to be any points that should be neglected as the results of wrong measurements.

(b) The theoretical relation between s and s' may be written as $s' = f(1 + s'/s)$. We enter the values of $x = 1 + s'/s$ in column E. In cell E1 we type **= 1 + B1/A1**. After **ENTER**, we fill down from E1 to E7.

We estimate the errors in x from the errors in s and s'. From the theory of the propagation of errors we have

$$\delta x = (x - 1)\sqrt{\left(\frac{\delta s}{s}\right)^2 + \left(\frac{\delta s'}{s'}\right)^2}.$$

In cell F1 we type **= (E1 − 1) * sqrt((C1/A1)^2 + (D1/B1)^2)**. After **ENTER**, we fill down from F1 to F7. Column F now contains the values of δx.

We will plot the points (x, s') with their errors. We copy column B, which contains s', in column G. We highlight columns E and G and through **Insert** we choose a scatter plot for these two variables.

We delete the **Chart Title** text box. We double-click on a point and change the color of the points to black. We open , **Error Bars**, **More options**, to open the **Format Error Bars** window. In , **Horizontal Error Bars**, we select **Direction:**

Both and **End Style: Cup**. We also select **Error Amount: Custom** and tick the **Specify Value** box. In the window that opens, we type **= Sheet1!$F$1:$F$7** in both **Positive** and **Negative Error Value**. We repeat for the **Vertical Error Bars**. In the last step, we type **= Sheet1!$D$1:$D$7** in both **Positive** and **Negative Error Value**.

In ⊞, we select **Axis Titles**. We change the **X-Axis Title** to $\boldsymbol{x = 1 + s'/s}$ and black color. We change the **Y-Axis Title** to $\boldsymbol{s'}$ **(cm)** and black color.

We double-click on the X-Axis and open the **Format Axis** window. In ▮, **Axis Options**, we select **Bounds Minimum** 0, **Maximum** 5, **Units Major** 1, **Minor** 0.5. In **Tick Marks**, we select **Cross** for both **Major** and **Minor Type**.

For the Y-Axis, in ▮, **Axis Options**, we select **Bounds Minimum** 0, **Maximum** 50, **Units Major** 10, **Minor** 5. In **Tick Marks**, we select **Cross** for both **Major** and **Minor Type**.

In **Format Axis > Text Options > Text Fill**, we select color black for both axes.

In ⊞, **Grid Lines**, we also tick **Primary Minor Horizontal** and **Primary Minor Vertical** grid lines.

We open ⊞, **Trendline**, **More Options**. We choose **Linear** and **Set Intercept** to (0, 0). We also choose for the equation of the line to be shown in the plot. The final result is shown above by the figure on the right above.

The equation of the line is given as $s' = 10.001x$. This means that the value of the focal length is $f = 10.001$ cm. It is obvious that the error in f as derived from the points of the plot would be about 10 times smaller than that expected from the errors in x and s' of the points.

Example 12.17 [O]

Solve Example 12.16 [E] using Origin®.

We fill columns A and B with the values of s and s' and label them $\boldsymbol{s}$ **(cm)** and $\boldsymbol{s'}$ **(cm)**, respectively. The values of δs and $\delta s'$ are entered in columns C and D. We select column C and then

Column > Set As > X Error
We select column D and then,

Column > Set As > Y Error
We select columns A, B, C and D and

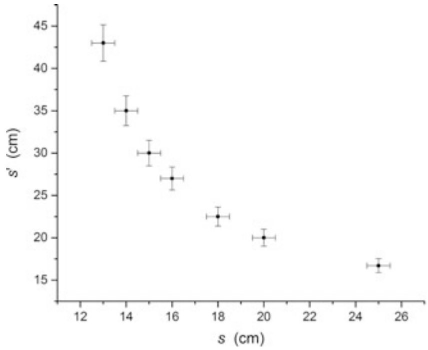
Plot > Symbol > Scatter
A graph appears, showing the points (s, s') with their error bars (see below, left). We will improve the appearance of the graph.

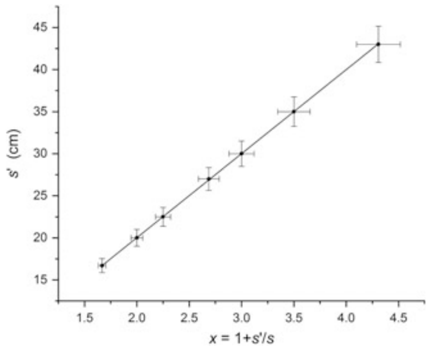
1. We delete the two text box given in the figure.
2. The labels and ranges of the two axes are in an acceptable form. The ranges are 12–26 cm for the s-axis and 15–45 cm for the s'-axis.

We export the graph using

File > Export Graphs > Open Dialog…

Select **Image Type** (say jpg), **File Name** and **Path**. The figure may be imported in the text here. The result is shown in the figure on the left below. It is seen that all the points lie on a smooth curve and no point needs to be rejected.





In column E we enter the values of $x = 1 + s'/s$ as follows:

Column > Set Column Values

and entering **1 + col(B)/col(A)**. Pressing **OK** fills column E with the x values.

We estimate the errors in x from the errors in s and s'. From the theory of the propagation of errors we have

$$\delta x = (x-1)\sqrt{\left(\frac{\delta s}{s}\right)^2 + \left(\frac{\delta s'}{s'}\right)^2}.$$

We highlight column F. Then

Column > Set Column Values

and type **(col(E) − 1) * sqrt((col(C)/col(A))^2 + (col(D)/col(B))^2)** and then **OK**. The errors in x are entered in column F.

We arrange the data in columns in preparation for plotting $s'(x)$. Column E contains the values of x. We highlight column E and then

Column > Set As > X

Column F contains the values of δx. We highlight column F and then

Column > Set As > X Error

We copy column B in column G. We highlight column G and then

Column > Set As > Y

We copy column D in column H. We highlight column H and then

Column > Set As > Y Error

We highlight columns E, F, G and H. Then

Plot > Symbol > Scatter

A graph appears, showing the points (x, s') with their error bars.
We will improve the appearance of the graph.

1. We delete the text box given in the figure.
2. We double-click on a point and change the shape and size of the points from square and 8 pts to circular and 5 points.
3. We change the labels of the axes by double-clicking on them and writing ***x* = 1 + *s′*/*s*** in place of x and ***s′* (cm)** for the Y axis. We use the default font of Arial 22 pts.
4. The ranges of the axes are satisfactory.

While in the graph of (x, s') we select
Analysis > Fitting > Linear Fit > Open Dialog
Press **OK**. A straight line is drawn between the experimental points.
Double-click on the line and change its color from red to black and its thickness from 0.5 to 1. Press **OK**.
The result is shown in the figure on the right above.
The straight line has a slope of 9.98841 ± 0.02756 cm. The focal length of the lens is, therefore, equal to $f = 9.99 \pm 0.03$ cm.

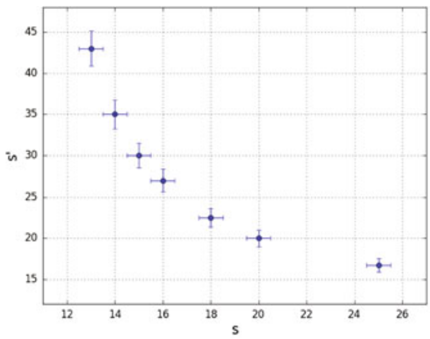
Example 12.18 [P]

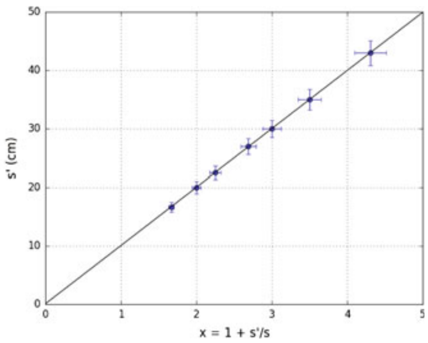
Solve Example 12.16 [E] using Python.

(a) We first plot the graph $s'(s)$.

```
import numpy as np
import matplotlib.pyplot as plt
s1 = np.array([13, 14, 15, 16, 18, 20, 25])
s2 = np.array([43, 35, 30, 27, 22.5, 20, 16.7])
errs1 = np.array([0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5])

errs2 = 0.05*s2

plt.errorbar(s1, s2, xerr=errs1, yerr=errs2, fmt=‘o’, color=‘b’)
plt.xlim(11, 27)
plt.ylim(12,48)
plt.xlabel("s")
plt.ylabel("s'")
plt.grid(True)
plt.show()
```

The resulting scatter plot is shown in the figure below, on the left.





(b) We will now plot $s'(x)$.

We evaluate $x = 1 + s'/s$ and its errors $\delta x = (x-1)\sqrt{\left(\frac{\delta s}{s}\right)^2 + \left(\frac{\delta s'}{s'}\right)^2}$.

```
x = 1 + s2/s1
errx = (x-1)*np.sqrt((errs1/s1)**2 + (errs2/s2)**2)
```

We plot a scatter plot of x versus s2. We also add the best-fit least-squares straight line:

```
plt.errorbar(x, s2, xerr=errx, yerr=errs2, fmt='o', color='b')
plt.xlim(0, 5)
plt.ylim(0, 50)
plt.xlabel("x = 1 + s'/s")
plt.ylabel("s' (cm)")
plt.grid(True)
fit = np.polyfit(x, s2, 1)
p = np.poly1d(fit)
xp = np.linspace(0, 5, 200)
plt.plot(xp, p(xp), '-', color="black")
plt.show()
```

The resulting graph is shown in the figure on the right, above. By looking at the contents of the p object (the polynomial of the straight line fit to the data) we get:

```
In[]: p
Out[]: poly1d([ 9.97003572,  0.0928256 ])
```

Therefore, we calculated intercept = 0.09283 cm and slope = 9.97004 cm. This means that the focal length of the lens is $f = 9.97$ cm.

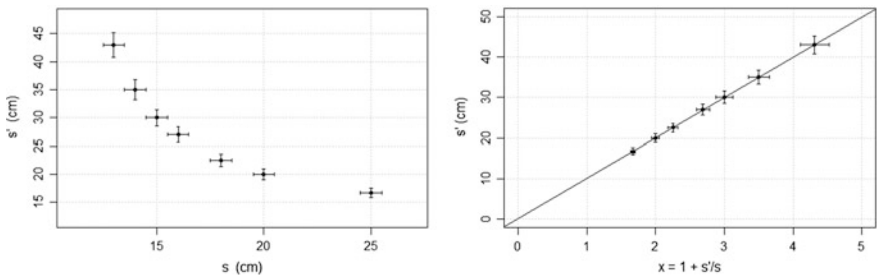
Example 12.19 [R]

Solve Example 12.16 [E] using R.

(a) We first plot the graph $s'(s)$.

```
# data vectors:
s1 <- c(13, 14, 15, 16, 18, 20, 25)
s2 <- c(43, 35, 30, 27, 22.5, 20, 16.7)
errs1 <- c(0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5)
errs2 = 0.05*s2

# scatter plot of data with s1 and s2 axes labels, lengths and grid:
plot(s1, s2, pch=20, xlab="s", ylab="s'", xlim=c(11, 27),
ylim=c(12, 48), grid())

# add s1 error bars
arrows(s1-errs1, s2, s1+errs1, s2, length=0.02, angle=90, code=3)
# add s2 error bars
arrows(s1, s2-errs2, s1, s2+errs2, length=0.02, angle=90, code=3)
```

The resulting scatter plot is shown in the figure below, on the left.



We see that there do not appear to be any points that should be neglected as the results of wrong measurements.

(b) We will now plot $s'(x)$.

We evaluate $x = 1 + s'/s$ and its errors $\delta x = (x-1)\sqrt{\left(\frac{\delta s}{s}\right)^2 + \left(\frac{\delta s'}{s'}\right)^2}$.

```
# evaluate x and its errors:
x = 1+s2/s1
errx = (x-1)*sqrt((errs1/s1)^2+(errs2/s2)^2)
```

```
#scatter plot of data with x and s2 axes labels, lengths and grid:
plot(x, s2, pch=20, xlab="x = 1 + s'/s", ylab="s' (cm)", xlim=c(0, 5),
```

```
ylim=c(0, 50), grid())

# add x error bars
arrows(x-errx, s2, x+errx, s2, length=0.02, angle=90, code=3)
# add s2 error bars
arrows(x, s2-errs2, x, s2+errs2, length=0.02, angle=90, code=3)
```

The resulting graph is shown in the figure on the right, above.
We find the best-fit least-squares straight line through the points:

```
# find least-squares best-fit straight line:
fit <- lm(s2~x)
fit

# Plot least-squares best-fit straight line:
abline(fit)
```

The best-fit least-squares straight line is also drawn in the figure.

The least-squares fit gives: intercept = 0.09283 cm and slope = 9.97004 cm. This means that the focal length of the lens is f = 9.97 cm. An estimate of the error in f can be made, based on the scatter of the experimental points about the straight line. It appears to be less than 1%.

12.5.2 The Evaluation of Slopes of Straight Lines in Graphs with Logarithmic Scales

We will now discuss the way to evaluate the slopes of straight lines in graphs with one or both the scales being logarithmic.

12.5.2.1 Two Logarithmic Scales

In the case of a straight line in a graph with two logarithmic scales, if the actual values of the logarithms are marked on the axes, the slope is found as described above for a straight line in a graph with linear axes. We must remember that, in this case, the slope, being the ratio of two differences of logarithms, is a dimensionless number. Also, since the common logarithm and the natural logarithm are related by the expressions

$$z = 10^{\log z}, \quad \ln z = (\ln 10) \log z, \quad \ln z = (2.3026\ldots) \times \log z, \tag{12.16}$$

$$z = e^{\ln z}, \quad \log z = (\log e) \ln z, \quad \log z = (0.4343\ldots) \times \ln z, \tag{12.17}$$

i.e. through a multiplying factor, the value of the slope is the same both in the case of two scales of common logarithms and in the case of two scales of natural logarithms.

More frequently, however, although the scales are logarithmic, log x and log y, the numbers x and y are marked on them, as for example in Fig. 12.29. In these cases, if A and B are two geometrical (not experimental) points on the straight line, the slope is given by the ratio

$$\kappa \equiv \frac{\Delta \log y}{\Delta \log x} = \frac{\log y_{\mathrm{B}} - \log y_{\mathrm{A}}}{\log x_{\mathrm{B}} - \log x_{\mathrm{A}}} \quad \text{or} \quad \kappa \equiv \frac{\Delta \ln y}{\Delta \ln x} = \frac{\ln y_{\mathrm{B}} - \ln y_{\mathrm{A}}}{\ln x_{\mathrm{B}} - \ln x_{\mathrm{A}}} \tag{12.18}$$

After we read the values of x and y off the graph, we must substitute their logarithms, common or natural, in Eq. (12.18). The slope κ, being the ratio of two differences of logarithms, is a dimensionless quantity (pure number).

Points A and B must be chosen to be at a distance between them as large as possible, but also at such positions that the reading of the values of x and y may be done with as much accuracy as possible, especially since the scales are not linear. Of course, on the logarithmic paper usually used, there are lines at as many subdivisions as possible, more than in our figures, and the reading of values is easier.

As a numerical example, we evaluate the slope of the straight line of Fig. 12.9, which we reproduce here as Fig. 12.30. We choose points A and B to have $V_a = 1$ and 100 V, respectively. The corresponding values of I_a are 5.0×10^{-5} and 5.0×10^{-2} A. Thus, we have for the slope

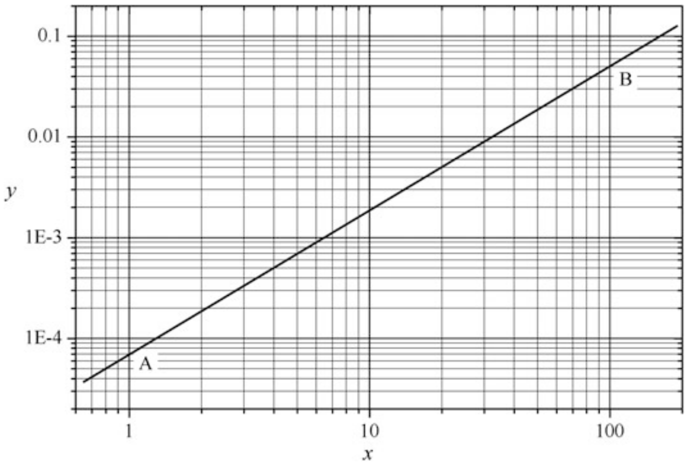


Fig. 12.29 The evaluation of the slope of a straight line in a graph with two logarithmic scales

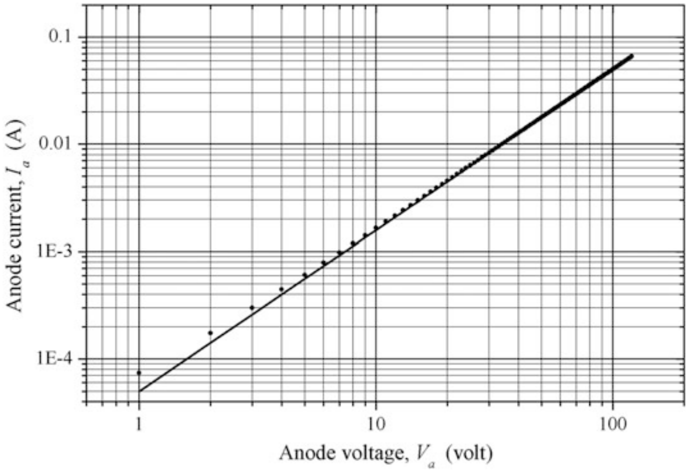


Fig. 12.30 A numerical example of the evaluation of the slope of a straight line in a graph with two logarithmic scales. The notation 1E-3 = 10^{-3} etc. is used for powers of 10

$$\kappa = \frac{\log I_B - \log I_A}{\log V_B - \log V_A} = \frac{\log(5.0 \times 10^{-2}) - \log(5.0 \times 10^{-5})}{\log(100) - \log(1)}$$
$$= \frac{(-1.301) - (-4.301)}{2 - 0} = \frac{3}{2}, \tag{12.19}$$

as predicted by theory.

A second example is given in Fig. 12.31, in which the periods of revolution of the planets around the Sun, T, are plotted as a function of their distances from it, a,

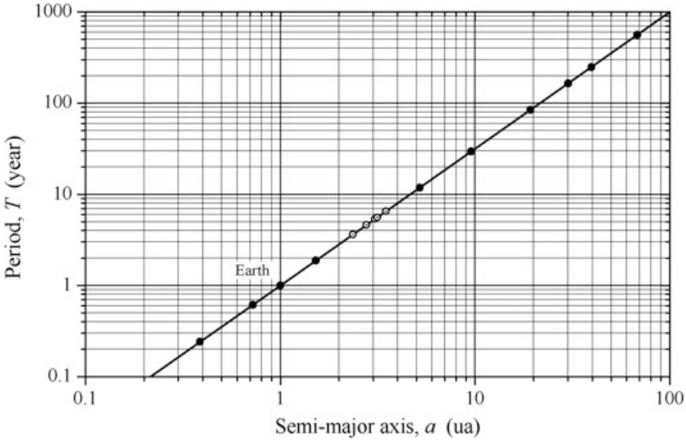


Fig. 12.31 Kepler's third law (Fig. 12.10). A numerical example of the evaluation of the slope of a straight line, in a graph with two logarithmic scales

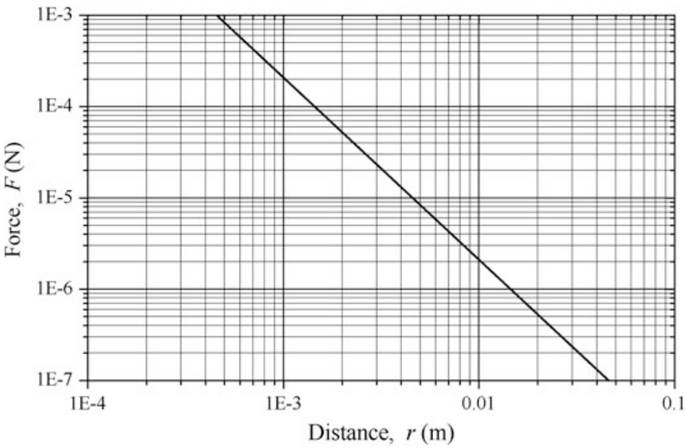


Fig. 12.32 The force F between two point charges as a function of their distance r

using logarithmic scales. The relationship is seen to be linear. For the calculation of the slope we use the points Earth: (1, 1) and B: (100, 1000). The slope of the line is

$$\kappa = \frac{\log T_{\mathrm{B}} - \log T_{\mathrm{Earth}}}{\log a_{\mathrm{B}} - \log a_{\mathrm{Earth}}} = \frac{\log(1000) - \log(1)}{\log(100) - \log(1)} = \frac{3}{2}, \tag{12.20}$$

verifying Kepler's third law, $\mathrm{T} \propto a^{3/2}$.

A third example is shown in Fig. 12.32, in which the force F between two point electric charges is plotted as a function of the distance between them, r. We use the points

$$\mathrm{A}{:}(4.5 \times 10^{-4}\ \mathrm{m},\quad 1 \times 10^{-3}\,\mathrm{N}) \quad \text{and} \quad \mathrm{B}:(0.045\ \mathrm{m},\quad 1 \times 10^{-7}\,\mathrm{N}) \tag{12.21}$$

and find for the slope of the straight line the value

$$\kappa = \frac{\log(1 \times 10^{-7}) - \log(1 \times 10^{-3})}{\log(0.045) - \log(4.5 \times 10^{-4})} = \frac{(-7) - (-3)}{(-1.347) - (-3.347)} = \frac{-4}{2} = -2. \tag{12.22}$$

Therefore, it is

$$\log F = K - 2 \log r \quad \text{or} \quad F = A/r^2, \tag{12.23}$$

as expected from Coulomb's law.

12.5.2.2 *One Linear Scale and One Logarithmic Scale*

We will assume that the logarithmic scale is that of the dependent variable (y), as in Fig. 12.33. If A and B are two (geometrical) points on the straight line, the slope is given by the ratio

$$\kappa \equiv \frac{\Delta \ln y}{\Delta x} = \frac{\ln y_B - \ln y_A}{x_B - x_A} \tag{12.24}$$

and, after we read the values of x and y, we must substitute for the natural logarithms of the y values in Eq. (12.23). This presupposes that the relation connecting x and y is of the form

$$y = A\,e^{\kappa x}. \tag{12.25}$$

This is no limitation, as the relation may be transformed to an exponential of any other base, through the relation $y = A\,a^{(\kappa/\ln a)\,x}$. It should be noted that, since the difference in the logarithms for y_A and y_B is the logarithm of their ratio, which is a dimensionless quantity, the units of the slope κ are the inverse of those of x:

$$[\text{Units of the slope } \kappa] = 1/[\text{Units of } x]. \tag{12.26}$$

Points A and B must be chosen to be at as great a distance from each other as possible and in such positions that they make the reading of the values of y as accurate as possible, also taking into account the fact that the scale is not linear. A good choice of points would be those with $y_A = 10$ and $y_B = 10\ 000$. We start with values of y, since we can choose such values that are easy to read off the graph

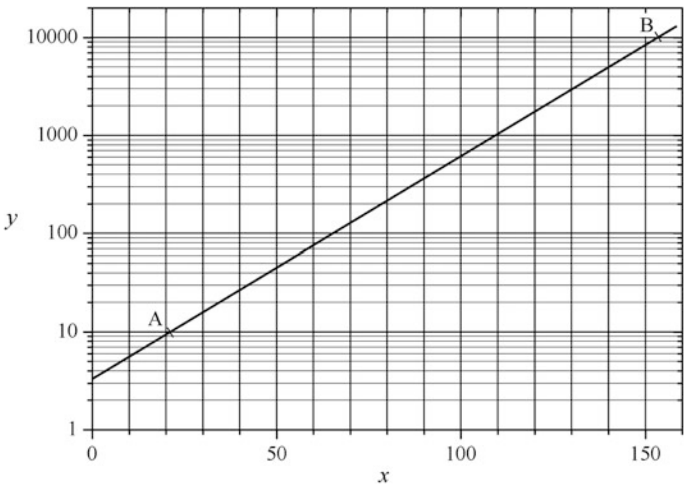


Fig. 12.33 The evaluation of the slope of a straight line in a graph with one linear scale and one logarithmic scale

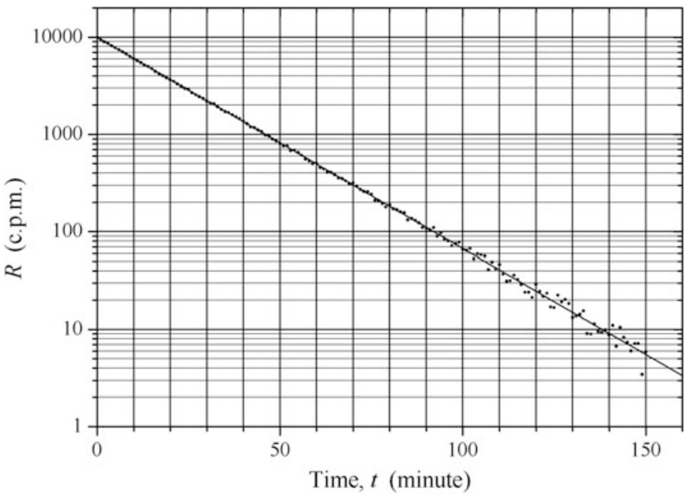


Fig. 12.34 A numerical example of the evaluation of the slope of a straight line in a graph with one linear scale and one logarithmic scale. The graph shows the variation with time of the counting rate of a radioactive sample

than other values of y which do not correspond to a grid line of the logarithmic scale. This would have been the difficulty, had we started by choosing two values of x.

A numerical example is given in Fig. 12.34, in which we reproduce the variation as a function of time of the disintegration counting rate, R, of a radioactive sample of Fig. 8.2.

As points A and B we choose those with $R_A = 10\,000$ and $R_B = 10$ c.p.m. The co-ordinates of these points are A: (0, 10^4 c.p.m.) and B: (138 min, 10 c.p.m.). Then,

$$\kappa = \frac{\ln R_B - \ln R_A}{t_B - t_A} = \frac{\ln(10) - \ln(10^4)}{138 - 0} = \frac{2.303 - 9.210}{138} = -0.0501\,\mathrm{min}^{-1}. \tag{12.27}$$

Alternatively, we may choose as points A and B those with $t = 0$ and $t = 160$ min. The co-ordinates of these points are A: (0, 10^4 c.p.m.) and B: (160 min, 3.3 c.p.m.). The slope of the straight line is:

$$\kappa = \frac{\ln R_B - \ln R_A}{t_B - t_A} = \frac{\ln(3.3) - \ln(10^4)}{160 - 0} = \frac{1.194 - 9.210}{160} = -0.0501\,\mathrm{min}^{-1}. \tag{12.28}$$

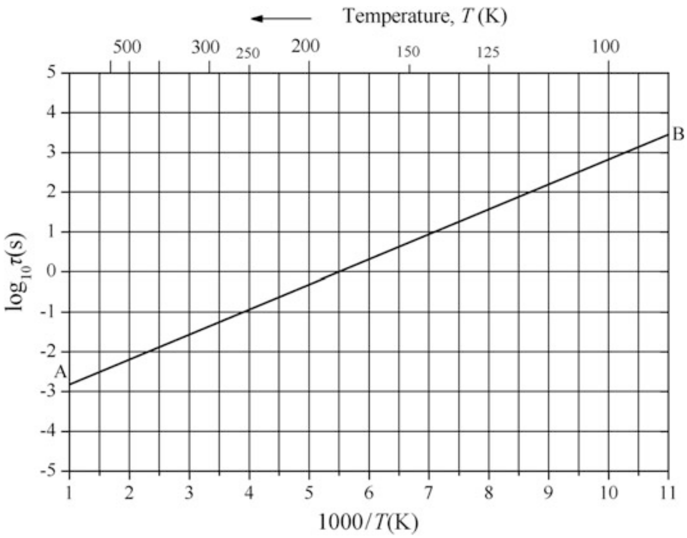


Fig. 12.35 An Arrhenius diagram ($\log_{10}\tau$ – 1/T)

From the law of radioactivity, $R(t) = R_0 e^{-\lambda t}$, we expect

$$\ln R = \ln R_0 - \lambda t \tag{12.29}$$

so that the slope found is $\kappa = -\lambda$. It follows from these measurements that for the particular radioisotope it is $\lambda = 0.050\,\mathrm{min}^{-1}$. The mean life-time of the isotope is $\tau = 1/\lambda = 20\,\mathrm{min}$. The errors in these values could be found either by a graphical method or the method of least squares applied to the straight line $\ln R(t)$. It can be seen that these errors are mostly due to the dispersion of values at low R's.

Another example is given in Fig. 12.35, which shows a series of measurements similar to that of Fig. 12.15. The slope of the straight line may be found from the points

$$\text{A:}\,(1/T = 1 \times 10^{-3}\ \mathrm{K}^{-1}, \log_{10}\tau(\mathrm{s}) = -2.8)$$
$$\text{and}\ \ \text{B:}\,(1/T = 11 \times 10^{-3}\ \mathrm{K}^{-1}, \log_{10}\tau(\mathrm{s}) = 3.5).$$

We change the common logarithms to natural logarithms by multiplying then by 2.3026:

$$\text{A:}\ (1/T = 1 \times 10^{-3}\ \mathrm{K}^{-1}, \ln\tau(\mathrm{s}) = -6.45)$$
$$\text{and}\ \ \text{B:}\ (1/T = 11 \times 10^{-3}\ \mathrm{K}^{-1}, \ln\tau(\mathrm{s}) = 8.06).$$

The slope of the straight line is

$$\kappa = \frac{[\ln\ \tau(\mathrm{s})]_{\mathrm{B}} - [\ln\ \tau(\mathrm{s})]_{\mathrm{A}}}{1/T_{\mathrm{B}} - 1/T_{\mathrm{A}}} = \frac{8.06 - (-6.45)}{11 \times 10^{-3} - 1 \times 10^{-3}} = \frac{14.51}{10^{-2}} = 1451\,\mathrm{K}. \quad (12.30)$$

Assuming a relation of the form $\tau = \tau_0 \mathrm{e}^{E/kT}$ [Eq. (12.8)], where k is Boltzmann's constant, τ_0 a time constant and E an energy, we find that the straight line

$$\ln\ \tau = \ln\ \tau_0 + \frac{E}{k}\frac{1}{T} \quad (12.31)$$

which results from plotting the values of $\ln\ \tau$ as a function of $1/T$ has a slope of

$$\kappa = \frac{E}{k}. \quad (12.32)$$

Therefore, the experimental values give

$$\kappa = \frac{E}{k} = 1451\,\mathrm{K}. \quad (12.33)$$

As it is $1/k = 11604.5$ K/eV, the value of the energy E is

$$E = \frac{1451}{11604.5} = 0.125\,\mathrm{eV}. \quad (12.34)$$

Given that the accuracy with which values are read off the graph cannot be better than about 2%, we accept that $\pm 2\,\%$ is a reasonable estimate for the fractional error in the value of E. Thus,

$$E = 0.125 \pm 0.003\,\mathrm{eV}, \quad (12.35)$$

unless we have another indication from the dispersion of the experimental points.

12.5.2.3 *The Graphical Resolution of the Activity of a Radioactive Sample into Components*

It is often the case that the activity of a radioactive sample is due to two or more radioisotopes, with different decay constants. In the case of two radioisotopes with comparable activities but having decay constants which are quite different the separation of their activities is possible by a graphical method.

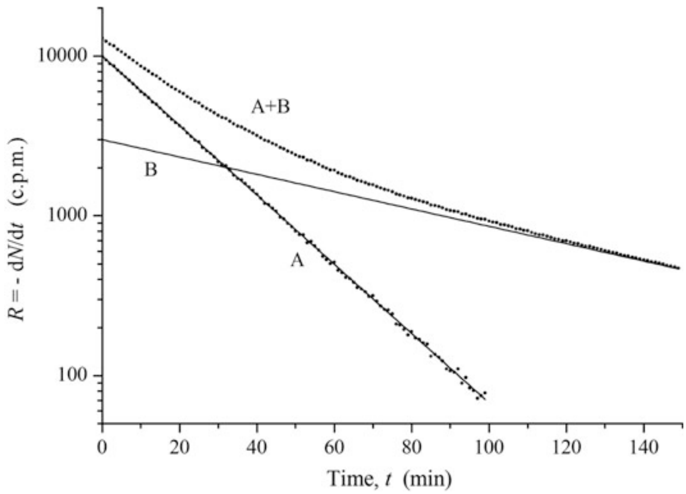


Fig. 12.36 The graphical resolution of the activity of a radioactive sample into two components, A and B, due to two different radioisotopes

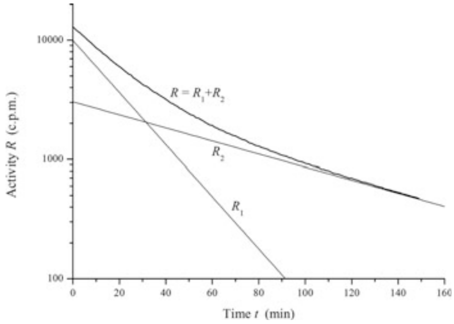
If the total activity of the sample is

$$R = R_A + R_B = R_{0A} e^{-\lambda_A t} + R_{0B} e^{-\lambda_B t} \tag{12.36}$$

and we have at our disposal a large enough number of measurements at different times t, the curve of the total activity as a function of time will be as the curve A + B of Fig. 12.36. At large values of time, the activity of the shorter-lived of the two radioisotopes is reduced to negligible values compared to the activity of the other radioisotope. If the plot is $\log R - t$, and the background, which is independent of time, has been subtracted, the straight line B may be determined, which gives the activity of radioisotope B. Subtracting this from the total activity, we find the activity of radioisotope A. In this way we may determine the quantities $\lambda_A, \lambda_B, R_{0A}$ and R_{0B}. In the example of Fig. 12.35, it is found that $\tau_A = 1/\lambda_A = 20$ min, $\tau_B = 1/\lambda_B = 80$ min, $R_{0A} = 10,000$ c.p.m. and $R_{0B} =$ 3000 c.p.m. Obviously, the graphical method has a limited accuracy. More accurate results may be obtained by numerical methods. The method is very rarely useful for more than two radioisotopes.

Example 12.20 [O]

Measurements of the activity of a radioactive sample, R, are given for $0 \leq t \leq 150$ min (They are the same as those of Example 8.5). Plot $\log R(t)$ and verify that the activity seems to be due to two isotopes with different decay constants. Assume that the activity is given by $R = R_{10} e^{-t/\tau_1} + R_{20} e^{-t/\tau_2}$ and find the parameters of the functions.



The values of t are entered in column A and those of R in column B. We plot log$R$ (t). While in the graph environment, we press

Analysis > Fitting > Nonlinear Curve Fit > Open Dialog…

In the window that opens we select

Settings > Function Selection > Origin Basic Functions > Exponential > ExpDec2

The curve fitted is y = A1 * exp(−x/t1) + A2 * exp(−x/t2) + y0. We want to set y0 = 0.

Open **Parameters**. Tick **y0** and set its value to zero. Press **Fit**. The results are:

$$\mathrm{A1} = R_{10} = 9950 \pm 48 \text{ c.p.m.}, \quad \mathrm{t1} = \tau_1 = 19.88 \pm 0.10\,\text{min}$$
$$\mathrm{A2} = R_{20} = 3036 \pm 51 \text{ c.p.m.}, \quad \mathrm{t2} = \tau_2 = 79.43 \pm 0.95\text{ min}$$

The curve $R = R_{10}\mathrm{e}^{-t/\tau_1} + R_{20}\mathrm{e}^{-t/\tau_2}$ is plotted in the graph. We also plot $R_1 = R_{10}\mathrm{e}^{-t/\tau_1}$ and $R_2 = R_{20}\mathrm{e}^{-t/\tau_2}$.

The graph is shown in the figure presented above.

Is a resolution into three components possible? A radioactive sample contains three radioisotopes, A, B and C, which contribute counting rates $R_\mathrm{A}(t) = 10\,000\ \mathrm{e}^{-t/100}$, $R_\mathrm{B}(t) = 2000\ \mathrm{e}^{-t/300}$ and $R_\mathrm{C}(t) = 300\ \mathrm{e}^{-t/900}$, respectively, in c.p.m. when t is in min. The total counting rate of the sample is $R(t) = R_\mathrm{A}(t) + R_\mathrm{B}(t) + R_\mathrm{C}(t)$. Discuss the possibility of performing a graphical analysis of this curve into its three components. Assume that the counting rates are evaluated using the results of 10-minute measurements Fig. 12.37.

From the figure, it is seen that drawing a tangent to the $R(t)$ curve is not easy. It should be noted that at the lower point of the curve, a 10-min measurement will give the result of $20 \times 10 = 200 \pm 14$ counts, corresponding to a counting rate of 20 ± 1.4 c.p.m. We see that there is an error of 7% in the counting rates at times near 2500 min. This makes drawing a tangent to the curve even more difficult. The best we can do is to draw two straight lines, C1 and C2 as the limiting cases

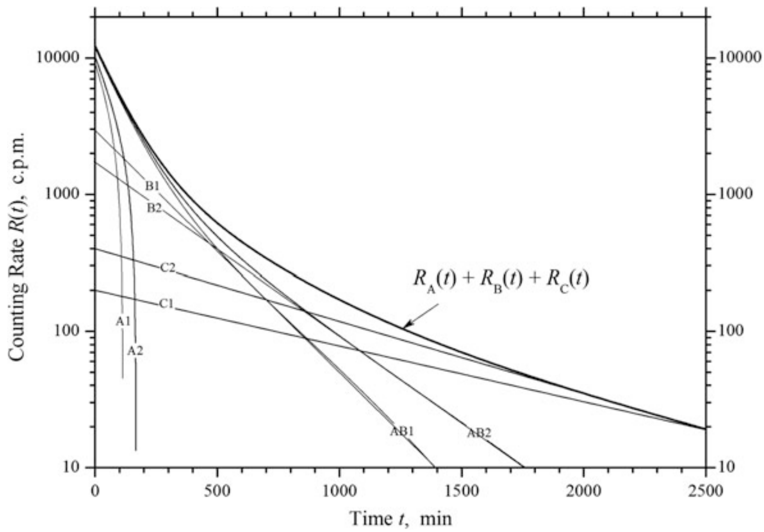


Fig. 12.37 The graphical resolution of the activity of a radioactive sample into three components, A, B and C, due to three different radioisotopes

between which the correct tangent lies. We find that these lines cut the R axis at $R_{C1}(0) = 200$ and $R_{C2}(0) = 400$ c.p.m., respectively, and correspond to lifetimes equal to $\tau_{C1} = 1062\,\text{min}$ and $\tau_{C2} = 820\,\text{min}$. The limiting cases for the counting rate from isotope C are $R_{C1}(t) = 200\,\mathrm{e}^{-t/1062}$ and $R_{C2}(t) = 400\,\mathrm{e}^{-t/820}$ c.p.m.

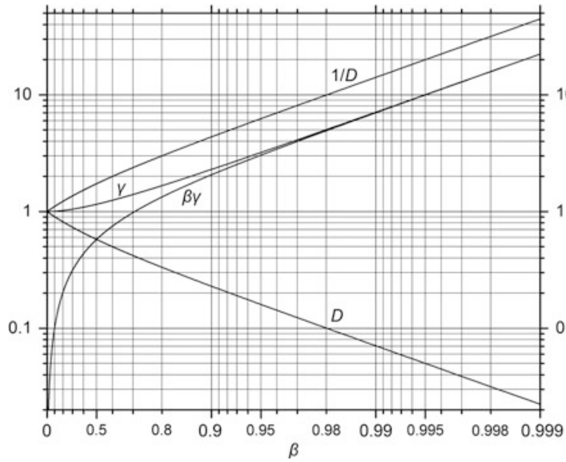
Subtracting the straight lines C1 and C2 from the total $R(t)$, we have the curves AB1 and AB2, which are the limits within which $R_A(t) + R_B(t)$ lies. The tangents to these curves at high times give the straight lines B1 and B2 for isotope B. The equations of these lines are found to be $R_{B1}(t) = 2900\,\mathrm{e}^{-t/245}$ and $R_{B2}(t) = 1700\,\mathrm{e}^{-t/341}$.

Subtracting the sums B1 + C1 and B2 + C2 from the total $R(t)$, we have the curves A1 and A2, which are supposed to be giving the counting rate from isotope A. These curves are far from being straight. The method is seen to fail for the analysis of a curve with counting rates from three isotopes.

Example 12.21 [O]

In the special theory of relativity, we are interested in the quantities $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta\gamma$, $D = \sqrt{\frac{1-\beta}{1+\beta}}$ and $1/D = \sqrt{\frac{1+\beta}{1-\beta}}$, especially at values of the speed approaching that of light in vacuum, c, since then the reduced speed $\beta = \upsilon/c$ approaches unity and some of these quantities diverge. We wish to find suitable scales that will show the detail in the above quantities at values of β approaching unity.

We plot these quantities on a logarithmic scale from 0.02 to 50, as functions of the variable $1 - \beta$, for values of β increasing from 0 to 0.999. We use a logarithmic scale for $1 - \beta$, but instead of setting the range of this from 0.001 to 1 we invert this scale by setting the range to be from 1 to 0.001. This means that the leftmost point of the X-axis will correspond to the value $\beta = 0$ and the one on the extreme right will correspond to $\beta = 0.999$, increasing from left to right. On the X-axis, instead of giving the values of $1 - \beta$, we give those of β. This means that the values of β are not entered automatically but have to be inserted 'by hand'. The results are shown in the graph that follows.



It is seen that, due to the method used, the scale for β can be made as detailed as we like for $\beta \to 1$, by adding more decades to the logarithmic scale of $1 - \beta$.

Programs
Excel
Ch. 12. *Excel—Column and Label*
Ch. 12. *Excel—Histogram*
Ch. 12. *Excel—Scatter Plot—Linear Scales*
Ch. 12. *Excel—Scatter Plot—Linear Scales—Errors*
Ch. 12. *Excel—Scatter Plot—Linear Scales—Multiple*
Ch. 12. *Excel—Scatter Plot—Linear-Log Scales*
Ch. 12. *Excel—Scatter Plot—Linear-Log Scales—Errors*
Ch. 12. *Excel—Scatter Plot—Linear-Log Scales—Multiple*
Ch. 12. *Excel—Scatter Plot—Log-Linear Scales*
Ch. 12. *Excel—Scatter Plot—Log-Linear Scales—Errors*
Ch. 12. *Excel—Scatter Plot—Log-Linear Scales—Multiple*
Ch. 12. *Excel—Scatter Plot—Log-Log Scales*
Ch. 12. *Excel—Scatter Plot—Log-Log Scales—Errors*
Ch. 12. *Excel—Scatter Plot—Log-Log Scales—Multiple*

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Programs
Origin
Ch. 12. *Origin—Column Plot* *Ch.* 12. *Origin—Column and Label* *Ch.* 12. *Origin—Histogram* *Ch.* 12. *Origin—Scatter Plot—Linear Scales* *Ch.* 12. *Origin—Scatter Plot—Linear Scales—Errors* *Ch.* 12. *Origin—Scatter Plot—Linear Scales—Multiple* *Ch.* 12. *Origin—Scatter Plot—Linear Scales—Weights* *Ch.* 12. *Origin—Scatter Plot—Linear Scales—Errors—Weights* *Ch.* 12. *Origin—Scatter Plot—Linear Scales—Errors—Weights from Errors* *Ch.* 12. *Origin—Scatter Plot—Linear-Log Scales* *Ch.* 12. *Origin—Scatter Plot—Linear-Log Scales—Errors* *Ch.* 12. *Origin—Scatter Plot—Linear-Log Scales—Multiple* *Ch.* 12. *Origin—Scatter Plot—Linear-Log Scales—Weights* *Ch.* 12. *Origin—Scatter Plot—Linear-Log Scales—Errors—Weights* *Ch.* 12. *Origin—Scatter Plot—Linear-Log Scales—Errors—Weights from Errors* *Ch.* 12. *Origin—Scatter Plot—Log-Linear Scales* *Ch.* 12. *Origin—Scatter Plot—Log-Linear Scales—Errors* *Ch.* 12. *Origin—Scatter Plot—Log-Linear Scales—Multiple* *Ch.* 12. *Origin—Scatter Plot—Log-Linear Scales—Weights* *Ch.* 12. *Origin—Scatter Plot—Log-Linear Scales—Errors—Weights* *Ch.* 12. *Origin—Scatter Plot—Log-Linear Scales—Errors—Weights from Errors* *Ch.* 12. *Origin—Scatter Plot—Log-Log Scales* *Ch.* 12. *Origin—Scatter Plot—Log-Log Scales—Errors* *Ch.* 12. *Origin—Scatter Plot—Log-Log Scales—Multiple* *Ch.* 12. *Origin—Scatter Plot—Log-Log Scales—Weights* *Ch.* 12. *Origin—Scatter Plot—Log-Log Scales—Errors—Weights* *Ch.* 12. *Origin—Scatter Plot—Log-Log Scales—Errors—Weights from Errors* *Ch.* 12. *Origin—Polar Diagram* *Ch.* 12. *Origin—Polar Diagram—Errors* *Ch.* 12. *Origin—Polar Diagram—Weights* *Ch.* 12. *Origin—Polar Diagram—Errors—Weights* *Ch.* 12. *Origin—Polar Diagram—Errors—Weights from Errors*
Python
Ch. 12. *Python—Histogram* *Ch.* 12. *Python—Column and Label* *Ch.* 12. *Python—Scatter Plot—Linear Scales* *Ch.* 12. *Python—Scatter Plot—Linear Scales—Errors* *Ch.* 12. *Python—Scatter Plot—Linear Scales—Multiple* *Ch.* 12. *Python—Scatter Plot—Linear-Log Scales* *Ch.* 12. *Python—Scatter Plot—Linear-Log Scales—Errors* *Ch.* 12. *Python—Scatter Plot—Linear-Log Scales—Multiple* *Ch.* 12. *Python—Scatter Plot—Log-Linear Scales* *Ch.* 12. *Python—Scatter Plot—Log-Linear Scales—Errors* *Ch.* 12. *Python—Scatter Plot—Log-Linear Scales—Multiple* *Ch.* 12. *Python—Scatter Plot—Log-Log Scales* *Ch.* 12. *Python—Scatter Plot—Log-Log Scales—Errors* *Ch.* 12. *Python—Scatter Plot—Log-Log Scales—Multiple*

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Programs
R
Ch. 12. *R—Histogram*
Ch. 12. *R—Column and Label*
Ch. 12. *R—Scatter Plot—Linear Scales*
Ch. 12. *R—Scatter Plot—Linear Scales—Errors*
Ch. 12. *R—Scatter Plot—Linear Scales—Multiple*
Ch. 12. *R—Scatter Plot—Linear-Log Scales*
Ch. 12. *R—Scatter Plot—Linear-Log Scales—Errors*
Ch. 12. *R—Scatter Plot—Linear-Log Scales—Multiple*
Ch. 12. *R—Scatter Plot—Log-Linear Scales*
Ch. 12. *R—Scatter Plot—Log-Linear Scales—Errors*
Ch. 12. *R—Scatter Plot—Log-Linear Scales—Multiple*
Ch. 12. *R—Scatter Plot—Log-Log Scales—Errors*
Ch. 12. *R—Scatter Plot—Log-Log Scales*
Ch. 12. R—*Scatter Plot—Log-Log Scales—Multiple*

Problems

12.1 **[E.O.P.R.]** The position y of a moving body is given as a function of time in the table that follows:

t (s)	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
y (m)	1.1	2.6	4.1	4.4	7.1	7.6	9.1	10.0	10.5

Mark the points (t, y) in a graph.
Make sure that a straight line would adequately describe the relation $y(t)$ and that there are no points that should be rejected as due to measurements with excessive errors.
Assume that a relation of the form $y = \alpha + \lambda t$ holds for the variables y and t.
Complete a table and use the method of least squares in order to determine the parameters α, λ and their errors $\delta\alpha$ and $\delta\lambda$.
Mark the point K:$(\bar{t},\ \bar{y})$ on the graph.
Draw the following straight lines on the graph:

1. the straight line $y = \alpha + \lambda t$, using the values found for α and λ
2. the straight line passing through point K and having a slope equal to $\lambda - \delta\lambda$
3. the straight line passing through point K and having a slope equal to $\lambda + \delta\lambda$.

12.2 **[E.O.P.R.]** Given in the table that follows, as a function of time t, are the measurements of the distance y of a body from the origin, as this moves on the y-axis.

t (s)	1.0	1.5	1.8	2.0	2.2	2.5
y (m)	30	50	68	73	100	110

(a) Mark the points (t, y) on a graph. Suppose that the values of t are known with errors equal to ± 0.2 s and that the values of y have fractional errors equal to $\pm 7\%$ and draw the errors $\pm\delta t$ and $\pm\delta y$ for each point. Investigate whether there are any points that you think should be neglected as the results of wrong measurements. Draw with a pencil the best (smooth) curve passing between the points of the graph (perfection is desirable but unattainable!).

(b) Assume that for y and t the relation $y = y_0 + \frac{1}{2}\gamma t^2$ holds. Construct a table in which you record the values of t, y and the corresponding values of the variable $x = t^2$. Evaluate also and enter in the table the errors δt, δy and δx in these magnitudes.

t (s)	δt (s)	y (m)	δy (m)	$x = t^2$ (s^2)	δx (s^2)

Mark on a graph the points (x, y) and draw their errors $\pm\delta x$ and $\pm\delta y$. Verify that the relation $y(x)$ is linear to a good enough approximation and draw the best in your judgment straight line between the points of the graph. Without marking anything on the graph, find the coordinates of two points of the straight line (geometrical points, *not* experimental) and evaluate its slope. For better results, these points must be at a distance between them which is as large as possible. From the intercept of the line with the y-axis and from its slope, determine the values of y_0 and γ.

[E.O.P.R.] Use the methods of non-linear curve fitting to fit a curve of the form $y = y_0 + \frac{1}{2}\gamma t^2$ to the experimental points.

12.3 **[E.O.P.R.]** Given in the table that follows, as a function of time t, are the measurements of the distance y of a body from the origin, as this moves on the y-axis.

t (s)	1.0	2.0	3.0	4.0	5.0	6.0
y (m)	1.4	4.0	7.2	12.0	17.0	24.0

(a) Mark on a graph the points (t, y). Assume that the values of t are known with great accuracy but in the values of y there are fractional errors equal to $\pm 7\%$ and draw the errors $\pm\delta y$ for all the points. Investigate whether there are any points that you think should be neglected as the results of wrong measurements. Draw with a pencil and by hand the best (smooth) curve passing between the points of the graph.

(b) Assume that the theoretical relation between y and t is the expression $y = \upsilon_0 t + \frac{1}{2}\gamma t^2$. Make a table in which you record the values of t and y and the corresponding values of the variable $z = y/t$.

t (s)	y (m)	$z = y/t$ (m/s)

Plot in a graph the points (t, z). Satisfy yourselves that the function $z(t)$ is linear. Draw the best, in your opinion, straight line that passes between these points. Determine the coordinates of two points of the line (geometrical points, *not* experimental) and evaluate its slope. From the intercept of the line with the z-axis and its slope determine the values of υ_0 and γ.

12.4 **[E.O.P.R.]** A mass m is connected to the free end of a spring which has a constant k, and the other end of which is fixed. Theory predicts that the motion of the mass, if this is displaced from its equilibrium point and let free, is simple harmonic, with a period equal to $T = 2\pi\sqrt{m/k}$.

The table below shows the values of T for various masses m connected to the same spring.

m (kg)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
T (s)	1.00	1.50	1.80	2.05	2.15	2.55	2.70

(a) Mark the points (m, T) on a graph. Assume that the values of m are known with great accuracy and that the values of T have fractional errors equal to $\pm 5\%$ and draw the errors $\pm \delta T$ of each point. Investigate whether there are any points that you think should be neglected as the results of wrong measurements. Draw with a pencil and by hand the best (smooth) curve passing between the points of the graph.

(b) Use the theoretical relation $T = 2\pi\sqrt{m/k}$ for the determination of the constant k as follows: From the relation it follows that $T^2 = (4\pi^2/k)\, m$. Construct a table in which you record the values of m and T and the corresponding values of the variable $y = T^2$.

m (kg)	T (s)	$y = T^2$ (s^2)

Mark in a graph the points $(m, y = T^2)$ and the best, in your opinion, straight line passing between them. Read off the straight line the coordinates of two points (geometrical points, *not* experimental) and evaluate the slope of the straight line. From the value of the slope evaluate the constant k.

Reference

1. The Royal Society, *Quantities, Units, and Symbols*, 1975