Undergraduate Lecture Notes in Physics

Costas Christodoulides
George Christodoulides

Analysis and Presentation of Experimental Results

With Examples, Problems and Programs

Springer

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Code and examples for the book "Analysis and Presentation of Experimental Results": https://github.com/aperbook/APER.

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Preface

The importance of the analysis and the presentation of experimental data cannot be overstated. Good experimental results may be rendered useless by failing to keep to certain rules in their presentation, either to an audience or in written form. It is our intention to present in this book these methods at an introductory university level. Those working in the experimental sciences, but also anyone involved in the analysis of numerical data, may find the book useful.

The book is intended to be used as a textbook and this has determined its characteristics: The theoretical proofs are given in considerable detail, many figures are used, as well as a large number of examples and problems to be solved by the reader. The vast majority of the examples are solved using four software packages: Excel®, Origin®, Python and R. Most of the problems may also be solved using these programs. Excel® is used due to its wide availability as a program for data analysis, Origin® because it is an excellent program for creating graphical presentations of data. Python and R are used because they are free, open-source programming languages, widely used in data science. Reference to these programs is made using the symbols [E], [O], [P] and [R], respectively. The same symbols also indicate that a certain problem may be solved using the corresponding program.

The book may be used as a textbook for an introductory course on Data Analysis and Presentation. It is hoped that it will provide a useful addition to the existing literature.

Athens, Greece
January 2017

Costas Christodoulides
George Christodoulides

Contents

Instructions on the Use of the Programs and Code Samples

This book contains more than a hundred programs and code samples that demonstrate how to analyse and present experimental data using Excel®, Origin®, R and Python. You can download all these programs (or the ones you're interested in) from https://github.com/aperbook/APER. The companion website is organized by software package, and then by book chapter. Additional programs and scripts, including extensions to existing programs, which could not be included in the book for space reasons, are available on the website.

The programs have been tested on the Windows®, Mac® and Linux platforms. The Microsoft Excel® spreadsheets have been created using version 2016, and will work with subsequent versions of the software. The OriginLab® Origin® files have been created using version 2016 Pro of the software. Both Microsoft Excel (products.office.com/en/excel) and OriginLab Origin (www.originlab.com/Origin) are proprietary software, and you will need a licensed version in order to open the files provided with this book. The licensing costs will depend on the number of copies of the software you wish to install; there are also academic and institutional licenses available. The Excel spreadsheets can also be opened using an open-source, free package such as LibreOffice Calc (www.libreoffice.org).

In recent years, open-source software projects have changed the landscape of scientific and data analysis software. We have chosen to focus on two of the most popular solutions, the R Project for Statistical Computing and the Python general-purpose programming language.

R is a special-purpose programming language and an environment for statistical computing. It is freely available (www.r-project.org) under the GNU General Public License. A very large number of contributed packages extend R's functionality and can be automatically downloaded through the CRAN archive. To make the most out of the R code provided with this book, we suggest that you install R along with the RStudio graphical user interface, which is also free and open-source. You have to install R before installing RStudio. To install R, visit https://cran.rstudio.com/ and chose your platform (Windows, Mac or Linux). You only need to install the "base" distribution to run the code in this book. Binary distributions (executable

files) are available for Windows and Mac. Linux users will usually install R using their operating system's package manager (e.g. the "apt-get" command in Ubuntu). After installing R, visit https://www.rstudio.com/products/rstudio/download3/ and select the appropriate installer for your platform. At the time of writing, the current version of R is 3.3.1 and the current version of RStudio is 0.99.903.

Python is a general-purpose, interpreted programming language (www.python.org). Python interpreters are available for several platforms, allowing the same platform code to run on different operating systems. Three major Python projects provide support for scientific programming, data analysis and graphics: NumPy (www.numpy.org), SciPy (www.scipy.org) and Matplotlib (www.matplotlib.org). In order to use the code provided in this book we suggest that you install a complete Python distribution, containing all the necessary packages. The Anaconda distribution has integrated over 720 packages to cover the needs of data scientists using Python. Download the distribution from https://www.continuum.io/downloads by choosing your platform (Windows, Mac or Linux, and 32- or 64-bit). The code in this book has been tested to work with both version 2.x and version 3.x of Python; however, we suggest that you download the installers for the latest version (Python 3.5). After installing the Anaconda distribution of Python, you can launch the IPython Qt Interactive Console, and run the code provided with this book interactively.

Reference to these programs is made using the symbols [E], [O], [P] and [R], for Excel, Origin, R and Python, respectively. The same symbols also indicate that a certain problem may be solved using the corresponding program.

General Bibliography

J. Topping, *Errors of Observation and their Treatment* (Springer, 4th ed., 1972).

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J.R. Taylor, *An Introduction to Error Analysis* (*The Study of Uncertainties in Physical Measurements*) (University Science Books, Sausalito, California, 2nd ed. 1997).

L. Kirkup, *Data Analysis with Excel®* (*An Introduction for Physical Scientists*) (Cambridge University Press, 2002).

Chapter 1
Basic Statistical Concepts

In order to understand the theory of errors, as well as the methods of analysis and presentation of experimental results, knowledge of the basic concepts of Probability Theory and Statistics is required. The purpose of this chapter is the concise presentation of the basic concepts from these fields that will be used in the book. Obviously, the treatment will not be in great detail and the reader should consult the relevant bibliography for this purpose. Those readers who have a satisfactory knowledge of these subjects may omit this chapter and consult it whenever necessary.

1.1 Tables, Distributions and Histograms

Let us assume that we have performed N measurements x_i $(i = 1, 2, \ldots, N)$ of a magnitude **x**, keeping the experimental conditions constant as far as possible. These measurements may be considered to be a sample which was taken from a parent population of the infinite number of measurements of this magnitude that may be performed under the particular conditions. In Table 1.1, the results of 100 such measurements have been recorded. This is the form in which the results appear immediately after the measurements, although the number of measurements is very rarely as high as this, except perhaps when they are acquired by an automated method.

If the number of the measurements is large enough, there may be values which appear more than once. If we count the frequency of appearance of each value, we may construct a table of this number, n_r, as a function of the numerical value x_r of the measurement, such as Table 1.2. In this table, the values x_r have been recorded in order of increasing magnitude and the number r is given in the first column of the table. The value of n_r is the *frequency* of appearance of the r-th value. Obviously,

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Table 1.1 100 measurements x_i of the magnitude **x** (in mm)

i	x_i	i	x_i	i	x_i	i	x_i	i	x_i
1	9.5	*21*	9.9	*41*	10.2	*61*	9.9	*81*	10.1
2	10.3	*22*	10.2	*42*	9.9	*62*	10.2	*82*	9.5
3	10.1	*23*	9.9	*43*	10.1	*63*	9.5	*83*	9.6
4	9.9	*24*	10.2	*44*	9.6	*64*	9.6	*84*	10.4
5	10.0	*25*	10.0	*45*	10.0	*65*	10.2	*85*	10.2
6	10.3	*26*	9.9	*46*	9.9	*66*	9.5	*86*	9.9
7	9.7	*27*	10.1	*47*	10.0	*67*	10.1	*87*	9.6
8	10.0	*28*	9.7	*48*	9.7	*68*	10.0	*88*	9.7
9	10.3	*29*	10.1	*49*	10.3	*69*	9.7	*89*	9.8
10	9.8	*30*	10.0	*50*	9.8	*70*	9.8	*90*	10.2
11	10.2	*31*	9.9	*51*	10.0	*71*	9.8	*91*	10.4
12	9.8	*32*	10.4	*52*	9.8	*72*	10.3	*92*	9.9
13	10.2	*33*	10.3	*53*	10.0	*73*	9.9	*93*	10.3
14	10.2	*34*	10.4	*54*	10.3	*74*	10.4	*94*	9.9
15	9.7	*35*	9.6	*55*	10.1	*75*	10.0	*95*	9.5
16	10.1	*36*	9.7	*56*	9.7	*76*	10.1	*96*	10.0
17	10.1	*37*	10.0	*57*	10.4	*77*	9.6	*97*	10.0
18	10.2	*38*	9.8	*58*	10.0	*78*	10.4	*98*	9.8
19	9.8	*39*	10.1	*59*	10.0	*79*	9.6	*99*	9.9
20	10.4	*40*	9.9	*60*	10.1	*80*	10.2	*100*	10.3

Table 1.2 The values of Table 1.1 grouped in classes

r	x_r (mm)	n_r	f_r
1	9.5	5	0.05
2	9.6	7	0.07
3	9.7	8	0.08
4	9.8	10	0.10
5	9.9	14	0.14
6	10.0	15	0.15
7	10.1	12	0.12
8	10.2	12	0.12
9	10.3	9	0.09
10	10.4	8	0.08
	$\Sigma =$	100	1

it is $\Sigma n_r = N$. Table 1.2 is said to give the *frequency distribution* of the measurements. If the frequencies n_r are divided by the total number of measurements N, the *relative frequencies* f_r of the values x_r are obtained and these are also given in the table. By definition, it is $\Sigma f_r = 1$. Due to this fact, f_r is said to be *normalized*.

If the variable **x** can only take discrete values (e.g. number of objects, exam grades between 1 and 10 etc.), the distribution is said to be *discrete*. If **x** varies in a

Table 1.3 The values of Table 1.1 grouped in classes of width 0.2 mm

Class (mm)	n_r	f_r
9.40–9.59	5	0.05
9.60–9.79	15	0.15
9.80–9.99	24	0.24
10.00–10.19	27	0.27
10.20–10.39	21	0.21
10.40–10.59	8	0.08
$\Sigma =$	100	1

continuous manner (e.g. length, time, temperature etc.), the distribution is said to be *continuous*.

In many cases, the N values x_i are grouped together in a certain number of *classes*, which have *boundaries* and *width* Δx, also known as *bin width*. For example, the measurements of Table 1.1 may be distributed in classes of width $\Delta x = 0.2$ mm, having the following boundaries (in mm): 9.40–9.59, 9.60–9.79, 9.80–9.99, 10.00–10.19, 10.20–10.39, 10.40–10.59. The result is shown in Table 1.3.

The boundaries of the classes must be stated in a clear manner, in order to avoid mistakes during the distribution of the values in the classes. In Table 1.3 the boundaries of the classes are given with two decimal points, while the values are given with one. Another way of determining the boundaries of the classes is the following:

$$[9.4-9.6) \quad [9.6-9.8) \quad [9.8-10.0) \quad [10.0-10.2) \quad [10.2-10.4) \quad [10.4-10.6)$$

where the symbol [9.4 or 9.4], for example, means that in both cases the value 9.4 is included in the class, whereas the symbol (9.6 or 9.6) means that the value 9.6 is not included. A third way to write the class boundaries is:

$$9.4-9.6^- \quad 9.6-9.8^- \quad 9.8-10.0^- \quad 10.0-10.2^- \quad 10.2-10.4^- \quad 10.4-10.6^-$$

where by writing 9.4 as a boundary we mean that the value 9.4 is included in the class, whereas by writing 9.6^- we mean that the value 9.6 is not included.

If the points (x_r, n_r) are plotted in a diagram such as the one in Fig. 1.1. we have a *frequency polygon*. If we plot the points (x_r, f_r), we have a *relative frequency polygon* (Fig. 1.1, right axis).

The frequency (or the relative frequency) of the classes may be used in the drawing of a *histogram* of the measurements, as follows: The x-axis is divided into regions which correspond to the classes of the distribution of the values and in each region a column is raised, having a height proportional to the frequency x_r (or the relative frequency f_r). The result is a *frequency histogram* (or a *histogram of relative frequencies*, respectively). Shown in Fig. 1.2 is the frequency histogram and the relative frequency histogram of the values of Table 1.3, in which the width of the classes is $\Delta x = 0.2$ mm.

Obviously, we need as many columns as possible in a histogram, so that we detect any fine structure it may possess. At the same time, we need a large enough

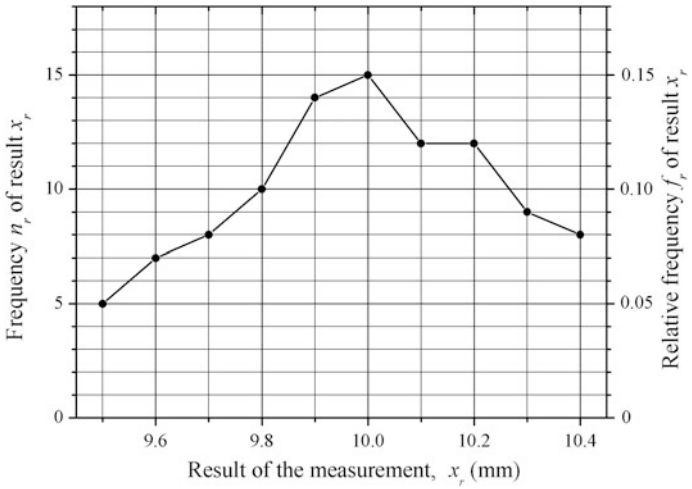


Fig. 1.1 A polygon of the frequencies n_r and of the relative frequencies f_r of the 100 measurements of Table 1.1

number of measurements in the majority of the columns, so as to have a 'smooth statistical behavior'. To achieve this, the width of the classes may be decided using the following practical rule:

> If N is the number of the values to be recorded in the histogram and R is the range of these values, then the width of the classes is chosen approximately equal to $R/\sqrt{N}$. Obviously this figure is suitably rounded.

As an example, if we have $N = 100$ measurements, the values of which have a range of $R = 32$ mm, the suggested value for the width of the classes is $\Delta x = 32/\sqrt{100} = 3.2$ mm, which should be rounded to 3 mm.

The values given by this rule may not always be used since the value of Δx is decided after the particular facts of a situation are taken into account. For example, it is usually desirable to have classes with widths equal to 1, 2 or 5 units, their multiples or their sub-multiples.

Example 1.1 [E]

Given 100 random numbers, use Excel® in order to plot a histogram of them.

For the purposes of this example and using the random number generation functionality of Excel® we enter, in column A, 100 random numbers, normally distributed with a mean of 5 and a standard deviation of 1. The details need not concern us here. We will explain how this is done in later stages of the book. All we need to know now is that we have 100 numbers in column A, in cells A1–A100. As in Fig. 1.2, we will assume that the numbers given are the results of measurements of a length in mm. It should be noted that the histogram is to be drawn using the raw data of the measurements, and not after they have been counted and grouped in classes as in Table 1.3.

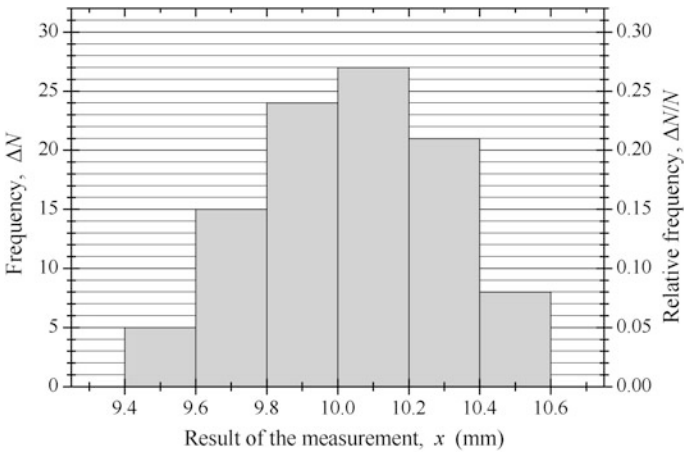


Fig. 1.2 The frequency histogram and the histogram of the relative frequencies of the measurements of Table 1.3. The width of the classes is $\Delta x = 0.2$ mm

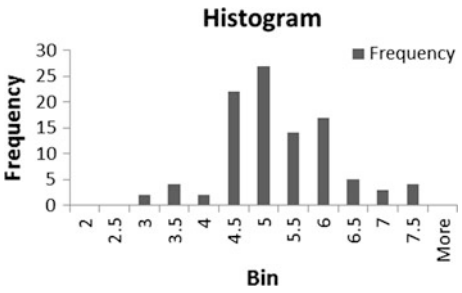
We note that the numbers lie within the range of 2.5 and 7.5 mm. In preparation for plotting the histogram we enter the numbers 2.5, 3, 3.5, …, 6, 6.5, 7, 7.5 in cells E1–E11. These will be the limits of the bins of the histogram. The bin width is thus taken to be equal to $\Delta x = 0.5$ mm.

In the window **Data** we open **Data Analysis**. This is an Add-in which has to be installed if it is not already installed (To install, follow: **File**, **Options**, **Add-ins**, **Analysis ToolPak**, **OK**). We open the **Histogram** dialog box.

With the cursor in **Input Range**: we move the cursor to A1 and, with the left hand mouse button held down, we drag the cursor to cell A100. When the mouse button is released, the range $A$1:$A$100 appears in the **Input Range**: box.

With the cursor in **Bin Range**: we move the cursor to E1 and, with the left hand mouse button held down, we drag the cursor to cell E11. When the mouse button is released, the range $E$1:$E$11 appears in the **Bin Range**: box.

We tick the **Chart Output** and press **OK**. The table giving the Frequency for each of the bins and the Histogram shown below appear.



Bin	Frequency
2	0
2.5	0
3	2
3.5	4
4	2
4.5	22
5	27
5.5	14
6	17
6.5	5
7	3
7.5	4
More	0

We will now modify the histogram.

We delete the words **Frequency** and **Histogram**. We may give a new title to the histogram.

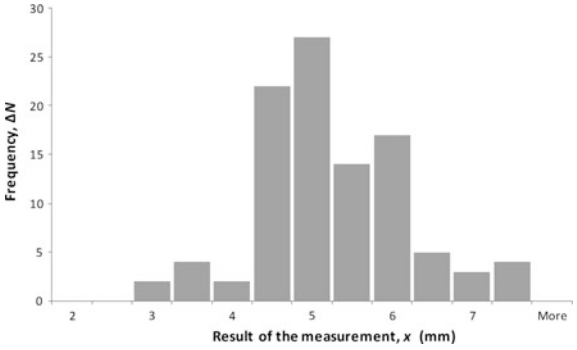
We change the X-label by double-clicking on the existing word Bin and changing it to **Result of the measurement**, **Δ*x* (mm)**. Similarly, we change the Y-label to **Frequency**, **Δ*N***.

Double-click on the *x*-axis and open the **Format Axis** dialog box. Then open **Labels > Specify interval unit** and set value to 2. The Δ*N* axis is satisfactory.

Double-click on one of the histogram's bars and open the **Format Data Series** dialog box. At **Series Options**, set the **Gap Width** to **10%**. This sets the empty gap between the columns to 10% of the bin width.

Clicking on the **Fill and Line** icon, , we open the **Fill**, **Border** dialog box. At **Fill > Color** we set the color to gray.

The final histogram is shown in the following figure.



Example 1.2 [O]

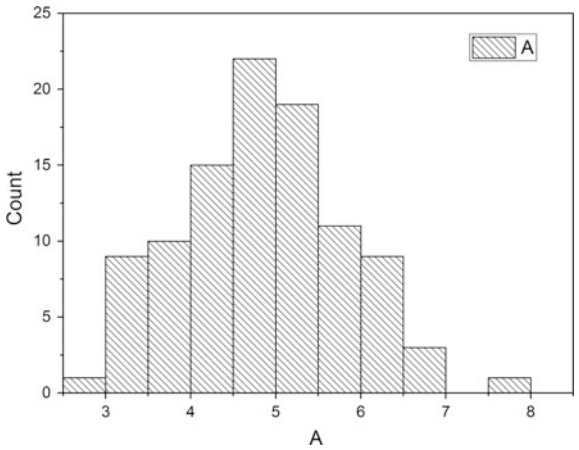
Given 100 random numbers, use Origin® in order to plot a histogram of them.

For the purpose of this example and using the random number generation functionality of Origin® we enter, in column A, 100 random numbers, normally distributed with a mean of 5 and a standard deviation of 1. The details need not concern us here. We will explain how this is done in later stages of the book. All we need to know now is that we have 100 numbers in column A. As in Fig. 1.2, we will assume that the numbers given are the results of measurements of a length in mm. It should be noted that the histogram is to be drawn using the raw data of the measurements, not after they have been counted and grouped in classes as in Table 1.3.

We highlight column A by left-clicking on label A. Then press

Plot > Statistics > Histogram

The histogram shown in the adjacent figure appears. We will modify this histogram.



1. Delete the text box containing A by left-clicking on the rectangle and then pressing **Delete**.
2. We change the color of the bars by double-clicking on one of them and in the window that opens pressing

   Pattern > Fill > Color > LT Gray

3. In the same window, we change the gap between the bars

   Spacing > Gaps Between Bars (in %) > 5

4. We change the labels of the axes: Double-click on the X-label and write **Result of the measurement, Δx (mm)**. Then Double-click on the Y-label and write **Frequency**, ΔN.
5. We change the tick labels of the axes by double-clicking on one of them and then

   Scale > Horizontal > From 2 to 8.5 > Major Ticks, Value 1 > Minor Ticks, Count 1

   Scale > Vertical > From 0 to 25 > Major Ticks, Value 5 > Minor Ticks, Count 4

   Press **Apply**.
   We want to show the tick labels on the right axis as well, so

   Tick Labels > Right > Tick **Show**

   Press **OK**.
6. We now draw the grid lines:

   Grids > Horizontal > Major Grid Lines, Tick **Show, Color Black, Style Solid, Thickness 0.5 >**

   > Minor Grid Lines, Tick **Show, Color Black, Style Solid, Thickness 0.5**

   Press **OK**. We will not draw vertical grid lines.
7. We modify Lines and Ticks:

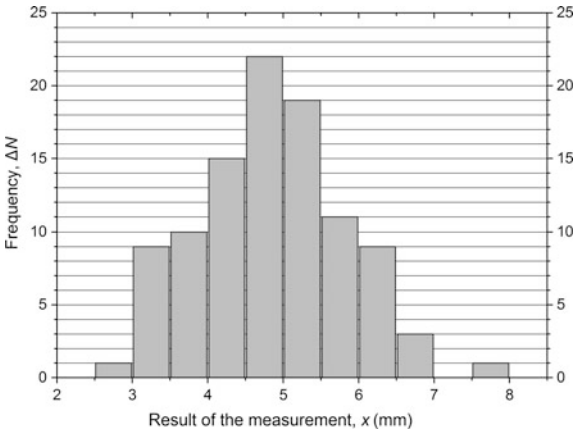
   Lines and Ticks > Top > Tick **Show Line and Ticks > Major Ticks,** Tick **Style In > Minor Ticks,** Tick **Style In**

   Then

   Lines and Ticks > Right > Tick **Show Line and Ticks > Major Ticks,** Tick **Style Out > Minor Ticks,** Tick **Style Out**

   Press **OK**. We will not draw vertical grid lines. The final histogram is shown in the figure below.

   We export the graph for use as a jpg figure:
   File > Export Graphs > Open Dialog… > Image Type, jpg > File Name, give name **> Path**, give directory in which figure is to be stored **> OK**



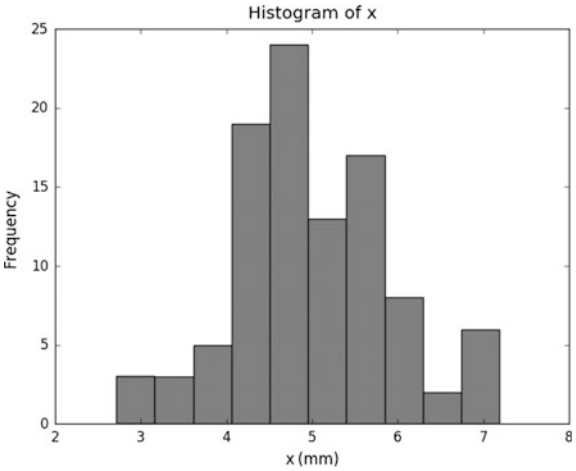
Example 1.3 [P]

Using Python, plot the histogram of the 100 random numbers of Example 1.1 [E].

```
# We import the Python modules necessary for data analysis and
# graphics (numpy and matplotlib).

import numpy as np
import matplotlib.pyplot as plt

# We form a vector x with components the 100 random numbers of
#Example 1.1 [E].

x = np.array([4.78887, 4.17608, 4.08729, 6.26209, 4.52593, 4.62893, 5.90426,
5.75323, 4.07328, 5.63953, 4.91056, 4.73250, 5.72151, 4.24848, 4.90887, 4.67623,
5.90012, 4.65317, 4.79153, 5.45268, 5.30632, 4.26952, 5.65727, 5.18226, 4.38460,
4.54494, 4.00306, 5.97430, 4.22142, 4.96759, 7.08994, 4.81065, 4.53098, 4.86568,
5.47072, 4.96407, 7.11077, 6.69075, 5.16618, 6.30068, 5.29543, 3.61960, 2.94918,
4.22886, 2.71753, 5.54133, 4.13987, 4.10216, 5.91200, 5.47286, 5.15642, 5.56657,
5.77732, 5.83059, 4.28017, 5.15317, 6.12074, 6.88621, 4.91502, 4.86923, 7.11721,
5.74695, 4.20977, 6.94411, 4.69592, 5.76876, 4.21269, 3.37941, 4.93466, 5.82413,
4.82099, 4.07058, 4.02249, 4.94776, 4.45813, 4.43090, 4.65317, 4.74413, 5.48435,
5.03837, 6.11916, 3.35769, 7.19028, 5.31314, 4.23431, 6.12275, 4.03023, 4.59844,
5.67686, 5.74261, 4.48146, 4.97754, 4.75857, 4.30989, 4.77602, 3.41128, 5.17985,
5.09158, 3.14031, 3.74785])
```

```
# The histogram is produced with the following set of commands.

plt.hist(x, bins=10, align='mid', facecolor='grey')
plt.xlim(2, 8)
plt.xlabel('x (mm)')
plt.ylabel('Frequency')
plt.title('Histogram of x')
plt.show()
```

The histogram produced is shown here.



Example 1.4 [R]

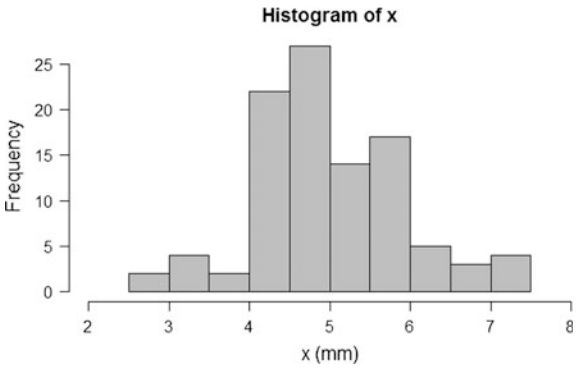
Using R, plot the histogram of the 100 random numbers of Example 1.1 [E].

We form a vector x with components the numbers of Example 1.1 [E]:

```
> x <- c(4.78887, 4.17608, 4.08729, 6.26209, 4.52593, 4.62893, 5.90426, 5.75323,
4.07328, 5.63953, 4.91056, 4.73250, 5.72151, 4.24848, 4.90887, 4.67623, 5.90012,
4.65317, 4.79153, 5.45268, 5.30632, 4.26952, 5.65727, 5.18226, 4.38460, 4.54494,
4.00306, 5.97430, 4.22142, 4.96759, 7.08994, 4.81065, 4.53098, 4.86568, 5.47072,
4.96407, 7.11077, 6.69075, 5.16618, 6.30068, 5.29543, 3.61960, 2.94918, 4.22886,
2.71753, 5.54133, 4.13987, 4.10216, 5.91200, 5.47286, 5.15642, 5.56657, 5.77732,
5.83059, 4.28017, 5.15317, 6.12074, 6.88621, 4.91502, 4.86923, 7.11721, 5.74695,
4.20977, 6.94411, 4.69592, 5.76876, 4.21269, 3.37941, 4.93466, 5.82413, 4.82099,
4.07058, 4.02249, 4.94776, 4.45813, 4.43090, 4.65317, 4.74413, 5.48435, 5.03837,
6.11916, 3.35769, 7.19028, 5.31314, 4.23431, 6.12275, 4.03023, 4.59844, 5.67686,
5.74261, 4.48146, 4.97754, 4.75857, 4.30989, 4.77602, 3.41128, 5.17985, 5.09158,
3.14031, 3.74785)
```

```
> hist(x, main="Histogram of x", xlab="x (mm)", col="grey",
xlim=c(2, 8), las=1, breaks=7)
```

On pressing **ENTER** the histogram shown is produced.

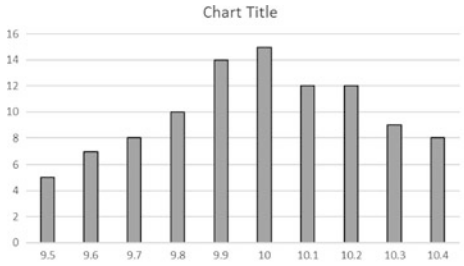


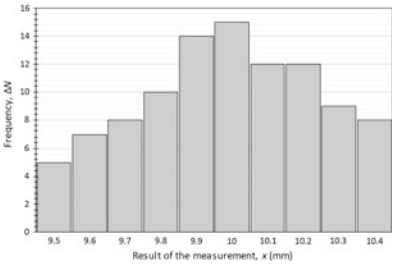
Example 1.5 [E]

Using Excel®, plot a bar chart of the data in Table 1.2.

We enter the data (x_r, n_r) in columns A and B respectively. We highlight the cells between A1 and B10.

We open the **Insert** window. From **Recommended Charts** we select the column chart. The result is shown in the figure on the left.





We modify the bar chart in the same way as we did in Example 1.X [E]. The final result is shown in the figure on the right.

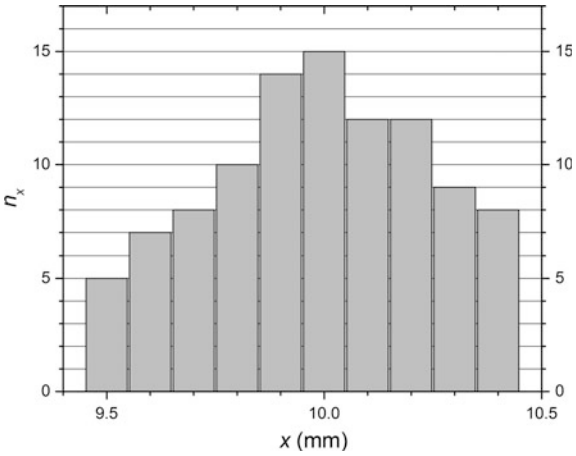
Example 1.6 [O]

Using Origin®, plot a bar chart of the data in Table 1.2.

We enter the data (x_r, n_r) in columns A and B respectively. We highlight columns A and B by left-clicking on label A and, holding the Shift or Control key down, left-clicking on label B. Then select

Plot > Column/Bar/Pie > Column

The bar chart of (x_r, n_r) appears. We modify this bar chart by following the procedure of Example 1.2 [O]. The final result is shown in the figure.



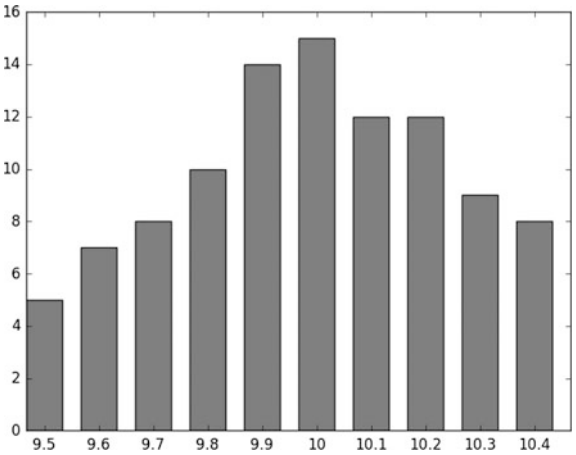
Example 1.7 [P]

Using Python, plot a bar chart of the data in Table 1.2.

The categories are stored in the vector `x` and the frequency data in the vector `y`. The width of the bar is defined in the variable `width`, and used in subsequent commands to center the x-axis labels to the bars.

```
import numpy as np
import matplotlib.pyplot as plt
x = (9.5, 9.6, 9.7, 9.8, 9.9, 10, 10.1, 10.2, 10.3, 10.4)
y = [5, 7, 8, 10, 14, 15, 12, 12, 9, 8]
ind = np.arange(len(y))
width = 1/1.5
fig, ax = plt.subplots()
plt.bar(ind, y, width, color='grey')
ax.set_xticks(ind + width/2)
ax.set_xticklabels(x, ha='center')
plt.show()
```

The histogram shown is produced.



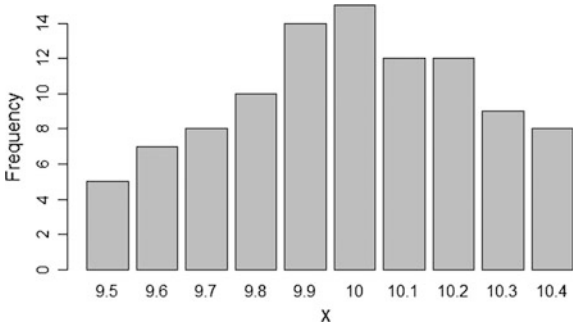
Example 1.8 [R]

Using R, plot a bar chart of the data in Table 1.2.

The categories of data are given in the second column, “9.5”, “9.6”, “9.7” etc. We form a vector, **x**, with these as components. We also form a vector, **Frequency**, with the frequencies of the x-values.

```
> xbarchart = data.frame(x=factor(c("9.5", "9.6", "9.7", "9.8", "9.9", "10",
  "10.1", "10.2", "10.3", "10.4")), levels = c("9.5", "9.6", "9.7", "9.8", "9.9",
  "10", "10.1", "10.2", "10.3", "10.4"), Frequency = c(5, 7, 8, 10, 14, 15, 12, 12,
  9, 8))
> barplot(xbarchart$Frequency, names = xbarchart$x, xlab="x", ylab="Frequency")
```

The chart shown on the right is produced.



1.2 Probability Density

If the height $\Delta N = n_r$ of each column of a histogram is divided by the total number of measurements, N, and by the width Δx of the column, the *probability density*

$$f(x) \equiv \frac{1}{N}\frac{\Delta N}{\Delta x} \tag{1.1}$$

is found. The same is achieved if the height of each column of a histogram of relative frequencies is divided by its width, Δx. The term *density* is used, because $f(x)$ gives the probability $\Delta N/N$ *per unit* x for a value to be found in the region of x. The units of $f(x)$ are obviously the inverse of the units of x. Due to its definition, if $f(x)$ is the probability density at a column of the histogram, then

$$f(x)\Delta x = \frac{\Delta N}{N} = (\text{fraction of the measurements in the particular column of the histogram}). \tag{1.2}$$

The sum $\Sigma f(x)\Delta x$ over all the columns of the histogram will be equal to unity:

$$\sum_r f(x)\Delta x = \frac{1}{N}\sum_r \Delta N = \frac{1}{N}N = 1. \tag{1.3}$$

The histogram of $f(x)$, therefore, has a total area equal to unity and is said to be *normalized*.

The sum $\Sigma f(x)\Delta x$ for the columns of the histogram between the limits of x_1 and x_2 gives the fraction of the values of x which lie between x_1 and x_2, or the probability for a value x of the magnitude $\mathbf{x}$ to lie between x_1 and x_2.

The shape of the histogram obviously depends on the width Δx of its classes. If we have a large number of values x of the magnitude $\mathbf{x}$, the width may be chosen to be very small and then the columns of the histogram tend to define a continuous function $f(x)$ for the probability density, which is known as the *probability density function* (also known as the *density function* or the *frequency function*, or, simply, the *density*) of the values x of the random variable $\mathbf{x}$. The graph of $f(x)$ is called (normalized) *frequency curve* (Fig. 1.3). The function $f(x)$, as probability per unit range, takes only positive values.

The normalization condition is now,

$$\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x = 1. \tag{1.4}$$

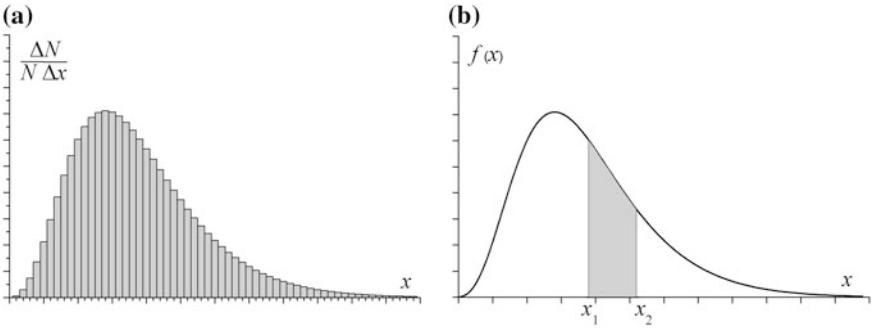


Fig. 1.3 The transition of a histogram **a** to a frequency curve $f(x)$ **b**, as the width of the classes tends to zero

Also, the partial sum of columns, mentioned above, takes the form

$$\int_{x_1}^{x_2} f(x)\mathrm{d}x \equiv \Pr\{x_1 \leq x < x_2\} \tag{1.5}$$

and gives the probability for a value of x to lie between x_1 and x_2. If the function $f(x)$ is proportional to a function $g(x)$, which is not normalized, then the normalized $f(x)$ is given by

$$f(x) = \frac{g(x)}{\int_{-\infty}^{\infty} g(x)\mathrm{d}x} \tag{1.6}$$

and

$$\Pr\{x_1 \leq x < x_2\} = \frac{\int_{x_1}^{x_2} g(x)\mathrm{d}x}{\int_{-\infty}^{\infty} g(x)\mathrm{d}x} \tag{1.7}$$

1.3 Mean Value

The *mean* of N measurements x_i $(i = 1, 2, \ldots, N)$ is defined as

$$\overline{x} \equiv \frac{1}{N}\sum_{i=1}^{N} x_i. \tag{1.8}$$

If the measurements can be grouped in a number of K classes in which the value x_r ($r = 1, 2, \ldots, K$) has a frequency n_r, then it will be $\sum_{i=1}^{N} x_i = \sum_{r=1}^{K} n_r x_r$ and the mean is given by

$$\overline{x} = \frac{1}{N} \sum_{r=1}^{K} n_r x_r. \tag{1.9}$$

Since $n_r/N = f_r$ is the relative frequency of the value x_r, this expression may also be written as

$$\overline{x} = \sum_{r=1}^{K} f_r x_r. \tag{1.10}$$

This quantity is also known as *expected value* or *mathematical expectation* $E(x)$ of the variable **x**, especially in the form of Eq. (1.10).

The weighted mean is examined in Sect. 9.4.

Example 1.9

Evaluate the mean of the measurements of Table 1.1, as these were grouped in Table 1.2.

We will find the result using both Eqs. (1.9) and (1.10), so we construct two tables with columns: r, x_r, n_r and f_r, for r = 1, 2, … 10. We evaluate the products $n_r x_r$ and $f_r x_r$ for each value of r.

r	x_r (mm)	n_r	$n_r x_r$ (mm)
1	9.5	5	47.5
2	9.6	7	67.2
3	9.7	8	77.6
4	9.8	10	98.0
5	9.9	14	138.6
6	10.0	15	150.0
7	10.1	12	121.2
8	10.2	12	122.4
9	10.3	9	92.7
10	10.4	8	83.2
	Σ =	100	998.4

r	x_r (mm)	f_r	$f_r x_r$ (mm)
1	9.5	0.05	0.475
2	9.6	0.07	0.672
3	9.7	0.08	0.776
4	9.8	0.10	0.980
5	9.9	0.14	1.386
6	10.0	0.15	1.500
7	10.1	0.12	1.212
8	10.2	0.12	1.224
9	10.3	0.09	0.927
10	10.4	0.08	0.832
	Σ =	1	9.984

The sum $\sum_{r=1}^{K} n_r x_r = 998.4$ mm gives $\overline{x} = \frac{1}{N} \sum_{r=1}^{K} n_r x_r = 9.984$ mm.

The sum $\sum_{r=1}^{K} f_r x_r = 9.984\,\text{mm}$ similarly gives $\overline{x} = \sum_{r=1}^{K} f_r x_r = 9.984\,\text{mm}$.

For reasons which we will examine later, it is more correct to round the result to one decimal digit, $\overline{x} = 10.0\,\text{mm}$.

Assume now that we have a continuous distribution with probability density $f(x)$. The proportion of measurements lying between the values of x and $x + \mathrm{d}x$ is equal to $f(x)\mathrm{d}x$. The sum $\sum f_r x_r$ for this region is equal to $f(x)x\mathrm{d}x$. For all the possible values of x, the mean is given by Eq. (1.10), which is now modified to

$$\overline{x} = \int_{-\infty}^{\infty} f(x)x\mathrm{d}x. \tag{1.11}$$

If x takes values only between some limits, then these will also be the limits of integration, since $f(x)$ vanishes outside these limits. In general, if we wish to find the mean value of those measurements lying between the limits a and b, then we will apply Eq. (1.11) between these limits, provided we have normalized $f(x)$ between a and b. So, instead of $f(x)$ in Eq. (1.11) we will now substitute the probability density

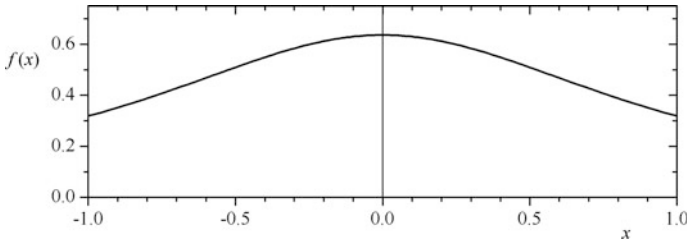
$$f_{a,b}(x) = \frac{f(x)}{\int_a^b f(x)\mathrm{d}x} \tag{1.12}$$

which is, indeed, normalized in the range a to b. The mean value of those measurements lying between a and b is, therefore,

$$\overline{x} = \frac{\int_a^b f(x)x\mathrm{d}x}{\int_a^b f(x)\mathrm{d}x}. \tag{1.13}$$

Example 1.10

The probability density of the random variable **x** for the distribution known as the *Cauchy distribution*, is given by the relation $f(x) = 2/[\pi(1+x^2)]$ for $-1 \le x \le 1$. Verify that $f(x)$ is normalized for the values of x between –1 and +1 and find the mean value of x in the ranges $(-1, 1)$ and (0, 1).



Since

$$\int_{-1}^{1} f(x)\mathrm{d}x = \frac{2}{\pi}\int_{-1}^{1}\frac{\mathrm{d}x}{1+x^2} = \frac{2}{\pi}\left[\arctan x\right]_{-1}^{1} = \frac{2}{\pi}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = 1$$

the function is normalized in the range $-1 \leq x \leq 1$.

The mean value of x in the range $(-1, 1)$ is, according to Eq. (1.11),

$$\overline{x} = \int_{-1}^{1} f(x)x\mathrm{d}x = \frac{2}{\pi}\int_{-1}^{1}\frac{x\mathrm{d}x}{1+x^2} = \frac{1}{\pi}\left[\ln(1+x^2)\right]_{-1}^{1} = 0,$$

as expected, since $f(x)$ is an even function.

$f(x)$ is not normalized in the range (0, 1). Using Eq. (1.13), therefore, we find

$$\overline{x} = \frac{\int_0^1 f(x)x\mathrm{d}x}{\int_0^1 f(x)\mathrm{d}x} = \frac{\int_0^1 \frac{2}{\pi}\frac{x\mathrm{d}x}{1+x^2}}{\int_0^1 \frac{2}{\pi}\frac{\mathrm{d}x}{1+x^2}} = \frac{\frac{1}{2}\left[\ln(1+x^2)\right]_0^1}{\left[\arctan x\right]_0^1} = \frac{\frac{1}{2}\ln 2}{\pi/4} = \frac{2}{\pi}\ln 2$$

or

$$\overline{x} = \frac{2}{\pi}\ln 2 = 0.441\ldots.$$

1.4 Measures of Dispersion

Questions which arise are '*by how much do the values of x differ from each other?*' and '*how scattered are the values of x relative to some particular value?*'. These are quantified by means of three *measures of dispersion*: the *range* of the distribution, the *mean absolute deviation* of the values from their mean and the *standard deviation* of the values from their mean.

1.4.1 Range

The range of a distribution of values is defined as the difference between the largest value of **x**, $x_{\max}$, and the smallest, $x_{\min}$:

$$R \equiv x_{\max} - x_{\min}. \tag{1.14}$$

For example, the range of the distribution of the 100 values of Table 1.1 (or 1.2) is 10.4 – 9.5 = 0.9 mm.

1.4.2 Deviation from the Mean and Mean Absolute Deviation from the Mean

If $x_1, x_2, \ldots, x_N$ are N values of the variable x and $\overline{x}$ their mean, the *deviation* of the value x_i from the mean $\overline{x}$ is defined as

$$d_i \equiv x_i - \overline{x}. \tag{1.15}$$

The deviations of the N values from their mean are $d_1 = x_1 - \overline{x}$, $d_2 = x_2 - \overline{x}$, … $d_N = x_N - \overline{x}$. Obviously, their sum is equal to zero:

$$\sum_i d_i = d_1 + d_2 + \ldots + d_N = x_1 + x_2 + \ldots + x_N - N\overline{x} = 0.$$

The sum of the absolute values of the deviations is a positive quantity, from which we get, by division by the number of the values, their *mean absolute deviation from the mean*:

$$\overline{|d|} = \frac{1}{N}(|d_1| + |d_2| + \ldots + |d_N|) = \frac{1}{N}\sum_{i=1}^{N} |d_i|. \tag{1.16}$$

If the values $x_1, x_2, \ldots, x_K$ have frequencies $n_1, n_2, \ldots, n_K$, respectively, then

$$\overline{|d|} = \frac{n_1|d_1| + n_2|d_2| + \ldots + n_K|d_K|}{n_1 + n_2 + \ldots + n_K} = \frac{1}{N}\sum_{r=1}^{K} n_r|x_r - \overline{x}| = \sum_{r=1}^{K} f_r|x_r - \overline{x}|, \tag{1.17}$$

where f_r is the relative frequency of the value x_r.

Example 1.11

Find the mean absolute deviation from the mean of the 100 measurements of Table 1.1.

We use the grouping of Table 1.2 and Eq. (1.17). We will apply both ways presented in Eq. (1.17), so we evaluate both the products $n_r|x_r - \overline{x}|$ and $f_r|x_r - \overline{x}|$. It has already been found that $\overline{x} = 9.984$ mm.

r	x_r (mm)	$x_r - \overline{x}$ (mm)	$\lvert x_r - \overline{x}\rvert$ (mm)	n_r	$n_r\lvert x_r - \overline{x}\rvert$ (mm)	f_r	$f_r\lvert x_r - \overline{x}\rvert$ (mm)
1	9.5	0.484	0.484	5	2.420	0.05	0.02420
2	9.6	0.384	0.384	7	2.688	0.07	0.02688
3	9.7	0.284	0.284	8	2.272	0.08	0.02272
4	9.8	0.184	0.184	10	1.840	0.10	0.01840
5	9.9	0.084	0.084	14	1.176	0.14	0.01176

(continued)

(continued)

r	x_r (mm)	$x_r - \bar{x}$ (mm)	$\lvert x_r - \bar{x}\rvert$ (mm)	n_r	$n_r\lvert x_r - \bar{x}\rvert$ (mm)	f_r	$f_r\lvert x_r - \bar{x}\rvert$ (mm)
6	10.0	–0.016	0.016	15	0.240	0.15	0.00240
7	10.1	–0.116	0.116	12	1.392	0.12	0.01392
8	10.2	–0.216	0.216	12	2.592	0.12	0.02592
9	10.3	–0.316	0.316	9	2.844	0.09	0.02844
10	10.4	–0.416	0.416	8	3.328	0.08	0.03328
			Sums, Σ =	100	20.792	1	0.20792

Thus, $\overline{|d|} = \frac{1}{N}\sum_{r=1}^{K} n_r|x_r - \bar{x}| = \frac{20.792}{100} = 0.20792 = 0.21$ mm and

$\overline{|d|} = \sum_{r=1}^{K} f_r|x_r - \bar{x}| = 0.20792 = 0.21$ mm.

The values of Table 1.1 have, therefore, a mean of 9.98 mm and a mean absolute deviation from their mean equal to 0.21 mm, or 2% (10.0 and 0.2 mm, respectively, might be a more realistic statement, as we will explain later).

1.4.3 Standard Deviation

The mean absolute deviation from the mean is a useful quantity as a measure of the dispersion of the values. It was found, however, that another quantity contains much more information about the statistics of the scatter of the values. This is the *standard deviation* or the *root mean square deviation* of the values from their mean, defined as:

$$s_x \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}. \tag{1.18}$$

If the values are grouped in K classes, Eq. (1.18) may also be written in the forms

$$s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r(x_r - \bar{x})^2} \tag{1.19}$$

and

$$s_x = \sqrt{\sum_{r=1}^{K} f_r(x_r - \bar{x})^2}. \tag{1.20}$$

The quantity s_x is called the *sample standard deviation*, because it is evaluated for a sample consisting of only N values, taken from the *parent population* of the infinite possible results of measurements of x.

For a continuous distribution, described by the normalized probability density function $f(x)$, Eq. (1.20) is modified to

$$\sigma_x = \sqrt{\int_{-\infty}^{\infty} (x - \overline{x})^2 f \mathrm{d}x} \tag{1.21}$$

where now the standard deviation is denoted by σ_x and not s_x, as it is a property of a parent population and not of a sample drawn from it.

The square of the standard deviation is called the *variance*:

$$\mathrm{Var}(x) = \sigma_x^2 = \frac{1}{N} \sum_{i=1}^{N} (x_i - \overline{x})^2, \tag{1.22}$$

$$\mathrm{Var}(x) = \sigma_x^2 = \int_{-\infty}^{\infty} (x - \overline{x})^2 f \mathrm{d}x. \tag{1.23}$$

If $f(x)$ is not normalized or if we wish to evaluate the standard deviation of only those values of x lying between some limits, say a and b, then Eq. (1.21) becomes

$$\sigma_x = \sqrt{\frac{\int_a^b (x - \overline{x})^2 f \mathrm{d}x}{\int_a^b f \mathrm{d}x}}. \tag{1.24}$$

The standard deviation is the *root mean square* (*rms*) of the deviations from the mean. From the definition, it is obvious that the standard deviation has the dimensions and the units of x.

Sometimes, the *coefficient of variation* is used, which is equal to the standard deviation expressed as a fraction of the mean,

$$C = \frac{s_x}{\overline{x}}. \tag{1.25}$$

It is also expressed as a percentage of the mean,

$$100 \frac{s_x}{\overline{x}} \%. \tag{1.26}$$

The coefficient of variation is a dimensionless quantity (pure number).

Note: The standard deviations s_x and σ_x are estimated using deviations from the mean. They are known as the standard deviations *from* the mean, not to be confused with the standard deviation *of* the mean, to be encountered in the chapters to follow.

The weighted standard deviation is examined in Sect. 9.4.

Example 1.12

Find the mean and the standard deviation of the 10 measurements of the second column of the table below.

i	x_i (mm)	$x_i - \bar{x}$ (mm)	$(x_i - \bar{x})^2$ (mm^2)
1	9.0	–0.7	0.49
2	8.6	–1.1	1.21
3	10.2	0.5	0.25
4	9.8	0.1	0.01
5	10.4	0.7	0.49
6	12.0	2.3	5.29
7	10.1	0.4	0.16
8	9.2	–0.5	0.25
9	10.3	0.6	0.36
10	7.4	–2.3	5.29
Σ =	97.0		13.80

It is found that $\sum_{i=1}^{10} x_i = 97.0\,\text{mm}.$, Therefore, $\bar{x} = 9.7\,\text{mm}$.

We evaluate $x_i - \bar{x}$ and $(x_i - \bar{x})^2$.

It is found that $\sum_{i=1}^{10} (x_i - \bar{x})^2 = 13.80\,\text{mm}^2$.

Therefore

$$s_x = \sqrt{\frac{1}{N}\sum_{i=1}^{10} (x_i - \bar{x})^2} = \sqrt{\frac{13.80}{10}} = \sqrt{1.380} = 1.17\,\text{mm}.$$

Example 1.13 [E]

Solve Example 1.10 using Excel®.

We enter the values of x_i in the cells A1–A10 of column A.

The function used to estimate the mean of these values is **AVERAGE()**. We double click on an empty cell, say C13, and type AVERAGE(. We then move the cursor to cell A1. Holding the left-hand side button of the mouse pressed down, we drag the cursor down to cell A10. Cell C13 now shows = AVERAGE(A1:A10. We close the parenthesis and press **ENTER**. The result appearing in cell C13 is 9.7. Therefore, the mean is $\bar{x} = 9.7\,\text{mm}$.

The standard deviation of the sample, s_x, is evaluated by the function **STDEV.P ()**. We double click on an empty cell, say D13, and type STDEV.P(. We then move the cursor to cell A1. Holding the left-hand side button of the mouse pressed down, we drag the cursor down to cell A10. Cell D13 now shows = STDEV.P(A1:A10. We close the parenthesis and press **ENTER**. The result appearing in cell D13 is 1.174734. Therefore, the mean is $s_x = 1.2\,\text{mm}$.

An alternative way by which to obtain these results is to use

Data > Data Analysis > Descriptive Statistics

When the dialog box opens, we click on **Input Range** and select cells A1–A10 by the method described above. The range $A$1:$A$10 will then appear in the box. Select **Summary statistics** and press **OK**.

Mean	9.7
Standard error	0.391578004
Median	9.95
Mode	#N/A
Standard deviation	1.238278375
Sample variance	1.533333333
Kurtosis	1.062759924
Skewness	−0.076807201
Range	4.6
Minimum	7.4
Maximum	12
Sum	97
Count	10

A table appears, which contains, among other statistical properties of the sample, the values: Mean 9.7 and Standard Deviation 1.238278.

The standard deviation given by the table is not that of the sample, $s_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2}$, but rather the best estimate for the standard deviation of the parent population, $\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}$ (see Sect. 4.2.4). We may evaluate s_x using the relation $s_x = \sqrt{\frac{N-1}{N}}\,\hat{\sigma}$. The result is $s_x = \sqrt{\frac{9}{10}}\,1.238278 = 1.174734$ mm, as expected.

Example 1.14 [O]

Solve Example 1.12 using Origin®.

We enter the values of x_i in column A. We highlight column A and then

Statistics > Descriptive Statistics > Statistics on Columns > Open Dialog…

In the window that opens, we tick the following:

Quantities > Tick **Mean, Standard Deviation**

Open **Computation Control > Weight Method > Direct Weight**
Then **Variance Divisor of Moment > N**

The last setting puts the number N in the denominator of Eq. (4.10) (The choice DF would put $N - 1$ in the denominator).

Pressing **OK** we obtain the results:

[Mean] = 9.7, [Standard Deviation] = 1.17473

We have found that $\overline{x} = 9.7$ mm and $s_x = 1.2$ mm, in agreement with the results of Example 1.10.

Example 1.15 [P]

Solve Example 1.12 using Python.

```
from __future__ import division
import numpy as np
import math

# Enter the values given as the components of the vector x:

x = np.array([9, 8.6, 10.2, 9.8, 10.4, 12, 10.1, 9.2, 10.3, 7.4])

# Evaluation of the parameters:
N = len(x)
mean_x = x.mean()
mean_abs_dev_mean = np.sum(np.abs(x-mean_x)) / N
std_dev_sample = x.std(ddof = 1) * math.sqrt((N-1)/N)

# Preparing the printout:
print ("Number of values N = ", N)
print ("Mean = ", mean_x)
print ("Mean absolute deviation from the mean = ", mean_abs_dev_mean)
print ("Standard deviation of the sample = ", std_dev_sample)

Running the program, returns:

Number of values N = 10
Mean = 9.7
Mean absolute deviation from the mean = 0.92
Standard deviation of the sample = 1.17473401245
```

Example 1.16 [R]

Solve Example 1.10 using R.

We will use the built-in functions **mean** and **sd** for the calculation of the mean $\overline{x}$ and the standard deviation of the data, s_x. By the term sd, R means the best estimate of the standard deviation of the parent population, Eq. (4.39). The standard deviation of the sample is given by $s_x = \mathrm{sd}(a)\sqrt{(N-1)/N}$.

We first form the vector a with components the results of the measurements:

```
> a <- c(9.0, 8.6, 10.2, 9.8, 10.4, 12.0, 10.1, 9.2, 10.3, 7.4)
```

Then,

```
> mean(a); sd(a)*sqrt(9/10)
[1] 9.7
[1] 1.174734
```

We have found that $\bar{x} = 9.7$ mm and $s_x = 1.2$ mm, in agreement with the results of Example 1.10.

Example 1.17

Find the standard deviation of the 100 measurements of Table 1.1.

We will make use of the grouping of Table 1.2 and evaluate the standard deviation in two ways, using Eqs. (1.19) and (1.20). For this purpose, we evaluate the products $n_r(x_r - \bar{x})^2$ and $f_r(x_r - \bar{x})^2$. It has already been found that $\bar{x} = 9.984$ mm.

r	x_r (mm)	$x_r - \bar{x}$ (mm)	$(x_r - \bar{x})^2$ (mm^2)	n_r	$n_r(x_r - \bar{x})^2$ (mm^2)	f_r	$f_r(x_r - \bar{x})^2$ (mm^2)
1	9.5	0.484	0.23426	5	1.17128	0.05	0.0117128
2	9.6	0.384	0.14746	7	1.03219	0.07	0.0103219
3	9.7	0.284	0.08066	8	0.64525	0.08	0.0064525
4	9.8	0.184	0.03386	10	0.33856	0.10	0.0033856
5	9.9	0.084	0.00706	14	0.09878	0.14	0.0009878
6	10.0	–0.016	0.00026	15	0.00384	0.15	0.0000384
7	10.1	–0.116	0.01346	12	0.16147	0.12	0.0016147
8	10.2	–0.216	0.04666	12	0.55987	0.12	0.0055987
9	10.3	–0.316	0.09986	9	0.89870	0.09	0.0089870
10	10.4	–0.416	0.17306	8	1.38445	0.08	0.0138445
			Sums, Σ =	100	6.29439	1	0.0629439

From Eq. (1.19) we find $s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r(x_r - \bar{x})^2} = \sqrt{\frac{6.29439}{100}} = \sqrt{0.06294} = 0.25$ mm and from Eq. (1.20) we also find $s_x = \sqrt{\sum_{r=1}^{K} f_r(x_r - \bar{x})^2} = \sqrt{0.06294} = 0.25$ mm.

The values of Table 1.1 have, therefore, a mean $\bar{x} = 9.98$ mm and a standard deviation $s_x = 0.25$ mm. The coefficient of variation of the values is equal to $s_x/\bar{x} = 25/9.98 = 0.025$ or 2.5%.

Example 1.18 [E]

Solve Example 1.17 using Excel®.

We enter the values of x_r and n_r in columns A and B, respectively. We need to evaluate the weighted standard deviation of x, with n as weights. The weighted mean and weighted standard deviation are defined in Sect. 9.4, but, essentially, we use Eq. (1.9), $\overline{x} = \frac{1}{N}\sum_{r=1}^{K} n_r x_r$ for the mean and Eq. (1.19), $s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r (x_r - \overline{x})^2}$ for the standard deviation, with n_r as weights and N as the sum of the weights.

We will first evaluate the weighted mean. Highlight an empty cell, say E1. Left-click on cell E1 and type:

=SUMPRODUCT(A1:A10;B1:B10)/SUM(B1:B10)

Pressing **ENTER** will return the number 9.9840 in cell E1. This is the required mean, $\overline{x} = 9.984$ mm.

We will give this number the name **M**. To do this, we right click on cell E1. In the dialog box that opens, we select **Define Name**… and in the cell for **Name** we write **M**.

We will now evaluate the weighted standard deviation. We first evaluate the terms $(x_r - \overline{x})^2$. We highlight cell C1 and type: **= (A1−M)^2**. Pressing **ENTER** returns the number `0.234256` in cell C1. To fill cells C1 to C10 with the values of $(x_r - \overline{x})^2$, we highlight cells C1–C10 and press

Fill > Down

To evaluate the standard deviation, we highlight an empty cell, say D13 and type

=SQRT(SUMPRODUCT(B1:B10;C1:C10)/SUM(B1:B10))

Pressing **ENTER** returns the number `0.250886`. We have found that the standard deviation of the sample in the Table of Example 1.14 is $s_x = 0.251$ mm, in agreement with the results of Example 1.17.

Example 1.19 [O]

Solve Example 1.17 using Origin®.

We enter the values of x_r and n_r in columns A and B, respectively. We need to evaluate the weighted standard deviation of x, with the n_r's as weights. The weighted mean and weighted standard deviation are defined in Sect. 9.4, but, essentially, we use Eq. (1.19), $s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r (x_r - \overline{x})^2}$, with n_r as weights and N as the sum of the weights.

We highlight columns A and B. Then,

Statistics > Descriptive Statistics > Statistics on Columns > Open Dialog…

In the dialog box that opens,

Input > Input data > Independent Columns

Range 1 > Data Range > A(X)

Weighting Range > B(Y)

Quantities > select **Standard Deviation**

Computation Control > Weight Method > Direct Weight

Variance > Divisor of Moment > WS

The last choice places the sum of the weights x_r in the denominator as in Eq. (1.19).

We press **OK**. The result given is [Standard Deviation] = 0.25089

We have found that $s_x = 0.25$ mm, in agreement with the results of Example 1.17.

Example 1.20 [P]

Solve Example 1.17 using Python.

We need to calculate $s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r (x_r - \overline{x})^2}$. First we need to evaluate the mean $\overline{x} = \frac{1}{N}\sum_{r=1}^{K} n_r x_r$. The task is equivalent to calculating the weighted mean and the weighted standard deviation of the sample, for the measurements x_r, with corresponding weights n_r, where $N = \sum_{r=1}^{K} n_r$. The weighted mean and weighted standard deviation are defined in Sect. 9.4. The equations derived there are the same as those given above, with the weights w_r replacing n_r, and $\sum_{r=1}^{K} w_r$ replacing N.

```
import math
import numpy as np
x = np.array([9.5, 9.6, 9.7, 9.8, 9.9, 10, 10.1, 10.2, 10.3, 10.4])
w = np.array([5, 7, 8, 10, 14, 15, 12, 12, 9, 8])
wmean = np.average(x, weights=w)
variance = np.average((x-wmean) ** 2, weights=w)
s = math.sqrt(variance)
```

```
print ("Weighted mean =", wmean)
print ("Weighted standard deviation s =", s)
```

The results are:

```
Weighted mean = 9.984
Weighted standard deviation s = 0.2508864284890676
```

Example 1.21 [R]

Solve Example 1.17 using R.

r	x_r (mm)	n_r
1	9.5	5
2	9.6	7
3	9.7	8
4	9.8	10
5	9.9	14
6	10.0	15
7	10.1	12
8	10.2	12
9	10.3	9
10	10.4	8

We need to calculate $s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r(x_r - \overline{x})^2}$. First we need to evaluate the mean $\overline{x} = \frac{1}{N}\sum_{r=1}^{K} n_r x_r$. The task is equivalent to calculating the weighted mean and the weighted standard deviation of the sample, for the measurements x_r, with corresponding weights n_r, where $N = \sum_{r=1}^{K} n_r$. The weighted mean and weighted standard deviation are defined in Sect. 9.4. The equations derived there are the same as those given above, with the weights w_r replacing n_r, and $\sum_{r=1}^{K} w_r$ replacing N. We may, therefore use the function **weighted.mean(x, w, …)** available in R.

We first define the vectors of the x and w = n values and then find the weighted mean:

```
> x <- c(9.5, 9.6, 9.7, 9.8, 9.9, 10, 10.1, 10.2, 10.3, 10.4)
> w <- c(5, 7, 8, 10, 14, 15, 12, 12, 9, 8)
> wmean = weighted.mean(x, w)
> wmean
[1] 9.984
```

We notice that the variance s_x^2 of the sample is simply the weighted mean of the quantity $(x_r - \overline{x})^2$. This is done as follows:

```
> variance = weighted.mean((x-wmean)^2, w)
> variance
[1] 0.062944
> sqrt(variance)
[1] 0.2508864
```

We find that $s_x = 0.25$ mm, in agreement with previous results.

Example 1.22

Find, in the range $[-1, 1]$, the standard deviation of the distribution of the variable x whose probability density is $f(x) = (2/\pi)/(1+x^2)$ (Cauchy distribution).

In Example 1.2, it was found that this probability density is normalized in the range $-1 \leq x \leq 1$ and that, over the same range, the distribution has a mean equal to $\overline{x} = 0$.

We will use expression (1.21), integrating between the limits –1 and 1. Thus,

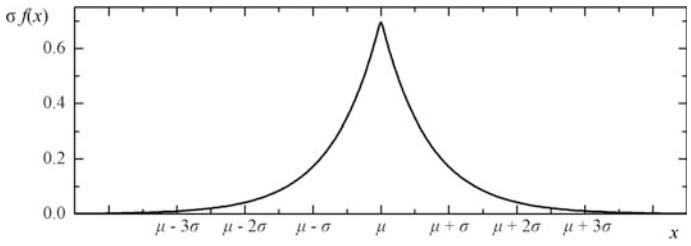
$$\sigma = \sqrt{\int_{-\infty}^{\infty} (x-\overline{x})^2 f dx} = \sqrt{\int_{-1}^{1} x^2 \frac{2}{\pi}\frac{1}{1+x^2} dx} = \sqrt{\frac{2}{\pi}}\sqrt{\int_{-1}^{1}\left(1 - \frac{1}{1+x^2}\right)dx}$$
$$= \sqrt{\frac{2}{\pi}}\sqrt{[x - \arctan x]_{-1}^{1}} = \sqrt{\frac{2}{\pi}}\sqrt{2 - \frac{\pi}{2}} = \sqrt{\frac{4}{\pi} - 1}$$

or

$$\sigma = \sqrt{\frac{4}{\pi} - 1} = 0.52272\ldots$$

Example 1.23

The Laplace distribution, has the probability density function $f(x) = Ae^{-\alpha|x-\mu|}$ in the range $-\infty < x < \infty$. Find: (a) The value of A which normalizes the distribution, (b) the mean $\overline{x}$ of the distribution and (c) its standard deviation σ.



We will use $z = x - \mu$ as variable. Then $\mathrm{d}z = \mathrm{d}x$.

(a) The normalization condition is: $\int_{-\infty}^{+\infty} f(x)\mathrm{d}x = 1$. Therefore,

$$\int_{-\infty}^{+\infty} A\mathrm{e}^{-\alpha|x-\mu|}\mathrm{d}x = \int_{-\infty}^{+\infty} A\mathrm{e}^{-\alpha|z|}\mathrm{d}z = A\int_{-\infty}^{0} \mathrm{e}^{\alpha x}\mathrm{d}x + A\int_{0}^{+\infty} \mathrm{e}^{-\alpha x}\mathrm{d}x$$
$$= 2A\int_{0}^{+\infty} \mathrm{e}^{-\alpha x}\mathrm{d}x = 2A\mathrm{e}^{\alpha\mu}\left[-\frac{\mathrm{e}^{-\alpha x}}{\alpha}\right]_{\mu}^{\infty} = 2A\mathrm{e}^{\alpha\mu}\frac{\mathrm{e}^{-\alpha\mu}}{\alpha} = \frac{2A}{\alpha} = 1$$

from which we have $A = \dfrac{\alpha}{2}$ and

$$f(x) = \frac{\alpha}{2}\mathrm{e}^{-\alpha|x-\mu|}$$

as the normalized probability density distribution function.

(b) The mean is given by the relation:

$$\bar{x} = \int_{-\infty}^{+\infty} x\, f(x)\,\mathrm{d}x = \frac{\alpha}{2}\int_{-\infty}^{+\infty} (\mu + x)\ \mathrm{e}^{-\alpha|z|}\,\mathrm{d}z$$
$$= \frac{\alpha}{2}\mu\int_{-\infty}^{+\infty} \mathrm{e}^{-\alpha|z|}\,\mathrm{d}z + \frac{\alpha}{2}\int_{-\infty}^{+\infty} z\ \mathrm{e}^{-\alpha|z|}\,\mathrm{d}z$$
$$= \frac{\alpha}{2}\left\{\mu\int_{-\infty}^{0} \mathrm{e}^{\alpha z}\,\mathrm{d}z + \mu\int_{0}^{\infty} \mathrm{e}^{-\alpha z}\,\mathrm{d}z + \cancel{\int_{-\infty}^{0} z\,\mathrm{e}^{\alpha z}\,\mathrm{d}z} + \cancel{\int_{0}^{\infty} z\,\mathrm{e}^{-\alpha z}\,\mathrm{d}z}\right\}$$
$$= \frac{\alpha}{2}\mu\left\{\left[\frac{\mathrm{e}^{\alpha x}}{\alpha}\right]_{-\infty}^{0} - \left[\frac{\mathrm{e}^{\alpha x}}{\alpha}\right]_{0}^{\infty}\right\} = \mu\,.$$

(c) The standard deviation of the distribution, σ, will be found using $\sigma^2 = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\mathrm{d}x$.

Therefore,

$$\sigma^2 = \frac{\alpha}{2}\int_{-\infty}^{+\infty} z^2\mathrm{e}^{-\alpha|z|}\mathrm{d}z = \frac{\alpha}{2}\left\{\int_{-\infty}^{0} z^2\mathrm{e}^{\alpha|z|}\mathrm{d}z + \int_{0}^{+\infty} z^2\mathrm{e}^{-\alpha|z|}\mathrm{d}z\right\}$$
$$= \alpha\int_{0}^{+\infty} z^2\mathrm{e}^{-\alpha z}\mathrm{d}z = \alpha\left[\left(-\frac{2}{\alpha^3} - \frac{2z}{\alpha^2} + \frac{z^2}{\alpha}\right)\mathrm{e}^{-\alpha z}\right]_{0}^{\infty} = \alpha\frac{2}{\alpha^3} = \frac{2}{\alpha^2}$$

Finally,

$$\sigma^2 = \frac{2}{\alpha^2} \quad \text{or} \quad \sigma = \frac{\sqrt{2}}{\alpha}.$$

The distribution has the form

$$f(x) = \frac{1}{\sqrt{2}\sigma} \mathrm{e}^{-\sqrt{2}|x-\mu|/\sigma}$$

with mean μ and standard deviation σ.

Example 1.24 [E]

An example of a Laplace distribution is $f(x) = e^{-2|x|}$ (see Example 1.23). The distribution is normalized in the interval $-\infty < x < \infty$. Using numerical integration, find (a) the mean and (b) the standard deviation of the distribution.

In the numerical calculations, we will consider as adequate the use of values of x between –10 and 10. The values will change in steps of $\delta x = 0.001$. The variable will be $x_n = n\delta x$ or $x_n = 0.001n$, for $(-10000 \leq n \leq 10000)$. We fill column A with numbers n between –10000 and 10000 in the following way:

Write the number –10000 in cell A1. Highlight cell A. Then

Fill > Series in Columns > Type Linear > Step Value = 1 > Stop value = 10000 > OK

Column A now contains the values of n.

Set B1 = A1/1000 and then **Fill down** to B10000. Column B now contains the values of x_n. Set C1 = 2*ABS(B1) and then Fill down to C10000. Set D1 = EXP(-C1) and then **Fill down** to D10000. Column D now contains the values of $\mathrm{e}^{-2|x_n|}$.

(a) *The mean*

The mean is given by

$$\overline{x} = \int_{-\infty}^{\infty} x\mathrm{e}^{-2|x|}\mathrm{d}x \approx \sum_{n=-10000}^{10000} x_n \mathrm{e}^{-2|x_n|}\delta x = 10^{-6} \sum_{n=-10000}^{10000} n\mathrm{e}^{-|0.002n|}.$$

Set E1 = B1*D1/1000 and then **Fill down** to E10000. Column E now contains the values of $x_n \mathrm{e}^{-2|x_n|}\delta x$. Set E20004 = SUM(E1:E20001). Cell E20004 now contains $\overline{x} = \int_{-\infty}^{\infty} x\mathrm{e}^{-2|x|}\mathrm{d}x \approx 10^{-6} \sum_{n=-10000}^{10000} n\mathrm{e}^{-|0.002n|}$. The result is `2.17694E-17`, which is taken to be close enough to zero, $\overline{x} = 0$.

(b) *The standard deviation*

The standard deviation σ is given by $\sigma^2 = \int_{-\infty}^{\infty} (x-\overline{x})^2 e^{-2|x|} dx \approx \sum_{n=-10000}^{10000} (x_n - \overline{x})^2 e^{-2|x_n|} \delta x$. In (a) it was found that $\overline{x} = 0$. Set F1 = B1*E1 and then **Fill down** to F10000. Column F now contains the values of $(x_n - \overline{x})^2 e^{-2|x_n|} \delta x$. Set F20004 = SUM(F1:F20001). Cell F20004 now contains $\sigma^2 \approx \sum_{n=-10000}^{10000} (x_n - \overline{x})^2 e^{-2|x_n|} \delta x$. The result is `0.499999772`, which is taken to be close to ½. It was therefore found that $\sigma = 1/\sqrt{2}$, in agreement with the result of Example 1.23.

Example 1.25 [O]

An example of a Laplace distribution is $f(x) = e^{-2|x|}$ (see Example 1.23). The distribution is normalized in the interval $-\infty < x < \infty$. Using numerical integration, find (a) the mean and (b) the standard deviation of the distribution.

In the numerical calculations, we will consider as adequate the use of values of x between – 10 and 10. The values will change in steps of δx = 0.001. The variable will be $x_n = n\delta x$ or $x_n = 0.001n$, for $(-10000 \le n \le 10000)$. We fill column A with numbers n between – 10000 and 10000 by highlighting column A and then

Column > Set Column Values

and typing **i−10001** for i between 1 and 20001. Column A now contains the values of n.

(a) *The mean*

The mean is given by

$$\overline{x} = \int_{-\infty}^{\infty} x e^{-2|x|} dx \approx \sum_{n=-10000}^{10000} x_n e^{-2|x_n|} \delta x = 10^{-6} \sum_{n=-10000}^{10000} n e^{-|0.002n|}.$$

Highlight column B. Then

Column > Set Column Values

In the window that opens, we type **col(A)*exp(-abs(col(A)/500))/10^6** and then press **OK**. Then sum the numbers in column B by highlighting the column and pressing **Σ**. The sum is given as $\overline{x} = 2.918 \times 10^{-16}$. This is taken as giving the result $\overline{x} = 0$ with a satisfactory accuracy.

(b) *The standard deviation*

The standard deviation σ is given by $\sigma^2 = \int_{-\infty}^{\infty} (x - \overline{x})^2 \mathrm{e}^{-2|x|} \mathrm{d}x \approx \sum_{n=-10000}^{10000} (x_n - \overline{x})^2 \mathrm{e}^{-2|x_n|} \delta x$. In (a) it was found that $\overline{x} = 0$. Therefore $\sigma^2 \approx 10^{-9} \sum_{n=-10000}^{10000} n^2 \mathrm{e}^{-|0.002n|}$.

Highlight column C. Then

Column > Set Column Values

In the window that opens, we type **((col(A))^2)*exp(-abs(col(A)/500))/10^9** and then press **OK**. Then sum the numbers in column B by highlighting the column and pressing **Σ**. The sum is given as $\sigma^2 = 0.5000$. This is taken as giving the result $\sigma = 1/\sqrt{2}$ with a satisfactory accuracy.

Example 1.26 [P]

An example of a Laplace distribution is $f(x) = e^{-2|x|}$ (see Example 1.23). The distribution is normalized in the interval $-\infty < x < \infty$. Using numerical integration, find (a) the mean and (b) the standard deviation of the distribution.

Numerical integration is performed using the `scipy.integrate` subpackage in Python. The `quad` function performs general-purpose integration. Its first argument is a callable function; we will use the lambda-calculus notation to quickly pass definition functions without separately defining them in our program. The return value is a tuple, with the first element holding the estimated value of the integral and the second element holding an upper bound on the error.

We begin with importing the necessary modules:

```
import numpy as np
import scipy.integrate as integrate
```

(a) *The mean*

The mean is given by $\overline{x} = \int_{-\infty}^{\infty} x \mathrm{e}^{-2|x|} \mathrm{d}x$.

```
mean = integrate.quad(lambda x: x*np.exp(-2*abs(x)), -np.inf, np.inf)
```

The result returned is (0, 0), therefore $\overline{x} = 0$.

(b) *The standard deviation*

The standard deviation σ is given by $\sigma^2 = \int_{-\infty}^{\infty} (x - \overline{x})^2 \mathrm{e}^{-2|x|} \mathrm{d}x = \int_{-\infty}^{\infty} x^2 \mathrm{e}^{-2|x|} \mathrm{d}x$.

```
stdev = integrate.quad(lambda x: x**2 * np.exp(-2*abs(x)), -np.inf, np.inf)
```

The result returned is `(0.4999, 6.81651e-11)`, therefore $\sigma = 1/\sqrt{2}$.

Example 1.27 [R]

An example of a Laplace distribution is $f(x) = e^{-2|x|}$ (see Example 1.19). The distribution is normalized in the interval $-\infty < x < \infty$. Using numerical integration, find (a) the mean and (b) the standard deviation of the distribution.

We will use the **integrate** command of R.

(a) *The mean*

The mean is given by $\bar{x} = \int_{-\infty}^{\infty} x e^{-2|x|} dx$.

```
> fun1 <- function(x) x*exp(-2*abs(x))
> integrate (fun1, lower=-30, upper=30)
0 with absolute error < 5.8e-17
```

The result is `5.8e-17`, which is taken to be close enough to zero, $\bar{x} = 0$.

(b) *The standard deviation*

The standard deviation σ is given by $\sigma^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 e^{-2|x|} dx = \int_{-\infty}^{\infty} x^2 e^{-2|x|} dx$.

```
> fun2 <- function(x) x^2*exp(-2*abs(x))
> integrate (fun2, lower=-30, upper=30)
0.5 with absolute error < 1.7e-07
```

The result is taken to be close enough to ½. It was therefore found that $\sigma = 1/\sqrt{2}$, in agreement with the result of Example 1.19.

Example 1.28

Show that the sum of the squares of the deviations of the values x_i $(i = 1, 2, \ldots, N)$ from a given value X is minimum when X is equal to $\bar{x}$, the mean of the values x_i, in which case the sum is equal to Ns_x^2, where s_x is the standard deviation of the values x_i from $\bar{x}$.

The sum is $A = \sum_{i=1}^{N} (x_i - X)^2$. Taking its derivative with respect to X, we have

$$\frac{dA}{dX} = \frac{d}{dX} \sum_{i=1}^{N} (x_i - X)^2 = -2 \sum_{i=1}^{N} (x_i - X) = 2\left(NX - \sum_{i=1}^{N} x_i \right) = 2(NX - N\bar{x})$$
$$= 2N(X - \bar{x}).$$

If $X = \bar{x}$, it is $\frac{dA}{dX} = 0$. Then $A = \sum_{i=1}^{N} (x_i - X)^2 = \sum_{i=1}^{N} (x_i - \bar{x})^2 = Ns_x^2$.

The value of A is minimum for $X = \bar{x}$, since then $\frac{d^2A}{dX^2} = 2N > 0$.

Problems

1.1 **[E.O.P.R.]** The measurement of a length x gave the following results:

23.302 23.273 23.310 23.304 23.263 23.321 23.295 23.270
23.284 23.255 cm.

Find: (a) The mean, $\bar{x}$, (b) the mean absolute deviation from the mean, $\overline{|d|}$, and (c) the standard deviation s_x of the measurements from the mean.

1.2 **[E.O.P.R.]** Measurements of an interval of time, T, gave the following results:

2.20 2.15 2.24 2.18 2.21 2.17 2.18 s.

Find: (a) The mean, $\overline{T}$, (b) the mean absolute deviation from the mean, $\overline{|d_T|}$, and (c) the standard deviation s_T of the measurements from the mean.

1.3 The heights of 90 students were measured and gave the results of the table below (i is the group number, h_i is the mean height of the students in group i and n_i is the frequency of the value h_i in the i-th group):

i	1	2	3	4	5	6	7	8	9	10
h_i (cm)	170	171	172	173	174	175	176	177	178	179
n_i	1	2	2	2	3	5	4	7	8	11

i	11	12	13	14	15	16	17	18	19	20
h_i (cm)	180	181	182	183	184	185	186	187	188	189
n_i	12	11	9	5	3	1	2	1	0	1

Note A height of 170 cm means one between 169.5 and 170.5^{-}, not including 170.5 cm. For computational purposes, all students in this group are considered to have a height of 170 cm exactly.

(a) **[E.O.P.R.]** Plot a histogram of these values first with a bin width of 1 cm and then with 2 cm. (b) Find the mean of the distribution. (c) Find the standard deviation of the distribution.

1.4 (a) If the grades x of an exam (x an integer between 0 and 10) are increased by 1 unit, what will the effect be on the mean $\bar{x}$ and on the standard deviation s_x of the grades?

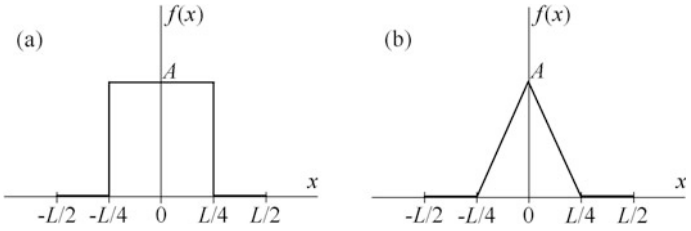
(b) If the grades x of an exam (x an integer between 0 and 10) are increased by 10%, what will the effect be on the mean $\bar{x}$ and on the standard deviation s_x of the grades?

1.5 The probability density function of a variable x is $f(x) = A[1 - \cos(2\pi x)]$ in the range [0, 1] and $f(x) = 0$ elsewhere. Find: (a) the normalizing factor A. (b) the mean and the standard deviation of x.

1.6 Find the mean and the standard deviation of the distribution whose probability density function is $f(x) = \frac{\pi}{8}\sin\left(\frac{\pi}{4}x\right)$ for $0 \leq x \leq 4$ and $f(x) = 0$ outside this range.

1.7 If the probability density function of the random variable $\mathbf{x}$ is $f(x) = Ce^{-x/a}$ for all positive values of x and zero for negative values of x, find the mean and the standard deviation of the distribution.

1.8 Two probability density functions have the shapes shown in the figure, in the range $-L/2 \leq x \leq L/2$.



In both cases, find the value of the normalizing factor A, the mean $\bar{x}$ and the standard deviation σ_x.

1.9 The position of a simple harmonic oscillator is given by the relation $x = a\sin\omega t$ and its velocity by the relation $\upsilon = a\omega\cos\omega t$. Find:

(a) The probability $f(x)\mathrm{d}x$ that the oscillator will, at a moment, be found between x and $x + \mathrm{d}x$.
(b) The probability $g(\upsilon)\mathrm{d}\upsilon$ that an instantaneous measurement of the magnitude of its speed will give a result between υ and $\upsilon + \mathrm{d}\upsilon$.
(c) The mean values of x and υ.
(d) The mean values of $|x|$ and $|\upsilon|$.
(e) The standard deviation σ_x of x and the standard deviation σ_υ of υ.

1.10 According to Quantum Mechanics, the harmonic oscillator of Problem 1.9, when it is in its ground state with energy $E = \frac{1}{2}\hbar\omega$, has a probability density for its position x which is given by the function $f_Q(x) = \left|\psi_0^2(x)\right| = A\exp\left(-\frac{x^2}{a^2}\right)$, where $a = \sqrt{\frac{\hbar}{m\omega}}$ is the classical amplitude of oscillation.

(a) Find the value of A which normalizes $f_Q(x)$ in the range $-\infty < x < \infty$.
(b) Draw $f(x)$ of Problem 1.9 and $f_Q(x)$ and compare.
(c) What is the probability, according to Quantum Mechanics, that the harmonic oscillator will be found outside the region $-a \leq x \leq a$, allowed by Classical Mechanics?

1.11 According to the kinetic theory of gases, the velocities of the molecules of a gas are such that the components of their velocities in any one direction are normally distributed (Maxwell distribution). The probability, therefore, that a molecule has an x component of velocity with a value between u and $u + \mathrm{d}u$ is $A\mathrm{e}^{-hmu^2}\mathrm{d}u$, where A, h and m are constant. Find the normalizing value of A and the mean $\overline{|u|}$ of the absolute value of the x component of the molecules' velocities.

1.12 **[E.O.P.R.]** Using numerical methods, find, in the range $[-1, 1]$, the mean and the standard deviation of the distribution of the variable x whose probability density is $f(x) = (2/\pi)/(1 + x^2)$ (the Cauchy distribution). The function $f(x)$ is normalized in the range $[-1, 1]$. Compare the results with those of Example 1.17.

Chapter 2
Measurement Errors

2.1 Errors of Measurements

All experimental measurements are inaccurate to some degree. By this statement we mean that the measured quantity (e.g. length, weight, mass, time interval, speed, temperature etc.) *has* a *real value*, which we wish to determine, but any measurement we perform of this magnitude, direct or indirect, does not have as its result this real value but some value which differs from it by an unknown amount. We call the difference between the numerical result of the measurement and the real value of the magnitude being measured, *error of the measurement*.

Measurement errors are classified in two categories: *accidental* and *systematic*. There is no clear definition of the exact difference between them. Neither do they obey any simple law. It is difficult to distinguish them; the error of a measurement is usually a combination of errors of both kinds.

2.1.1 Accidental or Random Errors

Accidental or *random errors* are due to many unpredictable factors and their presence may be revealed by repetitions of the measurement. The main characteristics of accidental errors are that they have no regularity in successive measurements of the same magnitude and that their sign is equally probable to be positive or negative. The basic property of accidental errors to be positive or negative with equal probabilities, as well as the fact that small deviations from the real value are more probable than large

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ones, make possible the determination of an estimate of the uncertainty with which the measured magnitude is known. This is achieved by repeating the measurement, under identical experimental conditions, many times, so that, on average, accidental errors mutually cancel out to a certain degree.

Classical examples of random errors are those due to *thermal noise*. The inevitable thermal noise affects all systems, mechanical and electrical. The indications of a torsion balance or a galvanometer, for example, are always non-zero and vary with time (*Brownian motion*). This fact is due to their thermal interaction with their surroundings through their collisions with air molecules or even the photons of the ambient electromagnetic radiation. The noise in electronic instruments is of the same (thermal) origin. This noise (known as *Johnson* or *Nyquist* noise) is due to the random thermal motion of the electrons in the components of the instrument, such as resistors, which leads to the appearance of small potential differences across them. These signals, after being processed by the instrument, appear at its output as random variations of its reading. There exist, of course, means of minimizing thermal noise (suitable filtering of electric signal, lowering the temperature since the noise depends on it etc.). It is, however, both practically and theoretically impossible to eliminate this noise completely. The noise is added to the signal being measured changing its value. The often-used term *signal to noise ratio* describes precisely this situation.

2.1.2 Systematic Errors

Systematic errors may be due to imperfections of the instruments or the method used, and to the observer. They are the most difficult to deal with, as the repetition of the measurement does not reveal their existence. Some examples of sources of systematic errors will help clarify this statement.

Zero error is one of the commonest of systematic errors. If, for example, the pointer of an instrument (for those instruments that still have pointers!) has been shifted relative to its scale, in such a way that for zero input signal the instrument shows a non-zero output signal x_z, then all the instrument's readings will differ by x_z from what they should be. In this example, the systematic error is constant.

If a ruler was marked so that a length of 999 mm was subdivided into 1000 equal parts, which are supposed to have a length of one mm, then each measurement of length using this ruler will give results which are systematically larger by 0.1%. In this example, the systematic error is equal to a constant proportion of the measured quantity. In addition, if the subdivision of the length into equal parts was not performed with the necessary precision, this will add more systematic errors.

A mercury thermometer, whose column does not have a constant cross section, will give systematically and irregularly wrong values for the temperature. This, of course, is true assuming that the scale of the thermometer was drawn using the usual method of establishing the points corresponding to 0 and 100 °C and subdividing the distance between them into 100 equal intervals, each corresponding to 1 °C.

An instrument which needs a certain time to reach a state of equilibrium (e.g. thermal) before it can function normally, will systematically give erroneous readings during its transition period. In this case the systematic error will be a function of time.

The use of a wrong numerical value in the processing of experimental results, or an approximate theoretical relation involving the measured magnitudes, will lead to systematic errors. It is of course arguable whether these errors may be considered to be experimental errors. A classical example is the case of the measurement of the electronic charge by Millikan. The value found by Millikan was $e = 1.591 \times 10^{-19}$ C, with an estimated probable uncertainty of the order of 0.002×10^{-19} C, or about 0.1%. The accepted value today is $e = 1.602 \times 10^{-19}$ C, accurate to the significant digits given. It is seen that the real error of Millikan's value was greater than 0.7%, which is five times that given by Millikan. The problem arose from the fact that the value for viscosity of air available to Millikan was wrong. Due to this error, all values of the atomic constants, such as Planck's constant and Avogadro's constant, the determination of which depends on the value of e, were wrong by errors larger than 0.7% until 1930.

The avoidance of systematic errors depends mainly on the observer's experience. Systematic errors are difficult to detect and usually are the most important errors present in measurements. The most common way of detecting systematic errors is the *calibration* of the measuring instrument, by comparing it with another instrument which is known to have greater precision and negligible systematic errors. Another way of testing for systematic errors in an instrument or procedure is to use it in the measurement of a *standard*. A balance, for example, may be tested by weighing a standard of known weight. A voltmeter may be calibrated by the measurement of a standard of *emf*. A radioactive source of well known activity may be used in the calibration of an arrangement for the measurement of radioactivity.

Figure 2.1 illustrates the relationship between random and systematic errors.

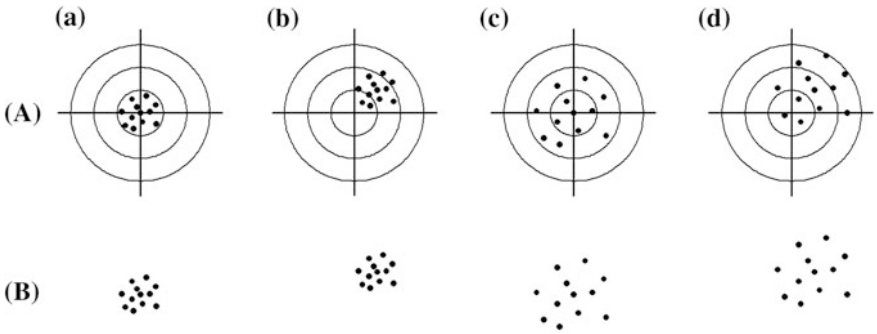

Fig. 2.1 A schematic illustration of random and systematic errors: ***a*** Random errors only. ***b*** Random and systematic errors. ***c*** Only random errors, but larger than those of (***a***). ***d*** Random errors larger than those of (***a***) and systematic errors. In line **A** the real value of the measured quantity is also shown (*center of the circles*) and the distinction between random and systematic errors is possible. In practice, however, the real value is not known [line **B**] and the detection of systematic errors is difficult

2.1.3 Personal Errors

The habits of experimentalists differ and it has been adequately documented, mainly from astronomical observations, that, in measurements where subjective judgment is important, some systematic errors are characteristic of the observer or the instrument-observer combination. Bessel had examined the positions of stars, as these were determined by leading astronomers of his age by measuring the times of passage across the meridians of various observatories and found systematic differences between them. Another good example refers to the estimation of the sunspot activity. This is measured by a number *R,* which was proposed by R. Wolf of the Zurich Observatory and is defined as:

$$R = k[10 \times (\text{number of visible groups of sunspots}) + (\text{number of all sunspots})].$$

The coefficient k in the definition of this so-called *Wolf number*, depends on the combination of the observer and the telescope used. It is determined by comparison with some such combination which is used as standard for $k = 1$. It is found that it has values which differ, in some cases, by up to 20% from unity. The k coefficients are known for many combinations of telescope-observer, a fact that makes possible the coordinated observation of solar activity by many observatories simultaneously.

2.1.4 Occasional Errors

In some experimental arrangements it is possible for signals to be detected and measured which occur very rarely and cannot be considered to be a permanent source of noise. For example, in a system which measures pulses resulting from some process under investigation, a false event may be recorded, such as one due to the relatively rare high-energy cosmic ray showers. It must be stressed here that some of science's greatest discoveries were made possible when an experienced observer realized that such a signal was not a spurious noise signal but was caused by an unknown effect.

More pedestrian causes may lead to similar mistakes; an example from real life is that of the postgraduate student who was tormented for days trying to explain a small peak that appeared in the curve of light intensity versus time he was recording in his studies, before he realized that the source of the signal was the building's elevator arriving at the fourth floor where his laboratory was situated! Apart from working in a hut outside the main building, the student learned the advantages of good shielding of his apparatus from external electrical signals.

The first author had a similar, very striking experience during his participation in a project to detect gravitational radiation back in the 70s. The detector, a cylinder of mass over 600 kg, was so sensitive that it could detect minute sound signals before the chamber in which it was situated was evacuated to a low enough pressure.

This resulted in pulses appearing at the output, which one could either interpret as gravitational radiation pulses emitted when a star fell into a black hole at the center of the Galaxy or to the sound waves created when someone was walking along the corridor of the floor above. The final site for the detector was actually a farm belonging to the university, outside the city, where the farm's cows could not cause such pulses.

The problem with occasional errors is mainly that they are so rare that we cannot predict the number of expected events during our experiment. This number is small, in any case. Occasional errors, however, may be very significant and cause a modification of the results of a sensitive experiment when they occur. We might call them *parasitic*, although this term fits any unwanted signal in general.

2.1.5 The Errors in Reading the Indications of Instruments

Despite the fact that in modern instruments the participation of the observer in the taking of readings becomes more and more rare, there are still many instances in which the subjectivity factor and the habits of the observer play an important role in the reading of the indications of instruments. Most uncertainties which are due to the observer may be minimized as the observer's experience increases, both in general and due to the fact that a procedure is repeated many times. In all cases, however, it is necessary for the observer to be fully conscious of both his capabilities and of the instruments he is using. Overestimation of these capabilities may lead to problems.

For example, if we measure a length 5 times using a ruler and record the results

$$17 \quad 17 \quad 17 \quad 17 \quad 17 \text{ mm}$$

are we justified in stating that the measured length is equal to 17 mm *exactly*? Obviously not. If the length was actually 17.01 mm would we be able to measure this? The answer is 'no', because the method we used did not allow us to detect the difference between 17.00 and 17.01 mm. The *accuracy with which we read the scale* of the instrument we are using is a basic quantity that we must always have in mind. In a well designed experiment, the accuracy of the method should manifest itself in the differences among the numerical results of repeated measurements. By this we mean that the accuracy with which the instrument's scale can be read should be such that the random errors of the measurements become apparent. Such a set of measurements would be, for example,

$$17.2 \quad 17.0 \quad 16.8 \quad 17.1 \quad 16.9 \text{ mm}.$$

This would mean, however, that we have the ability to take readings with an accuracy of 0.1 mm, which is not the case when we measure lengths with a ruler

and the naked eye. The belief that we have this ability is over-ambitious in this particular case. The solution is obvious:

In those cases in which the random errors are not apparent in our measurements, the uncertainty in our result should be based on the accuracy with which we can read the indications of the instrument used.

Examples are given below:

Let us assume that we are measuring the length of an object using a ruler and the naked eye. We naturally assume that the ends of the object are well defined, so that it makes sense to talk of its *exact length*. If the smallest subdivisions on the ruler correspond to mm, then the procedure would be to place one end of the object next to an incision on the ruler and then see with which incision of the ruler the other end coincides. The difference of the two readings on the ruler will be the length of the object. The positioning of the one end of the object next to an incision on the ruler may usually be achieved with an accuracy of 1/5 mm. The reading of the position of the other end has the same uncertainty. The *reading error* for the length of the object will, therefore, be of the order of $2 \times 1/5 = 0.4$ mm. A more realistic estimate of the reading error for measurements with a common ruler would usually be 1 mm. So, for the 5 measurements we had above, all of which gave the result of 17 mm, it follows that:

The length measured is 17 mm, with a possible error of the order of 1 mm.

What we have said above also apply to the case when we have only one measurement of a quantity. In this case no estimate of the random error can be made and the reading error should be considered to be a lower estimate of the error of the measurement.

Another example is that of the measurement of time. We assume that we use a chronometer of the traditional kind, with a circular scale and subdivisions of 1/5 of a second. If we measure the time that passes between two events and we read an indication of, say, 15.8 s, just how sure are we of this result? We assume that the two events are well defined in time so that it makes sense to talk of accuracy in the timing of their occurrence equal to 1/5 s. Our reaction time in pressing the chronometer's knob at the right time is not zero. It might be that a reaction time of 1/10 s is possible for some people. However, do the mechanical parts of the chronometer's knob react at such a speed? Something else that must be taken into account is that the chronometer's pointer does not change position continuously but in steps of 1/5 s. In the best of cases, therefore, the accuracy of our measurements cannot be smaller than about 1/5 s, assuming of course that the chronometer is that well manufactured.

The problem is also present in measurements performed with instruments having digital indications. If the instrument has a 4-digit display, we must assume that the error corresponds to one unit in the last digit. Most digital instruments do not round to the nearest previous digit but simply reject all digits beyond those shown in the display. The *round-off error* must thus be taken to be equal to one unit in the last

digit of the display. So, if the indication of a digital instrument is, for example, 1.245, the reading error must be taken as being equal to 0.001 units.

Good quality instruments are usually accompanied by instructions on how to estimate the highest possible systematic error, something that is determined by the manufacturer by calibrating the instrument. A voltmeter may, for example be accompanied by a certificate stating that:

The maximum possible error in the measured voltage is equal to:

0.005 $\times$ (*maximum indication of the scale used*) + *0.010* $\times$ (*indication of the instrument*).

If, for example we are using the scale of 0–3 mV and the indication of the instrument is 2.45 mV, the maximum possible systematic error due to the instrument is $0.005 \times 3 + 0.010 \times 2.45 = 0.015 + 0.025 = 0.04$ mV.

While it is certain that an accurate value of the magnitude being measured is desirable, it is not always necessary for us to do *everything we can* in order to lower the error as much as possible; in most cases, a *reasonable* error is tolerable. However, what is certainly needed is for us to have a good estimate of the possible error in our measurement. The use of our measurements so as to derive from them the most accurate estimate for the real value of the magnitude being measured, as well as of the possible error in this value, is the main purpose of the first part of this book. The theory to be used in the mathematical analysis is valid only for random errors. We must never forget, therefore, that our measurements may contain systematic errors which are much larger than the random errors and which will be definitive for the usefulness of our results.

2.2 Errors in Compound Quantities

We will now examine the methods of evaluating the error in a compound quantity $Q = Q(x, y, \ldots)$ which is a function of the quantities x, y, …, which we have measured. If x_0, y_0, … are the *real values* of the quantities x, y, … and x_m, y_m, … the numerical values that resulted from their measurement, then, the *errors* in these magnitudes are defined as

$$e_x \equiv x_m - x_0, \quad e_y \equiv y_m - y_0, \ldots. \tag{2.1}$$

The *reduced or fractional error* in the quantity x is defined as

$$f_x \equiv \frac{e_x}{x_0}. \tag{2.2}$$

Obviously, it is

$$x_{\mathrm{m}} = x_0 + e_x = x_0(1 + f_x). \tag{2.3}$$

The *percentage error* is also defined as $100e_x/x_0\%$.

If the fractional error in x is small compared to unity ($f_x \ll 1$), the following approximate relations hold

$$\frac{x_0}{x_{\mathrm{m}}} = \frac{1}{1 + f_x} \approx 1 - f_x \tag{2.4}$$

and

$$\frac{e_x}{x_0} = \frac{e_x}{x_{\mathrm{m}}}\frac{x_{\mathrm{m}}}{x_0} = \frac{e_x}{x_{\mathrm{m}}}(1 + f_x) \approx \frac{e_x}{x_{\mathrm{m}}}. \tag{2.5}$$

The errors e_x, e_y, … are unknown to us, since we do not know the real values x_0, y_0, … of the magnitudes x, y, …. As a consequence, the error in the compound quantity Q will also be unknown to us. We could say that the numerical results of the examples to follow would be known only to somebody who knew, apart for our experimental results, the real values of the quantities being measured as well. We will, however, examine the way in which the errors in x, y, … affect the estimated value Q_{m} of the compound quantity Q, because this will help us understand the concept of *propagation of errors*, i.e. the evaluation of the deviation of the value Q_{m} from the real value Q_0, due to the errors in x_{m}, y_{m},….

2.2.1 Error in a Sum or a Difference

If it is $Q = x + y$, then $Q_0 = x_0 + y_0$ and if the measurements of x and y gave the results x_{m} and y_{m}, it will be

$$Q_{\mathrm{m}} = x_{\mathrm{m}} + y_{\mathrm{m}} = x_0 + y_0 + e_x + e_y = Q_0 + e_Q, \tag{2.6}$$

where e_Q is the error in Q_{m}. The fractional error in $Q = x + y$ is, therefore,

$$f_Q = \frac{e_Q}{Q_0} = \frac{e_x + e_y}{x_0 + y_0} = \frac{x_0 f_x + y_0 f_y}{x_0 + y_0}. \tag{2.7}$$

The relation also holds when either x_0 or y_0 is negative, in which case Eq. (2.7) gives the error in the difference of the two quantities.

We observe that, if f_x and f_y are comparable and $x_0 \gg y_0$, then $f_Q \approx f_x$, while, if $y_0 \gg x_0$, then $f_Q \approx f_y$.

The result may be generalized to give the fractional error in $Q_0 = x_0 + y_0 + z_0 + \ldots$ as

$$f_Q = \frac{e_Q}{Q_0} = \frac{e_x + e_y + e_z + \ldots}{x_0 + y_0 + z_0 + \ldots} = \frac{x_0 f_x + y_0 f_y + z_0 f_z + \ldots}{x_0 + y_0 + z_0 + \ldots}. \tag{2.8}$$

In particular, if it is $Q_0 = kx_0$ where k is an integer, putting $x_0 = y_0 = z_0 = \ldots$ (k terms) in Eq. (2.8) we have

$$f_Q = \frac{e_Q}{Q_0} = \frac{ke_x}{kx_0} = \frac{e_x}{x_0} = f_x \quad \text{or} \quad f_Q = f_x. \tag{2.9}$$

The result is true for every k: Since $Q_0 = kx_0$, $Q_\mathrm{m} = kx_\mathrm{m}$ and $x_\mathrm{m} = x_0(1 + f_x)$, we have

$$Q_m = kx_0(1 + f_x) = Q_0(1 + f_x),$$

and, because $Q_m = Q_0(1 + f_Q)$, it follows that $f_Q = f_x$.

Example 2.1

The measurements of x and y gave the results $x_\mathrm{m} = 6.2$ cm and $y_\mathrm{m} = 3.6$ cm. The real values of these quantities are $x_0 = 6.1$ cm and $y_0 = 3.4$ cm. What is the error in the sum $Q = x + y$?

Obviously, the real value of Q is $Q_0 = x_0 + y_0 = 6.1 + 3.4 = 9.5$ cm. The value determined by the measurements is $Q_\mathrm{m} = x_\mathrm{m} + y_\mathrm{m} = 6.2 + 3.6 = 9.8$ cm. It is immediately seen that the error in Q is equal to $e_Q = 9.8 - 9.5 = 0.3$ cm and the fractional error is $f_Q = e_Q/Q_0 = 0.3/9.5 = 0.03$, or 3%.

Using Eq. (2.7), we find again

$$f_Q = \frac{x_0 f_x + y_0 f_y}{x_0 + y_0} = \frac{(x_\mathrm{m} - x_0) + (y_\mathrm{m} - y_0)}{x_0 + y_0} = \frac{x_\mathrm{m} + y_\mathrm{m}}{x_0 + y_0} - 1 = 0.03.$$

Example 2.2

The quantities x and y were measured with fractional errors $f_x = 0.01$ and $f_y = 0.02$. If the real values of these quantities are $x_0 = 15$ m and $y_0 = 5$ m, what is the fractional error in the sum $Q = x + y$?

From Eq. (2.7) we have

$$f_Q = \frac{x_0 f_x + y_0 f_y}{x_0 + y_0} = \frac{15 \times 0.01 + 5 \times 0.02}{15 + 5} = \frac{0.15 + 0.10}{20} = 0.0125, \text{ or } 1.25\%.$$

2.2.2 Error in a Product

If $Q = xy$, then $Q_0 = x_0 y_0$ and if the measurements of x and y gave the results x_m and y_m, it will be

$$\begin{aligned} Q_\mathrm{m} &= x_\mathrm{m} y_\mathrm{m} = (x_0 + e_x) \times (y_0 + e_y) = x_0(1 + f_x) \times y_0(1 + f_y) \\ &= x_0 y_0 (1 + f_x)(1 + f_y). \end{aligned} \tag{2.10}$$

Since for small f_x and f_y it is $(1 + f_x)(1 + f_y) \approx 1 + f_x + f_y$, it follows that

$$Q_\mathrm{m} \approx Q_0(1 + f_x + f_y) \tag{2.11}$$

and the fractional error in Q is

$$f_Q = \frac{Q_\mathrm{m} - Q_0}{Q_0} = f_x + f_y, \tag{2.12}$$

i.e., *the fractional error in $Q = xy$ is equal to the sum of the fractional errors in x and y.*

The result may be generalized and in the case of $Q = xyz\ldots$. We have

$$Q_\mathrm{m} = x_\mathrm{m} y_\mathrm{m} z_\mathrm{m} \ldots = x_0 y_0 z_0 \ldots (1 + f_x)(1 + f_y)(1 + f_z)\ldots \approx Q_0(1 + f_x + f_y + f_z + \ldots) \tag{2.13}$$

and, therefore,

$$f_Q = \frac{Q_\mathrm{m} - Q_0}{Q_0} = f_x + f_y + f_z + \ldots, \tag{2.14}$$

i.e., *the fractional error in $Q = xyz\ldots$ is equal to the algebraic sum of the fractional errors in x, y, z, ….*

Example 2.3

The measurements of the quantities x and y gave results with fractional errors $f_x = 0.01$ and $f_y = 0.02$, respectively. Which is the fractional error in the product $Q = xy$?

Equation (2.12) gives $f_Q = f_x + f_y = 0.01 + 0.02 = 0.03$ or 3%.

2.2.3 Error in a Power

For the special case of $Q = x^n$ where n is a positive integer, Eq. (2.14) gives $f_Q = n f_x$ or that *the fractional error of the power x^n is equal to n times the fractional error in x.*

Generally, let $Q = kx^n$, where n is any real number and k a constant. Since it is $Q_m = Q_0(1+f_Q)$, $x_m = x_0(1+f_x)$, $Q_0 = kx_0^n$ and $Q_m = kx_m^n$, we have

$$Q_m = Q_0(1+f_Q) = kx_m^n = kx_0^n(1+f_x)^n \approx Q_0(1+nf_x) \tag{2.15}$$

for $f_x \ll 1$. From the equality of the first term and the last term, it follows that

$$f_Q = nf_x \tag{2.16}$$

or that *the fractional error in any multiple of the n-th power of x is equal to n times the fractional error in x.*

Special cases: If $Q = x^2$ it is $f_Q = 2f_x$ and if $Q = \sqrt{x}$ it is $f_Q = \frac{1}{2}f_x$.

Example 2.4

If the result of the measurement of x has a fractional error $f_x = 0.005$, what is the fractional error in the quantity $Q = 7x^{3/2}$?

From Eq. (2.16), $f_Q = nf_x = \frac{3}{2} \times 0.005 = 0.0075 \approx 0.008$.

2.2.4 Error in a Quotient

If it is $Q = x/y$ and the measurements of x and y gave the results x_m and y_m, then

$$Q_m = \frac{x_m}{y_m} = \frac{x_0 + e_x}{y_0 + e_y} = \frac{x_0(1+f_x)}{y_0(1+f_y)} \approx \frac{x_0}{y_0}(1+f_x)(1-f_y) \approx Q_0(1+f_x-f_y) \tag{2.17}$$

for small f_x and f_y, and the fractional error in Q is

$$f_Q = \frac{Q_m - Q_0}{Q_0} = f_x - f_y, \tag{2.18}$$

i.e. *the fractional error in $Q = x/y$ is equal to the difference of the fractional errors of x and y.*

The result may be generalized and in the case when it is $Q = \frac{x'y'z'\ldots}{x''y''z''\ldots}$ we have

$$\begin{aligned} Q_m &= \frac{x'_m y'_m z'_m \ldots}{x''_m y''_m z''_m \ldots} = \frac{x'_0 y'_0 z'_0 \ldots}{x''_0 y''_0 z''_0 \ldots} \times \frac{(1+f_{x'})(1+f_{y'})(1+f_{z'})\ldots}{(1+f_{x''})(1+f_{y''})(1+f_{z''})\ldots} \\ &\approx Q_0(1+f_{x'}+f_{y'}+f_{z'}+\ldots-f_{x''}-f_{y''}-f_{z''}-\ldots) \end{aligned} \tag{2.19}$$

and

$$f_Q = \frac{Q_m - Q_0}{Q_0} = (f_{x'}+f_{y'}+f_{z'}+\ldots) - (f_{x''}+f_{y''}+f_{z''}+\ldots), \tag{2.20}$$

i.e. *the fractional error in Q is equal to the algebraic sum of the fractional errors in $x', y', z', \ldots$, minus the algebraic sum of the fractional errors in $x'', y'', z'', \ldots$.*

Example 2.5

If the quantities x and y were measured with fractional errors $f_x = -0.015$ and $f_y = 0.02$, respectively, what will the fractional error in $Q = x^2/y$ be?

We initially evaluate the fractional error in x^2. Equation (2.16) gives $f_{x^2} = 2f_x = -0.03$. Then Eq. (2.18) gives $f_Q = f_{x^2} - f_y = -0.03 - 0.02 = -0.05$.

2.2.5 The Use of Differentials

2.2.5.1 Functions of One Variable

If $Q(x)$ is a function of one variable, x, its derivative is $\frac{dQ}{dx}$. From the definition of the derivative $\lim_{\delta x \to 0} \frac{\delta Q}{\delta x} = \frac{dQ}{dx}$, it follows that for small δx it is, approximately,

$$\delta Q \approx \frac{dQ}{dx} \delta x. \tag{2.21}$$

Equation (2.21) gives the change δQ in Q due to a small change δx in x. If now it is $x_m = x_0 + e_x$ and $Q_m = Q_0 + e_Q$, and we put $\delta x \equiv x_m - x_0 = e_x$ and $\delta Q \equiv Q_m - Q_0 = e_Q$ in Eq. (2.21), we will have, to a good approximation for small e_x,

$$e_Q = \frac{dQ}{dx} e_x, \tag{2.22}$$

a relationship which correlates the error in Q to the error in x.

The geometrical interpretation of the relations (2.21) and (2.22) is given in Fig. 2.2. Assuming a linear relationship between δQ and δx, which is shown in the figure by the tangent to the curve $Q(x)$ at x, we evaluate the error in δQ. The dashed line gives a better value for δQ, because it takes into account the non-linearity of $Q(x)$. We would also have a better value for δQ by taking point x at the center of δx. These, however, are second-order corrections, which are not important for small values of δx.

For example, if $Q = x^2$, then $\frac{dQ}{dx} = 2x$ and, therefore, $e_Q = 2xe_x$. Dividing on the left by Q and on the right by x^2, we find that $\frac{e_Q}{Q} = 2\frac{e_x}{x}$, or $f_Q = 2f_x$, as we found above. In the same way we may verify that, if it is $Q = \sqrt{x}$, then $f_Q = \frac{1}{2} f_x$.

Strictly speaking, the derivative should be evaluated at $x = x_0$. However, since this value is not known to us and because we assume that x_m does not differ by much from x_0, we can do nothing else but evaluate the derivative at the point $x = x_m$.

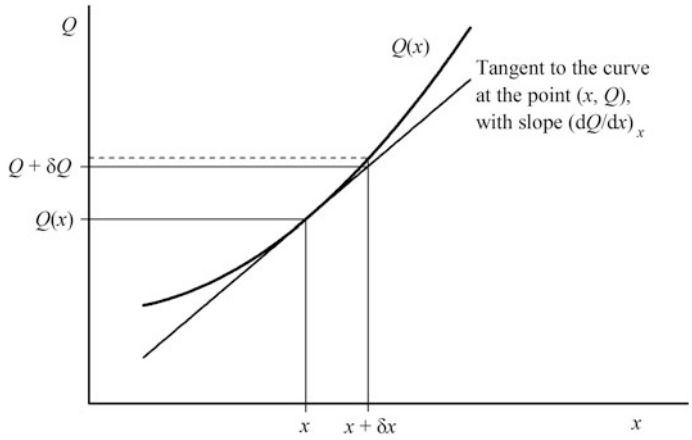

Fig. 2.2 The relationship between the error δx in the variable x and the corresponding error δQ in the function $Q(x)$

Warning: When angles are involved in Eqs. (2.21) and (2.22), we must bear in mind that the relations $\delta(\sin\theta) \approx \dfrac{\mathrm{d}(\sin\theta)}{\mathrm{d}\theta}\delta\theta = \cos\theta\,\delta\theta$, $\delta(\cos\theta) \approx \dfrac{\mathrm{d}(\cos\theta)}{\mathrm{d}\theta}\delta\theta = -\sin\theta\,\delta\theta$, as well as other similar trigonometric relations, are valid only if the error in the angle, $\delta\theta$, is given in radians. This is a common source of errors when one meets such problems for the first time.

Example 2.6

On measuring the radius of a sphere, the value $r_\mathrm{m} = 10.1$ mm was found, instead of the real value $r_0 = 10$ mm. What will the error be in the volume of the sphere, if this is evaluated using the value r_m?

Since the volume of the sphere is given by $V = \frac{4}{3}\pi r^3$ and $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$, it will be $e_V = \delta V = 4\pi r^2 \delta r = 4\pi r^2 e_r$, where $e_r = 10.1 - 10 = 0.1$ mm. If in the evaluation of $\dfrac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$ we use the real value $r_0 = 10$ mm, we find that

$$e_V = \delta V = 4\pi r_0^2 e_r = 4\pi \times (10)^2 \times 0.1 = 126\,\mathrm{mm}^3.$$

The real volume of the sphere is $V_0 = \frac{4}{3}\pi r_0^3 = 4189$ mm^3.

Using the value $r_\mathrm{m} = 10.1$ mm, we find $V_\mathrm{m} = \frac{4}{3}\pi r_\mathrm{m}^3 = 4316$ mm^3, which is larger than V_0 by 127 mm^3.

The fractional error in the volume is equal to 127/4189 = 0.030 (or 3%), which is three times the fractional error 0.1/10 = 0.010 (or 1%) in the radius. This is expected, since it is $V \propto r^3$.

2.2.5.2 Functions of Many Variables

If $Q(x, y, z, \ldots)$ is a function of the variables x, y, z,..., then it is known from differential calculus that the *differential* of the function is

$$\mathrm{d}Q = \frac{\partial Q}{\partial x}\mathrm{d}x + \frac{\partial Q}{\partial y}\mathrm{d}y + \frac{\partial Q}{\partial z}\mathrm{d}z + \ldots, \tag{2.23}$$

where $\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}, \ldots$ are the *partial derivatives* of the function $Q(x, y, z, \ldots)$ with respect to the variables x, y, z, ..., respectively. (The concept of the partial derivative is simple: the partial derivative $\frac{\partial Q}{\partial x}$ of Q with respect to x is found by differentiating $Q(x, y, z, \ldots)$ with respect to x, keeping all the other variables, y, z, ... constant. $\frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}, \ldots$ are found in a similar way.)

For small changes $\delta x, \delta y, \delta z, \ldots$ in $x, y, z, \ldots$, the change in $Q(x, y, z, ..)$ is given by

$$\delta Q = \frac{\partial Q}{\partial x}\delta x + \frac{\partial Q}{\partial y}\delta y + \frac{\partial Q}{\partial z}\delta z + \ldots, \tag{2.24}$$

which contains an infinite number of terms of higher order in $\delta x, \delta y, \delta z, \ldots$, of the form $\frac{\partial^2 Q}{\partial x^2}(\delta x)^2$, $\frac{\partial^2 Q}{\partial y^2}(\delta y)^2$, $\frac{\partial^2 Q}{\partial y \partial x}(\delta x)(\delta y)$ etc., which have been omitted as negligible.

If $\delta x \equiv x_\mathrm{m} - x_0 = e_x, \delta y \equiv y_\mathrm{m} - y_0 = e_y, \delta z \equiv z_\mathrm{m} - z_0 = e_z, \ldots$ are the errors in the values of x, y, z, ..., then

$$e_Q = \frac{\partial Q}{\partial x}e_x + \frac{\partial Q}{\partial y}e_y + \frac{\partial Q}{\partial z}e_z + \ldots \tag{2.25}$$

is the error in Q.

Strictly speaking, the evaluation of the partial derivatives $\frac{\partial Q}{\partial x}, \frac{\partial Q}{\partial y}, \frac{\partial Q}{\partial z}, \ldots$ should be done using the real values of $x, y, z, \ldots$, which are not known. However, if the fractional errors in these, $f_x, f_y, f_z, \ldots$, are small enough, the measured values $x_\mathrm{m}, y_\mathrm{m}, z_\mathrm{m}, \ldots$ may be used without introducing a significant error in the calculations. This method is adopted in most of the examples that follow.

The relations (2.24) and (2.25) are two equivalent formulations of the *principle of superposition of errors*. Its physical interpretation is evident if we consider the term $\frac{\partial Q}{\partial x}e_x$ to be the error in Q which is due to the error e_x of x etc. Of course, in the evaluation of the error in Q, due to the propagation of errors, we will find that we can only make statistical predictions for $e_Q = \delta Q$, since we can only make statistical predictions concerning the values of $e_x = \delta x$, $e_y = \delta y$, $e_z = \delta z, \ldots$. So, if $\delta x, \delta y, \delta z, \ldots$ are the *probable errors* in x, y, z, ..., we will prove that the probable error in Q is

$$\delta Q = \sqrt{\left(\frac{\partial Q}{\partial x}\delta x\right)^2 + \left(\frac{\partial Q}{\partial y}\delta y\right)^2 + \left(\frac{\partial Q}{\partial z}\delta z\right)^2 + \ldots}, \tag{2.26}$$

i.e. we will prove that *the probable error in Q is equal to the square root of the sum of the squares of the contributions of the probable errors in x, y, z, ... to the error in Q.*

Example 2.7

The real values of the quantities x, y and z are $x_0 = 1$, $y_0 = 2$ and $z_0 = 3$. They were measured with fractional errors $f_x = 0.01$, $f_y = -0.02$ and $f_z = 0.01$, respectively. What is the fractional error in the function $Q = 2x^2y + 5z/y$?

The real value of Q is $Q_0 = 2x_0^2y_0 + 5z_0/y_0 = 2 \times 1^2 \times 2 + 5 \times 3/2 = 11.5$. Also, the errors in x, y and z are

$$\delta x = 1 \times 0.01 = 0.01, \quad \delta y = 2 \times (-0.02) = -0.04, \quad \delta z = 3 \times 0.01 = 0.03.$$

The partial derivatives of $Q = 2x^2y + 5z/y$ are $\frac{\partial Q}{\partial x} = 4xy, \frac{\partial Q}{\partial y} = 2x^2 - \frac{5z}{y^2}, \frac{\partial Q}{\partial z} = \frac{5}{y}$. From Eq. (2.24), it is $\delta Q = \frac{\partial Q}{\partial x}\delta x + \frac{\partial Q}{\partial y}\delta y + \frac{\partial Q}{\partial z}\delta z$ and we have

$$\delta Q = 4xy\delta x + \left(2x^2 - \frac{5z}{y^2}\right)\delta y + \frac{5}{y}\delta z.$$

Substituting, we find $\delta Q = 4 \times 1 \times 2 \times 0.01 + \left(2 \times 1^2 - 5 \times \frac{3}{2^2}\right) \times (-0.04) + \frac{5}{2} \times 0.03 = 0.225$ and $f_Q = \delta Q/Q_0 = 0.225/11.5 = 0.0196 = 0.02$ or 2%.

Example 2.8

The hypotenuse of a right-angled triangle was measured to be equal to $a = 10.3$ m, with fractional error $f_a = 0.01$, and one of the triangle's angles was measured to be $B = 56.3°$, with fractional error $f_B = 0.02$. Using the results for a and B, find the values and the fractional errors of the other elements of the triangle (angle C and sides b and c).

The other elements of the triangle are given by $C = 180° - A - B$, $b = a \sin B$, $c = a \cos B$.

(a) *The other acute angle*: $C = 180° - A - B$.

We take angle A of the triangle to be a right angle and, therefore, $A = 90°$ exactly. Therefore, $C = 90° - B = 90° - 56.3° = 33.7°$
which has an error equal to $e_C = e_{90°} - e_B = 0° - 56.3° \times 0.02 = -1.1°$
and a fractional error $f_C = e_C/C_0 \approx e_C/C = -1.1°/33.7° = -0.033$ or –3.3%.

(b) *The opposite side*: $b = a \sin B$.

Since $B = 56.3°$ and $a = 10.3$ m, we have $b = a \sin B = 10.3 \times \sin 56.3° = 8.57$ m.
Here, $\frac{\partial b}{\partial a} = \sin B$ and $\frac{\partial b}{\partial B} = a \cos B$ and, therefore, $\delta b = \frac{\partial b}{\partial a}\delta a + \frac{\partial b}{\partial B}\delta B = \sin B \delta a + a \cos B \delta B$.
The error in a is $\delta a \approx 10.3 \times 0.01 = 0.10$ m.

Also, $\delta B = 56.3^\circ \times 0.02 = 1.13^\circ$ and therefore $\delta B = \frac{2\pi}{360^\circ} \times 1.13^\circ = 0.0197 = 0.020$ rad.

So, $\delta b = \sin 56.3^\circ \times 0.103 + 10.3 \times \cos 56.3^\circ \times 0.020 = 0.0857 + 0.1143 = 0.20$ m.

(c) *The adjacent side*: $c = a \cos B$.

The length of the side is $c = 10.3 \cos 56.3^\circ = 5.71$ m.

Here $\frac{\partial c}{\partial a} = \cos B$ and $\frac{\partial c}{\partial B} = -a \sin B$ and, therefore, $\delta c = \frac{\partial c}{\partial a}\delta a + \frac{\partial c}{\partial B}\delta B = \cos B \delta a - a \sin B \delta B$.

So, $\delta c = \cos 56.3^\circ \times 0.103 - 10.3 \times \sin 56.3^\circ \times 0.020 = 0.0571 - 0.1714 = -0.11$ m.

Example 2.9

The acceleration of gravity g may be determined by measuring the period T of a pendulum of length l and using the relation $g = 4\pi^2 l/T^2$. In one such experiment, with a pendulum of length l = 1.000 m, whose fractional error is $f_l = -0.005$, a period of T = 2.01 s was measured, with a fractional error $f_T = 0.01$. Find the value of g and its fractional error f_g.

The value of g found by using the measured values of l and T is equal to $g = 4\pi^2 l/T^2 = 4 \times (3.1416)^2 \times 1.000/(2.01)^2 = 9.76$ m/s^2.

Because it is $\delta g = \frac{\partial g}{\partial l}\delta l + \frac{\partial g}{\partial T}\delta T = 4\pi^2\left(\frac{1}{T^2}\delta l - 2\frac{l}{T^3}\delta T\right) = 4\pi^2 \frac{l}{T^2}\left(\frac{\delta l}{l} - 2\frac{\delta T}{T}\right) = g\left(\frac{\delta l}{l} - 2\frac{\delta T}{T}\right)$, the fractional error in g is equal to

$$f_g = \frac{\delta g}{g} = \frac{\delta l}{l} - 2\frac{\delta T}{T} = f_l - 2f_T = -0.005 - 2 \times 0.01 = -0.025.$$

This is equivalent to an error in g equal to $\delta g = f_g g = -0.025 \times 9.76 = -0.24$ m/s^2.

It is worth examining the following question: What portion of the error in g is due to the fact that we have used the approximate value $\pi \approx 3.14$ instead of the exact value?

If we consider π to be a variable with error $\delta\pi = 3.14 - \pi = 3.14 - 3.14159\ldots = -0.0016$, then the contribution of $\delta\pi$ to the error δg in g will be equal to

$$\delta g_\pi = \frac{\partial g}{\partial \pi}\delta\pi = \frac{8\pi l}{T^2}\delta\pi = 2g\frac{\delta\pi}{\pi} = 2 \times 9.76 \times \frac{(-0.0016)}{3.14159} = -0.00994 = -0.01\,\text{m/s}^2,$$

which is negligible compared to the error $\delta g = -0.24$ m/s^2 due to the errors in l and T. This should have been expected, since the fractional error in π is only $f_\pi = \delta\pi/\pi = -0.0016/3.14159 = -0.0005$ or -0.05%, while the fractional errors in l and T are -0.5% and 1%, respectively.

Example 2.10

Find the relationship between the errors in the variables $x, y, z, \ldots$ and the error in the function $Q(x, y, x, \ldots) = Ax^{\alpha}y^{\beta}z^{\gamma}\ldots$, where A is a constant.

The natural logarithm of the function is $\ln Q = \ln A + \alpha \ln x + \beta \ln y + \gamma \ln z + \ldots$. From this relation, by taking differentials, we get $\frac{\mathrm{d}Q}{Q} = \alpha\frac{\mathrm{d}x}{x} + \beta\frac{\mathrm{d}y}{y} + \gamma\frac{\mathrm{d}z}{z} + \ldots$. Therefore, for small δx, δy, $\delta z, \ldots$ the approximation $\frac{\delta Q}{Q} = \alpha\frac{\delta x}{x} + \beta\frac{\delta y}{y} + \gamma\frac{\delta z}{z} + \ldots$ holds.

In Example 2.9, the function was $g = 4\pi^2 lT^{-2}$. Therefore, the last relation gives $\frac{\delta g}{g} = \frac{\delta l}{l} - 2\frac{\delta T}{T}$. Assuming that π is also a variable, it is $\frac{\delta g}{g} = 2\frac{\delta \pi}{\pi} + \frac{\delta l}{l} - 2\frac{\delta T}{T}$, in agreement with the last results.

Example 2.11

If it is difficult to differentiate an expression, the error of which we require, it is possible to use numerical methods for this purpose. We will apply this technique to the expression $F = x^{y^z}$.

Let $x = 2.00$, $y = 1.50$, $z = 1.20$ and $\delta x = 0.10$, $\delta y = -0.15$, $\delta z = 0.20$. We need to find the corresponding δF.

Given F, it is true that:

$$\frac{\partial F}{\partial x}\delta x \approx F(x+\delta x, y, z) - F(x, y, z) = (x+\delta x)^{y^z} - x^{y^z} = 2.10^{1.50^{1.20}} - 2.00^{1.50^{1.20}} = 0.2535$$

$$\frac{\partial F}{\partial y}\delta y \approx F(x, y+\delta y, z) - F(x, y, z) = x^{(y+\delta y)^z} - x^{y^z} = 2.00^{1.35^{1.20}} - 2.00^{1.50^{1.20}} = -0.3870$$

$$\frac{\partial F}{\partial z}\delta z \approx F(x, y, z+\delta z) - F(x, y, z) = x^{y^{z+\delta z}} - x^{y^z} = 2.00^{1.50^{1.40}} - 2.00^{1.50^{1.20}} = 0.3086$$

Summing, we have $\delta F = 0.175$.

Completing Sect. 2.2, we repeat that most of what were mentioned are usually of no use in arithmetical applications, since the errors we referred to are not known. The topics we examined, however, are of great theoretical importance and constitute the background for the understanding of the theory of errors, something which will become obvious in the following chapters.

Problems

2.1 When you stand on the bathroom scales, its reading is 70.5 kg. When you get off it, it shows –1.5 kg. How much do you weigh?

2.2 The fractional errors in the lengths of the sides a and b of a rectangle are –2 and 3%, respectively. Find the fractional error in its area.

2.3 The lengths of the edges of a rectangular parallelepiped have real values $a = 1$ m, $b = 2$ m and $c = 3$ m. The lengths of these edges were measured and found to be $a_m = 1.02$ m, $b_m = 1.99$ m and $c_m = 3.05$ m. Find:

(a) the volume of the parallelepiped using first a, b and c, and then a_m, b_m and c_m.
(b) the fractional error in the volume of the parallelepiped using the results of (a).
(c) the fractional errors in a, b and c.
(d) the fractional error in the volume of the parallelepiped using the formula for the evaluation of the fractional error of a compound quantity in terms of the fractional errors of the variables on which it depends. Compare with the result of (b).

2.4 Find the fractional error in $Q = x^2yz^{-2}$ in terms of the fractional errors in the variables.

2.5 Find the fractional error in $Q = x^2(y+2)z^{-2}$ in terms of the fractional errors in the variables.

2.6 For the determination of the focal length f of a lens, the distances $s = 0.53$ m and $s' = 0.32$ m of the object and the image from the lens are measured and the formula $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$ is used. If the errors in s and s' are $\delta s = 0.01$ m and $\delta s' = 0.02$ m, find the fractional error in the value of f calculated.

2.7 The rate of flow, $\phi = dV/dt$, of a fluid with viscosity η through a cylindrical pipe of length l and radius r is $\phi = \frac{\pi p r^4}{8l\eta}$, where p is the pressure difference between the two ends of the pipe (Poiseuille's formula). (a) Find the fractional error in ϕ in terms of the fractional errors in η, l, r and p. (b) Which quantity must be measured with the greatest accuracy if we want to have a small error in ϕ?

2.8 The relativistic mass of a body moving with speed V, is given by the relation $m = \frac{m_0}{\sqrt{1-V^2/c^2}}$, where m_0 is a constant of the body, known as its rest mass. If the ratio V/c is very much smaller than unity, find the fractional error in m, in terms of the fractional error in V.

2.9 The displacement x of a simple harmonic oscillator as a function of time t is given by the relation $x = a\sin(\omega t)$, where a and ω are constants. If measurements of a and ω gave the results a_m and ω_m, which have fractional errors δa and $\delta\omega$, respectively, find the fractional error in x as a function of time.

Chapter 3
A Thought Experiment

In order to describe certain processes and introduce some useful terms, we will examine in detail the results of a thought experiment. By the term 'experiment' we mean here a series of successive measurements of the same magnitude, under identical experimental conditions. To be specific, let us assume that we are using a certain instrument to measure the distance x between two points. We will assume that the *real* distance between the points is $x_0 = 100\,\text{mm}$, exactly. Obviously, this is the quantity we do not know and which we will try to determine with our measurements, with as small an error as possible. In the example to follow, the results of our measurements have been selected in such a way that they are subjected to some limitations, which will be taken into account in our arithmetic results to be given below, but which the reader is not required to know at this stage understanding these conditions is, after all, one of the main aims of this book! The 'results' of our measurements in this thought experiment were found using random numbers for the purpose of deciding what the numerical result of each one of these assumed measurements should be. The simulation of experiments using random numbers in order to decide what the outcome of a certain process (e.g. a measurement) should be, is one of the many applications of the so-called *Monte Carlo method*.

3.1 The Thought Experiment

Commencing our measurements, let us assume that the first measurement gives a result of $x_1 = 101.82\,\text{mm}$. This is the only measurement we have of x and is, therefore, the best estimate we can have for x_0. We have, however, no estimate of the amount by which this value is *possible* to differ from x_0. We repeat the measurement under exactly identical conditions which, let us assume, gives the result of $x_2 = 100.49\,\text{mm}$. The values of x_1 and x_2 differ from each other by 0.33 mm. The first thing we notice is that the reading error of our instrument, i.e. 0.01 mm, is

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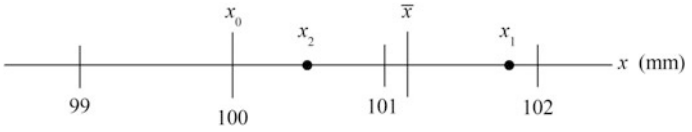


Fig. 3.1 The positions of the first 2 measurements of the magnitude x on the x-axis. Also shown is the real value $x_0 = 100\,\text{mm}$ of the magnitude being measured and the mean value $\overline{x} = 101.16\,\text{mm}$ of the 2 measurements

much smaller than needed, given that our measurements differ from each other by a difference of the order of 1 mm. This makes it clear to us that giving the results with two decimal digits is rather unjustified. Let us ignore this fact for the moment. Something else we can say is that, since the two measurements have a difference of the order of 1 mm, their difference from the real value x_0 is, most probably, of the same order of magnitude. The reader must have noticed that we continuously refer to possible estimates for the various values. This is natural and will be a characteristic of all our conclusions; we may, however, expect that, as we increase the number of our measurements, the confidence we will have in the various numerical values will increase. We mark the numerical results of our two measurements, x_1 and x_2, on the x-axis, as shown in Fig. 3.1.

Having only two measurements, which differ from each other, the only thing we can be almost certain of is that one of them will be nearer to the real value than the other, without knowing which one this value is. It is also certain that, if we take the *mean value* of these two measurements,

$$\overline{x} = \frac{1}{2}(x_1 + x_2), \tag{3.1}$$

this will differ from the real value x_0 by less than the worst one of the two measurements, whichever that might be. We can prove this as follows: The *error* in the measurement x_i is

$$e_i \equiv x_i - x_0. \tag{3.2}$$

For the two measurements it is $x_1 = x_0 + e_1$, $x_2 = x_0 + e_2$ and

$$\overline{x} = \frac{1}{2}(x_1 + x_2) = x_0 + \frac{1}{2}(e_1 + e_2). \tag{3.3}$$

The difference between the mean and the real value is

$$e_\mu = e_{\overline{x}} = \overline{x} - x_0 = \frac{1}{2}(e_1 + e_2) \tag{3.4}$$

which has a lower absolute value than the larger of the absolute values of e_1 and e_2. There is also the possibility for $\overline{x}$ to be nearer to x_0 than either of x_1 and x_2. For the

Table 3.1 Ten measurements of length

i	1	2	3	4	5	6	7	8	9	10
x_i (mm)	101.82	100.49	100.05	99.69	101.31	98.70	98.68	98.75	98.51	100.87

two values of Fig. 3.1 the mean is $\bar{x} = 101.16$ mm and this is also marked in the figure.

If we continue taking measurements, we may have a table of the successive results x_i, such as Table 3.1 in which the results of the first 10 measurements are given, in the order in which they were acquired.

The *mean* of N measurements x_i $(i = 1, 2, \ldots, N)$ is defined as

$$\bar{x} \equiv \frac{1}{N} \sum_{i=1}^{N} x_i. \tag{3.5}$$

For the 10 values of our example we have

$$\bar{x} = \frac{1}{10} \sum_{i=1}^{10} x_i = \frac{1}{10} (101.82 + 100.49 + 100.05 + 99.69 + 101.31 + 98.70 + 98.68 + 98.75 + 98.51 + 100.87) = 99.89 \text{ mm}.$$

As we will explain later, in the particular example we are justified in giving the results with only one decimal digit, so we write:

$$\bar{x} = 99.9 \text{ mm}.$$

In Fig. 3.2 these 10 measurements are marked on the x-axis just as was done in Fig. 3.1 for the first two.

The distribution of the results is better visualized, especially if their number is large, if they are plotted in a *histogram*. To construct a histogram, we follow the procedure described below (with reference to Fig. 3.2): we divide the x-axis into equal intervals of some width Δx, we measure the number ΔN of the measurements found in each interval and raise in each such bin a column with height proportional to ΔN. Figure 3.3 shows the histogram of the measurements of Table 3.1 and Fig. 3.2. A bin width of $\Delta x = 0.1$ mm was chosen. The centers of the intervals were

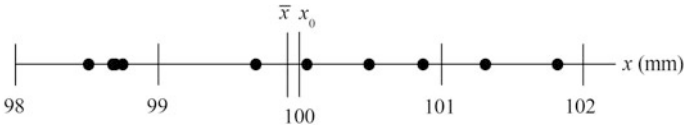


Fig. 3.2 The distribution on the x-axis of the results of the first 10 measurements of quantity x. The real value x_0 of the magnitude being measured is also shown, as well as the mean $\bar{x}$ of the 10 measurements

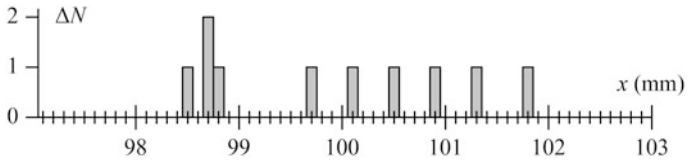


Fig. 3.3 A histogram of the 10 measurements x_i of Table 3.1. The bin width has been taken equal to $\Delta x = 0.1$ mm

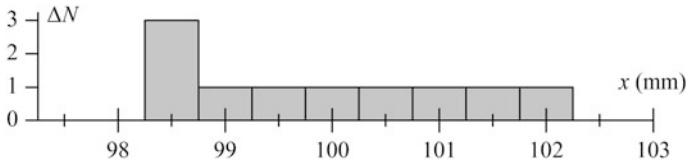


Fig. 3.4 A histogram of the 10 measurements x_i of Table 3.1. The bin width has been taken equal to $\Delta x = 0.5$ mm

taken to be those points of the x-axis which correspond to tenths of the unit. Thus, the first interval in which a measurement appears is the interval $98.45 < x_i \leq 98.55$. The next two measurements appear in the interval $98.65 < x_i \leq 98.75$ and so on.

If the histogram of Fig. 3.3 is redrawn with a bin width of $\Delta x = 0.5$ mm, we obtain the histogram of Fig. 3.4. The choice of the suitable bin width is dictated by the total number of measurements and their distribution. If the width is very small, one only measurement will appear in most bins, as in Fig. 3.3. At the other end, a Δx which is too large would result in almost all the measurements being classified in one bin. In both these extreme cases, the resulting histogram does not convey the maximum possible information about the distribution of the values. It is desirable to choose an intermediate value of Δx, so that the accumulation of events in one region is shown by columns of adequate height, but in such a way that the information about the detailed distribution of the measurements on the x-axis is not lost. These will become apparent in the histograms shown below.

The advantages of a histogram are apparent when the number of measurements is large. To demonstrate this fact, we suppose that we have performed 2500 measurements of the quantity x. The results x_i of the first 1000 measurements are recorded in Fig. 3.5 as a function of the order in which each measurement was performed, i. If we assume that the time interval between successive measurements is constant, the horizontal axis of the figure is that of time.

[For the reader who already has the necessary knowledge, we mention that the choice of the values was done, using random numbers, in such a way that the statistical distribution of the values of x_i is *Gaussian*, with mean $\bar{x} = 100$ mm and parent standard deviation $\sigma = 1$ mm. For reference, the values $\bar{x} - \sigma = 99$ mm, $\bar{x} = 100$ mm and $\bar{x} + \sigma = 101$ mm are marked in Fig. 3.5 by horizontal lines.

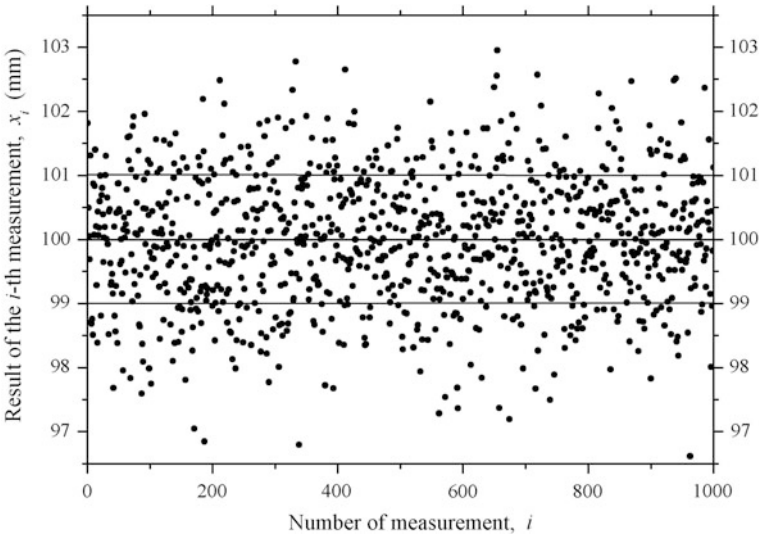


Fig. 3.5 1000 measurements of the magnitude x. The results x_i of the measurements are recorded as a function of the increasing number of the order, i, in which they were acquired. The mean of the measurements is 100 mm

The conditions under which real values behave in this way will be examined in a subsequent chapter.]

In Table 3.2 the first 100 measurements of Fig. 3.5 are given.

Histograms of the first 50, 100, 500 and 1000 measurements are shown in Figs. 3.6, 3.7, 3.8 and 3.9. In these histograms, the bin width was taken equal to $\Delta x = 0.2$ mm.

From Figs. 3.6, 3.7, 3.8 and 3.9, it becomes evident that, as the number of measurements increases, the nature of the statistical distribution of the measurements becomes clearer. Of course, in our example, the shape of the distribution was adopted when the results of the measurements were chosen. However, the same distribution applies for real measurements, for which we can only make assumptions about the statistical distribution of their parent population, which may be one of the required magnitudes to be derived from the series of measurements. In order to make the comparison of the histograms of Figs. 3.6, 3.7, 3.8 and 3.9 with each other easier, the total (shaded) area covered by the columns is approximately the same in all the figures. Consequently, the scales of the histograms may differ from each other, but in all cases the sum of the heights of all the columns is, equal to the total number of measurements N. If we plot all our 2500 'theoretical' measurements in a histogram, as in Fig. 3.10, we have the maximum possible information that can be extracted from the given experiment. Obviously, one has the ability to choose a small bin width Δx, in order to bring out a fine structure in the distribution, if it exists.

Table 3.2 100 measurements x_i of the magnitude x, in mm

101.82	100.49	100.05	99.69	101.31	98.70	98.68	98.75	98.51	100.87
100.07	100.83	101.40	100.20	99.29	98.39	100.43	100.07	100.18	100.60
100.99	98.81	100.69	100.80	100.39	99.64	100.03	100.19	101.00	101.30
100.01	100.65	99.92	98.52	100.44	100.47	99.63	99.29	99.33	99.46
99.15	97.68	100.20	98.80	98.56	99.27	100.41	98.38	100.87	100.87
100.47	100.43	99.62	100.00	99.66	99.15	97.95	100.16	100.34	100.03
99.48	100.24	100.78	100.12	101.60	101.23	100.41	101.61	97.83	101.09
99.76	99.03	101.77	101.92	99.51	99.30	100.10	100.97	100.57	100.07
99.12	101.39	98.67	98.64	99.44	100.15	97.59	98.37	98.09	100.48
100.93	101.96	99.92	100.48	99.56	101.04	101.24	100.38	97.98	99.80

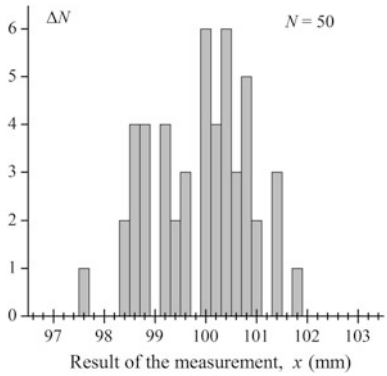


Fig. 3.6 A histogram of the first 50 measurements x_i of Fig. 3.5, with a bin width of $\Delta x = 0.2\,\text{mm}$

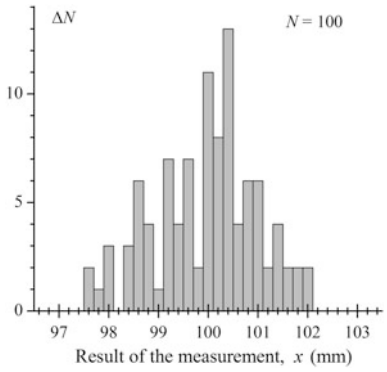


Fig. 3.7 A histogram of the first 100 measurements x_i of Fig. 3.5, with a bin width of $\Delta x = 0.2\,\text{mm}$

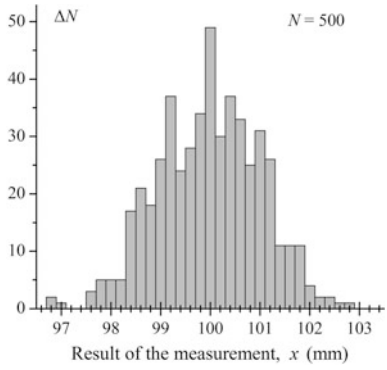


Fig. 3.8 A histogram of the first 500 measurements x_i of Fig. 3.5, with a bin width of $\Delta x = 0.2\,\text{mm}$

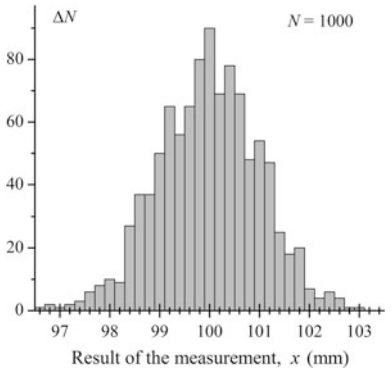


Fig. 3.9 A histogram of all the 1000 measurements x_i of Fig. 3.5, with a bin width of $\Delta x = 0.2\,\text{mm}$

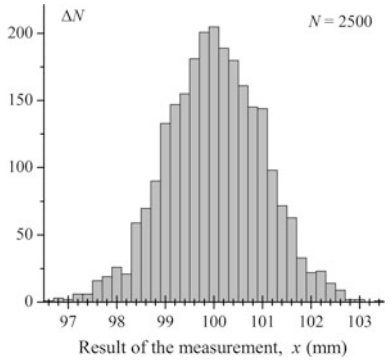


Fig. 3.10 A histogram of 2500 measurements x_i, with a bin width of $\Delta x = 0.2\,\text{mm}$

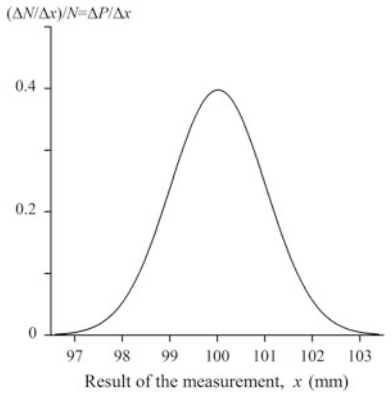


Fig. 3.11 The normalized 'histogram' of a very large number of measurements x_i

If it were possible to have a much larger number of measurements of the same quantity, our histogram would have the form of the curve in Fig. 3.11. In this histogram, the division into columns is not visible, due to their large number. The curve of Fig. 3.11 differs from the others in a few very important characteristics. Having a very large number of measurements at our disposal, we have the freedom to choose a very small interval Δx. This will have as a consequence the histogram to be transformed into a continuous curve. In fact, the general shape of the curve is not changed if we choose an even smaller Δx. Given these facts, we will choose the scale of ΔN in a different manner, so that the graph is easier to use. If for some value of Δx, we have between $x - \Delta x/2$ and $x + \Delta x/2$ a number of measurements equal to ΔN and the total number of measurements is N, then in the interval Δx around the value of x we have a proportion $\Delta N/N$ of the measurements. We may, therefore, say that the *probability* for a measurement to be found between the values $x - \Delta x/2$ and $x + \Delta x/2$ is

$$\Delta P(x) = \frac{\Delta N}{N}. \tag{3.6}$$

If we divide by Δx, we have

$$\frac{\Delta P}{\Delta x} = \frac{1}{N}\frac{\Delta N}{\Delta x} \tag{3.7}$$

as *the probability per unit x-interval* for a measurement to have a value near a certain x. This quantity has been plotted as a function of x in Fig. 3.11. The use of the curve is now much easier, being independent of N. For example, for $x = 100$ we read from the curve the value of $\frac{\Delta P}{\Delta x} = \frac{1}{N}\frac{\Delta N}{\Delta x} = 0.4$, approximately. Because it is $N = 2500$, if we take the interval $\Delta x = 0.2$ mm about $x = 100$, we will have $\Delta N = 0.4 \times 2500 \times 0.2 = 200$ as the number of measurements with values in the region $99.9 < x < 100.1$. This is in satisfactory agreement with the indication of Fig. 3.10 for $x = 100$.

In the limit $\Delta x \to 0$, this *probability density* is equal to

$$f(x) \equiv \frac{\mathrm{d}P}{\mathrm{d}x} = \frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}x}. \tag{3.8}$$

Plotting the curve is easily done as follows: we count the number $\mathrm{d}N$ of the measurements found in the small interval $\mathrm{d}x$ around the value x and divide by $\mathrm{d}x$ and by the total number N of the measurements. The result gives us $\mathrm{d}P/\mathrm{d}x$ and so we have the point $(x,\ \mathrm{d}P/\mathrm{d}x)$ of the curve. Figure 3.12 gives one such *probability density function* for the variable x. Its shape will not concern us at the moment. We will only mention that it has the general shape of the histograms we examined, with

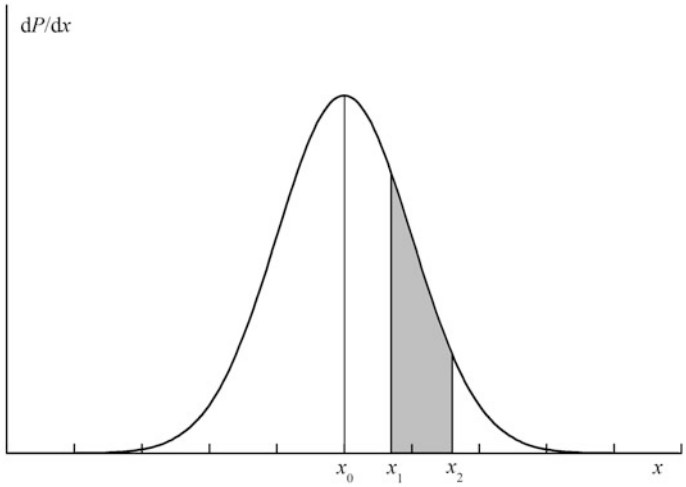


Fig. 3.12 A probability density function for the variable x

maximum at some value x_0 and most values of x being found in a region near the maximum.

The relation (3.8) may be used in the extraction of some conclusions concerning the properties of a curve such as that of Fig. 3.12. If the probability density $f(x) = \mathrm{d}P/\mathrm{d}x$ is multiplied by the width $\mathrm{d}x$ of an interval of x values, the product

$$f(x)\mathrm{d}x = \left(\frac{\mathrm{d}P}{\mathrm{d}x}\right)\mathrm{d}x = \mathrm{d}P \tag{3.9}$$

gives the probability for the variable to take a value in an interval of width $\mathrm{d}x$ about x, say between x and $x + \mathrm{d}x$. The probability for x to have a value between x_1 and x_2 ($x_2 > x_1$) is, therefore,

$$P\{x_1 \leq x \leq x_2\} = \int_{x_1}^{x_2} \mathrm{d}P = \int_{x_1}^{x_2} \left(\frac{\mathrm{d}P}{\mathrm{d}x}\right)\mathrm{d}x = \int_{x_1}^{x_2} f(x)\mathrm{d}x. \tag{3.10}$$

In Fig. 3.12, this probability is given by the area of the shaded surface under the curve between the points x_1 and x_2. The probability for x to have any value is, obviously, equal to unity and is given by the integral

$$P\{-\infty < x < +\infty\} = \int_{-\infty}^{+\infty} f(x)\mathrm{d}x = 1. \tag{3.11}$$

This is seen from

$$P\{-\infty < x < +\infty\} = \int_{-\infty}^{+\infty} \left(\frac{\mathrm{d}P}{\mathrm{d}x}\right)\mathrm{d}x = \int_{-\infty}^{+\infty} \frac{1}{N_{\mathrm{tot}}}\frac{\mathrm{d}N}{\mathrm{d}x}\mathrm{d}x = \frac{1}{N_{\mathrm{tot}}}\int_{x=-\infty}^{x=+\infty} \mathrm{d}N$$
$$= \frac{1}{N_{\mathrm{tot}}} N_{\mathrm{tot}} = 1. \tag{3.12}$$

Equation (3.11) simply states the fact that the probability density function $f(x)$ is, due to the way it was defined, *normalized*. The division by N in Eqs. (3.6) and (3.7) was done for exactly this purpose.

For the measurements we performed to be useful, certain conditions must be satisfied. The measurements must have been all made under identical conditions and the magnitude being measured must have remained unchanged during the duration of the measurements. Systematic errors must also be absent. This last condition is very difficult to verify and we must bear in mind that even in what appear to be the best of measurements, considerable systematic errors may be present.

If there are variations with time, in the magnitude being measured or the systematic errors that may be present, the series of measurements shown in Fig. 3.5 will be modified. Two such examples are shown in Figs. 3.13 and 3.14. In Fig. 3.13, apart from the random errors, there is also present in the measurements a

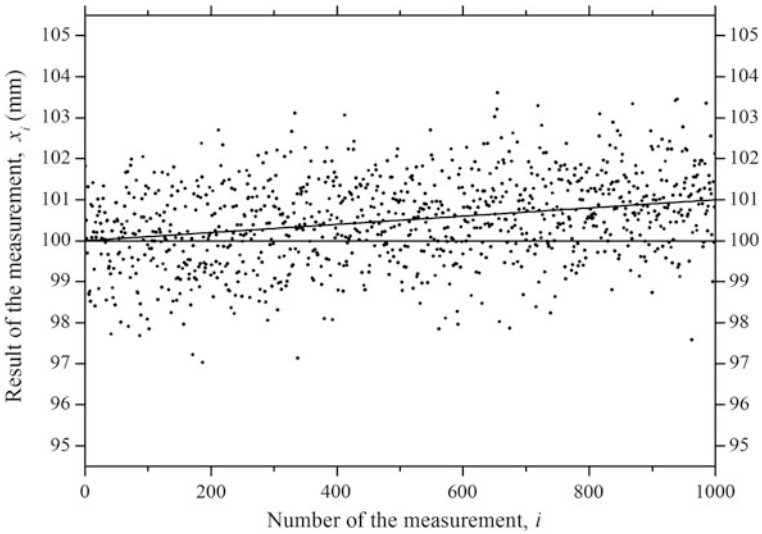


Fig. 3.13 The measurements of Fig. 3.5, to which a systematic error increasing linearly with time has been added

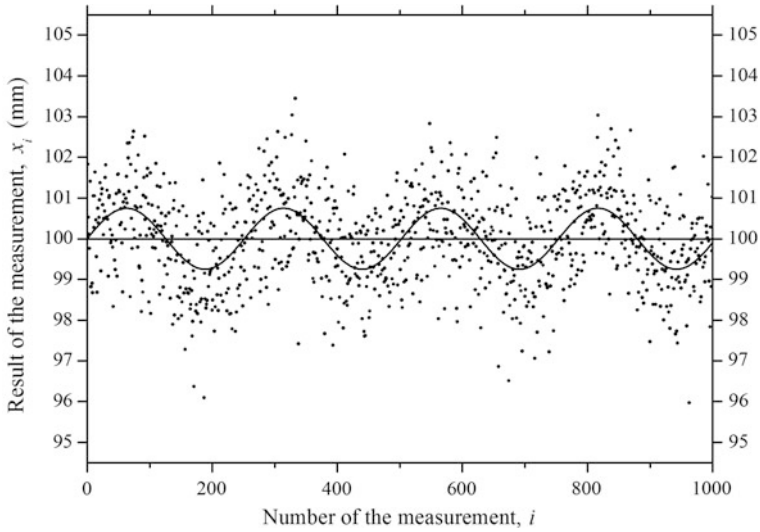


Fig. 3.14 The measurements of Fig. 3.5, to which a systematic error varying periodically with time has been added

systematic error which increases linearly with time. If the same time interval intervenes between successive measurements, the results will have the distribution shown in the figure. Something similar is true in the measurements of Fig. 3.14, with the difference that in this case the systematic error varies periodically with time. Situations like these are very difficult to detect, especially since the number of measurements is never as large as that of our examples.

We will now use the measurements made in order to derive an estimate for the value of the magnitude measured. Let the *real value* of the measured quantity be x_0, which of course is unknown to us. If the result of the i-th measurement is x_i, the *error* in this measurement is defined as

$$e_i \equiv x_i - x_0. \tag{3.13}$$

If a total of N measurements have been made, the *mean* of their results, x_i $(i = 1, 2, \ldots, N)$, is defined as

$$\overline{x} \equiv \frac{1}{N} \sum_{i=1}^{N} x_i. \tag{3.14}$$

Which is the behavior of the mean $\overline{x}$ as the number of measurements increases? To answer this question, we will examine the variation with the number of measurements of the difference between the mean and the real value, which is defined as

$$e_\mu \equiv \bar{x} - x_0. \tag{3.15}$$

Combining Eqs. (3.13) and (3.14), we have

$$e_\mu = \bar{x} - x_0 = \frac{1}{N}\sum_{i=1}^{N} x_i - x_0 = \frac{1}{N}\sum_{i=1}^{N} (x_0 + e_i) - x_0 = \frac{1}{N}\sum_{i=1}^{N} e_i. \tag{3.16}$$

The error in the mean,

$$e_\mu = \frac{1}{N}(e_1 + e_2 + \ldots + e_i + \ldots + e_N), \tag{3.17}$$

is, therefore, equal to the mean value of the errors in the measurements. Given that the errors e_i are assumed to be positive or negative with the same probability, the absolute value of e_μ will be smaller than the largest absolute value of the errors e_i. In general, therefore, the mean $\bar{x}$ will be nearer to x_0 than the worst measurement. Statistically, we expect that the mean value e_μ of the errors will be smaller the bigger the number of measurements N. Thus we accept that the mean $\bar{x}$ is the best estimate we have for x_0.

We will demonstrate what we have said above, using again the measurements of our thought experiment. Figure 3.15 shows the variation with n of the mean $\bar{x}(n)$

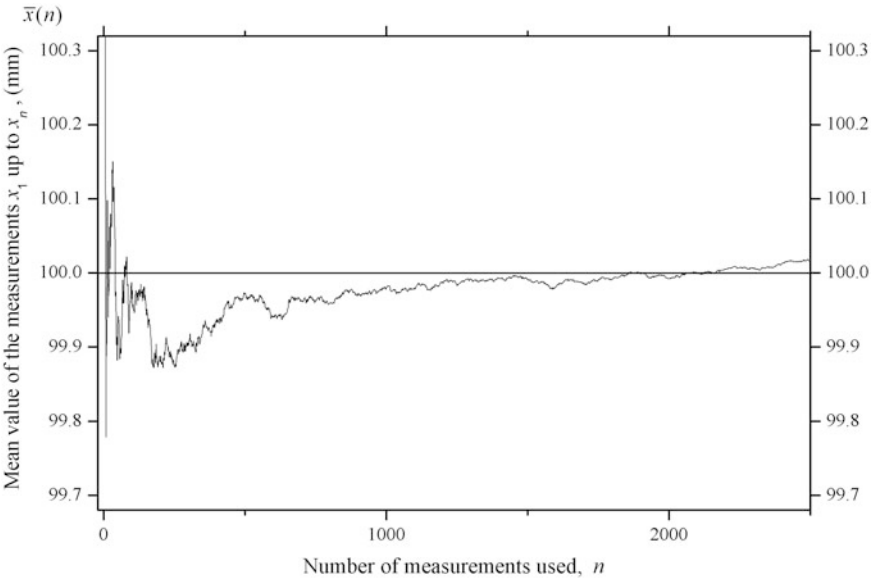


Fig. 3.15 The variation with n of the mean $\bar{x}(n)$ evaluated using the first n results x_i of the measurements of our thought experiment (*running average*). The real value of the magnitude being measured was taken to be $x_0 = 100$ mm

evaluated using the first n results x_i of the measurements of our thought experiment (a *running average*). The first value is, according to Table 3.1, $\overline{x}(1) = 101.82$ mm and this is the first point of the graph (which happens to be off scale). The next point is evaluated using the first two measurements and so on. As expected, it is seen in the figure that the mean has a general tendency to approach the real value (taken to be $x_0 = 100$ mm in the example) as the number of measurements increases. After the expected relatively large initial fluctuations, the variations in $\overline{x}(n)$ become smaller and smaller. This is natural, since for a large number of measurements, $\overline{x}(n)$ has acquired a kind of 'inertia', resisting change. This is due to the fact that the addition of one more measurement (with error e_i) to the n which are already present does not affect significantly neither the mean $\overline{x}(n)$ nor the error $e_\mu(n)$ of the mean.

Although these effects will be examined rigorously in the next chapter, we can give a short mathematical interpretation of what we stated above. By definition, it is

$$\overline{x}(n) = \frac{1}{n}(x_1 + x_2 + \ldots + x_i + \ldots + x_n). \tag{3.18}$$

Since

$$x_i = x_0 + e_i, \tag{3.19}$$

we have

$$\overline{x}(n) = x_0 + \frac{1}{n}(e_1 + e_2 + \ldots + e_i + \ldots + e_n) = x_0 + e_\mu(n) \tag{3.20}$$

where

$$e_\mu(n) = \frac{1}{n}(e_1 + e_2 + \ldots + e_i + \ldots + e_n) \tag{3.21}$$

is the error in $\overline{x}(n)$. Adding one more measurement will change $e_\mu(n)$ and, therefore $\overline{x}(n)$ as well. The error in the mean when $n+1$ values have been used in its evaluation will be

$$\begin{aligned} e_\mu(n+1) &= \frac{1}{n+1}(e_1 + e_2 + \ldots + e_i + \ldots + e_{n+1}) \\ &= \frac{1}{n+1}(e_1 + e_2 + \ldots + e_i + \ldots + e_n) + \frac{e_{n+1}}{n+1}. \end{aligned} \tag{3.22}$$

Therefore, it is

$$e_\mu(n+1) = \frac{n}{n+1} e_\mu(n) + \frac{e_{n+1}}{n+1}. \tag{3.23}$$

The fractional change in the error in the mean caused by the addition of one more measurement to the n already used is

$$\frac{\Delta e_\mu(n)}{e_\mu(n)} = \frac{e_\mu(n+1) - e_\mu(n)}{e_\mu(n)} = \frac{1}{n+1}\left(\frac{e_{n+1}}{e_\mu(n)} - 1\right). \tag{3.24}$$

The value of the expression in brackets is of order unity:

$$\left(\frac{e_{n+1}}{e_\mu(n)} - 1\right) \sim 1. \tag{3.25}$$

Therefore,

$$\frac{\Delta e_\mu(n)}{e_\mu(n)} \sim \frac{1}{n+1} \tag{3.26}$$

and

$$\Delta e_\mu(n) \sim \frac{e_\mu(n)}{n+1}. \tag{3.27}$$

The fractional change in the error in the mean, $e_\mu(n)$, caused by the addition of one more measurement in the evaluation of the mean, is of the order of $1/(n+1)$. As n increases, this change tends to zero. The change in $\overline{x}(n)$ also tends to zero and $\overline{x}(n) \to x_0$, something that is obvious in Fig. 3.15. We must remember that for the values of our numerical example the errors e_i in x_i are of the order of 1 mm (see Fig. 3.5). It will be proved below that, if the dispersion of the measurements is such that they differ from the real value by 1 mm on the average, then the difference of the mean from the real value will be, approximately, equal to 1 mm/$\sqrt{N}$, where N is the number of measurements in the calculation of the mean. In Fig. 3.15, for the maximum value $N = 2500$, the deviation of the mean from the real value is expected to be 1 mm/$\sqrt{2500} = 0.02$ mm, approximately. The deviations seen in the figure for N near 2500 are actually of this order of magnitude.

The variation with n of the standard deviation $s_x(n)$ from the mean $\overline{x}(n)$ evaluated using the first n results x_i of the measurements of our thought experiment is shown in Fig. 3.16. The standard deviation of the measurements is seen to tend to that of their parent population, which was chosen to be 1 mm in our thought experiment.

Qualitatively, we may make the following comments: as the number of the measurements is increased, the dispersion of the results about the mean does not vary drastically. This is seen in the histograms of Figs. 3.4, 3.6, 3.7, 3.8, 3.9 and 3.10, in which the width of the distribution remains virtually constant. This, in any case, depends on the distribution of the random errors, which in its turn depends on

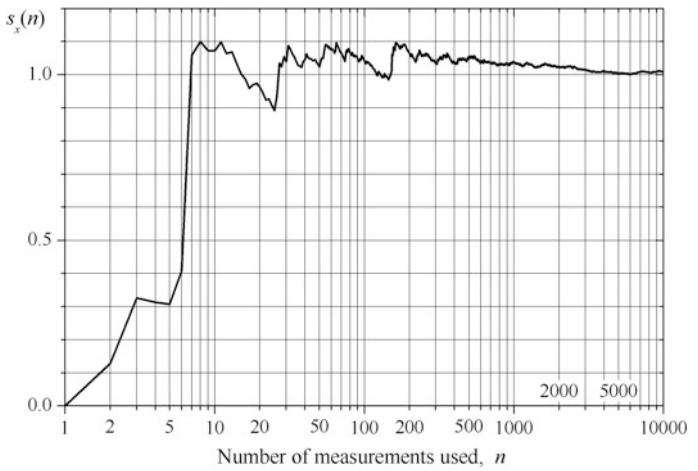
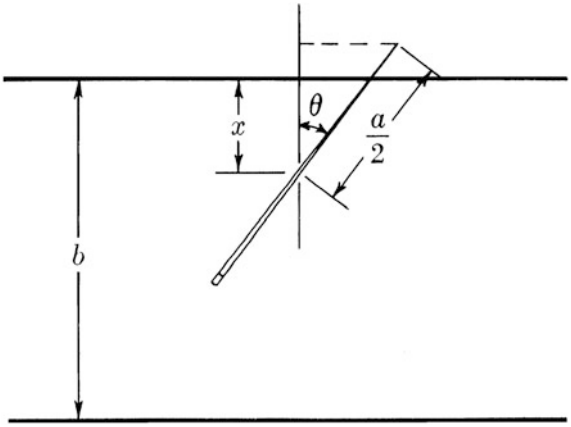


Fig. 3.16 The variation with n of the standard deviation $s_x(n)$ from the mean $\bar{x}(n)$ evaluated using the first n results x_i of the measurements of our thought experiment. The results of the measurements were selected so that $\sigma = 1$ mm. The scale for n is logarithmic

the accuracy of the method used. As the number of measurements increases, the shape of the distribution is defined with greater accuracy. This makes possible the more accurate determination of the mean. For the measurements of our thought experiment, in which negative and positive deviations of the measurements from the real value were taken to be equally probable, the determination of the mean actually depends on the determination of the axis of symmetry of the histogram of the distribution of the results. As the shape of the histogram is established with greater accuracy, the mean is known with smaller deviation from the real value. The quantitative demonstration of this effect will be given in the next chapter.

Example 3.1

Buffon's needle. For an experimental determination of the value of π using the method of Buffon's needle, many parallel straight lines are drawn on a horizontal surface, at distances b from each other. A needle having length equal to $a < b$ is dropped on the surface, many times, successively (see figure). It is proved that the probability that the needle will fall in such a position and orientation that it cuts one of the lines is $P = 2a/\pi b$. In a 'thought experiment', with $b = a$, Ukrainian *idiot savant* (learned idiot) Kerenii 'threw' the needle 355 times and observed that it cut the lines 226 times. The value for π he found by this method is $\pi = 2 \times 355/226 = 3.141\,592\,920$, which is correct in its first 7 significant figures. Discuss the 'experiment'.



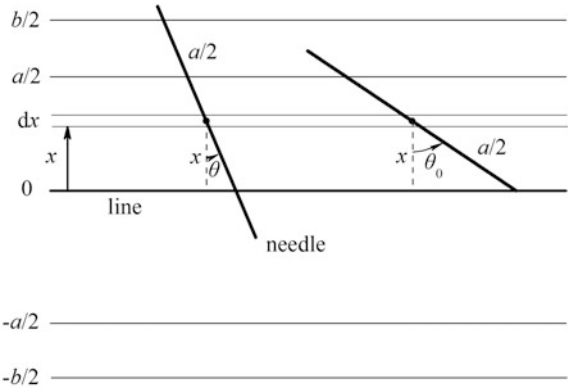
The Probability that the Needle Cuts a Line

Shown in the figure below is a line at $x = 0$. For the needle to be possible to cut the line, its center must fall at points for which it is $-a/2 \leq x \leq a/2$. The probability of this happening is $P_1 = a/b$. For a needle whose center falls in the range $-a/2 \leq x \leq a/2$, the probability of having its center fall in a strip between x and $x + \mathrm{d}x$ is $\mathrm{d}P_2 = \mathrm{d}x/a$. Let the angle the needle forms with the normal on the line be θ. Now, for a needle with its center in the strip between x and $x + \mathrm{d}x$ to cut the nearest line, it must form an angle θ in the range $-\theta_0 \leq \theta \leq \theta_0$ or $\pi/2 - \theta_0 \leq \theta \leq \pi/2 + \theta_0$, where $\cos\,\theta_0 = x/(a/2)$. The probability, therefore, that a needle whose center lies in the strip between x and $x + \mathrm{d}x$ cuts the line is $P_3 = 4\theta_0/2\pi = 2\theta_0/\pi$. The combined probability that the needle's center lies in the region $-a/2 \leq x \leq a/2$ and between x and $x + \mathrm{d}x$ and cuts the line is

$$\mathrm{d}P = P_1\,\mathrm{d}P_2\,P_3 = \frac{a}{b}\frac{\mathrm{d}x}{a}\frac{2\theta_0}{\pi} = \frac{2}{\pi b}\arccos(2x/a)\mathrm{d}x$$

The total probability for the needle to cut a line is

$$P = \frac{2}{\pi b}\int_{-a/2}^{a/2} \arccos\,(2x/a)\,\mathrm{d}x = \frac{a}{\pi b}\int_{-1}^{1} \arccos\,z\,\mathrm{d}z = \frac{2a}{\pi b}.$$



Discussion of the Experiment

To find out what the accuracy of the 'experiment' is, we will assume that the needle (for which it is $a = b$) is thrown one more time. There are two possibilities:

(a) If the needle cuts a line, the new value of π will be $\pi = 2 \times 356/227 = 3.136\,56 \approx 3.14$, which, even after being rounded, is correct to only 3 significant figures.

(b) If the needle does not cut a line, the new value of π will be $\pi = 2 \times 356/226 = 3.150\,44$, which is correct only to 2 significant figures.

Obviously, the great accuracy of the result is due to the fact that Kerenii 'happened to observe' 226 events in which the needle cut the lines in 355 tries and that he stopped the 'experiment' exactly at that point. The proximity of its value to the real one is, therefore, due to a coincidence. Besides, if we had no knowledge of the real value of π and we calculated the expected error $\delta\pi$ in π, this would have been much larger than the one achieved by Kerenii with the numbers he chose (see below).

It should be mentioned that the Italian mathematician Lazarini had estimated [1], before Kerenii, that with a needle of length 5 cm and a line distance of 6 cm, if the needle cuts the lines in 1808 out of 3408 trials, then it follows that

$$\pi = 2 \times \frac{5}{6} \times \frac{3408}{1808} = 3.1415929,$$

a value that is exact to 7 significant figures. Kerenii achieved the same result with a tenth of the trials.

The reliability of the value of π evaluated in this way may be found. From

$$P = \frac{2a}{\pi b} = \frac{N_1}{N}, \text{ it follows that } \pi = \frac{2a}{b}\frac{N}{N_1}.$$

If we assume that a, b and N are known with a satisfactory accuracy, then π may be considered to be a function of only one variable and its error, $\delta\pi$, will be due solely on the error in N_1. Therefore,

$$d\pi = \frac{2a}{b} N \left(-\frac{dN_1}{N_1^2} \right), \quad \frac{\delta\pi}{\pi} = -\frac{\delta N_1}{N_1}$$

if $\delta\pi$ is the error in π due to the error δN_1 in N_1. It will be proved later that if the expected number of events is N_1, the deviations of the measured values from N_1 is of the order of $\sqrt{N_1}$ (Poisson statistics). Then

$$\frac{\delta\pi}{\pi} = \frac{\delta N_1}{N_1} = \frac{\sqrt{N_1}}{N_1} = \frac{1}{\sqrt{N_1}}.$$

For the $N_1 = 226$ of the Kerenii 'experiment' we expect $\delta\pi/\pi = 1/\sqrt{N_1} = 1/\sqrt{226} = 0.067$ and $\delta\pi = 0.067 \times \pi = 0.21$ or 6.7%. For the $N_1 = 1808$ of the Lazarini 'experiment' we expect $\delta\pi/\pi = 1/\sqrt{N_1} = 1/\sqrt{1808} = 0.024$ and $\delta\pi = 0.024 \times \pi = 0.074$ or 2.4%. The accuracy which can be achieved by the method is not enough to justify the results of Kerenii and Lazarini, which, obviously are due to numerical coincidences.

From a purely experimental point of view, if we wish to determine π with an accuracy of 1 part in 3×10^6 using the method of Buffon's needle, we must measure the lengths a and b with at least this accuracy. We must also be able to judge that the needle cuts a line with such a power of discrimination, i.e. on the length scale of the order of one third of a millionth of a or b. If these lengths are about 3 cm, this resolving power must be better than $3 \times 10^{-2}/3 \times 10^6 = 10^{-8}$ m or 10 nm. This length is approximately equal to 100 atomic diameters. In addition, the thickness of the lines must be of the same order of magnitude, or smaller, and the lines must be straight to this accuracy. Anyone planning such an experiment must achieve experimental conditions such as these (in the real world and not the 'thought' world).

Reference

1. T.H. O'Beirne, *Puzzles and Paradoxes* (Oxford University Press, 1965)

Chapter 4
The Statistical Analysis of Experimental Results

We will now examine the way in which we can best use the results of experimental measurements to reach conclusions relating to the magnitude measured. Understanding the concepts and methods presented in this chapter possibly constitutes the main benefit the reader may derive from studying this book.

4.1 The Mean and the Dispersion of the Results of Measurements

Let us assume that we have measured a quantity N times, under exactly the same experimental conditions. We will use the results of the measurements in order to get an estimate of the value of the magnitude measured. Let the *real value* of the measured quantity be x_0, which is, of course, unknown to us. If the result of the i-th measurement is x_i, the *error* in this measurement is defined as

$$e_i \equiv x_i - x_0. \tag{4.1}$$

If a total of N measurements have been performed, the *mean* of their results, x_i ($i = 1, 2, \ldots, N$), is defined as

$$\overline{x} \equiv \frac{1}{N}\sum_{i=1}^{N} x_i. \tag{4.2}$$

The difference of the mean from the true value is defined as the *error in the mean*,

$$e_\mu = e_{\overline{x}} \equiv \overline{x} - x_0. \tag{4.3}$$

C. Christodoulides and G. Christodoulides, *Analysis and Presentation of Experimental Results*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-53345-2_4

Combining these equations, we have

$$e_\mu = e_{\overline{x}} = \overline{x} - x_0 = \frac{1}{N}\sum_{i=1}^{N} x_i - x_0 = \frac{1}{N}\sum_{i=1}^{N} (x_0 + e_i) - x_0 = \frac{1}{N}\sum_{i=1}^{N} e_i. \qquad (4.4)$$

The error in the mean,

$$e_\mu = e_{\overline{x}} = \frac{1}{N}(e_1 + e_2 + \ldots + e_i + \ldots + e_N), \qquad (4.5)$$

is, therefore, equal to the mean of the errors of the results. Given that the errors e_i are both negative and positive, and that we usually assume that it is equally probable for an error to be negative or positive, the absolute value of e_μ will be smaller than the largest absolute value among the errors e_i. In general, therefore, the mean $\overline{x}$ will be nearer to x_0 than the worst result of the measurements. We actually expect the mean e_μ of the errors to decrease as we increase the number of measurements N. Thus, we accept that the mean $\overline{x}$ is the best estimate we have for x_0 after N measurements of the magnitude x. A rigorous proof of this statement, based on the theory of errors, will be given below.

Example 4.1

Let the unknown real length of a rod be 100 mm and that 10 measurements of the length of the rod gave the following results (in mm):

100.1 100.2 99.8 100.3 99.9 100.2 99.9 100.4 100.0 100.3.

We want to find $\overline{x}$ and e_μ.

We construct a table with columns showing the corresponding values of i, x_i and e_i, for $N = 10$ ($i = 1, 2, \ldots, 10$).

i	x_i (mm)	e_i (mm)
1	100.1	0.1
2	100.2	0.2
3	99.8	−0.2
4	100.3	0.3
5	99.9	−0.1
6	100.2	0.2
7	99.9	−0.1
8	100.4	0.4
9	100.0	0.0
10	100.3	0.3
Σ =	1001.1	1.1

The sum of the x_i's is: $\sum_{i=1}^{10} x_i = 1001.1$ mm.

The value of their mean is: $\overline{x} = \frac{1}{10}\sum_{i=1}^{10} x_i = 100.11 \approx 100.1\,\text{mm}$.
The error in the mean is equal to $e_\mu = \overline{x} - x_0 = 0.11 \approx 0.1\,\text{mm}$.
This error is not known to us.

Since the real value x_0 of the quantity being measured is unknown to us, the errors e_i and e_μ are also unknown. It is therefore impossible for us to examine the dispersion of the measurements x_i relative to the real value. We can, however, examine the dispersion of the measurements relative to the mean $\overline{x}$ of the measurements, which is known to us. We define the difference of the measurement x_i from the mean of all the measurements as

$$d_i \equiv x_i - \overline{x}, \tag{4.6}$$

which is known as the *deviation of measurement x_i from the mean* or as the *residual* of the measurement.

From the definitions $e_i = x_i - x_0$ and $d_i = x_i - \overline{x}$, it is $x_i = x_0 + e_i = \overline{x} + d_i$ and

$$e_i - d_i = \overline{x} - x_0. \tag{4.7}$$

Therefore, $e_1 + e_2 + \ldots + e_N = (x_1 - x_0) + (x_2 - x_0) + \ldots + (x_N - x_0) = N(\overline{x} - x_0)$

or

$$\sum_{i=1}^{N} e_i = N(\overline{x} - x_0) \tag{4.8}$$

and $d_1 + d_2 + \ldots + d_N = (x_1 - \overline{x}) + (x_2 - \overline{x}) + \ldots + (x_N - \overline{x}) = N\overline{x} - N\overline{x} = 0$

or

$$\sum_{i=1}^{N} d_i = 0. \tag{4.9}$$

Example 4.2

In the table of Example 4.1 we now also record the values of $|e_i|$, d_i and $|d_i|$.

i	x_i (mm)	e_i (mm)	$\lvert e_i\rvert$ (mm)	d_i (mm)	$\lvert d_i\rvert$ (mm)
1	100.1	0.1	0.1	−0.01	0.01
2	100.2	0.2	0.2	0.09	0.09
3	99.8	−0.2	0.2	−0.31	0.31
4	100.3	0.3	0.3	0.19	0.19
5	99.9	−0.1	0.1	−0.21	0.21
6	100.2	0.2	0.2	0.09	0.09

(continued)

(continued)

i	x_i (mm)	e_i (mm)	$\lvert e_i\rvert$ (mm)	d_i (mm)	$\lvert d_i\rvert$ (mm)
7	99.9	−0.1	0.1	−0.21	0.21
8	100.4	0.4	0.4	0.29	0.29
9	100.0	0.0	0.0	−0.11	0.11
10	100.3	0.3	0.3	0.19	0.19
Σ =	1001.1	1.1	1.9	0	1.70

We use the value $\overline{x} = 100.11$ mm in the estimation of the d_i and $|d_i|$.

As expected, we find that $\sum_{i=1}^{N} d_i = 0$.

Also, $\overline{|e_i|} = \frac{1}{N}\sum_{i=1}^{N} |e_i| = \frac{1.9}{10} = 0.19 \approx 0.2$ mm, $\sum_{i=1}^{N} |d_i| = 1.70$ mm and $\overline{|d|} = \frac{1}{N}\sum_{i=1}^{N} |d_i| = 0.17 \approx 0.2$ mm.

4.2 The Standard Deviations

4.2.1 The Standard Deviation of the Measurements

The dispersion of the results of the measurements about their mean, is described by the *standard deviation from the mean* of the measurements. The deviation $d_i = x_i - \overline{x}$ is measured from the mean of the measurements. Thus, the standard deviation from the mean $\overline{x}$ of a series of measurements consisting of N measurements x_i ($i = 1, 2, \ldots, N$), is defined as

$$s_x \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N} (x_i - \overline{x})^2}. \tag{4.10}$$

This is the standard deviation of a series of N measurements, which constitute a *sample* from the infinite measurements of the magnitude x which might be performed. These infinite possible measurements are the *parent population*, or simply the *population*, from which we have taken a sample consisting of N random values. We must bear in mind that $\overline{x}$ is the *sample mean* and s_x is the *sample standard deviation*. The quantity s_x is also referred to as the *standard deviation of a single measurement in the sample*. Our aim is to derive as much information as possible about the properties of the statistical distribution of the parent population. The degree to which we can achieve this goal by performing only N measurements, will be discussed below.

As already mentioned in Chap. 1, if the N measurements are distributed in K classes, where the r-th class contains n_r measurements that gave a result x_r, Eq. (4.10) may also be written in the forms

$$s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r (x_r - \bar{x})^2} \tag{4.11}$$

and

$$s_x = \sqrt{\sum_{r=1}^{K} f_r (x_r - \bar{x})^2}, \tag{4.12}$$

where $f_r = \dfrac{n_r}{N}$.

A relation often used is derived in the following manner:

$$\begin{aligned} s_x^2 &= \frac{1}{N}\sum_{r=1}^{K} n_r (x_r - \bar{x})^2 = \frac{1}{N}\sum_{r=1}^{K} (x_r^2 - 2\bar{x}x_r + \bar{x}^2) n_r \\ &= \frac{1}{N}\sum_{r=1}^{K} x_r^2 n_r - \frac{2}{N}\bar{x}\sum_{r=1}^{K} x_r n_r + \frac{\bar{x}^2}{N}\sum_{r=1}^{K} n_r = \frac{1}{N}\sum_{r=1}^{K} x_r^2 n_r - \bar{x}^2 \end{aligned} \tag{4.13}$$

and, therefore,

$$s_x^2 = \overline{x^2} - \bar{x}^2 \quad \text{or} \quad \overline{(\Delta x)^2} = \overline{x^2} - \bar{x}^2. \tag{4.14}$$

Example 4.3

Find the standard deviation from the mean of the values of Example 4.1.

We use the value of $\bar{x}$ = 100.11 in evaluating $x_i - \bar{x}$.

i	x_i (mm)	$x_i - \bar{x}$ (mm)	$(x_i - \bar{x})^2$ (mm^2)
1	100.1	−0.01	0.0001
2	100.2	0.09	0.0081
3	99.8	−0.31	0.0961
4	100.3	0.19	0.0361
5	99.9	−0.21	0.0441
6	100.2	0.09	0.0081
7	99.9	−0.21	0.0441
8	100.4	0.29	0.0841
9	100.0	−0.11	0.0121
10	100.3	0.19	0.0361
Σ =	1001.1	0.00	0.3690

We find that $\sum_{i=1}^{N} (x_i - \bar{x})^2 = 0.369$ mm^2.

Therefore, from Eq. (4.10), $s_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2} = \sqrt{\frac{0.369}{10}} = \sqrt{0.0369} = 0.192$ mm. The standard deviation of the 10 values is $s_x = 0.19$ mm.

Example 4.4 [E]

For the data of Example 4.3, using Excel®, find the mean, $\overline{x}$, the sample standard deviation, s_x, and the mean absolute deviation, $\overline{|d|}$.

We enter the values of x_i in cells A1 to A10. We highlight cells A1 to A10. Then,

Data > Data Analysis > Descriptive Statistics > OK

In the dialog box that opens, we set **Input, Input Range > $A$1:$A$10**, **Grouped by > Columns** and tick the box for **Summary statistics**. Press **OK**. The program returns a table, from which we read:

[Mean] = 100.11, [Standard Deviation] = 0.202485.

It must be remembered that Excel returns as **Standard Deviation** not the value of the standard deviation of the sample, $s_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2}$, but rather the best estimate for the standard deviation of the parent population, $\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}$ (see Sect. 4.2.4). We may evaluate s_x using the relation $s_x = \sqrt{\frac{N-1}{N}}\,\hat{\sigma}$. The result is $s_x = \sqrt{\frac{9}{10}}\,0.202485 = 0.192094$ mm, as expected.

To calculate the mean absolute deviation, $\overline{|d|}$, we proceed as follows:

Set cell B1 **= ABS(A1-100.11)**. Fill Down cells B2 to B10. Column B now contains the values of $|x_i - \overline{x}|$. Highlight cells A1 to A10. Open the **Autosum** dialog box and press **Sum**. The result returned is 1.7. Dividing by $N = 10$, we have the result: $\overline{|d|} = 0.17$ mm.

Summarizing, $\overline{x} = 100.11$ mm and $s_x = 0.19$ mm and $\overline{|d|} = 0.17$ mm.

Example 4.5 [O]

For the data of Example 4.3, using Origin®, find the mean, $\overline{x}$, the sample standard deviation, s_x, and the mean absolute deviation, $\overline{|d|}$.

We enter the numbers in column A. We highlight column A. Then,

Statistics > Descriptive Statistics > Statistics on Columns > Open Dialog…

In the window that opens, we tick the following:

Input > Input Data > Range 1 > Data Range > A(X)
Quantities > Tick **Mean, Standard Deviation, Mean Absolute Deviation**

Open **Computation Control > Weight Method > Direct Weight**
Then **Variance Divisor of Moment > N**

The last setting puts the number N in the denominator of Eq. (4.10) (The choice DF would put $N - 1$ in the denominator).

Pressing **OK** we obtain the results:

[Mean] = 100.11, [Standard Deviation] = 0.19209, [Mean Absolute Deviation] = 0.17

Summarizing, $\bar{x} = 100.11$ mm and $s_x = 0.19$ mm and $\overline{|d|} = 0.17$ mm.

Example 4.6 [P]

For the data of Example 4.3, find the mean, $\bar{x}$, the sample standard deviation, s_x, and the mean absolute deviation, $\overline{|d|}$, using Python.

```
from __future__ import division
import numpy as np
import math

# Enter the values given as the components of the vector x:

x = np.array([100.1, 100.2, 99.8, 100.3, 99.9, 100.2, 99.9, 100.4, 100.0,
100.3])

# Evaluation of the parameters:

N = len(x)
mean_x = x.mean()
mean_abs_dev_mean = np.sum(np.abs(x-mean_x)) / N
std_dev_sample = x.std(ddof = 1) * math.sqrt((N-1)/N)

# Preparing the printout:

print ("Number of values N =", N)
print ("Mean =", mean_x)
print ("Standard deviation of the sample =", std_dev_sample)
print ("Mean absolute deviation from the mean =", mean_abs_dev_mean)
```

Running the program, returns:

```
Number of values N = 10
Mean = 100.11
Standard deviation of the sample = 0.192093727123
Mean absolute deviation from the mean = 0.17
```

Example 4.7 [R]

For the data of Example 4.3, find the mean, $\bar{x}$, the sample standard deviation, s_x, and the mean absolute deviation, $\overline{|d|}$, using R.

We first find the mean, $\bar{x}$, and the standard deviation, **s.d.**

```
> x <- c(100.1, 100.2, 99.8, 100.3, 99.9, 100.2, 99.9, 100.4, 100.0, 100.3)
> meanx = mean(x)
> meanx
[1] 100.11
> sd(x)
[1] 0.2024846
```

The mean was found to be $\bar{x} = 100.11$ mm.

It should be pointed out R returns as **sd** not the value of the standard deviation of the sample, $s_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2}$, but rather the best estimate for the standard deviation of the parent population, $\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \bar{x})^2}$ (see Sect. 4.2.4). We may evaluate s_x using the relation $s_x = \sqrt{\frac{N-1}{N}}\,\hat{\sigma}$. The result is $s_x = \sqrt{\frac{9}{10}}\,0.202485 = 0.192094$ mm or $s_x = 0.192$ mm, as expected.

The mean absolute deviation, $\overline{|d|}$, is found by

```
> sum(abs(x-meanx))/10
[1] 0.17
```

We have found that $\bar{x} = 100.11$ mm $s_x = 0.19$ mm, $\overline{|d|} = 0.17$ mm.

Example 4.8

A total of 33 measurements are classified into 10 classes (of 10 different values of the result) as seen in the table below. Find the mean and the standard deviation from the mean of the measurements.

r	x_r (mm)	n_r	$n_r x_r$ (mm)	$x_r - \bar{x}$ (mm)	$(x_r - \bar{x})^2$ (mm^2)	$n_r(x_r - \bar{x})^2$ (mm^2)
1	9.4	1	9.4	−0.4848	0.2350	0.2350
2	9.5	1	9.5	−0.3848	0.1481	0.1481
3	9.6	4	38.4	−0.2848	0.0811	0.3244
4	9.7	3	29.1	−0.1848	0.0342	0.1025
5	9.8	5	49.0	−0.0848	0.0072	0.0360
6	9.9	5	49.5	0.0152	0.0002	0.0012
7	10.0	6	60.0	0.1152	0.0133	0.0796

(continued)

(continued)

r	x_r (mm)	n_r	$n_r x_r$ (mm)	$x_r - \bar{x}$ (mm)	$(x_r - \bar{x})^2$ (mm^2)	$n_r(x_r - \bar{x})^2$ (mm^2)
8	10.1	4	40.4	0.2152	0.0463	0.1852
9	10.2	3	30.6	0.3152	0.0994	0.2981
10	10.3	1	10.3	0.4152	0.1724	0.1724
	Σ =	33	326.2			1.5824

We find the mean $\bar{x} = \frac{1}{33}\sum_{r=1}^{10} n_r x_r = \frac{326.2}{33} = 9.8848 \approx 9.88$ mm.

Since $\sum_{r=1}^{K} n_r(x_r - \bar{x})^2 = 1.5833$ mm^2, the standard deviation of the 33 values from the mean is $s_x = \sqrt{\frac{1.5833}{33}} = \sqrt{0.04795} = 0.2190 \approx 0.22$ mm.

Example 4.9 [E]

Solve Example 4.8 using Excel®.

We enter the values of x_r and n_r in columns A and B, respectively. We need to evaluate the weighted standard deviation of x, with n as weights. The weighted mean and weighted standard deviation are defined in Sect. 9.4, but, essentially, we use Eq. (1.9), $\bar{x} = \frac{1}{N}\sum_{r=1}^{K} n_r x_r$ for the mean and Eq. (1.19), $s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r(x_r - \bar{x})^2}$ for the standard deviation, with n_r as weights and N as the sum of the weights.

We will first evaluate the weighted mean. Highlight an empty cell, say E1. Left click on cell E1 and type:

=SUMPRODUCT(A1:A10;B1:B10)/SUM(B1:B10)

Pressing **ENTER** will return the number 9.8848 in cell E1. This is the required mean, $\bar{x} = 9.88$ mm.

We will give this number the name . To do this, we right click on cell E1. In the dialog box that opens, we select **Define Name…** and in the cell for **NameM** we write **M**.

We will now evaluate the weighted standard deviation. We first evaluate the terms $(x_r - \bar{x})^2$. We highlight cell C1 and type: **= (A1-M)^2**. Pressing **ENTER** returns the number `0.235078` in cell C1. To fill cells C1 to C10 with the values of $(x_r - \bar{x})^2$, we highlight cells C1-C10 and press

Fill > Down

To evaluate the standard deviation, we highlight an empty cell, say D13 and type

=SQRT(SUMPRODUCT(B1:B10;C1:C10)/SUM(B1:B10))

Pressing **ENTER** returns the number `0.21898`. We have found that the standard deviation of the sample is $s_x = 0.22$ mm, in agreement with the results of Example 4.8.

Example 4.10 [O]

Solve Example 4.8 using Origin®.

We enter x_r and n_r in columns A and B. We highlight columns A and B. Then,

Statistics > Descriptive Statistics > Statistics on Columns > Open Dialog…

In the window that opens, we tick the following:

Input > Input Data > Range 1 > Data Range > A(X) > Weighting Range > B(Y)
Quantities > Tick **Mean, Standard Deviation**

Open **Computation Control > Weight Method > Direct Weight**
The last choice ensures that the numbers n_r will act as weights.
Then **Variance Divisor of Moment > WS**
The last setting puts the quantity number $\sum_r n_r = N$ in the denominator of Eq. (4.11).
Pressing **OK** we obtain the results:
[Mean] = 9.88485, [Standard Deviation] = 0.21898
Summarizing, $\bar{x} = 9.88$ mm, $s_x = 0.22$ mm.

Example 4.11 [P]

Solve Example 4.8 using Python.

r	x_r (mm)	n_r
1	9.4	1
2	9.5	1
3	9.6	4
4	9.7	3
5	9.8	5
6	9.9	5
7	10.0	6
8	10.1	4
9	10.2	3
10	10.3	1

We need to calculate $s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r (x_r - \bar{x})^2}$. First we need to evaluate the mean $\bar{x} = \frac{1}{N}\sum_{r=1}^{K} n_r x_r$. The task is equivalent to calculating the weighted mean and the weighted standard deviation of the sample, for the measurements x_r, with corresponding weights n_r, where $N = \sum_{r=1}^{K} n_r$. The weighted mean and weighted standard deviation are defined in Sect. 9.4. The equations derived there are the same as those given above, with the weights w_r replacing n_r, and $\sum_{r=1}^{K} w_r$ replacing N.

We will use the weighted average function from the numpy package. We also import the math module in order to use the square root function. The measurement values are stored in the vector x and the corresponding weights in the vector w.

```
import math
import numpy as np
x = np.array([9.4, 9.5, 9.6, 9.7, 9.8, 9.9, 10, 10.1, 10.2, 10.3])
w = np.array([1,1, 4, 3, 5, 5, 6, 4, 3, 1])
wmean = np.average(x, weights=w)
variance = np.average((x-wmean) ** 2, weights=w)
s = math.sqrt(variance)

# Preparing the printout:
print ("Weighted mean =", wmean)
print ("Weighted standard deviation of the sample =", s)
```

Running the program returns:

```
Weighted mean = 9.88484848485
Weighted standard deviation of the sample = 0.21898002139563225
```

Example 4.12 [R]

Solve Example 4.8 using R.

r	x_r (mm)	n_r
1	9.4	1
2	9.5	1
3	9.6	4
4	9.7	3
5	9.8	5
6	9.9	5
7	10.0	6
8	10.1	4
9	10.2	3
10	10.3	1

We need to calculate $s_x = \sqrt{\frac{1}{N}\sum_{r=1}^{K} n_r (x_r - \overline{x})^2}$. First we need to evaluate the mean $\overline{x} = \frac{1}{N}\sum_{r=1}^{K} n_r x_r$. The task is equivalent to calculating the weighted mean and the weighted standard deviation of the sample, for the measurements x_r, with corresponding weights n_r, where $N = \sum_{r=1}^{K} n_r$. The weighted mean and weighted standard

deviation are defined in Sect. 9.4. The equations derived there are the same as those given above, with the weights w_r replacing n_r, and $\sum_{r=1}^{K} w_r$ replacing N. We may, therefore use the function **weighted.mean(x, w, …)** available in R.

We first define the vectors of the x and w = n values and then find the weighted mean:

```
> x <- c(9.4, 9.5, 9.6, 9.7, 9.8, 9.9, 10, 10.1, 10.2, 10.3)
> w <- c(1, 1, 4, 3, 5, 5, 6, 4, 3, 1)
> wmean = weighted.mean(x, w)
> wmean
[1] 9.884848
```

We notice that the variance s_x^2 of the sample is simply the weighted mean of the quantity $(x_r - \overline{x})^2$. Therefore,

```
> variance = weighted.mean((x-wmean)^2, w)
> variance
[1] 0.04795225
> sqrt(variance)
[1] 0.21898
```

Summarizing, we have found that $\overline{x} = 9.88$ mm and $s_x = 0.22$ mm.

4.2.1.1 Use of a Working Mean in Order to Minimize Arithmetical Calculations

It is sometimes convenient to use a suitable working mean in evaluating the standard deviation, in order to minimize the work involved. If m is the working mean selected, then

$$\sum_{i=1}^{N} \frac{1}{N}(x_i - m) = \overline{x} - m. \tag{4.15}$$

Defining

$$\overline{x}_m = \sum_{i=1}^{N} \frac{1}{N}(x_i - m), \tag{4.16}$$

it follows that

$$\overline{x} = \overline{x}_m + m. \tag{4.17}$$

The standard deviation is found from

$$\begin{aligned} s_x^2 &= \frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - m + m - \overline{x})^2 \\ &= \frac{1}{N}\sum_{i=1}^{N}\left[(x_i - m)^2 + 2(x_i - m)(m - \overline{x}) + (m - \overline{x})^2\right] \end{aligned} \tag{4.18}$$

which gives

$$\begin{aligned} s_x^2 &= \frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2 + 2(m - \overline{x})\frac{1}{N}\sum_{i=1}^{N}(x_i - m) + (m - \overline{x})^2 \\ &= \frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2 - 2(m - \overline{x})^2 + (m - \overline{x})^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2 - (m - \overline{x})^2 \end{aligned} \tag{4.19}$$

Defining

$$s_m^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2, \tag{4.20}$$

we have

$$s_x^2 = s_m^2 - (m - \overline{x})^2 \tag{4.21}$$

Summarizing:
If

$$\overline{x}_m = \sum_{i=1}^{N}\frac{1}{N}(x_i - m) \quad \text{and} \quad s_m = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - m)^2}, \tag{4.22}$$

it is

$$\overline{x} = m + \overline{x}_m \quad \text{and} \quad s_x = \sqrt{s_m^2 - (m - \overline{x})^2}. \tag{4.23}$$

Example 4.13

Five measurements of the speed of light, c, gave the following results (in m/s):

299 792 459.2 299 792 457.4 299 792 457.1 299 792 458.8 299 792 457.8.

Find the mean and the standard deviation of the measurements.

We use $m = 299\,792\,457$ m/s as a working average and evaluate $(c_i - m)$ and $(c_i - m)^2$ and their sums:

i	c_i (m/s)	$(c_i - m)$ (m/s)	$(c_i - m)^2$ (m^2/s^2)
1	299 792 459.2	2.2	4.84
2	299 792 457.4	0.4	0.16
3	299 792 457.1	0.1	0.01
4	299 792 458.8	1.8	3.24
5	299 792 457.8	0.8	0.64
	$\Sigma =$	5.3	8.89

We find $\overline{c}_m = \sum_{i=1}^{N} \frac{1}{N}(c_i - m) = \frac{5.3}{5} = 1.06$ m/s and $s_m = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(c_i - m)^2}$ $= \sqrt{\frac{8.89}{5}} = 1.33$ m/s.

Therefore, $\overline{c} = m + \overline{c}_m = 299\,792\,457 + 1.06 = 299\,792\,458.06$ m/s

and $s_c = \sqrt{s_m^2 - (m - \overline{c})^2} = \sqrt{1.33^2 - 1.06^2} = \sqrt{0.645} = 0.80$ m/s.

The final results are: $\overline{c} = 299\,792\,458.1$ m/s and $s_c = 0.8$ m/s.

4.2.2 The Standard Deviation of the Mean

Assume that we perform M (for $k = 1, 2, \ldots, M$) identical series of measurements of the quantity x, each consisting of N (for $i = 1, 2, \ldots, N$) measurements, as seen in Table 4.1.

The value $< x >_i$ is the mean of the M values of a given i,

$$< x >_i = \frac{1}{M}\sum_{k=1}^{M} x_{k,i}. \tag{4.24}$$

We will return to these mean values later.

For each series of measurements ($k = 1, 2, \ldots, M$) we evaluate the mean of the measurements

Table 4.1 M series of measurements of the quantity x, consisting of N measurements each

$i =$	1	2	…	i	…	N	$\overline{x}_k,\ s_{xk}$
$k = 1$	$x_{1,1}$	$x_{1,2}$	…	$x_{1,i}$	…	$x_{1,N}$	$\overline{x}_1,\ s_{x,1}$
2	$x_{2,1}$	$x_{2,2}$	…	$x_{2,i}$	…	$x_{2,N}$	$\overline{x}_2,\ s_{x,2}$
…	…	…	…	…	…	…	…
k	$x_{k,1}$	$x_{k,2}$	…	$x_{k,i}$	…	$x_{k,N}$	$\overline{x}_k,\ s_{x,k}$
…	…	…	…	…	…	…	…
M	$x_{M,1}$	$x_{M,2}$	…	$x_{M,i}$	…	$x_{M,N}$	$\overline{x}_M,\ s_{x,M}$
	$<x>_1$	$<x>_2$	…	$<x>_i$	…	$<x>_N$	

$$\overline{x}_k = \frac{1}{N}\sum_{i=1}^{N} x_{k,i} \tag{4.25}$$

and their standard deviation

$$s_{x,k} = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_{k,i} - \overline{x}_k)^2}. \tag{4.26}$$

The values of $\overline{x}_k$ and $s_{x,k}$ are given in the last column of the table. The M results $\overline{x}_k$ will, in general, differ from each other and we will thus have a distribution of the mean values. The $M \times N$ values of x in the table have a mean $\overline{X}$ and a standard deviation S, which are given by

$$\overline{X} = \frac{1}{MN}\sum_{k=1}^{M}\sum_{i=1}^{N} x_{k,i} \tag{4.27}$$

and

$$S = \sqrt{\frac{1}{MN}\sum_{k=1}^{M}\sum_{i=1}^{N}(x_{k,i} - \overline{X})^2}. \tag{4.28}$$

As the number of the series of measurements M tends to infinity, these values tend to the corresponding values of the total of the infinite measurements which it is possible to perform. These are the parent population from which each sample consisting of N measurements is taken. The mean value of x for the infinite population is denoted by μ and its standard deviation with σ (Greek letters are used, in general, for the parent population and Latin letters for the sample). Thus, we have

$$\lim_{M\to\infty} \overline{X} = \mu \quad \text{and} \quad \lim_{M\to\infty} S = \sigma. \tag{4.29}$$

The various values of $\overline{x}_k$ estimated from the M series of measurements have, themselves, a distribution about the real value x_0 of the quantity x, which is characterized by a standard deviation, known as the *standard deviation of the mean* and is denoted by $\sigma_{\overline{x}}$. Our aim is to find relations which will enable us to make estimates for σ and $\sigma_{\overline{x}}$, when we know s_x for one series of N measurements of x.

In the mathematical analysis that will follow, the sample means are symbolized as up to now, by a line over the symbol, e.g. $\overline{x}$. The mean values evaluated for the whole of the parent population will be symbolized as $< x >$, $< e >$, $< e^2 >$ etc. These values are calculated for the $M \times N$ values of x, which result from the M series of N measurements each, as $M \to \infty$. Thus, for example, for each column of Table 4.1, the values $< x >_i$, as mean values of x which result from infinite measurements, will be equal to the real value x_0. The same is true for the totality of the table's values. For the same reasons, the standard deviation of the values of each column, or of the whole table, will tend to σ, while for the last column of the table, the mean of the values of $\overline{x}_k$ will tend to x_0, and the mean of their standard deviations $s_{x,k}$ to σ.

Example 4.14

$M = 8$ series of the quantity x, each consisting of $N = 6$ measurements, gave the results shown in the table that follows. Find the mean and the standard deviation for each series of measurements Then, find the mean of all the measurements and the standard deviation of the 8 mean values of the series of measurements.

	$x_{k,i}$						$\overline{x}_k$	$s_{x,k}$
i =	1	2	3	4	5	6		
k = 1	9.5	10.3	10.1	9.9	10.0	10.3	10.02	0.273
2	9.7	10.0	10.3	9.8	10.2	9.8	9.97	0.221
3	10.2	10.2	9.7	10.1	10.1	10.2	10.08	0.177
4	9.8	10.4	9.9	10.2	9.9	10.2	10.07	0.213
5	10.0	9.9	10.1	9.7	10.1	10.0	9.97	0.137
6	9.9	10.4	10.3	10.4	9.6	9.7	10.05	0.330
7	10.0	9.8	10.1	9.9	10.2	9.9	9.98	0.134
8	10.1	9.6	10.0	9.9	10.0	9.7	9.88	0.177
						Sums Σ =	80.02	1.662
						Mean =	10.00	0.208
						Standard deviation =	0.06	

The mean for each series of measurements is given in the column of $\overline{x}_k$.

The standard deviation for each series of measurements is given in the column of $s_{x,k}$. Their mean is 0.208.

Because the sum of the means $\overline{x}_k$ of the 8 series of measurements is 80.02, the mean of all the measurements is $\overline{X} = 80.02/8 = 10.00$.

The standard deviation of the 8 mean values $\overline{x}_k$ is 0.06.

The standard deviation of all the 48 values is $S = 0.23$.

4.2.3 The Relationship Between σ and $\sigma_{\overline{x}}$

If we have N measurements x_i ($i = 1, 2, \ldots, N$) of the quantity x, whose real value is x_0, the error in x_i is $e_i = x_i - x_0$ and the error in the mean $\overline{x}$ is $e_\mu = \overline{x} - x_0$. Since

$$e_\mu = \overline{x} - x_0 = \frac{1}{N}\sum_i e_i, \tag{4.30}$$

it will be

$$e_\mu^2 = \frac{1}{N^2}\left(\sum_i e_i\right)^2 = \frac{1}{N^2}\sum_i e_i^2 + \frac{1}{N^2}\sum_i \sum_{j,j\neq i} e_i e_j, \tag{4.31}$$

where the squares of e_i are summed in the first sum, while the products of different e_i are summed in the second sum. We will now assume that we have a large number M of series of N measurements each and we will take the (population) means of these two sums as $M \to \infty$.

The population mean of e_μ^2 is denoted by $<e_\mu^2>$, while the population mean $\frac{1}{N}\sum_i e_i^2$ is $<e^2>$, i.e. the mean of the square of the error. However, by definition, it is $<e_\mu^2> = \sigma_\mu^2 = \sigma_{\overline{x}}^2$, where $\sigma_\mu = \sigma_{\overline{x}}$ is the standard deviation of the mean and $<e^2> = \sigma^2$, where σ is the standard deviation of the population of the infinite measurements that may be performed.

The population mean of the sum $\sum_i \sum_{j,j\neq i} e_i e_j$ tends to zero, being the average of the products of a large number ($M \to \infty$) of mutually independent quantities, which are symmetrically distributed around zero. Equation (4.31) gives, therefore,

$$\sigma_\mu = \sigma_{\overline{x}} = \frac{\sigma}{\sqrt{N}}. \tag{4.32}$$

A different proof of this relation will be given in Example 6.5 of Chap. 6.

At present, we have no knowledge regarding σ, which describes the distribution about the real value x_0 of the infinite measurements x_i that can be made. In the next subsection we will find an estimate for this value, based on the known quantity s_x, the standard deviation of the N measurements we have performed. In this way, it will also be possible to have an estimate for the value of $\sigma_\mu = \sigma_{\overline{x}}$.

4.2.4 The Relationship Between s_x and σ and $\sigma_{\overline{x}}$

From the definition of the standard deviation of the measurements x_i

$$s_x^2 \equiv \frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2 \tag{4.33}$$

and the fact that

$$x_i - \overline{x} = e_i - e_\mu, \tag{4.34}$$

we have the relation

$$s_x^2 = \frac{1}{N}\sum_{i=1}^{N}(e_i - e_\mu)^2 = \frac{1}{N}\sum_{i=1}^{N}e_i^2 - 2e_\mu\frac{1}{N}\sum_{i=1}^{N}e_i + e_\mu^2 = \frac{1}{N}\sum_{i=1}^{N}e_i^2 - e_\mu^2 \tag{4.35}$$

Evaluating the population means we have

$$< s_x^2 > \; = \sigma^2 - \sigma_\mu^2 \tag{4.36}$$

Combined with Eqs. (4.32) and (4.36) gives

$$\sigma^2 = \frac{N}{N-1} < s_x^2 > \quad \text{or} \quad \sigma = \sqrt{\frac{N}{N-1} < s_x^2 >} \tag{4.37}$$

and

$$\sigma_\mu^2 = \frac{1}{N-1} < s_x^2 > \quad \text{or} \quad \sigma_\mu = \sqrt{\frac{1}{N-1} < s_x^2 >}\,. \tag{4.38}$$

The quantity $< s_x^2 >$ is unknown to us, since, in theory, we need an infinite number of measurements for it to be determined with absolute accuracy. The best *estimate* that we have for it is s_x^2, which results from the N measurements we have made. Therefore, the best estimates we have at our disposal for σ and σ_μ are, respectively,

$$\hat{\sigma} = \sqrt{\frac{N}{N-1}}\, s_x \tag{4.39}$$

and

$$\hat{\sigma}_\mu = \frac{s_x}{\sqrt{N-1}}, \tag{4.40}$$

where the carets (hats) above σ and σ_μ state the fact that, strictly speaking, we do not have an equation but that the magnitude on the right is the best estimate for the magnitude on the left. The carets are usually omitted.

Using the fact that $s_x \equiv \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \overline{x})^2}$ and omitting the carets, we have

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2} \tag{4.41}$$

and

$$\sigma_{\overline{x}} = \sigma_\mu = \sqrt{\frac{1}{N(N-1)}\sum_{i=1}^{N}(x_i - \overline{x})^2} \tag{4.42}$$

for the standard deviation of the parent population of the infinite possible results of the measurements of the magnitude x that may be performed and for the standard deviation of the mean, respectively.

Example 4.15

Find the (best estimates of the) standard deviations of the means for the values of Examples 4.3 and 4.8.

Example 4.3: From $s_x = 0.192$ mm and $N = 10$, we find $\sigma_{\overline{x}} = \sigma_\mu \approx \hat{\sigma} = \frac{s_x}{\sqrt{N-1}}$ and, therefore,

$$\sigma_{\overline{x}} = \sigma_\mu \approx \hat{\sigma} = \frac{0.192}{3} = 0.062 \text{ mm}.$$

Example 4.8: From $s_x = 0.22$ mm and $N = 33$, it is $\sigma_{\overline{x}} = \sigma_\mu \approx \hat{\sigma} = \frac{0.22}{\sqrt{32}} = 0.039$ mm.

Example 4.16 [E]

Given the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, find their mean $\overline{x}$, sample standard deviation s_x, the best estimate for the standard deviation of the parent population, $\hat{\sigma}$, the standard deviation of the mean $\sigma_{\overline{x}}$ and mean absolute deviation $\overline{|d|}$.

We enter the values of x_i in cells A1 to A10. We highlight cells A1 to A10. Then,

Data > Data Analysis > Descriptive Statistics > OK

In the dialog box that opens, we set **Input, Input Range > $A$1:$A$10**, **Grouped by > Columns** and tick the box for **Summary statistics**. Press **OK**. The program returns a table, from which we read:

[Mean] = 5.500, [Standard Error] = 0.957427 [Standard Deviation] = 3.027650. This is also the best estimate of the standard deviation of the parent population.

By the term [Standard Error], Excel means the standard error of the mean or the standard deviation of the mean, $\sigma_{\overline{x}}$. Therefore, $\sigma_{\overline{x}} = 0.96$ mm.

Bearing in mind the comments made in Example 4.4, the [Standard Deviation] given by Excel is $\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}$. From this, we find the standard deviation of the sample as $s_x = \sqrt{\frac{N-1}{N}}\,\hat{\sigma}$. The result is $s_x = \sqrt{\frac{9}{10}}\,3.027650 = 2.87228$ mm.

To calculate the mean absolute deviation, $\overline{|d|}$, we proceed as follows:

Set cell B1 **= ABS(A1-5.5)**. Fill Down cells B2 to B10. Column B now contains the values of $|x_i - \overline{x}|$. Highlight cells A1 to A10. Open the **Autosum** dialog box and press **Sum**. The result returned is 25.00. Dividing by $N = 10$, we have the result: $\overline{|d|} = 2.5$ mm.

Summarizing, $\overline{x} = 5.5$ mm, $s_x = 2.9$ mm, $\hat{\sigma} = 3.0$ mm, $\overline{|d|} = 2.5$ mm and $\sigma_{\overline{x}} = 0.96$ mm.

Example 4.17 [O]

Given the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, find their mean $\overline{x}$, sample standard deviation s_x, the best estimate for the standard deviation of the parent population, $\hat{\sigma}$, the standard deviation of the mean $\sigma_{\overline{x}}$ and mean absolute deviation $\overline{|d|}$.

We enter the numbers in column A. We select column A. Then,

Statistics > Descriptive Statistics > Statistics on Columns > Open Dialog…

In the window that opens, we tick the following:

Input > Input Data > Range 1 > Data Range > A(X)
Quantities > Tick **Mean, Standard Deviation, Mean Absolute Deviation**

Open **Computation Control > Weight Method > Direct Weight**
Then **Variance Divisor of Moment > N**

The last setting puts the number N in the denominator of Eq. (4.10) (The choice DF would put $N - 1$ in the denominator).

Pressing **OK** we obtain the results:

[Mean] = 5.5, [Standard Deviation] = 2.8723, [Mean Absolute Deviation] = 2.5

Summarizing, $\overline{x} = 5.5$ mm, $s_x = 2.9$ mm, $\hat{\sigma} = \sqrt{N/(N-1)}\,s_x = 3.0$ mm, $\overline{|d|} = 2.5$ mm and $\sigma_{\overline{x}} = s_x/\sqrt{N-1} = 0.96$ mm.

Example 4.18 [P]

Given the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, find their mean $\overline{x}$, sample standard deviation s_x, the best estimate for the standard deviation of the parent population, $\hat{\sigma}$, the standard deviation of the mean $\sigma_{\overline{x}}$ and mean absolute deviation $\overline{|d|}$.

```
from __future__ import division
import numpy as np
import math

# Enter the values given as the components of the vector x
x = np.array([1, 2, 3, 4, 5, 6, 7, 8, 9, 10])

# Evaluation:
N = len(x)
mean_x = x.mean()
std_dev_sample = x.std(ddof = 1) * math.sqrt((N-1)/N)
std_dev_popul = x.std(ddof = 1)
std_dev_mean = std_dev_sample * math.sqrt(1/(N-1))
mean_abs_dev_mean = np.sum(np.abs(x-mean_x)) / N

# Preparation for printout:
print ("Number of values N =", N)
print ("Mean =", mean_x)
print ("Standard deviation of the sample =", std_dev_sample)
print ("Standard deviation of the population =", std_dev_popul)
print ("Standard deviation of the mean =", std_dev_mean)
print ("Mean absolute deviation from the mean =", mean_abs_dev_mean)
```

Running the program returns the results:

```
Number of values N = 10
Mean = 5.5
Standard deviation of the sample = 2.87228132327
Standard deviation of the population = 3.0276503541
Standard deviation of the mean = 0.957427107756
Mean absolute deviation from the mean = 2.5
```

Example 4.19 [R]

Given the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, find their mean $\overline{x}$, sample standard deviation s_x, the best estimate for the standard deviation of the parent population, $\hat{\sigma}$, the standard deviation of the mean $\sigma_{\overline{x}}$ and mean absolute deviation $\overline{|d|}$.

The mean, $\overline{x}$, and mean absolute deviation $\overline{|d|}$ are:

```
> x <- c(1,2,3,4,5,6,7,8,9,10)
> mean(x)
[1] 5.5
> sum(abs(x - mean(x)))/10
[1] 2.5
```

R calculates the best estimate for the standard deviation of the parent population, $\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}$ as **sd(x)**. From this, we find the standard deviation of the sample as $s_x = \sqrt{\frac{N-1}{N}}\,\hat{\sigma}$ and the standard deviation or error of the mean as $\sigma_{\overline{x}} = \frac{s_x}{\sqrt{N-1}}$:

```
> sd(x)
[1] 3.02765
> sx = sqrt(9/10)*sd(x)
> sx
[1] 2.872281
err = sx/sqrt(9)
> err
[1] 0.9574271
```

We have found that $\overline{x} = 5.5\,\text{mm}$, $\overline{|d|} = 2.5\,\text{mm}$, $\hat{\sigma} = 3.0\,\text{mm}$, $s_x = 2.9\,\text{mm}$, and $\sigma_{\overline{x}} = 0.96\,\text{mm}$.

4.3 The Standard Deviation of the Standard Deviation of the Mean

The value of the standard deviation $\sigma_{\overline{x}}$ of the mean $\overline{x}$ of a series of measurements was determined using the N measurements performed. If we perform another series of N measurements, what will the difference be between the two standard deviations? And finally, if we perform a large number M of series with N measurements each, what kind of dispersion will there be in the standard deviations $\sigma_{\overline{x},k}$ ($k = 1, 2, \ldots, M$) of the means of the M series of measurements? Having made only one series of measurements, the best estimate that we have for the mean of these standard deviations is $\sigma_{\overline{x}}$. The dispersion of the values $\sigma_{\overline{x},k}$ around the mean is expressed by a *standard deviation of the standard deviation of the mean* $\sigma_{\overline{x}}$. We will denote this by $\sigma(\sigma_{\overline{x}})$ and its *fractional* value by α, in which case it will be $\sigma(\sigma_{\overline{x}}) = \alpha\sigma_{\overline{x}}$.

It is proved that for N measurements, and under the same assumptions for the statistical behavior of the $\sigma_{\overline{x},k}$ that we accepted to hold for the measurements x_i and their errors, it is, to a good approximation,

$$\alpha(N) = \frac{1}{\sqrt{2(N-1)}}. \tag{4.43}$$

The values of α are given in the table that follows for different values of N.

N	5	6	8	10	15	20	30	50	100
α	0.354	0.316	0.267	0.236	0.189	0.162	0.131	0.101	0.071

The function $\alpha(N)$ is plotted in Fig. 4.1.

The value of α is useful as an estimate of the accuracy with which we know the standard deviation of the mean, $\sigma_{\bar{x}}$. We will see, for example, in the chapter for the presentation of numerical results, that, for the usual number of measurements, which is of the order of 10, the standard deviation is known with an uncertainty of about 24% and that it makes no sense to give the numerical value of $\sigma_{\bar{x}}$ with more than one or, at most, two significant figures. As a consequence it makes no sense to give the numerical value of the mean with greater accuracy.

Shown in Fig. 4.2 are the standard deviations s_x of the N measurements of the quantity x, and of their mean, $\sigma_{\bar{x}}$, as a function of N, as this increases from 2 to 500. These 500 values of 'measurements' were taken at random from a parent population similar to that of our 'thought experiment' of Chap. 3, which was chosen to have $\sigma = 1$. In order to cover the wide range of values of N without loss of detail in the behavior at low N, the scale of N in the figure was taken to be logarithmic. As N increases, s_x is seen to approach the value of $\sigma = 1$, and $\sigma_{\bar{x}}$ tends to the value $\sigma/\sqrt{N}$. Also drawn in the figure are the curves for $\sigma\left(1 \pm 1/\sqrt{2(N-1)}\right)$ and $\frac{\sigma}{\sqrt{N}}\left(1 \pm 1/\sqrt{2(N-1)}\right)$, between which s_x and $\sigma_{\bar{x}}$ are seen to lie for most values of N. The fluctuations in s_x and $\sigma_{\bar{x}}$ appear to be of the order of magnitude predicted by Eq. (4.43).

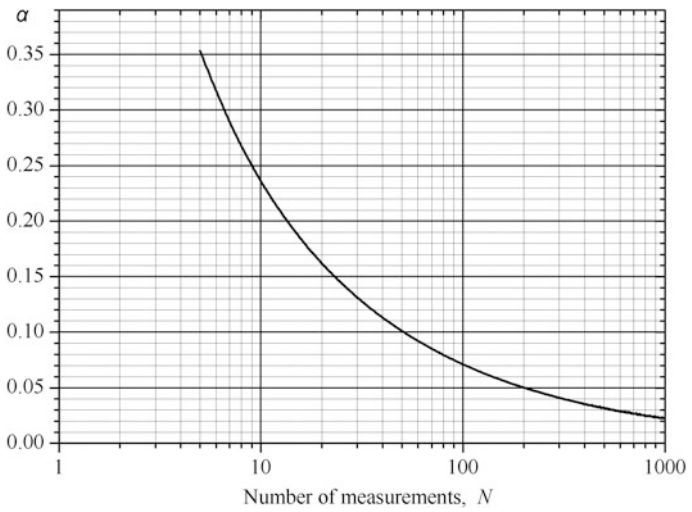


Fig. 4.1 The variation with the number N of the measurements of the fractional standard deviation, α, of the standard deviation of the mean. The scale of N is logarithmic

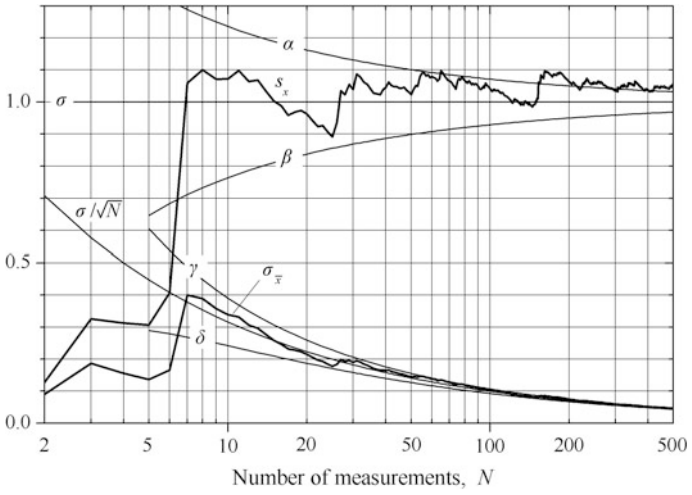


Fig. 4.2 The standard deviations s_x of the N measurements of the quantity x, and of their mean $\sigma_{\bar{x}}$, as a function of N, as this increases from 2 to 500 (the scale for N is logarithmic). Also drawn are the curves $\sigma/\sqrt{N}$, $\sigma\left(1 \pm 1/\sqrt{2(N-1)}\right)$ (α and β) and $\frac{\sigma}{\sqrt{N}}\left(1 \pm 1/\sqrt{2(N-1)}\right)$ (γ and δ)

The reader might be relieved to know that we have absolutely no use for the standard deviation of the standard deviation of the standard deviation.

4.4 Information Derived from the Measurement of $\bar{x}$ and s_x

From the values of $\bar{x}$ and s_x we have from the N measurements of x we made, we may extract some useful information regarding the distribution of the values of x in the parent population of all the possible results and the real value x_0 of x.

4.4.1 The Mean Value of the Results of the Measurements and Its Standard Deviation

We have already explained, qualitatively, why the mean $\bar{x}$ is the best estimate we have for the real value x_0, under the assumption that the parent population is symmetrical relative to x_0, i.e. that positive and negative errors are equally probable. A more rigorous proof will be given in Chap. 9, which deals with the Theory of Errors.

Using the definition

$$s_x = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})^2} \tag{4.44}$$

of the standard deviation of the results of the measurements, we will show that we can conclude that s_x tends to a constant value as the number of measurements increases. This value is the standard deviation of the parent population of the possible results of the measurements:

$$\lim_{N\to\infty} s_x = \sigma. \tag{4.45}$$

If we look at the successive values of s_x^2 as N increases, we have

$$\frac{(x_1 - \bar{x}_1)^2}{1}, \frac{(x_1 - \bar{x}_2)^2 + (x_2 - \bar{x}_2)^2}{1+1}, \frac{(x_1 - \bar{x}_3)^2 + (x_2 - \bar{x}_3)^2 + (x_3 - \bar{x}_3)^2}{1+1+1}, \ldots \tag{4.46}$$

for N equal to 1, 2, 3, …, respectively, where $\bar{x}_N$ is the mean after the first N measurements. We see that making another measurement, x_N, leads to the increase of the numerator by the quantity $(x_N - \bar{x}_N)^2$ and of the denominator by unity. As the mean value of the results gradually tends to a constant value, the same happens to the mean value of the quantities $(x_N - \bar{x}_N)^2$. This simultaneous proportional increase of numerator and denominator has an effect on the value of the fraction which becomes gradually smaller and the value of s_x tends to a limit. Since the sample becomes, with increasing N, more and more representative of the parent population, this limit must be σ.

Due to the fact that the standard deviation of the mean is, according to Eq. (4.32), equal to

$$\sigma_{\bar{x}} = \sigma_\mu = \frac{\sigma}{\sqrt{N}}, \tag{4.47}$$

it follows that, for large values of N, $\sigma_{\bar{x}}$ is inversely proportional to $\sqrt{N}$ and tends to zero. The deviation of the mean $\bar{x}$ from the real value x_0 tends to zero and $\bar{x}$ is increasingly a better estimate for the real value x_0. This procedure is seen in Fig. 4.3. Shown in this figure is the variation of the mean $\bar{x}_N$ for the 'measurements' of our thought experiment of Chap. 3, with the number of measurements performed, N, as this number increases, finally reaching the value of 10,000. The curves for $x_0 + \sigma/\sqrt{N}$ and $x_0 - \sigma/\sqrt{N}$ are also drawn in the figure. These values are known to us here, as the results of the measurements x_i were specially selected for the 'experiment', using random numbers, so that they have $x_0 = 100\,\text{mm}$ and $\sigma = 1\,\text{mm}$.

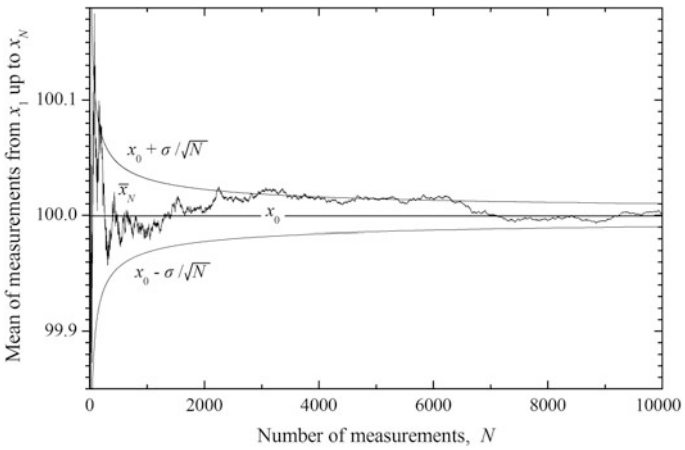


Fig. 4.3 The variation of the mean $\overline{x}_N$ for N measurements of the quantity x, as this number increases. The real value of x is $x_0 = 100\,\text{mm}$ and the standard deviation of the parent population of the possible measurements is $\sigma = 1\,\text{mm}$. The curves for $x_0 \pm \sigma/\sqrt{N}$ were also drawn in the figure

The standard deviation of the mean, $\sigma_{\overline{x}}$, being the best estimate we have for the root of the mean square of the deviations from the real value, of the means of many series of measurements of x, gives an estimate of the expected difference of the determined value from the real. We will see below that it helps us make predictions for the statistical distribution of the means of many series of measurements of x. Concerning the determination of the real value of x, we may say that, most probably, it lies between the limits $\overline{x} \pm \sigma_{\overline{x}}$. We state this by writing

$$x = \overline{x} \pm \sigma_{\overline{x}}, \tag{4.48}$$

when giving the numerical values of $\overline{x}$ and $\sigma_{\overline{x}}$. For the first 1000 values of Fig. 4.3, for example, we find that it is $\overline{x} = 99.99\,\text{mm}$ and $\sigma_{\overline{x}} = 0.03\,\text{mm}$, to an accuracy of two decimal digits. Thus, we write:

$$x = 99.99 \pm 0.03\,\text{mm}.$$

The presentation of numerical results will be examined in the next chapter.

The quantity

$$\delta x \equiv \sigma_{\overline{x}} = \sigma_{\mu} = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \overline{x})^2}{N(N-1)}} \tag{4.49}$$

is also called *standard error in the mean* or, simply, *error in the mean* $\overline{x}$.

Also used is the *relative or fractional standard deviation of the mean* or the *relative or fractional error in the mean*,

$$\frac{\delta x}{\overline{x}}, \tag{4.50}$$

which is also expressed as a percentage,

$$100\,\frac{\delta x}{\overline{x}}\,\%. \tag{4.51}$$

4.4.2 The Statistical Distribution of the Results of the Measurements

From the standard deviation s_x of the results of the measurements, we have an estimate $\hat{\sigma}$ for the standard deviation σ of the parent population [Eq. (4.39)]. If we knew the mathematical form of the probability density function $f(x)$ of the parent population of all the possible measurements that can be made, we would be able to make estimates for the parameters present in $f(x)$. For example, if the distribution had the form of the Laplace distribution,

$$f(x) = \frac{\alpha}{2}\mathrm{e}^{-\alpha|x-\mu|}, \tag{4.52}$$

we would have estimates for μ and α. In Example 1.6 we found that, for this normalized distribution, it is $\overline{x} = \mu$ and $\sigma = \sqrt{2}/\alpha$. Since the estimate we found for σ is

$$\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}, \tag{4.53}$$

we would find that it is

$$\alpha = \sqrt{2}/\hat{\sigma} \tag{4.54}$$

and, thus, have an estimate for $f(x)$. From this we would be able to derive quantitative conclusions regarding the distribution of the results of the measurements, such as, for example, the proportion of measurements expected to have values between certain limits, the probability for a result to exceed a certain value etc.

However, the function $f(x)$ is not known to us with certainty and the histogram of the measurements is usually too vague (due to the low number of measurements) to give us even an approximation for the form of $f(x)$. There are, however, well grounded reasons for us to believe that, under some very general conditions, the distribution of the results of the measurements is expressed by a probability density function which has the, so called, *Gaussian* form

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \, \mathrm{e}^{-(x-\mu)^2/2\sigma^2} \equiv G_{\mu,\sigma}(x). \tag{4.55}$$

It is easily proved that μ is the mean value and σ is the standard deviation of the results of the parent population of the measurements, x. The distribution is also called the *normal distribution.*

Strictly speaking, the distribution is termed normal when it is stated in the form

$$G_{0,1}(x) = \frac{1}{\sqrt{2\pi}} \, \mathrm{e}^{-x^2/2}, \tag{4.56}$$

i.e. when it has as mean the value of $\mu = 0$ and a standard deviation equal to $\sigma = 1$ (or, in other words, when the function has been normalized so that the deviations x are measured from the mean, in units of σ).

The conditions for this distribution to be valid are:

(a) the errors of the measurements are due to the superposition of a large number of small deviations from the real value and
(b) these deviations are equally probable to be positive or negative.

These conditions appear reasonable for the description of the behavior of the random errors of measurements, but it must not be taken for granted that the Gaussian distribution has general validity. In most cases, it is a satisfactory approximation of reality and it is used due to its mathematical simplicity. This will be discussed in more detail in Chap. 9.

The graph of the Gaussian distribution. is shown in Fig. 4.4 in a universal form. The quantity $\sigma f(x)$ has been plotted as a function of $(x - \mu)$, which is expressed in units of σ. Thus, the shape of the curve of the figure is independent of σ and μ.

The curve of the Gaussian function is symmetrical relative to the axis $x = \mu$. It has a maximum equal to $0.3989/\sigma$ at $x = \mu$, while for $x = \mu \pm \sigma$ it takes the value $0.2420/\sigma$. The points of the curve at $x = \mu \pm \sigma$ are points of inflection. For large values of $|x - \mu|$ the curve tends rapidly and asymptotically towards the x-axis.

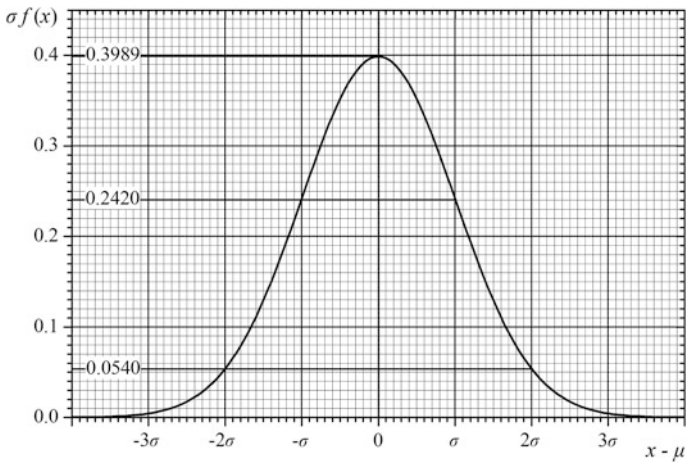


Fig. 4.4 The density function of the Gaussian distribution

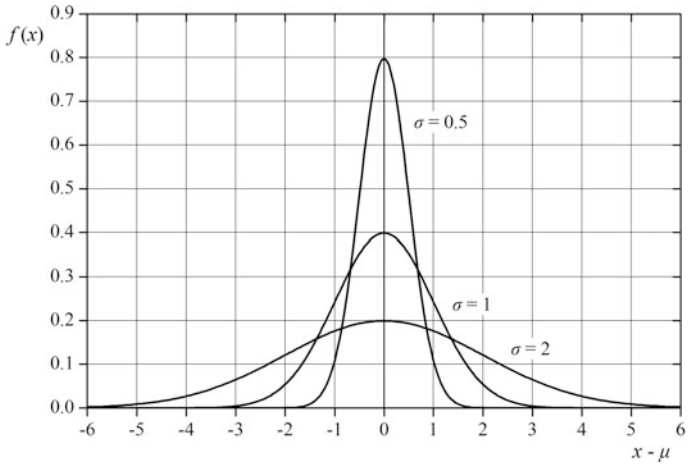


Fig. 4.5 Plots of the Gaussian function for σ equal to 0.5, 1 and 2

The significance of the parameter σ is seen in Fig. 4.5, where $f(x)$ was drawn as a function of $(x - \mu)$, for σ equal to 0.5, 1 and 2. It is immediately evident that large σ means a large dispersion of the values of x.

The total area between the curve and the x-axis is equal to unity:

$$\int_{-\infty}^{+\infty} f(x)\,dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{+\infty} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}\,dx = 1. \tag{4.57}$$

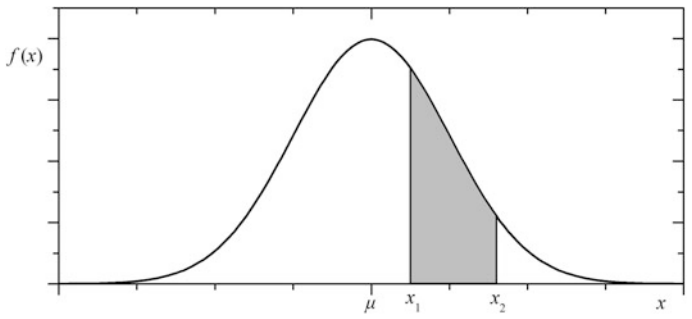


Fig. 4.6 The area between the Gaussian curve and the x-axis, in the range $[x_1, x_2]$

The function is, therefore, normalized. As a consequence, the area between the curve and the x-axis in the range $[x_1, x_2]$ gives the probability for a value of x to lie in the range $x_1 \leq x \leq x_2$ (see Fig. 4.6):

$$\Pr\{x_1 \leq x \leq x_2\} = \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} \mathrm{e}^{-(x-\mu)^2/2\sigma^2} dx. \tag{4.58}$$

There are detailed tables of the Gaussian function and its integral. The function

$$\mathrm{erf}(x) \equiv \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{e}^{-t^2} dt, \tag{4.59}$$

which is called *error function* (thus erf from **er***ror* **f***unction*), is used in the evaluation of the integral of the probability density function of the normal distribution [1].

Also defined is the function [2]

$$\Phi(x) \equiv \frac{1}{\sqrt{2\pi}} \int_0^x \mathrm{e}^{-t^2/2} dt. \tag{4.60}$$

It is

$$\Phi(x) = \frac{1}{2} \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right). \tag{4.61}$$

The functions $\mathrm{erf}(x)$ and $\Phi(x)$ are odd and, therefore, $\mathrm{erf}(-x) = -\mathrm{erf}(x)$ and $\Phi(-x) = -\Phi(x)$. Also, $\mathrm{erf}(0) = 0$, $\Phi(0) = 0$ and $\mathrm{erf}(\infty) = 1$, $\Phi(\infty) = \frac{1}{2}$.

Table 4.2 Values of the functions $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2}\, dt$

x	$G_{0,1}(x)$	$\Phi(x)$	x	$G_{0,1}(x)$	$\Phi(x)$
0	0.398 942	0	2.1	0.043 984	0.482 136
0.1	0.396 952	0.039 828	2.2	0.035 475	0.486 097
0.2	0.391 042	0.079 260	2.3	0.028 327	0.489 276
0.3	0.381 388	0.117 911	2.4	0.022 395	0.491 802
0.4	0.368 270	0.155 422	2.5	0.017 528	0.493 790
0.5	0.352 065	0.191 462	2.6	0.013 583	0.495 339
0.6	0.333 224	0.225 747	2.7	0.010 421	0.496 533
0.7	0.312 254	0.258 036	2.8	0.007 915	0.497 445
0.8	0.289 691	0.288 145	2.9	0.005 953	0.498 134
0.9	0.266 085	0.315 940	3.0	0.004 432	0.498 650
1.0	0.241 971	0.341 345	3.1	0.003 267	0.499 032
1.1	0.217 852	0.364 334	3.2	0.002 384	0.499 313
1.2	0.194 186	0.384 930	3.3	0.001 723	0.499 517
1.3	0.171 368	0.403 200	3.4	0.001 232	0.499 663
1.4	0.149 727	0.419 243	3.5	0.000 873	0.499 767
1.5	0.129 518	0.433 193	3.6	0.000 612	0.499 841
1.6	0.110 921	0.445 201	3.7	0.000 425	0.499 892
1.7	0.094 049	0.455 435	3.8	0.000 292	0.499 928
1.8	0.078 950	0.464 070	3.9	0.000 199	0.499 952
1.9	0.065 616	0.471 283	4.0	0.000 134	0.499 968
2.0	0.053 991	0.477 250	4.1	0.000 0893	0.499 979

Some values of the functions $G_{0,1}(x)$ and $\Phi(x)$ are given in Table 4.2.
The functions $G_{0,1}(x)$ and $\Phi(x)$ have been plotted in Fig. 4.7.
With the aid of the functions erf(x) and $\Phi(x)$ we find that, for $X \geq \mu$,

$$\Pr\{\mu \leq x \leq X\} = \frac{1}{2}\text{erf}\left(\frac{X-\mu}{\sigma\sqrt{2}}\right) = \Phi\left(\frac{X-\mu}{\sigma}\right) \tag{4.62}$$

is the probability that a value of x lies in the region $\mu \leq x \leq X$. Defining as $\chi \equiv X - \mu$ the deviation of x from the mean μ, and making use of the symmetry of the Gaussian distribution with respect to the mean, we have

$$\Pr\{\mu - \chi \leq x \leq \mu + \chi\} = \text{erf}\left(\frac{\chi}{\sigma\sqrt{2}}\right) = 2\,\Phi\left(\frac{\chi}{\sigma}\right) \tag{4.63}$$

as the probability for a value of x to differ from the mean by less than χ.

Measuring the deviation of x from the mean in multiples of the standard deviation, i.e. putting $\chi = v\sigma$, where v is a positive number, we have

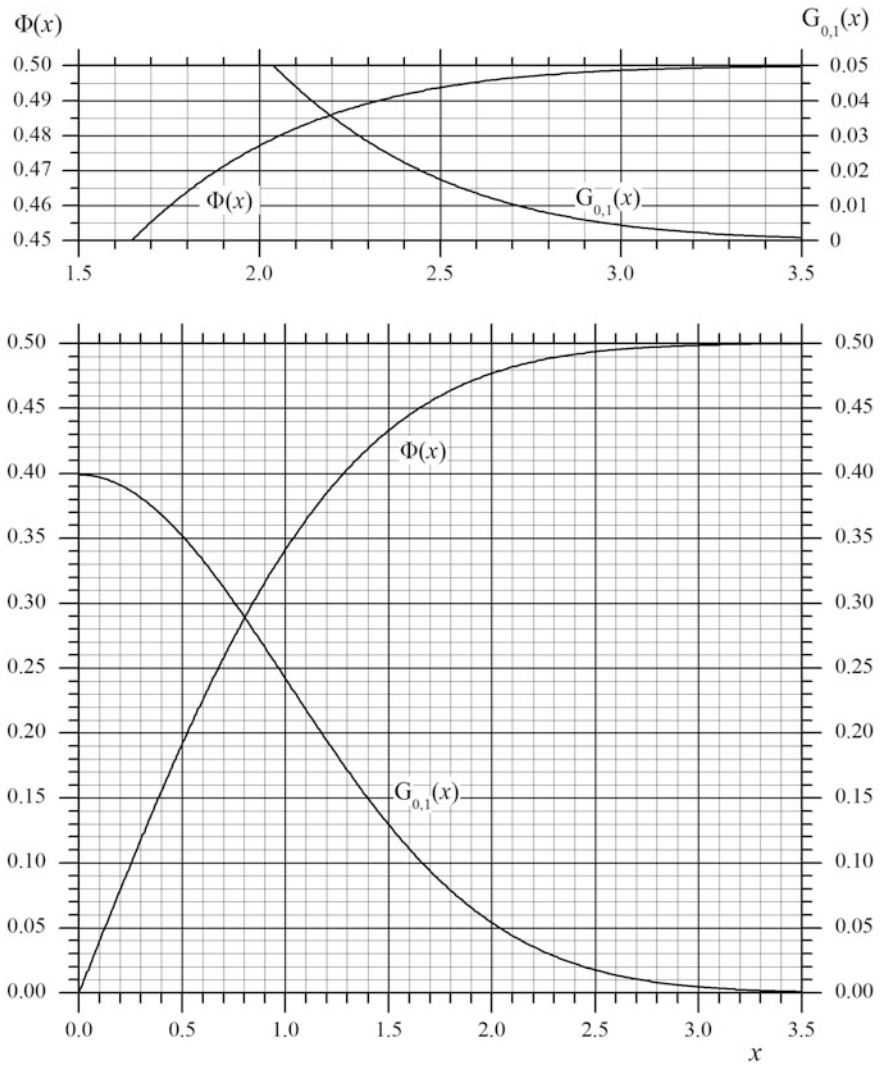


Fig. 4.7 The functions $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$ and $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt$

$$\Pr\{\mu - v\sigma \le x \le \mu + v\sigma\} = \text{erf}\left(\frac{v}{\sqrt{2}}\right) = 2\,\Phi(v) \tag{4.64}$$

as the probability for a value of x to differ from the mean by less than v times the standard deviation.

The probability for a value of x to differ from the mean by more than v times the standard deviation is given by the relation

Table 4.3 The probabilities that a value of x lies in the range $\mu - v\sigma \le x \le \mu + v\sigma$ or outside it

v	$\Pr\{\mu - v\sigma \le x \le \mu + v\sigma\}$	$\Pr\{x \le \mu - v\sigma \text{ or } x \ge \mu + v\sigma\}$
0	0	1
0.001253	0.001	0.999
0.01253	0.01	0.99
0.1257	0.1	0.9
0.6745	0.5	0.5
1	0.68269	0.31731
1.282	0.8	0.2
1.665	0.9	0.1
1.96	0.95	0.05
2	0.95450	0.04550
2.33	0.98	0.02
2.58	0.99	0.01
3	0.99730	0.00270
3.29	0.999	0.001
∞	1	0

$$\Pr\{x \le \mu - v\sigma \text{ or } x \ge \mu + v\sigma\} = 1 - \text{erf}\left(\frac{v}{\sqrt{2}}\right) \equiv \text{erfc}\left(\frac{v}{\sqrt{2}}\right) = 1 - 2\,\Phi(v), \qquad (4.65)$$

where $\text{erfc}\,(x) \equiv 1 - \text{erf}\,(x)$ is the *complementary error function*. The values of these probabilities for various values of v are given in Table 4.3.

The table shows that:

Half the values of x are expected to differ from the mean by more than 0.6745 times the standard deviation. The value $0.6745\,\sigma$ is called *probable error* in the results for x.

31.7% of the values, or about 1 in 3, are expected to differ from the mean by more than one standard deviation.

4.6% of the values, or 1 in 22, are expected to differ from the mean by more than 2σ.

0.27% of the values, or 1 in 370, are expected to differ from the mean by more than 3σ.

Other useful conclusions that follow from Eq. (4.65) are:

95% of the values lie in the region $\mu \pm 1.96\sigma$.

Only 1% of the values, or 1 in 100, are expected to differ from the mean by more than 2.58σ.

Only 0.1% of the values, or 1 in 1000, are expected to differ from the mean by more than 3.29σ.

In Fig. 4.8 the arrows show the regions, about 0, which contain 50, 68, 90, 95 and 99% of the values. Also shown are the regions of $x - \mu$ above which lie 1, 2.3, 5, 10, 16 and 25% of the values.

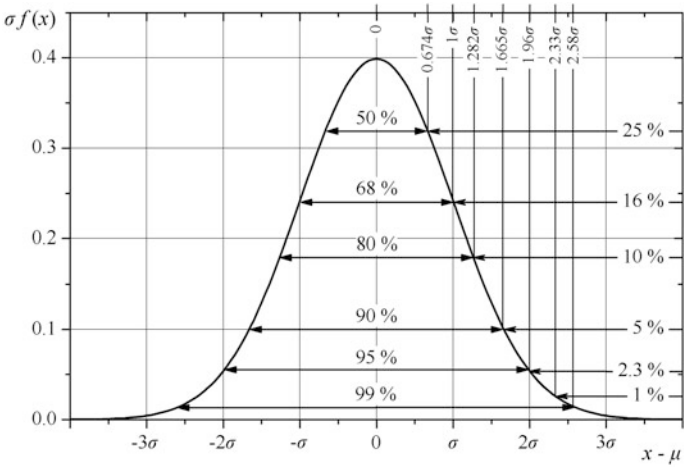


Fig. 4.8 The regions, about 0, which contain 50, 68, 90, 95 and 99% of the values, in the case of the Gaussian distribution. Also shown are the values of $x - \mu$ above which lie 1, 2.3, 5, 10, 16 and 25% of the values

These values are useful when we have to decide if a result that differs from the mean by a large difference should be accepted or rejected, since the difference is very improbable to be due to random errors. For example, since a measurement has a probability of 0.0027 to differ by more than 3σ from the mean, in 10 measurements, say, we expect $0.0027 \times 10 = 0.027$ measurements to differ by more than 3σ from the mean. If we actually have one such value, we might think that the probability of something like that happening as a result of random errors is too low and therefore we should exclude that particular value from the analysis of our measurements.

Having in mind the statistical estimates we mentioned above, we conventionally consider σ as an indicative value for the deviations of the measurements from the mean and, as a consequence, from the real value also. For this reason, σ is also called *standard deviation* (or *standard error* or simply *error*) *of a single measurement*. Having determined the value of s_x for a series of measurements of the quantity x and considering $\hat{\sigma}$, as this is derived from Eq. (4.39), as the best estimate we have for σ, we may say that the expected standard error of a single measurement of the quantity x that we may make, is equal to σ.

The statement 'the standard deviation of x is s_x' means that 68% of the results of the measurements of x are expected to lie in the region between $\overline{x} - s_x$ and $\overline{x} + s_x$.

Example 4.20 [E]

Given a Gaussian distribution with $\mu = 2$ and $\sigma = 1$, find the probability of a value between $x_1 = 3$ and $x_2 = 4$.

We follow the path:

Formulas > More Functions > Statistical

Select **NORM.DIST**. In the dialog box that opens, set

X = 4, Mean = 2, Standard dev = 1 and **Cumulative = TRUE**

Pressing **OK** returns the probability of x being smaller than 4, $\Pr\{x<4\} = 0.977250$.

Setting

X = 3, Mean = 2, Standard dev = 1 and **Cumulative = TRUE**

and pressing **OK** returns the probability of x being smaller than 3, $\Pr\{x<3\} = 0.841345$.

Taking the difference of the two probabilities, we have the probability of x having a value between $x_1 = 3$ and $x_2 = 4$ as being equal to $\Pr\{3<x<4\} = \Pr\{x<4\} - \Pr\{x<3\} = 0.977250 - 0.841345 = 0.135905$.

Example 4.21 [O]

Given a Gaussian distribution with $\mu = 2$ and $\sigma = 1$, find the probability of a value between $x_1 = 3$ and $x_2 = 4$.

The probability density distribution is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}$.

Here, it is $\frac{1}{\sqrt{2\pi}\sigma} = 0.39894$ and $\frac{1}{2\sigma^2} = 0.5$, so $f(x) = 0.39894\ \mathrm{e}^{-0.5(x-2)^2}$.

We will perform numerical integration of this function between $x_1 = 3$ and $x_2 = 4$.

We first fill the first 1000 cells of column A with values between 3.0005 and 3.9995, increasing in steps of $\delta x = 0.001$. This is done as follows:

We select column A by left-clicking on label A. Then, **Column > Set Column Values**.

Set col(A) equal to $3.0005 + (i-1) * 0.001$, for $i = 1$ to $i = 1000$.

We select column B by left-clicking on label B. Then, **Column > Set Column Values**.

Set col(B) equal to $0.39894 * \exp(-0.5 * (\mathrm{col(A)} - 2)\hat{}2)$, for $i = 1$ to $i = 1000$.

We add all the values in column B, using **Σ**. The result is 135.9043. Multiplying by $\delta x = 0.001$, we find the area under the curve between $x_1 = 3$ and $x_2 = 4$. This gives the value of 0.1359 as the probability of a value between $x_1 = 3$ and $x_2 = 4$.

It is not clear whether the accuracy with which the probability is given is justified. This can be checked by performing the numerical integration with a smaller δx, say $\delta x = 0.0005$. If the result is the same with 4 significant figures, then the result can be assumed to be accurate with the given significant figures.

Example 4.22 [P]

Given a Gaussian distribution with $\mu = 2$ and $\sigma = 1$, find the probability of a value between $x_1 = 3$ and $x_2 = 4$.

```
# http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.html
from scipy.stats import norm

# Enter the values of the distribution's parameters:

mean = 2            # the mean
stdev = 1           # and standard deviation of the distribution

# Enter the values of the limits of x:
x1 = 3
x2 = 4

# Evaluation:
# The probability of a value of x between x1 and x2:
print ("The probability of a value of x between 3 and 4 is =", norm.cdf(x2, mean, stdev) - norm.cdf(x1, mean, stdev))

# Result:
The probability of a value of x between 3 and 4 is = 0.135905121983
```

Example 4.23 [R]

Given a Gaussian distribution with $\mu = 2$ and $\sigma = 1$, find the probability of a value between $x_1 = 3$ and $x_2 = 4$.

The function **pnorm(q, μ, σ)** gives the probability that a value of x is smaller than q, $P\{x<q\}$. The probability of a value between $x_1 = 3$ and $x_2 = 4$ is $P\{x_1<x<x_2\} = P\{x<x_2\} - P\{x<x_1\}$:

```
> P = pnorm(4,2,1)-pnorm(3,2,1)
> P
[1] 0.1359051
```

We have found that $\Pr\{3<x<4\} = 0.135905$.

Example 4.24

Using Table 4.2 verify the result of Examples 4.20–4.23.

It is given that $\mu = 2$, $\sigma = 1$, $x_1 = 3$ and $x_2 = 4$. From Table 4.2, for $x = x_1 - \mu = 1$ we find that $\Phi(1) = 0.341\,345$ and for $x = x_2 - \mu = 2$ it is $\Phi(2) = 0.477\,250$. The difference gives the probability of a value between 3 and 4 as 0.135905. The result of Examples 4.20 and 4.21 is 0.1359.

Example 4.25 [E]

Given the Gaussian probability distribution function $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, find the value x_0 of x for which there is a probability $\Pr\{x \leq x_0\} = 0.9$ that it is $x \leq x_0$.

We follow the path:

Formulas > More Functions > Statistical

And select the function **NORM.INV**. In the dialog box that opens, we set:
Probability = 0.9, Mean = 0 and Standard dev = 1
Pressing OK returns the required value as being $x_0 = 1.281552$.
If the standard deviation of the distribution is σ and not 1, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$ we multiply x_0 by σ. If, for example, it is $\sigma = 2$, we multiply the value of x_0 by 2, obtaining $x_0' = 2.563103$. If also the mean is not 0 but it is μ, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$, we add to x_0' the value of μ. If, say it is $\mu = 3$, we obtain $x_0'' = 5.563103$.

Example 4.26 [O]

Given the Gaussian probability distribution function $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, find the value x_0 of x for which there is a probability $\Pr\{x \leq x_0\} = 0.9$ that it is $x \leq x_0$.

We highlight any empty cell, in which we want the result to be written, by left-clicking on it. Then,

Column > Set column values > Functions > Distributions > INV > norminv (p)

Substituting p = 0.9 and pressing **OK**, we get for the required value $x_0 = 1.28155$.

If the standard deviation of the distribution is σ and not 1, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}$ we multiply x_0 by σ. If, for example, it is $\sigma = 2$, we multiply the value of x_0 by 2, obtaining $x_0' = 2.56310$. If also the mean is not 0 but it is μ, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$, we add to x_0' the value of μ. If, say it is $\mu = 3$, we obtain $x_0'' = 5.56310$.

Example 4.27 [P]

Given the Gaussian probability distribution function $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, find the value x_0 of x for which there is a probability $\Pr\{x_0 \leq x\} = 0.9$ that it is $x_0 \leq x$.

```
# http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.norm.
html
from scipy.stats import norm
# Enter the following of the distribution's parameters:

mean = 0            # the mean
stdev = 1           # and standard deviation of the distribution

# Enter the value of the cumulative probability, p:
p = 0.9

# Evaluation:
# The value of x at which the cumulative probability is equal to p is:
x0 = norm.ppf(p, mean, stdev)

# Result:
print ("The value of x for which the cumulative probability is p, is x0 =", x0)

The value of x for which the cumulative probability is p, is x0 =
1.28155156554
```

Example 4.28 [R]

Given the Gaussian probability distribution function $G_{0,1}(x) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}$, find the value x_0 of x for which there is a probability $\Pr\{x \le x_0\} = 0.9$ that it is $x \le x_0$.

The function **pnorm(*q*, μ, *σ*)** gives the value x_0 of x for which there is a probability $\Pr\{x_0 \le x\} = q$ that it is $x_0 \le x$. Here, $\mu = 1$, $\sigma = 1$ and $q = 0.9$. Therefore,

```
> qnorm(0.9, 0, 1)
[1] 1.281552
```

and $x_0 = 1.28155$.

If the standard deviation of the distribution is σ and not 1, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-x^2/2\sigma^2}$ we multiply x_0 by σ. If, for example, it is $\sigma = 2$, we multiply the value of x_0 by 2, obtaining $x_0' = 2.56310$. If also the mean is not 0 but it is μ, i.e. it is $f(x) = \frac{1}{\sqrt{2\pi}\sigma}\,\mathrm{e}^{-(x-\mu)^2/2\sigma^2}$, we add to x_0' the value of μ. If, say it is $\mu = 3$, we obtain $x_0'' = 5.56310$.

4.4.3 Statistical Estimates for the Mean

If we perform M series of measurements, each with N measurements, we will have M pairs of values for the mean $\bar{x}$ of the N measurements and its standard deviation, $\sigma_{\bar{x}}$. It is proved that:

> *If samples consisting of N values are taken from a parent population whose mean is μ and its standard deviation σ, and if N is large, the distribution of the means of the samples tends to a normal (Gaussian) distribution, independently of the form of the distribution of the parent population.*

This is known as the *central limit theorem*. We will discuss this theorem in Chap. 9.

We have found that the relationship between the standard deviation of the mean, σ_μ, and the standard deviation of all the possible measurements of the parent population, σ, is

$$\sigma_\mu = \frac{\sigma}{\sqrt{N}}.$$

We therefore conclude, making use of the central limit theorem, that the mean values are normally distributed about the value of μ, with a standard deviation σ_μ. The distribution function of the means $\bar{x}$ is, therefore,

$$f(\bar{x}) = \frac{1}{\sqrt{2\pi}\,\sigma_\mu} \mathrm{e}^{-(\bar{x}-\mu)^2/2\sigma_\mu^2}. \tag{4.66}$$

Of course, μ and σ_μ are unknown. The best estimates we have for them, having performed only one series of N measurements, is the mean $\bar{x}$ and the standard deviation of the mean, $\sigma_{\bar{x}}$, of these N measurements. We know that for 'series' of measurements consisting of only one measurement each, σ_μ is equal to the standard deviation of the parent population, σ, while, according to the relation $\sigma_\mu = \sigma/\sqrt{N}$, as N increases, σ_μ tends to zero, as demonstrated in Fig. 4.9.

Figure 4.9 shows the distributions of the means, $\bar{x}$, of series of measurements consisting of 1, 5 and 25 measurements each. The standard deviation of the parent population of the measurements is σ. For $N = 1$ measurement, the means are the values x_i themselves and, therefore, their standard deviation will be that of the parent population, σ. For $N = 5$ measurements, the dispersion of the means about the real value μ is small and their standard deviation is $\sigma_\mu = \sigma/\sqrt{5} = 0.447\,\sigma$. Increasing the number of measurements to $N = 25$ has the result that the dispersion is further reduced and the standard deviation is reduced to $\sigma_\mu = \sigma/\sqrt{25} = 0.2\,\sigma$. The advantage achieved by increasing the number of measurements for a better determination of the unknown quantity μ is obvious. The larger the number of measurements used in the evaluation of the mean, the more probable it is that the mean is near the real value μ. The standard deviation of the mean, σ_μ, is a measure of the order of magnitude of the error present in the determination of μ using the N

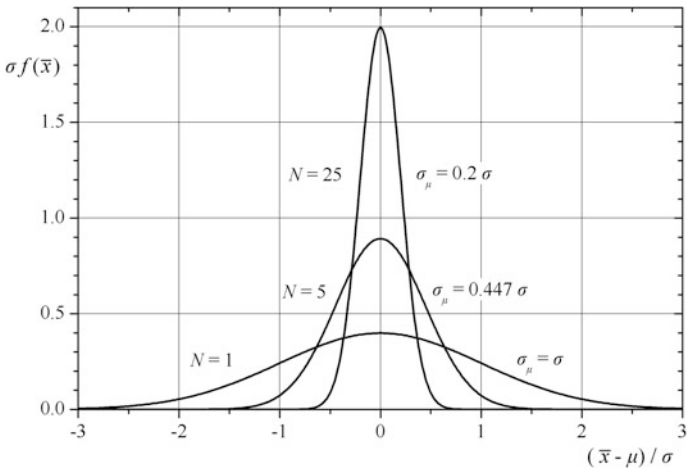


Fig. 4.9 Distributions of the means $\bar{x}$ derived from series of measurements of x consisting of $N = 1$, 5 and 25 measurements each. The real value of x is $x_0 = \mu$ and the standard deviation of the parent population of all the possible measurements x_i is equal to σ

measurements performed. The best estimate we have for the value of σ_μ is $\sigma_{\bar{x}}$. We can therefore consider the quantity

$$\delta x = \sigma_{\bar{x}} \tag{4.67}$$

as a measure of the error we have when we consider $\bar{x}$ to be an estimate of the real value $x_0 = \mu$ of the quantity measured.

It should finally be noted that, given that the distribution of the means is Gaussian, the statistical estimates valid for normally distributed variables, as these were presented in Sect. 4.4.2, are also true for the mean $\bar{x}$. The statement $x = \bar{x} \pm \sigma_{\bar{x}}$ means that there is a 68% probability that $\bar{x}$ differs from the real value x_0 of x by less than $\sigma_{\bar{x}}$.

4.4.4 *Summary of the Method of Analysis of the Results*

With reference to Fig. 4.10, we will now summarize the whole process of the statistical treatment of the results of the measurements of the quantity x.

(a) We suppose that the quantity x has a clearly defined real value x_0, which remains constant at least for the time needed for the measurements to be made. In our example, we will assume that it is $x_0 = 100$ mm exactly.

(b) The experimental procedure we will follow, the instruments we will use and the sources of noise determine the distribution of all the possible measurements that can be made, known as the parent population. This has a probability density $f(x)$. There are reasons for us to believe that this is Gaussian (normal). In our numerical example we take the standard deviation of the Gaussian to be $\sigma = 1$ mm.

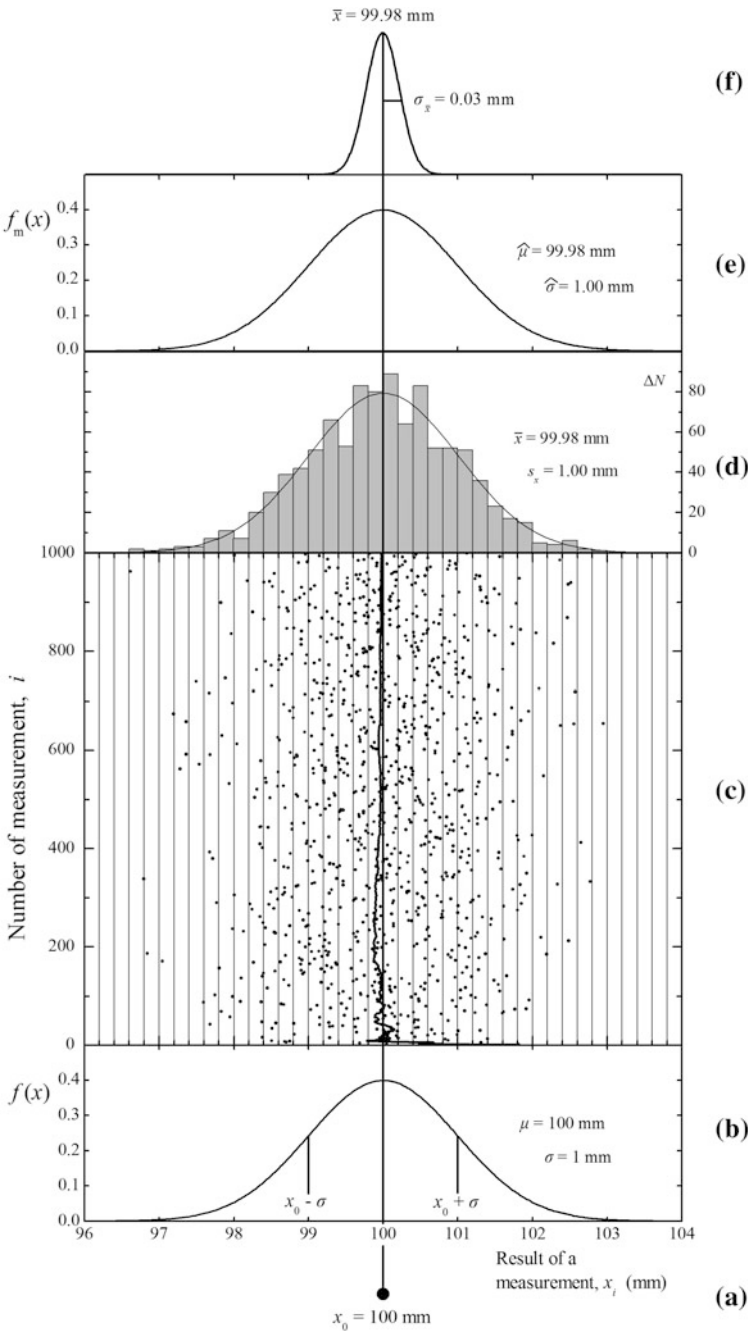


Fig. 4.10 Summary of the method of analysis of the results

(c) We perform, successively and under identical experimental conditions, a number N of measurements x_i of the quantity x. The results of the measurements have random deviations from the real value, whose statistical behavior is described by the function $f(x)$. The points in the figure show the results of these measurements in the order in which they were performed, while the thick line shows the variation of the mean $\overline{x}$ of the first N measurement, as the number of measurements increases from 1 to 1000. The asymptotic approach of $\overline{x}$ to x_0 is evident, with the random deviations being of the expected order of magnitude. We should point out that, almost always, the number of measurements performed is much smaller than used here. We use a large number of measurements, however, in order to have a sample large enough for its statistical properties to be clearly visible.

(d) The histogram of the measurements shows the grouping of the results around the real value and their dispersion according to the standard deviation of the parent population. We evaluate the mean $\overline{x}$ and the standard deviation, s_x, of the results. For our numerical result we find $\overline{x} = 99.98\,\text{mm}$ and $s_x = 1.00\,\text{mm}$. The continuous curve shows the best Gaussian distribution that may be fitted to the histogram (using a method we will describe in a later chapter).

(e) The Gaussian distribution that results from the histogram of the measurements, $f(x)_{\text{m}}$, has mean and standard deviation the estimates $\hat{\mu} = \overline{x} = 99.98\,\text{mm}$ and $\hat{\sigma} = \sqrt{\frac{N}{N-1}}\, s_x = 1.00\,\text{mm}$ we have for these quantities.

(f) The results of our measurements have a mean $\overline{x} = 99.98\,\text{mm}$ and a standard deviation of the mean $\sigma_{\overline{x}} = s_x/\sqrt{N-1} = 0.03\,\text{mm}$. The final result is given, therefore, as

$$x = 99.98 \pm 0.03\,\text{mm}\,(1000\,\text{measurements}).$$

We note that, in this particular example, the error in the mean,

$$e_\mu \equiv \overline{x} - x_0 = 99.98 - 100.00 = -0.02\,\text{mm},$$

happens to be smaller than one standard deviation of the mean.

The result $x = 99.98 \pm 0.03\,\text{mm}$ defines a Gaussian curve with mean $\overline{x} = 99.98\,\text{mm}$ and standard deviation $\sigma_{\overline{x}} = 0.03\,\text{mm}$, which gives the probability the real value x_0 of $\mathbf{x}$ to be in a certain range of values.

The reader may perhaps have reservations as to whether the example used, with its 1000 measurements, is realistic. Obviously, we used a large number of measurements in order to demonstrate their statistical behavior. The same analysis is used for smaller numbers of measurements, say 5–10, but in those cases the results must be considered to be less accurate.

Programs
Excel
Ch. 04. Excel—Mean and Standard Deviations
Origin
Ch. 04. Origin—Mean and Standard Deviations
Python
R
Ch. 04. R—Mean and Standard Deviations

Problems

4.1 **[E.O.P.R.]** Find the sample standard deviation of the values: 1.6, 1.4, 1.0, 2.4, 1.2, 2.0.

4.2 **[E.O.P.R.]** Find the mean, the standard deviation and the standard deviation of the mean of the measurements:

10 11 12 13 14 15 16 17 18 19.

4.3 Show that, for a number of measurements equal to 2, the standard deviation of their mean is equal to half the difference of the two measurements.

4.4 If an amount a, exactly, is added to the results x_i of a series of measurements, what will the change be in (a) the mean and (b) the standard deviation of the values?

4.5 If the results x_i of a series of measurements are multiplied by an exact factor of a, what will the change be in (a) the mean and (b) the standard deviation of the values?

4.6 **[E.O.P.R.]** Ten successive measurements of the period of a pendulum gave the following results:

i	1	2	3	4	5	6	7	8	9	10
T_i (s)	2.16	1.85	2.06	1.93	2.03	1.98	2.02	1.97	2.06	1.95

Calculate: (a) the mean value of T, (b) the standard deviation of the measurements and (c) the standard deviation of their mean.

4.7 Find the mean, the standard deviation and the standard deviation of the mean of the 30 measurements x_r of the table below, if their frequencies are n_r:

r	1	2	3	4	5	6	7	8	9	10
x_r	10	11	12	13	14	15	16	17	18	19
n_r	1	2	3	4	5	5	4	3	2	1

What will the values of these quantities be if (a) the values x_r are doubled and (b) the frequencies n_r are doubled?

4.8 A series of 51 measurements of x gave the following results x_r, with the frequencies n_r given:

r	1	2	3	4	5	6	7	8
x_r (cm)	125	126	127	128	129	130	131	132
n_r	2	3	9	16	11	7	2	1

Find the mean $\overline{x}$ and the standard deviation s_x of the measurements and the standard deviation $\sigma_{\overline{x}}$ of their mean.

4.9 **[E.O.P.R.]** For the values

$$x_i: \quad 10.3 \quad 10.1 \quad 10.5 \quad 10.4 \quad 10.7 \quad 10.4 \quad 10.2 \quad 10.5 \quad 10.3$$

find the mean value, $\overline{x}$, their standard deviation, s_x, the standard deviation of the mean, $\sigma_{\overline{x}}$, as well as the standard deviations of these two standard deviations.

What are the best estimates for the mean μ and the standard deviation σ of the parent population from which the sample was taken?

4.10 If the values of a magnitude **x**, as they are obtained by N measurements, have a standard deviation (of a single observation) $\sigma_x = 0.05$, for which values of N would the standard deviation of the mean be equal to (a) 0.03, (b) 0.02, (c) 0.005?

4.11 If we wish to have an estimate of the standard deviation of the parent population with an uncertainty of less than 5%, how many measurements must our series of measurements consist of?

4.12 **[E.O.P.R.]** If the random variable **x** is normally distributed with $\overline{x} = 10.0$ and $\sigma_x = 0.5$, what is the probability for the observation of a value (a) $x \leq 9$, (b) $9 \leq x \leq 11$, (c) $x \geq 11$, (d) $x \geq 11.5$?

4.13 From measurements performed on a large number of electric light bulbs produced by a certain factory, it was found that the durations of their lives had a mean of τ = 1200 h and a standard deviation of σ = 200 h.

What is the probability that a bulb from this factory will operate, before it fails, for less than (a) 800, (b) 1000, (c) 1200, (d) 1500 and (e) 1800 h?

(f) What is the probability that a bulb will fail after it has operated for a time between 800 and 1200 h?

4.14 For the measurement of the power consumed by electric light bulbs, a voltmeter was used which has a standard deviation of 0.20 V in its measurements and an ammeter with a standard deviation of 0.015 A in its measurements. What is the % standard deviation of a measurement of the power with these instruments in each of the following cases?

(a) A 500 W bulb operating at 115 V.
(b) A 60 W bulb operating at 115 V.
(c) A 60 W bulb operating at 32 V.
(d) A 60 W bulb operating at 8 V.

Note: The power consumed by a bulb is $P = IV$ where I is the current through the bulb and V is the potential difference across it.

4.15 What is the % accuracy in the determination of the density of a steel sphere which has a mass of 10 g and a density of about 7.85 g/cm^3, if the standard deviation in the measurement of its radius is 0.015 mm and in the measurement of its mass 0.05 mg?

4.16 **[E.O.P.R.]** A large number of measurements of the thermal conductivity of copper at the temperature of °C, have a Gaussian distribution with mean $k = 385$ W/(m·°C) and standard deviation σ = 15 W/(m·°C). What is the probability that a measurement lies between: (a) 370 and 400, (b) 355 and 415, and (c) 340 and 430 W/(m·°C)?

References

1. M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (*with Formulas, Graphs and Mathematical Tables*), (Dover Publications, Inc., New York, 1965). Ch. 7
2. Op. cit., Ch. 26, for related functions

Chapter 5
The Presentation of Numerical Results

We will now discuss the way in which numerical results should be presented. First, however, we will say a few things about significant figures and the rounding of numbers. In this chapter, the system of units known as S.I. is also presented, as well as the basic rules that must be followed in its use for the presentation of experimental results.

5.1 Significant Figures and Rounding of Numbers

Significant figures of a number are all its digits, except any consecutive leading zeros if the number is written in decimal form.

The position of the decimal point or a multiplying factor of a power of ten do not affect the number of significant figures. Some examples are given in Table 5.1. It should be noted that a zero at the end of the number counts as a significant figure. So, for example, the number 2.70 has three significant figures, while 2.7 has only two. The number of significant figures of integers may be designated if they are written in the form: (decimal) $\times$ (power of 10). For example, the number 12 300, which has 5 significant figures (s.f.), may be written with 4 s.f. as 1.230×10^4, with 3 s.f. as 1.23×10^4, with 2 s.f. as 1.2×10^4 and with 1 s.f. as 1×10^4. A slight problem arises with two-digit integers ending in 0. For example, it does not look natural for 40 to be written as 4×10. In these cases, the number may be written with two digits (40) and be stated that only one figure is significant, in the sense that the zero may not necessarily be zero but it could be 1, 2, 3 or 4.

In order to decrease the number of significant figures of a decimal number we use *rounding*. Rounding is performed as follows:

C. Christodoulides and G. Christodoulides, *Analysis and Presentation of Experimental Results*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-53345-2_5

Table 5.1 Examples of numbers with various numbers of significant figures

1 significant figure	2 significant figures	3 significant figures	4 significant figures
3	31	305	3050
0.3×10^2	0.31×10^2	3.05×10^2	3.050×10^3
0.2	0.23	0.234	0.2336
0.06	0.058	0.0582	0.05815
0.002	0.0016	0.00156	0.001558
7×10^{-4}	7.3×10^{-4}	7.30×10^{-4}	7.300×10^{-4}
6×10^6	6.1×10^6	6.06×10^6	6.063×10^6

If the least significant figure (the one on the right) is 0, 1, 2, 3 or 4, it is simply omitted. If it is 5, 6, 7, 8 or 9, the digit is omitted and the next digit on the left is increased by unity.

Examples: $1.42 \approx 1.4 \approx 1$, $3.6285 \approx 3.629 \approx 3.63 \approx 3.6 \approx 4$.

To round an integer, we first convert it into the form (decimal) $\times$ (power of 10) and then we round the decimal.

Example: $3506 = 3.506 \times 10^3 \approx 3.51 \times 10^3 \approx 3.5 \times 10^3 \approx 4 \times 10^3$.

If we round but keep the zeros, e.g. $3506 \approx 3510 \approx 3500 \approx 4000$, we give the wrong impression that the number of significant figures remains equal to 4.

5.2 The Presentation of a Numerical Result of a Series of Measurements

If we have performed N measurements x_i $(i = 1, 2, \ldots, N)$ of the magnitude x, the results of which have a mean $\overline{x}$ and a standard deviation of the mean $\sigma_{\overline{x}}$, the result of the measurements is usually presented in the form:

$$x = \overline{x} \pm \sigma_{\overline{x}} \text{ units.} \tag{5.1}$$

Examples:
The length of the rod was measured to be equal to $l = 18.25 \pm 0.13$ cm.
The electrical resistance is equal to $R = 103.6 \pm 1.2\ \Omega$.
During the measurement, $\Delta N = 169 \pm 13$ *fissions of nuclei were counted.*

It should be noted that the units given refer to both the mean value and its standard deviation. Parentheses which would stress this fact are not necessary.

By this we mean that the result of the first example should not necessarily be written as $l = (18.25 \pm 0.13)$ cm, without, of course, this being forbidden.[1]

When a multiplying factor of a power of ten is present in the numerical result, this power must be the same for both $\overline{x}$ and $\sigma_{\overline{x}}$. The way of presenting the result in this case is:

$$D = (1.49 \pm 0.15) \times 10^6 \,\text{km}$$

The parentheses state that the multiplying factor applies to both quantities.

An alternative way of presenting the error in the mean, especially if a large number of significant figures is used, is

$$c = 2.997\ 924\ 59(12) \times 10^8 \,\text{m/s}.$$

By this notation, it is meant that the two digits in parentheses state the standard deviation of the mean and appear at the same position as the last two digits of the mean value, i.e. it is:

$$c = 2.997\ 924\ 59(12) \times 10^8 \text{ m/s} \equiv (2.997\ 924\ 59 \pm 0.000\ 000\ 12) \times 10^8 \text{ m/s}$$

In the examples given, we applied the basic rule:

The numerical values of the mean and of the standard deviation of the mean are given with the same number of decimal digits (or, with the same accuracy).

Therefore, it would be wrong to write

$$32.263 \pm 0.14 \text{ instead of the correct } 32.26 \pm 0.14,$$

$$\text{or } 4.63 \pm 0.1348 \text{ instead of the correct } 4.63 \pm 0.13.$$

It is of course understood that value and error are given in the same units.

It is recommended that the use of commas as delimiters which separate the digits in triads in multi-digit numbers be avoided. Instead, a small empty space should be used. For example, instead of $c = (2.997{,}924{,}59 \pm 0.000{,}000{,}12) \times 10^8 \text{m/s}$ we write $c = (2.997\,924\,59 \pm 0.000\,000\,12) \times 10^8 \text{m/s}$ and instead of 32,467.63 we write 32 467.63.

This is done in order to avoid the confusion between dots and the commas used as decimal points in non-English speaking countries. The space is not necessary if the integral part of the number consists of four digits. For example, we write 4632, without a space.

[1] In fact, according to the suggestions of ISO 31-0: 1992 (E), the form $l = (18.25 \pm 0.13)$ cm is the correct one. This, however, has not known any significant acceptance in scientific work, possibly because the proliferation of parentheses is judged to be unnecessary.

So far we have said nothing about the accuracy with which $\overline{x}$ and $\sigma_{\overline{x}}$ should be presented. We will examine this question in the next section. For the moment we state that:

For the presentation of numerical results, we first decide on the accuracy with which the standard deviation of the mean will be given (i.e. the error in the mean), $\sigma_{\overline{x}}$, and then give the mean $\overline{x}$ with the same accuracy (i.e. with the same number of decimal figures). Any rounding of the numbers which is necessary must be taken into account.

5.3 The Number of Significant Figures Used in the Presentation of Numerical Results

The question of the number of significant figures with which numerical results should be presented is not as simple as it appears. Initially, we present the problem with a numerical example:

If our measurements gave the numerical results

$$\overline{x} = 876.12345, \quad \sigma_{\overline{x}} = 1.2345 \tag{5.2}$$

in suitable units, with how many decimal figures should each one of the results be finally stated?

Given that in the example the standard deviation of the mean is approximately equal to unity, the last integral digit as well as the decimal part of the mean are uncertain. Obviously, there is no sense in giving the mean with so many decimals. The same is true for the standard deviation. We remind that, according to what was said in Sect. 4.3, the standard deviation of the standard deviation is equal to

$$\sigma(\sigma_{\overline{x}}) = \frac{\sigma_{\overline{x}}}{\sqrt{2(N-1)}}, \tag{5.3}$$

where N is the number of the measurements used in the evaluation of $\overline{x}$ and $\sigma_{\overline{x}}$. If, for example, for the results (5.2) it is $N = 10$, the standard deviation of the standard deviation is

$$\sigma(\sigma_{\overline{x}}) = \frac{\sigma_{\overline{x}}}{\sqrt{2(N-1)}} = 0.236 \times \sigma_{\overline{x}} = 0.236 \times 1.2345 = 0.291 \tag{5.4}$$

Thus, the uncertainty in the value of $\sigma_{\overline{x}}$ is approximately equal to 24%, and the decimal digits are uncertain.

We will present and discuss the various views on the subject before we decide which one we adopt in this book:

(A) *The standard deviation of the mean is always given with one significant figure and the mean with the same accuracy.*

Thus, the numerical results (5.2) are presented as $x = 876 \pm 1$. Other examples are:

$$56 \pm 2 \quad 0.73 \pm 0.05 \quad 0.0069 \pm 0.0007 \quad (4.2 \pm 0.3) \times 10^{-5}$$
$$(8.0 \pm 0.1) \times 10^{6}.$$

The justification is the following: Given that, especially in an experiment in an educational laboratory, the number of measurements is rarely larger than 10 and often is about 5, the error in the standard deviation is equal to about 25% or even larger. The digits beyond the first are uncertain and are omitted (in our numerical example, the digits 0.2345 represent about 20% of $\sigma_{\overline{x}}$).

(B) *The standard deviation of the mean is given with one significant figure, unless its first two digits lie between 10 and 25, in which case it is given with two significant figures. The mean is given with the same accuracy.*

Thus, the numerical results (5.2) are presented as: $x = 876.1 \pm 1.2$.

Other examples are:

$$(4.2 \pm 0.3) \times 10^{-5} \quad (8.1 \pm 0.5) \times 10^{6} \quad 56 \pm 3 \quad 0.006\,97 \pm 0.000\,07,$$

$$(4.20 \pm 0.15) \times 10^{-5} \quad (8.05 \pm 0.23) \times 10^{6} \quad 56.4 \pm 1.2 \quad 0.006\,97 \pm 0.000\,17.$$

The justification is the following: If, for example, the standard deviation of the mean has 1 as its initial digit, a decimal part equal to 0.4 represents about 30% of the whole value and is larger than 25%, which is the percent error in the standard deviation for N between 5 and 10. For this reason, the second significant figure is not omitted. A similar argument holds for all cases where the first two digits lie between 10 and 25.

The general criterion to be used is whether, given the number of measurements used in extracting the numerical results, the standard deviation of the standard deviation of the mean justifies the presentation of the standard deviation with two significant figures or not.

(C) *The standard deviation of the mean is always given with two significant figures and the mean with the same accuracy.*

In an educational laboratory, the number of measurements used in the evaluation of the mean may not be large enough to justify the presentation of the standard deviation with two significant figures. In scientific research, however, things are different for two main reasons:

1. The number of measurements may be large enough
2. The result may be final for the researcher who performed the measurements but it must be considered as initial or intermediate result if it is to be used in calculations by other researchers.

With this last possibility in mind, the result must be given with enough accuracy so that no errors are introduced in the calculations of other magnitudes based on these results due to its rounding that has been performed. This practice of retaining one more significant figure than justified in intermediate results in order to evaluate the final result, should always be followed.

When, therefore, it is anticipated that the result of an experiment may be used for further calculations, the standard deviation of the mean should be given with two significant figures and the mean should be given with the same accuracy. Based on these facts, the well known mathematician Harold Jeffreys has commented: '*If one wants to render a series of good measurements useless, the best way in which one may achieve this is to give the value of the error with one significant figure and not mention the number of measurements*'.

We accept, therefore, that, in results of serious scientific research,

1. The standard deviation of the mean should be given with two significant figures,
2. The mean should be given with the same accuracy and
3. The number of measurements should be given.

Under these conditions, a numerical result must be presented in the form

$$x = \overline{x} \pm \sigma_{\overline{x}} \,\text{units}\,(N \,\text{measurements}) \tag{5.5}$$

as, for example, $x = 6.327 \pm 0.017\,\mu\text{m}$ (12 measurements).

Alternatively, instead of the number of measurements, the fractional standard deviation of the standard deviation $\sigma(\sigma_{\overline{x}})/\sigma_{\overline{x}}$ [see Eq. (4.43)] should be stated, based on the number of measurements performed. The numerical result is then presented in the form

$$x = \overline{x} \pm \sigma_{\overline{x}}(1 \pm \sigma(\sigma_{\overline{x}})/\sigma_{\overline{x}})\,\text{units} \tag{5.6}$$

as, for example, $x = 6.327 \pm 0.017(1 \pm 0.2)\,\mu\text{m}$.

Despite all this, the number of measurements is rarely given in scientific works, due mainly to the fact that, usually, the results are not obtained by repeating the same measurement many times under the same conditions but in other ways. Nevertheless, the standard deviation of the mean is given with two significant figures. Thus, one sees in tables the values of the fundamental physical constants to be given in the concise form [1]:

Newtonian constant of gravitation:	$G = 6.673\,84(80) \times 10^{-11}\,\mathrm{m^3 kg^{-1} s^{-2}}$
Planck constant:	$h = 6.626\,069\,59(27) \times 10^{-34}\,\mathrm{J\,s}$
Electron charge:	$e = -1.602\,176\,565(35) \times 10^{-19}\,\mathrm{C}$
Rydberg constant:	$R_\infty = 10\,973\,731.568\,539(55)\,\mathrm{m^{-1}}$

The values given here are those accepted since 2010. As new results for the constants become available, the values are modified to take into account both the old and the new values of the constants. For this purpose, it is necessary to have the standard deviations of the two values with adequate accuracy. Hence the justification for the two significant figures. By the way, included in the values given above are, on one hand, the constant known with the largest uncertainty, G, for which it is $\delta G/G = 1.2 \times 10^{-4}$ (δG is the error or the standard deviation of G) and, on the other hand, Rydberg's constant, R_∞, one of the constants known with the highest accuracy, $\delta R_\infty / R_\infty = 5.0 \times 10^{-12}$.

Conclusion

Having discussed the three possibilities for the number of significant figures in the presentation of $\sigma_{\overline{x}}$ and $\overline{x}$, in this book we will adopt the following:

1. The standard deviation of the mean, $\sigma_{\overline{x}}$, will be given with one significant figure, unless its two first digits lie between 10 and 25, included, in which cases it will be given with two significant figures.
2. The mean, $\overline{x}$, will be given with the same accuracy as $\sigma_{\overline{x}}$.
3. In cases when it is known that the number of measurements is large enough and in cases of reliable values of physical constants, we will give $\sigma_{\overline{x}}$ with two significant figures and the mean, $\overline{x}$, with the same accuracy.
4. In all cases, the intermediate results will be evaluated and presented with one more significant figure than justified in the final presentation of $\sigma_{\overline{x}}$.

5.4 The International System of Units (S.I.) and the Rules of Its Use

The International System of Units (S.I., *Système International d' Unités*) was adopted in 1960 by the 11th General Conference on Weights and Measures (CGPM, *Conférence Générale des Poids et Mesures*), under the auspices of The International Bureau of Weights and Measures (BIPM, *Bureau International des Poids et Mesures*). See http://physics.nist.gov/cuu/Units/index.html

It is based on seven *base units,* which are considered to be independent of each other as regards their dimensions. All the other units used are *derived units*, which

Table 5.2 The base units of S.I.

Quantity	Unit	
	Name	Symbol
Length	meter	m
Mass	kilogram	kg
Time	second	s
Electric current	ampere	A
Thermodynamic temperature	kelvin	K
Amount of substance	mole	mol
Luminous intensity	candela	cd

In the use of the unit mol, the kind of the entities being measured must be mentioned (atoms, molecules, ions, electrons, other particles or groups of such particles)

are formed by combinations of multiplications and divisions of powers of the base units, as these are dictated by the physical laws connecting the relative physical magnitudes. The base units are given in Table 5.2 while some derived units are given in Table 5.3. The names of the units are given in the tables. There are differences in the spelling of the names of the units in different languages (e.g. kilogramme and ampère in French, instead of the English kilogram and ampere), but the symbols of the units are the same in all languages.

Multiples and sub-multiples of the S.I. units are expressed with the use of prefixes. These represent powers of 10 in steps of the factor 10^3, with some smaller steps (of the factor) used near unity. These prefixes are given in Table 5.4. The prefixes *deci*, *deca* and *hecto* are not widely used.

We mention below the main rules that must be obeyed in the use of S.I. units.

1. The symbols for physical quantities are written in italics, while those of units in upright characters (e.g. $m = 1.3$ kg, $f = 1.034$ MHz, $V = 1.2$ V). An empty space is always left between the numerical value and the symbols of the units.
2. The symbols of units named after scientists, have their first letter in upper case (e.g. Pa, Bq, W, A, Hz, Wb, F, H, S). The names of the units, however, are written in lower case characters (e.g. joule, volt, siemens). The spelling and syntax of the names of units may vary from language to language. Only the use of the symbols is mandatory. For example, although the use of the symbol V is mandatory for the unit of volt, the use of the name *volt* is not.
3. The symbols of the units do not change in the plural. For example, we write 3 kg and not 3 kgs. However, the plurals of the names of the units are formed freely in each language, according to its rules. For example, in English we may write 5 kilograms or 3 volts. It should be noted that the word siemens is in the singular, despite the s at the end.

Table 5.3 The main derived units of S.I.

Quantity	Unit			
	Name	Symbol	Equivalent	
			In terms of other S.I. units	In terms of base S.I. units
Angle	radian	rad	$m/m = 1$	$m/m = 1$
Solid angle	steradian	sr	$m^2/m^2 = 1$	$m^2/m^2 = 1$
Velocity			m/s	m/s
Acceleration			m/s^2	m/s^2
Angular velocity			rad/s	rad/s
Angular acceleration			rad/s^2	rad/s^2
Frequency	hertz	Hz	s^{-1}	s^{-1}
Force, weight	newton	N	$kg \cdot m/s^2$	$kg \cdot m/s^2$
Pressure, stress	pascal	Pa	N/m^2	N/m^2
Energy, work, heat	joule	J	$N \cdot m$	$kg \cdot m^2/s^2$
Momentum, impulse			$N \cdot s$	$kg \cdot m/s$
Power, radiant flux	watt	W	J/s	$kg \cdot m^2/s^3$
Electric charge	coulomb	C	$A \cdot s$	$A \cdot s$
Electric potential, emf	volt	V	J/C, W/A	$kg \cdot m^2/(s^3 \cdot A)$
Electrical resistance	ohm	Ω	V/A	$kg \cdot m^2/(s^3 \cdot A^2)$
Electrical conductance	siemens	S	A/V, Ω^{-1}	$s^3 \cdot A^2/(kg \cdot m^2)$
Magnetic flux	weber	Wb	$V \cdot s$	$kg \cdot m^2/(s^2 \cdot A)$
Induction	henry	H	Wb/A	$kg \cdot m^2/(s^2 \cdot A^2)$
Capacity	farad	F	C/V	$s^4 \cdot A^2/(kg \cdot m^2)$
Strength of electric field			V/m, N/C	$kg \cdot m/(s^3 \cdot A)$
Magnetic flux density	tesla	T	Wb/m^2, $N/(A \cdot m)$	$kg/(s^2 \cdot A)$
Electric displacement			C/m^2	$A \cdot s/m^2$
Degree Celsius	degree Celsius	°C	K	K
Luminous flux	lumen	lm	$cd \cdot sr$	
Illuminance	lux	lx	lm/m^2	
Radioactivity (activity)	becquerel	Bq	s^{-1}	s^{-1}
Absorbed dose	gray	Gy	J/kg	m^2/s^2
Biologically equivalent dose	sievert	Sv	J/kg	m^2/s^2
Catalytic activity	katal	kat	mol/s	mol/s

4. The word ‘degree’ and the symbol ° are used for the Celsius scale but not for temperatures on the thermodynamic scale (Kelvin). Thus, we write 36 °C or 36 degrees Celsius, but 287 K (or, less often, 287 kelvin). For a temperature interval it is $1\,°C = 1$ K. The two scales are related through the exact equation $T(K) \equiv t(°C) + 273.15$.

Table 5.4 Prefixes used with S.I. units

Factor	Prefix	Symbol	Factor	Prefix	Symbol
10^{-1}	deci	d	10^{1}	deca	da
10^{-2}	centi	c	10^{2}	hecto	h
10^{-3}	milli	m	10^{3}	kilo	k
10^{-6}	micro	μ	10^{6}	mega	M
10^{-9}	nano	n	10^{9}	giga	G
10^{-12}	pico	p	10^{12}	tera	T
10^{-15}	femto	f	10^{15}	peta	P
10^{-18}	atto	a	10^{18}	exa	E
10^{-21}	zepto	z	10^{21}	zetta	Z
10^{-24}	yocto	y	10^{24}	yotta	Y

5. The prefixes for factors equal to or greater than 10^6 are written in capitals.[2] The rest are written in lower case letters. There is no empty space between the prefix and the symbol for the unit. Compound prefixes, such as mμ, μμ etc, should be avoided. An exponent on the symbol of the unit applies for the prefix as well: For example, $\mathrm{cm}^3 = (\mathrm{cm})^3 = (10^{-2} \times \mathrm{m})^3 = 10^{-6}\ \mathrm{m}^3$. When a multiple or sub-multiple of a unit is written in full, so should the prefix, in lower case letters: For example, megahertz, but not Mhertz or Megahertz.
6. The kilogram is, for historical reasons, the only base unit of the S.I. which is written with a prefix. Multiples and sub-multiples of kg are formed with the prefix to the symbol applied to g and not to kg, as, for example, μg, mg etc.
7. In multiplying units, a raised multiplication dot is placed between the two symbols. Alternatively, a small empty space is left between the symbols, as in $\mathrm{N} \cdot \mathrm{m}$, $\mathrm{N} \cdot \mathrm{s}$, $\mathrm{A} \cdot \mathrm{s}$ and $\mathrm{cd} \cdot \mathrm{sr}$ or N m, N s, A s and cd sr.
8. The division of units is denoted by forward slashes (e.g. m/s) or using negative exponents (e.g. $\mathrm{m\,s}^{-1}$). The multiple uses of the forward slash (m/s/s) is not allowed. When more than one units appear in the denominator, they should be enclosed in parentheses, as, for example, $\mathrm{N}/(\mathrm{A} \cdot \mathrm{m})$. In these cases, it is recommended that negative exponents are used, $\mathrm{NA}^{-1}\mathrm{m}^{-1}$. Units with prefixes can appear both in the numerator and the denominator as, for example, $\mathrm{m}\Omega/\mathrm{m}$, K/ms. Special care is needed to distinguish between the use of m as a symbol for the *meter* and its use as the prefix 10^{-3}. There is a difference between m N and mN. In cases like this the use of a multiplication dot helps in avoiding confusion $(\mathrm{m} \cdot \mathrm{N})$.

[2]It should be stressed that the prefix kilo- must be written as a lower case k. Symbols such as Kg and Km are wrong (they might be misunderstood to mean kelvin gram or kelvin meter, respectively).

Table 5.5 Units which do not belong to the S.I. but are used in practice, mainly for historical reasons

Quantity	Unit		
	Name	Symbol	Definition
Time	minute	min	1 min = 60 s
	hour	h	1 h = 60 min = 3600 s
	day	d	1 d = 24 h = 86 400 s
	year (Julian)	a (also y or yr)	365.25 d
Angle	degree	°	$1^\circ = (\pi/180)$ rad
	minute	′	$1' = (1/60)^\circ = (\pi/10\,800)$ rad
	second	″	$1'' = (1/60)' = (\pi/648\,000)$ rad
Volume	liter	L	1 L = 1 $dm^3 = 10^{-3}$ m^3
Mass	tonne, metric ton	t	1 t = 1000 kg
Ratio	neper	Np	(*)
	bel	B	(**)

(*) Two signals differ by N neper (Np), if it is $N = \ln \left|\frac{I}{I_0}\right|$ or $I = I_0\, e^N$

(**) It is 1 B = 10 dB (decibel). When referring to power, P, two signals differ by N db if it is $N = 10\ log\left|\frac{P}{P_0}\right|$ or $P = P_0 10^{N/10}$. When referring to field quantities, V, two signals differ by N db if it is $N = 20\ log\left|\frac{V}{V_0}\right|$ or $V = V_0 10^{N/20}$.

Table 5.6 Units which are acceptable for use in S.I and whose values are determined experimentally

Quantity	Unit		
	Name	Symbol	Value
Energy	electron volt	eV	$1.602\ 176\ 565(35) \times 10^{-19}$ J
Mass	unified atomic mass unit	u, Da*	$1.660\ 538\ 921(73) \times 10^{-27}$ kg

*Da = dalton

Also used, mainly for historical reasons, are a group of units which, strictly speaking, do not belong to the S.I. These units are shown in Table 5.5.

Given in Table 5.6 are some units which are accepted for use in the S.I. and which are determined experimentally.

Other units used in parallel to the S.I. units are:

astronomical unit 1 ua (also au or AU) = originally defined as the length of the semi-major axis of the Earth's orbit, is now defined as 149 597 870 700 m (exactly)

light year 1 ly = distance travelled by light in one Julian year $(365.25\ \text{d}) = 9.460\,730\,472\,580\,800 \times 10^{15}$ m (exactly)

parsec	1 pc = distance at which one astronomical unit subtends an angle of one arc second = $3.085\,677\,6 \times 10^{16}$ m = 3.2616 ly = 206 264.81 ua. (pc = **par**allax **sec**ond. A star having a parallax of 1″ is at a distance of 1 pc. For a parallax of 0.5″ the distance is 2 pc etc.)
nautical mile	=1852 m
knot	=1 nautical mile per hour
hectar	1 ha = 100 ar = 10^4 m^2
bar	1 bar = 1000 mbar = 10^5 N / m^2 = 10^5Pa
angstrom	1Å ≡ 10^{-8} cm = 10^{-10} m = 0.1 nm
barn[3]:	1 barn = 1 b ≡ 10^{-24} cm^2 = $10^{-28}m^z$

5.5 Recommendations on the Notation Used for Mathematical Constants, Algebraic Parameters, Variables, Indices, Mathematical Functions, Operators, Physical Units, Elementary Particles and Isotopes

The ISO recommendation for the way of writing constants, variables, operators, units, elementary particles and isotopes are as follows:

Magnitude	Examples
Mathematical constants	
should be written in upright letters	π, e, i, γ, ϕ
Algebraic parameters, variables	
should be written in italics	a, b, x, y
Indices	
numerical indices should be written in upright letters	a_0, x_3
indices taking numerical values should be written in italics	a_i, x_l
indices which are variables or functions should be written in italics	a_z, $y_{x=\bar{x}}$
non-numerical indices should be written in upright letters	f_{UV}, $x_{\mathrm{exper.}}$, $\iint_{\mathrm{ellipse}}$

(continued)

[3]Used as a unit of cross-section (area). Legend has it that in the early days of measuring the cross-sections of reactions of neutrons with nuclei, a value so large was once measured that somebody remarked 'this is as big as a barn!'. The unit for cross-section was thus christened. We take this opportunity to suggest that the barn be renamed the Rutherford (R or rd). The unit of radioactivity named the Rutherford (one million decays per second) is obsolete, as it has been replaced by the megabecquerel.

(continued)

Magnitude	Examples
Mathematical functions	
should be written in upright letters	$\sin x$, $\cos x$, e^x, $\exp(x)$, $\mathrm{erf}(x)$
special functions	$\Gamma(x)$, $\mathrm{J}_l(x)$, $\mathrm{P}_l(x)$
functions in the general sense should be written in italics	$f(x)$, $g(x)$, $V(t)$
Operators	
should be written in upright letters	δ, $\frac{\mathrm{d}}{\mathrm{d}}$, $\frac{\partial}{\partial}$, Δ, ∇, $\sum$ ($\frac{d}{d}$ frequent exceptions: δ and $\frac{d}{d}$)
Physical units and their prefixes	
should be written in upright letters	s, μs, m, km, V, eV, TeV, Ω
Elementary particles	
should be denoted by symbols in upright letters	α, β, γ, p, n, ν_e, π^0, K, τ
Isotopes	
should be denoted by upright letters	H, Au, $^{15}_{8}\mathrm{O}$, $^{12}_{6}\mathrm{C}^{2+}$, $\mathrm{H_2O}$

Problems

5.1 The following numbers are given:

$$0.0,017,624 \quad 8.14369 \quad 267980 \quad 1.27386 \times 10^5 \quad 8.13227 \times 10^{-8}.$$

Enter, in a table, all the numbers with 1, 2, 3 and 4 significant figures.

5.2 The following numbers are given:

$$4.7386 \times 10^4 \quad 5.13227 \times 10^{-8} \quad 0.00029764 \quad 3.14159 \quad 937980.$$

Enter, in a table, all the numbers with 1, 2, 3 and 4 significant figures.

5.3 Write the following results giving the standard deviation of the mean with 2 significant figures and the mean with the same accuracy:

$$523.5782 \pm 5.367 \quad 0.0078321 \pm 0.0000632 \quad 4.7301 \times 10^6 \pm 942105.$$

5.4 Write the following results correctly:

$$263.582 \pm 0.2467 \quad 0.003321 \pm 0.002572 \quad 4.6308 \times 10^3 \pm 1210.$$

5.5 Given that the results

$$823.6 \pm 5.4 \quad 0.007832 \pm 0.000043 \quad (4.73 \pm 0.44) \times 10^6$$

were obtained by performing 5, 10 and 25 measurements, respectively, express each one of them in the form $x = \overline{x} \pm \sigma_{\overline{x}}(1 \pm \sigma(\sigma_{\overline{x}})/\sigma_{\overline{x}})$.

5.6 Given the results

$$23.62 \pm 0.24 \times (1 \pm 0.35) \qquad 0.003321 \pm 0.0015 \times (1 \pm 0.24)$$
$$(4.6 \pm 1.2 \times (1 \pm 0.16)) \times 10^3,$$

find in each case the approximate number of measurements, N, used in the determination of the result.

Reference

1. *The NIST Reference on Constants, Units and Uncertainty* at http://physics.nist.gov/cuu/Constants/. For a comprehensive list of values, see Appendix 4

Chapter 6
The Propagation of Errors

Quite often, we have to calculate the value of a quantity that was not measured directly but is expressed in terms of one or more quantities which have been measured and whose means and standard deviations are known. The standard deviation of this quantity is evaluated taking into account the so-called *propagation of errors* from the quantities that have been measured to the derived quantity. It is our aim in this chapter to explain how this is done.

6.1 The Combination of Two Series of Measurements of the Same Physical Magnitude

We start, rather arbitrarily, with a topic that, strictly speaking, does not involve the propagation of errors but is related to it. It refers to the cases when we have two groups of measurements of the same quantity and wish to combine them in order to obtain a larger sample of results, from which a better estimate of the mean value of the measured quantity can be derived.

Let us assume that we have at our disposal two equally reliable series of measurements of the magnitude x, performed under identical experimental conditions:

The first, with N_α measurements $x_{\alpha,i}$ $(i = 1, 2, \ldots, N_\alpha)$, of which the mean is $\overline{x}_\alpha$, the standard deviation from the mean is s_α and the standard deviation of the mean is $\sigma_{\overline{x},\alpha}$ and
A second, with N_β measurements $x_{\beta,j}$ $(j = 1, 2, \ldots, N_\beta)$, of which the mean is $\overline{x}_\beta$, the standard deviation from the mean is s_β and the standard deviation of the mean is $\sigma_{\overline{x},\beta}$.

We wish to combine the two series of measurements into one, consisting of $N = N_\alpha + N_\beta$ measurements, of which we wish to find the mean $\overline{x}$, the standard deviation from the mean s_x and the standard deviation of the mean, $\sigma_{\overline{x}}$.

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6.1.1 The Mean $\overline{x}$ of All the Measurements

The mean of all the N measurements is

$$\overline{x} = \frac{1}{N_\alpha + N_\beta}(x_{\alpha,1} + x_{\alpha,2} + \ldots + x_{\alpha,N_\alpha} + x_{\beta,1} + x_{\beta,2} + \ldots + x_{\beta,N_\beta}) \tag{6.1}$$

$$\overline{x} = \frac{1}{N_\alpha + N_\beta}\left(N_\alpha \frac{x_{\alpha,1} + x_{\alpha,2} + \ldots + x_{\alpha,N_\alpha}}{N_\alpha} + N_\beta \frac{x_{\beta,1} + x_{\beta,2} + \ldots + x_{\beta,N_\beta}}{N_\beta}\right) \tag{6.2}$$

and, finally,

$$\overline{x} = \frac{N_\alpha \overline{x}_\alpha + N_\beta \overline{x}_\beta}{N_\alpha + N_\beta}. \tag{6.3}$$

6.1.2 The Standard Deviation s_x of All the Measurements

In order to find the standard deviation from the mean of the N measurements, we evaluate the sum

$$s_x^2 = \frac{1}{N_\alpha + N_\beta}\left(\sum_{i=1}^{N_\alpha}(x_{\alpha,i} - \overline{x})^2 + \sum_{j=1}^{N_\beta}(x_{\beta,j} - \overline{x})^2\right) \tag{6.4}$$

$$s_x^2 = \frac{1}{N_\alpha + N_\beta}\left(\sum_{i=1}^{N_\alpha}(x_{\alpha,i} - \overline{x}_\alpha + \overline{x}_\alpha - \overline{x})^2 + \sum_{j=1}^{N_\beta}(x_{\beta,j} - \overline{x}_\beta + \overline{x}_\beta - \overline{x})^2\right) \tag{6.5}$$

$$\begin{aligned} s_x^2 = \frac{1}{N_\alpha + N_\beta}\Bigg(&\sum_{i=1}^{N_\alpha}(x_{\alpha,i} - \overline{x}_\alpha)^2 + 2(\overline{x}_\alpha - \overline{x})\sum_{i=1}^{N_\alpha}(x_{\alpha,i} - \overline{x}_\alpha) + \sum_{i=1}^{N_\alpha}(\overline{x}_\alpha - \overline{x})^2 \\ &+ \sum_{j=1}^{N_\beta}(x_{\beta,j} - \overline{x}_\beta)^2 + 2(\overline{x}_\beta - \overline{x})\sum_{j=1}^{N_\beta}(x_{\beta,j} - \overline{x}_\beta) + \sum_{j=1}^{N_\beta}(\overline{x}_\beta - \overline{x})^2\Bigg) \end{aligned} \tag{6.6}$$

$$s_x^2 = \frac{1}{N_\alpha + N_\beta}\left(N_\alpha s_\alpha^2 + 0 + N_\alpha(\overline{x} - \overline{x}_\alpha)^2 + N_\beta s_\beta^2 + 0 + N_\beta(\overline{x} - \overline{x}_\beta)^2\right) \tag{6.7}$$

$$s_x^2 = \frac{N_\alpha s_\alpha^2 + N_\beta s_\beta^2}{N_\alpha + N_\beta} + \frac{1}{N_\alpha + N_\beta}\left(N_\alpha(\overline{x} - \overline{x}_\alpha)^2 + N_\beta(\overline{x} - \overline{x}_\beta)^2\right). \tag{6.8}$$

Since

$$\overline{x} - \overline{x}_\alpha = \frac{N_\alpha \overline{x}_\alpha + N_\beta \overline{x}_\beta}{N_\alpha + N_\beta} - \overline{x}_\alpha = \frac{N_\beta (\overline{x}_\beta - \overline{x}_\alpha)}{N_\alpha + N_\beta}, \tag{6.9}$$

it is

$$N_\alpha (\overline{x} - \overline{x}_\alpha)^2 = \frac{N_\alpha N_\beta^2}{(N_\alpha + N_\beta)^2} (\overline{x}_\beta - \overline{x}_\alpha)^2 \tag{6.10}$$

and, similarly,

$$N_\beta (\overline{x} - \overline{x}_\beta)^2 = \frac{N_\alpha^2 N_\beta}{(N_\alpha + N_\beta)^2} (\overline{x}_\beta - \overline{x}_\alpha)^2. \tag{6.11}$$

Therefore,

$$s_x^2 = \frac{N_\alpha s_\alpha^2 + N_\beta s_\beta^2}{N_\alpha + N_\beta} + \frac{N_\alpha N_\beta}{(N_\alpha + N_\beta)^2} (\overline{x}_\alpha - \overline{x}_\beta)^2. \tag{6.12}$$

Summarizing the results we have so far, the total of $N = N_\alpha + N_\beta$ measurements have mean and standard deviation from the mean

$$\overline{x} = \frac{N_\alpha \overline{x}_\alpha + N_\beta \overline{x}_\beta}{N_\alpha + N_\beta} \quad \text{and} \quad s_x = \sqrt{\frac{N_\alpha s_\alpha^2 + N_\beta s_\beta^2}{N_\alpha + N_\beta} + \frac{N_\alpha N_\beta}{(N_\alpha + N_\beta)^2} (\overline{x}_\alpha - \overline{x}_\beta)^2}. \tag{6.13}$$

Defining the ratios

$$\alpha \equiv \frac{N_\alpha}{N_\alpha + N_\beta} \quad \text{and} \quad \beta \equiv \frac{N_\beta}{N_\alpha + N_\beta}, \tag{6.14}$$

we have

$$\overline{x} = \alpha \overline{x}_\alpha + \beta \overline{x}_\beta \quad \text{and} \quad s_x = \sqrt{\alpha s_\alpha^2 + \beta s_\beta^2 + \alpha\beta (\overline{x}_\alpha - \overline{x}_\beta)^2}. \tag{6.15}$$

6.1.3 The Standard Deviation of the Mean $\sigma_{\overline{x}}$ of All the Measurements

The relations between the standard deviations from the means and the standard deviations of the means are:

$$\sigma_{\bar{x},\alpha} = \frac{s_\alpha}{\sqrt{N_\alpha - 1}}, \qquad \sigma_{\bar{x},\beta} = \frac{s_\beta}{\sqrt{N_\beta - 1}}, \qquad \sigma_{\bar{x}} = \frac{s_x}{\sqrt{N-1}} \tag{6.16}$$

Substituting in the second Eq. (6.13), we find

$$s_x = \sigma_{\bar{x}}\sqrt{N-1} = \sigma_{\bar{x}}\sqrt{N_\alpha + N_\beta - 1} = \sqrt{\frac{N_\alpha s_\alpha^2 + N_\beta s_\beta^2}{N_\alpha + N_\beta} + \frac{N_\alpha N_\beta}{(N_\alpha + N_\beta)^2}(\bar{x}_\alpha - \bar{x}_\beta)^2} \tag{6.17}$$

$$\sigma_{\bar{x}} = \sqrt{\frac{N_\alpha s_\alpha^2 + N_\beta s_\beta^2}{(N_\alpha + N_\beta)(N_\alpha + N_\beta - 1)} + \frac{N_\alpha N_\beta (\bar{x}_\alpha - \bar{x}_\beta)^2}{(N_\alpha + N_\beta)^2 (N_\alpha + N_\beta - 1)}} \tag{6.18}$$

or

$$\sigma_{\bar{x}} = \sqrt{\frac{1}{N-1}\left[\alpha s_\alpha^2 + \beta s_\beta^2 + \alpha\beta(\bar{x}_\alpha - \bar{x}_\beta)^2\right]}, \tag{6.19}$$

as would also follow directly from the second of Eq. (6.15).

In terms of $\sigma_{\bar{x},\alpha}$ and $\sigma_{\bar{x},\beta}$,

$$\sigma_{\bar{x}} = \sqrt{\frac{N_\alpha(N_\alpha - 1)}{N(N-1)}\sigma_{\bar{x},\alpha}^2 + \frac{N_\beta(N_\beta - 1)}{N(N-1)}\sigma_{\bar{x},\beta}^2 + \frac{N_\alpha N_\beta}{N^2(N-1)}(\bar{x}_\alpha - \bar{x}_\beta)^2} \tag{6.20}$$

or

$$\sigma_{\bar{x}} = \sqrt{\frac{1}{N-1}\left[\alpha(\alpha N - 1)\sigma_{\bar{x},\alpha}^2 + \beta(\beta N - 1)\sigma_{\bar{x},\beta}^2 + \alpha\beta(\bar{x}_\alpha - \bar{x}_\beta)^2\right]}. \tag{6.21}$$

Example 6.1

Let a series of $N_\alpha = 8$ measurements of the magnitude x have mean $\bar{x}_\alpha = 10.42$, standard deviation $s_\alpha = 0.24$ and standard deviation of the mean $\sigma_{\bar{x},\alpha} = s_\alpha/\sqrt{N_\alpha - 1} = 0.091$. Let also a second series of $N_\beta = 12$ measurements of the same magnitude have $\bar{x}_\beta = 10.65$, $s_\beta = 0.20$ and $\sigma_{\bar{x},\beta} = s_\beta/\sqrt{N_\beta - 1} = 0.060$.

Find: the mean $\bar{x}$, the standard deviation s_x and the standard deviation of the mean $\sigma_{\bar{x}}$ of the total of the $N = N_\alpha + N_\beta = 20$ measurements.

Here, it is $\alpha = \frac{N_\alpha}{N_\alpha + N_\beta} = \frac{8}{20} = 0.4$ and $\beta = \frac{N_\beta}{N_\alpha + N_\beta} = \frac{12}{20} = 0.6$.

Therefore, from Eq. (6.15) we find $\bar{x} = \alpha\bar{x}_\alpha + \beta\bar{x}_\beta = 0.4 \times 10.42 + 0.6 \times 10.65 = 10.56$ and $s_x = \sqrt{\alpha s_\alpha^2 + \beta s_\beta^2 + \alpha\beta(\bar{x}_\alpha - \bar{x}_\beta)^2} = \sqrt{0.4 \times (0.24)^2 + 0.6 \times (0.20)^2 + 0.4 \times 0.6 \times (10.42 - 10.65)^2}$ $= \sqrt{0.0230 + 0.0240 + 0.0127} = 0.244$ or $s_x = 0.24$.

The standard deviation of the mean is: $\sigma_{\overline{x}} = \frac{s_x}{\sqrt{N-1}} = \frac{0.244}{\sqrt{19}} = 0.056$ or $\sigma_{\overline{x}} = 0.06$.

The standard deviations of the mean were also given with two significant figures, in order to show the improvement of $\sigma_{\overline{x}} = 0.056$ compared to $\sigma_{\overline{x},\alpha} = 0.091$ and $\sigma_{\overline{x},\beta} = 0.060$.

6.2 The Mean and the Standard Deviation of a Function of Measured Quantities

Before we deal with the general problem, we will examine some simple cases.

6.2.1 The Mean and the Standard Deviations of a Function of One Variable

Let N measurements x_i of the variable x have a mean $\overline{x}$ and a standard deviation s_x and the magnitude z be a function $z(x)$ of x. Figure 6.1a shows the curve $z(x)$, a value x_i of x on the x-axis and the corresponding value of $z(x_i)$. In the region near $\overline{x}$, the curve $z(x)$ is approximated by the tangent to the curve at the point $\overline{x}$ (Fig. 6.1b). Using this linear relationship between x and z, we will show that the mean value $\overline{z(x)}$ of $z(x)$ is approximately equal to $z(\overline{x})$, i.e. the value of $z(x)$ which corresponds to the value $x = \overline{x}$.

Mean

For every measurement x_i $(i = 1, 2, \ldots, N)$, we evaluate the corresponding value $z_i = z(x_i)$ of z. The mean of these N values of z is

$$\overline{z} = \overline{z(x)} = \frac{1}{N}\sum_{i=1}^{N} z(x_i). \tag{6.22}$$

If we expand function $z(x)$ in a Taylor series in the region of $\overline{x}$, we have

$$z(x) = z(\overline{x}) + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{x=\overline{x}} (x - \overline{x}) + \frac{1}{2!}\left(\frac{\mathrm{d}^2 z}{\mathrm{d}x^2}\right)_{x=\overline{x}} (x - \overline{x})^2 + \ldots. \tag{6.23}$$

Assuming that the deviations from the mean are small enough so that terms involving the second or higher powers of $(x - \overline{x})$ are negligible, we have

$$z(x) \approx z(\overline{x}) + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{x=\overline{x}} (x - \overline{x}), \tag{6.24}$$

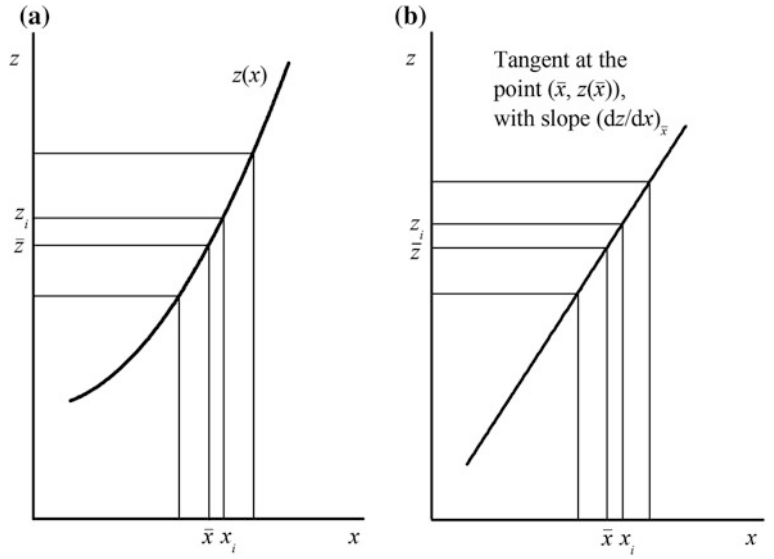

Fig. 6.1 **a** The relation $z(x)$ and the way of finding the point $z_i = z(x_i)$ which corresponds to a measurement x_i. **b** The approximate linear relationship between x and z in the region near $x = \bar{x}$

a relation which represents the straight line in Fig. 6.1b. So, for every value x_i of x, we have

$$z(x_i) \approx z(\bar{x}) + \left(\frac{dz}{dx}\right)_{x=\bar{x}} (x_i - \bar{x}). \tag{6.25}$$

Therefore,

$$\bar{z} = \frac{1}{N}\sum_{i=1}^{N} z(x_i) \approx \frac{1}{N}\sum_{i=1}^{N} z(\bar{x}) + \frac{1}{N}\sum_{i=1}^{N}\left(\frac{dz}{dx}\right)_{x=\bar{x}} (x_i - \bar{x}). \tag{6.26}$$

However,

$$\sum_{i=1}^{N} z(\bar{x}) = N z(\bar{x}) \tag{6.27}$$

and

$$\frac{1}{N}\sum_{i=1}^{N}\left(\frac{dz}{dx}\right)_{x=\bar{x}} (x_i - \bar{x}) = \frac{1}{N}\left(\frac{dz}{dx}\right)_{x=\bar{x}} \sum_{i=1}^{N} (x_i - \bar{x}) = \frac{1}{N}\left(\frac{dz}{dx}\right)_{x=\bar{x}} (N\bar{x} - N\bar{x}) = 0 \tag{6.28}$$

and so

$$\overline{z} \approx z(\overline{x}). \tag{6.29}$$

Therefore, the mean $\overline{z}$ of the values $z_i = z(x_i)$ is approximately equal to $z(\overline{x})$. In other words, the mean value of the z_i values corresponding to the x_i values is approximately equal to the value of z which corresponds to the mean $\overline{x}$ of the x_i. This important result will be generalized to functions of many variables in Sect. 6.2.3.

Example 6.2

Ten measurements of the radius r of a circle gave the following results:

10.0 9.9 9.8 10.3 9.7 10.2 10.1 10.4 9.9 9.7 mm.

Find the mean $\overline{r}$ and the standard deviation s_r of the measurements. If $S = \pi r^2$ is the area of the circle, check that the relation $\overline{S} = \pi \overline{r}^2$ is valid.

i	r_i (mm)	S_i (mm^2)
1	10.0	314.17
2	9.9	307.91
3	9.8	301.72
4	10.3	333.29
5	9.7	295.59
6	10.2	326.85
7	10.1	320.47
8	10.4	339.79
9	9.9	307.91
10	9.7	295.59
Σ =	100.0	3143.29

We find that $\overline{r} = \frac{100.0}{10} = 10.0$ mm.

From the values of S_i we find $\overline{S} = \frac{3143.29}{10} = 314.33$ mm^2.

From $\overline{S} = \pi \overline{r}^2$ we find $\overline{S} = \pi(10.0)^2 = 314.16$ mm^2.

The values of $\overline{S}$ found using the two methods differ by very little from each other. The numerical values were given with more accuracy than justified, in order to make possible the comparison of the two results.

Standard deviation

To evaluate the standard deviation from the mean of the values z_i, we note that, according to Eq. (6.25) and to a good approximation, it is

$$z_i - \overline{z} = z(x_i) - z(\overline{x}) = \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{x=\overline{x}} (x_i - \overline{x}) \tag{6.30}$$

and, therefore,

$$s_z^2 = \frac{1}{N}\sum_{i=1}^{N} (z_i - \overline{z})^2 = \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2_{x=\overline{x}} \frac{1}{N}\sum_{i=1}^{N} (x_i - \overline{x})^2 = \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2_{x=\overline{x}} s_x^2. \tag{6.31}$$

Thus, the standard deviation of the values $z_i = z(x_i)$ from their mean is

$$s_z = \left|\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{x=\overline{x}}\right| s_x. \tag{6.32}$$

The absolute value of the derivative is taken, since, by definition, the standard deviation is a positive magnitude.

Standard deviation of the mean

Knowing s_x and s_z, we may have the best estimates for the corresponding standard deviations, $\sigma_{\overline{x}}$ and $\sigma_{\overline{z}}$, of the means $\overline{x}$ and $\overline{z}$, respectively. From the general relation (4.40), it is $\sigma_{\overline{x}} = s_x/\sqrt{N-1}$ and $\sigma_{\overline{z}} = s_z/\sqrt{N-1}$. Thus, Eq. (6.32) gives

$$\sigma_{\overline{z}} = \left|\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{x=\overline{x}}\right| \sigma_{\overline{x}} \tag{6.33}$$

as the relationship between the standard deviations of the means.

Special cases:

Let the measurements of a quantity x have mean value $\overline{x}$ and standard deviation from the mean s_x.

1. If $z = kx$, where k is a constant, then $\overline{z} = k\overline{x}$ and, since $\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{x=\overline{x}} = k$, it is $s_z = |k| s_x$.
2. If $z = x^2$, then $\overline{z} = \overline{x}^2$ and, since $\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{x=\overline{x}} = 2\overline{x}$, it is $s_z = 2|\overline{x}| s_x$.
3. If $z = \sin(kx)$, where k is a constant, then $\overline{z} = \sin(k\overline{x})$ and, since $\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{x=\overline{x}} = k\cos(k\overline{x})$, it is $s_z = |k\cos(k\overline{x})| s_x$.

Figure 6.2 gives the geometrical interpretation of Eqs. (6.32) and (6.33). For Eq. (6.32), the pair of values $\overline{x}$ and s_x describe a distribution $f(x)$ of the measurements x_i about the mean $\overline{x}$, with standard deviation s_x. This distribution was drawn in the figure, on the x-axis. With the aid of the relation $z(x)$ the values of $z_i = z(x_i)$ are found, which correspond to the measurements x_i. If the distribution of the values x_i is narrow enough for the curve $z(x)$ to be considered to be linear in the region between the limits $\overline{x} \pm s_x$, then the distribution $f(z)$ of the z_i values, which is evaluated from

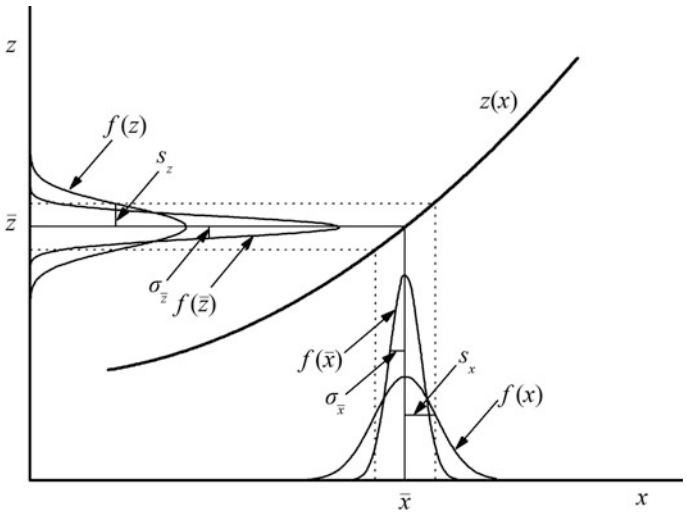

Fig. 6.2 The relationship between the distribution of the values x_i of the measurements of the magnitude x and the distribution of the corresponding values z_i of the derived magnitude $z(x)$. The distribution $f(x)$ of the measurements x_i, with a standard deviation s_x, is transformed into the distribution $f(z)$ of the values z_i, with standard deviation s_z. The distribution $f(\bar{x})$ of the means $\bar{x}$, with standard deviation $\sigma_{\bar{x}}$, is transformed into the distribution $f(\bar{z})$ of the values $\bar{z}$, with standard deviation $\sigma_{\bar{z}}$

$f(x)$ using the relation (6.24), has maximum at $\bar{z} = z(\bar{x})$ and standard deviation s_z, which is given by Eq. (6.32). This distribution was drawn on the z-axis. The same remarks apply to the distribution of the mean values. Equation (6.33) describes in the same way the transformation of $\bar{x}$ to $\bar{z}$. The distribution of the means $\bar{x}$, $f(\bar{x})$, which corresponds to the parameters $(\bar{x}, \sigma_{\bar{x}})$, is transformed to the distribution $f(\bar{z})$ of $\bar{z}$, with parameters $(\bar{z}, \sigma_{\bar{z}})$. The distributions $f(\bar{x})$ and $f(\bar{z})$ were also drawn in Fig. 6.2.

Example 6.3

Test the validity of the relation $s_z = \left|\left(\frac{dz}{dx}\right)_{x=\bar{x}}\right| s_x$ for the measurements of Example 6.2.

i	r_i (mm)	$r_i - \bar{r}$ (mm)	$(r_i - \bar{r})^2$ (mm^2)	S_i (mm^2)	$S_i - \bar{S}$ (mm^2)	$(S_i - \bar{S})^2$ (mm^4)
1	1.0	0.0	0.00	314.2	−0.1	0.0
2	9.9	−0.1	0.01	307.9	−6.4	41.0
3	9.8	−0.2	0.04	301.7	−12.6	158.8
4	10.3	0.3	0.09	333.3	19.0	361.0
5	9.7	−0.3	0.09	295.6	−18.7	349.7
6	10.2	0.2	0.04	326.9	12.6	158.8
7	10.1	0.1	0.01	320.5	6.2	38.4

(continued)

(continued)

i	r_i (mm)	$r_i - \overline{r}$ (mm)	$(r_i - \overline{r})^2$ (mm^2)	S_i (mm^2)	$S_i - \overline{S}$ (mm^2)	$(S_i - \overline{S})^2$ (mm^4)
8	10.4	0.4	0.16	339.8	25.5	650.3
9	9.9	−0.1	0.01	307.9	−6.4	41.0
10	9.7	−0.3	0.09	295.6	−18.7	349.7
Σ =	100.0		0.54			2148.7

From the values of r_i and the corresponding values of S_i we find, directly, that it is $s_r = \sqrt{\frac{0.54}{10}} = \sqrt{0.054} = 0.23$ mm and $s_S = \sqrt{\frac{2148.7}{10}} = \sqrt{214.87} = 14.66 = 14.7$ mm^2.

Since it is $s_S = \left|\left(\frac{dS}{dr}\right)_{r=\overline{r}}\right| s_r = 2\pi\overline{r}s_r = 2\pi \times 10.0 \times 0.23 = 14.45 = 14.5$ mm^2, we actually do find that it is $s_S = \left|\left(\frac{dS}{dr}\right)_{r=\overline{r}}\right| s_r$ to a good approximation.

6.2.2 The Mean and the Standard Deviation of an Algebraic Sum

Assume that $Q = x + y$ (where x and y may be positive or negative) and that we have performed N_x measurements of the magnitude x and N_y measurements of the magnitude y, with results x_i ($i = 1, 2, \ldots, N_x$) and y_j ($j = 1, 2, \ldots, N_y$), respectively. If the measurements of x have a mean $\overline{x}$ and a standard deviation from the mean s_x and the measurements of y have a mean $\overline{y}$ and a standard deviation from the mean s_y, we want to answer the question: what is the mean $\overline{Q}$ and the standard deviation from the mean s_Q of the derived magnitude Q?

If all the measurements x_i and y_j are equally reliable, then the combination $Q_{ij} = x_i + y_j$ of any value of x with any value of y gives an acceptable value for the sum. We have, therefore, a total of N_xN_y equivalent (equally acceptable) results for Q (see Table 6.1), from which we may evaluate the quantities $\overline{Q}$ and s_Q.

Mean

The sum of all the terms of Table 6.1 is equal to

$$\begin{aligned}\sum_{\substack{j \\ (N_xN_y\text{terms})}} \sum_i Q_{ij} &= \sum_j \sum_i (x_i + y_j) = \sum_j \sum_i x_i + \sum_i \sum_j y_j \\ &= \sum_j N_x\overline{x} + \sum_i N_y\overline{y} = N_yN_x\overline{x} + N_xN_y\overline{y}\end{aligned} \tag{6.34}$$

and, therefore, the mean of Q is

$$\overline{Q} = \frac{1}{N_xN_y} \sum_j \sum_i Q_{ij} = \overline{x} + \overline{y}. \tag{6.35}$$

Table 6.1 All the possible values of $Q_{ij} = x_i + y_j$

$i =$	1	2	⋯	i	⋯	N_x
$j = 1$	$x_1 + y_1$	$x_2 + y_1$	⋯	$x_i + y_1$	⋯	$x_{N_x} + y_1$
2	$x_1 + y_2$	$x_2 + y_2$	⋯	$x_i + y_2$	⋯	$x_{N_x} + y_2$
⋯	⋯	⋯	⋯	⋯	⋯	⋯
j	$x_1 + y_j$	$x_2 + y_j$	⋯	$x_i + y_j$	⋯	$x_{N_x} + y_j$
⋯	⋯	⋯	⋯	⋯	⋯	⋯
N_y	$x_1 + y_{N_y}$	$x_2 + y_{N_y}$	⋯	$x_i + y_{N_y}$	⋯	$x_{N_x} + y_{N_y}$

Standard deviation from the mean

In order to evaluate the standard deviation from the mean s_Q of Q, we first evaluate the magnitude

$$\begin{aligned} N_x N_y s_Q^2 &= \sum_j \sum_i (\overline{Q} - Q_{ij})^2 = \sum_j \sum_i (\overline{Q} - x_i - y_j)^2 = \sum_j \sum_i (\overline{x} - x_i + \overline{y} - y_j)^2 \\ &= \sum_j \sum_i (\overline{x} - x_i)^2 + \sum_i \sum_j (\overline{y} - y_j)^2 + 2 \sum_j \sum_i (\overline{x} - x_i)(\overline{y} - y_j) \end{aligned} \tag{6.36}$$

The first two sums are

$$\sum_j \sum_i (\overline{x} - x_i)^2 = \sum_j N_x s_x^2 = N_y N_x s_x^2 \quad \text{and}$$
$$\sum_i \sum_j (\overline{y} - y_j)^2 = \sum_i N_y s_y^2 = N_x N_y s_y^2.$$

The third sum is

$$\begin{aligned} \sum_j \sum_i (\overline{x} - x_i)(\overline{y} - y_j) &= \sum_j \left((\overline{y} - y_j) \sum_i (\overline{x} - x_i) \right) = \sum_j (\overline{y} - y_j)(N_x \overline{x} - N_x \overline{x}) \\ &= 0. \end{aligned}$$

Therefore, Eq. (6.36) becomes

$$N_x N_y s_Q^2 = N_y N_x s_x^2 + N_x N_y s_y^2 \tag{6.37}$$

and, finally,

$$s_Q^2 = s_x^2 + s_y^2, \quad \text{or} \quad s_Q = \sqrt{s_x^2 + s_y^2}. \tag{6.38}$$

If we have $Q = \alpha x + \beta y$ and define the quantities $x_1 \equiv \alpha x$ and $y_1 \equiv \beta y$, then $Q = x_1 + y_1$ and the results (6.35) and (6.38) give

$$\overline{Q} = \overline{x}_1 + \overline{y}_1 \quad \text{and} \quad s_Q = \sqrt{s_{x_1}^2 + s_{y_1}^2}. \tag{6.39}$$

From the results of the Sect. 6.2.1, however, it is $\overline{x}_1 = \alpha\overline{x}$, $\overline{y}_1 = \beta\overline{y}$, $s_{x_1} = \alpha s_x$ and $s_{y_1} = \beta s_y$. Thus, the relations (6.39) become

$$\overline{Q} = \alpha\,\overline{x} + \beta\,\overline{y} \quad \text{and} \quad s_Q = \sqrt{\alpha^2 s_x^2 + \beta^2 s_y^2}. \tag{6.40}$$

These results may be generalized for the sum of any number of terms. If it is $Q = \alpha x + \beta y + \gamma z$ and we put $R = \beta y + \gamma z$, it will be $Q = \alpha x + R$. From what is already known, we have $\overline{Q} = \alpha\overline{x} + \overline{R}$ and $s_Q^2 = \alpha^2 s_x^2 + s_R^2$, while $\overline{R} = \beta\overline{y} + \gamma\overline{z}$ and $s_R^2 = \beta^2 s_y^2 + \gamma^2 s_z^2$. Substituting, we have

$$\overline{Q} = \alpha\overline{x} + \beta\overline{y} + \gamma\overline{z} \quad \text{and} \quad s_Q = \sqrt{\alpha^2 s_x^2 + \beta^2 s_y^2 + \gamma^2 s_z^2}. \tag{6.41}$$

In the same way we find that, if it is $Q = \alpha x + \beta y + \gamma z + \ldots$, then

$$\overline{Q} = \alpha\overline{x} + \beta\overline{y} + \gamma\overline{z} + \ldots \quad \text{and} \quad s_Q = \sqrt{\alpha^2 s_x^2 + \beta^2 s_y^2 + \gamma^2 s_z^2 + \ldots}. \tag{6.42}$$

These results will also be proved using the Theory of Errors, which is based on the normal distribution of the errors. The reader may already have noticed that the method of using all the possible sums in the evaluation of $\overline{Q}$ and s_Q is equivalent to the *convolution* of the two distributions. This method will also be used in the proof to be given in Chap. 9, using continuous distributions of the errors.

The standard deviation of the mean $\overline{Q}$ will be discussed later, after we have proved the relations that give $\overline{Q}$ and s_Q in the general case.

Example 6.4

Three measurements of the magnitude x gave the results 10, 11 and 12. Also, three measurements of the magnitude y gave the results 6, 8 and 10. If it is $Q = x + y$, verify Eqs. (6.35) and (6.38).

For the measurements we find that $\overline{x} = 11$, $\overline{y} = 8$, $s_x = \sqrt{2/3}$, $s_y = \sqrt{8/3}$.

Therefore,

$$\overline{Q} = \overline{x} + \overline{y} = 11 + 8 = 19 \quad \text{and} \quad s_Q = \sqrt{s_x^2 + s_y^2} = \sqrt{2/3 + 8/3} = \sqrt{10/3}.$$

Taking all the combinations of x and y, the values of $Q = x + y$ which are produced are:

$$Q: \quad 16 \quad 18 \quad 20 \qquad 17 \quad 19 \quad 21 \qquad 18 \quad 20 \quad 22\,.$$

These values have mean value

$$\overline{Q} = \frac{1}{9}(16 + 18 + 20 + 17 + 19 + 21 + 18 + 20 + 22) = 19$$

and standard deviation from the mean

$$s_Q = \sqrt{\frac{1}{9}(3^2 + 1^2 + 1^2 + 2^2 + 0^2 + 2^2 + 1^2 + 1^2 + 3^2)} = \sqrt{\frac{30}{9}} = \sqrt{\frac{10}{3}},$$

in agreement with the results of Eqs. (6.35) and (6.38).

Example 6.5

Prove the relation $\sigma_{\overline{x}} = \sigma/\sqrt{N}$ for the standard deviation of the mean of N measurements of the magnitude x, if σ is the standard deviation from the mean of the parent population.

The mean of the N measurements is defined as $\overline{x} = \frac{1}{N}(x_1 + x_2 + \ldots + x_N)$.

Since successive measurements are considered to be independent from each other, $\overline{x}$ may be taken to be a function of the form $Q = \alpha x + \beta y + \gamma z + \ldots$, of the independent variables $x = x_1$, $y = x_2$, $z = x_3, \ldots$ with $\alpha = \beta = \gamma = \ldots = 1/N$. The standard deviation of the individual measurements $x_1, x_2, x_3, \ldots, x_N$ is, of course, equal to σ, the standard deviation of the parent population of all the possible measurements. Thus, $s_x = s_y = s_z = \ldots = \sigma$ and Eq. (6.42) gives

$$\sigma_{\overline{x}} = s_Q = \sqrt{\alpha^2 s_x^2 + \beta^2 s_y^2 + \gamma^2 s_z^2 + \ldots} = \sqrt{\frac{\sigma^2}{N^2} + \frac{\sigma^2}{N^2} + \frac{\sigma^2}{N^2} + \ldots + \frac{\sigma^2}{N^2}} = \sqrt{N\frac{\sigma^2}{N^2}}$$

and, finally, $\sigma_{\overline{x}} = \dfrac{\sigma}{\sqrt{N}}$.

6.2.3 *The Mean and the Standard Deviations of a General Function of Many Variables*

Let $Q = Q(x, y)$ be a function of the variables x and y and that from N_x measurements of the magnitude x and N_y measurements of the magnitude y, which gave results x_i $(i = 1, 2, \ldots, N_x)$ and y_j $(j = 1, 2, \ldots, N_y)$, it was found that the means and

the standard deviations from the means for x and y were $\overline{x}$, $\overline{y}$ and s_x, s_y, respectively. We wish to determine the mean $\overline{Q}$ and the standard deviation s_Q of Q from the mean, as these magnitudes are derived from the measurements of x and y.

From Calculus we know that the Taylor expansion of the function $Q = Q(x, y)$ in the region of the point $(\overline{x}, \overline{y})$ is

$$Q(x, y) = Q(\overline{x}, \overline{y}) + \left(\frac{\partial Q}{\partial x}\right)_{(\overline{x},\overline{y})} (x - \overline{x}) + \left(\frac{\partial Q}{\partial y}\right)_{(\overline{x},\overline{y})} (y - \overline{y}) + \ldots, \tag{6.43}$$

where the partial derivatives are evaluated at the point $(\overline{x}, \overline{y})$. The omitted terms consist of products of derivatives of higher order with higher powers of $(x - \overline{x})$ and $(y - \overline{y})$ and their mixed products, $(x - \overline{x})(y - \overline{y})$ etc. For small deviations $d_{x,i} = x_i - \overline{x}$ and $d_{y,j} = y_i - \overline{y}$ of the values x_i and y_j from their means $\overline{x}$ and $\overline{y}$, respectively, these terms may be neglected. Thus, for the values x_i and y_j, the corresponding value of Q is, approximately,

$$Q(x_i, y_j) = Q(\bar{x}, \bar{y}) + \left(\frac{\partial Q}{\partial x}\right)_{(\bar{x},\bar{y})} (x_i - \bar{x}) + \left(\frac{\partial Q}{\partial y}\right)_{(\bar{x},\bar{y})} (y_j - \bar{y}). \tag{6.44}$$

Mean

Taking the mean value of the $N_x N_y$ possible combinations of x_i and y_j, we have

$$\begin{aligned} \overline{Q} &= \frac{1}{N_x N_y} \sum_j \sum_i Q(x_i, y_j) \\ &= \frac{1}{N_x N_y} \left[\sum_j \sum_i Q(\overline{x}, \overline{y}) + \left(\frac{\partial Q}{\partial x}\right)_{(\overline{x},\overline{y})} \sum_j \sum_i (x_i - \overline{x}) + \left(\frac{\partial Q}{\partial y}\right)_{(\overline{x},\overline{y})} \sum_i \sum_j (y_j - \overline{y}) \right] \end{aligned} \tag{6.45}$$

and

$$\begin{aligned} \overline{Q} &= Q(\overline{x}, \overline{y}) + \frac{1}{N_x N_y} \left[\left(\frac{\partial Q}{\partial x}\right)_{(\overline{x},\overline{y})} \sum_j (N_x \overline{x} - N_x \overline{x}) + \left(\frac{\partial Q}{\partial y}\right)_{(\overline{x},\overline{y})} \sum_i (N_y \overline{y} - N_y \overline{y}) \right] \\ &= Q(\overline{x}, \overline{y}) \end{aligned}$$

or

$$\overline{Q(x, y)} = Q(\overline{x}, \overline{y}), \tag{6.46}$$

i.e. *the mean value* $\overline{Q}$ *of* Q *is equal, to a good approximation, to the value* $Q = Q(x, y)$ *takes at* $x = \overline{x}$ *and* $y = \overline{y}$.

Standard deviation

For the evaluation of the standard deviation s_Q of Q, we observe that Eq. (6.44) gives

$$Q(x_i, y_j) - Q(\overline{x}, \overline{y}) = \left(\frac{\partial Q}{\partial x}\right)_{(\overline{x},\overline{y})} (x_i - \overline{x}) + \left(\frac{\partial Q}{\partial y}\right)_{(\overline{x},\overline{y})} (y_j - \overline{y}). \tag{6.47}$$

Squaring and summing for all the $N_x N_y$ possible combinations of x_i and y_j and dividing by $N_x N_y$, we have

$$\begin{aligned} s_Q^2 &= \frac{1}{N_x N_y} \sum_j \sum_i \left[Q(x_i, y_j) - \overline{Q(x, y)} \right]^2 = \frac{1}{N_x N_y} \sum_j \sum_i \left[Q(x_i, y_j) - Q(\overline{x}, \overline{y}) \right]^2 \\ &= \frac{1}{N_x N_y} \sum_j \sum_i \left[\left(\frac{\partial Q}{\partial x}\right)_{(\overline{x},\overline{y})} (x_i - \overline{x}) + \left(\frac{\partial Q}{\partial y}\right)_{(\overline{x},\overline{y})} (y_j - \overline{y}) \right]^2 \end{aligned} \tag{6.48}$$

or

$$\begin{aligned} s_Q^2 = {} & \frac{1}{N_x N_y} \left(\frac{\partial Q}{\partial x}\right)^2_{(\overline{x},\overline{y})} \sum_j \sum_i (x_i - \overline{x})^2 + \frac{1}{N_x N_y} \left(\frac{\partial Q}{\partial y}\right)^2_{(\overline{x},\overline{y})} \sum_i \sum_j (y_j - \overline{y})^2 \\ & + \frac{2}{N_x N_y} \left(\frac{\partial Q}{\partial x}\right)_{(\overline{x},\overline{y})} \left(\frac{\partial Q}{\partial y}\right)_{(\overline{x},\overline{y})} \sum_j \sum_i (x_i - \overline{x})(y_j - \overline{y}) \end{aligned} \tag{6.49}$$

The first two sums are

$$\begin{aligned} \sum_j \sum_i (x_i - \overline{x})^2 &= \sum_j N_x s_x^2 = N_y N_x s_x^2 \\ \sum_i \sum_j (y_j - \overline{y})^2 &= \sum_i N_y s_y^2 = N_x N_y s_y^2. \end{aligned}$$

The third sum is

$$\begin{aligned} \sum_j \sum_i (x_i - \overline{x})(y_j - \overline{y}) &= \sum_j \left((y_j - \overline{y}) \sum_i (x_i - \overline{x}) \right) = \sum_j (y_j - \overline{y}) (N_x \overline{x} - N_x \overline{x}) \\ &= 0. \end{aligned}$$

Equation (6.49) may, therefore, be written as

$$s_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2_{(\overline{x},\overline{y})} s_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2_{(\overline{x},\overline{y})} s_y^2, \tag{6.50}$$

or

$$s_Q = \sqrt{\left(\frac{\partial Q}{\partial x}\right)^2 s_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2 s_y^2} \tag{6.51}$$

where the partial derivatives are evaluated for $x = \overline{x}$ and $y = \overline{y}$.

The method may be generalized to a function of more variables. Thus, if $Q = Q(x, y, z, \ldots)$ is a function of the variables $x, y, z, \ldots$, the means and the standard deviations of which are, after the execution of a series of measurements of each, $\overline{x}, \overline{y}, \overline{z}, \ldots$ and $s_x, s_x, s_x, \ldots$, then

$$\begin{aligned} Q(x, y, z, \ldots) = Q(\overline{x}, \overline{y}, \overline{z}, \ldots) + \left(\frac{\partial Q}{\partial x}\right)_{(\overline{x},\overline{y},\overline{z},\ldots)} (x - \overline{x}) + \\ + \left(\frac{\partial Q}{\partial y}\right)_{(\overline{x},\overline{y},\overline{z},\ldots)} (y - \overline{y}) + \left(\frac{\partial Q}{\partial z}\right)_{(\overline{x},\overline{y},\overline{z},\ldots)} (z - \overline{z}) + \ldots \end{aligned} \tag{6.52}$$

The mean is evaluated as above, summing over all the possible combinations of $x_i, y_j, z_k, \ldots$. For the evaluation of s_Q^2 we will have vanishing triple sums of products of the forms

$$\sum_k \sum_j \sum_i (x_i - \overline{x})(y_j - \overline{y}) = 0 \quad \sum_k \sum_j \sum_i (x_i - \overline{x})(z_k - \overline{z}) = 0$$
$$\sum_k \sum_j \sum_i (y_j - \overline{y})(z_k - \overline{z}) = 0.$$

The general results are:

$$\overline{Q(x, y, z)} = Q(\overline{x}, \overline{y}, \overline{z}, \ldots) \tag{6.53}$$

and

$$s_Q = \sqrt{\left(\frac{\partial Q}{\partial x}\right)^2_{\overline{x},\overline{y},\overline{z},\ldots} s_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2_{\overline{x},\overline{y},\overline{z},\ldots} s_y^2 + \left(\frac{\partial Q}{\partial z}\right)^2_{\overline{x},\overline{y},\overline{z},\ldots} s_z^2 + \ldots}, \tag{6.54}$$

where the partial derivatives are evaluated at the point $\overline{\mathbf{r}} = (\overline{x}, \overline{y}, \overline{z}, \ldots)$, as indicated in the equation.

Standard deviation of the mean

Of greater interest than the standard deviation of the values of Q is the standard deviation $\sigma_{\overline{Q}}$ of the mean $\overline{Q}$. If, instead of the N_x measurements of the magnitude x and the N_y measurements of the magnitude y etc., we had $N_{\overline{x}}$ estimates of the mean $\overline{x}$, $N_{\overline{y}}$ estimates of the mean $\overline{y}$ etc., we would be able to evaluate, apart from the mean $\overline{Q}$ of Q, the standard deviation $\sigma_{\overline{Q}}$ of the values of $\overline{Q}$, exactly as we did for s_Q above. The fact that we do not have at our disposal many *measurements* of $\overline{x}$, $\overline{y}$ etc. is not an obstacle. To the degree that, since knowing from measurements the pairs of values $(\overline{x}, s_x)$, $(\overline{y}, s_y)$ etc. we also know the distributions of the means of the variables $(\overline{x}, \sigma_{\overline{x}})$, $(\overline{y}, \sigma_{\overline{y}})$ etc., to a satisfactory precision, we may consider that we have at our disposal as many values as we need of $\overline{x}$, $\overline{y}$ etc. for the evaluation of $\overline{Q}$ and $\sigma_{\overline{Q}}$. The procedure to be followed is the same used above and the result for the standard deviation of the Q values is:

$$\sigma_{\overline{Q}} = \sqrt{\left(\frac{\partial Q}{\partial x}\right)^2_{\overline{x},\overline{y},\overline{z},\ldots} \sigma^2_{\overline{x}} + \left(\frac{\partial Q}{\partial y}\right)^2_{\overline{x},\overline{y},\overline{z},\ldots} \sigma^2_{\overline{y}} + \left(\frac{\partial Q}{\partial z}\right)^2_{\overline{x},\overline{y},\overline{z},\ldots} \sigma^2_{\overline{z}} + \ldots} \tag{6.55}$$

where the partial derivatives are evaluated at the point $\overline{\mathbf{r}} = (\overline{x}, \overline{y}, \overline{z}, \ldots)$.

The relations (6.53)–(6.55) may be used in any case of propagation of errors. In the examples that follow, we extract certain of the results already proved and examine some other cases.

Example 6.6

Find the mean, standard deviation from the mean and the standard deviation of the mean of the sum $Q = \alpha x + \beta y + \gamma z + \ldots$, if $\overline{x}, \overline{y}, \overline{z}, \ldots$, $s_x, s_y, s_z, \ldots$ and $\sigma_{\overline{x}}$, $\sigma_{\overline{y}}$, $\sigma_{\overline{z}}, \ldots$ are known.

From Eq. (6.53) we find $\overline{Q} = Q(\overline{x}, \overline{y}, \overline{z}, \ldots) = \alpha\overline{x} + \beta\overline{y} + \gamma\overline{z} + \ldots$.

For the function $Q = \alpha x + \beta y + \gamma z + \ldots$, it is $\frac{\partial Q}{\partial x} = \alpha$, $\frac{\partial Q}{\partial y} = \beta$, $\frac{\partial Q}{\partial z} = \gamma$, … and from Eq. (6.54) we get $s_Q = \sqrt{\alpha^2 s_x^2 + \beta^2 s_y^2 + \gamma^2 s_z^2 + \ldots}$, in agreement with Eq. (6.41).

Equation (6.55) gives $\sigma_{\overline{Q}} = \sqrt{\alpha^2\sigma^2_{\overline{x}} + \beta^2\sigma^2_{\overline{y}} + \gamma^2\sigma^2_{\overline{z}} + \ldots}$.

Example 6.7

Find the mean, the standard deviation of from the mean and the standard deviation of the mean of the function $Q = \alpha x^n$, if $\overline{x}$, s_x and $\sigma_{\overline{x}}$ are known.

Equation (6.53) gives $\overline{Q} = \alpha\overline{x}^n$.

Since $\frac{\partial Q}{\partial x} = \frac{dQ}{dx} = \alpha n x^{n-1}$ and $\left(\frac{\partial Q}{\partial x}\right)_{\overline{x}} = \alpha n(\overline{x})^{n-1}$, from Eq. (6.54) we have $s_Q = \sqrt{\left(\frac{\partial Q}{\partial x}\right)_{\overline{x}}^2 s_x^2} = \left|\left(\frac{\partial Q}{\partial x}\right)_{\overline{x}}\right| s_x = \left|\alpha n(\overline{x})^{n-1}\right| s_x$ (the standard deviation is, by definition, positive).

Equation (6.55) gives: $\sigma_{\overline{Q}} = \sqrt{\left(\frac{\partial Q}{\partial x}\right)_{\overline{x}}^2 \sigma_{\overline{x}}^2} = \left|\left(\frac{\partial Q}{\partial x}\right)_{\overline{x}}\right| \sigma_{\overline{x}} = \left|\alpha n(\overline{x})^{n-1}\right| \sigma_{\overline{x}}$.

It is worth noting that $\frac{s_Q}{|\overline{Q}|} = |n| \frac{s_x}{|\overline{x}|}$ and $\frac{\sigma_{\overline{Q}}}{|\overline{Q}|} = |n| \frac{\sigma_{\overline{x}}}{|\overline{x}|}$.

Example 6.8

Find the mean and the standard deviation of the mean of the function $Q = \cos\theta$, if $\overline{\theta}$ and $\sigma_{\overline{\theta}}$ are known.

The mean of Q is $\overline{Q} = \overline{\cos\theta} \approx \cos\overline{\theta}$.

Since $\frac{\partial Q}{\partial \theta} = \frac{\mathrm{d}Q}{\mathrm{d}\theta} = -\sin\theta$, the standard deviation of the mean of Q is

$$\sigma_{\overline{Q}} = \sqrt{\left(\frac{\partial Q}{\partial \theta}\sigma_{\overline{\theta}}\right)^2} = \left|\left(\frac{\mathrm{d}Q}{\mathrm{d}\theta}\right)_{\overline{\theta}}\right| \sigma_{\overline{\theta}} = \left|\sin\overline{\theta}\right| \sigma_{\overline{\theta}}.$$

It is understood that $\sigma_{\overline{\theta}}$ is expressed in radians.

Example 6.9

Find the mean and the standard deviation of the mean of the function $Q = xy^2/z$, if $\overline{x}$, $\overline{y}$, $\overline{z}$ and $\sigma_{\overline{x}}$, $\sigma_{\overline{y}}$, $\sigma_{\overline{z}}$, ... are known.

The mean is $\overline{Q} = \overline{xy^2}/\overline{z}$.

Since $\frac{\partial Q}{\partial x} = \frac{y^2}{z}$, $\frac{\partial Q}{\partial y} = \frac{2xy}{z}$ and $\frac{\partial Q}{\partial z} = -\frac{xy^2}{z^2}$, the standard deviation of the mean of Q is

$$\begin{aligned}\sigma_{\overline{Q}} &= \sqrt{\left(\frac{\partial Q}{\partial x}\right)^2_{\overline{x},\overline{y},\overline{z},\ldots} \sigma_{\overline{x}}^2 + \left(\frac{\partial Q}{\partial y}\right)^2_{\overline{x},\overline{y},\overline{z},\ldots} \sigma_{\overline{y}}^2 + \left(\frac{\partial Q}{\partial z}\right)^2_{\overline{x},\overline{y},\overline{z},\ldots} \sigma_{\overline{z}}^2} \\ &= \sqrt{\left(\frac{\overline{y}^2}{\overline{z}}\right)^2 \sigma_{\overline{x}}^2 + \left(\frac{2\overline{xy}}{\overline{z}}\right)^2 \sigma_{\overline{y}}^2 + \left(\frac{\overline{xy}^2}{\overline{z}^2}\right)^2 \sigma_{\overline{z}}^2}.\end{aligned}$$

Dividing by $|\overline{Q}|$, we find that $\frac{\sigma_{\overline{Q}}}{|\overline{Q}|} = \sqrt{\left(\frac{\sigma_{\overline{x}}}{\overline{x}}\right)^2 + \left(2\frac{\sigma_{\overline{y}}}{\overline{y}}\right)^2 + \left(\frac{\sigma_{\overline{z}}}{\overline{z}}\right)^2}$.

Example 6.10

Find the mean and the standard deviation of the mean of the function $Q = A\,x^{\alpha}y^{\beta}z^{\gamma}\ldots$, if $\overline{x},\ \overline{y},\ \overline{z},\ldots$ and $\sigma_{\overline{x}},\ \sigma_{\overline{y}},\ \sigma_{\overline{z}},\ldots$ are known.

The mean is $\overline{Q} = A\,\overline{x}^{\alpha}\overline{y}^{\beta}\overline{z}^{\gamma}\ldots$.

Since $\dfrac{\partial Q}{\partial x} = \alpha A x^{\alpha-1}y^{\beta}z^{\gamma} = \dfrac{\alpha Q}{x}$, $\dfrac{\partial Q}{\partial y} = \beta A x^{\alpha}y^{\beta-1}z^{\gamma} = \dfrac{\beta Q}{y}$, $\dfrac{\partial Q}{\partial z} = \gamma A x^{\alpha}y^{\beta}z^{\gamma-1} = \dfrac{\gamma Q}{z}$, …

it is

$$\frac{\sigma_{\overline{Q}}}{|\overline{Q}|} = \sqrt{\left(\frac{1}{Q}\frac{\partial Q}{\partial x}\sigma_{\overline{x}}\right)^2 + \left(\frac{1}{Q}\frac{\partial Q}{\partial y}\sigma_{\overline{y}}\right)^2 + \left(\frac{1}{Q}\frac{\partial Q}{\partial z}\sigma_{\overline{z}}\right)^2 + \ldots}$$

$$\frac{\sigma_{\overline{Q}}}{|\overline{Q}|} = \sqrt{\left(\alpha\frac{\sigma_{\overline{x}}}{\overline{x}}\right)^2 + \left(\beta\frac{\sigma_{\overline{y}}}{\overline{y}}\right)^2 + \left(\gamma\frac{\sigma_{\overline{z}}}{\overline{z}}\right)^2 + \ldots}$$

6.2.4 *Another Approach to the Evaluation of the Mean and the Standard Deviation of a Compound Quantity*

It is possible, instead of following the procedure of Sect. 6.2.3 in order to evaluate $\overline{Q}$ and s_Q of a function of many variables $Q = Q(x, y, z, \ldots)$, to work as follows:

We measure N sets of values of the variables of Q, i.e. $x_i, y_i, y_i, \ldots$ $(i = 1, 2, \ldots, N)$ and estimate Q for each such set, Q_i $(i = 1, 2, \ldots, N)$. We use the values of $x_i, y_i, y_i, \ldots$ in order to find $\overline{x}, \overline{y}, \overline{z}, \ldots$ and $s_x, s_y, s_z, \ldots$, and finally $\overline{Q}$ and s_Q.

Assuming that $\overline{x}, \overline{y}, \overline{z}, \ldots$ are the most probable values of $x, y, z, \ldots$, we expand $Q(x_i, y_i, z_i, \ldots)$ in a Taylor series

$$\begin{aligned} Q(x_i, y_i, z_i, \ldots) = {} & Q(\overline{x}, \overline{y}, \overline{z}, \ldots) + \left(\frac{\partial Q}{\partial x}\right)_{\overline{x},\overline{y},\overline{z},\ldots}(x_i - \overline{x}) \\ & + \left(\frac{\partial Q}{\partial y}\right)_{\overline{x},\overline{y},\overline{z},\ldots}(y_i - \overline{y}) + \left(\frac{\partial Q}{\partial z}\right)_{\overline{x},\overline{y},\overline{z},\ldots}(z_i - \overline{z}) + \ldots \end{aligned} \tag{6.56}$$

Mean

$$\begin{aligned} \overline{Q}(x, y, z, \ldots) = \frac{1}{N}\Bigg[& \sum_i Q(\overline{x}, \overline{y}, \overline{z}, \ldots) + \left(\frac{\partial Q}{\partial x}\right)_{\overline{x},\overline{y},\overline{z},\ldots}\sum_i (x_i - \overline{x}) \\ & + \left(\frac{\partial Q}{\partial y}\right)_{\overline{x},\overline{y},\overline{z},\ldots}\sum_i (y_i - \overline{y}) + \ldots \Bigg] \end{aligned} \tag{6.57}$$

The sums $\sum_i (x_i - \bar{x})$, $\sum_i (y_i - \bar{y})$ etc. are all equal to zero. Therefore,

$$\overline{Q}(x, y, z, \ldots) = Q(\bar{x}, \bar{y}, \bar{z}, \ldots) \tag{6.58}$$

Standard deviation

Expanding

$$s_Q^2 = \frac{1}{N}\sum_i \left[Q(x_i, y_i, z_i, \ldots) - Q(\bar{x}, \bar{y}, \bar{z}, \ldots)\right]^2 \tag{6.59}$$

we have

$$\begin{aligned} s_Q^2 = &\left(\frac{\partial Q}{\partial x}\right)^2_{\bar{x},\bar{y},\bar{z},\ldots} \frac{1}{N}\sum_i (x_i - \bar{x})^2 + \left(\frac{\partial Q}{\partial y}\right)^2_{\bar{x},\bar{y},\bar{z},\ldots} \frac{1}{N}\sum_i (y_i - \bar{y})^2 \\ &+ 2\left(\frac{\partial Q}{\partial x}\right)_{\bar{x},\bar{y},\bar{z},\ldots} \left(\frac{\partial Q}{\partial y}\right)_{\bar{x},\bar{y},\bar{z},\ldots} \frac{1}{N}\sum_i (x_i - \bar{x})(y_i - \bar{y}) \\ &+ 2\left(\frac{\partial Q}{\partial x}\right)_{\bar{x},\bar{y},\bar{z},\ldots} \left(\frac{\partial Q}{\partial z}\right)_{\bar{x},\bar{y},\bar{z},\ldots} \frac{1}{N}\sum_i (x_i - \bar{x})(z_i - \bar{z}) + \ldots \end{aligned} \tag{6.60}$$

We denote by $s_{xy}^2 = \frac{1}{N}\sum_i (x_i - \bar{x})(y_i - \bar{y})$, $s_{xz}^2 = \frac{1}{N}\sum_i (x_i - \bar{x})(z_i - \bar{z})$ etc.

These cross products depend on the correlation of the variables $x, y, z, \ldots$ with each other. If we assume that the variables are independent of each other, and for a very large N, given that the differences $(x_i - \bar{x})$ etc. are equally probable to be positive or negative, we can assume that these sums are negligible compared to the square terms. Therefore,

$$s_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2_{\bar{x},\bar{y},\bar{z},\ldots} \frac{1}{N}\sum_i (x_i - \bar{x})^2 + \left(\frac{\partial Q}{\partial y}\right)^2_{\bar{x},\bar{y},\bar{z},\ldots} \frac{1}{N}\sum_i (y_i - \bar{y})^2 + \ldots \tag{6.61}$$

or

$$s_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2_{\bar{x},\bar{y},\bar{z},\ldots} s_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2_{\bar{x},\bar{y},\bar{z},\ldots} s_y^2 + \left(\frac{\partial Q}{\partial z}\right)^2_{\bar{x},\bar{y},\bar{z},\ldots} s_z^2 + \ldots, \tag{6.62}$$

in agreement with Eq. (6.54). It must be pointed out that the methods of Sects. 6.2.3 and 6.2.4 are different in the sense that they correspond to two different experimental procedures. Namely:

In Sect. 6.2.3 the independent variables $x, y, z, \ldots$ are measured independently and with different numbers of readings each. The means $\overline{x}, \overline{y}, \overline{z}, \ldots$ are thus found and then $\overline{Q}$ and s_Q.

In Sect. 6.2.4, N independent sets of measurements $x_i, y_i, y_i, \ldots$ are performed, each resulting in a value Q_i. These N values of Q_i are then used in order to find $\overline{Q}$ and s_Q.

The method of Sect. 6.2.3 is the one most commonly used in experimental procedure. We have shown, however that the final results are approximately the same.

6.3 The Error in $\overline{Q}(\overline{x}, \overline{y}, \overline{z}, \ldots)$ Due to the Errors in $\overline{x}, \overline{y}, \overline{z}, \ldots$

Equation (6.55) gives the standard deviation of the mean of a function of many variables $\overline{Q}(\overline{x}, \overline{y}, \overline{z}, \ldots)$ in terms of the standard deviations of the means of these variables. Since we consider $\sigma_{\overline{x}} = \delta x$, $\sigma_{\overline{y}} = \delta y$, $\sigma_{\overline{z}} = \delta z$ etc. to be the errors in $\overline{x}, \overline{y}, \overline{z}, \ldots$, the corresponding error $\delta Q = \sigma_{\overline{Q}}$ in $\overline{Q}$ is

$$\delta Q = \sqrt{\left(\frac{\partial Q}{\partial x}\right)^2 (\delta x)^2 + \left(\frac{\partial Q}{\partial y}\right)^2 (\delta y)^2 + \left(\frac{\partial Q}{\partial z}\right)^2 (\delta z)^2 + \ldots} \tag{6.63}$$

where the partial derivatives are evaluated for $\overline{x}, \overline{y}, \overline{z}, \ldots$, which we consider to be the best estimates we have for $x_0, y_0, z_0, \ldots$, the real values of the variables. We give the numerical result for Q as

$$Q = \overline{Q} \pm \sigma_{\overline{Q}} = \overline{Q} \pm \delta Q. \tag{6.64}$$

For a function $z(x)$ of one variable x, Eq. (6.63) reduces to

$$\delta z = \sigma_{\overline{z}} = \left|\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)_{x=\overline{x}}\right| \delta x. \tag{6.65}$$

The geometrical relation of these two errors is shown in Fig. 6.2.

The general expression giving the errors in functions of one or more variables in terms of the errors in these variables is given in Table 6.2, together with the errors in some common functions.

Table 6.2 The propagation of errors.

The results of the measurements are $x = \overline{x} \pm \delta x,\ y = \overline{y} \pm \delta y, \ldots$, where $\overline{x}, \overline{y}, \ldots$ are the means and $\delta x = \sigma_{\overline{x}},\ \delta y = \sigma_{\overline{y}}, \ldots$ are the standard errors or standard deviations from the means, of the variables $x, y, \ldots$. By definition, $\delta x, \delta y, \ldots, \delta Q$ are positive. $\alpha, \beta, \ldots, k, n$ are positive or negative constants. The partial derivatives are evaluated for $x = \overline{x},\ y = \overline{y}, \ldots$

Function Q	Error δQ or standard deviation $\sigma_{\overline{Q}}$ of the mean $\overline{Q}$
General $Q = Q(x, y, \ldots)$	$\delta Q = \sigma_{\overline{Q}} = \sqrt{\left(\frac{\partial Q}{\partial x}\,\delta x\right)^2 + \left(\frac{\partial Q}{\partial y}\,\delta y\right)^2 + \ldots}$
Sum $Q = x + y$	$\delta Q = \sqrt{(\delta x)^2 + (\delta y)^2}$
Difference $Q = x - y$	$\delta Q = \sqrt{(\delta x)^2 + (\delta y)^2}$
General sum $Q = \alpha x \pm \beta y \pm \gamma z \pm \ldots$	$\delta Q = \sqrt{(\alpha\delta x)^2 + (\beta\delta y)^2 + (\gamma\delta z)^2 + \ldots}$
Multiplication with constant $Q = kx$	$\delta Q = \lvert k\rvert\delta x$
Power $Q = x^n$	$\delta Q = \lvert n(\overline{x})^{n-1}\rvert\,\delta x = \left\lvert n\frac{\overline{Q}}{\overline{x}}\right\rvert\delta x$
Product $Q = xy$	$\delta Q = \lvert\overline{x}\overline{y}\rvert\sqrt{\left(\frac{\delta x}{\overline{x}}\right)^2 + \left(\frac{\delta y}{\overline{y}}\right)^2}$
Ratio $Q = \frac{x}{y}$	$\delta Q = \left\lvert\frac{\overline{x}}{\overline{y}}\right\rvert\sqrt{\left(\frac{\delta x}{\overline{x}}\right)^2 + \left(\frac{\delta y}{\overline{y}}\right)^2}$
Product of powers $Q = x^\alpha y^\beta z^\gamma \ldots$ ($\alpha, \beta, \gamma, \ldots$ positive or negative)	$\delta Q = \lvert\overline{Q}\rvert\sqrt{\left(\alpha\frac{\delta x}{\overline{x}}\right)^2 + \left(\beta\frac{\delta y}{\overline{y}}\right)^2 + \left(\gamma\frac{\delta z}{\overline{z}}\right)^2 + \ldots}$
Exponential $Q = e^{\alpha x}$	$\delta Q = \lvert\alpha\rvert e^{\alpha\overline{x}}\,\delta x$
Natural Logarithm $Q = \ln(\alpha x)$	$\delta Q = \frac{\delta x}{\lvert\overline{x}\rvert}$
Common Logarithm $Q = \log(\alpha x)$	$\delta Q = \frac{\delta x}{\lvert\overline{x}\rvert\ln 10}$
Sine $Q = \sin x$	$\delta Q = \lvert\cos\overline{x}\rvert\,\delta x$ (δx in radians)
Cosine $Q = \cos x$	$\delta Q = \lvert\sin\overline{x}\rvert\,\delta x$ (δx in radians)
General sine $Q = \sin(\alpha x)$	$\delta Q = \lvert\alpha\cos\alpha\overline{x}\rvert\delta x$ ($\alpha\,\delta x$ in radians)
General cosine $Q = \cos(\alpha x)$	$\delta Q = \lvert\alpha\sin\alpha\overline{x}\rvert\delta x$ ($\alpha\,\delta x$ in radians)

6.3.1 The Case of Asymmetrical Errors

We will consider the case of a function $z(x)$ of one variable, x, for mathematical simplicity. The analysis is much more complicated for functions of more variables. There are cases in which $z(x)$ is rapidly varying in some region of x values and the error δx in $\overline{x}$ is so large that the approximation of Eq. (6.30) [Eq. (6.52) for more variables] is not adequate. Terms of order higher than the first in $(x_i - \overline{x})$ should be taken into account. The effect of this is that the value of

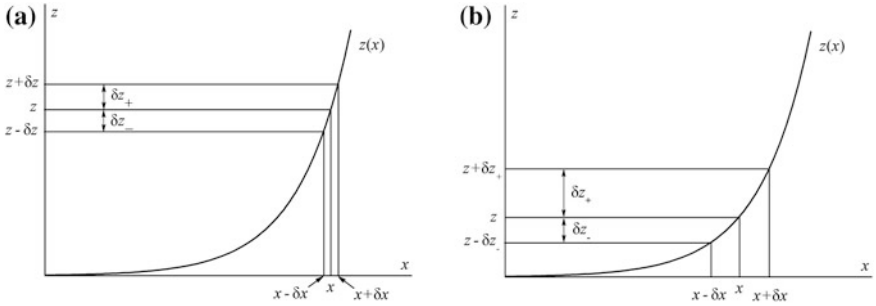

Fig. 6.3 The error δz in $z(x)$ corresponding to the error δx in x, in the cases in which $\delta z_- \equiv z(x-\delta x) - z(x)$ and $\delta z_+ \equiv z(x+\delta x) - z(x)$ are **a** comparable in magnitude and **b** significantly different

$$\delta z_- = z(x - \delta x) - z(x) \tag{6.66}$$

is widely different than

$$\delta z_+ = z(x + \delta x) - z(x). \tag{6.67}$$

This situation is demonstrated in Fig. 6.3.

In these cases, we sometimes give the result as

$$z = \bar{z}^{\,+\delta z_+}_{\,-\delta z_-}. \tag{6.68}$$

The same applies for functions of more variables than one. A numerical example is

$$E = 3.1^{\,+0.4}_{\,-0.3}\,\text{GeV}.$$

Problems

6.1 If it is $x = 3.4 \pm 0.1$, find the values of and the errors in the quantities:

(a) $\sqrt{x}$, (b) $5x$, (c) x^2, (d) $1/x$, (e) $1/x^2$.

6.2 For $x = 0.500 \pm 0.025$, find the value of y and its error in the following cases:

(a) $y = 5x^2$, (b) $y = 2x - 4x^2$, (c) $y = 2/x^3$ and (d) $y = 2(x-1)^2$.

6.3 If it is $y = \frac{x}{1+x}$ and the error in x is δx, find the relative error $\frac{\delta y}{y}$ in y.

6.4 If it is $\theta = 31.3^{\circ} \pm 0.2^{\circ}$, find the values of and the errors in the magnitudes: (a) $\sin\theta$, (b) $\cos\theta$, (c) $\tan^2\theta$, (d) $\sin^2\theta$.

6.5 Find the mean of f and its error:

(a) If $f(x,y) = 2x^2 + 3y$ and $x = 2.000 \pm 0.010$, $y = 1.500 \pm 0.020$.
(b) If $f(x,y) = 3xy^3$ and $x = 0.1250 \pm 0.0025$, $y = 0.1250 \pm 0.0025$.
(c) If $f(x,y) = x^2y^3$ and $x = 0.120 \pm 0.003$, $y = 0.150 \pm 0.004$.
(d) If $f(x,y) = 3x^2 - 6y$ and $x = 2.000 \pm 0.020$, $y = 1.250 \pm 0.015$.
(e) If $f(x,y) = 3ye^{4x}$ and $x = 0.500 \pm 0.025$, $y = 0.600 \pm 0.025$.
(f) If $f(x,y) = 5ye^{-2x}$ and $x = 0.1250 \pm 0.0025$, $y = 0.1250 \pm 0.0025$.

6.6 If it is $x = 4.6 \pm 0.3$ cm and $y = 7.2 \pm 0.5$ cm, find the values of and the errors in the magnitudes: (a) $x + y$, (b) $x - y$, (c) x/y and (d) xy.

6.7 Given the results $x = 2.23 \pm 0.05$ and $y = 1.87 \pm 0.03$, find:

(a) The % errors in x and y.
(b) The errors in $x + y$ and $x - y$.
(c) The % errors in $x + y$ and $x - y$.

6.8 From measurements, it was found that $x = 222 \pm 11$ and $y = 213 \pm 13$. Find the value of the magnitude $Q = x - y$ and its error δQ. Is the fact that it is $\delta Q > Q$ a problem?

6.9 If it is $x = a \cos\theta$, with $a = 2.00 \pm 0.13$ m and $\theta = 45.0^\circ \pm 0.3^\circ$, find x and its error.

6.10 If it is $x = a \sin\theta + b$, with $a = 1.00 \pm 0.02$ m, $b = -0.50 \pm 0.01$ m and $\theta = 45^\circ \pm 1^\circ$, find x and its error.

6.11 The position x of a body is given by the relation $x = a \sin\theta + b \cos\theta$, with $a = 1.00 \pm 0.07$ m, $b = -0.50 \pm 0.04$ m and $\theta = 45^\circ \pm 1^\circ$. Find x and its error.

6.12 If $x = a \sin\theta \cos\phi$, with $a = 1.50 \pm 0.01$ m, $\theta = 45.0^\circ \pm 0.3^\circ$ and $\phi = 30.0^\circ \pm 0.2^\circ$, find x and its error.

6.13 If $\delta\omega$ is the error in ω, $\delta\phi$ is the error in ϕ (in radians) and δA is the error in A, find the error δy in $y = A \sin(\omega t + \phi)$ as a function of t.

6.14 A right-angled triangle has hypotenuse equal to $c = 10.0 \pm 0.2$ m and one of its angles is $A = 30.0^\circ \pm 0.3^\circ$. Find, with their errors:

(a) The other angle of the triangle.
(b) The lengths a and b of the other sides.

6.15 In a triangle ABC, two sides have lengths $a = 5.00 \pm 0.12$ m and $b = 4.00 \pm 0.22$ m. The angle between them is equal to $\theta = 30^\circ \pm 1^\circ$. Find the length of the third side c of the triangle and its error. It is given that $c = (a^2 + b^2 - 2ab\ \cos\theta)^{1/2}$.

6.16 An isosceles triangle ABC has the sides AB and BC equal. The angle between them is equal to $\theta = 50.0^\circ \pm 0.5^\circ$. The height AD of the triangle has length $H = 1.00 \pm 0.02$ m. Find the lengths a, b and c of the triangle's sides and their errors.

6.17 A force $\mathbf{F}$ has magnitude equal to $F = 12.0 \pm 0.3$ N and direction which forms an angle $\theta = 60^\circ \pm 1^\circ$ with the x-axis. Find the components F_x and F_y of the force and their errors.

6.18 The index of refraction n of a substance is given, according to Snell's law, by the relation $n = \frac{\sin\theta_i}{\sin\theta_r}$. In an experiment, the angles $\theta_i = 62.10^{\circ} \pm 0.11^{\circ}$ and $\theta_r = 43.26^{\circ} \pm 0.08^{\circ}$ were measured. Evaluate n and its error δn.

6.19 The displacement of a simple harmonic oscillator is given by the relation $x = a\,\sin(2\pi f\,t + \phi)$, where $a = 1.00 \pm 0.03$ m, $f = 1.00 \pm 0.02$ Hz and $\phi = 45^{\circ} \pm 1^{\circ}$. Find the displacement of the oscillator and its error at the time $t = 1$ s.

6.20 If $x = x_0 + a\,\sin(2\pi f\,t)$, with $x_0 = 2.00 \pm 0.04$ m, $a = 1.00 \pm 0.03$ m, and $f = 1.00 \pm 0.03$ Hz, find x and its error at $t = 0.5$ s.

6.21 A cylindrical rod has radius $r = 1.80 \pm 0.01$ cm and length $l = 7.40 \pm 0.07$ cm. What is the relative error in the volume of the cylinder evaluated from these values?

If the mass of the cylinder was measured to be $M = 423 \pm 1$ g, what is the density ρ of the rod's material and its error $\delta\rho$?

6.22 The focal length f of a lens is related to the distances s of the object and s' of the image from the lens through the relation $\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$. If for $s = 12.6 \pm 0.2$ cm it was found to be $s' = 7.8 \pm 0.1$ cm, find the focal length of the lens and its error.

6.23 The period of a mathematical pendulum is $T = 2\pi\sqrt{l/g}$, where l is the length of the pendulum. If $g = 9.800$ m/s^2 with great accuracy and $l = 1.00 \pm 0.01$ m, what is the period T of the pendulum and its error δT?

6.24 If in an LC circuit L is known with an error $\pm 6\%$ and C with an error $\pm 8\%$, what is the % error in the resonance frequency $f = \frac{1/2\pi}{\sqrt{LC}}$ of the circuit?

6.25 The gain of an amplifier in db (decibel) is given by the relation $G = 20\,\log_{10}\left(\frac{V_o}{V_i}\right)$, where V_i is the input voltage and V_o the output voltage. In an amplifier, it is $V_i = 12.3 \pm 0.2$ mV and $V_o = 895 \pm 19$ mV. What is, in db, the gain G of the signal and its error? *Hint:* $\log_{10} x = \log_{10} e \times \ln x$.

6.26 For the determination of $Q = 3x + 5y^2$, measurements of x and y gave the following pairs:

x_i	12	13	11	12	10	14	13	12	14	13	12
y_i	35	37	34	37	34	37	36	35	38	34	35

Find the mean $\overline{Q}$ and its standard deviation $\sigma_{\overline{Q}}$,

(a) Assuming that x and y are independent of each other and, therefore, that the equations for the propagation of errors may be used, and

(b) Finding that there is some dependence of x and y from each other and evaluating first the value Q_i for each pair (x_i, y_i) and then $\overline{Q}$ and $\sigma_{\overline{Q}}$.

6.27 The mean of 100 measurements of x is 3.22 and their standard deviation from the mean 0.15. Another 200 measurements have respective values 3.56 and 0.22. Find the mean and the standard deviation from the mean of the total of the 300 measurements. What are the standard deviations of the means for the 100, 200 and 300 values?

6.28 The mean value of x for 500 measurements is 5.08 and their standard deviation from the mean is 0.12. If 200 of these measurements have a mean of 4.95 and standard deviation from the mean 0.15, what are the mean and the standard deviation from the mean for the other 300 measurements?

6.29 The minimum deviation D of monochromatic light in a prism, gives for the index of refraction of the material of the prism for the particular color the expression

$$n = \frac{\sin\frac{1}{2}(A+D)}{\sin\frac{1}{2}A},$$

where A is the angle of the prism. If measurements of A and D gave the mean values and their standard deviations $A = 60°5.2' \pm 0.2'$ and $D = 46° 36.6' \pm 0.4'$, find the refractive index and its error.

6.30 Poiseuille's equation for the volume flow rate ($\phi = dV/dt$) of a fluid with viscosity η, through a cylindrical pipe of length l and radius r is $\phi = \dfrac{\pi p r^4}{8l\eta}$, where p is the pressure difference between the two ends of the pipe. Find the error in ϕ in terms of the errors (standard deviations of the means) of η, l, r and p.

6.31 The relation $1/f = (n-1)(1/r_1 - 1/r_2)$ is known as the lens-maker formula. If it is $f = 50.1 \pm 0.2$ cm, $r_1 = 31.2 \pm 0.1$ cm and $r_2 = 149.0 \pm 0.5$ cm, find the index of refraction n and its error, for the glass from which the particular lens is made.

6.32 **[E.O.P.R.]** A function is defined as

$$f(x) = \sin\left\{\frac{1}{5}\exp\left[\sqrt{x} + \left[x^2 + \sqrt{x + \ln(1+x)}\right]^3\right]\right\}.$$

Evaluate $f(x)$ numerically for $0 \le x \le 1$ with x increasing in steps of $\delta x = 0.01$ and thus plot, as a function of x, the error δf in $f(x)$ for an error $\delta x = 0.01$ in x (see Example 2.11).

Chapter 7
The Three Basic Probability Distributions

In this chapter, we will examine briefly the three basic probability distributions. For more information, the reader may consult the appropriate bibliography.

7.1 Histograms of Experimental Probabilities

Suppose that we toss 10 coins simultaneously and when they fall we observe that 3 of them show 'tails' and 7 'heads'. The whole procedure may be considered to constitute a *measurement*, the *result* of which is: 3 coins show 'tails'. We intend to repeat the same procedure many times, so we enter in a table the order number i of the measurement and its result n_T (Table 7.1, $i = 1$, $n_T = 3$). We repeat the tossing again and let the result now be that 6 coins show 'tails'. We enter this measurement in the table ($i = 2$, $n_T = 6$). We repeat the experiment 100 times and complete the table.

We group these results as in Table 7.2, which shows for each result n_T its frequency in 100 tosses.

We draw a histogram of the frequencies (Fig. 7.1). We observe a tendency for the histogram to be symmetrical with respect to the central column (for $n_T = 5$), as we expected, since, with equal probabilities for a coin to show 'heads' or 'tails', we obviously have:

$$\begin{aligned}\text{Probability for } n \text{ coins to show 'tails'} &= \text{Probability for } n \text{ coins showing 'heads'} \\ &= \text{Probability for } (10 - n) \text{ coins showing 'tails'}\end{aligned}$$

If the bin width of the n_T-axis is equal to unity, the total area of the columns of the histogram will be equal to the total number of the measurements performed, i.e. 100. If we divide the vertical scale by the total number of the measurements, the

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Table 7.1 Numbers of coins showing 'tails', in 100 tosses of 10 coins. i = order number of toss n_T = number of coins in the toss showing 'tails'

i	n_T	i	n_T	i	n_T	i	n_T	i	n_T	i	n_T	i	n_T	i	n_T	i	n_T	i	n_T
1	3	*11*	5	*21*	4	*31*	6	*41*	3	*51*	3	*61*	5	*71*	4	*81*	6	*91*	5
2	6	*12*	3	*22*	2	*32*	4	*42*	5	*52*	4	*62*	6	*72*	5	*82*	4	*92*	6
3	2	*13*	6	*23*	5	*33*	1	*43*	7	*53*	5	*63*	4	*73*	7	*83*	4	*93*	3
4	5	*14*	6	*24*	3	*34*	5	*44*	4	*54*	6	*64*	9	*74*	4	*84*	5	*94*	5
5	7	*15*	4	*25*	6	*35*	6	*45*	5	*55*	4	*65*	5	*75*	2	*85*	8	*95*	7
6	0	*16*	5	*26*	5	*36*	4	*46*	2	*56*	4	*66*	3	*76*	5	*86*	3	*96*	4
7	9	*17*	7	*27*	3	*37*	5	*47*	7	*57*	6	*67*	6	*77*	7	*87*	7	*97*	7
8	4	*18*	4	*28*	5	*38*	4	*48*	8	*58*	3	*68*	7	*78*	4	*88*	5	*98*	2
9	5	*19*	7	*29*	7	*39*	7	*49*	4	*59*	7	*69*	7	*79*	8	*89*	6	*99*	6
10	4	*20*	5	*30*	6	*40*	5	*50*	6	*60*	5	*70*	5	*80*	6	*90*	5	*100*	5

Table 7.2 Number of tosses, in a total of 100 tosses of 10 coins, in which n_T coins showed 'tails'. $P(n_T)$ is the corresponding experimental probability

n_T	Number of events	Probability $P(n_T)$
0	1	0.01
1	1	0.01
2	5	0.05
3	10	0.10
4	20	0.20
5	26	0.26
6	17	0.17
7	15	0.15
8	3	0.03
9	2	0.02
10	0	0.00

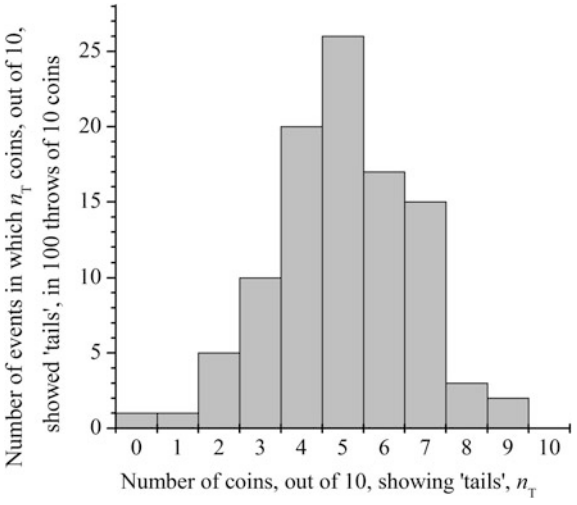


Fig. 7.1 Histogram of the number of tosses, in a total of 100 tosses of 10 coins, in which n_T of the 10 coins showed 'tails'

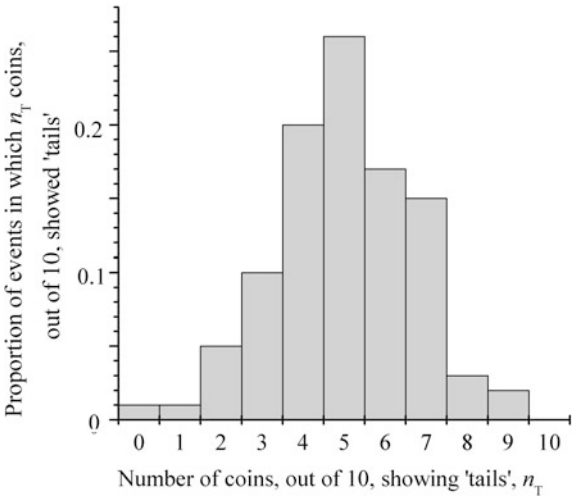


Fig. 7.2 Histogram of the (experimental) probabilities for the appearance of n_T coins showing 'tails' in a toss of 10 coins, as these were determined in a total of 100 tosses

vertical axis will now give the probability $P(n_T)$ for the result n_T to appear. We now have a frequency histogram (Fig. 7.2). The histogram is normalized since the total area of all the columns is equal to unity, which is the probability of any of the possible results appearing.

If we repeat the experiment for another (large) number of times, the resulting histogram is not expected to differ greatly from the first. We notice that the result with the highest probability to occur is 5 coins showing 'tails' (and the other 5 showing 'heads'). It is also less probable all 10 coins to show 'tails' (or all 10 coins showing 'heads'). Obviously, there is a theoretical prediction of the distribution of the frequencies and the results of our experiment will tend towards it as the number of measurements becomes larger. Before we examine the theory behind our experiment, let us perform another similar experiment.

We throw 10 dice in the air. Suppose that we measure the number n_A of the dice showing an ace. We repeat our experiment 100 times (Table 7.3) and we now obtain a histogram similar to that of Fig. 7.3. Since now the probability of the observed event happening, i.e. a die showing ace, is 1/6 (and is 5/6 for showing any other number), we do not expect symmetry in the histogram. We would have a similar histogram for any of the other 5 numbers present on the die. We notice that the result with the highest probability to occur is for 1 die showing ace. It is also very improbable for all 10 dice to show an ace (or even 9, 8 or 7).

We now examine theoretically the general problem.

Table 7.3 The number of throws, in 100 throws of 10 dice, in which n_A dice showed an ace. $P(n_A)$ is the corresponding experimental probability

n_A	Number of events	Probability $P(n_A)$
0	16	0.16
1	32	0.32
2	29	0.29
3	16	0.16
4	6	0.06
5	1	0.01
6	0	0.00
7	0	0.00
8	0	0.00
9	0	0.00
10	0	0.00

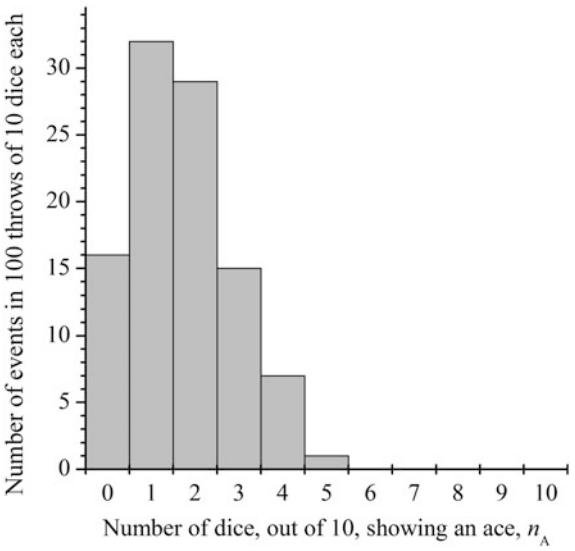


Fig. 7.3 Histogram of the number of throws, in a total of 100 throws of 10 dice, in which n_A of the 10 dice showed an 'ace'

7.2 The Binomial or Bernoulli Distribution

Assume that an experiment can only give two results: P, with probability p for occurring, and Q (= not P), with probability $q = 1 - p$. For example, in the case of the coin we examined, is: P = *the coin shows 'tails'*, $p = \frac{1}{2}$, Q = *the coin shows 'heads'* and $q = 1 - p = \frac{1}{2}$. If n statistically independent trials are performed, it is required that we find the probability of result P occurring a total of x times (and, consequently, Q being occurring a total of $n - x$ times).

Let us first examine the probability of x results P occurring one after the other, followed by $n - x$ consecutive results Q. This is only one possible outcome of the experiment and, given that the trials are statistically independent, the probability of this happening is $p^x q^{n-x}$. The same is the probability for exactly x results P and $n - x$ results Q happening in any order. The number of different ways in which this can happen (Table 7.4) is equal to the number of possible combinations of the x results P and the $n - x$ results Q, i.e.

$$\binom{n}{x} = \frac{n!}{x!(n-x)!} \tag{7.1}$$

Therefore, the probability of observing any one of these statistically independent events (x results P and $n - x$ results Q, in any order), each of which has a probability $p^x q^{n-x}$ to happen, is

$$P_n(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}. \tag{7.2}$$

The example for $n = 6$ and $x = 4$ is shown in detail in Table 7.5. There are $\binom{n}{x} = \frac{n!}{x!\,(n-x)!} = \frac{6!}{4!\,(6-4)!} = 15$ different combinations of P and Q with no regard to the order in which they happen.

This is the *binomial distribution* or the *Bernoulli distribution* (James Bernoulli, 1654–1705). It is a *discrete distribution* since the variable x may only take positive integral values. The name binomial stems from the fact that in the binomial expansion

$$(q+p)^n = q^n + nq^{n-1}p + \frac{n(n-1)}{2} q^{n-2}p^2 + \ldots + p^n, \tag{7.3}$$

successive terms give the probabilities for observing the result P x = 0, 1, 2, …, n times in n trials.

In Figs. 7.4 and 7.5 are shown the theoretical predictions of relation (7.2), for the two problems we examined above, the toss of 10 coins and the throw of 10 dice, respectively.

In the limit, as the number n becomes very large, the differences between successive columns of the histogram become proportionally smaller and the histogram appears as an almost continuous distribution (Fig. 7.6).

Table 7.4 All the different possible results of a series of n trials

	The various ways in which results P may occur in n trials, if the order in which the events P and Q appear is taken into account															Probability
	1	2	3	⋯	x–1	x	x+1	x+2	⋯	n–x	⋯	n–3	n–2	n–1	n	
1	**P**	**P**	**P**		**P**	**P**	**Q**	**Q**	…	**Q**	…	**Q**	**Q**	**Q**	**Q**	$p^x q^{n-x}$
2	**P**	**P**	**P**		**P**	**Q**	**P**	**Q**	…	**Q**	…	**Q**	**Q**	**Q**	**Q**	$p^x q^{n-x}$
3	**P**	**P**	**P**	…	**Q**	**P**	**P**	**Q**	…	**Q**	…	**Q**	**Q**	**Q**	**Q**	$p^x q^{n-x}$
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…
k	**Q**	**P**	**Q**	…	**P**	**P**	**Q**	**P**	…	**P**	…	**Q**	**P**	**P**	**P**	$p^x q^{n-x}$
…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…	…
N–2	**Q**	**Q**	**Q**	…	**Q**	**Q**	**Q**	**Q**	…	**Q**	…	**P**	**P**	**P**	**P**	$p^x q^{n-x}$
N–1	**Q**	**Q**	**Q**	…	**Q**	**Q**	**Q**	**Q**	…	**P**		**P**	**P**	**P**	**P**	$p^x q^{n-x}$
$N = \frac{n!}{x!\,(n-x)!}$												Total probability for x events P : $P_n(x) = \frac{n!}{x!\,(n-x)!}\ p^x q^{n-x}$				

Table 7.5 All the different results of a series of 6 trials Pr. = probability of a particular combination occurring

The 15 different ways in which $x = 4$ results P may occur in $n = 6$ trials, if we take into account the order in which the results P and Q appear																
	1	2	3	4	5	6	Pr.			1	2	3	4	5	6	Pr.
1	**P**	**P**	**P**	**P**	**Q**	**Q**	p^4q^2		9	**P**	**Q**	**P**	**Q**	**P**	**P**	p^4q^2
2	**P**	**P**	**P**	**Q**	**P**	**Q**	p^4q^2		10	**P**	**Q**	**Q**	**P**	**P**	**P**	p^4q^2
3	**P**	**P**	**P**	**Q**	**Q**	**P**	p^4q^2		11	**Q**	**P**	**P**	**P**	**P**	**Q**	p^4q^2
4	**P**	**P**	**Q**	**P**	**P**	**Q**	p^4q^2		12	**Q**	**P**	**P**	**P**	**Q**	**P**	p^4q^2
5	**P**	**P**	**Q**	**P**	**Q**	**P**	p^4q^2		13	**Q**	**P**	**P**	**Q**	**P**	**P**	p^4q^2
6	**P**	**P**	**Q**	**Q**	**P**	**P**	p^4q^2		14	**Q**	**P**	**Q**	**P**	**P**	**P**	p^4q^2
7	**P**	**Q**	**P**	**P**	**P**	**Q**	p^4q^2		15	**Q**	**Q**	**P**	**P**	**P**	**P**	p^4q^2
8	**P**	**Q**	**P**	**P**	**Q**	**P**	p^4q^2									

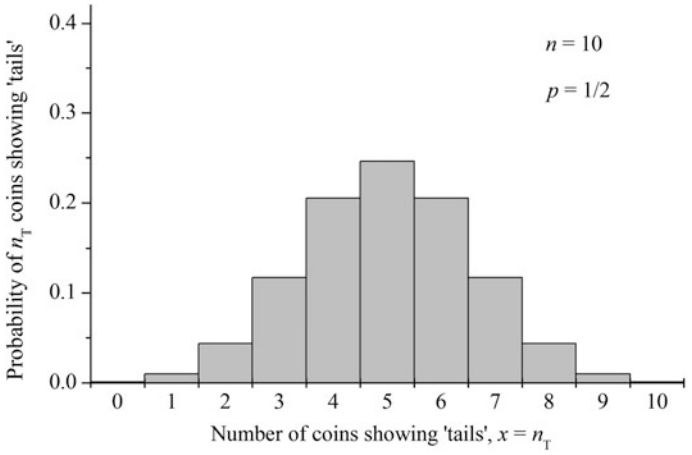


Fig. 7.4 Theoretical values of the probability of appearance of n_T coins showing 'tails' in the toss of 10 coins

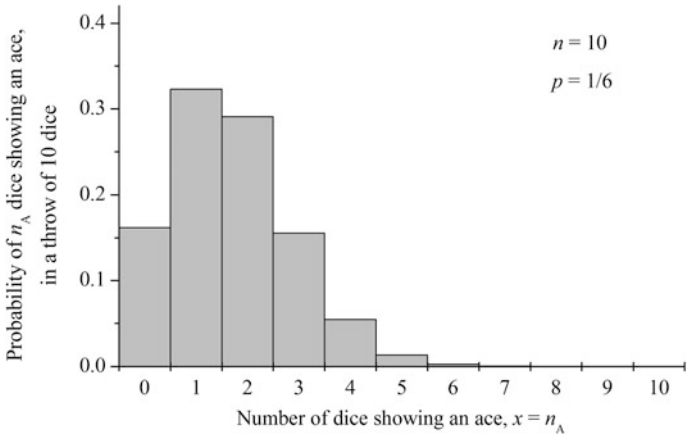


Fig. 7.5 Theoretical values for the probabilities of n_A dice showing an ace in the throw of 10 dice

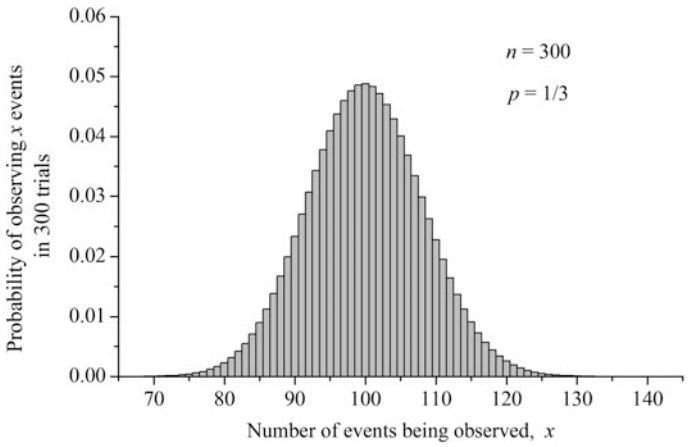


Fig. 7.6 The probabilities of observing x events, according to the binomial distribution, for $n = 300$ trials, when the probability of an event happening is $p = 1/3$

Example 7.1 [E]

Find the probability for the value $x = 5$ for the binomial distribution with $n = 10,\ p = 1/3$.

We select any empty cell, say A1, in which we want the result to be written, by left-clicking on it. Type in the cell: **= BINOM.DIST(5;10;1/3;FALSE)**

Pressing **ENTER** returns the required probability in cell A1: $P_{10}(5) = 0.13656$.

Example 7.2 [O]

Find the probability for the value $x = 5$ for the binomial distribution with $n = 10,\ p = 1/3$.

We select any empty cell, in which we want the result to be written, by left-clicking on it. Then,

Column > Set Column Values > Functions > Distributions > PDF > binopdf (x,nt,p)

Substituting x = 5, nt = 10 and p = 1/3 and pressing **OK**, we get for the required value $P_{10}(5) = 0.13656$.

Example 7.3 [P]

Find the probability for the value $x = 5$ for the binomial distribution with $n = 10,\ p = 1/3$.

```
# http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.
binom.html
from scipy.stats import binom

# Enter the values of n, p and x:
n = 10       # number of trials
p = 1/3      # probability of success
x = 5        # number of successes

# Evaluation:
P = binom.pmf(x, n, p)
print ("Probability of x successes, P = ", P)

# Result:
Probability of x successes, P = 0.136564548087
```

Example 7.4 [R]

Find the probability for the value $x = 5$ for the binomial distribution with $n = 10,\ p = 1/3$.

The function

```
dbinom(x, n, p)
```

gives the probability for the value x for the binomial distribution with total number of trials n and probability of success p. Therefore,

```
> dbinom(5, 10, 1/3)
[1] 0.1365645
```

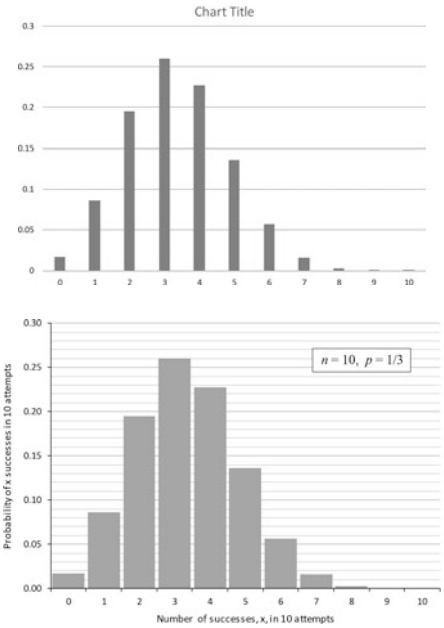
gives the probability for the value $x = 5$ for the binomial distribution with $n = 10$, and $p = 1/3$ as $P_{10}(5) = 0.13656$.

Example 7.5 [E]

Plot the histogram of probabilities for the binomial distribution with $n = 10,\ p = 1/3$.

Label column A as **x** and column B as **Probability of x successes**. Fill cells A4 to A14 with the numbers 0 to 10. Click cell B4 and type **= BINOM.DIST(A4;10;1/3; FALSE)**. Press **ENTER**. Fill down to cell B14.

Highlight cells A4 to B14. The choice **Insert > Recommended Charts** gives the **Clustered Column** chart of the selected values. Pressing **OK** produces the histogram shown in the first figure below.



By double-clicking on an axis opens the corresponding **Axis Options** window. In **Line** we select **Solid Line** and set the **Color** to **black**.

Pressing the ⊞ key at the top right hand side of the graph opens the **Chart Elements** window. We choose **Axis Titles**. The words Axis Title appear near the two axes. Left-clicking on one of them opens a window. We choose **Format Axis Title…** . In **Text Options > Text Fill**, set **Color Black**. In the title box type **Number of successes, x, in 10 attempts**. Repeat with the **Y axis** and type **Probability of x successes in 10 attempts**.

Pressing the ⊞ key at the top right hand side of the graph opens the **Chart Elements** window. We choose **Gridlines** and tick **Primary Minor Horizontal**. The minor horizontal gridlines appear now on the graph.

On the **Insert** tab, select **Text Box**. Draw a text box and type in it **n = 10 p = 1/3**. Drag the text box and place it on the graph.

The final result is shown in the lower figure, above.

Example 7.6 [O]

Plot the histogram of probabilities for the binomial distribution with $n = 10,\ p = 1/3$.

Set column A as the x variable and fill with the values 0, 1, 2, …10.
Highlight a column, say B, by left-clicking on its label. Then,

Column > Set column values > Functions > Distributions > PDF > binopdf (x,nt,p)

Here, it is binopdf(col(A),10,1/3). Press **OK**.

Cells B1 to B11 will be filled with the binomial probabilities $P_n(x) = \frac{n!}{x!(n-x)!}p^x(1-p)^{n-x}$.

Note that, instead of col(A) above, we may write (i−1).

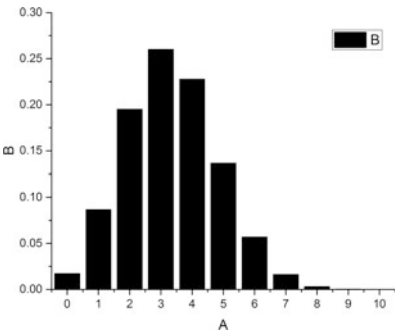
By summing the values in cells B1 to B11, we find that the sum of the probabilities equals unity, as expected.

We will plot a bar chart of these probabilities:

Highlight A1 to A11 and B1 to B11 by left-clicking on cell A1, holding the Shift key down and left-clicking on cell B11. Then,

Column > Column/Bar/Pie > Column

The bar chart of the $P_n(x)$ values is drawn (see figure).

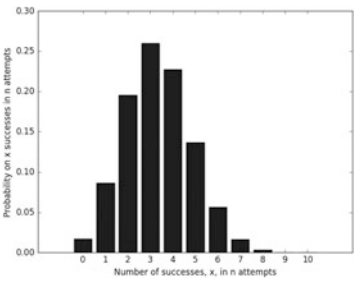


Example 7.7 [P]

Plot the histogram of probabilities for the binomial distribution with $n = 10,\ p = 1/3$.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import binom

# Enter the values of n and p:
n = 10      # number of trials
p = 1/3     # probability of success

# Create a vector x with the values 0 ... n:
x = np.array(range(0, n + 1, 1))

# Evaluation:
# The function binom.pmf(x, n, p) returns the probabilities for all x
#values of the
# binomial distribution with total number of attempts n and probability of
```

```
# success p. The probability mass function (pmf) is equivalent
#to R's dbinom function.
# We find:

dbinom = binom.pmf(x, n, p)

# Create a bar plot of these probabilities:
plt.bar(x, dbinom, align="center")
plt.xlabel('Number of successes, x, in n attempts')
plt.ylabel('Probability on x successes in n attempts')
plt.xticks(x)
plt.show()
```




```
# Show the values of the probabilities:
dbinom

# Results:
array([ 1.73415299e-02, 8.67076496e-02,
   1.95092212e-01, 2.60122949e-01, 2.27607580e-01,
   1.36564548e-01, 5.69018950e-02, 1.62576843e-02,
   3.04831581e-03, 3.38701756e-04, 1.69350878e-05
```

Example 7.8 [R]

Plot the histogram of probabilities for the binomial distribution with $n = 10,\ p = 1/3$.

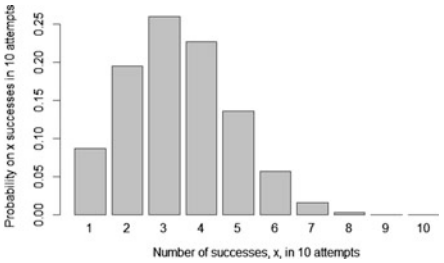
The function

```
dbinom(x, n, p)
```

returns the probabilities for all x values of the binomial distribution with total number of attempts n and probability of success p. In this case we find:

```
> dbinom(x,10,1/3)
[1] 8.670765e-02 1.950922e-01 2.601229e-01 2.276076e-01 1.365645e-01
5.690190e-02 1.625768e-02 3.048316e-03
[9] 3.387018e-04 1.693509e-05
```

```
#We create a bar plot of these probabilities:

> barplot(dbinom(x,10,1/3), names.arg=x, xlab="Number of successes, x,
in 10 attempts", ylab="Probability on x successes in 10 attempts")
```



Example 7.9 [E]

For the binomial distribution with $n = 10$ and $p = 0.4$, find the probability that the number of events, x, is: (a) $0 \leq x \leq 3$, (b) $4 \leq x \leq 6$ and (c) $7 \leq x \leq 10$.

Following the path **Formulas > More Functions > Statistical**, we choose the function **BINOM.DIST.RANGE**. The function **BINOM.DIST.RANGE**, has the arguments Trials = n, Probability_s = p, Number_s = x_1 and Number_s2 = x_2. It gives the probability that, given the probability of a success p and the number of trials n, the number of successes lies between the integers x_1 and x_2, both included.

(a) $p = 0.4$, $n = 10$, $x_1 = 0$ and $x_2 = 3$. The probability is 0.382281.
(b) $p = 0.4$, $n = 10$, $x_1 = 4$ and $x_2 = 6$. The probability is 0.562958.
(c) $p = 0.4$, $n = 10$, $x_1 = 7$ and $x_2 = 10$. The probability is 0.054762.

As expected, the sum of the probabilities is equal to 1.

Alternatively, the command **BINOM.DIST.RANGE(n,p,x_1,x_2)** may be used directly.

Example 7.10 [O]

For the binomial distribution with $n = 10$ and $p = 0.4$, find the probability that the number of events, x, is: (a) $0 \leq x \leq 3$, (b) $4 \leq x \leq 6$ and (c) $7 \leq x \leq 10$.

The function **binocdf(k,n,p)** gives, for a given number of trials n and probability p, the probability $P\{x \leq k\}$ that the number of successes x is less than or equal to k. For the three cases we are interested in, we must evaluate:

(a) $P\{x \leq 3\} = \text{binocdf}(3, 10, 0.4)$
(b) $P\{4 \leq x \leq 6\} = P\{x \leq 6\} - P\{x \leq 3\} = \text{binocdf}(6, 10, 0.4) - \text{binocdf}(3, 10, 0.4)$
(c) $P\{7 \leq x \leq 10\} = P\{x \leq 10\} - P\{x \leq 7\} = \text{binocdf}(10, 10, 0.4) - \text{binocdf}(6, 10, 0.4)$

To evaluate (a) we click on an empty cell, say A1, and follow the path:

Column > Set Column Values > Function > Distributions > CDF > binocdf (k,n,p)

We type **binocdf**$(\mathbf{3}, \mathbf{10}, \mathbf{0.4})$ and press **OK**. Cell A1 now contains the probability $P\{x \leq 3\} = 0.38228$.

For (b), in the window that opens we type **binocdf**$(\mathbf{6}, \mathbf{10}, \mathbf{0.4})$ $-$**binocdf**$(\mathbf{3}, \mathbf{10}, \mathbf{0.4})$ and press **OK**. Cell A1 contains $P\{4 \leq x \leq 6\} = 0.56296$. For (c) we type **binocdf**$(\mathbf{10}, \mathbf{10}, \mathbf{0.4})$ $-$ **binocdf**$(\mathbf{6}, \mathbf{10}, \mathbf{0.4})$. In cell A1 we find $P\{7 \leq x \leq 10\} = 0.05476$. The sum of the three probabilities is unity, as expected.

Example 7.11 [P]

For the binomial distribution with $n = 10$ and $p = 0.4$, find the probability that the number of events, x, is: (a) $0 \leq x \leq 3$, (b) $4 \leq x \leq 6$ and (c) $7 \leq x \leq 10$.

```
# http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.
binom.html
from scipy.stats import binom

# Enter the following values:
n = 10        # number of trials
p = 0.4       # probability of success

# Evaluation:
# (a)
x1 = 0        # minimum number of successes
x2 = 3        # maximum number of successes

P = binom.cdf(x2, n, p) - binom.cdf(x1, n, p) + binom.pmf(x1, n, p)
print ("(a) Probability of x between 0 and 3, P =", P)

# (b)
x1 = 4        # minimum number of successes
x2 = 6        # maximum number of successes

P = binom.cdf(x2, n, p) - binom.cdf(x1, n, p) + binom.pmf(x1, n, p)
print ("(b) Probability of x between 4 and 6, P =", P)

# (c)
x1 = 7        # minimum number of successes
x2 = 10       # maximum number of successes

P = binom.cdf(x2, n, p) - binom.cdf(x1, n, p) + binom.pmf(x1, n, p)
print ("(c) Probability of x between 7 and 10, P =", P)

# Results:
(a) Probability of x between 0 and 3, P = 0.3822806016
(b) Probability of x between 4 and 6, P = 0.5629575168
(c) Probability of x between 7 and 10, P = 0.0547618816
```

Example 7.12 [R]

For the binomial distribution with $n = 10$ and $p = 0.4$, find the probability that the number of events, x, is: (a) $0 \leq x \leq 3$, (b) $4 \leq x \leq 6$ and (c) $7 \leq x \leq 10$.

The function **pbinom(k, n, p)** returns, for a given number of trials n and probability p, the probability $P\{x \leq k\}$ that the number of successes x is less than or equal to k. For the three cases we are interested in, we must evaluate:

(a) $P\{x \leq 3\} = \text{pbinom}(3, 10, 0.4)$
(b) $P\{4 \leq x \leq 6\} = P\{x \leq 6\} - P\{x \leq 3\} = \text{pbinom}(6, 10, 0.4) - \text{pbinom}(3, 10, 0.4)$
(c) $P\{7 \leq x \leq 10\} = P\{x \leq 10\} - P\{x \leq 7\} = \text{pbinom}(10, 10, 0.4) - \text{pbinom}(6, 10, 0.4)$

The results are:

```
> pbinom(3, 10, 0.4)
[1] 0.3822806
> pbinom(6, 10, 0.4)-pbinom(3, 10, 0.4)
[1] 0.5629575
> pbinom(10, 10, 0.4)-pbinom(6, 10, 0.4)
[1] 0.05476188
```

We have found that (a) $P\{x \leq 3\} = 0.38228$, (b) $P\{4 \leq x \leq 6\} = 0.56296$ and (c) $P\{7 \leq x \leq 10\} = 0.05476$

7.2.1 The Normalization of the Binomial Distribution

The sum of all the probabilities predicted by the binomial distribution is, according to Eq. (7.3),

$$\sum_{x=0}^{n} P_n(x) = q^n + nq^{n-1}p + \frac{n(n-1)}{2}q^{n-2}p^2 + \ldots + p^n = (q+p)^n = 1, \quad (7.4)$$

since it is $q + p = 1$. The distribution is, therefore, normalized.

7.2.2 The Mean Value of x for the Binomial Distribution

The function

$$g(z) \equiv (q+pz)^n = q^n + nq^{n-1}pz + \frac{n(n-1)}{2}q^{n-2}p^2z^2 + \ldots + p^nz^n \quad (7.5)$$

has as coefficient of the power z^x the probability

$$P_n(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x}. \tag{7.6}$$

Thus, it is

$$g(z) = P_n(0) + zP_n(1) + z^2 P_n(2) + \ldots + z^n P_n(n) = \sum_{x=0}^{n} z^x P_n(x). \tag{7.7}$$

Differentiating with respect to z

$$\frac{\mathrm{d}g}{\mathrm{d}z} = np(q+pz)^{n-1} = P_n(1) + 2zP_n(2) + 3z^2 P_n(3) + \ldots + nz^{n-1} P_n(n) \tag{7.8}$$

and substituting $z = 1$ we have

$$np = P_n(1) + 2P_n(2) + 3P_n(3) + \ldots + xP_n(x) + \ldots + nP_n(n). \tag{7.9}$$

However,

$$P_n(1) + 2P_n(2) + 3P_n(3) + \ldots + xP_n(x) + \ldots + nP_n(n) = \sum_{x=0}^{n} xP_n(x) = \bar{x}. \tag{7.10}$$

Therefore,

$$\bar{x} = np \tag{7.11}$$

is the expected or the mean value of x for the binomial distribution.

7.2.3 The Standard Deviation of x from the Mean for a Binomial Distribution

From the relation

$$\begin{aligned} \sigma^2 &= \sum_{x=0}^{n} P_n(x)(x-\bar{x})^2 = \sum_{x=0}^{n} P_n(x)(x^2 - 2x\bar{x} + \bar{x}^2) \\ &= \sum_{x=0}^{n} P_n(x)\left[x(x-1) + x - 2x\bar{x} + \bar{x}^2\right] \end{aligned} \tag{7.12}$$

we have

$$\sigma^2 = \sum_{x=0}^{n} P_n(x)x(x-1) + (1-2\bar{x})\sum_{x=0}^{n} P_n(x)x + \bar{x}^2 \sum_{x=0}^{n} P_n(x) \tag{7.13}$$

and

$$\sigma^2 = \sum_{x=0}^{n} P_n(x)x(x-1) + \bar{x} - \bar{x}^2. \tag{7.14}$$

Differentiating $g(z)$ of Eq. (7.5) twice with respect to z, we get the relation

$$\begin{aligned} \frac{\mathrm{d}^2 g}{\mathrm{d}z^2} &= n(n-1)p^2(q+pz)^{n-2} \\ &= 1 \cdot 2P_n(2) + 2 \cdot 3zP_n(3) + \ldots + n(n-1)z^{n-2}P_n(n) \end{aligned} \tag{7.15}$$

which for $z = 1$ gives

$$n(n-1)p^2 = 1 \cdot 2P_n(2) + 2 \cdot 3P_n(3) + \ldots + n(n-1)P_n(n) = \sum_{x=0}^{n} P_n(x)x(x-1). \tag{7.16}$$

Substituting in Eq. (7.14), we find that

$$\sigma^2 = n(n-1)p^2 + \bar{x} - \bar{x}^2 = n(n-1)p^2 + np - n^2p^2 = np - np^2 = np(1-p) \tag{7.17}$$

and, finally, that

$$\sigma = \sqrt{np(1-p)} = \sqrt{npq} \tag{7.18}$$

is the standard deviation of x from the mean for the binomial distribution.

7.3 The Poisson Distribution

If the number of trials n is large, the evaluation of the probabilities becomes laborious. There is, however, an approximation which can give accurate results in the cases of large n (say $n > 100$) and $\mu = np$ relatively small ($p < 0.05$). Starting from the relation (7.2),

$$P_n(x) = \frac{n(n-1)\ldots(n-x+1)}{x!} p^x q^{n-x} \tag{7.19}$$

and using the notation $\mu = np$,

$$P_n(x) = \frac{n(n-1)\ldots(n-x+1)}{n^x x!}(np)^x(1-p)^{n-x} \tag{7.20}$$

$$= \frac{n(n-1)\ldots(n-x+1)}{n\cdot n\cdot\ldots\cdot n}\frac{\mu^x}{x!}(1-p)^{n-x} \tag{7.21}$$

$$= \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)\frac{\mu^x}{x!}(1-p)^{n-x} \tag{7.22}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)}{(1-p)^x}\frac{\mu^x}{x!}(1-p)^n. \tag{7.23}$$

Since, for small p, it is

$$\lim_{n\to\infty}\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{x-1}{n}\right)}{(1-p)^x} = 1 \tag{7.24}$$

and

$$\lim_{p\to 0}(1-p)^n = \lim_{p\to 0}\left[(1-p)^{-1/p}\right]^{-\mu} = \mathrm{e}^{-\mu}. \tag{7.25}$$

For $p \to 0$ and $n \to \infty$, with $\mu = np$ remaining finite, we have

$$\lim_{n\to\infty} P_n(x) = \frac{\mu^x}{x!}\mathrm{e}^{-\mu}. \tag{7.26}$$

The conclusion is: If the probability p of an event happening is very small and the number of trials n is very large, so that the product $\mu = np$ is not negligible, the binomial distribution takes the form

$$P_\mu(x) = \frac{\mu^x}{x!}\mathrm{e}^{-\mu}, \tag{7.27}$$

which is known as the *Poisson distribution*. It was formulated in 1837 by the French mathematician Siméon Denis Poisson (1781–1840).

$P_\mu(x)$ gives the probability of observing x events, in a process described by the Poisson distribution, characterized by the parameter μ. The parameter μ is the only parameter appearing in the function for the Poisson distribution and has a very simple physical meaning: since p is the probability of an event happening and n is the number of trials, the product $\mu = np$ is the mean number of events expected to happen, as we will prove below. The Poisson distribution is important not only as an approximation to the binomial distribution, but also because it describes a large

number of physical phenomena and procedures of everyday life. The independent extraction of relation (7.27) may be based on certain assumptions. Let the values t_i be distributed on the t-axis. If the following conditions are satisfied,

1. the probability for a certain number of points appearing in a given interval depends only on the length Δt of the interval,
2. if $P_{\geq 2}(\Delta t)$ is the probability that 2 or more points are found in an interval of length Δt, then $P_{\geq 2}(\Delta t)/\Delta t \to 0$ as $\Delta t \to 0$,
3. if $P_1(\Delta t)$ is the probability that 1 point is found in an interval of length Δt, then, as $\Delta t \to 0$, it is $P_1(\Delta t)/\Delta t \to \Lambda$, a constant,

then it follows that the probability of x points to be found in an interval of length t is given by Poisson's law [1]

$$P_{(\Lambda t)}(x) = \frac{(\Lambda t)^x}{x!} \mathrm{e}^{-\Lambda t}. \tag{7.28}$$

The proof will be given in Chap. 8, in the context of studying the statistics of radioactivity.

Figure 7.7 shows the probability histogram of the Poisson distribution for the case in which $\mu = 10/6$. The histogram presents the solution to the problem we have already examined using the binomial distribution, in which 10 ten dice are thrown and we wish to know the probabilities that $x = n_A = 0, 1, \ldots, 10$ aces are shown.

Table 7.6 shows the comparison of the predictions of the Poisson distribution with those of the binomial distribution. Figure 7.7 for the Poisson distribution must be compared with Fig. 7.5 for the binomial distribution. The agreement is satisfactory in the region of maximum probability but not so good at the tails, for the small values of probability.

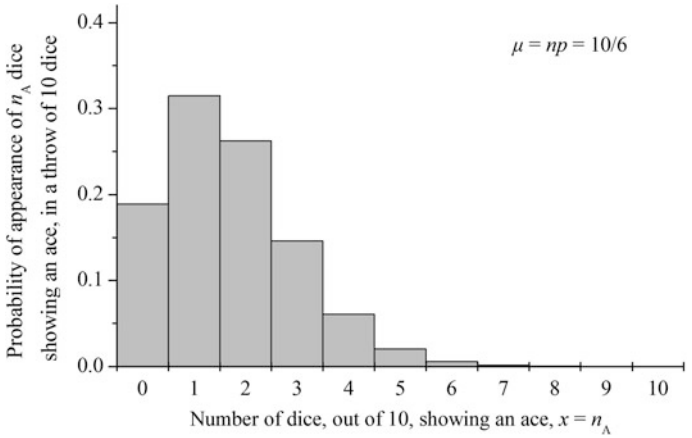


Fig. 7.7 The probabilities predicted by the Poisson distribution when the expected value is $\mu = 10/6$. The histogram gives the answers to the problem of 10 dice being thrown and the probabilities are required for the dice showing $x = n_A = 0, 1, \ldots, 10$ aces

Table 7.6 Comparison of the predictions of the binomial distribution for $n = 10$, $p = 1/6$, with those of the Poisson distribution for $\mu = np = 10/6$

x	Binomial distribution	Poisson distribution
0	0.1615	0.1889
1	0.3230	0.3148
2	0.2907	0.2623
3	0.1550	0.1457
4	0.0543	0.0607
5	0.0130	0.0202
6	0.0022	0.0056
7	0.00025	0.00134
8	0.00002	0.00028

Example 7.13 [E]

Find the probability for the value $x = 1$ for the Poisson distribution with $\mu = 5/3$. Also the probability for x to be less than or equal to 1.

We highlight any empty cell, say A1, in which we want the result to be written, by left-clicking on it. Then,

Formulas > More Functions > Statistical > POISSON.DIST

In the window that opens, we substitute X = 1, Mean = 1.66667 and **Cumulative = FALSE** and press **OK**. We get for the required value $P_{5/3}(1) = 0.314793$.

For the cumulative probability that x is less than or equal to 1 (i.e. it is either 0 or 1) we substitute X = 1, Mean = 1.66667 and **Cumulative = TRUE** and press **OK**. We get for the required value $P_{5/3}(\leq 1) = 0.503668$.

Example 7.14 [O]

Find the probability for the value $x = 1$ for the Poisson distribution with $\mu = 5/3$. Also the probability for x to be less than or equal to 1.

To calculate the probability that x is equal to 1, we highlight any empty cell, in which we want the result to be written, by left-clicking on it. Then,

Column > Set column values > Functions > Distributions > PDF > poisspdf (x,lambda)

Substituting x = 1 and lambda = μ = 5/3 and pressing **OK**, we get for the required value $P_{5/3}(1) = 0.31479$.

For the cumulative probability that x is less than or equal to 1, we highlight any empty cell, in which we want the result to be written, by left-clicking on it. Then,

Column > Set column values > Functions > Distributions > CDF > poisscdf (x,lambda)

Substituting x = 1 and lambda = μ = 5/3 and pressing **OK**, we get for the required value $P_{5/3}(\leq 1) = 0.503669$.

Example 7.15 [P]

Find the probability for the value $x = 1$ for the Poisson distribution with $\mu = 5/3$. Also the probability for x to be less than or equal to 1.

```
# To find the probability for x = 1

# http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.
poisson.html

from scipy.stats import poisson

# Enter the value of the mean of the distribution:
mean = 5/3

# Enter the value of x at which the value of the density function is required:
#x = 1

# Evaluation:
# The value at x of the probability density function f(x) is:
print (poisson.pmf(x, mean))

# Result:
0.314792671396

# To find the cumulative probability for x ≤ 1

# http://docs.scipy.org/doc/scipy-0.16.0/reference/generated/scipy.
stats.poisson.html

from scipy.stats import poisson

# Enter the value of the mean of the distribution:
mean = 5/3

# Enter the value of the maximum value of x:
x = 1

# Evaluation:
# The probability P(x) that the number of successes is less than or equal to

# x is:
print (poisson.cdf(x, mean))

# Result:
0.503668274233
```

Example 7.16 [R]

Find the probability for the value $x = 1$ for the Poisson distribution with $\mu = 5/3$. Also the probability for x to be less than or equal to 1.

The probability for the value x for the Poisson distribution with given μ is given by the function: dpois(x,μ).

For $x = 1$:

```
> dpois(1,5/3)
[1] 0.3147927
```

and $P_{5/3}(1) = 0.3147927$

For the Poisson distribution with given μ, the probability for x to be less than or equal to q is given by the function: ppois(q, μ).

For $x = 1$:

```
> ppois(1, 5/3)
[1] 0.5036683
```

and $P_{5/3}(\leq 1) = 0.5036683$. This is also equal to

`dpois(0,5/3)+dpois(1,5/3). It is ppois(0,5/3) = 0.1888756`

Example 7.17 [E]

Plot the histogram of probabilities for the Poisson distribution with mean $\mu = 10/3$.

The function **= POISSON.DIST(x;m;FALSE)** returns the probability that x events occur when the expected mean of the Poisson distribution is m.

We highlight any empty cell, say A1, in which we type 0 and press **ENTER**. In cell A2 we type **A1 + 1** and press **ENTER**. We **Fill Down** to (say) cell A16. Column A now contains the integers from 0 to 15.

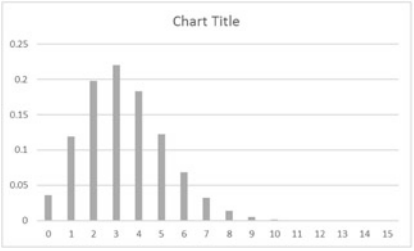
We click in cell B1 and type

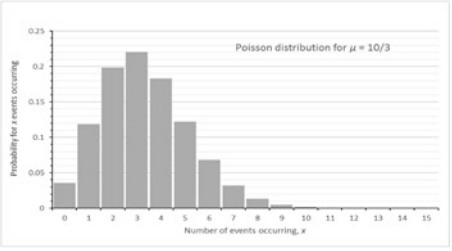
=POISSON.DIST(A1;3.33333333;FALSE)

This evaluates the expression $P_{\mu}(x) = \frac{\mu^x}{x!}e^{-\mu}$ for $\mu = 10/3$ and $x = 0$.

We **Fill Down** to cell B16. Column B now contains the probabilities $P_{10/3}(x)$ for x between 0 and 15.

We highlight columns A and B and in **Insert** and from the **Recommended Charts** we select **Clustered Column**. The graph shown in the figure on the left is drawn. We format the graph so that it finally looks like the graph on the right.





Example 7.18 [O]

Plot the histogram of probabilities for the Poisson distribution with mean $\mu = 10/3$.

Set column A as the x variable and fill with the values 0, 1, 2, …12.

Highlight a column, say B, by left-clicking on its label. Then,

Column > Set column values > Function > Distributions > PDF > poisspdf(x, lambda)

Here, x = col(A) and lambda = μ = 10/3. Press OK.

Cells B1 to B11 will be filled with the Poisson probabilities $P_\mu(x) = \frac{\mu^x}{x!}\mathrm{e}^{-\mu}$, with μ in place of λ.

Note that, instead of col(A) above, we may write **(i−1)**.

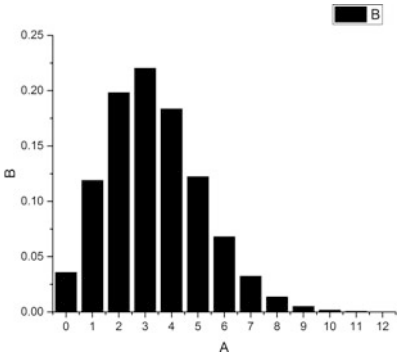
By summing the values in cells B1 to B11, we find that the sum of the probabilities equals unity, as expected.

We will plot a histogram of these probabilities:

Highlight cells A1 to A11 and B1 to B11 by left-clicking on cell A1, holding the Shift key down and left-clicking on cell B11. Then,

Column > Column/Bar/Pie > Column

The histogram of the $P_\mu(x)$ values is drawn (see figure).

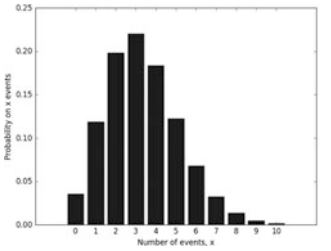


Example 7.19 [P]

Plot the histogram of probabilities for the Poisson distribution with mean $\mu = 10/3$.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.stats import poisson

# Enter the value of the mean of the distribution:
mean = 10/3

# Values of x (non-negative integers) to be used (0, 1, 2,... 11):
x = np.array(range(0, 11, 1))
```

```
# The function pmf(x, mean) r1eturns the probabilities for
# all x values of the Poisson distribution:
  dpois = poisson.pmf(x, mean)

# Create a bar plot of these probabilities:
plt.bar(x, dpois, align="center")
plt.xlabel('Number of events, x')
plt.ylabel('Probability on x events')
plt.xticks(x)
plt.show()
```



Example 7.20 [R]

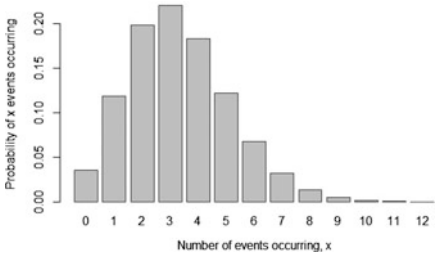
Plot the histogram of probabilities for the Poisson distribution with mean $\mu = 10/3$.

We calculate the Poisson probabilities for x between 0 and 12:

```
x<-c(0,1,2,3,4,5,6,7,8,9,10,11,12)
> dpois(x, 10/3)
[1] 0.0356739933 0.1189133112 0.1981888519 0.2202098355 0.1835081962
0.1223387975 0.0679659986 0.0323647612 0.0134853172 0.0049945619
0.0016648540 0.0005045012 0.0001401392
```

We draw a bar plot of the probabilities as a function of x:

```
> barplot(dpois(x, 10/3), names.arg=x, xlab="Number of events occurring,
x", ylab="Probability of x events occurring")
```



7.3.1 The Normalization of the Poisson Distribution

The Poisson distribution, Eq. (7.27), is normalized, since

$$\sum_{x=0}^{\infty} P_\mu(x) = \sum_{x=0}^{\infty} \frac{\mu^x}{x!} \mathrm{e}^{-\mu} = \mathrm{e}^{-\mu} \sum_{x=0}^{\infty} \frac{\mu^x}{x!} = \mathrm{e}^{-\mu} \mathrm{e}^{\mu} = 1. \tag{7.29}$$

7.3.2 The Mean Value of x for the Poisson Distribution

The mean value of x for the Poisson distribution is, according to Eq. (1.21), equal to

$$\bar{x} = \sum_{x=0}^{\infty} x P_\mu(x) = \mathrm{e}^{-\mu} \left(0 \frac{\mu^0}{0!} + 1 \frac{\mu}{1!} + 2 \frac{\mu^2}{2!} + 3 \frac{\mu^3}{3!} + \ldots \right) = \mathrm{e}^{-\mu} \mu \, \mathrm{e}^{\mu} = \mu \tag{7.30}$$

and, therefore,

$$\bar{x} = \mu. \tag{7.31}$$

7.3.3 The Standard Deviation from the Mean of x for the Poisson Distribution

The standard deviation of x from the mean for the Poisson distribution may be found, by analogy to Eq. (4.12), from the relation

$$\begin{aligned} \sigma^2 &= \sum_{x=0}^{\infty} (x - \bar{x})^2 P_\mu(x) = \sum_{x=0}^{\infty} (x^2 - 2x\bar{x} + \bar{x}^2) P_\mu(x) \\ &= \sum_{x=0}^{\infty} x^2 P_\mu(x) - 2\bar{x} \sum_{x=0}^{\infty} x P_\mu(x) + \bar{x}^2 \sum_{x=0}^{\infty} P_\mu(x) \end{aligned} \tag{7.32}$$

or

$$\sigma^2 = \sum_{x=0}^{\infty} x^2 \frac{\mu^x}{x!} \mathrm{e}^{-\mu} - 2\bar{x}^2 + \bar{x}^2 = \sum_{x=0}^{\infty} x^2 \frac{\mu^x}{x!} \mathrm{e}^{-\mu} - \bar{x}^2. \tag{7.33}$$

However,

$$\sum_{x=0}^{\infty} x^2 \frac{\mu^x}{x!} e^{-\mu} = e^{-\mu}\left(0^2 \frac{\mu^0}{0!} + 1^2 \frac{\mu}{1!}\mu + 2^2 \frac{\mu^2}{2!} + 3^2 \frac{\mu^3}{3!} + 4^2 \frac{\mu^4}{4!} + \ldots\right)$$
$$= e^{-\mu}\mu \frac{d}{d\mu}\left(\mu + 2\frac{\mu^2}{2!} + 3\frac{\mu^3}{3!} + 4\frac{\mu^4}{4!} + \ldots\right) = e^{-\mu}\mu \frac{d}{d\mu}(\mu e^{\mu}) \tag{7.34}$$

and, finally,

$$\sum_{x=0}^{\infty} x^2 \frac{\mu^x}{x!} e^{-\mu} = e^{-\mu}\mu\,(1+\mu)\, e^{\mu} = \mu + \mu^2. \tag{7.35}$$

Substituting in Eq. (7.33), with $\bar{x} = \mu$, we have

$$\sigma^2 = \mu \quad \text{and} \quad \sigma = \sqrt{\mu} \tag{7.36}$$

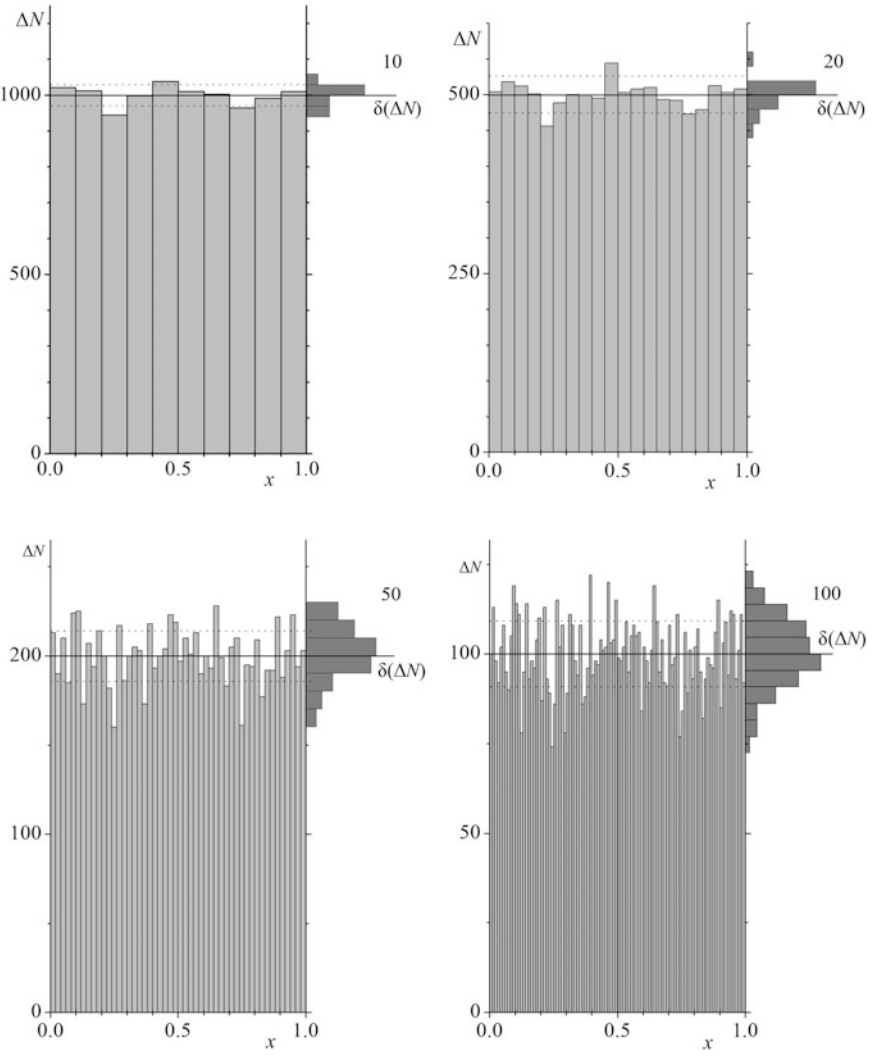
as the standard deviation of x from the mean for the Poisson distribution.

Example 7.21

The Poisson distribution in a histogram of random numbers.

The first 50 000 of the decimal digits of π are divided into groups of 5 and divided by 100 000, so that we have 10 000 five-digit numbers in the range [0, 1).

In the first histogram of the figure below, the numbers are divided into 10 classes. Each class is expected to consist of $\overline{\Delta N} = 1000$ numbers. The expected fluctuation in this number is, according to the Poisson distribution, $\delta(\overline{\Delta N}) = \sqrt{\overline{\Delta N}} = \sqrt{1000} = 32$ or 3.2%. Also drawn in the histogram are straight lines at the values $\overline{\Delta N} = 1000$ and $\overline{\Delta N} \pm \sqrt{\overline{\Delta N}} = 1000 \pm 32$ (dashed lines). It is seen that the numbers ΔN vary around the mean value and that most of them differ from it by less than one standard deviation from the mean, $\delta(\overline{\Delta N})$. At the upper right end of the histogram, the histogram of the values ΔN was drawn. Their distribution is better seen in this histogram.



In the second histogram of the figure, the random numbers are divided into 20 classes. Each class is expected to consist of $\overline{\Delta N} = 500$ numbers. The expected fluctuation in this number is, according to the Poisson distribution, $\delta(\overline{\Delta N}) = \sqrt{\overline{\Delta N}} = \sqrt{500} = 22.4$ or 4.5%. Also drawn in the histogram are straight lines at the values $\overline{\Delta N} = 500$ and $\overline{\Delta N} \pm \sqrt{\overline{\Delta N}} = 500 \pm 22$ (dashed lines).

In the third histogram of the figure, the random numbers are divided into 50 classes. Each class is expected to consist of $\overline{\Delta N} = 200$ numbers. The expected fluctuation in this number is, according to the Poisson distribution, $\delta(\overline{\Delta N}) =$

$\sqrt{\overline{\Delta N}} = \sqrt{200} = 14$ or 7%. Also drawn in the histogram are straight lines at the values $\overline{\Delta N} = 200$ and $\overline{\Delta N} \pm \sqrt{\overline{\Delta N}} = 200 \pm 14$ (dashed lines).

In the fourth histogram of the figure, the random numbers are divided into 100 classes. Each class is expected to consist of $\overline{\Delta N} = 100$ numbers. The expected fluctuation in this number is, according to the Poisson distribution, $\delta(\overline{\Delta N}) = \sqrt{\overline{\Delta N}} = \sqrt{100} = 10$ or 10%. Also drawn in the histogram are straight lines at the values $\overline{\Delta N} = 100$ and $\overline{\Delta N} \pm \sqrt{\overline{\Delta N}} = 100 \pm 10$ (dashed lines).

It is seen that the random numbers are distributed in the manner predicted by the Poisson distribution. This also partly supports the assumption originally made, namely that the decimal digits of π are randomly distributed!

Example 7.22

The corpuscular nature of light and the twinkling of stars.

John Michel (known for introducing to cosmology the concept of a black hole), argued in the 18th century that, given the corpuscular nature of light, the rate of arrival of these particles (photons) from a star to the eye of the observer must exhibit statistical fluctuations, which must be visible in faint stars. The twinkling (astronomical scintillation) of the faint stars must be due, at least in part, to this phenomenon.

The human eye is a very sensitive instrument. It responds to variations of light in time intervals of the order of $\tau = 1/20$ s (*afterimage*) and may detect a light source if it receives from the source only a few photons in this time interval.

A star of magnitude $V = 6$, which is just visible to the naked eye of most people, emits so much energy in the visible spectrum that at the Earth it causes an energy flux density of $\phi = 1 \times 10^{-8}$ erg/cm^2 s, approximately. With a mean energy of 5×10^{-12} erg for each visible photon, this energy flux is equivalent to a flux of 2000 photons per square centimeter and per second. If the area of the eye's pupil is about 0.5 cm^2, then, on the average, 1000 photons enter the eye per second. During the 'time constant' τ, the number of photons expected to enter the eye is $x = 50$ whose standard deviation is $\sigma = 7$. We see that $\sigma/x = 0.14$.

A fluctuation of the order of 14% is expected in the photon flux, as detected by the eye. This would clearly be detectable. For observation from inside the atmosphere most of the scintillation is due to air currents, which change the density and the index of the refraction of the atmosphere with the result that photons are scattered out of their path to the eye. This is supported by the fact that twinkling is more intense when the stars are low in the sky and their light has to travel a greater thickness of it. We must conclude that only a part of the twinkling of the stars is due to its corpuscular nature. Naturally, in observations outside the atmosphere, in space, star twinkling must be due entirely to the statistical variation of the number of photons arriving to the eye, as suggested by Michel.

7.4 The Normal or Gaussian Distribution

We have seen how the binomial distribution takes the form of a Poisson distribution when the number of trials n is large while the product np is not large, because the probability p has a low value. The binomial distribution takes a particularly useful form when the number of trials is large (say $n > 30$) and the probability p is not very small ($p > 0.05$). This approximation is known as *Laplace's approximation* and is satisfactory for $np > 5$ when it is $p \leq \frac{1}{2}$ and for $n(1-p) > 5$ when $p \geq \frac{1}{2}$. The approximation is based on the known formula of Stirling, which gives an asymptotic approximation for the factorial of a number, when this number is large,

$$n! \sim \sqrt{2\pi n}\, \mathrm{e}^{-n} n^n. \tag{7.37}$$

The symbol $\sim$ means that the ratio of the two quantities tends to unity, while their difference does not tend to zero.

Making use of this approximation in the expression of the binomial distribution

$$P_n(x) = \frac{n!}{x!(n-x)!} p^x q^{n-x} \tag{7.38}$$

which gives the probability of x events occurring in n trials, we have the approximate relation

$$P_n(x) \approx \left(\frac{np}{x}\right)^x \left(\frac{nq}{n-x}\right)^{n-x} \sqrt{\frac{n}{2\pi x(n-x)}} \tag{7.39}$$

or

$$AP_n(x) \approx \left(\frac{x}{np}\right)^{-x} \left(\frac{n-x}{nq}\right)^{-(n-x)} \quad \text{where} \quad A \equiv \sqrt{\frac{2\pi x(n-x)}{n}}. \tag{7.40}$$

If we denote with

$$\delta = x - np \tag{7.41}$$

the deviation of the number of observed events x from its expected value, np, then, taking into account the fact that $p + q = 1$, we also have

$$x = np + \delta \quad n - x = nq - \delta. \tag{7.42}$$

Therefore,

$$AP_n(x) \approx \left(1+\frac{\delta}{np}\right)^{-(np+\delta)}\left(1-\frac{\delta}{nq}\right)^{-(nq-\delta)}, \tag{7.43}$$

and

$$\ln[AP_n(x)] \approx -(np+\delta)\ln\left(1+\frac{\delta}{np}\right)-(nq-\delta)\ln\left(1-\frac{\delta}{nq}\right). \tag{7.44}$$

If we assume that it is

$$\left|\frac{\delta}{np}\right|<1 \quad \text{and} \quad \left|\frac{\delta}{nq}\right|<1, \tag{7.45}$$

we may expand the logarithms in powers of $\frac{\delta}{np}$ and $\frac{\delta}{nq}$:

$$\ln\left(1+\frac{\delta}{np}\right)=\frac{\delta}{np}-\frac{\delta^2}{2n^2p^2}+\frac{\delta^3}{3n^3p^3}+\ldots \tag{7.46}$$

and

$$\ln\left(1-\frac{\delta}{nq}\right)=-\frac{\delta}{nq}-\frac{\delta^2}{2n^2q^2}-\frac{\delta^3}{3n^3q^3}+\ldots. \tag{7.47}$$

Thus, we find that it is

$$\ln[AP_n(x)] \approx -\frac{\delta^2}{2npq}-\frac{\delta^3(p^2-q^2)}{2\cdot 3n^2p^2q^2}-\frac{\delta^4(p^3+q^3)}{3\cdot 4n^3p^3q^3}-\ldots. \tag{7.48}$$

If $\delta \ll npq$ so that we may neglect all the terms except the first, we have, approximately,

$$\ln[AP_n(x)] = \ln[P_n(x)] + \ln A \approx -\frac{\delta^2}{2npq}. \tag{7.49}$$

With the same assumptions we have

$$A=\sqrt{\frac{2\pi x(n-x)}{n}}=\sqrt{2\pi npq\left(1+\frac{\delta}{np}\right)\left(1-\frac{\delta}{nq}\right)} \approx \sqrt{2\pi npq}. \tag{7.50}$$

Therefore,

$$P_n(x) \approx \frac{1}{\sqrt{2\pi\, npq}} \mathrm{e}^{-\delta^2/2npq}. \tag{7.51}$$

Writing

$$\mu = np, \quad \sigma = \sqrt{npq} \tag{7.52}$$

and because it is

$$\delta = x - np = x - \mu, \tag{7.53}$$

we finally have

$$P_n(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-(x-\mu)^2/2\sigma^2} \tag{7.54}$$

as the probability of observing x events when the expected number is μ.

If we consider x to be a continuous variable, then, we may say that $P_n(x) = \Delta P(x)$ is the probability of observing a number of events between $x - \frac{1}{2}\Delta x = x - \frac{1}{2}$ and $x + \frac{1}{2}\Delta x = x + \frac{1}{2}$. The probability density is, therefore,

$$f(x) = \frac{\Delta P(x)}{\Delta x} = \frac{P_n(x)}{1}, \tag{7.55}$$

or

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}, \tag{7.56}$$

This is the *Gaussian* or *normal distribution*. It was initially proved by de Moivre in 1733, when he was studying the problem of coin tossing. Laplace and Gauss, independently of each other, derived the distribution again. The use of the distribution by Gauss in the study of the distribution of errors in astronomical observations, led to it being known as the *Gaussian distribution* or *Gauss' distribution for errors*. Its importance in Statistics is so great that it has been said that '*it plays in Statistics a role similar to that of the straight line in Geometry*' [2]. The distribution is important because:

1. It is the distribution towards which most probability density functions tend.
2. The distribution of values of most physical quantities are satisfactorily described by it.
3. The measurements having random errors are distributed, to a satisfactory degree, normally about the true value of the measured quantity.

Figure 7.8 shows the Gaussian or normal distribution. The curve has been made universal by drawing the values of $\sigma f(x)$ as a function of $x - \mu$ expressed in units of σ.

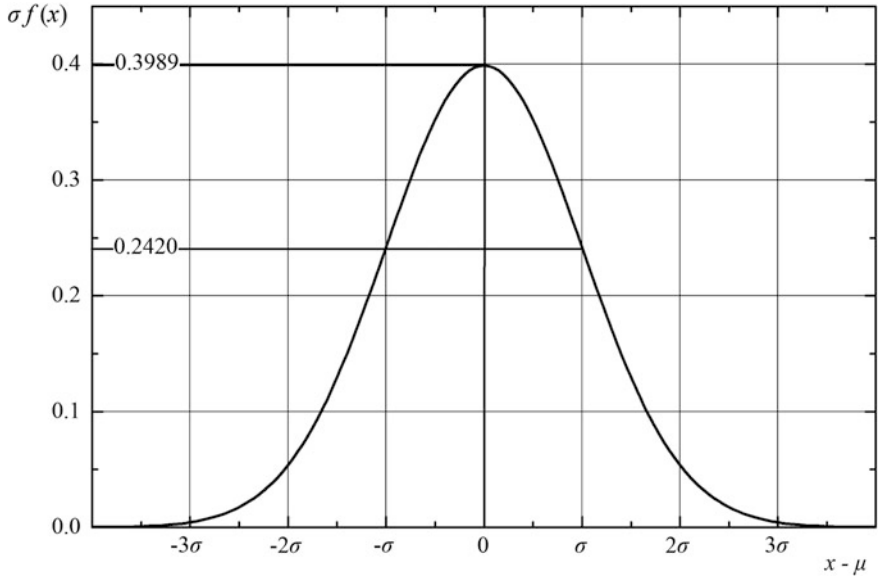


Fig. 7.8 The curve of the Gaussian or normal distribution

Example 7.23 [E]

Find the probability density at the value $x = 1.5$ for the normal distribution with $\mu = 0$ and $\sigma = 1$.

We highlight any empty cell, say A1, in which we want the result to be written, by left-clicking on it. Then,

Formulas > More Functions > Statistical > NORM.DIST

In the window that opens, we enter **x** = 1.5, **Mean** = 0, **Standard_dev** = 1 and **Cumulative = FALSE**. Pressing **OK**, we get for the required value $f(1.5) = 0.129518$.

Example 7.24 [O]

Find the probability density at the value $x = 1.5$ for the normal distribution with $\mu = 0$ and $\sigma = 1$.

We highlight any empty cell, in which we want the result to be written, by left-clicking on it. Then,

Column > Set column values > Functions > Distributions > PDF > normpdf (x,mu,sigma)

Substituting x = 1.5, mu = μ = 0, sigma = σ = 1 and pressing **OK**, we get for the required value $f(1.5) = 0.12952$.

Example 7.25 [P]

Find the probability density at the value $x = 1.5$ for the normal distribution with $\mu = 0$ and $\sigma = 1$.

```
# http://docs.scipy.org/doc/scipy/reference/generated/scipy.stats.
norm.html

from scipy.stats import norm

# Enter the following values:
mean = 0    # the mean
stdev = 1    # and standard deviation of the distribution

# Enter the value of x at which the value of the probability density
# function is required:
x = 1.5

# Evaluation:
# The value at x of the probability density function f(x) is:
print (norm.pdf(x, mean, stdev))

# Result:
0.129517595666
```

Example 7.26 [R]

Find the probability density at the value $x = 1.5$ for the normal distribution with $\mu = 0$ and $\sigma = 1$.

For a Normal distribution with mean μ and standard deviation σ, the probability density is given by the function

```
dnorm(x, μ, σ)
```

Here, it is:

```
> dnorm(1.5, 0, 1)
[1] 0.1295176
and f(1.5) = 0.12952.
```

Example 7.27 [E]

Plot the probability density function for the normal distribution with $\mu = 5$ and $\sigma = 1$.

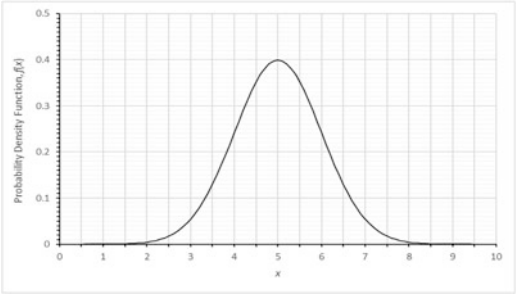
Set column A as the x variable and fill with the values 0–10, increasing in steps of 0.1. Cells B1 to B101 will be filled with values of the normal probability density function $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$. Here, $\frac{1}{\sqrt{2\pi}\,\sigma} = 0.39894$ and $\frac{1}{2\,\sigma^2} = 0.5$. Highlight cell B1 and type

=0.39894*EXP(−0.5*(A1-5)^2)

We press. **ENTER**. We **Fill Down** to B101. Column B now contains the values of $f(x)$.

We highlight the cells A1 to B101. In **Insert**, **Recommended Charts**, we select the line plot.

After suitable formatting, we obtain the graph shown.



Example 7.28 [O]

Plot the probability density function for the normal distribution with $\mu = 5$ and $\sigma = 1$.

Set column A as the x variable and fill with the values 0–10, increasing in steps of 0.01.

This is done as follows:

Highlight column A by left-clicking on its label. Then,

Column > Set column values

and enter **col(A) = (i−1)/100** for **i** between 1 and 1001.

Select a column, say B, by left-clicking on its label. Then,

Column > Set column values > Functions > Distributions > PDF > normpdf (x,mu,sigma)

Here, it is normpdf (col(A);5;1). Press **OK**.

Cells B1–B1001 will be filled with values of the normal probability density function $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$.

Note that, instead of **col(A)** above, we may write **(i-1)/100**.

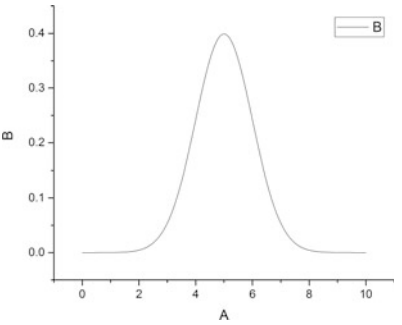
By summing the values in cells B1–B1001, we find the sum of 100, which, when multiplied by the interval $\delta x = 0.01$, gives unity as expected.

We will plot a graph of the probability density:

Highlight columns A and B by left-clicking on the label of A and, holding the Shift key down, left-clicking on the label of B. Then,

Plot > Line > Line

The graph of $f(x)$ is drawn (see figure).



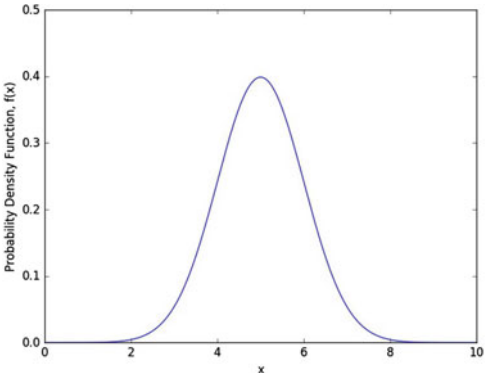
Example 7.29 [P]

Plot the probability density function for the normal distribution with $\mu = 5$ and $\sigma = 1$.

We use the scipy.stats sub-package that contains the definition of the probability density function for the normal distribution. We create a vector x of 1000 values between 0 and 10 and calculate PDF for the normal distribution with $\mu = 5$ and $\sigma = 1$ into the vector y.

```
from scipy.stats import norm
import numpy as np
import matplotlib.pyplot as plt
mean = 5
stdev = 1
x = np.linspace(0, 10, 1000)
y = norm.pdf(x, mean, stdev)
plt.plot(x, y, '-')
plt.xlim(0, 10)
plt.ylim(0, 0.5)
plt.xlabel("x")
plt.ylabel("Probability Density Function, f(x)")
plt.show()
```

The result is shown here:



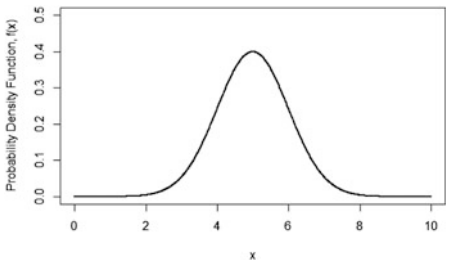
Example 7.30 [R]

Plot the probability density function for the normal distribution with $\mu = 5$ and $\sigma = 1$.

We create a vector of x values between 0 and 10, in steps of 0.01. Then, for these values, we plot the probability density function for the normal distribution with $\mu = 5$ and $\sigma = 1$.

```
> x <- seq(0,10,by=0.01)
> plot(x, dnorm(x,5,1), pch=20, cex=0.5, xlab="x", ylab=
"Probability Density Function, f(x)", xlim=c(0, 10), ylim=c(0, 0.5))
```

The resulting curve is shown in the figure below.



More examples related to the Gaussian distribution were presented in Chap. 4 (Examples 4.17–4.19 and 4.20–4.23).

7.4.1 The Normalization of the Gaussian Distribution

The Gaussian probability density, in the form of Eq. (7.56), is normalized, because

$$\int_{-\infty}^{+\infty} f(x)\mathrm{d}x = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}\mathrm{d}x = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-z^2}\mathrm{d}z = 1 \qquad (7.57)$$

where, as is customary, the limits of integration are taken to be $-\infty$ and $+\infty$, even when x cannot take negative values. This does not create a problem if it is $\sigma \ll \mu$.

7.4.2 The Mean Value of x for the Gaussian Distribution

The mean value of x for $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\mathrm{e}^{-(x-\mu)^2/2\sigma^2}$ is:

$$\bar{x} = \int_{-\infty}^{+\infty} xf(x)\mathrm{d}x = \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} x\mathrm{e}^{-(x-\mu)^2/2\sigma^2}\mathrm{d}x \qquad (7.58)$$

$$\begin{aligned}\bar{x} &= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} (x-\mu+\mu)\mathrm{e}^{-(x-\mu)^2/2\sigma^2}\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} (x-\mu)\mathrm{e}^{-(x-\mu)^2/2\sigma^2}\mathrm{d}x + \frac{\mu}{\sqrt{2\pi}\,\sigma}\int_{-\infty}^{+\infty} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}\mathrm{d}x \\ &= 0 + \frac{\mu}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \mathrm{e}^{-t^2/2}dt = \frac{\mu}{\sqrt{2\pi}}\sqrt{2\pi} = \mu.\end{aligned} \qquad (7.59)$$

The mean value of x for the Gaussian distribution is, therefore, $\bar{x} = \mu$.

7.4.3 The Standard Deviation of x from the Mean for the Gaussian Distribution

The standard deviation of x from the mean for $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\mathrm{e}^{-(x-\mu)^2/2\sigma^2}$ is found using the formula:

$$s_x^2 = \int_{-\infty}^{+\infty} (x - \bar{x})^2 f(x)\mathrm{d}x = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} (x - \mu)^2 \mathrm{e}^{-(x-\mu)^2/2\sigma^2}\mathrm{d}x \tag{7.60}$$

$$s_x^2 = \frac{2\sigma^2\sqrt{2}\,\sigma}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{+\infty} t^2 \mathrm{e}^{-t^2}\mathrm{d}t = \sigma^2. \tag{7.61}$$

The standard deviation of x from the mean for the Gaussian distribution is, therefore, equal to σ.

7.4.4 Testing Whether a Set of Data Has a Gaussian Distribution

Given a set of values with their frequencies, we may test whether they can be described by a Gaussian distribution by using special probability graph paper (Fig. 7.9). The ordinate scale of this graph paper is such that the cumulative frequency of the data gives a straight line when plotted using this paper, if it is described by a Gaussian distribution (Fig. 7.8). Given the values x_r and their respective relative frequencies f_r, we evaluate the cumulative relative frequency $F_r = f_1 + f_2 + \ldots + f_r$ up to each value (see Table 7.7). Potting F_r as a function of x_r on the special probability graph paper should give a straight line if the x_r are normally distributed.

What we are in fact doing is to plot the data on graph paper that linearizes the curve of the integral of the normal curve [or the function $\Phi(x)$ of Sect. 4.4.2]. From the straight line obtained, we find μ and σ. Since it is $F(\mu) = 0.5$, the intersection of the straight line of the distribution with the line $F(x) = 0.5$ gives μ. The intersections of the straight line with the lines $F(x = \mu - \sigma) = 0.158$ and $F(x = \mu + \sigma) = 0.842$ give σ. Example 7.31 demonstrates the method.

Table 7.7 The cumulative relative frequencies for testing whether a distribution is Gaussian

r	1	2	…	r	…	n
x_r	x_1	x_2	…	x_r	…	x_n
f_r	f_1	f_2	…	f_r	…	f_n
F_r	f_1	$f_1 + f_2$	…	$f_1 + f_2 + \ldots + f_r$	…	$f_1 + f_2 + \ldots + f_n = 1$

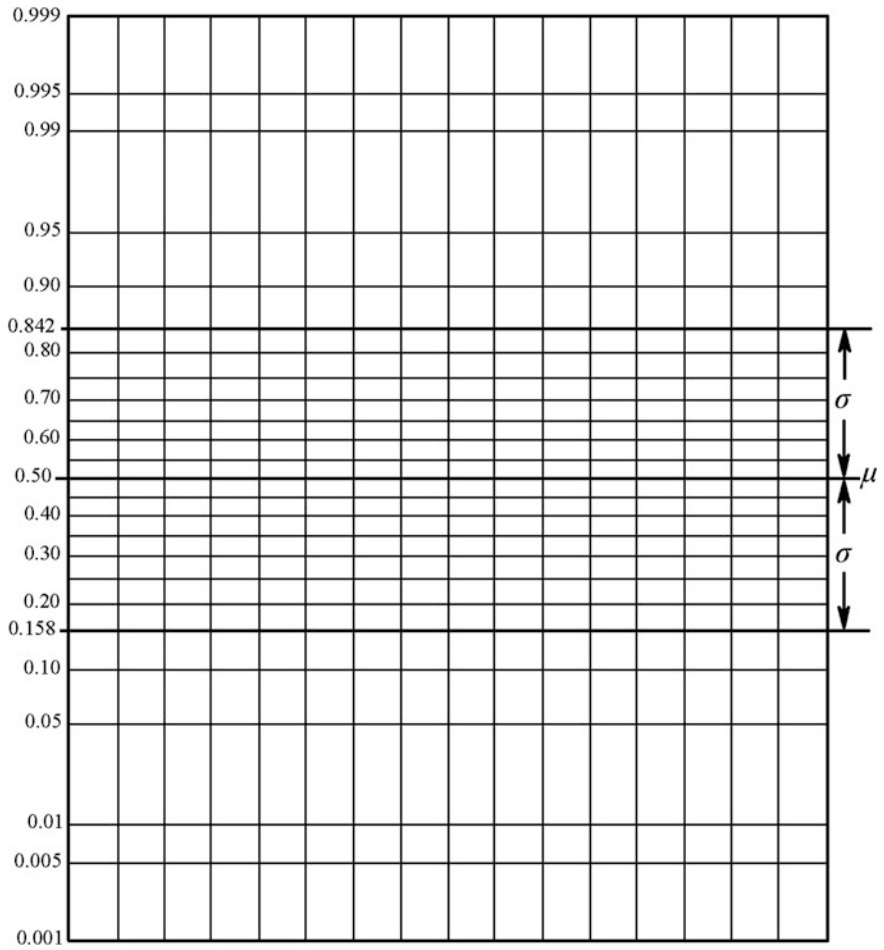


Fig. 7.9 Probability graph paper for testing whether a set of data are described by a Gaussian distribution

Example 7.31

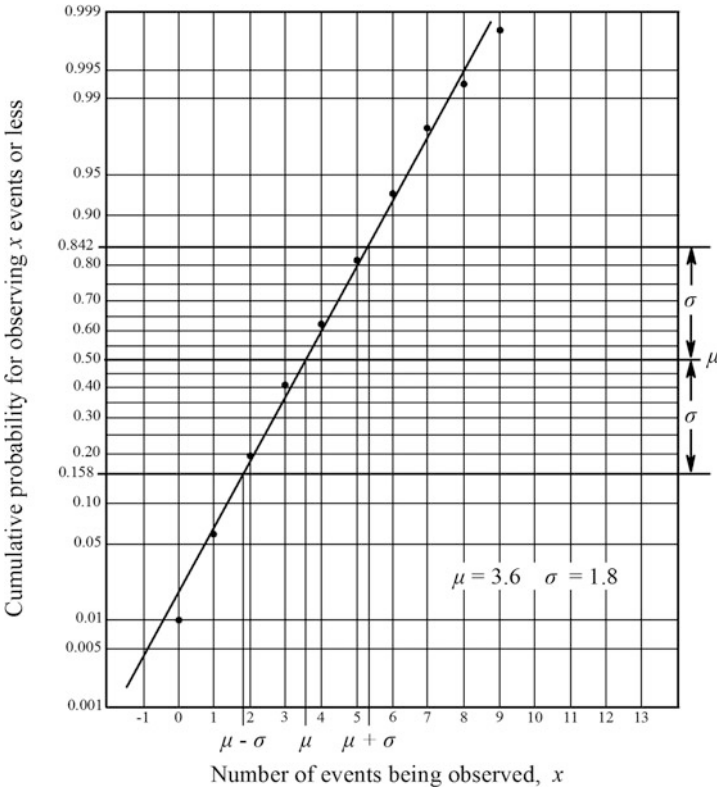
Evaluate the probabilities given by the binomial distribution for $n = 16$ and $p = 1/4$. Test whether these can be satisfactorily be represented by a Gaussian and find its parameters.

For $n = 16$ and $p = 1/4$ the binomial distribution gives the relative probabilities shown in the second row of the table given below. The cumulative probabilities F_r are evaluated.

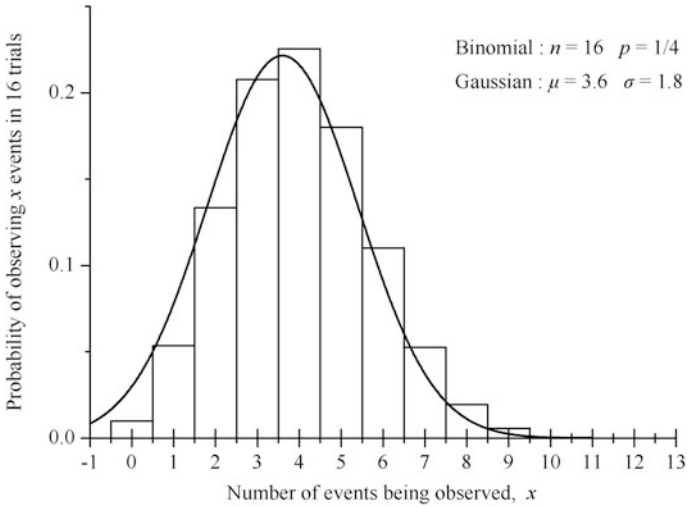
r	0	1	2	3	4	5	6
f_r	0.010 023	0.053 454	0.133 635	0.207 877	0.225 199	0.180 160	0.110 098
F_r	0.010 023	0.063 476	0.197 111	0.404 988	0.630 188	0.810 348	0.920 446

r	7	8	9	10	11	12	13
f_r	0.052 427	0.019 660	0.005 825	0.001 359	0.000 247	0.000 034	0.000 003
F_r	0.972 873	0.992 533	0.998 358	0.999 717	0.999 964	0.999 998	1

F_r is plotted as a function of $x = r$ in the figure shown below. Here, obviously, the role of x is played by the discrete variable r.



It is seen that the points fall on a straight line to a satisfactory degree so that we may conclude that the binomial probabilities for $n = 16$ and $p = 1/4$ are, approximately, normally distributed. We find that $F(3.6) = 0.5$, $F(1.75) = 0.158$ and $F(5.35) = 0.842$. These give $\mu = 3.6$ and $\sigma = \frac{1}{2}(5.35 - 1.75) = 1.8$. The Gaussian curve with these parameters is plotted in the figure below, together with the histogram of the probabilities given by the binomial distribution. The agreement is seen to be quite good.



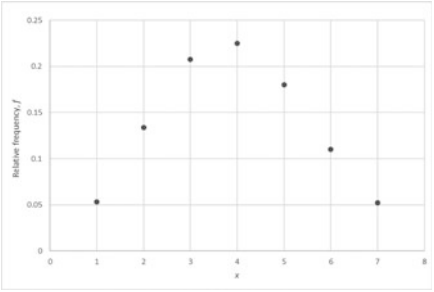
Example 7.32 [E]

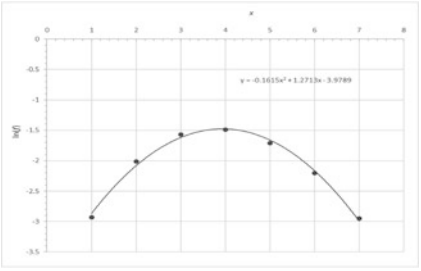
Find the best Gaussian fit to the data of Example 7.31.

Excel® is not able to fit a Gaussian curve $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}$ to a given set of data. We will achieve this by fitting a parabola to the logarithms of the relative frequencies f_r of the table of Example 7.31. To avoid giving too much weight to the low values of f_r, we will use only those values above 0.05.

We enter the values of r from 1 to 7 in the first 7 cells of column A. We enter the corresponding values of f_r from the table in the first 7 cells of column B.

We highlight columns cells A1 to B7. In **Insert** we choose the **scatter plot** for the data. The result is shown by the first figure below.





In column C we evaluate the natural logarithms of the values of f_r. We plot $y = \ln f_r$ as a function of $x = r$. The result is shown by the scatter plot in the last figure above. In **Trendline**, we choose the fit of a polynomial of 2nd order. This is also shown in the figure, as well as its equation,

$$y = -0.1615x^2 + 1.2713x - 3.9789.$$

This gives

$$f(x) = 0.2282\ \mathrm{e}^{-0.1615\ (x-3.94)^2}.$$

The fit is seen to be quite good.
The value of the mean is found to be $\mu = 3.94$
We have two ways of evaluating the standard deviation σ.
From $\frac{1}{\sqrt{2\pi}\,\sigma} = 0.2282$ we get $\sigma = 1.75$. From $\frac{1}{2\sigma^2} = 0.1615$ we get $\sigma = 1.76$.
We have, therefore, found that $\mu = 3.94$ and $\sigma = 1.76$.
These seem to be in reasonable agreement with the values $\mu = 3.6$ and $\sigma = 1.8$ found in Example 7.31.

Example 7.33 [O]

Find the best Gaussian fit to the data of Example 7.31.

We enter the values of r from the table in the first 14 cells of column A.

We enter the values of f_r from the table in the first 14 cells of column B.
We highlight columns A and B by left-clicking on label A and then, holding the Shift or Control keys down, by left-clicking on label B. Then,

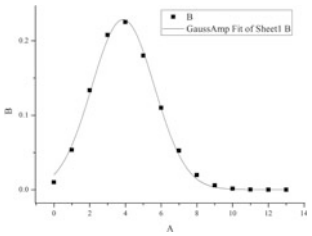
Analysis > Fitting> Nonlinear Curve Fit > Open Dialog…

Open **Origin Basic Functions**. Select **GaussAmp**.

The curve to be fitted to the data is $y = y_0 + Ae^{-\frac{(x-x_c)^2}{2w^2}}$.
Pressing **OK** gives the results:

$$y_0 = -0.000320 \pm 0.00266,\ x_c = 3.89799 \pm 0.03733$$
$$w = 1.7792 \pm 0.04629 \quad \text{and} \quad A = 0.22865 \pm 0.00462.$$

These results compare well with those of Example 7.25.
By double-clicking on the graph of the Gaussian, we have the figure shown.



Example 7.34 [P]

Find the best Gaussian fit to the data of Example 7.31.

We perform a non-linear least-squares fit, by first defining a Python function for $y = Ae^{-B(x-M)^2}$ and then using the `curve_fit` function of the `scipy.optimize` sub-package.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import curve_fit

x = np.array([0,1,2,3,4,5,6,7,8,9,10,11,12,13])
y = np.array([0.010023, 0.053454, 0.133635, 0.207877, 0.225199, 0.180160,
0.110098, 0.052427, 0.019660, 0.005825, 0.001359, 0.000247, 0.000034,
0.000003])

def y_func(x, A, B, M):
   return A*np.exp(-B*(x-M)**2)

popt, pcov = curve_fit(y_func, x, y, p0=(0.2, 0.05, 3))
```

The function curve_fit applies non-linear least squares optimization and returns the optimal values for the parameters in the array popt. The returned array pcov contains the estimated covariance of popt; we can calculate one standard deviation errors on the parameters using the command:

```
perr = np.sqrt(np.diag(pcov)).
```

We obtain the following results:

```
A = 0.22842 ± 0.00366
B = 0.16251 ± 0.00604
M = 3.87025 ± 0.03239
```

Example 7.35 [R]

Find the best Gaussian fit to the data of Example 7.31.

We perform a non-linear least-squares fit of the function $y = Ae^{-B(x-M)^2}$ to the data $y(x)$:

```
> x<- c(0,1,2,3,4,5,6,7,8,9,10,11,12,13)
> y<- c(0.010023, 0.053454, 0.133635, 0.207877, 0.225199, 0.180160,
! 0.110098, 0.052427, 0.019660, 0.005825, 0.001359, 0.000247, 0.000034,
0.000003)
> fm1<- nls(y~A*exp(-B*(x-M)^2), start=list(A=0.2, B=0.05, M=3))
# The starting values of the parameters were taken to be A=0.2, B=0.05, M=3.
# The results are:
> fm1
Nonlinear regression model
 model: y ~ A * exp(-B * (x - M)^2)
  data: parent.frame()
   A    B    M
0.2284 0.1625 3.8702
residual sum-of-squares: 0.0003035

Number of iterations to convergence: 8
Achieved convergence tolerance: 8.388e-07
```

The expression found is: $y = 0.2284\ \mathrm{e}^{-0.1625\ (x-3.8702)^2}$. This is in reasonable agreement with the results found above.

It should be mentioned that applying the Shapiro-Wilk normality test to these data results in:

```
> x<- c(0.010023, 0.053454, 0.133635, 0.207877, 0.225199, 0.180160,
0.110098, 0.052427, 0.019660, 0.005825, 0.001359, 0.000247, 0.000034,
0.000003)
```

```
> shapiro.test(x)

 Shapiro-Wilk normality test

data: x
W = 0.81345, p-value = 0.007356
```

Since the p-value is much smaller than the accepted limit of 0.1, the distribution of the data is concluded not to be normal.

Example 7.36 [O]

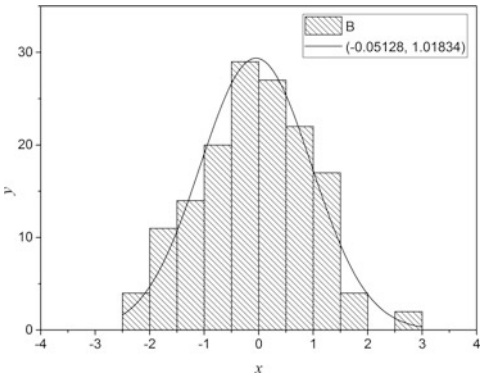
A number (150) of experimental results are given:

```
−0.36227, −0.72336, 0.25829, −0.63828, −0.72004, 0.43179, 2.76151,
−1.40337, 0.86314, −1.15092, 1.39139, 0.67595, 1.32693, 0.57761, −0.1579,
−0.55005, 0.35054, −1.02895, −1.00314, 0.81742, 0.46314, −0.21819,
−0.94712, −0.16706, −0.23703, −1.76221, 2.52329, −0.61768, 1.80028,
−0.20479, −0.07768, 0.68996, 0.40924, 0.83458, 0.37919, −0.1395, 1.05981,
−0.82333, 1.65204, −1.18495, −1.57975, −0.27824, 0.97032, −1.01357,
−0.78813, 1.04406, −1.16846, 0.43493, −0.74256, −0.49802, −1.7116,
1.49943, −1.01954, −2.20054, −1.68864, 0.97479, −0.34184, 1.24138,
−0.06232, −1.35798, 0.75658, 0.8391, −0.15863, 1.35233, −2.2624, −0.55066,
0.09311, 0.18774, 0.20881, −1.10017, 0.5552, 0.24363, −0.02832, −0.13308,
−2.26402, −1.83235, −0.35377, −1.87406, 0.14683, −0.41459, −0.95635,
−0.73978, 1.43312, 0.20272, −0.23276, 1.21751, 0.46918, 1.2172, −0.87965,
−0.48266, −0.18738, −0.63683, 1.13841, 1.49612, −0.49139, 1.92343,
0.27571, 0.5419, 0.43918, 0.1276, 0.04161, 0.72672, −1.56058, 0.97689,
0.35745, −1.25339, 1.08251, −0.66593, −0.49742, −0.57648, 1.22971,
0.83159, −0.13528, −0.44435, −0.81729, 0.93361, 0.39369, 0.47159, 0.62175,
−0.12871, −0.0014, 0.48777, −1.67127, 1.54825, 0.58673, 0.79441, 0.64989,
0.09791, 0.71776, −0.4637, −0.37636, −1.33489, −0.59725, −1.00688,
−1.8187, 1.37006, −0.26607, 0.19439, 0.88053, 0.27277, −1.4066, 0.10998,
−0.5615, −0.58129, 0.34726, −1.57048, 1.29247, 1.12755, −1.99822, −0.2519
```

Find the distribution that best fits the data.

We import the data and place them in column B. Then,
Statistics > Descriptive Statistics > Distribution Fit > Open Dialog…
In the window that opens, we select:
Distributions.
Distribution Type: Continuous Distribution.
Pressing **OK** returns the results that the best fit is Normal, with $\mu = -0.05128$ and $\sigma = 1.01834$.

The results also include a histogram of the data, together with the normal curve fitted. Double-clicking on the histogram, we obtain the graph presented in the figure.



The test for normality can also be performed as follows:

Statistics > Descriptive Statistics >Normality Test > Open Dialog…

On pressing **OK**, the results given are $\mu = -0.05128$ and $\sigma = 1.01834$, as before.

7.4.5 *The Gaussian Distribution and the Phenomenon of Diffusion*

We will now generalize the term 'event', to which we referred rather vaguely. If we are describing the tossing of a number of coins, which we consider to be an experiment, 'event' could be the showing of 'tails' by x coins. We could, however, examine another kind of process, such as, for example, the motion of a particle on a straight line. We assume that the particle makes random steps of constant length l and that the probability of the particle moving towards the positive direction is p and towards the negative direction it is q. Here, experiment would be the performance by the particle of n steps, in total, and a result would be x of these steps to be in the positive direction and $n - x$ in the negative. The particle will then have moved by $x - (n - x) = 2x - n$ steps in the positive direction and its position will be $X(n) = (2x - n)l$. Equation (7.54) gives, therefore, the probability for the particle to be, after n steps, at the position $X(n) = (2x - n)l$. Because it is

$$x - \mu = x - np = \frac{X}{2l} + \frac{n}{2} - np = \frac{1}{2l}[X - (2p - 1)nl], \tag{7.62}$$

the probability the particle being at the position $X(n)$ after n steps is, according to Eq. (7.54),

$$P_n(X) = \frac{1}{\sqrt{2\pi}\sqrt{npq}} \mathrm{e}^{-[X-(2p-1)nl]^2/8npql^2}. \tag{7.63}$$

Putting

$$\chi \equiv (2p-1)nl \quad \text{and} \quad s \equiv \sqrt{4npql} \tag{7.64}$$

we have

$$P_n(X) = \frac{2l}{\sqrt{2\pi}s} \mathrm{e}^{-(X-\chi)^2/2s^2}. \tag{7.65}$$

If we have initially at the origin N non-interacting particles moving according to the manner we described above, after each one of them has taken n steps, $NP_n(X)$ of these particles will be at the position $X(n)$. Now, if n is odd, $x-(n-x) = 2x-n$ will also be odd, while if n is even, $x-(n-x) = 2x-n$ will also be even. Therefore, neighboring particles will be at a distance of $2l$ from each other. The mean density of particles in the region of $X(n)$ will be, therefore, equal to $NP_n(X)/2l$ particles per unit length. The probability per unit length for a particle to be found in the region of $X(n)$ is, therefore, $P_n(X)/2l$. This is the probability density of the distribution of the particles on the straight line,

$$f(X) = \frac{1}{\sqrt{2\pi}\,s} \mathrm{e}^{-(X-\chi)^2/2s^2}. \tag{7.66}$$

As we have seen in Chap. 4, the quantities $\chi = (2p-1)nl$ and $s = \sqrt{4npq}\,l$, determine the position of the maximum and the width of the distribution, respectively. If the particles move at a constant rate of α steps per unit time, it will be $n = \alpha t$ and the quantity $\upsilon = \alpha l$ must be considered to be the speed of the particles. Then, it is

$$\chi(t) = (2p-1)\upsilon t \quad \text{and} \quad s(t) = \sqrt{4pql\upsilon t} \tag{7.67}$$

and the maximum of the particle's concentration will be at the position $\chi(t)$, while the width of the distribution varies as $s(t)$. The phenomenon described by Eq. (7.66) is a process of asymmetrical diffusion of particles in one dimension [3]. If it is $p = q = \frac{1}{2}$, then $\chi = 0$, $s(t) = \sqrt{l\upsilon t}$ and the distribution is symmetrical, with the maximum of the density remaining at the original position of the particles and the width of the distribution increasing with time in such a way that $s(t) \propto \sqrt{t}$ or, equivalently, $s \propto \sqrt{n}$.

Figure 7.10 shows the development with time of a distribution of particles diffusing symmetrically ($p = q = \frac{1}{2}$) in one dimension. The distributions at three values of the time, $t = l/\upsilon$, $t = 5l/\upsilon$ and $t = 25l/\upsilon$ are given. Figure 7.11 shows, for the same values of time, an asymmetrical diffusion ($p = \frac{2}{3}$, $q = \frac{1}{3}$). We remind

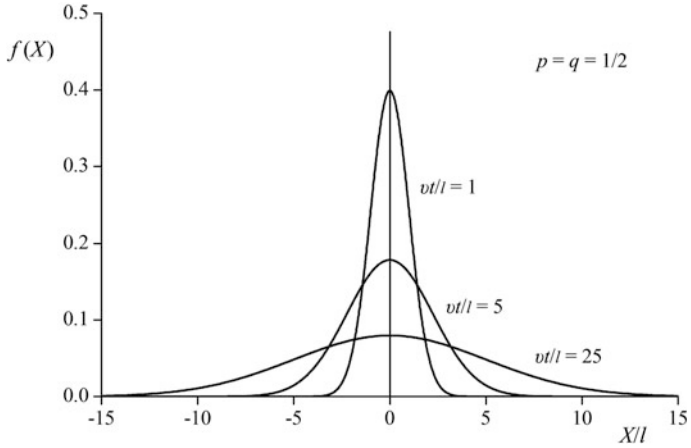


Fig. 7.10 Symmetrical diffusion ($p = q = \frac{1}{2}$) of particles in one dimension. Development with time. p is the probability for a step of length l towards positive X and q is the probability for a step of length l towards negative X

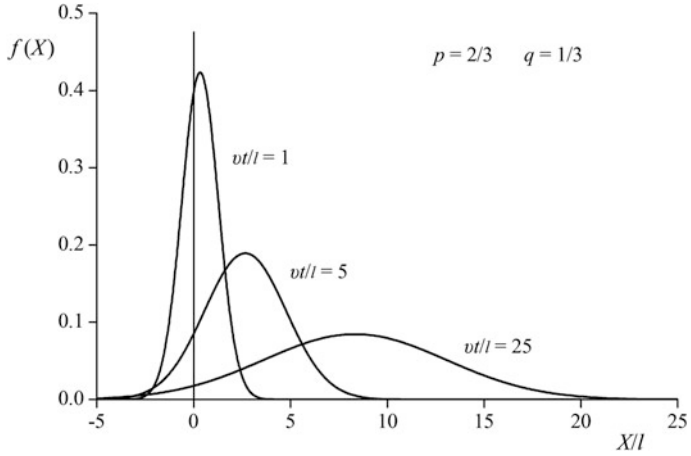


Fig. 7.11 Asymmetrical diffusion ($p = \frac{2}{3}$, $q = \frac{1}{3}$) of particles in one dimension. Development with time

the reader that p is the probability for a step towards positive values of X and q is the probability for a step towards negative values of X.

In general, it is possible that the probability density

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}, \tag{7.68}$$

under suitable conditions, describes the distribution of values of any quantity. We will see later that the Gaussian function describes, under certain assumptions, the distribution of errors or, equivalently, the distribution of the results of the measurements of a physical magnitude.

Programs
Excel
Ch. 07. *Excel—Binomial Distribution—Values and Histogram*
Ch. 07. *Excel—Binomial Distribution—Probability of x successes*
Ch. 07. *Excel—Binomial Distribution—Cumulative Probability*
Ch. 07. *Excel—Binomial Distribution—Probability of x between x1 and x2*
Ch. 07. *Excel—Binomial Distribution—Inverse*
Ch. 07. *Excel—Poisson Distribution—Values and Histogram*
Ch. 07. *Excel—Poisson Distribution—Probability Density Function*
Ch. 07. *Excel—Poisson Distribution—Cumulative Probability*
Ch. 07. *Excel—Poisson Distribution—Probability of x between x1 and x2*
Ch. 07. *Excel—Normal Distribution—Probability Density Function*
Ch. 07. *Excel—Normal Distribution—Cumulative Probability*
Ch. 07. *Excel—Normal Distribution—Probability of x between x1 and x2*
Ch. 07. *Excel—Normal Distribution—Inverse*
Origin
Ch. 07. *Origin—Binomial Distribution—Values and Histogram*
Ch. 07. *Origin—Binomial Distribution—Probability of x successes*
Ch. 07. *Origin—Binomial Distribution—Cumulative Probability*
Ch. 07. *Origin—Binomial Distribution—Probability of x between x1 and x2*
Ch. 07. *Origin—Poisson Distribution—Values and Histogram*
Ch. 07. *Origin—Poisson Distribution—Probability Density Function*
Ch. 07. *Origin—Poisson Distribution—Cumulative Probability*
Ch. 07. *Origin—Poisson Distribution—Probability of x between x1 and x2*
Ch. 07. *Origin—Normal Distribution—Probability Density Function*
Ch. 07. *Origin—Normal Distribution—Cumulative Probability*
Ch. 07. *Origin—Normal Distribution—Probability of x between x1 and x2*
Ch. 07. *Origin—Normal Distribution—Inverse*
Python
Ch. 07. *Python—Binomial Distribution—Values and Histogram*
Ch. 07. *Python—Binomial Distribution—Probability of x successes*
Ch. 07. *Python—Binomial Distribution—Cumulative Probability*
Ch. 07. *Python—Binomial Distribution—Probability of x between x1 and x2*
Ch. 07. *Python—Normal Distribution—Probability Density Function*
Ch. 07. *Python—Normal Distribution—Cumulative Probability*
Ch. 07. *Python—Normal Distribution—Probability of x between x1 and x2*
Ch. 07. *Python—Normal Distribution—Inverse*
Ch. 07. *Python—Poisson Distribution—Values and Histogram*
Ch. 07. *Python—Poisson Distribution—Probability Density Function*
Ch. 07. *Python—Poisson Distribution—Cumulative Probability*
Ch. 07. *Python—Poisson Distribution—Probability of x between x1 and x2*

(continued)

(continued)

Programs
R
Ch. 07. R—Binomial Distribution—Values and Histogram
Ch. 07. R—Binomial Distribution—Probability of x successes
Ch. 07. R—Binomial Distribution—Cumulative Probability
Ch. 07. R—Binomial Distribution—Probability of x between x1 and x2
Ch. 07. R—Binomial Distribution—Inverse
Ch. 07. R—Normal Distribution—Cumulative Probability
Ch. 07. R—Normal Distribution—Inverse
Ch. 07. R—Normal Distribution—Probability Density Function
Ch. 07. R—Normal Distribution—Probability of x between x1 and x2
Ch. 07. R—Poisson Distribution—Cumulative Probability
Ch. 07. R—Poisson Distribution—Probability Density Function
Ch. 07. R—Poisson Distribution—Probability of x between x1 and x2
Ch. 07. R—Poisson Distribution—Values and Histogram

Problems

7.1 Find the probabilities $P_n(x)$ of the binomial distribution for $p = \frac{1}{3}$ and $n = 6$. Evaluate from first principles the mean $\bar{x}$ and the standard deviation σ of the distribution and verify that they agree with the values $\bar{x} = np$ and $\sigma = \sqrt{np(1-p)}$.

7.2 **[E.O.P.R.]** Evaluate the probabilities of the binomial distribution for $n = 10$ and $p = 0.3$.

7.3 **[E.O.P.R.]** We throw 5 coins in the air and count how many of them fall showing 'tails'. (a) What are the probabilities for 0, 1, 2, 3, 4 or 5 coins to show 'tails'? (b) If this procedure is carried out 128 times, what is the expected number of cases with 0, 1, 2, 3, 4 or 5 coins showing 'tails'?

7.4 For a binomial distribution, the mean is $\bar{x} = 6$ and the standard deviation $\sigma = 2$. Find the values of n, p and q.

7.5 A book with 600 pages contains 200 typographical errors. What is the probability for a certain page to have 2 errors? What is the probability of a page having more than 1 error?

7.6 **[E.O.P.R.]** Find the probabilities for enough values of x for a Poisson distribution with $\mu = 3$, so that the mean $\bar{x}$ and the standard deviation σ are evaluated and verify the relation $\sigma = \sqrt{\bar{x}}$.

7.7 **[E.O.P.R.]** The following distribution is considered to be approximately Gaussian

x_r	25	26	27	28	29	30	31	32	33	34
n_r	1	5	17	49	85	52	25	11	4	1

Draw the histogram of the values and find their mean $\bar{x}$ and their standard deviation σ. Together with the histogram draw the curve $\frac{N}{\sqrt{2\pi}\sigma} e^{-(x-\bar{x})^2/2\sigma^2}$, where $N = \sum_r n_r$. Is there an agreement between the histogram and the curve?

7.8 **[E.O.P.R.]** Fifty students were asked to guess the length of a straight line to the nearest millimeter. The numbers n_r of students who gave each one of the answers, x_r, are given in the table below:

x_r(mm)	42	43	44	45	46	47	48	49	50
n_r	1	2	5	11	12	10	7	1	1

Determine the characteristic parameters, $\bar{x}$ and σ, of a Gaussian curve for these data and find the probability density function $f(x)$. Draw, together, the curve $50\, f(x)$ and the histogram of the values $n_r(x)$.

References

1. I.S. Sokolnikoff and R.M. Redheffer, *Mathematics of Physics and Modern Engineering* (McGraw-Hill Co. Inc., 1966). Ch. 9, Section 11
2. A.C. Aitken, *Statistical Mathematics* (Oliver and Boyd, 1952)
3. S. Chandrasekhar, 'Stochastic Problems in Physics and Astronomy', *Reviews of Modern Physics*, **15**, 1–89, 1943. Reprinted in N. Wax (ed.), *Selected Papers on Noise and Stochastic Processes* (Dover, 1954)

Chapter 8
The Statistics of Radioactivity

The laws of Statistics find applications in the phenomenon of radioactivity. The disintegration of a nucleus is a random event which is not affected by the history of the nucleus or the conditions external to the nucleus. In this chapter we will evaluate the probabilities that govern radioactive decay. The results we will derive will be of great importance to experimental practice.

8.1 The Behavior of Large Samples. The Law of Radioactivity

Soon after the discovery of radioactivity, it was found experimentally that the *activity* (rate of decay) of a sample decreases exponentially with time [1]. In 1905, E. von Schweidler proved theoretically the law of radioactivity, considering the process of nuclear decay as a purely statistical effect. The basic assumption he made is that the probability ΔP for a certain nucleus to decay during a sufficiently small interval of time Δt is proportional to this interval,

$$\Delta P = \lambda \Delta t, \tag{8.1}$$

with the coefficient of proportionality λ being characteristic of a certain kind of nucleus and mode of decay, independent of the nucleus' history or of any other influence from neighboring nuclei or the environment. The probability for a nucleus not to decay during a certain time interval $0 \leq t < \Delta t$ is

$$1 - \Delta P = 1 - \lambda \Delta t. \tag{8.2}$$

The probability the nucleus will not decay during the interval $\Delta t \leq t < 2\Delta t$ is exactly the same. The combined probability that the nucleus will not decay during the time interval $0 \leq t < 2\Delta t$ is, therefore,

C. Christodoulides and G. Christodoulides, *Analysis and Presentation of Experimental Results*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-53345-2_8

$$(1 - \Delta P)^2 = (1 - \lambda \Delta t)^2. \tag{8.3}$$

In general, the probability a nucleus will not decay during the time interval between $t = 0$ and $t = n\,\Delta t$ is

$$(1 - \Delta P)^n = (1 - \lambda \Delta t)^n = (1 - \lambda t/n)^n, \tag{8.4}$$

which, in the limit $\Delta t \to 0, \quad n \to \infty$, tends to $\mathrm{e}^{-\lambda t}$. Thus, if initially ($t = 0$) there were N_0 nuclei of the particular isotope, at time t the surviving nuclei will be

$$N(t) = N_0 \mathrm{e}^{-\lambda t}. \tag{8.5}$$

This is the law of radioactivity.

Alternatively, if at time t there exist $N(t)$ nuclei, in a time interval $\mathrm{d}t$ there will be $\lambda N(t)\,\mathrm{d}t$ decays of nuclei (we assume that the number of nuclei is large enough so that the function $N(t)$ may be considered to be continuous, as was done here, and $\mathrm{d}t$ is small compared to the duration of our measurement but large enough so that the difference between the real number of decays during $\mathrm{d}t$ from the theoretically expected would be negligible). Thus,

$$\mathrm{d}N = -\lambda N\,\mathrm{d}t \tag{8.6}$$

the solution of which is Eq. (8.5).

The constant λ is characteristic of the isotope for the particular mode of decay (if there are more than one) and is called *decay constant*. It is proved that the *mean lifetime* of the nuclei is equal to

$$\tau = \frac{1}{\lambda}. \tag{8.7}$$

The time needed for half of the initial nuclei to decay is equal to

$$\tau_{1/2} = \frac{\ln 2}{\lambda}. \tag{8.8}$$

and is called *half-life*.

The variation of the number of surviving nuclei with time is shown in Fig. 8.1.

The *activity* of a sample (number of decays per unit time), which is the magnitude measured directly, is

$$A \equiv -\frac{\mathrm{d}N}{\mathrm{d}t} = \lambda N(t) = \lambda N_0 \mathrm{e}^{-\lambda t} \tag{8.9}$$

and is seen to decrease exponentially with time, in agreement with experiment.

Figure 8.2 shows, as a function of time, the rate $R(t)$ at which the disintegrations of the nuclei of a radioactive sample are counted, by an experimental arrangement

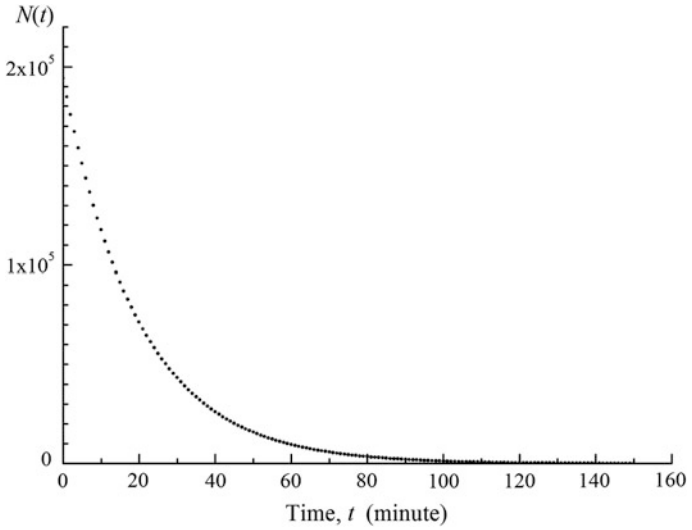


Fig. 8.1 The decay with time of the number of the nuclei of a radioactive sample. The points are at a time distance of one minute from each other. For this particular isotope it is $\tau = 20$ min

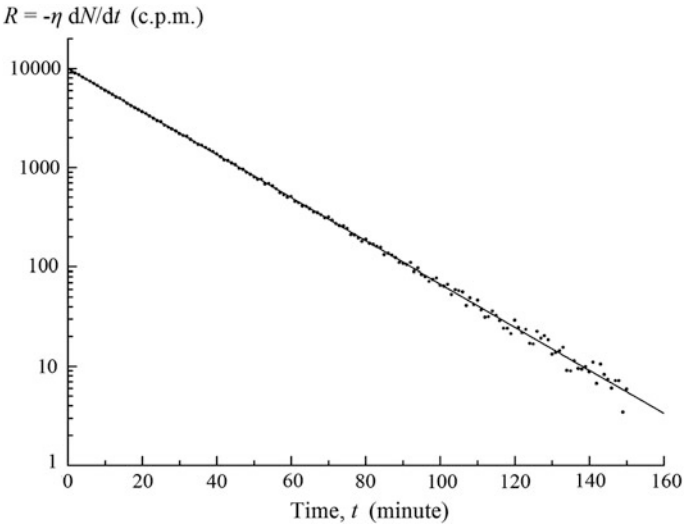


Fig. 8.2 Plot, as a function of time, of the rate of counting $R(t)$ of the disintegrations of a radioactive sample, with an experimental arrangement capable of counting only 10% of the decays $(\eta = 0.1)$. The scale of $R(t)$ is logarithmic. The continuous line shows the theoretically predicted rate, according to Eqs. (8.9) and (8.10)

which counts only 10% of the disintegrations (*efficiency* $\eta = 0.1$). Each measurement is, therefore, equal to

$$R(t) = -\eta \frac{\mathrm{d}N}{\mathrm{d}t} = \eta A \tag{8.10}$$

The units of $R(t)$ are c.p.m. (counts per minute) while those of the activity $A = -\mathrm{d}N/\mathrm{d}t$ are, usually, d.p.s (disintegrations per second). The scale of $R(t)$ in Fig. 8.2 is logarithmic. Because it is

$$\ln R(t) = \ln(\lambda N_0) - \lambda t, \tag{8.11}$$

the relationship between $\ln R(t)$ and t is linear. The same is true for the activity of the sample. The method used for the determination of the decay constant is based on plotting $\log R(t)$ as a function of time (from the slope of the straight line).

The statistical nature of the process of radioactivity must be stressed. This is something that was also highlighted theoretically by the successful interpretation that Quantum Mechanics gave to the phenomenon of α decay via the tunnel effect. It is this statistical nature of the phenomenon we will try to describe below in this chapter. At present we simply mention that, according to the relation (8.6), the average expected number of decays in a relatively small interval of time, Δt, is

$$\Delta N \approx \lambda N(t)\, \Delta t \approx \lambda N_0 \mathrm{e}^{-\lambda t} \Delta t. \tag{8.12}$$

Due to the nature of radioactive decay, the number which will be measured in practice will have fluctuations about this value. It will be proved that the fractional fluctuation is greater for small values of ΔN. This is obvious in Fig. 8.2, where, for large values of t, when the rate of disintegration is small, the differences between the measured and the theoretically predicted rate which has no fluctuations (continuous line) are proportionally large. It must be kept in mind that the scale for the rate is logarithmic and this brings out the *fractional* variations in this magnitude.

8.2 Nuclear Disintegrations and the Binomial Distribution

Assume that, initially ($t = 0$), we have in a sample N_0 radioactive nuclei and that we wish to know what is the probability that in a time t we will have exactly x decays. We may consider that we observe N_0 objects, each one of which has a probability $p(t)$ to suffer something (decay) and a probability $q(t) = 1 - p(t)$ that it will remain unchanged in the time interval between $t = 0$ and t. From the law of radioactivity, we know that it is

$$q(t) = \frac{N(t)}{N_0} = \mathrm{e}^{-\lambda t} \tag{8.13}$$

and, therefore, also

$$p(t) = 1 - \mathrm{e}^{-\lambda t}. \tag{8.14}$$

According to the binomial distribution, therefore, the probability exactly x of the N_0 nuclei to disintegrate in the time interval between $t = 0$ and t is

$$P_{N_0}(x) = \frac{N_0!}{(N_0 - x)!\,x!} p^x (1 - p)^{N_0 - x}. \tag{8.15}$$

Substituting for $p(t)$ from Eq. (8.14), we have

$$P_{N_0}(x) = \frac{N_0!}{(N_0 - x)!\,x!} (1 - \mathrm{e}^{-\lambda t})^x (\mathrm{e}^{-\lambda t})^{N_0 - x}. \tag{8.16}$$

This is the exact relation for the probabilities, independently of any restrictions on the values of N_0, x and λ.

From the properties of the binomial distribution (Subsection 7.2.2), we know that the expected or mean value of the number of disintegrations x in the time interval from $t = 0$ to t is

$$\bar{x} = N_0 p = N_0 (1 - \mathrm{e}^{-\lambda t}) \tag{8.17}$$

and its standard deviation

$$\sigma_{\bar{x}} = \sqrt{N_0 pq} = \sqrt{N_0 (1 - \mathrm{e}^{-\lambda t}) \mathrm{e}^{-\lambda t}}. \tag{8.18}$$

We note that it is

$$\sigma_{\bar{x}} = \sqrt{\bar{x} \mathrm{e}^{-\lambda t}}. \tag{8.19}$$

For small values of t, i.e. for $t \ll \tau\ (= 1/\lambda)$, the mean or expected value of the number of disintegrations in the time interval from $t = 0$ to t is equal to

$$\bar{x} = \lambda N_0 t \tag{8.20}$$

and the standard deviation of the mean $\bar{x}$ is equal to

$$\sigma_{\bar{x}} = \sqrt{\bar{x}}. \tag{8.21}$$

This is a very important characteristic of the phenomenon of radioactivity. The measurement of a number of events (disintegrations) also gives the standard deviation of this number (equal to its square root).

Example 8.1

A sample of radioactive material initially contains $N_0 = 10^9$ nuclei, whose decay constant is $\lambda = 10^{-6}$ s^{-1}. What is the expected number of disintegrations in the time interval between $t = 0$ and $t = 10$ s ?

The mean lifetime of these radioactive nuclei is $\tau = 1/\lambda = 10^6$ s, which is much larger than the duration of the measurement, $t = 10$ s. We may, therefore, use Eqs. (8.20) and (8.21). We find $\bar{x} = \lambda N_0 t = 10^{-6} \times 10^9 \times 10 = 10\,000$ disintegrations and $\sigma_{\bar{x}} = \sqrt{\bar{x}} = 100$ disintegrations.

The expected number of disintegrations is $\bar{x} = 10\,000 \pm 100$.

The mean activity of the sample will therefore be $\bar{A} = -\overline{\mathrm{d}N/\mathrm{d}t} = \bar{x}/t = 1000 \pm 10$ d.p.s. (disintegrations per second). It therefore follows that measurements lasting for $t = 10$ s will give us the activity of this sample with an error of 1%.

If we perform many measurements of duration $t = 10$ s each, while the number N of the nuclei has not changed appreciably from the initial, N_0, the distribution of the results is expected to be given by the Gaussian distribution function

$$f(x) = \frac{1}{\sqrt{2\pi}\ 100} \mathrm{e}^{-(x-10000)^2/2\times 10000},$$

where x is the total number of disintegrations measured in 10 s.

Example 8.2 [E]

The number of nuclei of isotope 1 varies with time according to the relation $N_1 = N_{10}\mathrm{e}^{-t/\tau_1}$. The nuclei of the isotope 2 produced also decays with a mean life of τ_2. Plot the concentration $N_2(t)$ of the daughter isotope as a function of time. Given: $N_{10} = 10^6$, $\tau_1 = 10$ min, $N_{20} = 0$, $\tau_2 = 5$ min.

In a time interval $\mathrm{d}t$, the number of nuclei of the daughter isotopes produced is $(\mathrm{d}N_1/\mathrm{d}t)\mathrm{d}t$, while that of the nuclei decaying is $(N_2/\tau_2)\mathrm{d}t$. The net change of the nuclei of isotope 2 is:

$$\mathrm{d}N_2 = \frac{N_{10}}{\tau_1}\mathrm{e}^{-t/\tau_1}\mathrm{d}t - \frac{N_2}{\tau_2}\mathrm{d}t \quad \text{or} \quad \frac{\mathrm{d}N_2}{\mathrm{d}t} + \frac{N_2}{\tau_2} = \frac{N_{10}}{\tau_1}\mathrm{e}^{-t/\tau_1}$$

The solution of this differential equation with $N_{20} = 0$ is

$$N_2(t) = \frac{\tau_2}{\tau_1 - \tau_2} N_{10}\left(\mathrm{e}^{-t/\tau_1} - \mathrm{e}^{-t/\tau_2}\right).$$

Substituting in this equation

$$N_2(t) = 10^6\left(\mathrm{e}^{-t/10} - \mathrm{e}^{-t/5}\right).$$

with t in min.

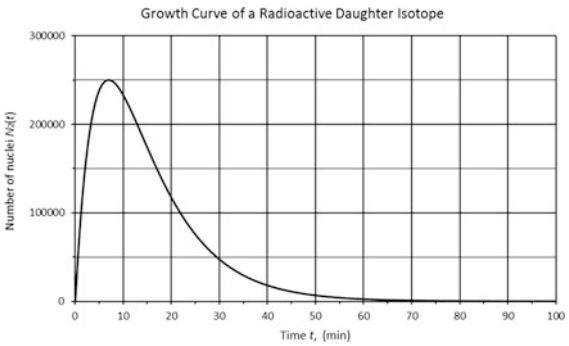
We will plot this function for $0 \leq t \leq 100$ min, with t increasing in steps of 0.1 min.

We place the t values in column A by the following procedure. Highlight cell A1 by left-clicking on it. Type **0** in the cell. Press **ENTER** and in cell A2 type **= A1 + 0.1**. **Fill Down** to cell A1001.

Column B will contain the values of $N_2(t)$. Highlight cell B1 by left-clicking on it. Type in this cell: **= 10^6*(exp(-A1/10)- exp(-A1/5))**. Then **Fill Down** to cell B1001.

Highlight columns A and B by left-clicking on the label A and then, holding the Shift or Control key down, left-clicking on label B. From **Insert > Recommended Charts** we select the **line chart**. The plot of $N_2(t)$ appears. We will format this graph:

1. In the chart title box type **Growth Curve of a Radioactive Daughter Isotope**.
2. Right-click on the curve. In the window that opens select **color black**. This changes the color of the curve to black. Set the thickness of the curve to 1.25 pts.
3. We left-click on a number of the Y-scale and open the **Format Axis, Axis Options** window. We set **Bounds Minimum** 0 and **Maximum** 300000, **Units Major** 100000 and **Minor** 50000. **Tick Marks**, **Major Type Outside** and **Minor Type Outside**. For the X-scale we select **Bounds Minimum** 0 and **Maximum** 100, **Units Major** 10 and **Minor** 5. **Tick Marks**, **Major Type Outside** and **Minor Type Outside**.
4. Pressing the $\boxed{+}$ key opens the **Chart Elements** dialog box. We choose **Axis Titles**. For X-Axis we write **Time, *t* (min)**. For Y-Axis we write **Number of nuclei, $N_2(t)$**.



5. Pressing the [+] key to open the **Chart Elements** dialog box, we choose for the X-axis major gridlines to be visible. For the Y-axis, we choose both major and minor gridlines to appear.

Example 8.3 [O]

The number of nuclei of isotope 1 varies with time according to the relation $N_1 = N_{10}e^{-t/\tau_1}$. The nuclei of the isotope 2 produced also decays with a mean life of τ_2. Plot the concentration $N_2(t)$ of the daughter isotope as a function of time. Given: $N_{10} = 10^6$, $\tau_1 = 10$ min, $N_{20} = 0$, $\tau_2 = 5$ min.

In a time interval dt, the number of nuclei of the daughter isotopes produced is $(dN_1/dt)dt$, while that of the nuclei decaying is $(N_2/\tau_2)dt$. The net change of the nuclei of isotope 2 is:

$$dN_2 = \frac{N_{10}}{\tau_1}e^{-t/\tau_1}dt - \frac{N_2}{\tau_2}dt \quad \text{or} \quad \frac{dN_2}{dt} + \frac{N_2}{\tau_2} = \frac{N_{10}}{\tau_1}e^{-t/\tau_1}$$

The solution of this differential equation with $N_{20} = 0$ is

$$N_2(t) = \frac{\tau_2}{\tau_1 - \tau_2}N_{10}\left(e^{-t/\tau_1} - e^{-t/\tau_2}\right).$$

Substituting in this equation

$$N_2(t) = 10^6\left(e^{-t/10} - e^{-t/5}\right)$$

with t in min.

We will plot this function for $0 \leq t \leq 100$min, with t increasing in steps of 0.1 min.

We place the t values in column A by the following procedure. Highlight column A by left-clicking on its label, A. Then

Column > Set Column Values

typing **(i-1)/10**, with i from 1 to 1001, in the window that opens. Then press **OK**.

Column B will contain the values of $N_2(t)$. Highlight column B by left-clicking on its label, B. Then

Column > Set Column Values

typing **10^6*(exp(col(A)/10)-exp(col(A)/5))**, with i from 1 to 1001, in the window that opens. Then press **OK**.

Highlight columns A and B by left-clicking on the label A and then, holding the Shift or Control key down, left-clicking on label B. Then

Plot > Line > Line

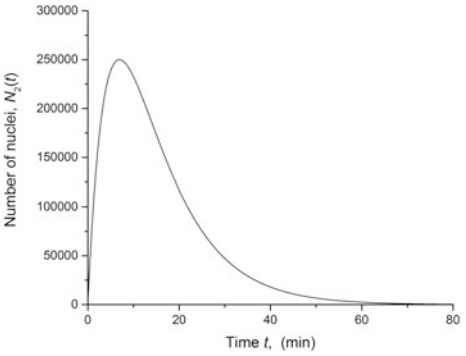
The plot of $N_2(t)$ appears. We will format this graph.

1. Delete the text box.
2. Double-click on the line and set

Line: Connect Straight, Style Solid, Width 1, Color Black Press **OK**.

3. Double-click on one of the axes and set
   Scale Horizontal, from 0 to 80, Type linear, Major Ticks: Type By Increments, Value 20, Minor Ticks: Type By Counts, Count 1
   Vertical from 0 to 300000, Type linear, Major Ticks: Type By Increments, Value 50000, Minor Ticks: Type By Counts, Count 1
4. We change the labels of the axes:
   Double-click on the X label and write **Time *t* (min)**. Double-click on the Y label and write **Number of nuclei, $N_2(t)$**.
5. Save the project. Export graph as jpg (say).

The final graph is shown here.



Example 8.4 [P]

The number of nuclei of isotope 1 varies with time according to the relation $N_1 = N_{10}e^{-t/\tau_1}$. The nuclei of the isotope 2 produced also decays with a mean life of τ_2. Plot the concentration $N_2(t)$ of the daughter isotope as a function of time. Given: $N_{10} = 10^6$, $\tau_1 = 10$ min, $N_{20} = 0$, $\tau_2 = 5$ min.

In a time interval dt, the number of nuclei of the daughter isotopes produced is $(dN_1/dt)dt$, while that of the nuclei decaying is $(N_2/\tau_2)dt$. The net change of the nuclei of isotope 2 is:

$$dN_2 = \frac{N_{10}}{\tau_1}e^{-t/\tau_1}dt - \frac{N_2}{\tau_2}dt \quad \text{or} \quad \frac{dN_2}{dt} + \frac{N_2}{\tau_2} = \frac{N_{10}}{\tau_1}e^{-t/\tau_1}$$

The solution of this differential equation with $N_{20} = 0$ is

$$N_2(t) = \frac{\tau_2}{\tau_1 - \tau_2}N_{10}\left(e^{-t/\tau_1} - e^{-t/\tau_2}\right).$$

Substituting in this equation

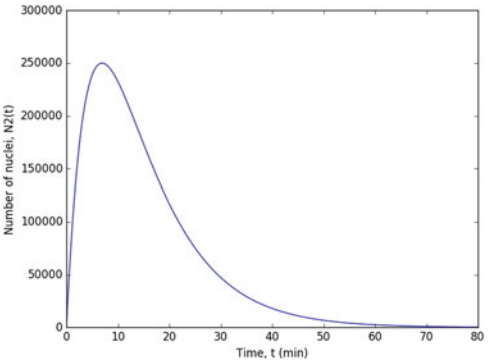
$$N_2(t) = 10^6\left(\mathrm{e}^{-t/10} - \mathrm{e}^{-t/5}\right)$$

with t in min.

Using matplotlib, we will plot this function for $0 \leq t \leq 100$ min, with t increasing in steps of 0.1 min.

```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt
t = np.linspace(0.0, 100.0, 1000)
N2 = 10**6 * (np.exp(-t/10.0)-np.exp(-t/5.0))
plt.plot(t, N2, '-')
plt.xlim(0, 80)
plt.ylim(0, 300000)
plt.ylabel("Number of nuclei, N2(t)")
plt.show()
```

The resulting curve is shown here.



Example 8.5 [R]

The number of nuclei of isotope 1 varies with time according to the relation $N_1 = N_{10}\mathrm{e}^{-t/\tau_1}$. The nuclei of the isotope 2 produced also decays with a mean life of τ_2. Plot the concentration $N_2(t)$ of the daughter isotope as a function of time. Given: $N_{10} = 10^6$, $\tau_1 = 10$ min, $N_{20} = 0$, $\tau_2 = 5$ min.

In a time interval dt, the number of nuclei of the daughter isotopes produced is $(\mathrm{d}N_1/\mathrm{d}t)\mathrm{d}t$, while that of the nuclei decaying is $(N_2/\tau_2)\mathrm{d}t$. The net change of the nuclei of isotope 2 is:

$$dN_2 = \frac{N_{10}}{\tau_1} e^{-t/\tau_1} dt - \frac{N_2}{\tau_2} dt \quad \text{or} \quad \frac{dN_2}{dt} + \frac{N_2}{\tau_2} = \frac{N_{10}}{\tau_1} e^{-t/\tau_1}$$

The solution of this differential equation with $N_{20} = 0$ is

$$N_2(t) = \frac{\tau_2}{\tau_1 - \tau_2} N_{10} \left(e^{-t/\tau_1} - e^{-t/\tau_2} \right).$$

Substituting in this equation

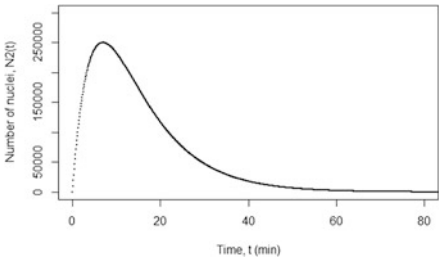
$$N_2(t) = 10^6 \left(e^{-t/10} - e^{-t/5} \right).$$

with t in min.

Just as we did above, we will plot this function for $0 \leq t \leq 100$ min, with t increasing in steps of 0.1 min.

```
> t<- seq(0, 100, by = 0.1)
> N2 <- 10^6*(exp(-t/10)-exp(-t/5))
>
> plot(t, N2, pch=20, cex=0.5, xlim=c(0, 80), ylim=c(0, 300000),
  xlab="Time, t (min)", ylab="Number of nuclei, N2(t)")
```

The graph shown in the figure appears.



Example 8.6 [E]

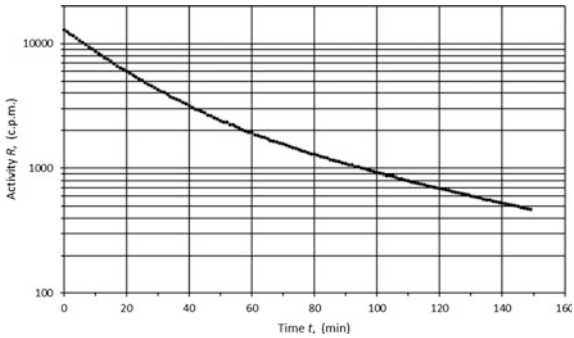
Measurements of the activity of a radioactive sample, R, are given every minute for $0 \leq t \leq 150$ min:

```
12993, 12414, 11882, 11566, 11023, 10623, 10207, 9813, 9428, 9026, 8639,
8353, 8058, 7709, 7517, 7218, 6904, 6637.86466, 6406, 6198, 5995, 5820,
5579, 5393, 5196, 5098, 4841, 4689, 4564, 4424, 4246, 4135, 4072, 3912,
3759, 3648, 3594, 3480, 3380, 3287, 3187, 3085, 2969, 2925, 2843, 2778,
2669, 2624, 2542, 1823, 1774, 1753, 1714, 1670, 1647, 1616, 1578, 1566,
1527, 1491, 1463, 1446, 1417, 1370, 1353, 1325, 1297, 1291, 1261, 1244,
```

1222, 1206, 1168, 1159, 1141, 1122, 1096, 1081, 1067, 1059, 1027, 1023, 998, 983, 964, 956, 948, 924, 913, 905, 880, 876, 865, 853, 828, 826, 810, 805, 786, 771, 762, 757, 745, 732, 719, 710, 563, 559, 550, 543, 537, 530, 525, 515, 512, 504, 497, 489, 484, 478, 469

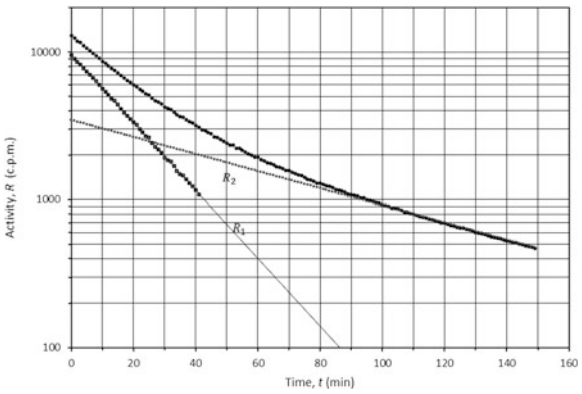
Plot log$R(t)$ and verify that the activity seems to be due to two isotopes with different decay constants. Analyze the curve $R(t)$ into two decay curves and find the two decay constants.

The values of t are entered in column A and those of R in column B. We plot log$R$ (t): Highlight columns A and B by left-clicking on label A and, holding the Shift or Control key down, left-click on label B. Then, from **Insert**, **Recommended Charts**, we choose the **Scatter** plot. After some basic formatting, the graph shown in the figure below is produced.



We assume that the activity in the interval $120 \leq t \leq 150$ min is due almost entirely to isotope 2. From columns A and B we copy the data for $120 \leq t \leq 150$ min and paste them in columns C and D. We plot these points. Pressing the $\boxed{+}$ key to open the **Chart Elements** dialog box, we choose **Trendline** and an **Exponential** fit. The equation given is y = 3456.9 $e^{-0.0133x}$, which corresponds to the activity of isotope 2 being given by $R_2(t) = R_{20}e^{-t/\tau_2}$, where $R_{20} = 3457$ c.p.m. and $\tau_2 = 1/0.0133 = 75$ min.

In cell E1 we type **= 3456.9*EXP(-0.0133*A1)**. We press **ENTER** and **Fill Down** to E150. Column E now contains the values of $R_2(t)$. In cell F1 we type **= B1-E1**. We press **ENTER** and **Fill Down** to F150. Column F now contains $R_1(t)$. We fit an exponential to these data, using only those values that are greater than 1000, as these have small proportional errors. The equation given is y = 9594.2$e^{-0.053x}$, which corresponds to the activity of isotope 1 is given by $R_1(t) = R_{10}e^{-t/\tau_1}$, where $R_{10} = 9594$ c.p.m. and $\tau_1 = 1/0.053 = 18.9$ min. In a graph we plot R_1, R_2 and $R_1 + R_2$. This is shown in the figure below.



Example 8.7 [O]

Measurements of the activity of a radioactive sample, R, are given for $0 \leq t \leq 150$ min (see Example 8.6 [E]). Plot log$R(t)$ and verify that the activity seems to be due to two isotopes with different decay constants. Analyze the curve $R(t)$ into two decay curves and find the two decay constants.

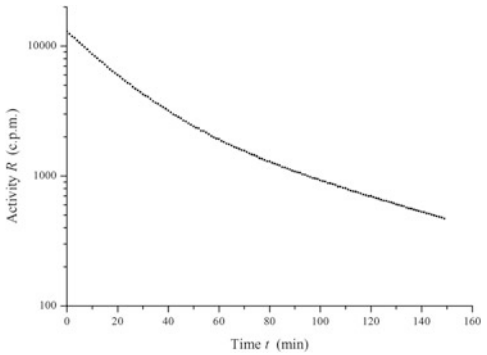
The values of t are entered in column A and those of R in column B. We plot log$R$ (t): Highlight columns A and B by left-clicking on label A and, holding the Shift or Control key down, left-click on label B. Then

Plot > Symbol > Scatter

A plot of $R(t)$ is produced. We will modify the plot to suit our requirements:

1. Delete the text box in the plot.
2. Double-click on a point and in **Plot Details > Symbol** change the 9 pt. squares to 3 pt. circles.
3. We change the labels of the axes:
   Double-click on the X label and write **Time *t* (min)**. Double-click on the Y label and write **Activity *R* (c.p.m)**.
4. 4. Double click on the t-axis. In the **Scale** window that opens, select, for Horizontal **From 0 to 160** and for Vertical **From 100 to 20 000** and **Type Log10**.

The graph produced is shown below.



We assume that the activity in the interval $120 \leq t \leq 150$ min is due almost entirely to isotope 2. From columns A and B we copy the data for $120 \leq t \leq 150$ min and paste them in columns C and D. We highlight column C by left-clicking on its label. We then set column C as an X axis by selecting **Column** and then **Set as X**. We plot a graph of these points exactly as above.

We will plot a best fit exponential curve between these points. While in the graph, we select

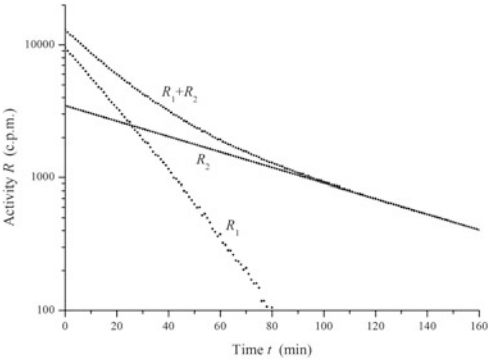
Analysis > Fitting > Exponential Fit

In the window that opens

Settings > Function Selection > Category, Exponential > Function, ExpDec1

We go to **Parameters**. We tick **Fixed** for **y0** and set its value to zero. Press **Fit**. The best fit for these points is given as y = A1*exp(-x/t1), or $R_2 = R_{20}\mathrm{e}^{-t/\tau_2}$, where $R_{20} = 3472.8 \pm 3.9$ c.p.m. and $\tau_2 = 74.52 \pm 0.47$ min.

For $0 \leq t \leq 150$, we enter $R_2 = R_{20}\mathrm{e}^{-t/\tau_2}$ in column E. The difference of col(B) – col(E) is evaluated in column F. This is the activity of the first isotope, $R_1 = R_{10}\mathrm{e}^{-t/\tau_1}$. A best fit performed on R_1 as above gives $R_{10} = 9580 \pm 14$ c.p.m. and $\tau_1 = 18.82 \pm 0.04$ min. In a graph we plot R_1, R_2 and $R_1 + R_2$. This is shown in the figure below.



Example 8.8 [P]

Measurements of the activity of a radioactive sample, R, are given for $0 \leq t \leq 150$ min (see Example 8.6 [E]). Plot $\log R(t)$ and verify that the activity seems to be due to two isotopes with different decay constants. Analyze the curve $R(t)$ into two decay curves and find the two decay constants.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.optimize import curve_fit

t = np.arange(0, 150)
R = np.array([12993, 12414, 11882, 11566, 11023, 10623, 10207, 9813, 9428, 9026,
8639, 8353, 8058, 7709, 7517, 7218, 6904, 6637.86466, 6406, 6198, 5995, 5820, 5579,
5393, 5196, 5098, 4841, 4689, 4564, 4424, 4246, 4135, 4072, 3912, 3759, 3648, 3594,
3480, 3380, 3287, 3187, 3085, 2969, 2925, 2843, 2778, 2669, 2624, 2542, 2477, 2408,
2348, 2327, 2226, 2218, 2157, 2098, 2028, 1985, 1935, 1928, 1855, 1823, 1774, 1753,
1714, 1670, 1647, 1616, 1578, 1566, 1527, 1491, 1463, 1446, 1417, 1370, 1353, 1325,
1297, 1291, 1261, 1244, 1222, 1206, 1168, 1159, 1141, 1122, 1096, 1081, 1067, 1059,
1027, 1023, 998, 983, 964, 956, 948, 924, 913, 905, 880, 876, 865, 853, 828, 826,
810, 805, 786, 771, 762, 757, 745, 732, 719, 710, 699, 698, 685, 674, 668, 653,
645, 643, 632, 626, 616, 603, 597, 590, 584, 570, 563, 559, 550, 543, 537, 530,
525, 515, 512, 504, 497, 489, 484, 478, 469])

# We define the function R(t) = R10e^(-t/τ1) + R20e^(-t/τ2) in Python as follows:
def R_func(t, A1, t1, A2, t2):
   return A1*np.exp(-t/t1) + A2*np.exp(-t/t2)

# We then use the curve_fit function of the scipy.optimize sub-package to
# perform non-linear least squares fitting to the data:
popt, pcov = curve_fit(R_func, t, R, p0 = (12000, 20, 3000, 100))

# By examining the popt array,
Popt
array([ 9949.25831076, 19.8784205, 3036.47784508, 79.38078197])

# we obtain the results of the fitting:
R10 = 9949.258 c.p.m, R20 = 3036.478 c.p.m, τ1 = 19.878, τ2 = 79.381
```

Example 8.9 [R]

Measurements of the activity of a radioactive sample, R, are given for $0 \leq t \leq 150$ min. Plot $\log R(t)$ and verify that the activity seems to be due to two isotopes with different decay constants. Analyze the curve $R(t)$ into two decay curves and find the two decay constants.

We will use non-linear least squares in order to fit a curve of the form $R(t) = R_{10}e^{-t/\tau_1} + R_{20}e^{-t/\tau_2}$ to the values of $R(t)$.

```
> t<- seq(0, 149, by=1)
>
> R<- c(12993, 12414, 11882, 11566, 11023, 10623, 10207, 9813, 9428, 9026, 8639,
8353, 8058, 7709, 7517, 7218, 6904, 6637.86466, 6406, 6198, 5995, 5820, 5579, 5393,
5196, 5098, 4841, 4689, 4564, 4424, 4246, 4135, 4072, 3912, 3759, 3648, 3594, 3480,
3380, 3287, 3187, 3085, 2969, 2925, 2843, 2778, 2669, 2624, 2542, 2477, 2408, 2348,
2327, 2226, 2218, 2157, 2098, 2028, 1985, 1935, 1928, 1855, 1823, 1774, 1753, 1714,
1670, 1647, 1616, 1578, 1566, 1527, 1491, 1463, 1446, 1417, 1370, 1353, 1325, 1297,
1291, 1261, 1244, 1222, 1206, 1168, 1159, 1141, 1122, 1096, 1081, 1067, 1059, 1027,
1023, 998, 983, 964, 956, 948, 924, 913, 905, 880, 876, 865, 853, 828, 826, 810,
805, 786, 771, 762, 757, 745, 732, 719, 710, 699, 698, 685, 674, 668, 653, 645,
643, 632, 626, 616, 603, 597, 590, 584, 570, 563, 559, 550, 543, 537, 530, 525,
515, 512, 504, 497, 489, 484, 478, 469)
>
> fm1 < - nls(R ~ A1*exp(-t/t1) + A2*exp(-t/t2), start = list
(A1 = 12000, t1 = 20, A2 = 3000, t2 = 100))
> fm1
  Nonlinear regression model
  model: R ~ A1 * exp(-t/t1) + A2 * exp(-t/t2)
  data: parent.frame()
  A1    t1    A2    t2
  9949.26  19.88 3036.48  79.38
  residual sum-of-squares: 56850
  Number of iterations to convergence: 3
  Achieved convergence tolerance: 5.324e-06
>
```

The results of the fitting are:

$$R_{10} = 9949 \text{ c.p.m.}, R_{20} = 3036 \text{ c.p.m.}, \tau_1 = 19.88 \text{ min and } \tau_2 = 79.38 \text{ min.}$$

```
> R1<-9949*exp(-t/19.88)
> R2<-3036*exp(-t/79.38)
>
```

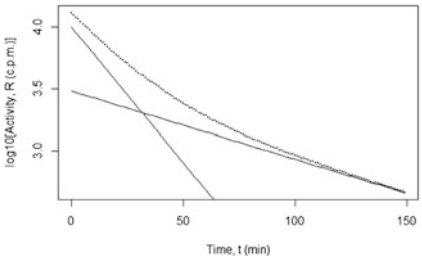
We plot $R(t)$

```
> plot(t,log10(R), pch=20, cex=0.5, xlab="Time, t (min)", ylab="log10
[Activity, R (c.p.m.)]")
>
```

and then add the curves (straight lines) for R_1 and R_2:

```
> lines(t, log10(R1))
> lines(t, log10(R2))
```

The graph shown in the figure is produced.



8.3 Radioactivity and the Poisson Distribution

Assume that, in a radioactive sample, the probability for a disintegration (i.e. an event) to occur in the small time interval dt is Λdt. The probability of having 0 events (i.e. no disintegration) in the time integral dt is, therefore, equal to $1 - \Lambda$dt. The parameter Λ, which expresses a probability per unit time, is considered to remain constant during the experiment. According to Sect. 8.1, if the sample consists of N nuclei, it will be $\Lambda = \lambda N$. For Λ to remain constant, the number N of the nuclei must not change significantly during the whole of the experiment.

We will denote by $P_r(t)$ the probability that r disintegrations will occur in the time interval $(0, t)$. (We point out that the notation for the probabilities in this section is different from that of Chap. 7).

Let us first evaluate the probability $P_0(t)$ of no disintegration in the time interval $(0, t)$. For no disintegration to happen in the time interval $(0, t + \mathrm{d}t)$, no disintegrations must happen either in the interval $(0, t)$ or the interval $(t, t + \mathrm{d}t)$. Since the probabilities are independent of each other, the probability that no disintegration will occur in the time interval $(0, t + \mathrm{d}t)$ will be equal to the product of the probabilities that no disintegrations should happen either in the interval $(0, t)$ or the interval $(t, t + \mathrm{d}t)$:

$$P_0(t + \mathrm{d}t) = P_0(t)\,(1 - \Lambda \mathrm{d}t). \tag{8.22}$$

Therefore,

$$\frac{P_0(t+\mathrm{d}t)-P_0(t)}{\mathrm{d}t}=-\Lambda P_0(t) \tag{8.23}$$

Or

$$\frac{\mathrm{d}P_0(t)}{\mathrm{d}t}=-\Lambda P_0(t). \tag{8.24}$$

The solution of this differential equation is:

$$P_0(t)=A\,\mathrm{e}^{-\Lambda t} \tag{8.25}$$

where A is a constant. Given that it is certain that no disintegration will happen in the interval (0, 0), it is $P_0(0)=1$. Therefore, $A=1$ and

$$P_0(t)=\mathrm{e}^{-\Lambda t}. \tag{8.26}$$

We will evaluate the probability $P_1(t)$ of exactly one disintegration to occur in the time interval (0, t). There are two ways in which exactly one disintegration will occur in the interval (0, $t+\mathrm{d}t$). The first is to have one disintegration occurring in the interval (0, t) and none in (t, $t+\mathrm{d}t$). The second is to have no disintegration occurring in the interval (0, t) and one in the interval (t, $t+\mathrm{d}t$). The total probability is, therefore,

$$P_1(t+\mathrm{d}t)=P_1(t)\,(1-\Lambda\mathrm{d}t)+P_0(t)\Lambda\mathrm{d}t \tag{8.27}$$

or

$$\frac{P_1(t+\mathrm{d}t)-P_1(t)}{\mathrm{d}t}=\frac{\mathrm{d}P_1(t)}{\mathrm{d}t}=-\Lambda P_1(t)+\Lambda P_0(t) \tag{8.28}$$

and the differential equation for $P_1(t)$ is

$$\frac{\mathrm{d}P_1(t)}{\mathrm{d}t}=-\Lambda P_1(t)+\Lambda\mathrm{e}^{-\Lambda t}. \tag{8.29}$$

whose solution, satisfying the condition $P_1(0)=0$, is

$$P_1(t)=\Lambda t\mathrm{e}^{-\Lambda t}. \tag{8.30}$$

Generalizing, we will now evaluate the probability $P_r(t)$ for exactly r disintegrations to occur in the time interval (0, t). There are two ways in which exactly r disintegrations occur in the time interval (t, $t+\mathrm{d}t$). The first is that r disintegrations occur in the interval (0, t) and none in the interval (t, $t+\mathrm{d}t$). The second is that

$r-1$ disintegrations occur in (0, t) and one in (t, $t+\mathrm{d}t$). The total probability is, therefore,

$$P_r(t+\mathrm{d}t) = P_r(t)\,(1-\Lambda \mathrm{d}t) + P_{r-1}(t)\Lambda \mathrm{d}t \tag{8.31}$$

from which we have the recurrence differential equation

$$\frac{\mathrm{d}P_r(t)}{\mathrm{d}t} = -\Lambda P_r(t) + \Lambda P_{r-1}(t) \tag{8.32}$$

for $P_r(t)$. It is easily verified, by substitution, that the function

$$P_r(t) = \frac{(\Lambda t)^r}{r!}\mathrm{e}^{-\Lambda t} \tag{8.33}$$

is a solution of the recurrence differential equation, which gives the already known results for $P_0(t)$ and $P_1(t)$.

From the known facts for the phenomenon of radioactivity, when the sample consists of N_0 nuclei whose decay constant is λ, it will be $\Lambda = \lambda N_0$. It has therefore been found that, for a given time interval t, the probabilities that $r = 0, 1, 2, \ldots$ disintegrations will occur, are given by the Poisson distribution

$$P_r(t) = \frac{(\lambda N_0 t)^r}{r!}\mathrm{e}^{-\lambda N_0 t}. \tag{8.34}$$

The mean or expected number of disintegrations in the time interval (0, t) will be

$$\mu = \lambda N_0 t \quad (t \ll \tau) \tag{8.35}$$

Returning to the notation of Chap. 7, the probability for x disintegrations occurring in the time interval (0, t), for which the expected number of disintegrations is $\mu = \lambda N_0 t$, is given by the relation

$$P_\mu(x) = \frac{\mu^x}{x!}\mathrm{e}^{-\mu}. \tag{8.36}$$

As it is known, for the Poisson distribution the mean or expected value of x is $\bar{x} = \mu$ and its standard deviation is $\sigma_{\bar{x}} = \sqrt{\mu} = \sqrt{\bar{x}}$. These are in agreement with what has been said in Sect. 8.2.

8.4 The Counting Rate of Nuclear Disintegrations and Its Error

In an experimental arrangement for the measurement of radioactivity, only a fraction of the total number of disintegrations is measured. This fraction is called *efficiency* η of the arrangement. Thus, if the activity of the sample being measured is $A = -\mathrm{d}N/\mathrm{d}t$, the counting rate of the disintegrations will be

$$R = \eta\,(-\mathrm{d}N/\mathrm{d}t) = \eta A \tag{8.37}$$

The problem we will now examine is this: If in a time interval t, the number of nuclei disintegrating is x, then ηx of them will be counted by our experimental arrangement. The counting rate will be $R = \eta x/t$. What is the expected or mean value $\bar{R}$ of R and which is its standard deviation $\sigma_{\bar{R}}$?

For the solution of the problem, we return to the beginning of the extraction of the relation for the probabilities for x disintegrations to occur in the time interval (0, t). Since the probability of a disintegration occurring in the time interval dt is $\Lambda\mathrm{d}t$, the probability of a disintegration being counted in the time interval dt is $\eta\Lambda dt$. Substituting $\eta\Lambda$ in place of Λ in Eq. (8.33) and since it is $\Lambda = \lambda N_0$, we find

$$P_x(t) = \frac{(\eta\lambda N_0 t)^x}{x!}\mathrm{e}^{-\eta\lambda N_0 t} \tag{8.38}$$

as the probability of $x = 0, 1, 2, \ldots$ disintegrations being counted in the time interval (0, t). The counting rate of the disintegrations in the time interval (0, t) is $R = x/t$. The possible values of the counting rate are

$$R_0 = 0, \quad R_1 = \frac{1}{t}, \quad R_2 = \frac{2}{t}, \ldots, \quad R_x = \frac{x}{t}, \ldots$$

and, therefore, the mean of the counting rate is

$$\bar{R} = \sum_{x=0}^{\infty}\frac{x}{t}P_x(t) = \frac{1}{t}\sum_{x=0}^{\infty}x\,P_x(t) = \frac{\bar{x}}{t} = \frac{\eta\lambda N_0 t}{t} = \eta\lambda N_0. \tag{8.39}$$

The standard deviation or the standard error of the mean $\bar{R}$ is

$$\sigma_{\bar{R}} = \sqrt{\sum_{x=0}^{\infty}(R-\bar{R})^2\,P_x(t)} = \sqrt{\sum_{x=0}^{\infty}\left(\frac{x}{t}-\frac{\bar{x}}{t}\right)^2 P_x(t)} = \frac{1}{t}\sqrt{\sum_{x=0}^{\infty}(x-\bar{x})^2\,P_x(t)} = \frac{\sigma_{\bar{x}}}{t}$$
$$= \frac{\sqrt{\eta\lambda N_0 t}}{t} \tag{8.40}$$

$$\sigma_{\bar{R}} = \frac{\sigma_{\bar{x}}}{t} = \sqrt{\frac{\bar{R}}{t}} = \frac{\sqrt{\bar{x}}}{t}. \tag{8.41}$$

Summarizing, if in a time interval equal to t we count M nuclear disintegrations, then we conclude that, for measurements lasting for time t, the expected or mean value of the number of the disintegrations counted, x, is

$$\bar{x} = M \pm \sqrt{M} \tag{8.42}$$

and, therefore, the expected or mean rate of counting is

$$\bar{R} = \frac{M}{t} \pm \frac{\sqrt{M}}{t}. \tag{8.43}$$

From the relation $\sigma_{\bar{R}} = \sqrt{\frac{\bar{R}}{t}}$ we see that, for a given counting rate R, in order to reduce the error in the measured rate $\bar{R}$ by a factor of 2 we must quadruple the duration of the measurement.

To avoid mistakes, it must be noted that the equation $\bar{x} = M \pm \sqrt{M}$ makes sense only if the magnitude M is a pure (i.e. dimensionless) number. Otherwise the equation would be dimensionally wrong, since the dimensions (and the units) of M and $\sqrt{M}$ would not be the same (see Appendix 2). It would be wrong, for example, to evaluate first the value of the rate $\bar{R}$ and then take the square root of this quantity as being its standard deviation!

Example 8.10

In a measurement that lasted for 8 min, 1685 nuclear disintegrations were counted in a radioactive sample. What is the counting rate and its standard deviation? If the efficiency of the measuring arrangement is equal to 10%, with negligible error, what is the estimate for the activity A of the sample?

It is $t = 8$ min and $M = 1685$. The mean value of the number of counts in an 8-minute measurement is, therefore, $\bar{x} = 1685 \pm \sqrt{1685} = 1685 \pm 40$ counts.

The counting rate is equal to $R = \frac{1685}{8} \pm \frac{40}{8} = 211 \pm 5$ c.p.m.

Given that $\eta = 0.1$ with good accuracy, the activity of the sample is $A = \frac{R}{\eta} = \frac{211 \pm 5}{0.1} = 2110 \pm 50$ d.p.m. (disintegrations per minute) or $A = \frac{2110 \pm 50}{60} = 35 \pm 1$ d.p.s. (disintegrations per second).

Example 8.11

The measurement of a radioactive sample for 100 s resulted in the recording of 635 counts. Taking the sample away and counting for 30 s, resulted in 98 counts (these counts are due to the radioactivity of the environment and is called *background*). Find the clear counting rate which is due to the sample alone, as well as its standard deviation.

The background counting rate is: $R_B = \frac{98 \pm \sqrt{98}}{30} = 3.27 \pm 0.33$ c.p.s.

The total counting rate of source and background is: $R_T = \frac{635 \pm \sqrt{635}}{100} = 6.35 \pm 0.25$ c.p.s.

The clear counting rate of the source alone is given by the difference

$$R_S = R_T - R_B = 6.35 - 3.27 = 3.08 \text{ c.p.s.}$$

which has a standard deviation $\sigma_S = \sqrt{\sigma_T^2 + \sigma_B^2} = \sqrt{0.25^2 + 0.33^2} = 0.41$ c.p.s.

Therefore, $R_S = 3.1 \pm 0.4$ c.p.s.

Example 8.12

If R_T is the total counting rate for source and background and R_B is the counting rate for the background alone, which must the division of the available time be among the two measurements in order to obtain the best accuracy in the measurement of the rate $R_S = R_T - R_B$ of the source?

If the available time is equal to t and a time t_T is used for the measurement of R_T and time $t_B = t - t_T$ for the measurement of R_B, we will have a total of $M_T = R_T t_T$ counts when we measure the source plus background and $M_B = R_B t_B$ counts when we measure the background alone.

The standard deviation σ_S of R_S is given by the relation $\sigma_S^2 = \sigma_T^2 + \sigma_B^2$, where $\sigma_T^2 = R_T/t_T$ and $\sigma_B^2 = R_B/t_B$.

Therefore, it is $\sigma_S^2 = \frac{R_T}{t_T} + \frac{R_B}{t - t_T}$. The value of σ_S, which is always positive, has a minimum when σ_S^2 has a minimum.

This happens when it is $\frac{d\sigma_B^2}{dt_T} = \frac{d}{dt_T}\left(\frac{R_T}{t_T} + \frac{R_B}{t - t_T}\right) = 0$ or $-\frac{R_T}{t_T^2} + \frac{R_B}{(t - t_T)^2} = 0$.

It follows that, for the smallest possible error σ_S in R_S, the time must be divided according to the relation $\frac{t_T}{t_B} = \sqrt{\frac{R_T}{R_B}}$.

Substituting, we find that, in this case, the value of the minimum standard deviation of R_S is:

$$\sigma_{S,\min} = \sqrt{\frac{R_T}{t}} + \sqrt{\frac{R_B}{t}}.$$

Example 8.13

Apply the conclusions of Example 8.12 to the measurements of Example 8.11.

For $R_T = 6.35$ c.p.s. and $R_B = 3.27$ c.p.s., the best division of the total time for the measurement will be $\frac{t_T}{t_B} = \sqrt{\frac{R_T}{R_B}} = \sqrt{\frac{6.35}{3.27}} = 1.39$.

Since it is $\frac{t_T + t_B}{t_B} = 2.39$, it follows that $t_B = \frac{t}{2.39} = 0.42\,t$ and $t_T = 0.58\,t$.

The best division of time would be 42% for the measurement of the background and 58% for the measurement of source plus background. If this division had been done in Example 8.9, in which it was $t = 130$ s, we would have $\sigma_{S,\min} = 0.38$ d.p.s. instead of 0.41 d.p.s. The difference is small.

Example 8.14

The counting rate for a radioactive source was measured to be $R = 160 \pm 4$ c.p.m. What was the approximate duration t of the measurement? The background may be considered to be negligible.

If the measurement lasted for time t, the number of counts recorded was $M = Rt$, with a standard deviation $\sqrt{M} = \sqrt{Rt}$. By dividing by t, it was found that

$$\frac{M \pm \sqrt{M}}{t} = \frac{Rt \pm \sqrt{Rt}}{t} \quad \text{or} \quad R \pm \sqrt{\frac{R}{t}} = 160 \pm 4 \text{ c.p.m.}$$

$$\Rightarrow \quad R = 160, \qquad \sqrt{\frac{R}{t}} = \sqrt{\frac{160}{t}} = 4, \quad \Rightarrow \quad t = 10 \text{ minutes.}$$

Alternatively, the fractional standard deviation of the rate is $\frac{4}{160} = 0.025$. If M counts were counted in total, it will be $\frac{\sqrt{M}}{M} = 0.025$ or $M = 40^2 = 1600$.

Since it is $R = 160$ c.p.m., it follows that $t = \frac{1600}{160} = 10$ min.

Problems

8.1 Two identical samples of a long-lived radioisotope are prepared at the same time. The first sample is monitored for 5 min and is found to give a total of $N_1 = 493$ counts. The second sample gives $N_2 = 1935$ counts in 20 min. (a) What are the counting rates R_1 and R_2 and their errors for the two samples? (b) By what factor is the fractional error in the counting rate decreased in increasing the counting time from 5 to 20 min?

8.2 A sample of a long-lived radioisotope emits an average of 10 α particles per hour.
- (a) What is the expected number of particles to be emitted in 10 min?
- (b) What is the probability that no particle will be emitted in a time interval of 10 min?

8.3 In an activation experiment, a sample is bombarded with neutrons. Immediately after, the activity of the sample is measured. During the first minute $n_1 = 256$ counts are recorded and during the second minute, $n_2 = 49$. Assuming that the counts are due to only one radioisotope and neglecting the background, find the decay constant λ of the isotope and its error, $\delta\lambda$.

8.4 Preliminary measurements of radioactivity gave for the background a counting rate approximately equal to $R_B = 1.0$ c.p.s. and for the radioactive source plus background a counting rate of $R_T = 4.0$ c.p.s. If we have at our disposal 9 min in which to measure these two magnitudes, (a) how must the time be divided between the two measurements for best results in the evaluation the net counting rate from the source, R_S? (b) approximately what will the standard deviation of R_S be?

8.5 In an experiment similar to that performed by C.S. Wu et al. in 1957 in order to prove the violation of parity in the decay of ^{60}Co, the spins of the nuclei of the radioisotope are aligned in a magnetic field, at low temperatures. Over a period of time, the number of photons emitted in the direction of the nuclear spins was measured to be $N_+ = 363$ counts and the number of photons emitted in the opposite direction was measured to be $N_- = 561$ counts. What is the value of the polarization ratio $r = N_-/N_+$ and what is its error, δr ?

8.6 A radioactive sample contains three radioisotopes, A, B and C. The initial numbers of nuclei and the mean lifetimes of these isotopes are $N_A = 1\,000\,000$, $N_B = 600\,000$, $N_C = 270\,000$ and $\tau_A = 100$ s, $\tau_B = 300$ s, $\tau_C = 900$ s, respectively. Find the counting rates $R_A(t)$, $R_B(t)$ and $R_C(t)$ due to the three isotopes, assuming that all the decays are counted. Plot, as a function of time, using a logarithmic R-scale, these counting rates and the total counting rate of the sample, $R(t) = R_A(t) + R_B(t) + R_C(t)$.

Reference

1. See, for example, G. Friedlander, J.W. Kennedy, E.S. Macias and J.M. Miller, *Nuclear and Radiochemistry* (J. Wiley and Sons, New York, 3rd ed., 1981). Ch. 9

Chapter 9
Elements from the Theory of Errors

We will now examine the general principles of the Theory of Errors. Apart from the a posteriori justification of some assumptions we have already made and used, this chapter will give us new theoretical tools which we will use in the development of new techniques in the analysis of experimental results.

9.1 The Normal or Gaussian Law of Errors

> *All believe in the exponential law of errors; the experimentalists because they think it can be proved with mathematics and the mathematicians because they believe it has been established experimentally.*
>
> E. Whittaker and G. Robinson [1]

The law known as *Gauss' law of errors*, was first formulated by Laplace in 1783. Laplace based his derivation on the assumption that the deviation of a measurement belonging to a group of measurements from the group's mean is due to a large number of small deviations, due to causes which are independent of each other. He assumed that deviations of the same magnitude are equally probable to be positive or negative. Gauss, later, proved the law based on the assumption that the numerical mean is the most probable value of a number of equivalent measurements.

The mathematical form of the law defines the *normal or Gaussian probability density function*, which gives the distribution of the results x of the measurements of a physical magnitude $\mathbf{x}$, as

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}. \tag{9.1}$$

C. Christodoulides and G. Christodoulides, *Analysis and Presentation of Experimental Results*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-53345-2_9

The probability density depends on two parameters, μ and σ. It is proved that μ is the mean of a large number of measurements or the value towards which the mean $\overline{x}$ of a series of measurements tends as the number of measurements tends to infinity and σ is the standard deviation of the measurements from μ. The function (9.1) describes the distribution of a very large number ($N \to \infty$) of measurement results, i.e. what we call the *parent population (or universe) of all the possible results x of the measurements*. A series of measurements of x constitutes a sample taken at random from this population. From this finite number of experimental results, we may deduce the best estimates for μ and σ, according to what has been said in Chap. 4.

Having in mind what we have shown for the binomial and the Gaussian distributions in Chap. 7, the extraction of the expression (9.1) for the probability density is simple. Let us suppose that the magnitude being measured has real value equal to x_0 and that the final deviation of a measurement x from x_0 is due to N independent sources of error. To simplify the arguments, we assume that the random errors from these sources all have the same magnitude ε. This limitation in not necessary and the law may also be extracted for a general distribution of these elementary errors [2]. The probabilities for the errors to have values $\pm\varepsilon$ are both equal to $p = \frac{1}{2}$. For N errors of magnitude ε, the final results will lie between the values $x = x_0 - N\varepsilon$ and $x = x_0 + N\varepsilon$.

If, in a measurement, n errors are positive and, therefore, $N - n$ errors are negative, the result of the measurement will be

$$x = x_0 + n\varepsilon - (N - n)\varepsilon = x_0 + (2n - N)\varepsilon. \tag{9.2}$$

The probability of this occurring is, according to the binomial distribution,

$$P_N(n) = \frac{N!}{n!(N-n)!} p^n (1-p)^{N-n} = \frac{N!}{n!(N-n)!} \left(\frac{1}{2}\right)^N, \tag{9.3}$$

and this is the probability of a measurement having as its result the value $x = x_0 + (2n - N)\varepsilon$.

In the limit, in which the binomial distribution approaches the Gaussian distribution, this probability becomes (see Sect. 7.4)

$$P_N(n) = \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-(n-\mu)^2/2\sigma^2} \tag{9.4}$$

with

$$\mu = Np = \frac{N}{2}, \quad \sigma = \sqrt{Npq} = \frac{\sqrt{N}}{2}. \tag{9.5}$$

When we take as variable the continuous variable x [see Eqs. (7.54)–(7.56)], we have the probability distribution

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-(x-\mu)^2/2\sigma^2}, \tag{9.6}$$

with

$$\mu = x_0, \quad \sigma = \sqrt{Npq}\varepsilon = \frac{\sqrt{N}}{2}\varepsilon. \tag{9.7}$$

The Gaussian curve is shown in Fig. 9.1. This distribution was, as we have already mentioned, originally derived by de Moivre for the results of coin tossing.

The relation between the binomial distribution and the error of a measurement, according to the assumptions we have made, is seen in Fig. 9.2. A large enough number of small balls fall, colliding with regularly arranged cylindrical obstacles, such that they force the balls to deviate to the left or to the right by a given constant step $\varepsilon/2$. The probabilities for a deviation to the right and for a deviation to the left are equal, $p = \frac{1}{2}$. The balls do not interact with each other during their descent. Shown in the figure is the distribution of the final positions of the balls after $N = 24$ such deviations. The distribution is binomial and, in the limit, Gaussian. The symmetry of the distribution is seen, as well as the fact that small deviations are more probable than large ones, since they are attained in a larger number of possible ways (paths).

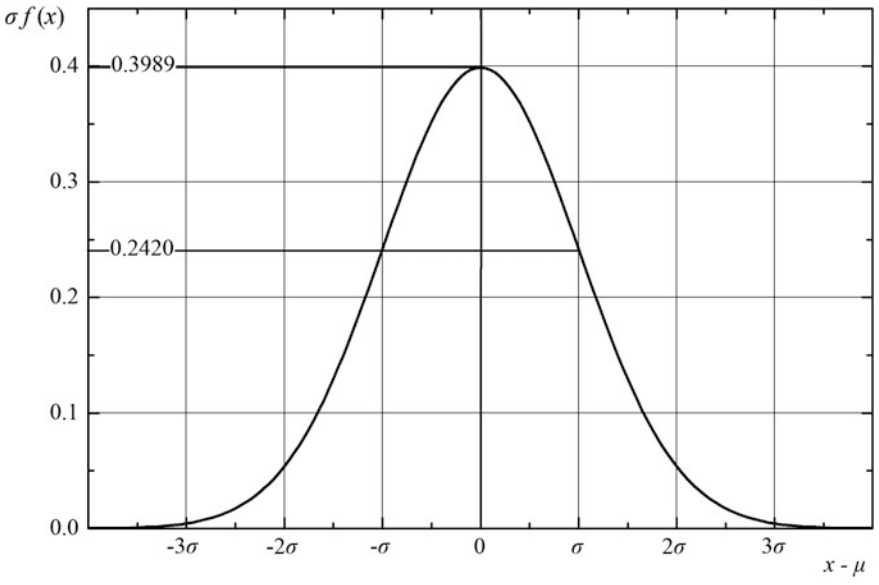


Fig. 9.1 The Gaussian function

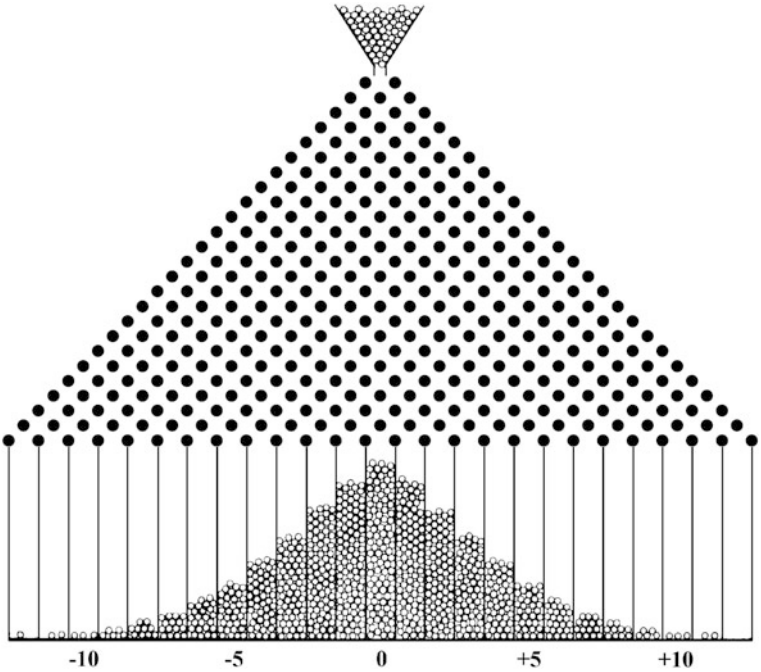


Fig. 9.2 The distribution of the deviations suffered by falling small balls which are forced 24 times to deviate to the *left* or to the *right*, by equal steps and with equal probabilities. Their number in each region of deviation is given by the binomial distribution

The arrangement is due to Francis Galton and is known by its Latin name *quincunx*, which describes the way fruit-bearing trees are planted. Since in the figure it is $N = 24$, the deviations are even multiples of $\varepsilon/2$. Thus, we see that the deviations have values which are integral multiples of ε between -12ε and $+12\varepsilon$. Each ball may be considered to be the result of a measurement, which is subjected to 24 small deviations of magnitude $\varepsilon/2$, with equally probable positive or negative sign.

Laplace and Gauss assumed that the normal distribution has universal validity, based on their own studies of experimental results. Today, this is known not to be true. Apart from the cases in which the deviations from the normal distribution are too striking to neglect, the distribution is used because, as put by Jeffreys [3]: '*The normal law of errors cannot be proved theoretically. The justification for its use is that the law represents measurements of many kinds without being too wrong and that it is much easier to use than other laws, which would be more accurate in certain cases*'.

Despite all these, even when the parent population does not have a normal distribution, the distribution of the means of a series of a finite number of measurements is nearer to the normal distribution than the parent population. This follows from a very important theorem, the *central limit theorem*, which we will discuss in detail below.

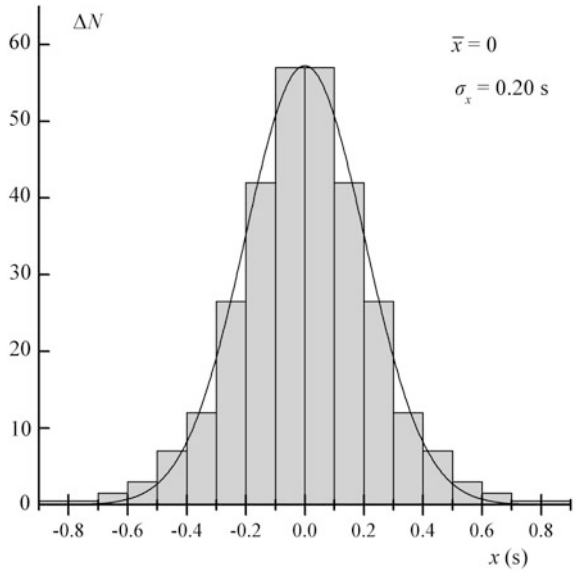


Fig. 9.3 The histogram of Bessel for the errors of 300 measurements of the right ascension of stars. The Gaussian fitted to the histogram has a mean $\bar{x} = 0$ and a standard deviation $\sigma_x = 0.20$ s

The first example of measurements appearing to be normally distributed was given by Bessel, who grouped the results of 300 measurements of the right ascension of stars. The errors were given in seconds of arc and lied between –1 and +1 s. The histogram of the errors (Fig. 9.3) is symmetrical with respect to zero, since Bessel grouped together the positive and the negative errors. As shown in the figure, a Gaussian curve with mean $\bar{x} = 0$ (as expected) and standard deviation from the mean $\sigma_x = 0.20$ s, is fitted to the histogram. The fit of the Gaussian to the data is very good. In fact, it so good that there were suggestions that a selection of values was made, in order to get a better agreement with the normal distribution of errors.

Birge [4] performed, with a spectrometer, a series of 500 adjustments in which he placed the vertical wire of the cross wire in the optical field of the instrument as near as he could to the center of a wide but symmetrical spectral line in the solar spectrum. This is the procedure followed in the measurement of the wavelength of the spectral line. He recorded the readings on the instrument's scale, in μm. The frequencies of the measurements' residuals from their mean value, $\upsilon_i = x_i - \bar{x}$, are presented in the histogram of Fig. 9.4. The agreement of the distribution of the errors with a normal distribution is very good. We may check whether the deviations of the values of Birge's histogram are near the expected ones. If ΔN_G are the values of the Gaussian curve fitted to the histogram of Fig. 9.4 (thick curve), then the expected deviations will be, according to the Poisson distribution, of the order of $\sqrt{\Delta N_G}$. The curves ΔN_G and $\Delta N_G \pm \sqrt{\Delta N_G}$ are also drawn in the figure. We note that the deviations of the histogram's columns from the Gaussian curve are within or near the expected limits.

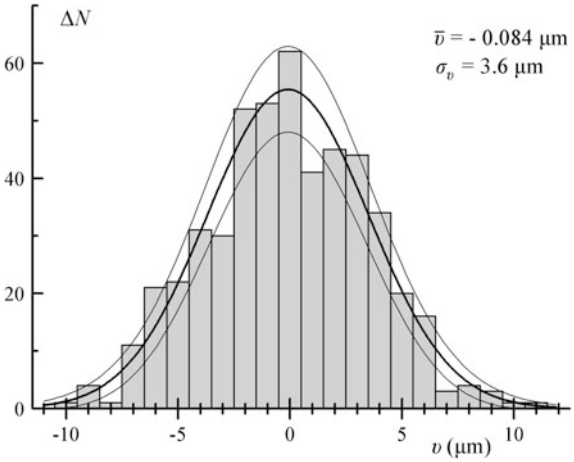


Fig. 9.4 Histogram of the residuals $\upsilon_i = x_i - \bar{x}$ of the 500 measurements performed by Birge with a spectrometer in order to test the normal law of errors. The Gaussian fitted to the histogram has a standard deviation of $\sigma_\upsilon = 3.6$ μm. Apart from the Gaussian curve ΔN_G (*thick line*) the curves $\Delta N_G \pm \sqrt{\Delta N_G}$ are also drawn. The expected standard deviation of ΔN_G is, according to the Poisson distribution, equal to $\sqrt{\Delta N_G}$

In other cases, the errors in the measurements of other experiments are not described so satisfactorily by a Gaussian curve. In some cases, the data required the use of a sum of two normal curves with different standard deviations. Naturally, it is obvious that the use of two curves, with four parameters to be determined instead of the two of a single curve, will always give better agreement than a single curve. On the other hand, the need for the use of two curves for a better fit, might be an indication of either that the errors are due to two widely different sources or that the measurements were not performed under exactly the same experimental conditions (i.e. they are derived from two different parent populations).

9.2 The Lyapunov Central Limit Theorem

A theorem of very great importance in the Theory of Probability and Statistics is *the Lyapunov central limit theorem*, which we will now discuss, without proving it [5]. An elementary formulation of the theorem, which is adequate for the purposes of this book, is the following:

If $x_1, x_2, \ldots x_N$ are the N values of a random sample taken from a parent population of the random variable **x**, *which has mean μ and standard deviation σ, then, as the number N tends to infinity, the distribution of the means $\bar{x}$ of the x_i approaches a normal distribution with mean and standard deviation*

$$\overline{(\bar{x})} = \mu_{\bar{x}} = \mu \quad \text{and} \quad \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}, \tag{9.8}$$

respectively. In other words, the probability density of the means $\bar{x} = \frac{1}{N}(x_1 + x_2 + \ldots + x_N)$ tends to

$$f(\bar{x}) = \frac{1}{\sqrt{2\pi}\,\sigma_{\bar{x}}} \mathrm{e}^{-(\bar{x}-\mu_{\bar{x}})^2/2\sigma_{\bar{x}}^2} = \frac{\sqrt{N}}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-N(\bar{x}-\mu)^2/2\sigma^2}. \tag{9.9}$$

It must be noted that the distribution of the parent population does not need to be normal.

We will explain the meaning of the theorem with the aid of Fig. 9.5. Figure 9.5a shows the probability density of the parent distribution of all the possible results of the measurements of x. To make the description of the sampling method easier, the area under the curve has been divided into a finite number of identical rectangular cells. In the limit, this number will be considered to tend to infinity. To each cell there corresponds a small region of x values, given by the projection of the cell on the x-axis. The process of *sampling* is simply the random picking of a number N of these cells. The cells have the same probability of being picked during the sampling. At points of large $f(x)$, the vertical column consists of a larger number of cells and the values of x corresponding to this column are more likely to be the result of a measurement. The 10 black cells in the figure could be the ones selected in a sampling with $N = 10$ values (measurements). Their projection on the x-axis results in the histogram of the 10 measurements (Fig. 9.5b). The mean $\bar{x}$ of these N values is evaluated. The central limit theorem states that, independently of the shape of the distribution of the parent population, these means, $\bar{x}$, which result from different series of N measurements each, have a distribution which, for large N, tends to a normal distribution with mean equal to the mean of the parent population, $\overline{(\bar{x})} = \mu_{\bar{x}} = \mu$, and standard deviation $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}}$, where σ is the standard deviation of the parent population.

We will demonstrate the central limit theorem with a few examples, in which the sampling is done from parent populations of known distributions.

Example 9.1

Use the Monte Carlo method to check the validity of the central limit theorem for measurement numbers N = 1, 2, 4, 8 and 16, when the parent distribution of the measurements has probability density: $f(x) = 0$ everywhere, except in the interval $0 \leq x \leq 1$, in which it is $f(x) = 1$.

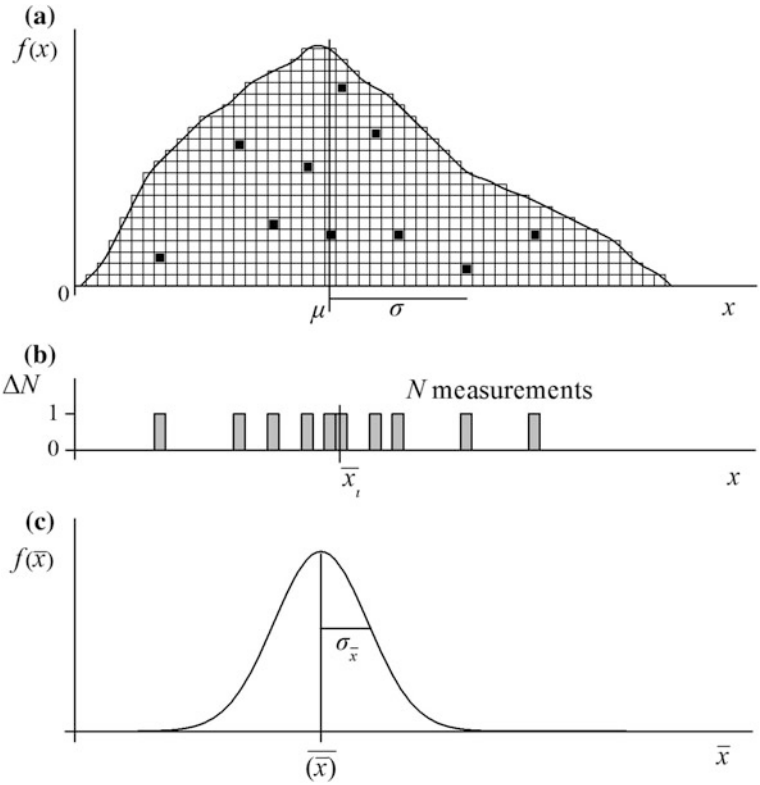
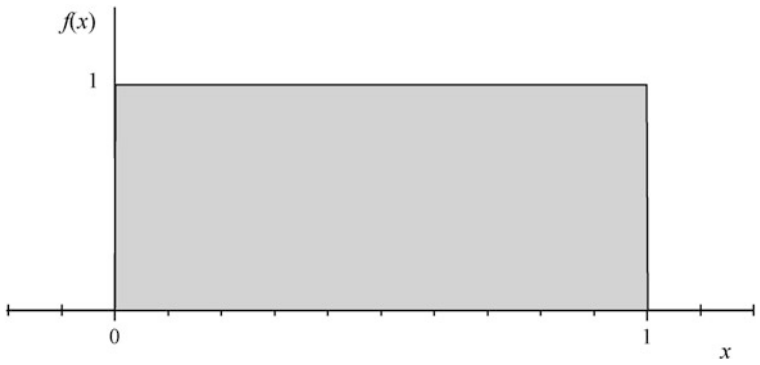


Fig. 9.5 The central limit theorem. **a** The parent population, with mean μ and standard deviation σ. **b** One series of N measurements with mean $\bar{x}_i$. **c** The distribution of the means, $\bar{x}_i$. It has a mean $\overline{(\bar{x})} = \mu_{\bar{x}}$, which tends to μ for large N and a standard deviation $\sigma_{\bar{x}}$, which tends to $\sigma/\sqrt{N}$

The probability density $f(x)$ has been drawn in the figure that follows.



It has a mean value $\mu = 0.5$ and a standard deviation given by the relation

$$\sigma = \sqrt{\int_0^1 (x - 0.5)^2 f(x)\, dx} = \sqrt{\left[\frac{1}{3}(x - 0.5)^3\right]_0^1} = \sqrt{\frac{1}{12}} = 0.289.$$

The Monte Carlo method will be used in a simulation of the experimental process in order to 'find' the results x_i of the measurements. The first 50 000 of the decimal digits of π were used as the random numbers required for the application of the method. They were divided into groups of 5 digits and divided by 10^5, thus giving 10 000 numbers between 0 and 1, with 5 significant figures each (from 0.00000 to 0.99999).

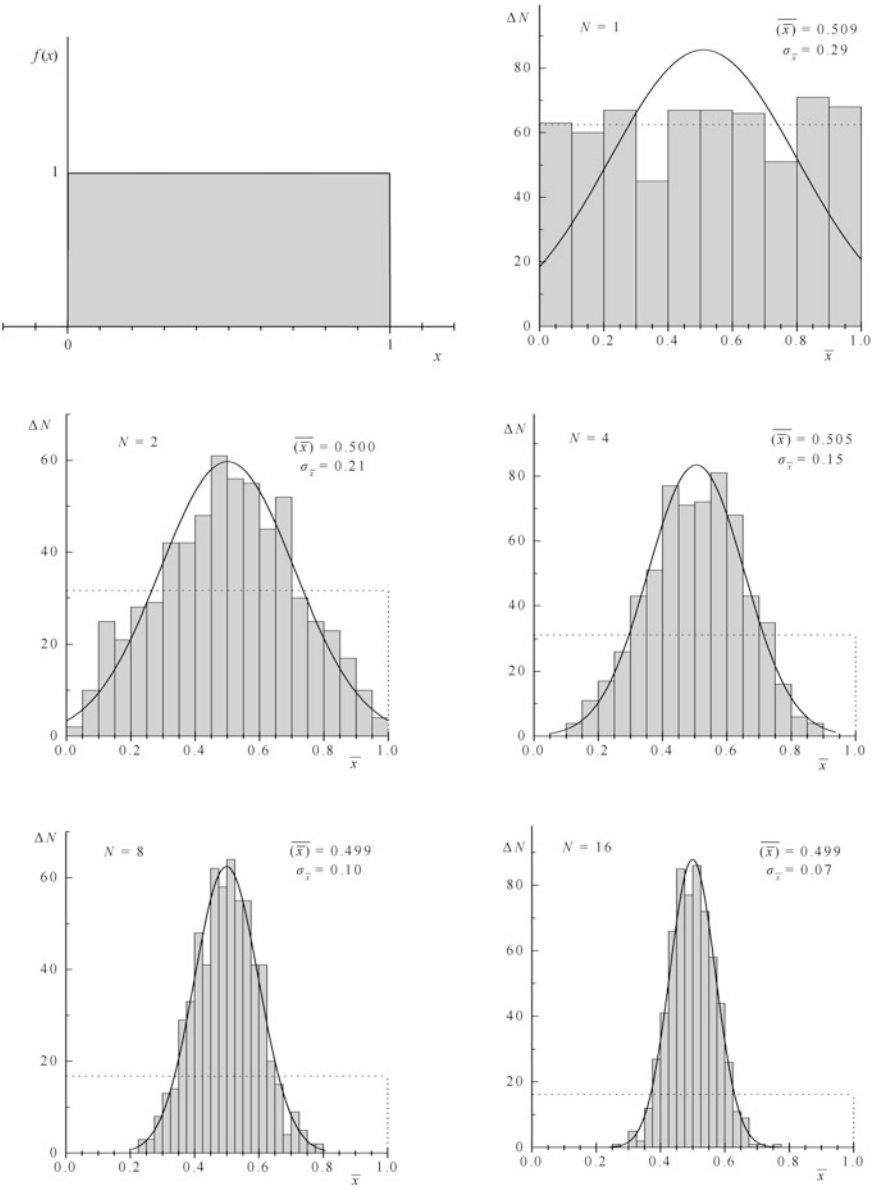
Since the results x_i are uniformly distributed between 0 and 1 (constant probability density) and the same must be true for random numbers, if they are indeed random, the random numbers found as described above are taken directly to be the values of x_i.

Thus, the first 625 numbers gave 625 results x_i, which are recorded in the first histogram of the figure given below ($N = 1$). The variable is denoted by $\bar{x}$ for uniformity with the rest of the histograms, but these 'mean' values consist of one measurement each ($N = 1$). As a consequence, this histogram must reproduce, approximately, the probability density $f(x)$. The Gaussian which is fitted to the histogram, with some dose of exaggeration, has a mean of 0.509 and $\sigma_{\bar{x}} = 0.29$.

Next, the first $2 \times 625 = 1250$ random numbers gave 625 pairs of values, the means of which, $\bar{x}$, were found and are given in the second histogram ($N = 2$). Even with N being just two, the fitting of a Gaussian curve to the distribution of the means is very satisfactory.

The procedure is continued for $N = 4$, 8 and 16. For $N = 16$, the $16 \times 625 = 10\,000$ random numbers give 625 series of 16 results of measurement x_i each. The distribution of the means of these groups of 16 values is given by the last histogram. The Gaussian approximation to this histogram is seen to be very good.

Given with all the histograms, in dashed line, is the mean number of values corresponding to each class, taking into account the width of the classes and the total number of means $\bar{x}$, which is 625 in all cases.



The table below gives, for comparison, the theoretically expected values of the means $\overline{(\overline{x})}$ of the $\overline{x}$ and the standard deviations $\sigma_{\overline{x}} = \sigma/\sqrt{N}$, as well as the values determined with the simulation of the experiment which was performed.

N	$\overline{(\bar{x})}$		$\sigma_{\bar{x}}$	
	Theoretically expected	'Experimental' result	Theoretically expected	'Experimental' result
1	0.5	0.509	0.289	0.29
2	0.5	0.500	0.204	0.21
4	0.5	0.505	0.144	0.15
8	0.5	0.499	0.102	0.10
16	0.5	0.499	0.072	0.07

As a conclusion, we note that the mean value of the means, $\overline{(\bar{x})}$, tends very quickly towards the expected value according to the central limit theorem (as should be expected for the relatively large number of 625 measurements!). The distribution of $\bar{x}$ also tends towards a normal distribution very quickly, with its standard deviation of the Gaussian being $\sigma_{\bar{x}} \propto 1/\sqrt{N}$.

In the next example, a distribution of x will be used which is very far from being normal, a kind of 'anti-Gaussian' distribution. The example will also give us the opportunity of a better understanding of the Monte Carlo method.

Example 9.2

Use the Monte Carlo method in order to test the validity of the central limit theorem for numbers of measurements $N = 1, 2, 4, 8$ and 16, when the parent distribution of the measurements is given by the probability density:

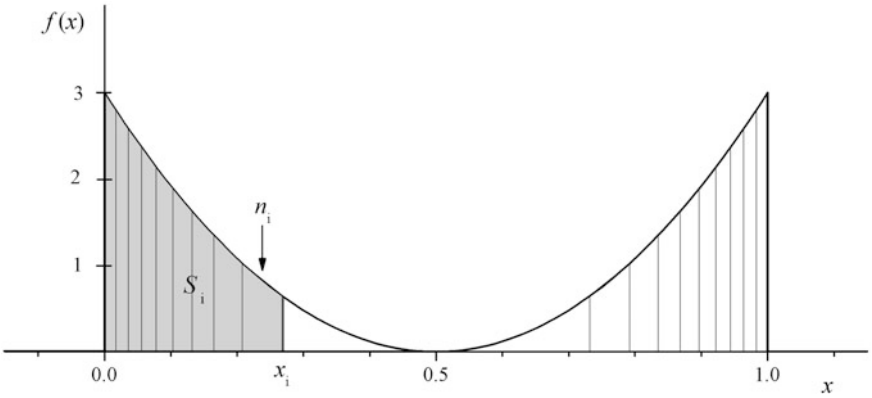
$f(x) = 0$ everywhere except in the region $0 \leq x \leq 1$, where it is $f(x) = 3(1-2x)^2$.

The function of the probability density is normalized. It has a parabolic shape (see figure below) with a minimum equal to 0 at $x = 0.5$. Due to the symmetry of the distribution, the mean value of x is $\bar{x} = 0.5$. This distribution was chosen as an example of an *anti-Gaussian* distribution since results near the mean have very small probability of being observed, while the opposite happens for values near the edges of the distribution.

The standard deviation of this distribution is

$$\sigma = \sqrt{\int_0^1 (x-0.5)^2 f(x)dx} = \sqrt{12\int_0^1 (x-0.5)^4\,dx} = \sqrt{\left[\frac{12}{5}(x-0.5)^5\right]_0^1}$$
$$= \sqrt{\frac{3}{20}} = 0.387.$$

The Monte Carlo method will be used again in the simulation of the measurement procedure. 10 000 random numbers, n_i, between 0 and 1, with 5 significant figures each, were found exactly as in the last example, using the first 50 000 decimal digits of π. The correspondence of these random numbers n_i to values of x_i is a little more difficult than in the last example, because now the values of x_i are not uniformly distributed between 0 and 1. As the way in which this will be achieved is of general importance for the Monte Carlo method, we will describe it in some detail.



In the present example we need to attribute values x_i of x to 10^4 random numbers. It should be noted that the 10^4 random numbers may take any one of 10^4 possible values, between 0.0000 and 0.9999. We divide the interval [0, 1] into 10^4 strips (see figure), each with different width Δx, choosing these Δx in such a way that the area under the curve between x and $x + \Delta x$ has the value of $f(x)\,\Delta x = 10^{-4}$. The area corresponds to the probability that a value lies between x and $x + \Delta x$. We now have 10^4 surface elements of equal areas and 10^4 possible values of the random numbers. The 10^4 random numbers each have the same probability to appear and the result of a measurement is equally probable to lie in one of the 10^4 surface elements.

We will correspond the 10^4 surface elements to the 10^4 random numbers in increasing order, so that the random number 0.0000 corresponds to the first Δx interval, the number 0.0001 to the second and so on, up to the number 0.9999 which corresponds to the 10 000th interval. We may then say that the appearance of a random number is equivalent to the result of the corresponding measurement lying in the interval between x and $x + \Delta x$ that is covered by the corresponding surface element. Let the increasing order numbers of the strips be N_i, between 0 and 9999. The appearance of a random number n_i, which according to the convention we adopted belongs to the strip with order number $N_i \equiv n_i \times 10^4$, is interpreted to mean that one 'measurement' gave a result lying in this strip. The area of the surface under the curve and to the left of the N_i-th strip is equal to $S_i = N_i \times 10^{-4} = n_i$. We conclude that the corresponding value x_i of x is such that the area under the curve from $x = 0$ to x_i is equal to $S_i = n_i$. Thus, the random number n_i uniquely defines a probability between 0 and 1, which corresponds to an area S_i which, in its turn, corresponds to a value x_i such that

$$S_i = \int_0^{x_i} f(x)\,\mathrm{d}x = n_i.$$

This equation may be solved for $x_i(n_i)$.

For the probability density $f(x) = 3(1-2x)^2$ $(0 \le x \le 1)$, we find that it is $S_i = \frac{1}{2}\left[1 + (2x_i - 1)^3\right]$ and, therefore, $x_i = \frac{1}{2} + \frac{1}{2}(2S_i - 1)^{1/3}$, from which relation we have the correspondence of x_i to the random number n_i:

$$x_i = \frac{1}{2} + \frac{1}{2}(2n_i - 1)^{1/3}.$$

In order to apply the Monte Carlo method to the present example, the random numbers of the previous example were used. Thus, the first 625 numbers gave, via the last relation, 625 results x_i, which are recorded in the first histogram of the figure that follows (for $N = 1$). Again the variable is denoted by $\overline{x}$ for the purposes of uniformity with the rest of the histograms, although these 'mean values' are actually single values. This histogram must reproduce the probability density $f(x)$, something which is seen to happen to a satisfactory degree. The process is simply sampling and the result shows the degree to which a sample of 625 measurements may determine the parent population.

The next histogram (for $N = 2$) is based on the first $2 \times 625 = 1250$ random numbers, which gave 625 pairs of values, the means of which, $\overline{x}$, were found. As expected, a maximum is observed in the region of $x = 0.5$, which has its origin in the means of pairs of values, one of which originates from the region around the center of the distribution ($\approx$0) and the other from the regions of its edges ($\approx$1). Striking maxima and minima due to similar combinations are still visible in the histogram for $N = 4$, but the Gaussian shape of the distribution is already clearly visible. The mean for $N = 4$ is 0.511 (instead of the expected 0.5) and the standard deviation is 0.22 (instead of the expected 0.19).

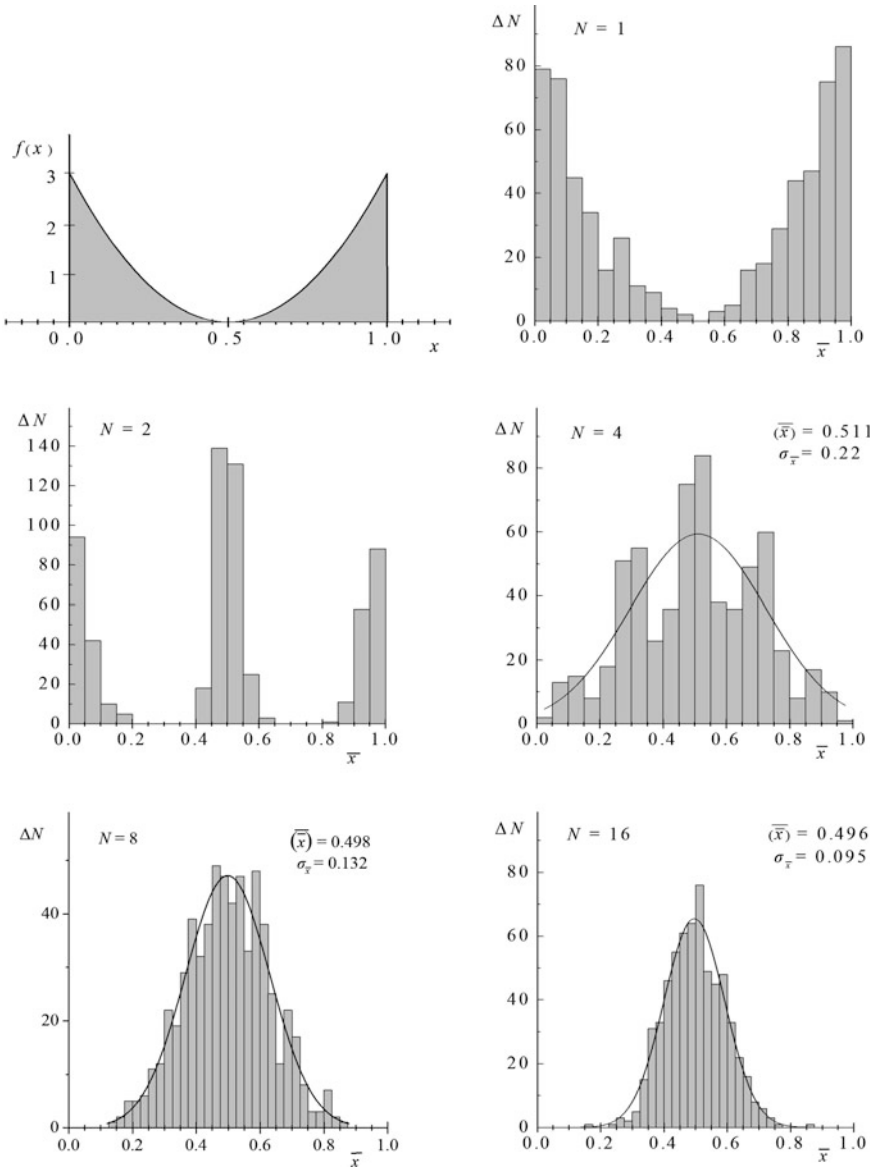
The normal shape of the histograms is more evident for means evaluated from $N = 8$ and $N = 16$ values of x_i (last two histograms).

The table that follows gives the theoretically expected values of the mean values $\overline{(\overline{x})}$ of the means $\overline{x}$ and of the standard deviations $\sigma_{\overline{x}} = \sigma/\sqrt{N}$, as well as the values which were determined through the simulation of the experiment we have performed.

N	$\overline{(\overline{x})}$		$\sigma_{\overline{x}}$	
	Theoretically expected	'Experimental' result	Theoretically expected	'Experimental' result
1	0.5	–	0.387	–
2	0.5	–	0.274	–
4	0.5	0.511	0.194	0.220
8	0.5	0.498	0.137	0.132
16	0.5	0.496	0.097	0.095

We note that, although the distribution of the measurements of the parent population is far from being normal, the distribution of the means $\overline{x}$ tends, relatively

quickly, to the normal form, in agreement with the central limit theorem. The standard deviations are again observed to be in satisfactory agreement with the theoretical relation $\sigma_{\overline{x}} = \sigma/\sqrt{N}$.



One more (purely theoretical) example demonstrating the validity of the central limit theorem will be given in Sect. 9.6 (Example 9.5), in the study of convolution and the calculation of the means and the standard deviations of sums of numbers picked from a certain distribution.

A note regarding random numbers. The subject of random numbers is of enormous importance in applications of the Monte Carlo method and in simulation in general. There are large tables of random numbers [6], as well as algorithms for their production [7] (pseudorandom numbers). In this book, we usually choose to use the decimal digits of π as a source of random numbers. The absolute randomness of a series of digits is not easy to prove and is actually impossible to prove beyond any doubt. The only thing one can say is that the series of these particular digits have passed successfully certain basic tests, such as, for example, that the variations of the frequencies of appearance of the 10 digits (0, 1, …, 9) are within the statistically expected limits, the same holds for the 100 two-digit combinations (00, 01, 02, …, 99) etc. The digits of π have successfully passed these tests [8]. Of course, a large number of people, ranging from professional mathematicians to amateurs interested in the theory of numbers, continuously search and find many coincidences, the occurrence of which is expected to be much rarer than observed in practice. Yasumasa Kanada, for example, having calculated the first 206.1 billion decimal digits of π, found out that the sequence 01234567891 appears 5 times, instead of the expected two times. Not paying any attention to such 'strange phenomena', the decimal digits of π are considered to be adequately random for the purposes of this book.

9.3 The Best Estimate that May Be Made for the Real Value of a Magnitude, Based on the Results of *N* Measurements of It

Assume that we have N values (measurements), x_i $(i = 1, 2, \ldots, N)$, of a random variable, which we will denote by $\mathbf{x}$. We take these values to be normally distributed about the real value x_0 of $\mathbf{x}$, with standard deviation σ. Then, referring to Fig. 9.6, we may say that

the probability for the first value of $\mathbf{x}$ to lie between x_1 and $x_1 + \mathrm{d}x_1$ is $\frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-(x_1 - x_0)^2/2\sigma^2} \mathrm{d}x_1$,
the probability for the second value of $\mathbf{x}$ to lie between x_2 and $x_2 + \mathrm{d}x_2$ is $\frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-(x_2 - x_0)^2/2\sigma^2} \mathrm{d}x_2$ etc., and
the probability for the N-th value of $\mathbf{x}$ to lie between x_N and $x_N + \mathrm{d}x_N$ is $\frac{1}{\sqrt{2\pi}\,\sigma} \mathrm{e}^{-(x_N - x_0)^2/2\sigma^2} \mathrm{d}x_N$.

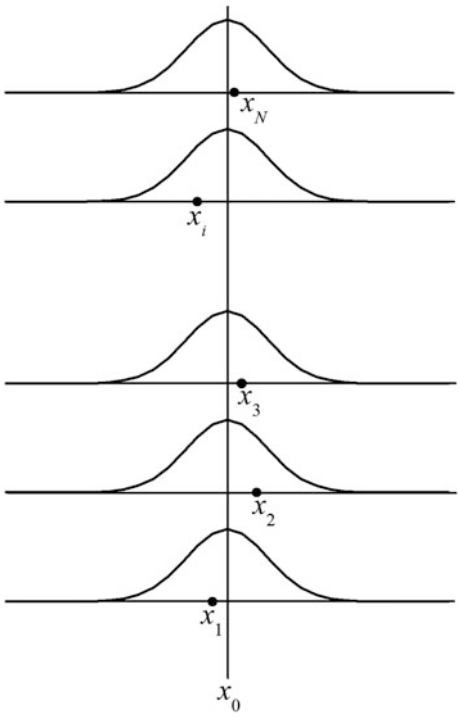


Fig. 9.6 The probability density function for a result x, and the results x_i of N measurements

The compound probability for all the measurements to lie within the limits mentioned is:

$$\mathrm{d}^N P = \frac{1}{\left(\sqrt{2\pi}\,\sigma\right)^N} \exp\left\{-\frac{1}{2\sigma^2}\left[(x_1 - x_0)^2 + (x_2 - x_0)^2 + \ldots + (x_N - x_0)^2\right]\right\} \times \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_N, \tag{9.10}$$

or

$$\mathrm{d}^N P = \frac{1}{\left(\sqrt{2\pi}\,\sigma\right)^N} \mathrm{e}^{-\chi^2/2}\, \mathrm{d}^N \upsilon, \tag{9.11}$$

where

$$\chi^2 \equiv \frac{1}{\sigma^2}\left[(x_1 - x_0)^2 + (x_2 - x_0)^2 + \ldots + (x_N - x_0)^2\right] \tag{9.12}$$

and

$$\mathrm{d}^N \upsilon = \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_N \tag{9.13}$$

may be considered to be the element of the N-dimensional volume about the point $(x_1, x_2 \ldots x_N)$.

The real value x_0 is not known. We consider that the best estimate we can make for it is the value $\hat{x}_0$ of x_0 which maximizes the probability for the results of the measurements of $\mathbf{x}$ we have to occur. For given limits dx_1, dx_2, …, dx_N, this happens when the quantity χ^2 is minimum, i.e. for

$$\frac{\partial \chi^2}{\partial x_0} = \frac{2}{\sigma^2}\left[(x_1 - x_0) + (x_2 - x_0) + \ldots + (x_N - x_0)\right]_{x_0 = \hat{x}_0} = 0 \tag{9.14}$$

or

$$N\hat{x}_0 - (x_1 + x_2 + \ldots + x_N) = 0. \tag{9.15}$$

The best estimate for x_0 is, therefore,

$$\hat{x}_0 = \frac{1}{N}(x_1 + x_2 + \ldots + x_N), \tag{9.16}$$

i.e. the mean of the N values of x.

The result may be considered to be a proof of *the principle of least squares*, first formulated by Legendre. The principle states that:

> *The most probable value of a magnitude being measured is that which minimizes the sum of the squares of the deviations of the results of the measurements from this value.*

Both Gauss and Laplace studied this principle. Gauss, assuming the mean of the measurements to be the most probable value of the magnitude being measured, derived the normal law of errors. Inversely, the normal law of errors may be used in order to prove the principle of the most probable value, as we have done.

9.4 The Weighting of Values

If the values we have at our disposal come from different parent populations with different standard deviations (i.e. distributions relative to the real value) due to the different accuracies in the determination of each value, then we will have

$$\mathrm{d}^N P = \frac{1}{\left(\sqrt{2\pi}\right)^N \sigma_1 \sigma_2 \ldots \sigma_N} \exp\left\{-\left[\frac{(x_1 - x_0)^2}{2\sigma_1^2} + \frac{(x_2 - x_0)^2}{2\sigma_2^2} + \ldots + \frac{(x_N - x_0)^2}{2\sigma_N^2}\right]\right\} \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_N, \tag{9.17}$$

and the minimization of

$$\chi^2 = \frac{(x_1 - x_0)^2}{\sigma_1^2} + \frac{(x_2 - x_0)^2}{\sigma_2^2} + \ldots + \frac{(x_N - x_0)^2}{\sigma_N^2} \tag{9.18}$$

gives the relation

$$\hat{x}_0 = \frac{(x_1/\sigma_1^2) + (x_2/\sigma_2^2) + \ldots + (x_N/\sigma_N^2)}{1/\sigma_1^2 + 1/\sigma_2^2 + \ldots + 1/\sigma_N^2} \tag{9.19}$$

as the best estimate for the real value x_0.

Equation (9.19) may also be written as

$$\hat{x}_0 = \frac{\sum_{i=1}^{N} w_i x_i}{\sum_{i=1}^{N} w_i} = \overline{x} \tag{9.20}$$

which is the *weighted mean* of x, with *statistical weight* for value x_i equal to

$$w_i = \frac{1}{\sigma_i^2}. \tag{9.21}$$

The importance of a value in the determination of the mean is, therefore, inversely proportional to the square of its standard deviation. The bigger the standard deviation of a measurement, the smaller the weight it is given in the determination of the mean, something that appears qualitatively reasonable. When all the measurements have the same weight, it is $w_i / \sum_i w_i = 1/N$ and the equations reduce to the known ones.

More generally, if the weights $w_1, w_2, \ldots, w_N$ are attributed, for whatever reason, to the values of measurements $x_1, x_2, \ldots, x_N$, respectively, the weighted mean of these values is given by

$$\overline{x} = \frac{\sum_{i=1}^{N} w_i x_i}{\sum_{i=1}^{N} w_i}. \tag{9.22}$$

Equations (9.18) and (9.19) show that the magnitude $\sum w_i (x_i - x_0)^2$ has a minimum when $\overline{x} = x_0$. In order to evaluate the standard deviation of the weighted values $x_1, x_2, \ldots, x_N$, we normalize the statistical weights w_i so that their sum is equal to unity, by dividing each one with $\sum w_i$. Defining the *normalized statistical weights*

$$\beta_i \equiv \frac{w_i}{\sum_{i=1}^{N} w_i}, \tag{9.23}$$

for which it is true that

$$\sum_i \beta_i = 1, \tag{9.24}$$

we have the weighted mean of x,

$$\overline{x} = \sum_{i=1}^{N} \beta_i x_i. \tag{9.25}$$

When all the measurements have the same weight, it is $\beta_i = \frac{1}{N}$.

The *weighted standard deviation* of the values x_i is defined as

$$s_x \equiv \sqrt{\sum_{i=1}^{N} \beta_i (x_i - \overline{x})^2} \tag{9.26}$$

and the *weighted standard deviation of the mean* $\overline{x}$ is given by

$$\sigma_{\overline{x}} \equiv \sqrt{\frac{1}{(N-1)} \sum_{i=1}^{N} \beta_i (x_i - \overline{x})^2}, \tag{9.27}$$

where N here is the number of x values with non-zero weight. It should be noted that in the evaluation of the standard deviation, β_i is used as statistical weight and not β_i^2.

Example 9.3

The results of 5 measurements, x_i, with their statistical weights w_i are given in columns 2 and 3 of the table below. Find the weighted mean of the results and its standard deviation.

i	x_i	w_i	β_i	$\beta_i x_i$	$x_i - \overline{x}$	$(x_i - \overline{x})^2$	$\beta_i (x_i - \overline{x})^2$
1	5.05	2	0.182	0.919	−0.074	0.00548	0.000997
2	5.25	1	0.091	0.478	0.126	0.01588	0.001445
3	5.16	3	0.273	1.409	0.036	0.00130	0.000355
4	5.09	4	0.363	1.848	−0.034	0.00116	0.000420
5	5.17	1	0.091	0.470	0.046	0.00212	0.000193
Sums		11	1	5.124			0.003410

The weighted mean is $\overline{x} = \sum_{i=1}^{N} \beta_i x_i = 5.124$.

The weighted standard deviation of the measurements is $s_x = \sqrt{\sum_{i=1}^{N} \beta_i (x_i - \overline{x})^2} = \sqrt{0.003410} = 0.0584$ and the weighted standard deviation of the mean is

$$\sigma_{\overline{x}} = \sqrt{\frac{1}{(N-1)} \sum_{i=1}^{N} \beta_i (x_i - \overline{x})^2} = \frac{s_x}{\sqrt{N-1}} = \frac{0.0584}{\sqrt{4}} = 0.0292.$$

The final result is:

$$x = 5.124 \pm 0.029.$$

Without weighting, these quantities are $\overline{x} = 5.144$, $s_x = 0.069$ and $\sigma_{\overline{x}} = 0.035$.

Example 9.4 [E]

Solve Example 9.3 using Excel®.

We will first evaluate the weighted mean. We enter the values of x_i and w_i in columns A and B, respectively. Highlight an empty cell, say E1. Left click on cell E1 and type:

$$=\mathbf{SUMPRODUCT}(\mathbf{A1}{:}\mathbf{A5}; \mathbf{B1}{:}\mathbf{B5})/\mathbf{SUM}(\mathbf{B1}{:}\mathbf{B5})$$

Pressing **ENTER** will return the number 5.123636 in cell E1. This is the required mean, $\overline{x} = 5.1236$ mm.

We will give this number the name **M**. To do this, we right click on cell E1. In the dialog box that opens, we select **Define Name…** and in the cell for **Name** we write **M**. Press **ENTER**.

We will now evaluate the weighted standard deviation. We first evaluate the terms $(x_r - \overline{x})^2$. We highlight cell C1 and type: **=(A1-M)^2**. Pressing **ENTER** returns the number 0.005422 in cell C1. To fill cells C1 to C5 with the values of $(x_r - \overline{x})^2$, we highlight cells C1-C5 and press **Fill > Down**.

To evaluate the standard deviation, we highlight an empty cell, say E2 and type

$$=\mathbf{SQRT}(\mathbf{SUMPRODUCT}(\mathbf{B1}{:}\mathbf{B5}; \mathbf{C1}{:}\mathbf{C5})/\mathbf{SUM}(\mathbf{B1}{:}\mathbf{B5}))$$

Pressing **ENTER** returns the number 0.058352.

The weighted standard deviation of the measurements is $s_x = \sqrt{\frac{1}{\sum_{i=1}^{N} w_i} \sum_{i=1}^{N} w_i (x_i - \overline{x})^2} =$ 0.058352 and the weighted standard deviation of the mean is

$$\sigma_{\overline{x}} = \frac{s_x}{\sqrt{N-1}} = \frac{0.058352}{\sqrt{4}} = 0.029176.$$

The final result is:

$$x = 5.124 \pm 0.029.$$

Example 9.5 [O]

Solve Example 9.3 using Origin®.

We enter x_i and w_i in columns A(X) and B(Y). Highlight column A by left-clicking on its label. Then

Statistics > **Descriptive Statistics** > **Statistics on Columns** > **Open Dialog**...

In the window that opens, in **Input Data**, **Range 1**, **Data Range**, column A is already selected. In **Weighting Range**, we select column B(Y).

In **Quantities**, we click **Mean** and **Standard Deviation**.

We open the window **Computation Control**. We select **Weight Method, Direct Weight** and **Variant Divisor of Moment, WS**. We press **OK**. The results are:

Mean = $\overline{x} = 5.12364$, Standard Deviation = $s_x = 0.05835$.

We calculate $\sigma_{\overline{x}} = \frac{s_x}{\sqrt{N-1}} = \frac{0.05835}{\sqrt{5-1}} = 0.02918$

Summarizing, $\overline{x} = 5.124$, $s_x = 0.058$ and $\sigma_{\overline{x}} = 0.029$, in agreement with the results of Example 9.3.

Example 9.6 [P]

Solve Example 9.3 using Python.

```
from __future__ import division
import numpy as np
import math

# Enter the values given as the components of the vector x:
x = np.array([5.05, 5.25, 5.16, 5.09, 5.17])

# Enter the corresponding weights w of the x values:
w = np.array([2, 1, 3, 4, 1])

# Evaluation
N = len(x)
wmean = np.average(x, weights = w)
variance = np.average((x-wmean)**2, weights = w)
stdev = math.sqrt(variance)
```

```
# Presentation of the results:
print ("Number of values, N =", N)
print ("Weighted mean = ", wmean)
print ("Weighted standard deviation of the sample =", stdev)
print ("Weighted standard deviation of the mean =", stdev/math.sqrt(N-1))

# Results:
Number of values, N = 5
Weighted mean = 5.12363636364
Weighted standard deviation of the sample = 0.058352023766840865
Weighted standard deviation of the mean = 0.029176011883420432
```

Example 9.7 [R]

Solve Example 9.3 using R.

The vectors x and w have as their components the values of x and w, respectively:

```
> x <- c(5.05, 5.25, 5.16, 5.09, 5.17)
> w <- c(2, 1, 3, 4, 1)
```

The weighted mean is found as

```
> wmean = weighted.mean(x,w)
> wmean
[1] 5.123636
```

The variance of the sample, s_x^2, is the weighted mean of the quantity $(x_i - \overline{x})^2$. This is found to be:

```
> variance = weighted.mean((x-wmean)^2, w)
> variance
[1] 0.003404959
```

The standard deviation of the sample is $s_x = \sqrt{s_x^2}$

```
> sqrt(variance)
[1] 0.05835202
```

and the standard deviation of the mean is $\sigma_{\overline{x}} = \frac{s_x}{\sqrt{N-1}} = \frac{0.0584}{\sqrt{4}} = 0.0292$.

Summarizing, $\overline{x} = 5.124$, $s_x = 0.058$ and $\sigma_{\overline{x}} = 0.029$, in agreement with the results of Example 9.3.

Let us suppose that we have N results of measurements, x_i, each with its weight w_i and that they can be grouped in a number of K classes each of which consists of measurements with the same values of x and w:

$$\overbrace{\underbrace{w_1x_1 + w_1x_1 + \ldots + w_1x_1}_{k=1}}^{n_1 \text{ terms}} + \overbrace{\underbrace{w_2x_2}_{k=2}}^{n_2 \text{ terms}} + \overbrace{\underbrace{w_3x_3 + \ldots + w_3x_3}_{k=3}}^{n_3 \text{ terms}} + \ldots + \overbrace{\underbrace{w_Kx_K + w_Kx_K + \ldots + w_Kx_K}_{k=K}}^{n_K \text{ terms}}. \tag{9.28}$$

It should be noted that any two x's or any two w's may be the same in two different groups but not *both x and w* may be the same in any two different groups. In such a case the two terms would be placed in the same group.

In this case, the numerator of Eq. (9.22) may be written as

$$\sum_{i=1}^{N} w_i x_i = n_1 w_1 x_1 + n_2 w_2 x_2 + \ldots + n_K w_K x_K = \sum_{k=1}^{K} n_k w_k x_k. \tag{9.29}$$

Similarly,

$$\sum_{i=1}^{N} w_i = n_1 w_1 + n_2 w_2 + \ldots + n_K w_K = \sum_{k=1}^{K} n_k w_k. \tag{9.30}$$

Therefore, the *weighted mean* is

$$\overline{x} = \frac{\sum_{k=1}^{K} n_k w_k x_k}{\sum_{k=1}^{K} n_k w_k} \tag{9.31}$$

It is seen that the product $n_k w_k$ replaces the weight w_i in estimating the weighted mean. In this sense it may be considered to be an *active weight*.

In a similar way, the *weighted sample standard deviation* is

$$s_x = \sqrt{\frac{\sum_{k=1}^{K} n_k w_k (x_k - \overline{x})^2}{\sum_{k=1}^{K} n_k w_k}}. \tag{9.32}$$

The *weighted population standard deviation* is

$$\sigma_{\overline{x}} = \sqrt{\frac{\sum_{k=1}^{K} n_k w_k (x_k - \overline{x})^2}{(N-1)\sum_{k=1}^{K} n_k w_k}}, \tag{9.33}$$

where N is the number of x values with non-zero weight.

In a way similar to that of Eq. (9.23) we may define the *normalized statistical weights*

$$\beta_k \equiv \frac{n_k w_k}{\sum_{k=1}^{K} n_k w_k}, \tag{9.34}$$

for which it is true that

$$\sum_k \beta_k = 1, \tag{9.35}$$

Equations (9.31)–(9.33) reduce to Eqs. (9.25)–(9.27), respectively.

Example 9.8

The results of $N = 30$ measurements x_i, with their statistical weights w_i, are grouped in $K = 9$ classes as shown in the table below.

k	1	2	3	4	5	6	7	8	9
n_k	2	1	4	7	6	3	4	1	2
x_k	4.60	4.70	4.80	4.90	5.00	5.10	5.20	5.30	5.40
w_k	3	2	4	4	3	4	1	2	2

Find the weighted mean of the results and its standard deviation.

We construct a table with the quantities k, n_k, x_k, w_k, $n_k w_k$, $n_k w_k x_k$, $(x_k - \overline{x})^2$ and $n_k w_k (x_k - \overline{x})^2$.

k	n_k	x_k	w_k	$n_k w_k$	$n_k w_k x_k$	$(x_k - \overline{x})^2$	$n_k w_k (x_k - \overline{x})^2$
1	2	4.6	3	6	27.6	0.1267	0.7602
2	1	4.7	2	2	9.40	0.0655	0.1310
3	4	4.8	4	16	76.8	0.0243	0.3888
4	7	4.9	4	28	137.2	0.0031	0.0868
5	6	5.0	3	18	90.0	0.0019	0.0342
6	3	5.1	4	12	61.2	0.0208	0.2496
7	4	5.2	1	4	20.8	0.0596	0.2384
8	1	5.3	2	2	10.6	0.1184	0.2368
9	2	5.4	2	4	21.6	0.1972	0.7888
Sums:	30			92	455.2		2.9146

The weighted mean is $\overline{x} = \frac{\sum_{k=1}^{K} n_k w_k x_k}{\sum_{k=1}^{K} n_k w_k} = \frac{455.2}{92} = 4.948.$

The weighted standard deviation of the sample is

$$s_x = \sqrt{\frac{\sum_{k=1}^{K} n_k w_k (x_k - \overline{x})^2}{\sum_{k=1}^{K} n_k w_k}} = \sqrt{\frac{2.9146}{92}} = 0.1778.$$

The weighted standard deviation of the mean is $\sigma_{\overline{x}} = \frac{s_x}{\sqrt{N-1}}$, where $N = \sum_{k=1}^{9} n_k = 30$. Therefore, $\sigma_{\overline{x}} = \frac{0.1778}{\sqrt{29}} = 0.0330$. The final result is: $x = 4.948 \pm 0.033$.

Example 9.9 [E]

Solve Example 9.8 using Excel®.

Comparing Eqs. (9.31) and (9.32) with Eqs. (9.22) and (9.26), it is obvious that this example is the same as Example 9.4 [E] if we replace w_i with $n_i w_i$. We enter the values of n_i, x_i and w_i in columns A, B and C, respectively. We calculate the values of $n_i w_i$: In cell D1 we type = A1*C1 and press **ENTER**. We fill down to cell D9. Column D now contains the values of $n_i w_i$.

We will first evaluate the weighted mean. Highlight an empty cell, say E1. Left click on cell E1 and write:

$$=\mathbf{SUMPRODUCT}(\mathbf{B1}{:}\mathbf{B9};\mathbf{D1}{:}\mathbf{D9})/\mathbf{SUM}(\mathbf{D1}{:}\mathbf{D9})$$

Pressing **ENTER** will return the number 4.9478 in cell E1. This is the required mean, $\overline{x} = 4.9478$ mm.

We will give this number the name **M**. To do this, we right click on cell E1. In the dialog box that opens, we select **Define Name…** and in the cell for **Name** we write **M**.

We will now evaluate the weighted standard deviation. We first evaluate the terms $(x_r - \overline{x})^2$. We highlight cell F1 and type: **=(B1-M)^2**. Pressing **ENTER** returns the number 0.120983 in cell F1. To fill cells F1 to F9 with the values of $(x_r - \overline{x})^2$, we highlight cells F1-F9 and press **Fill > Down** .

To evaluate the standard deviation, we highlight an empty cell, say G1 and type

$$=\mathbf{SQRT}(\mathbf{SUMPRODUCT}(\mathbf{D1}{:}\mathbf{D9};\mathbf{F1}{:}\mathbf{F9})/\mathbf{SUM}(\mathbf{D1}{:}\mathbf{D9}))$$

Pressing **ENTER** returns the number 0.177836.

The weighted standard deviation of the measurements is $s_x = \sqrt{\frac{1}{\sum_{i=1}^{N} w_i} \sum_{i=1}^{N} w_i (x_i - \overline{x})^2} = 0.177836$ and the weighted standard deviation of the

mean is $\sigma_{\overline{x}} = \frac{s_x}{\sqrt{N-1}}$, where $N = \sum_{k=1}^{9} n_k = 30$. Therefore, $\sigma_{\overline{x}} = \frac{0.1778}{\sqrt{29}} = 0.0330$. The final result is: $x = 4.948 \pm 0.033$.

Example 9.10 [O]

Solve Example 9.8 using Origin®.

We enter n_i, x_i and w_i in columns A(X), B(Y) and C(Y). Highlight column B by left-clicking on its label. Then

Statistics > Descriptive Statistics > Statistics on Columns > Open Dialog...

In the window that opens, in **Input Data**, **Range 1**, **Data Range**, column B is already selected. In **Weighting Range**, we select column C(Y).

In **Quantities**, we click **Mean** and **Standard Deviation**.

We open the window **Computation Control**. We select **Weight Method, Direct Weight** and **Variance Divisor of Moment, WS**. We press **OK**. The results are:

Weighted Mean $\overline{x} = 4.94783$, Weighted Standard Deviation of the Sample $s_x = 0.17784$.

We calculate the weighted standard deviation of the mean using the equation $\sigma_{\overline{x}} = \frac{s_x}{\sqrt{N-1}}$, where $N = \sum_{k=1}^{9} n_k = 30$. Therefore, $\sigma_{\overline{x}} = \frac{0.1778}{\sqrt{29}} = 0.0330$. The final result is: $x = 4.948 \pm 0.033$.

Example 9.11 [P]

Solve Example 9.8 using Python.

```
from __future__ import division
import numpy as np
import math

# Enter values of members of the groups:
n = np.array([2, 1, 4, 7, 6, 3, 4, 1, 2])

# Enter the values given as the components of the vector x:
x = np.array([4.6, 4.7, 4.8, 4.9, 5, 5.1, 5.2, 5.3, 5.4])

# Enter the corresponding weights w of the x values:
wt = np.array([3, 2, 4, 4, 3, 4, 1, 2, 2])

# "Active" weights:
w = n*wt
```

```
# Evaluation
G = len(x)
N = sum(n)
wmean = np.average(x, weights = w)
variance = np.average((x-wmean)**2, weights = w)
stdev = math.sqrt(variance)

# Presentation of the results
print ("Number of groups, G =", G)
print ("Number of measurements, N =", N)
print ("Weighted mean =", wmean)
print ("Weighted standard deviation of the sample =", stdev)
print ("Weighted standard deviation of the mean =", stdev/math.sqrt(N-1))

# Results:
Number of groups, G = 9
Number of measurements, N = 30
Weighted mean = 4.94782608696
Weighted standard deviation of the sample = 0.17783618553232663
Weighted standard deviation of the mean = 0.033023350612542336
```

Example 9.12 [R]

Solve Example 9.8 using R.

Comparing Eqs. (9.31) and (9.32) with Eqs. (9.22) and (9.26), it is obvious that this example is the same as Example 9.4 [E] if we use as weights the values $W_k = n_k w_k$.

k	1	2	3	4	5	6	7	8	9
n_k	2	1	4	7	6	3	4	1	2
x_k	4.60	4.70	4.80	4.90	5.00	5.10	5.20	5.30	5.40
w_k	3	2	4	4	3	4	1	2	2
$W_k = n_k w_k$	6	2	16	28	18	12	4	2	4

We define the vectors

```
> x <- c(4.60, 4.70, 4.80, 4.90, 5, 5.10, 5.20, 5.30, 5.40)
> W <- c(6, 2, 16, 28, 18, 12, 4, 2, 4)
>
```

and find the weighted mean

```
> Wmean = weighted.mean(x,W)
> Wmean
[1] 4.947826
>
```

The variance of the sample, s_x^2, is the weighted mean of the quantity $(x_i - \overline{x})^2$. This is found to be:

```
> variance = weighted.mean((x-Wmean)^2, W)
> variance
[1] 0.03162571
>
```

The standard deviation of the sample is $s_x = \sqrt{s_x^2}$

```
> sqrt(variance)
[1] 0.1778362
```

We calculate the weighted standard deviation of the mean using the equation $\sigma_{\overline{x}} = \frac{s_x}{\sqrt{N-1}}$, where $N = \sum_{k=1}^{9} n_k = 30$. Therefore, $\sigma_{\overline{x}} = \frac{0.1778}{\sqrt{29}} = 0.0330$.
Summarizing the results: $\overline{x} = 4.948$, $s_x = 0.1778$ and $\sigma_{\overline{x}} = 0.033$.

In the case when the statistical weight of measurement x_i is, according to Eq. (9.21), equal to $w_i = \frac{1}{\sigma_i^2}$, then, from Eq. (9.22) we have for the weighted mean

$$\overline{x} = \frac{\sum_{i=1}^{N} x_i/\sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2}, \tag{9.28}$$

for the weighted standard deviation of the measurements

$$s_x = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \overline{x})^2/\sigma_i^2}{\sum_{i=1}^{N} 1/\sigma_i^2}} \tag{9.29}$$

and for the weighted standard deviation of the mean

$$\sigma_{\overline{x}} = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \overline{x})^2/\sigma_i^2}{(N-1)\sum_{i=1}^{N} 1/\sigma_i^2}} \tag{9.30}$$

The same is true when we have the mean values $\overline{x}_1$, $\overline{x}_2$, …, $\overline{x}_r$, …, $\overline{x}_M$, of M different series of measurements, which have standard deviations $\sigma_{\overline{x}_1}$, $\sigma_{\overline{x}_2}, \ldots, \sigma_{\overline{x}_r}, \ldots, \sigma_{\overline{x}_M}$, respectively. In this case, the statistical weight of each mean is the inverse of the square of its standard deviation. The means have a (general) mean

$$\overline{(\overline{x})} = \frac{\sum_{r=1}^{M} \overline{x}_r/\sigma_{\overline{x}_r}^2}{\sum_{r=1}^{M} 1/\sigma_{\overline{x}_r}^2} \tag{9.31}$$

while the standard deviation of this general mean is

$$\sigma_{\overline{(\overline{x})}} = \sigma\left(\overline{(\overline{x})}\right) = \sqrt{\frac{\sum_{r=1}^{M} \left[\overline{x}_r - \overline{(\overline{x})}\right]^2/\sigma_{\overline{x}_r}^2}{\sum_{r=1}^{M} 1/\sigma_{\overline{x}_r}^2}}. \tag{9.32}$$

Example 9.13

Three experiments for the determination of the speed of light in vacuum gave the following results, in m/s:

$$c_1 = 299\,792\,459.3 \pm 1.6, \quad c_2 = 299\,792\,457.82 \pm 0.86,$$
$$c_3 = 299\,792\,458.4 \pm 1.1.$$

Find the weighted mean of these results and its standard deviation, taking as weights the inverses of the square of the standard deviation in each case.

Since the numbers are given with many digits, to avoid loss of accuracy, we subtract from all of them the number $c_0 = 299\,792\,40$ m/s and work with the smaller numbers that remain, x.

i	c_i (m/s)	x_i (m/s)	σ_i (m/s)	$1/\sigma_i^2$ $(\text{m/s})^2$	β_i	$\beta_i x_i$ (m/s)	$c_i - \overline{c}$ (m/s)	$(c_i - \overline{c})^2$	$\beta_i(c_i - \overline{c})^2$ $(\text{m/s})^2$
1	299 792 459.3	9.3	1.60	0.3906	0.1520	1.414	1.068	1.141	0.1734
2	299 792 457.82	7.82	0.86	1.3521	0.5263	4.116	–0.412	0.170	0.0895
3	299 792 458.4	8.4	1.10	0.8264	0.3217	2.702	–0.168	0.0282	0.0091
		Sums		2.5691	1	8.232			0.272

The weighted mean of the results is $\overline{c} = c_0 + \overline{x} = c_0 + \sum_i \beta_i x_i = 299\,792\,450 + 8.232 = 299\,792\,458.232$ m/s.

From the sum $\sum_i \beta_i (c_i - \overline{c})^2 = 0.272$ $(\text{m/s})^2$, we find that $s_c = \sqrt{\sum_i \beta_i (c_i - \overline{c})^2} = \sqrt{0.272} = 0.52$ m/s.

By Eq. 9.27, the standard deviation of the mean is $\sigma_{\overline{c}} = \frac{s_c}{\sqrt{N-1}} = \sqrt{\frac{0.272}{2}} = 0.37$ m/s.

Therefore, $c = 299\,792\,458.23 \pm 0.37$ m/s.

Example 9.14 [E]

Solve Example 9.13 using Excel®.

i	c_i (m/s)	x_i (m/s)	σ_i (m/s)
1	299 792 459.3	9.3	1.60
2	299 792 457.82	7.82	0.86
3	299 792 458.4	8.4	1.10

Acting as above, we subtract from all the values the quantity $c_0 = 299\,792\,450$ m/s and work with the smaller numbers that remain, x. We enter x_i and σ_i in cells A2-A4 and B2-B4, respectively. We will evaluate the weights to be used, $w_i = 1/\sigma_i^2$. We highlight cell C2 and type in it **=1/(B2)^2**. We **Fill Down** to cell C4. Column C now contains the values of w_i.

We will first evaluate the weighted mean. Highlight an empty cell, say E2. Left click on cell E2 and write:

$$=\mathbf{SUMPRODUCT}(\mathbf{A2{:}A4};\mathbf{C2{:}C4})/\mathbf{SUM}(\mathbf{C2{:}C4})$$

Pressing **ENTER** will return the number 8.2316 in cell E2. We will give this number the name **M**. To do this, we right click on cell E2. In the dialog box that opens, we select **Define Name…** and in the cell for **Name** we write **M**.

The weighted mean of the results is $\overline{c} = c_0 + \overline{x} = 299\,792\,450 + 8.2316 = 299\,792\,458.232$ m/s.

We will now evaluate the weighted standard deviation. We first evaluate the terms $(x_i - \overline{x})^2$. We highlight cell F2 and type: **=(A2-M)^2**. Pressing **ENTER** returns the number 0.141478 in cell F2. To fill cells F2 to F4 with the values of $(x_i - \overline{x})^2$, we highlight cells F2-F4 and press **Fill > Down**.

To evaluate the standard deviation, we highlight an empty cell, say G2 and type

$$=\mathbf{SQRT}(\mathbf{SUMPRODUCT}(\mathbf{C2}{:}\mathbf{C4};\mathbf{F2}{:}\mathbf{F4})/\mathbf{SUM}(\mathbf{C2}{:}\mathbf{C4}))$$

Pressing **ENTER** returns the number 0.5214. The weighted standard deviation of the measurements is $s_c = \sqrt{\frac{1}{\sum_{i=1}^{N} w_i} \sum_{i=1}^{N} w_i (x_i - \overline{x})^2} = 0.5214$. The standard deviation of the mean is $\sigma_{\overline{c}} = \frac{s_c}{\sqrt{N-1}} = \frac{0.5214}{\sqrt{2}} = 0.3687$ m/s. Therefore, c = 299 792 458.23 ± 0.37 m/s.

Example 9.15 [O]

Solve Example 9.13 using Origin®.

i	c_i (m/s)	x_i (m/s)	σ_i (m/s)
1	299 792 459.3	9.3	1.60
2	299 792 457.82	7.82	0.86
3	299 792 458.4	8.4	1.10

Acting as above, we subtract from all the measurements the quantity $c_0 = 299\,792\,450$ m/s and work with the smaller numbers that remain, x. We enter x_i and σ_i in columns A and B, respectively.

Highlight column A and then: **Column > Set As > Y**
Highlight column B and then: **Column > Set As > Y Error**

Highlight columns A and B and then,

Statistics > Descriptive Statistics > Statistics on Columns > Open Dialog...

In the window that opens,

Input > Input Data > Range 1 > Weighting Range > B(E)

Open the **Quantities** window and tick: **Mean, Standard Deviation**

Open the **Computation Control** window and **Weight Method > Instrumental**

The last choice sets the weight of each measurement x_i equal to $w_i = 1/\sigma_i^2$, where σ_i is the error in x_i. Then,

Variance Divisor of Moment > WS

The last choice sets the denominator of Eq. (9.29) equal to $w = \sum_i 1/\sigma_i^2$.

Pressing **OK** we obtain the results (for column A):

[Mean] $= \overline{x} = 8.2316$ m/s, [Standard Deviation] $= s_c = s_x = 0.52138$ m/s

By Eq. 9.27, the standard deviation of the mean is $\sigma_{\overline{c}} = \frac{s_c}{\sqrt{N-1}} = \frac{0.52138}{\sqrt{2}} = 0.36867$ m/s.

The final result is $c = 299\,792\,458.23 \pm 0.37$ m/s, in agreement with the results of Example 9.4.

Example 9.16 [P]

Three experiments for the determination of the speed of light in vacuum gave the following results, in m/s:

$$c_1 = 299\,792\,459.3 \pm 1.6, \quad c_2 = 299\,792\,457.82 \pm 0.86,$$
$$c_3 = 299\,792\,458.4 \pm 1.1.$$

Find the weighted mean of these results and its standard deviation, taking as weights the inverses of the square of the standard deviation in each case.

```
from __future__ import division
import numpy as np
import math

# Enter the values given as the components of the vector x:
x = np.array([299792459.3, 299792457.82, 299792458.4])

# Enter the values of the errors corresponding to the values of x:
s = np.array([1.6, 0.86, 1.1])

# Evaluation:
# Evaluate the corresponding weights w of the x values:
w = 1/(s*s)
N = len(x)
wmean = np.average(x, weights = w)
variance = np.average((x-wmean)**2, weights = w)
stdev = math.sqrt(variance)
```

```
# Presentation of the results:
print ("Number of values, N =", N)
print ("Weighted mean =", wmean)
print ("Weighted standard deviation of the mean =", stdev/math.sqrt(N-1))

# Results:
Number of values, N = 3
Weighted mean = 299792458.232
Weighted standard deviation of the mean = 0.36867082350704317
```

The final result is $c = 299\ 792\ 458.23 \pm 0.37$ m/s.

Example 9.17 [R]

Three experiments for the determination of the speed of light in vacuum gave the following results, in m/s:

$$c_1 = 299\,792\,459.3 \pm 1.6, \quad c_2 = 299\,792\,457.82 \pm 0.86,$$
$$c_3 = 299\,792\,458.4 \pm 1.1.$$

Find the weighted mean of these results and its standard deviation, taking as weights the inverses of the square of the standard deviation in each case.

i	c_i (m/s)	x_i (m/s)	σ_i (m/s)
1	299 792 459.3	9.3	1.60
2	299 792 457.82	7.82	0.86
3	299 792 458.4	8.4	1.10

We form the vector x with the values of $x_i = c_i - c_0$ as components and s with the probable errors in c_i (or x_i).

```
> x <- c(9.3, 7.82, 8.4)
> s <- c(1.60, 0.86, 1.10)
> w <- c(1/s^2)
> w
[1] 0.3906250 1.3520822 0.8264463
```

The weighted mean of x is:

```
> weighted.mean(x, w)
[1] 8.2316
> wmean = weighted.mean(x, w)
```

The variance of the sample, s_x^2, is the weighted mean of the quantity $(x_i - \overline{x})^2$, and the standard deviation of the sample is $s_x = \sqrt{s_x^2}$:

```
> variance <- weighted.mean((x-wmean)^2, w)
> variance
[1] 0.2718363
> stdev <- sqrt(variance)
> stdev
[1] 0.5213793
```

We calculate the weighted standard deviation of the mean using the equation $\sigma_{\overline{x}} = \frac{s_x}{\sqrt{N-1}}$. Therefore, $\sigma_{\overline{x}} = \frac{0.5213793}{\sqrt{2}} = 0.368671$. The final result is $c = 299\ 792\ 458.23 \pm 0.37$ m/s.

9.5 The Joint Probability Density for Two Random Variables

We will now examine the probability density of a random variable which is a function of two other random variables. To avoid confusion, we will adopt the following notation [9]:

A random variable is denoted by a bold letter, $\mathbf{x}$, and the values it takes by italic x.
The probability density of the random variable $\mathbf{x}$ is denoted by $f_{\mathbf{x}}(x)$.
The probability for the random variable $\mathbf{x}$ to take a value which is smaller than or equal to x is denoted by $P\{\mathbf{x} \leq x\}$.
The distribution function of the random variable $\mathbf{x}$ is denoted by $F_{\mathbf{x}}(x)$ and is equal to the probability for the random variable $\mathbf{x}$ to take a value which is equal to or smaller than x. Obviously,

$$F_{\mathbf{x}}(x) = P\{\mathbf{x} \leq x\} \tag{9.33}$$

It is

$$f_{\mathbf{x}}(x) = \frac{\mathrm{d}F_{\mathbf{x}}(x)}{\mathrm{d}x}. \tag{9.34}$$

The probability for the random variable $\mathbf{x}$ to have a value larger than x_1 and smaller than or equal to x_2, where $x_1 < x_2$, is denoted by $P\{x_1 < \mathbf{x} \leq x_2\}$. Obviously, it is

$$P\{x_1 < \mathbf{x} \leq x_2\} = P\{\mathbf{x} \leq x_2\} - P\{\mathbf{x} \leq x_1\}. \tag{9.35}$$

The *joint or common probability density function* of the variables $\mathbf{x}$ and $\mathbf{y}$, denoted by $f_{\mathbf{x},\mathbf{y}}(x, y)$, is such that the probability for the random variable $\mathbf{x}$ to have a

value between x and $x + \mathrm{d}x$ and the random variable $\mathbf{y}$ to have a value between y and $y + \mathrm{d}y$ is equal to $f_{\mathbf{x},\mathbf{y}}(x, y)\,\mathrm{d}x\,\mathrm{d}y$.

The function $f_{\mathbf{x},\mathbf{y}}(x, y)$ is said to be *normalized* if it is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{\boldsymbol{x},\boldsymbol{y}}(x, y)\,\mathrm{d}x\,\mathrm{d}y = 1. \tag{9.36}$$

The *joint or common distribution function* of the variables $\mathbf{x}$ and $\mathbf{y}$ is denoted by $F_{\mathbf{x},\mathbf{y}}(x, y)$ and is equal to the probability for the random variable $\mathbf{x}$ to have a value smaller than or equal to x and the random variable $\mathbf{y}$ to have a value smaller than or equal to y. It is

$$F_{\mathbf{x},\mathbf{y}}(x, y) \equiv P\{\mathbf{x} \le x,\ \mathbf{y} \le y\}. \tag{9.37}$$

The following relations are considered to be obvious [10]:

$$f_{\mathbf{x},\mathbf{y}}(x, y) = \frac{\partial^2 F_{\mathbf{x},\mathbf{y}}(x, y)}{\partial x\,\partial y} \tag{9.38}$$

$$f_{\mathbf{x}}(x) = \int_{-\infty}^{\infty} f_{\mathbf{x},\mathbf{y}}(x, y)\,\mathrm{d}y \quad f_{\mathbf{y}}(y) = \int_{-\infty}^{\infty} f_{\mathbf{x},\mathbf{y}}(x, y)\,\mathrm{d}x \tag{9.39}$$

$$F_{\mathbf{x}}(x) = F_{\mathbf{x},\mathbf{y}}(x,\ \infty) = \int_{-\infty}^{\infty}\mathrm{d}y \int_{-\infty}^{x} f_{\mathbf{x},\mathbf{y}}(\chi, y)\,\mathrm{d}\chi \tag{9.40}$$

$$F_{\mathbf{y}}(x) = F_{\mathbf{x},\mathbf{y}}(\infty,\ y) = \int_{-\infty}^{\infty}\mathrm{d}x \int_{-\infty}^{y} f_{\mathbf{x},\mathbf{y}}(x, \psi)\,\mathrm{d}\psi. \tag{9.41}$$

Let the random variable $\mathbf{x}$ have probability density $f_{\mathbf{x}}(x)$ and the variable $\mathbf{y}$ have probability density $f_{\mathbf{y}}(y)$. These are known as *marginal probability densities*. If the variables $\mathbf{x}$ and $\mathbf{y}$ are independent of each other, then the probability for the random variable $\mathbf{x}$ to have a value between x and $x + \mathrm{d}x$ and the random variable $\mathbf{y}$ to have value between y and $y + \mathrm{d}y$ is equal to the product of the two separate probabilities, i.e.

$$f_{\mathbf{x},\mathbf{y}}(x, y)\,\mathrm{d}x\,\mathrm{d}y = f_{\mathbf{x}}(x) f_{\mathbf{y}}(y)\,\mathrm{d}x\,\mathrm{d}y. \tag{9.42}$$

If $\mathbf{x}$ and $\mathbf{y}$ are normally distributed, with means and standard deviations $\mu_x,\ \sigma_x$ and $\mu_y,\ \sigma_y$, respectively, then

$$f_{\mathbf{x},\mathbf{y}}(x, y) = \frac{1}{2\pi\sigma_x\sigma_y}\,\mathrm{e}^{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}}. \tag{9.43}$$

This function of the two variables, x and y, has been drawn, in contour form, in Fig. 9.7.

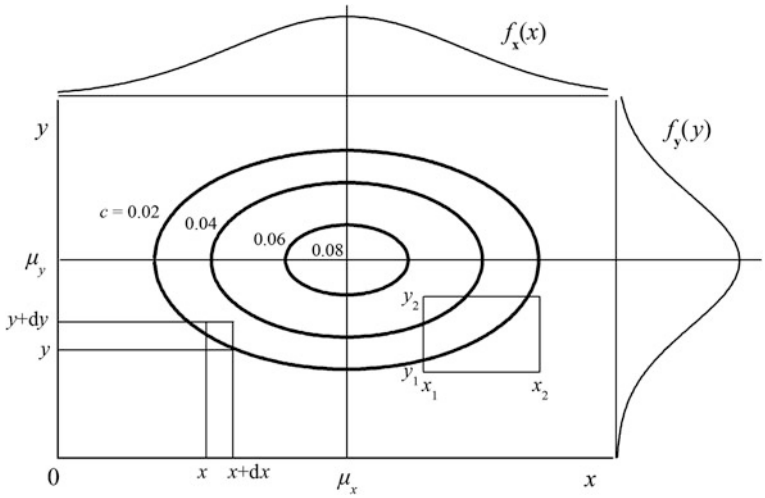


Fig. 9.7 The joint probability density function $f_{\mathbf{x},\mathbf{y}}(x, y)$ for the normally distributed random variables **x** and **y**, which have means and standard deviations μ_x, σ_x and μ_y, σ_y, respectively. Three ellipses of constant $f_{\mathbf{x},\mathbf{y}}(x, y) = c$ are shown in the figure. Also shown are the marginal probability densities $f_{\mathbf{x}}(x)$ and $f_{\mathbf{y}}(y)$

Drawn in the figure are:

1. Curves of constant values $f_{\mathbf{x},\mathbf{y}}(x, y) = c$, which are ellipses with center at the point $(\mu_x,\ \mu_y)$. Putting $f_{\mathbf{x},\mathbf{y}}(x, y) = c$ in Eq. (9.43), we find the equations of these ellipses to be

$$\frac{(x-\mu_x)^2}{\sigma_x^2} + \frac{(y-\mu_y)^2}{\sigma_y^2} = -2\,\ln(2\pi\sigma_x\sigma_y c). \tag{9.44}$$

2. The marginal probability densities

$$f_{\mathbf{x}}(x) = \frac{1}{\sqrt{2\pi}\sigma_x}\,\mathrm{e}^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \quad \text{and} \quad f_{\mathbf{y}}(y) = \frac{1}{\sqrt{2\pi}\sigma_y}\,\mathrm{e}^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}, \tag{9.45}$$

   situated at the upper part of the figure and the right-hand side, respectively.
3. The surface element $\mathrm{d}x\,\mathrm{d}y$ used in the evaluation of the probability $f_{\mathbf{x},\mathbf{y}}(x, y)\,\mathrm{d}x\,\mathrm{d}y$ for the random variable **x** to have a value between x and $x + \mathrm{d}x$ and the random variable **y** to have a value between y and $y + \mathrm{d}y$.
4. The parallelogram lying between the values of x_1 and x_2, and y_1 and y_2, over which the function $f_{\mathbf{x},\mathbf{y}}(x, y)$ must be integrated for the evaluation of the probability for the random variable **x** to have a value greater than x_1 and smaller than

or equal to x_2 and the random variable $\mathbf{y}$ to have a value greater than y_1 and smaller than or equal to y_2. This probability is equal to

$$P\{x_1 < \mathbf{x} \le x_2,\ y_1 < \mathbf{y} \le y_2\} = \frac{1}{2\pi\sigma_x\sigma_y} \int_{x_1}^{x_2} \mathrm{e}^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \mathrm{d}x \int_{y_1}^{y_2} \mathrm{e}^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}} \mathrm{d}y. \quad (9.46)$$

Example 9.18

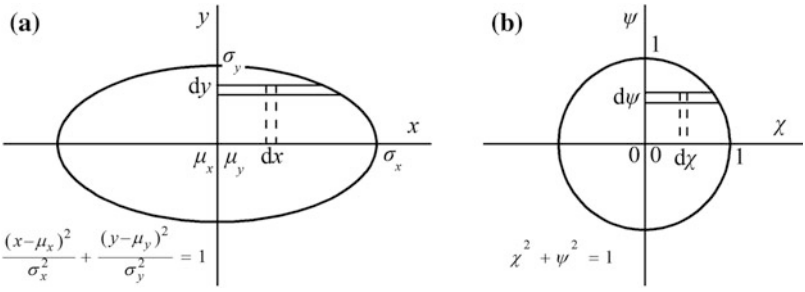
For the distribution of Eq. (9.43) and Fig. 9.7, find the probability that a point (x, y) lies within the ellipse with center the point (μ_x, μ_y) and semi-axes equal to σ_x and σ_y, along the respective axes.

The joint probability density function is $f_{\mathbf{x},\mathbf{y}}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} \mathrm{e}^{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}}$. The probability that a point lies in the surface element $\mathrm{d}x\,\mathrm{d}y$ about the point (x, y) is

$$\mathrm{d}^2P = \frac{1}{2\pi\sigma_x\sigma_y} \mathrm{e}^{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}} \mathrm{d}x\,\mathrm{d}y.$$

The probability that a point (x, y) lies within the ellipse with center the point (μ_x, μ_y) and semi-axes equal to σ_x and σ_y, along the respective axes is found by integrating this over the surface of the ellipse, as sown in figure (a):

$$P_{\sigma_x,\sigma_y} = \frac{1}{2\pi\sigma_x\sigma_y} \iint_{\text{ellipse}} \mathrm{e}^{-\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(y-\mu_y)^2}{2\sigma_y^2}} \mathrm{d}x\,\mathrm{d}y.$$



We change the variables to $\chi = \frac{x-\mu_x}{\sigma_x}$ and $\psi = \frac{y-\mu_y}{\sigma_y}$. Then

$$\mathrm{d}^2P = \frac{1}{2\pi} \mathrm{e}^{-\chi^2/2 - \psi^2/2} \mathrm{d}\chi\,\mathrm{d}\psi$$

and the ellipse transforms into the circle $\chi^2 + \psi^2 = 1$. The surface integral is now, as shown in figure (b),

$$P_{\sigma_x,\,\sigma_y} = \frac{1}{2\pi} \iint\limits_{\text{circle}} \mathrm{e}^{-\chi^2/2-\psi^2/2} \mathrm{d}\chi\, \mathrm{d}\psi = 4 \times \frac{1}{2\pi} \int_0^1 \mathrm{e}^{-\psi^2/2}\, \mathrm{d}\psi \int_0^{\sqrt{1-\psi^2}} \mathrm{e}^{-\chi^2/2} \mathrm{d}\chi.$$

Since

$$\int_0^{\sqrt{1-\psi^2}} \mathrm{e}^{-\chi^2/2} \mathrm{d}\chi = \sqrt{\frac{2}{\pi}}\, \mathrm{erf}\left(\frac{\sqrt{1-\psi^2}}{\sqrt{2}}\right),$$

we have

$$P_{\sigma_x,\,\sigma_y} = \sqrt{\frac{2}{\pi}} \int_0^1 \mathrm{e}^{-\psi^2/2}\, \mathrm{erf}\left(\frac{\sqrt{1-\psi^2}}{\sqrt{2}}\right) \mathrm{d}\psi.$$

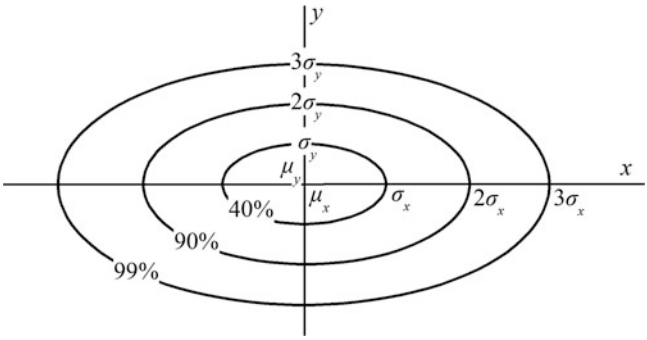
We could not find this integral in the tables, so we resorted to numerical integration. This gave:

$$P_{\sigma_x,\,\sigma_y} = 0.394.$$

The probability that a point $(x,\ y)$ lies within the ellipse with center the point $(\mu_x,\ \mu_y)$ and semi-axes equal to σ_x and σ_y, along the respective axes is, therefore 39.4%.

For the $2\sigma_x$, $2\sigma_y$ ellipse it is $P_{2\sigma_x,\,2\sigma_y} = 0.865$.
For the $3\sigma_x$, $3\sigma_y$ ellipse it is $P_{3\sigma_x,\,3\sigma_y} = 0.989$.

This last result states that 99% of the points lie within the ellipse with center the point $(\mu_x,\ \mu_y)$ and semi-axes equal to $3\sigma_x$ and $3\sigma_y$, along the respective axes. The percentages may be remembered as 40-90-99. These results are illustrated in the figure below.



9.6 The Probability Density of the Sum of Two Random Variables

Let the random variable $\mathbf{z} = \mathbf{x} + \mathbf{y}$ take values $z = x + y$, where the random variables $\mathbf{x}$ and $\mathbf{y}$ are normally distributed. The probability density $f_{\mathbf{z}}(z) = f_{\mathbf{x}+\mathbf{y}}(x+y)$ is required.

Figure 9.8 shows the probability densities $f_{\mathbf{x}}(x)$ and $f_{\mathbf{y}}(y)$ of $\mathbf{x}$ and $\mathbf{y}$ [(a) and (b) respectively]. In Fig. 9.8a, the area of the strip under the curve $f_{\mathbf{x}}(x)$ between x and $x + \mathrm{d}x$ gives the probability for $\mathbf{x}$ to have a value between x and $x + \mathrm{d}x$. In Fig. 9.8b, the area of the region under the curve $f_{\mathbf{y}}(y)$ and in the region of values $-\infty < \mathbf{y} \le y$, gives the probability for $\mathbf{y}$ to have a value in the region $-\infty < \mathbf{y} \le y$.

Due to the independence of the random variables $\mathbf{x}$ and $\mathbf{y}$ from each other, the probability for $\mathbf{x}$ to have a value between x and $x + \mathrm{d}x$ *and* $\mathbf{y}$ to have a value smaller than or equal to y is

$$P\{x < \mathbf{x} \le x + \mathrm{d}x,\ \mathbf{y} \le y\} = P\{x < \mathbf{x} \le x + \mathrm{d}x\}\ P\{\mathbf{y} \le y\}$$
$$= f_{\mathbf{x}}(x)\,\mathrm{d}x\,F_{\mathbf{y}}(y) = f_{\mathbf{x}}(x)\,\mathrm{d}x \int_{-\infty}^{y} f_{\mathbf{y}}(y)\,\mathrm{d}y. \qquad (9.47)$$

If it is $z = x + y$, this is the probability for the random variable $\mathbf{x}$ to have a value between x and $x + dx$ and the sum $\mathbf{z} = \mathbf{x} + \mathbf{y}$ to have a value smaller than or equal to z:

$$P\{x < \mathbf{x} \le x + \mathrm{d}x,\ \mathbf{z} \le z\} = f_{\mathbf{x}}(x)\,dx \int_{-\infty}^{z-x} f_{\mathbf{y}}(y)\,\mathrm{d}y. \qquad (9.48)$$

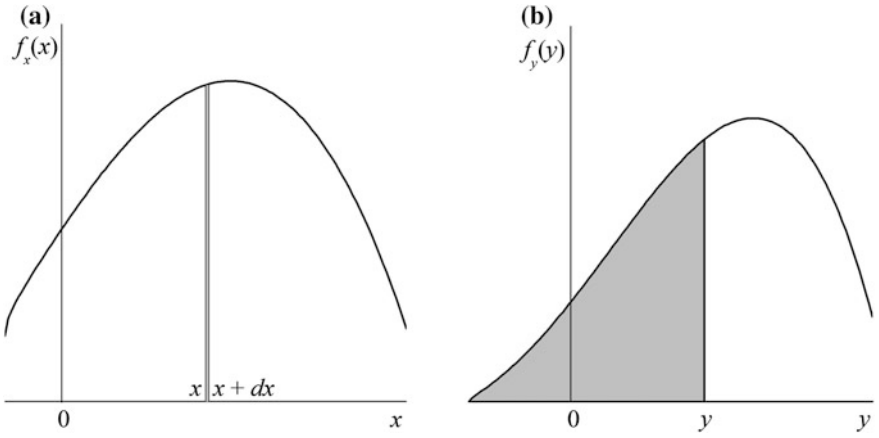


Fig. 9.8 The probability densities $f_{\mathbf{x}}(x)$ and $f_{\mathbf{y}}(y)$ of the random variables **x** and **y**

The probability for the random variable **x** to have *any* value and the sum $\mathbf{z} = \mathbf{x} + \mathbf{y}$ to have a value which is smaller than or equal to z, i.e. the probability for **z** to have a value which is smaller than or equal to z, is

$$F_{\mathbf{z}}(z) = \int_{-\infty}^{\infty} f_{\mathbf{x}}(x)\,\mathrm{d}x \int_{-\infty}^{z-x} f_{\mathbf{y}}(y)\,\mathrm{d}y. \tag{9.49}$$

The geometrical interpretation of this relation is seen in Fig. 9.9. The double integral of Eq. (9.49) gives the probability which corresponds to the integration of the function $f_{\mathbf{x}}(x) f_{\mathbf{y}}(y)$ over the shaded region and under the straight line $x + y = z$. In the shaded region, it is $x + y \le z$. The magnitude $\mathrm{d}x \int_{-\infty}^{z-x} f_{\mathbf{y}}(y)\,\mathrm{d}y$ is evaluated in the strip of Fig. 9.9 between x and $x + \mathrm{d}x$. Integrating then for all the values of x, we cover all the shaded region of the figure $(x + y \le z)$ and we find $F_{\mathbf{z}}(z)$.

The probability density for **z** is found by differentiating of $F_{\mathbf{z}}(z)$ with respect to z:

$$f_{\mathbf{z}}(z) = \frac{\mathrm{d}F_{\mathbf{z}}(z)}{\mathrm{d}z} = \int_{-\infty}^{\infty} f_{\mathbf{x}}(x)\,\mathrm{d}x \left\{ \frac{\partial}{\partial z} \int_{-\infty}^{z-x} f_{\mathbf{y}}(y)\mathrm{d}y \right\} = \int_{-\infty}^{\infty} f_{\mathbf{x}}(x)\,\mathrm{d}x \left\{ f_{\mathbf{y}}(z-x) \right\}. \tag{9.50}$$

Therefore,

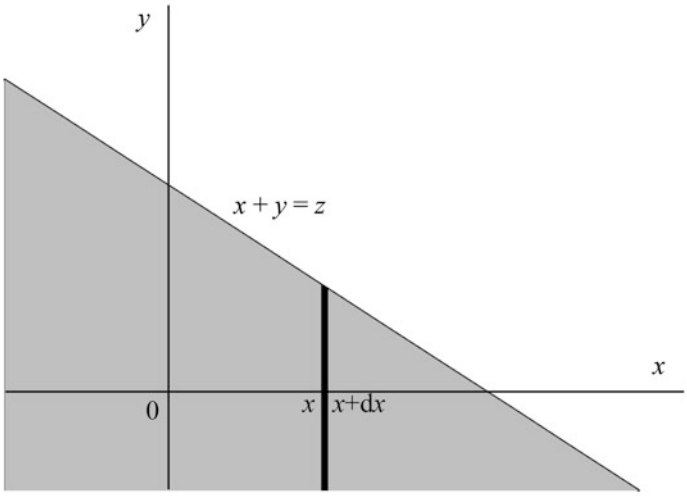


Fig. 9.9 The region of integration of the function $f_{\mathbf{x}}(x)f_{\mathbf{y}}(y)$ (*shaded region*) for the evaluation of the distribution function $F_{\mathbf{z}}(z)$ of the sum $\mathbf{z} = \mathbf{x} + \mathbf{y}$

$$f_{\mathbf{z}}(z) = \int_{-\infty}^{\infty} f_{\mathbf{x}}(x) f_{\mathbf{y}}(z - x)\,\mathrm{d}x \tag{9.51}$$

and, due to symmetry,

$$f_{\mathbf{z}}(z) = \int_{-\infty}^{\infty} f_{\mathbf{x}}(z - y) f_{\mathbf{y}}(y)\,\mathrm{d}y. \tag{9.52}$$

These integrals express the *convolution* of the functions $f_{\mathbf{x}}(x)$ and $f_{\mathbf{y}}(y)$, which is denoted by $f_{\mathbf{x}}(x) * f_{\mathbf{y}}(y)$ or, equivalently, by $f_{\mathbf{y}}(y) * f_{\mathbf{x}}(x)$.

If $\mathbf{x}$ and $\mathbf{y}$ take only positive values, then Eqs. (9.51) and (9.52) simplify to

$$f_{\mathbf{z}}(z) = \int_{0}^{z} f_{\mathbf{x}}(x) f_{\mathbf{y}}(z - x)\,\mathrm{d}x, \quad (z > 0) \tag{9.53}$$

and

$$f_{\mathbf{z}}(z) = \int_0^z f_{\mathbf{x}}(z-y) f_{\mathbf{y}}(y)\,\mathrm{d}y, \quad (z > 0). \tag{9.54}$$

The proof is simple: writing Eq. (9.51) as

$$f_{\mathbf{z}}(z) = \int_{-\infty}^{0} f_{\mathbf{x}}(x) f_{\mathbf{y}}(z-x)\,\mathrm{d}x + \int_0^z f_{\mathbf{x}}(x) f_{\mathbf{y}}(z-x)\,\mathrm{d}x + \int_z^{\infty} f_{\mathbf{x}}(x) f_{\mathbf{y}}(z-x)\,\mathrm{d}x, \tag{9.55}$$

we see that the first integral is equal to zero because in the region of integration $(-\infty, 0)$ the function $f_{\mathbf{x}}(x)$ is equal to zero, while the third integral is also equal to zero because for $x > z$ the function $f_{\mathbf{y}}(z-x)$ is equal to zero. Only the second integral remains, which gives Eq. (9.53). Equation (9.54) is proved in the same way.

9.6.1 The Probability Density of the Sum of Two Normally Distributed Random Variables

Let $\mathbf{x}$ and $\mathbf{y}$ be two mutually independent random variables with means and standard deviations μ_x, σ_x and μ_y, σ_y, respectively. Then, it will be

$$f_{\mathbf{x}}(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \mathrm{e}^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \quad \text{and} \quad f_{\mathbf{y}}(y) = \frac{1}{\sqrt{2\pi}\sigma_y} \mathrm{e}^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}} \tag{9.56}$$

and the probability density for their sum, $\mathbf{z} = \mathbf{x} + \mathbf{y}$, will be, according to Eq. (9.53),

$$f_{\mathbf{z}}(z) = \frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} \exp\left\{ -\frac{(x-\mu_x)^2}{2\sigma_x^2} - \frac{(z-x-\mu_y)^2}{2\sigma_y^2} \right\} \mathrm{d}x \tag{9.57}$$

After some algebraic manipulation, the exponent may be written in the form:

$$\{\} = -\frac{(z-\mu_x-\mu_y)^2}{2(\sigma_x^2+\sigma_y^2)} - \frac{\left(x - \frac{\sigma_y^2\mu_x + \sigma_x^2(\mu_y - z)}{\sigma_x^2+\sigma_y^2}\right)^2}{2\frac{\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2}}. \tag{9.58}$$

Therefore,

$$f_{\mathbf{z}}(z) = \frac{1}{2\pi\,\sigma_x\sigma_y}\exp\left[-\frac{(z-\mu_x-\mu_y)^2}{2(\sigma_x^2+\sigma_y^2)}\right]\int_{-\infty}^{\infty}\exp\left[-\frac{\left(x-\frac{\sigma_y^2\mu_x+\sigma_x^2(\mu_y-z)}{\sigma_x^2+\sigma_y^2}\right)^2}{2\frac{\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2}}\right]\mathrm{d}x. \tag{9.59}$$

The value of the integral is simply $\frac{\sqrt{2\pi}\sigma_x\sigma_y}{\sqrt{\sigma_x^2+\sigma_y^2}}$. Thus, (9.59) becomes

$$f_{\mathbf{z}}(z) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma_x^2+\sigma_y^2}}\exp\left[-\frac{(z-\mu_x-\mu_y)^2}{2(\sigma_x^2+\sigma_y^2)}\right], \tag{9.60}$$

or

$$f_{\mathbf{z}}(z) = \frac{1}{\sqrt{2\pi}\,\sigma_z}\mathrm{e}^{-\frac{(z-\mu_z)^2}{2\sigma_z^2}}, \tag{9.61}$$

which is a normal (Gaussian) distribution with mean and standard deviation

$$\mu_z = \mu_x + \mu_y \quad \text{and} \quad \sigma_z = \sqrt{\sigma_x^2+\sigma_y^2}, \tag{9.62}$$

respectively.

This is the same result as that found in Sect. 6.2.2. The result may be generalized to more added terms and, obviously, **x** and **y** may also take negative values. The result is valid, therefore, for the algebraic sum of any number of normally distributed variables.

Example 9.19

The magnitude **x** has a real value x_0 and a series of measurements of it have a mean value $\overline{x}$ and standard deviation of the mean $\sigma_{\overline{x}}$. The magnitude **y**, of the same nature as **x**, has a real value y_0 and a series of measurements of it have a mean value $\overline{y}$ and standard deviation of the mean $\sigma_{\overline{y}}$. What is the probability for y_0 to be greater than x_0?

The probability density of the *mean* $\overline{\mathbf{z}}$ of the difference $\mathbf{z} = \mathbf{y} - \mathbf{x}$ was found to be equal to

$$f_{\overline{\mathbf{z}}}(\overline{z}) = \frac{1}{\sqrt{2\pi}\,\sigma_{\overline{z}}}\,\mathrm{e}^{-\frac{(\overline{z}-\mu_{\overline{z}})^2}{2\sigma_{\overline{z}}^2}}, \quad \text{where} \quad \mu_{\overline{z}} = \overline{y} - \overline{x} \quad \text{and} \quad \sigma_{\overline{z}} = \sqrt{\sigma_{\overline{x}}^2+\sigma_{\overline{y}}^2}.$$

The probability for the random variable $\overline{\mathbf{z}}$ to have a value greater than $\overline{z}$ is (Sect. 4.4.2)

$$\Pr\{\overline{\mathbf{z}} > \overline{z}\} = \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{\overline{z} - \mu_{\overline{z}}}{\sqrt{2}\,\sigma_{\overline{z}}}\right) = \frac{1}{2} - \Phi\left(\frac{\overline{z} - \mu_{\overline{z}}}{\sigma_{\overline{z}}}\right).$$

The values x_0 and y_0 are the real values of the magnitudes $\mathbf{x}$ and $\mathbf{y}$. We wish to find the probability for y_0 to be greater than x_0. The best estimates we have for x_0 and y_0 are $\overline{x}$ and $\overline{y}$, respectively. Therefore, the best estimate we can have for the probability for y_0 to be greater than x_0, is equal to the probability for the value of the magnitude $\overline{\mathbf{z}}$ to be greater than 0,

$$\Pr\{y_0 > x_0\} = \frac{1}{2} - \frac{1}{2}\operatorname{erf}\left(\frac{-\mu_{\overline{z}}}{\sqrt{2}\,\sigma_{\overline{z}}}\right) = \frac{1}{2} - \Phi\left(\frac{-\mu_{\overline{z}}}{\sigma_{\overline{z}}}\right)$$

and finally, since it is $\operatorname{erf}(-z) = -\operatorname{erf}(z)$ and $\Phi(-z) = -\Phi(z)$,

$$\Pr\{y_0 > x_0\} = \frac{1}{2} + \frac{1}{2}\operatorname{erf}\left(\frac{\overline{y} - \overline{x}}{\sqrt{2}\sqrt{\sigma_{\overline{x}}^2 + \sigma_{\overline{y}}^2}}\right) = \frac{1}{2} + \Phi\left(\frac{\overline{y} - \overline{x}}{\sqrt{\sigma_{\overline{x}}^2 + \sigma_{\overline{y}}^2}}\right).$$

If it is $\overline{y} - \overline{x} = 3\sigma_{\overline{z}}$, then $\Pr\{y_0 > x_0\} = \frac{1}{2} + \Phi(3) = 0.999$.
If it is $\overline{y} - \overline{x} = 2\sigma_{\overline{z}}$, then $\Pr\{y_0 > x_0\} = \frac{1}{2} + \Phi(2) = 0.977$.
If it is $\overline{y} - \overline{x} = \sigma_{\overline{z}}$, then $\Pr\{y_0 > x_0\} = \frac{1}{2} + \Phi(1) = 0.84$.
If it is $\overline{y} = \overline{x}$, then $\Pr\{y_0 > x_0\} = \frac{1}{2} + \Phi(0) = 0.5$.

Also,

If it is $\overline{y} - \overline{x} = -\sigma_{\overline{z}}$, then $\Pr\{y_0 > x_0\} = \frac{1}{2} - \Phi(1) = 0.16$.
If it is $\overline{y} - \overline{x} = -2\sigma_{\overline{z}}$, then $\Pr\{y_0 > x_0\} = \frac{1}{2} - \Phi(2) = 0.023$.
If it is $\overline{y} - \overline{x} = -3\sigma_{\overline{z}}$, then $\Pr\{y_0 > x_0\} = \frac{1}{2} - \Phi(3) = 0.001$.

The example that follows is rather extensive and may be omitted without any consequences in the understanding of what will follow. It is, however, useful, since it deals with many topics we have already discussed and uses the last theoretical results.

Example 9.20

Find the probability densities for the sums of n random values of x, in the case of the distribution of the measurements having a probability density $f_{\mathbf{x}}(x) = 0$ for $x < 0$ and $f_{\mathbf{x}}(x) = \alpha\,\mathrm{e}^{-\alpha x}$ $(0 < \alpha,\ \ 0 \le x)$ and check the validity of the central limit theorem.

The characteristics of the distribution

The probability density is normalized because $\alpha \int_0^\infty \mathrm{e}^{-\alpha x}\mathrm{d}x = 1$.

We will first find the mean and the standard deviation of x.

The mean is

$$\overline{x} = \alpha \int_0^\infty x\mathrm{e}^{-\alpha x}\mathrm{d}x = \frac{1}{\alpha}.$$

The standard deviation is

$$\sigma_x = \sqrt{\int_0^\infty \left(x - \frac{1}{\alpha}\right)^2 \alpha \mathrm{e}^{-\alpha x}\mathrm{d}x}$$

or

$$\sigma_x = \sqrt{\alpha \int_0^\infty x^2\mathrm{e}^{-\alpha x}\mathrm{d}x - \frac{2}{\alpha}\int_0^\infty x\mathrm{e}^{-\alpha x}\mathrm{d}x + \frac{1}{\alpha^2}\int_0^\infty \mathrm{e}^{-\alpha x}\mathrm{d}x} = \sqrt{\frac{1}{\alpha^2} - \frac{1}{\alpha^2} + \frac{1}{\alpha^2}} = \frac{1}{\alpha}$$

$$\sigma_x = \frac{1}{\alpha}.$$

The probability densities of the sum of n values of $\mathbf{x}$

We now denote as $\mathbf{z}_n = \mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_i + \ldots + \mathbf{x}_n$ the sum of n values of the variable $\mathbf{x}$ (e.g. measurements of the magnitude $\mathbf{x}$).

The probability density of single values $\mathbf{z}_1 = \mathbf{x}$ is the given function

$$f_{\mathbf{x}}(x) = \alpha\,\mathrm{e}^{-\alpha x} \equiv f_{z_1}(z_1) \quad (0 \le x).$$

According to Eq. (9.53) the probability density for the sum of two values of $\mathbf{x}$, is given by the convolution of $f_{\mathbf{z_1}}(z_1)$ with itself:

$$f_{\mathbf{z}_2}(z_2) = \int_0^{z_2} f_{\mathbf{z}_1}(z_1) f_{\mathbf{z}_1}(z_2 - z_1)\,\mathrm{d}z_1.$$

Substituting, we find

$$f_{\mathbf{z}_2}(z_2) = \int_0^{z_2} (\alpha\mathrm{e}^{-\alpha z_1})(\alpha\mathrm{e}^{-\alpha z_2 + \alpha z_1})\,\mathrm{d}z_1 = \alpha^2\mathrm{e}^{-\alpha z_2}\int_0^{z_2}\mathrm{d}z_1 = \alpha^2 z_2\mathrm{e}^{-\alpha z_2}.$$

Knowing the probability density for the sum of two values of $\mathbf{x}$, we may find the probability density for the sum of three values of $\mathbf{x}$. This will be equal to the convolution of the probability density for the sum of two values with the probability density for the result of a measurement:

$$f_{\mathbf{z}_3}(z_3) = \int_0^{z_3} f_{\mathbf{z}_2}(z_2) f_{\mathbf{z}_1}(z_3 - z_2)\,\mathrm{d}z_2$$
$$f_{\mathbf{z}_3}(z_3) = \int_0^{z_3} (\alpha^2 z_2 \mathrm{e}^{-\alpha z_2})(\alpha \mathrm{e}^{-\alpha z_3 + \alpha z_2})\,\mathrm{d}z_2$$
$$= \alpha^3 \mathrm{e}^{-\alpha z_3} \int_0^{z_3} z_2\,\mathrm{d}z_2 = \alpha^3 \frac{z_3^2}{2!} \mathrm{e}^{-\alpha z_3}.$$

The general relation for the probability density $f_{\mathbf{z}_{n+1}}(z_{n+1})$ is given by the convolution of the probability density for n measurements, $f_{\mathbf{z}_n}(z_n)$, with the probability density for the result of a measurement, $f_{\mathbf{z}_1}(z_1)$,

$$f_{\mathbf{z}_{n+1}}(z_{n+1}) = \int_0^{z_{n+1}} f_{\mathbf{z}_n}(z_n) f_{\mathbf{z}_1}(z_{n+1} - z_n)\,\mathrm{d}z_n.$$

We test the *assumption* that it is $f_{\mathbf{z}_n}(z_n) = \alpha^n \frac{z_n^{n-1}}{(n-1)!} \mathrm{e}^{-\alpha z_n}$. Substituting in the last relation, we find

$$f_{\mathbf{z}_{n+1}}(z_{n+1}) = \int_0^{z_{n+1}} \left(\alpha^n \frac{z_n^{n-1}}{(n-1)!} \mathrm{e}^{-\alpha z_n} \right) (\alpha \mathrm{e}^{-\alpha z_{n+1} + \alpha z_n})\,\mathrm{d}z_n$$
$$f_{\mathbf{z}_{n+1}}(z_{n+1}) = \alpha^{n+1} \frac{\mathrm{e}^{-\alpha z_{n+1}}}{(n-1)!} \int_0^{z_{n+1}} z_n^{n-1} \mathrm{d}z_n = \alpha^{n+1} \frac{z_{n+1}^n}{n!} \mathrm{e}^{-\alpha z_{n+1}},$$

which is in agreement with the assumption we made for the form of $f_{\mathbf{z}_n}(z_n)$. Since the formula $f_{\mathbf{z}_n}(z_n) = \alpha^n \frac{z_n^{n-1}}{(n-1)!} \mathrm{e}^{-\alpha z_n}$ gives the correct results for $n = 1$, 2 and 3 (which we already know) and, since, as we have just proved, if it is valid for $f_{\mathbf{z}_n}(z_n)$ then it will be valid for $f_{\mathbf{z}_{n+1}}(z_{n+1})$ also, we reach the conclusion that it is valid for all values of n.

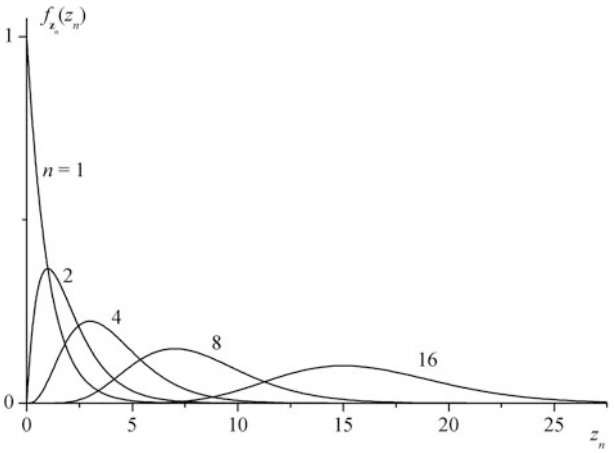
The probability densities for the sums $\mathbf{z}_n = \mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_i + \ldots + \mathbf{x}_n$ are, therefore,

$$f_{\mathbf{z}_1}(z_1) \equiv f_{\mathbf{x}}(x) = \alpha\,\mathrm{e}^{-\alpha x}, \quad f_{\mathbf{z}_2}(z_2) = \alpha^2 z_2 \mathrm{e}^{-\alpha z_2}, \quad f_{\mathbf{z}_3}(z_3) = \alpha^3 \frac{z_3^2}{2!} \mathrm{e}^{-\alpha z_3}, \quad \ldots$$

and are given by the general formula

$$f_{\mathbf{z}_n}(z_n) = \alpha^n \frac{z_n^{n-1}}{(n-1)!} \mathrm{e}^{-\alpha z_n}.$$

In the figure that follows, the curves $f_{\mathbf{z}_n}(z_n)$ were drawn for $\alpha = 1$ and the values of $n = 1, 2, 4, 8$ and 16.



The probability densities for the sum of a number of $n = 2, 4, 8$ or 16 values, taken at random from a parent population with probability density $f_{\mathbf{z}}(z) = \mathrm{e}^{-z}$ $(z \geq 0)$

The probability densities of the means $\overline{x}$ of n measurements of **x**.

Knowing the probability densities $f_{\mathbf{z}_n}(z_n)$ of the sums of n measurements of **x** and if $\overline{x} = z_n/n$ is the mean of n measurements of **x**, we wish to find the probability density $f_n(\overline{x})$ of the values $\overline{x}$.

The relation between the functions $f_{\mathbf{z}_n}(z_n)$ and $f_n(\overline{x})$ is found as follows: Because it is

(Probability for the mean $\overline{\mathbf{x}}$ of the n measurements of **x** to lie between $\overline{x}$ and $\overline{x} + \mathrm{d}\overline{x}$)
= (Probability for the sum $\mathbf{z}_n$ of the n measurements of **x** to lie between z_n and $z_n + \mathrm{d}z_n$)

we have

$$f_n(\overline{x})\,\mathrm{d}\overline{x} = f_{\mathbf{z}_n}(z_n)\,\mathrm{d}z_n$$

and

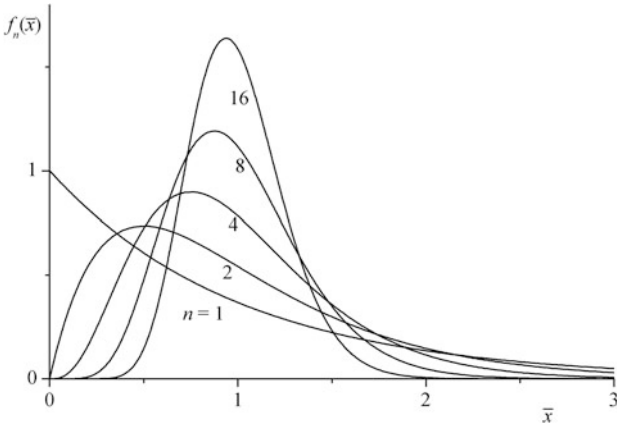
$$f_n(\overline{x}) = f_{\mathbf{z}_n}(z_n)\left|\frac{\mathrm{d}z_n}{\mathrm{d}\overline{x}}\right|,$$

where the absolute value is taken as, by definition, the probability densities are positive.

Taking into account the fact that $\overline{x} = z_n/n$, we have, $\dfrac{\mathrm{d}z_n}{\mathrm{d}\overline{x}} = n$, $f_n(\overline{x}) = n f_{\mathbf{z}_n}(z_n)$ and so the probability densities of the means $\overline{x}$ of n measurements of $\mathbf{x}$ is:

$$f_n(\overline{x}) = n\alpha^n \frac{(n\overline{x})^{n-1}}{(n-1)!}\,\mathrm{e}^{-n\alpha\overline{x}} \quad \text{or} \quad f_n(\overline{x}) = \frac{(n\alpha)^n}{(n-1)!}\overline{x}^{n-1}\,\mathrm{e}^{-n\alpha\overline{x}}.$$

These functions have been drawn in the figure that follows, for $\alpha = 1$ and $n = 1$, 2, 4, 8 and 16.



The probability densities of the means of a number $n = 2$, 4, 8 or 16 values, which are taken at random from a parent population with probability density $f_{\mathbf{z}}(z) = e^{-z}$ $(z \geq 0)$

Since

$$\int_0^\infty \frac{(n\alpha)^n}{(n-1)!}\overline{x}^{n-1}\,\mathrm{e}^{-n\alpha\overline{x}}\mathrm{d}\overline{x} = \frac{1}{(n-1)!}\int_0^\infty (n\alpha\overline{x})^{n-1}\,\mathrm{e}^{-n\alpha\overline{x}}\mathrm{d}(n\alpha\overline{x})$$
$$= \frac{1}{(n-1)!}\int_0^\infty t^{n-1}\,\mathrm{e}^{-t}\mathrm{d}t = 1,$$

the probability densities $f_n(\overline{x})$ are normalized, as expected.

The asymptotic approach to the Gaussian curve

The maximum of $f_n(\overline{x})$ appears at the value of $\overline{x}$ for which it is $\mathrm{d}f_n(\overline{x})/\mathrm{d}\overline{x} = 0$, i.e. for $\overline{x} = \frac{n-1}{\alpha n}$. Substituting in $f_n(\overline{x})$, we find its maximum value: $\hat{f}_n = \frac{(n-1)^{n-1}}{(n-1)!}\,\alpha n\,\mathrm{e}^{-(n-1)}$.

In terms of $\hat{f}_n$, we have $f_n(\overline{x}) = \hat{f}_n \left(\mathrm{e}\,\frac{n\alpha\overline{x}}{n-1}\right)^{n-1}\mathrm{e}^{-n\alpha\overline{x}}$.

Let

$$\alpha\overline{x} \equiv \frac{n-1}{n} + \delta,$$

where δ expresses, in units of $1/\alpha$, the distance along the $\overline{x}$-axis from the point $\overline{x} = \frac{n-1}{\alpha n}$ which corresponds to the curve's maximum. Then,

$$f_n(\overline{x}) = \hat{f}_n \left(1 + \frac{n\delta}{n-1}\right)^{n-1} \mathrm{e}^{-n\delta}$$

Taking logarithms,

$$\ln\left(f_n(\overline{x})/\hat{f}_n\right) = (n-1)\ \ln\left(1 + \frac{n\delta}{n-1}\right) - n\delta.$$

For small values of δ,

$$\ln\left(f_n(\overline{x})/\hat{f}_n\right) = (n-1)\left[\frac{n}{n-1}\delta - \frac{1}{2}\left(\frac{n}{n-1}\right)^2\delta^2 + \ldots\right] - n\delta$$

$$\ln\left(f_n(\overline{x})/\hat{f}_n\right) = -\frac{1}{2}\frac{n^2}{n-1}\delta^2$$

and, therefore,

$$f_n(\overline{x}) = \hat{f}_n\, \mathrm{e}^{-\frac{n^2\delta^2}{2(n-1)}}.$$

Returning to $\overline{x}$ via the relation $\alpha\overline{x} = \frac{n-1}{n} + \delta$, we have $f_n(\overline{x}) = \hat{f}_n\ \exp\left\{-\frac{\left(\overline{x} - \frac{n-1}{\alpha n}\right)^2}{2\left(\frac{\sqrt{n-1}}{\alpha n}\right)^2}\right\}$.

For large n, Stirling's formula gives $\hat{f}_n \approx \frac{1}{\sqrt{2\pi}}\frac{\alpha n}{\sqrt{n-1}}$ and, therefore, it is

$$f_n(\overline{x}) = \frac{1}{\sqrt{2\pi}}\frac{\alpha n}{\sqrt{n-1}}\ \exp\left\{-\frac{\left(\overline{x} - \frac{n-1}{\alpha n}\right)^2}{2\left(\frac{\sqrt{n-1}}{\alpha n}\right)^2}\right\},$$

which is a Gaussian with mean $\overline{(\overline{x})} = \frac{n-1}{\alpha n}$ and standard deviation $\sigma_{\overline{x}} = \frac{\sqrt{n-1}}{\alpha n}$.
We notice that $\sigma_{\overline{x}} = \frac{\sqrt{n-1}}{\alpha n} = \frac{1/\alpha}{\sqrt{n}}\sqrt{\frac{n-1}{n}} = \frac{\sigma_x}{\sqrt{n}}\sqrt{\frac{n-1}{n}}$ and $\sigma_{\overline{x}} \to \frac{\sigma_x}{\sqrt{n}}$ as $n \to \infty$.

We see that the central limit theorem applies. In the figure it is seen that the curve for $n = 16$ is already very similar to a Gaussian.

Programs
Excel
Ch. 09. *Excel—Weighted Mean and Standard Deviations*
Origin
Ch. 09. *Origin—Weighted Mean and Standard Deviations*
Python
Ch. 09. *Python—Weighted Mean and Standard Deviations*
R
Ch. 09. *R—Weighted Mean and Standard Deviations*

Problems

9.1 **[E.O.P.R.]** The results of 8 measurements of a magnitude are

$$x_i: \quad 5.24 \quad 5.42 \quad 5.20 \quad 5.00 \quad 5.15 \quad 5.32 \quad 5.24 \quad 5.37\,.$$

Find the mean $\overline{x}$ of the measurements and its error, $\sigma_{\overline{x}}$,

(a) if equal weights are attributed to the results,
(b) if the weights given to the results are, respectively,

$$w_i \quad 2 \quad 1 \quad 1 \quad 3 \quad 3 \quad 2 \quad 1 \quad 2\,.$$

9.2 **[E.O.P.R.]** A series of 10 measurements of the quantity x gave the result $x_1 = 8.65 \pm 0.12$, while another series of 20 measurements of the same quantity gave $x_2 = 8.45 \pm 0.08$. Find the value of x and its error for the total of the 30 measurements, if to the two values are attributed the weights:

(a) equal to the number of measurements in each result and
(b) inversely proportional to the square of the error of each result.

9.3 The probability of observation of the discrete values $x_1, x_2, \ldots, x_N$ is proportional to

$$P = \alpha^N \, \mathrm{e}^{-\alpha^2 \sum_i (x_i - \overline{x})^2}$$

where $\overline{x}$ is the mean of the x's. Find the value of α which maximizes P.

References

1. E. Whittaker, G. Robinson, *Calculus of observations* (Blackie and Son Ltd, London, 1929), p. 179
2. See, for example, Chandrasekhar, Stochastic problems in physics and astronomy, *Reviews of Modern Physics*, **15**, 1–89, 1943. Reprinted in N. Wax (ed.), *Selected Papers on Noise and Stochastic Processes* (Dover, 1954). In this article, which is of quite an advanced level, many problems are examined, among which the problem of particle diffusion in 1, 2 and 3 dimensions, with steps that are not constant but are characterized by any distribution function
3. H. Jeffreys, *Theory of Probability* (Clarendon Press, Oxford, 1948)
4. R.T. Birge, Phys. Rev. **40**, 207–227 (1932)
5. For an outline of the proof, see e.g. P.G. Hoel, *Introduction to Mathematical Statistics* (J. Wiley and Son Inc., New York, 1971). 4th ed., p. 359. For a complete proof, see W. Feller, *An Introduction to Probability Theory and its Applications* (J. Wiley and Son Inc., New York, 1968). 3rd ed. vol. 1, p. 244, vol. 2, p. 252
6. See, for example, Rand Corporation, *A Million Random Digits with 100 000 Normal Deviates* (The Free Press, 1955)
7. See, for example, C.A. Whitney, *Random Processes in Physical Systems* (J. Wiley and Sons, 1990) p. 20
8. See, for example, J. Arnt and C. Haenel, π *Unleashed* (Springer, 2000). This book is accompanied by a CD which contains, among other things, a table with the first 400 million digits of π!
9. Athanasios Papoulis and S. Unnikrishna Pillai, *Probability, Random Variables, and Stochastic Processes* (McGraw-Hill Book Company, 4th ed., 2002). Sects. 4.2 and 6.1
10. For proofs, see, for example A. Papoulis and S. Unnikrishna Pillai, op. cit., Chap. 6

Chapter 10
Comparison and Rejection of Measurements

The question of whether to reject a measurement arises very often. The question is then whether the observed large deviation of a measurement from the mean of the series of measurements to which it belongs is due to expected and acceptable random errors, or is the result of a mistake during measurement. It must be borne in mind that many researchers are of the opinion that no measurement should be rejected, as a matter of principle, because this would alter the results based on subjective criteria. Nevertheless, the question frequently arises and we will present here the criteria by which a measurement, even if not finally rejected, is placed on our list as the result of some unusual and unknown sequence of events.

10.1 The Problem of the Rejection of Measurements

Assume that we have N results x_i $(i = 1, 2, \ldots, N)$ of measurements of the magnitude **x**. These results have a mean value $\bar{x}$ and sample standard deviation s_x. If the distribution of the errors in x is Gaussian, the results will be distributed about their mean with this standard deviation. If we had a large number of measurements, we would expect some values to differ by a large amount from the mean.

The question that arises is: given the number N of measurements, by how much must a result differ from the mean for us to conclude that the difference is improbable to be due to random errors, but is rather the result of a mistake during the experimental procedure and, therefore, this measurement must be rejected as unacceptable?

Figure 10.1 shows the results of 13 measurements of the magnitude **x**, in the order in which they were obtained. The results are also given in Table 10.1. We find that $\sum x_i = 655$ and, therefore, the mean of the measurements is $\bar{x} = 50.4$. The mean value is marked in Fig. 10.1 by a horizontal (full) straight line at $x = 50.4$.

C. Christodoulides and G. Christodoulides, *Analysis and Presentation of Experimental Results*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-53345-2_10

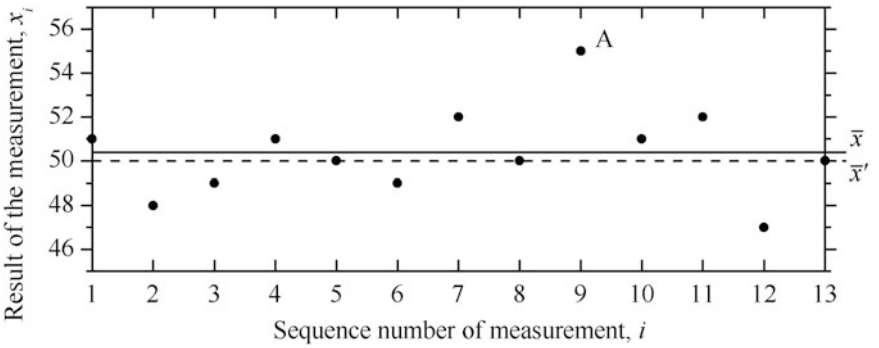


Fig. 10.1 A series of 13 measurements, among which there is one considered for rejection (A)

Table 10.1 The results of 13 measurements, one of which is a candidate for rejection

i	x_i	$x_i - \bar{x}$	$(x_i - \bar{x})^2$
1	51	0.62	0.384
2	48	−2.38	5.664
3	50	−0.38	0.144
4	51	0.62	0.384
5	50	−0.38	0.144
6	48	−2.38	5.664
7	52	1.62	2.624
8	50	−0.38	0.144
9	55	4.62	21.344
10	51	0.62	0.384
11	52	1.62	2.624
12	47	−3.38	11.424
13	50	−0.38	0.144
Σ =	655		51.08

From the sum $\sum (x_i - \bar{x})^2 = 51.08$, the standard deviation of the measurements is calculated to be $s_x = \sqrt{51.08/13} = 2.0$.

The 9th measurement differs from the mean by $d_9 = 55 - 50.4 = 4.6$. The difference is equal to $4.6/2.0 = 2.3$ standard deviations.

Assuming that the distribution of the measurements about the mean is normal, we may calculate the probability for a measurement to differ from the mean by at least 2.3 standard deviations either in the positive or the negative direction. From Table 4.2, we find that this probability is equal to $1 - 2 \times 0.4893 = 0.021$ or 2.1%. We expect that, due to random errors, one measurement in 48 will differ from the mean by more than 2.3 standard deviations. In our measurements we find one such measurement in only 13. The expected number would be $0.021 \times 13 = 0.27$. The observed number, 1, is almost 4 times the expected. We therefore conclude that,

most probably, this measurement differs from the mean by as much as it does not because of random errors but due to other causes and we reject it.

Having rejected the 9th measurement, we recalculate the mean and the standard deviation of the remaining 12 values. We find $\bar{x}' = 600/12 = 50$. This new mean value is also marked in Fig. 10.1 with a horizontal dashed line. The sum $\sum (x_i - \bar{x}')^2 = 28$, gives a standard deviation of the 12 measurements $s'_x = \sqrt{28/12} = 1.5$.

Among the remaining 12 measurements, the value with initial order number 12 is now the candidate for rejection. This measurement differs from the new mean by $|47 - 50|/1.5 = 2$ new standard deviations. From Table 4.2 we find that the probability of finding a measurement which differs from the mean by at least 2 standard deviations, either in the positive or the negative direction, is equal to $1 - 2 \times 0.4773 = 0.045$ or 4.5%. The expected number of such results in our 12 measurements is 0.54. Obviously this measurement must not be rejected, as the ratio of the observed to the expected number is near unity.

There seems to be a need for a criterion which, although not entirely objective, would at least be commonly agreed to, thus reducing to a certain degree the subjective factor. One such criterion we will examine below.

10.2 Chauvenet's Criterion

Chauvenet proposed the following criterion for the rejection of measurements:

A measurement belonging to a group of N measurements is rejected if its difference from the mean of the measurements is such that the probability of observation of such a difference or greater is less than $1/(2N)$.

In other words, Chauvenet's criterion rejects a measurement if the expected number of such measurements with a differences from the mean equal to or larger than its deviation is less than ½. Obviously, the number ½ is arbitrary and this is one of the objections against this criterion or any other criterion.

For use with Chauvenet's criterion, Table 10.2 gives the probability that the absolute difference of a value of x from the mean $\bar{x}$ is equal to or greater than v times the standard deviation s_x of the measurements, as a function of v. The difference of x from the mean $\bar{x}$ is expressed, in units of s_x, as $|x - \bar{x}|/s_x$ and the probability for this difference to be equal to or greater than v is denoted by $\Pr\left\{\frac{|x-\bar{x}|}{s_x} \geq v\right\}$.

According to Eq. (4.65), the probability for a value of x to differ from the mean $\bar{x}$ by more than v times the standard deviation s_x, is given by the relation

$$\Pr\{x \leq \bar{x} - v s_x \text{ or } x \geq \bar{x} + v s_x\} = 1 - \operatorname{erf}\left(\frac{v}{\sqrt{2}}\right) \equiv \operatorname{erfc}\left(\frac{v}{\sqrt{2}}\right) = 1 - 2\,\Phi(v). \tag{10.1}$$

Table 10.2 Probability for the absolute difference of a value of x from the mean $\bar{x}$ being equal to or greater than v times the standard deviation of the measurements, s_x

v	$\Pr\left\{\frac{\lvert x-\bar{x}\rvert}{s_x} \geq v\right\}$	v	$\Pr\left\{\frac{\lvert x-\bar{x}\rvert}{s_x} \geq v\right\}$	v	$\Pr\left\{\frac{\lvert x-\bar{x}\rvert}{s_x} \geq v\right\}$	v	$\Pr\left\{\frac{\lvert x-\bar{x}\rvert}{s_x} \geq v\right\}$
1.6	0.1096	2.1	0.0357	2.6	0.00932	3.1	0.001936
1.7	0.0891	2.2	0.0278	2.7	0.00694	3.2	0.001374
1.8	0.0719	2.3	0.0214	2.8	0.00512	3.3	0.000966
1.9	0.0574	2.4	0.01640	2.9	0.00374	3.4	0.000674
2	0.0455	2.5	0.01242	3	0.00270	3.5	0.000466

We wish to find the limit for rejection of values, $v_C \equiv \frac{|x-\bar{x}|}{s_x}$, according to Chauvenet's criterion. From Eq. (10.1) we have $\Pr\{x \leq \bar{x} - v_C s_x \text{ or } x \geq \bar{x} + v_C s_x\} = \frac{1}{2N}$ or

$$\text{erf}\left(\frac{v_C}{\sqrt{2}}\right) = 1 - \frac{1}{2N} \quad \text{and} \quad \Phi(v_C) = \frac{1}{2} - \frac{1}{4N} \tag{10.2}$$

from which, for a given N, we may find the corresponding value of v_C.

Given in the Table 10.3, for various values of the number N of the measurements, is the limit for rejection, $v_C \equiv |x - \bar{x}|/s_x$, according to Chauvenet's criterion. A value, out of a total of N values, is rejected if the absolute value of its difference from the sample mean, $|x - \bar{x}|$, is larger than $v_C s_x$, where the value of v_C is found from the table for the corresponding N. For example, one measurement in a series of 10 measurements is rejected if it differs from the sample mean by more than 1.96 s_x.

Example 10.1

Apply Chauvenet's criterion to the measurements of Table 10.1.

The N = 13 measurements of the table have a mean equal to $\bar{x} = 50.4$ and a standard deviation $s_x = 2.0$. The 9th measurement differs from the mean by

$$\frac{|x_9 - \bar{x}|}{s_x} = \frac{|55 - 50.4|}{2.0} = \frac{4.6}{2.0} = 2.3 \text{ standard deviations.}$$

From Table 10.2 we find that the probability for a difference from the mean greater than or equal to $2.3 s_x$ is $\Pr\left\{\frac{|x-\bar{x}|}{s_x} \geq 2.3\right\} = 0.0214$. For N = 13, it is $1/(2N)$ = 0.0385.

Since it is $\Pr\left\{\frac{|x-\bar{x}|}{s_x} \geq 2.3\right\} < 1/(2N)$, the 9th measurement is rejected.

Table 10.3 The rejection limit of a value, $v_C \equiv |x - \bar{x}|/s_x$, according to Chauvenet's criterion, as a function of the number N of measurements

N	v_C	N	v_C	N	v_C	N	v_C
–	–	21	2.26	41	2.51	110	2.84
–	–	22	2.28	42	2.51	120	2.87
–	–	23	2.29	43	2.52	130	2.89
4	1.53	24	2.31	44	2.53	140	2.92
5	1.65	25	2.33	45	2.54	150	2.93
6	1.73	26	2.34	46	2.55	160	2.94
7	1.80	27	2.35	47	2.55	170	2.96
8	1.86	28	2.37	48	2.56	180	2.98
9	1.91	29	2.38	49	2.57	190	3.00
10	1.96	30	2.39	50	2.58	200	3.02
11	2.00	31	2.41	55	2.61	218	3.05
12	2.04	32	2.42	60	2.64	258	3.10
13	2.07	33	2.43	65	2.67	306	3.15
14	2.10	34	2.44	70	2.69	364	3.20
15	2.13	35	2.45	75	2.72	433	3.25
16	2.16	36	2.46	80	2.74	517	3.30
17	2.18	37	2.47	85	2.76	619	3.35
18	2.20	38	2.48	90	2.77	742	3.40
19	2.22	39	2.49	95	2.79	892	3.45
20	2.24	40	2.50	100	2.81	1075	3.50

Alternatively, from Table 10.3 we find that, for 13 measurements, a measurement that differs from the mean by more than 2.07 s_x is rejected. For a difference of 2.3 s_x, the rejection is justified.

The remaining 12 measurements now have a mean $\bar{x}' = 50$ and a standard deviation of $s'_x = 1.5$.

The measurement with number 12 differs from the new mean by

$$\frac{|x_{12} - \bar{x}'|}{s'_x} = \frac{|47 - 50|}{1.5} = \frac{3}{1.5} = 2.0 \text{ standard deviations.}$$

From Table 10.2 we find that the probability for a difference from the mean greater or equal to $2.0s_x$ is $\Pr\left\{\frac{|x-\bar{x}|}{s_x} \geq 2.0\right\} = 0.0455$. For $N = 12$, it is $1/(2N) = 0.0417$.

Since it is $\Pr\left\{\frac{|x-\bar{x}|}{s_x} \geq 2.0\right\} > 1/(2N)$, the 12th measurement is not rejected.

Alternatively, from Table 10.3 we find that, for $N = 12$, a measurement that differs from the mean by up to 2.04 s_x is not rejected. With a difference of $2.0s_x$, the 12th measurement should not be rejected.

10.3 Comments Concerning the Rejection of Measurements

Before we take a stand on the subject of the 'rejection' of measurements, we must clarify exactly what we mean by this term. We examine the two main different possibilities below:

In the case we have examined (Fig. 10.1), we had a series of measurements of the same quantity, which were taken under identical experimental conditions, as much as this was possible. Of course, it is not possible to maintain the conditions unchanged. It is good practice, in all experimental work, to keep detailed notes for everything happening and resort to these in an effort to find what had possibly changed during the taking of the 'suspect' measurement and which led to the difference observed. It is, however, rather improbable that the causes have been recorded, given that, if changes in the experimental conditions had been noted, they would have been corrected before the execution of the measurement. The opinion that the measurement must be repeated is not the solution. In a series of N measurements, the measurement has been repeated $N - 1$ times! If the rejection of results alters significantly the final result, then it might be necessary, if it is still possible, for more measurements to be made. Of course, the danger exists here that we keep making measurements until we have a result we like. This would have much more serious consequences than the rejection of a measurement. Summarizing, however, we would say that, in cases such as this, the use of Chauvenet's criterion is justified. In no case, however, should the criterion be used for the rejection of two or more measurements, even when the number of measurements is large, in which case the rejection of a measurement would not alter the final result significantly. In cases when the criterion suggests the rejection of two or more measurements, the possibility should be seriously examined that the distribution of the parent population is not normal, the deviations being more important at its tails. Naturally, when a measurement has been rejected, this should be stated.

Someone using systematically Chauvenet's criterion in his work, sooner or later is bound to reject measurements which he should not reject. Large deviations are improbable but not impossible!

The second case is that in which we are dealing with measurements which are not expected to give the same result, and one differs significantly from its expected value. For example, if we measure the values of variable y as a function of another, x, and by the use of some method (such as the method of least squares, to be developed in the next chapter) we find the best mathematical relation between the two magnitudes (Fig. 10.2), then some point may deviate by so much from the expected value, that it is probable that this value is the result of a mistake (point A in Fig. 10.2). If the method used in the fitting of a curve to the results also gives the

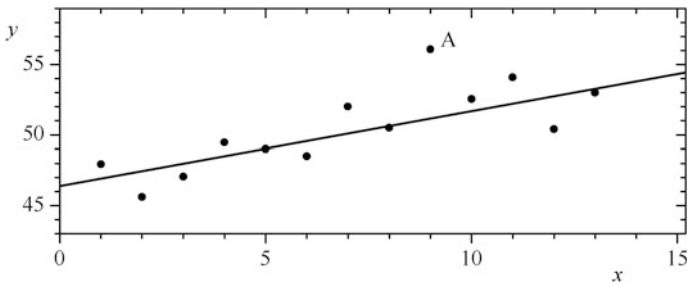


Fig. 10.2 A series of 13 measurements of y as a function of x, among which there is one candidate for rejection (A)

expected error in y for every value of x, it is possible to apply criteria for the rejection of some value.

In this case, greater attention is needed to the rejection of a measurement. The reason we might wish to reject 'erroneous' results is so that we then apply the curve-fitting method again to the remaining results and find more accurate values for the parameters of the function relating x and y. In contrast to the last case, however, we do not have other measurements taken under the same experimental conditions, with which we would compare the result under investigation. The right way to face the problem is to return to the laboratory (assuming that, quite wrongly, we have left it before detecting the problem!) and perform more measurements in the region of the suspect point. In the example of Fig. 10.2 this would mean the region between $x = 8$ and $x = 10$. Only then will we be in a position to decide whether the measurement should be rejected or that the relation $y(x)$ does not behave as assumed in this region and, perhaps, that the deviation is due to a hitherto unknown phenomenon.

In cases that the problem cannot be resolved in the way described above, the criterion will indicate whether something unusual is happening in that region of values, by giving an estimate of how probable it is for the deviation to be due to random errors. We must not forget, however, that for $|x - \bar{x}|/s_x > 2.5$ our confidence that the normal distribution accurately describes the deviations is low.

We will conclude our discussion by mentioning one of the many cases in the history of science when not rejecting a value that appeared wrong lead to important discoveries. Rayleigh and Ramsay, in 1894, noticed that nitrogen produced in the laboratory was lighter than atmospheric nitrogen by 0.5%. The fact that they did not interpret the difference as being the result of random errors but considered the deviation to be real, lead them to conclude that an unknown gas was present in the sample of what was thought to be pure atmospheric nitrogen. Thus argon was discovered. Of course, in support of the statistical analysis of experimental results, it must be said that it was the knowledge of the possible errors in the measurements that lead the two scientists to suspect that the deviation was statistically significant.

10.4 Comparison of the Means of Two Series of Measurements of the Same Quantity

The need frequently arises for the comparison of the results of two series of measurements of the same quantity. The two series of measurements were, possibly, performed by the same experimenter at different times or were performed by different researchers. It is also possible that the same quantity was measured using two different experimental methods. A classical example is the measurement, in an educational laboratory, of a universal constant or the property of a material and the comparison of the results with the generally accepted values for these magnitudes, found in tables.

We suppose that we have two means, $\bar{x}_1$ and $\bar{x}_2$, and their corresponding standard deviations, $\sigma_{\bar{x}_1}$ and $\sigma_{\bar{x}_2}$. The two series must be considered to have been taken from the parent population of the infinite possible values that may result in the measurement of the magnitude **x**. Being finite samples from the same parent population, they are not expected to agree completely. The question of whether the two samples originated from the same parent population is answered in Statistics by Student's *t*-test. Criteria which are simpler to apply are used for experimental results in the laboratory, of which we will describe only two:

1. The two results are considered to be in agreement with each other (or, better, that there are no serious indications for the presence of systematic errors), if the difference of their two means is smaller than or equal to the sum of the standard deviations of the two values:

$$|\bar{x}_2 - \bar{x}_1| \leq \sigma_{\bar{x}_1} + \sigma_{\bar{x}_2}. \tag{10.3}$$

2. The two results are considered to be in agreement with each other if the difference of their two means is smaller than or equal to the standard deviation of the difference of the two values:

$$|\bar{x}_2 - \bar{x}_1| \leq \sqrt{\sigma_{\bar{x}_1}^2 + \sigma_{\bar{x}_2}^2}. \tag{10.4}$$

We will consider the second criterion as somewhat more correct and we will use it in the example that follows.

Example 10.2

In an educational laboratory, two students determined experimentally, by two different methods, the absolute value e of the charge of the electron and found the following values:

$$e_1 = (1.62 \pm 0.02) \times 10^{-19}\ \text{C} \quad \text{and} \quad e_2 = (1.59 \pm 0.03) \times 10^{-19}\ \text{C}$$

Check whether the two results are consistent with each other and if they agree with the value of e generally accepted today.

The absolute value of the difference of the two results is $|e_2 - e_1| = 0.03 \times 10^{-19}$ C.

The standard deviation of the difference of the two mean values is

$$\sigma_{e_2-e_1} = \sqrt{(0.02)^2 + (0.03)^2} \times 10^{-19} = 0.036 \times 10^{-19} = 0.04 \times 10^{-19}\ \text{C}.$$

Since it is $|e_2 - e_1| < \sigma_{e_2-e_1}$, the two results are considered to agree with each other.

The absolute value of the electronic charge is given in tables as $e = 1.602\ 176\ 565(35) \times 10^{-19}$ C.

The standard error in this value is relatively negligible compared to the errors in the values obtained by the students. Therefore, as standard deviation of the difference of e and e_1 will be taken to be the standard deviation of e_1, i.e. 0.02×10^{-19} C. The difference between the values of e and e_1 is somewhat smaller than the standard deviation of their difference. We therefore conclude that the value e_1 is in agreement with the generally accepted today value of e, within the limits of the errors of the measurements. The absolute difference of e_2 from e is 0.01×10^{-19} C, which is quite smaller than the standard deviation of the difference of e_2 and e, which is equal to 0.03×10^{-19} C. Thus, e_2 is considered to be in agreement with the generally accepted today value of e, within the limits of the errors of the measurements.

Problems

10.1 The results of 13 measurements of the quantity x are:

$$9 \quad 6 \quad 5 \quad 9 \quad 7 \quad 9 \quad 6 \quad 10 \quad 4 \quad 7 \quad 8 \quad 5 \quad 13.$$

(a) Find the mean $\bar{x}$ and the standard deviation s_x of the measurements.
(b) Should the 13th measurement, $x = 13$, be rejected according to Chauvenet's criterion?

10.2 The results of 10 measurements are:

$$126 \quad 72 \quad 162 \quad 144 \quad 252 \quad 162 \quad 135 \quad 135 \quad 153 \quad 117.$$

Is there a result that should be rejected according to Chauvenet's criterion?

10.3 A series of 37 measurements resulted in the values x_r with frequencies n_r

x_r	31.9	32.0	32.2	32.3	32.4	32.5	32.6	33.0
n_r	1	3	7	12	6	6	1	1

Use Chauvenet's criterion in order to decide whether the last measurement should be rejected.

10.4 Two series of measurements of the same quantity gave the results $x_1 = 1.518 \pm 0.012$ and $x_2 = 1.535 \pm 0.015$. Are the two results mutually compatible?

10.5 Two series of measurements of the same quantity gave the results $x_1 = 163 \pm 6$ and $x_2 = 180 \pm 4$. Are the two results mutually compatible?

Chapter 11
The Method of Least Squares

11.1 Introduction

The *method of least squares* has many applications. For the purposes of this book, we will examine mainly its application in the fitting of the best straight line or curve to a series of experimental results, with the aim of determining the relationship existing between two variables. The method was originally developed by Legendre in 1805, while Gauss mentions that he had already used the method in 1794, at the age of 17, to determine the orbit of the asteroid Ceres. The problem which Legendre solved is the following:

> Let us assume that we have a certain number ($n > 2$) of linear equations $A_r x + B_r y = K_r$, in which A_r, B_r and K_r are constant. We may find pairs of values (x, y) which satisfy any one of the n equations, but these pairs do not satisfy all the equations simultaneously. In other words, the equations are not consistent with each other. They form an overdetermined system of equations. The problem that arises is to find the values of x and y which satisfy all the equations in the best possible way.

The answer depends, of course, on what we mean by the phrase '*the best possible way*'. Legendre stated the following principle: *The most probable value of a magnitude being measured is that for which the sum of the squares of the deviations of the measurements from this value is a minimum*. As we have seen in Chap. 9 (Sect. 9.3), the normal law of errors may be used in order to prove the principle of the most probable value. Inversely, Gauss derived the normal law of errors assuming that the mean value of a series is the most probable value of the magnitude being measured. It turns out that the sum of the squares of the deviations of the measurements from their mean value is the least possible (compared to the sum of the squares of the deviations from any other value).

Similar arguments may also be used to solve the problem of fitting the best straight line or curve to a series of experimental results, as we will show below. Strictly speaking, the method of least squares is valid only in those cases where the

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results of the measurements are normally distributed relative to the real values of the quantities being measured. It is, however, also applied to cases in which the distribution (if and when this is known) is only approximately normal, but, also, in general, when the relative errors are small.

11.2 The Theoretical Foundation of the Method of Least Squares

Let the random variable **y** be a function of only one independent random variable **x**. We will assume that the true values of the two variables are related through the mathematical expression

$$y_0 = y_0(x, \alpha_0, \beta_0, \ldots), \tag{11.1}$$

which gives the real value of y for a given x. The parameters $\alpha_0, \beta_0, \ldots$ are unknown to us. Our aim is to determine best estimates for these parameters, using the results of N measurements by which we have found for every value x_i of **x** the corresponding value y_i of **y**. We will assume that errors occur only in the values y_i and that the values x_i are known with absolute certainty. The problem becomes much more complicated if we assume that both x_i and y_i have errors.

We define

$$\delta y_{0i} = y_i - y_0(x_i) \tag{11.2}$$

to be the deviation of the measured value y_i from the true value $y_0(x_i)$ which is predicted by the function $y_0 = y_0(x, \alpha_0, \beta_0, \ldots)$ for the value x_i. The probability densities of the deviations are normal, with corresponding standard deviations σ_{0i} (also unknown). Thus, the probability that the value of **y** corresponding to x_i lies between y_i and $y_i + \delta y_{0i}$ is

$$\delta P_0\{y_i < \mathbf{y} \le y_i + \delta y_{0i}\} = \frac{\delta y_{0i}}{\sqrt{2\pi}\,\sigma_{0i}} \exp\left\{-\frac{[y_i - y_0(x_i)]^2}{2\sigma_{0i}^2}\right\}. \tag{11.3}$$

These are illustrated for the case of a linear relationship between y and x in Fig. 11.1a.

If the deviations are mutually independent, the composite probability that the result of the first measurement lies between y_1 and $y_1 + \delta y_{01}$, the result of the second measurement lies between y_2 and $y_2 + \delta y_{02}$ etc. for all the N measurements is

$$\mathrm{d}^N P_0 = \frac{1}{\left(\sqrt{2\pi}\right)^N \sigma_{01}\sigma_{02}\ldots\sigma_{0N}} \exp\left\{-\left[\frac{(\delta y_{01})^2}{2\sigma_{01}^2} + \frac{(\delta y_{02})^2}{2\sigma_{02}^2} + \ldots + \frac{(\delta y_{0N})^2}{2\sigma_{0N}^2}\right]\right\} \delta y_{01}\delta y_{02}\ldots\delta y_{0N}. \tag{11.4}$$

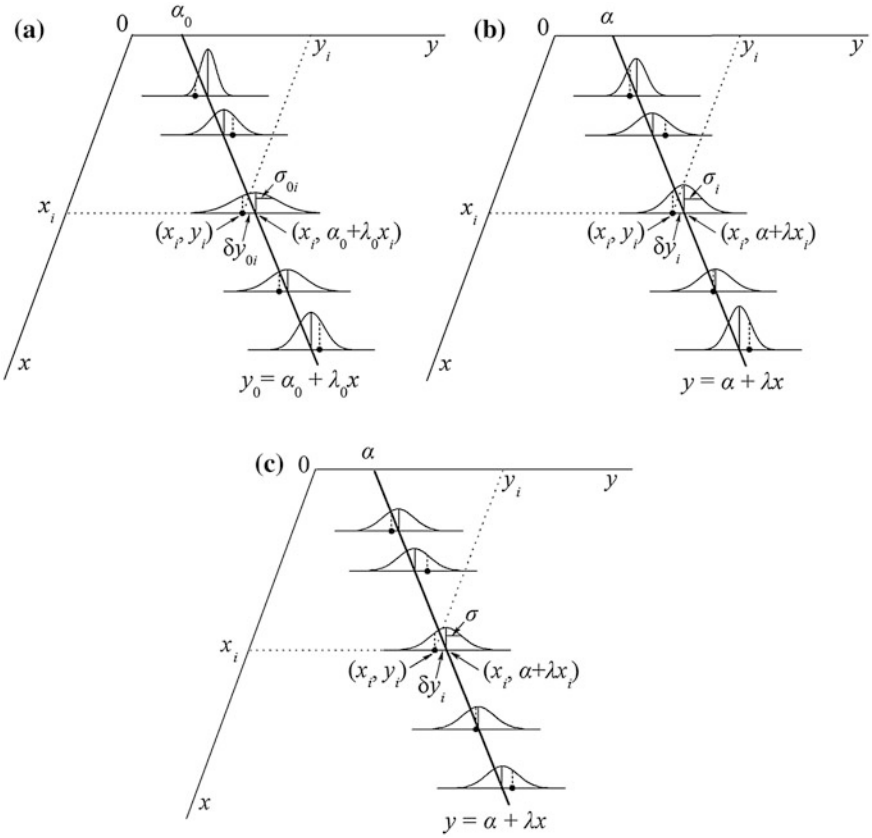


Fig. 11.1 Illustrating the method of least squares

In an *N-dimensional space of errors*, this probability is written as

$$\delta^N P_0 = \frac{1}{\left(\sqrt{2\pi}\right)^N \sigma_0^N} \mathrm{e}^{-\chi_0^2/2} \delta^N \upsilon_0, \tag{11.5}$$

where

$$\chi_0^2 \equiv \frac{(\delta y_{01})^2}{\sigma_{01}^2} + \frac{(\delta y_{02})^2}{\sigma_{02}^2} + \ldots + \frac{(\delta y_{0N})^2}{\sigma_{0N}^2} = \sum_{i=1}^{N} \frac{(\delta y_{0i})^2}{\sigma_{0i}^2}, \tag{11.6}$$

$$\sigma_0^N \equiv \sigma_{01}\sigma_{02}\ldots\sigma_{0N} \tag{11.7}$$

and the magnitude

$$\delta^N \upsilon_0 \equiv \delta y_{01} \delta y_{02} \ldots \delta y_{0N} \tag{11.8}$$

may be considered to be the element of N-dimensional volume around the point $(y_{01}, y_{02}, \ldots y_{0N})$.

The values of the parameters $\alpha_0, \beta_0, \ldots$ which maximize the probability $\delta^N P_0$ are those minimizing the quantity χ_0^2. Thus, a number of equations equal to the number of the parameters $\alpha_0, \beta_0, \ldots$ are derived,

$$\frac{\partial \chi_0^2}{\partial \alpha_0} = 0, \quad \frac{\partial \chi_0^2}{\partial \beta_0} = 0, \quad \ldots \quad , \tag{11.9}$$

from which we would determine the parameters.

However, the true values of $\delta y_{0i} = y_i - y_0(x_i)$ are not known. Neither do we know the true standard deviations σ_{0i}. We will assume a relationship between x and y,

$$y = y(x, \alpha, \beta, \ldots), \tag{11.10}$$

where $\alpha, \beta, \ldots$ are parameters which we wish to determine. These values will be the best estimates for $\alpha_0, \beta_0, \ldots$. Instead of the deviations from the true values of y, $\delta y_{0i} = y_i - y_0(x_i)$, we will use the deviations from the values given by the relation $y = y(x, \alpha, \beta, \ldots)$,

$$\delta y_i = y_i - y(x_i), \tag{11.11}$$

Furthermore, the standard deviations σ_{0i} will be replaced by the standard deviations σ_i estimated for the various values of y_i determined for a given value x_i. We then have

$$\chi^2 \equiv \frac{(\delta y_1)^2}{\sigma_1^2} + \frac{(\delta y_2)^2}{\sigma_2^2} + \ldots + \frac{(\delta y_N)^2}{\sigma_N^2} = \sum_{i=1}^{N} \frac{(\delta y_i)^2}{\sigma_i^2}, \tag{11.12}$$

instead of χ_0^2. These are illustrated in Fig. 11.1b.

The values of the parameters $\alpha, \beta, \ldots$ which maximize the probability $\delta^N P$, which is an estimate of $\mathrm{d}^N P_0$, are those minimizing the quantity χ^2. Thus, a number of equations equal to the number of the parameters $\alpha, \beta, \ldots$ are derived,

$$\frac{\partial \chi^2}{\partial \alpha} = 0, \quad \frac{\partial \chi^2}{\partial \beta} = 0, \quad \ldots \quad , \tag{11.13}$$

from which we determine the parameters.

It is noted that in the quantity χ^2 the deviations are weighted, with weights equal to $1/\sigma_i^2$. In all but in very rare occasions, however, the values of $1/\sigma_i^2$ are not known. We then consider that a good approximation is that all the σ_i may be

substituted by one common one (also usually unknown) σ [see Fig. 11.1c]. The quantity to be minimized is then

$$S \equiv (\sigma\chi)^2 = \sum_{i=1}^{N} (\delta y_i)^2 = \sum_{i=1}^{N} [y_i - y(x_i)]^2. \tag{11.14}$$

11.3 The Fitting of Curves to Experimental Points

We will now use the theoretical result of Eq. (11.14) in specific applications.

11.3.1 Straight Line

Assume that from measurements we have acquired the values of the magnitude y_i corresponding to N values of x_i $(i = 1, 2, \ldots, N)$. We assume that the relation between x and y is of the form

$$y = \alpha + \lambda x \tag{11.15}$$

and wish to determine the optimum values of the parameters α and λ.

We assume that the values of the independent variable x are known with absolute accuracy. During the experimental procedure it is usually true that the variable x may be adjusted with adequate accuracy, and therefore this assumption is practically justified. The deviation of y_i from the real value $y_{0,i}$ corresponding to the particular value x_i is governed by a Gaussian distribution with standard deviation σ, common to all measurements. In Fig. 11.1 (a) the true line connecting y to x is drawn, as well as the N experimental points (x_i, y_i). For each one of them, the Gaussian distribution for the corresponding value of y_i is also drawn. The best estimates for the parameters α and λ, according to the theory presented, are such that they maximize the probability of occurrence of the results obtained with the measurements. In Fig. 11.1 (c) the best straight line through the points is the one that will maximize the total length of the dashed lines.

Figure 11.2 shows the N points and the straight line $y = \alpha + \lambda x$. For the general point $(x_i,\ y_i)$, also drawn is the difference $d_i = y_i - (\alpha + \lambda x_i)$ between the measured value y_i and the value predicted by the relation $y = \alpha + \lambda x$ for $x = x_i$. The method of least squares requires the minimization of the sum

$$S \equiv \sum_{i=1}^{N} (y_i - y(x_i))^2 = \sum_{i=1}^{N} (y_i - \alpha - \lambda x_i)^2. \tag{11.16}$$

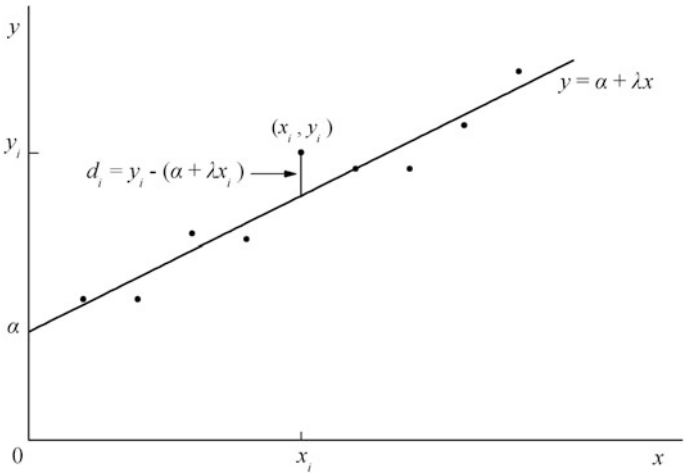


Fig. 11.2 The fitting of a straight line to experimental results with the method of least squares. The N experimental points and the straight line $y = \alpha + \lambda x$ are drawn. For the general point (x_i, y_i), also shown is the difference $d_i = y_i - (\alpha + \lambda x_i)$ between the measured value y_i and the value predicted by the relation $y = \alpha + \lambda x$ for $x = x_i$

This condition, which is the condition of Eq. (11.14), assumes that the weights in Eq. (11.12) are all taken to be the same. The more general case of measurements with different weights will be examined in Sect. 11.3.1.1.

Equating to zero the two partial derivatives of S with respect to α and λ, we have the two equations:

$$\frac{\partial S}{\partial \alpha} = \frac{\partial}{\partial \alpha} \sum_{i=1}^{N} (y_i - \alpha - \lambda x_i)^2 = -2 \sum_{i=1}^{N} (y_i - \alpha - \lambda x_i) = 0 \tag{11.17}$$

$$\frac{\partial S}{\partial \lambda} = \frac{\partial}{\partial \lambda} \sum_{i=1}^{N} (y_i - \alpha - \lambda x_i)^2 = -2 \sum_{i=1}^{N} x_i (y_i - \alpha - \lambda x_i) = 0. \tag{11.18}$$

These are rewritten as

$$\alpha N + \lambda \sum_{i=1}^{N} x_i = \sum_{i=1}^{N} y_i \tag{11.19}$$

and

$$\alpha \sum_{i=1}^{N} x_i + \lambda \sum_{i=1}^{N} x_i^2 = \sum_{i=1}^{N} x_i y_i. \tag{11.20}$$

They are known as the *normal equations*. For convenience, we adopt the notation

$$\sum_{i=1}^{N} x_i \equiv [x] \quad \sum_{i=1}^{N} y_i \equiv [y] \quad \sum_{i=1}^{N} x_i^2 \equiv [x^2] \quad \sum_{i=1}^{N} x_i y_i \equiv [xy]. \tag{11.21}$$

Equations (11.19) and (11.20) now have the form

$$\alpha N + \lambda\,[x] = [y] \quad \text{and} \quad \alpha\,[x] + \lambda\,[x^2] = [xy]. \tag{11.22}$$

They are solved to give:

$$\alpha = \frac{[y][x^2] - [x][xy]}{N[x^2] - [x]^2} \tag{11.23}$$

$$\lambda = \frac{N[xy] - [x][y]}{N[x^2] - [x]^2}. \tag{11.24}$$

From Eq. (11.19), we notice that it is

$$\alpha + \lambda \frac{[x]}{N} = \frac{[y]}{N}, \tag{11.25}$$

which states that the straight line of least squares passes through the point $\left(x = \frac{[x]}{N} = \bar{x}, \quad y = \frac{[y]}{N} = \bar{y}\right)$. The point K: $(\bar{x}, \bar{y})$, where $\bar{x}$ and $\bar{y}$ are the means of x and y respectively, may be considered to be the *center of the measurements*.

The accuracy with which we know α and λ is a useful magnitude. We will give here the results without proof. A complete analysis is given in Appendix 1. In order to find the errors $\delta\,\alpha$ and $\delta\,\lambda$ in α and λ, respectively, the standard deviation of the values y_i from the straight line must be evaluated. The best estimate for this quantity is:

$$\sigma_y = \sqrt{\frac{1}{N-2} \sum_{i=1}^{N} (y_i - \alpha - \lambda x_i)^2} \quad \text{or} \quad \sigma_y = \sqrt{\frac{[d^2]}{N-2}}, \tag{11.26}$$

where

$$d_i \equiv y_i - \alpha - \lambda x_i. \tag{11.27}$$

In terms of σ_y, the standard deviations of or the errors in α and λ are, respectively,

$$\delta\alpha = \sigma_\alpha = \sigma_y \sqrt{\frac{[x^2]}{N[x^2] - [x]^2}} \tag{11.28}$$

and

$$\delta\lambda = \sigma_\lambda = \sigma_y \sqrt{\frac{N}{N[x^2] - [x]^2}}. \tag{11.29}$$

To make calculations easier, we note that it is

$$\delta\lambda = \delta\alpha \sqrt{\frac{N}{[x^2]}}. \tag{11.30}$$

Thus, the final value for the intercept of the y-axis by the straight line and the line's slope are:

$$\alpha \pm \delta\alpha \quad \text{and} \quad \lambda \pm \delta\lambda.$$

Having determined α and λ, we may calculate the value of y for every value of x within the region of the validity of the law $y = \alpha + \lambda x$. We also need to know the error δy in this value. As explained in Appendix 1, the magnitudes α and λ are not independent from each other. It would, therefore, be wrong to write the equation of the straight line as

$$y = (\alpha \pm \delta\alpha) + (\lambda \pm \delta\lambda)x \tag{11.31}$$

and the error in y as

$$\delta y = \sqrt{(\delta\alpha)^2 + (x\delta\lambda)^2}, \tag{11.32}$$

combining $\delta\alpha$ and $\delta\lambda$ as if they were independent of each other. In fact, the magnitudes that are mutually independent are the position $(\bar{x}, \bar{y})$ of the center K of the least-squares straight line and its slope λ. The straight line is defined by its center $(\bar{x}, \bar{y})$ and the independent from it orientation of the line, which is thought to rotate about its center.

Taking these into account, the error δy in y, for some value of x, is given by

$$\delta y = \frac{\sigma_y}{\sqrt{N}} \sqrt{1 + \frac{N^2}{N[x^2] - [x]^2}(x - \bar{x})^2}. \tag{11.33}$$

Finally, if it is assumed that the straight line passes through the origin, i.e. it is

$$y = \lambda x, \tag{11.34}$$

then λ is given by the relations

$$\lambda = \frac{[xy]}{[x^2]} = \frac{[y]}{[x]}, \tag{11.35}$$

where the second one is found by adding the terms of Eq. (11.34) over all the values of i. Thus, in this case, the least-squares straight line passes through the origin (0, 0) and the center of the measurements $(\bar{x}, \bar{y})$. The error in λ is evaluated using the relations:

$$\sigma_y = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(y_i - \lambda x_i)^2} \tag{11.36}$$

$$\delta\lambda = \frac{\sigma_y}{\sqrt{[x^2]}}. \tag{11.37}$$

Also, if the straight line is parallel to the x-axis, i.e. it is

$$y = a, \tag{11.38}$$

then

$$a = \frac{[y]}{N}, \tag{11.39}$$

the mean value of y. The error in a will, therefore, be equal to the standard deviation of the mean of the y values,

$$\delta a = \sigma_{\bar{y}}. \tag{11.40}$$

Example 11.1

Apply the method of least squares to the measurements given in the first three columns of the table below, in order to fit to them a straight line $y = \alpha + \lambda x$. Find the value of y for $x = 1.5$.

It is $N = 11$.

The central point of the curve is K: $(\bar{x}, \bar{y})$, where

$$\bar{x} = [x]/N = 11/11 = 1.00 \quad \text{and} \quad \bar{y} = [y]/N = 35.44/11 = 3.22.$$

i	x_i	y_i	$x_i y_i$	x_i^2	d_i	d_i^2
1	0.0	0.92	0.000	0.00	−0.05090	0.00259
2	0.2	1.48	0.296	0.04	0.05892	0.00347
3	0.4	1.96	0.784	0.16	0.08874	0.00787
4	0.6	2.27	1.362	0.36	−0.05144	0.00265
5	0.8	2.61	2.088	0.64	−0.16162	0.02612
6	1.0	3.18	3.180	1.00	−0.04180	0.00175
7	1.2	3.80	4.560	1.44	0.12802	0.01639
8	1.4	4.01	5.614	1.96	−0.11216	0.01258
9	1.6	4.85	7.760	2.56	0.27766	0.07710
10	1.8	5.10	9.180	3.24	0.07748	0.00600
11	2.0	5.26	1.520	4.00	−0.21270	0.04524
Sums	11.0 $=[x]$	35.44 $=[y]$	45.344 $=[xy]$	15.40 $=\|[x^2]$		0.20176 $=[d^2]$

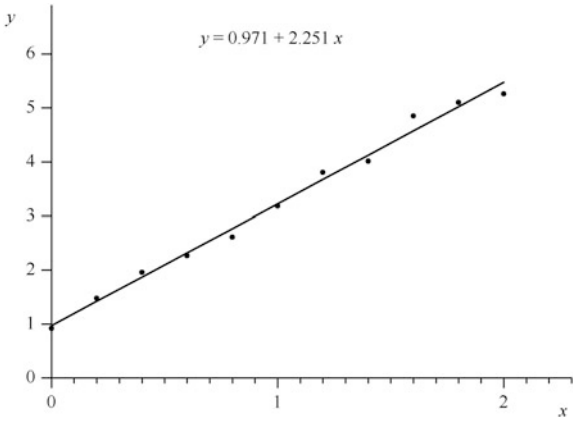
Thus,

$$\alpha = \frac{[y][x^2] - [x][xy]}{N[x^2] - [x]^2} = \frac{35.44 \times 15.40 - 11 \times 45.344}{11 \times 15.40 - 11^2} = 0.9709$$

$$\lambda = \frac{N[xy] - [x][y]}{N[x^2] - [x]^2} = \frac{11 \times 45.344 - 11 \times 35.44}{11 \times 15.40 - 11^2} = 2.2509$$

and the required straight line is $y = 0.971 + 2.251x$.

The experimental points and the straight line found have been drawn in the figure below.



To find the errors in α and λ we first evaluate σ_y. In the table, we have calculated the deviations $d_i \equiv y_i - \alpha - \lambda x_i$ and their squares. Thus, we find that

$$\sigma_y = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}(y_i - \alpha - \lambda x_i)^2} = \sqrt{\frac{0.20176}{9}} = 0.150.$$

The standard deviations of, or the errors in, α and λ are, respectively,

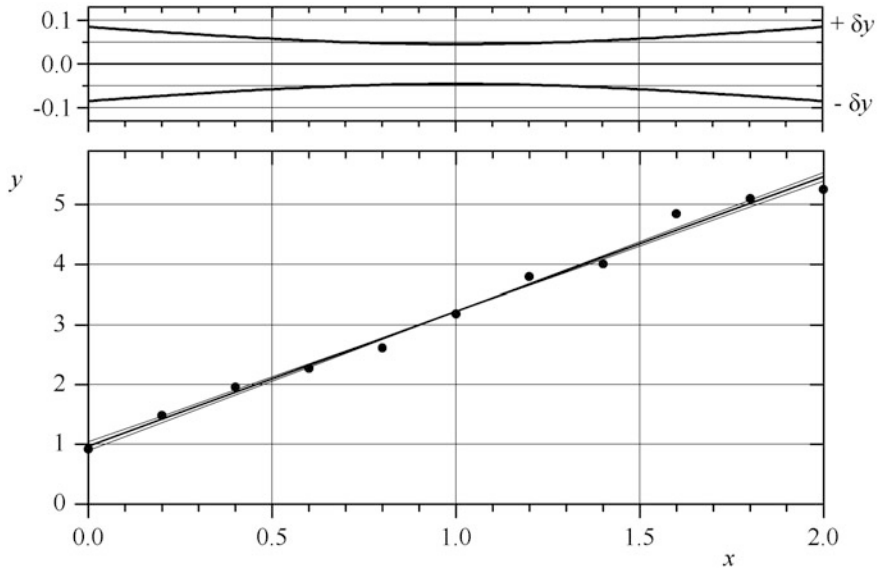
$$\delta\alpha = \sigma_\alpha = \sigma_y\sqrt{\frac{[x^2]}{N[x^2] - [x]^2}} = 0.150\sqrt{\frac{15.40}{11 \times 15.40 - 11^2}} = 0.084$$

and

$$\delta\lambda = \sigma_\lambda = \sigma_y\sqrt{\frac{N}{N[x^2] - [x]^2}} = 0.150\sqrt{\frac{11}{11 \times 15.40 - 11^2}} = 0.071.$$

Therefore, we have found that $\alpha = 0.971 \pm 0.084$ and $\lambda = 2.251 \pm 0.071$.

In the figure below, apart from the least-squares straight line $y = 0.971 + 2.251x$, also given are the straight lines passing through the central point of the measurements ($\bar{x} = 1.00$, $\bar{y} = 3.22$) and having slopes $\lambda = 2.251 \pm 0.071$.



The equation $y = 0.9709 + 2.2509\,x$ found, gives the value of y for every value of x.

The error in y is given by Eq. (11.29) as

$$\delta y = \frac{\sigma_y}{\sqrt{N}}\sqrt{1 + \frac{N^2}{N[x^2] - [x]^2}(x - \bar{x})^2} = 0.0452\sqrt{1 + 2.50 \times (x-1)^2}.$$

The variation of δy with x is shown in the figure. The error in y is minimum and equal to $\delta y = 0.045$ for $x = 1$. For $x = 0$ and for $x = 2$, the error is $\delta y = 0.085$.

From the relation $y = 0.97 + 2.25\,x$ we find that for $x = 1.5$ it is $y = 4.35$. The error in y is given by the equation $\delta y = 0.0452\sqrt{1 + 2.50 \times (x-1)^2}$ as $\delta y = 0.06$. Therefore, for $x = 1.5$ it is $y = 4.35 \pm 0.06$.

Example 11.2 [E]

Solve the problem of Example 11.1 using Excel®.

We place the data of columns x_i and y_i in cells A1-A11 and B1-B11, respectively. We highlight these cells by left-clicking on cell A1 and then, holding the **SHIFT** key down, we draw the cursor down to cell B11. In **Insert**, **Charts** we select **Scatter**. A scatter chart is created, with the points (x_i, y_i).

We press the ⊞ key at the top right corner of the plot's frame, opening **Chart Elements**. We select

Trendline, **More Options** and tick **Linear** and **Display equation on chart**. The result is y = 2.2509x + 0.9709. No errors are available for the coefficients.

Substituting x = 1.5 in the equation of the line, we obtain the result 4.34727. This is $y(1.5)$.

Example 11.3 [O]

Solve the problem of Example 11.1 using Origin®.

We place the data of columns x_i and y_i in columns A and B, respectively. We highlight both columns by left-clicking on the label of column A and then, holding the **Shift** key down, left-clicking on the label of column B. Then

Analysis > **Fitting** > **Linear Fit** > **Open Dialog**...

In the window that opens, we press **OK**. The program returns the results

Intercept$(=a) = 0.97091 \pm 0.08446$ and **Slope** $(=\lambda) = 2.25091 \pm 0.07138$.

These are the results found in Example 11.1.

A graph such as the one shown in Example 11.1 is also given.

If we also want to find y for a given x, in the dialog box that opens we have to select **Find X/Y** and then tick **Find Y from X**. With the results, there is a page titled **FitLinearFindYfromX1**. We go to this page and, in a cell of the column labeled **Enter X values:** we enter the value of $x = 1.5$. In the adjacent column, labeled **Y value**, the result 4.34727 appears. This is $y(1.5)$.

The errors in the values of y may also be taken into account in the fitting. The errors should be entered in a third column which is also selected in the analysis.

Example 11.4 [P]

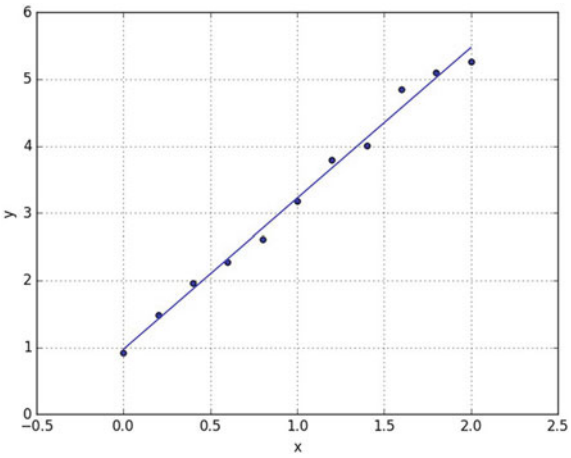
Solve the problem of Example 11.1 using Python.

```
from __future__ import division
import math
import numpy as np
import matplotlib.pyplot as plt

# Enter the values of x, y:

x = np.array([0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2])
y = np.array
([0.92, 1.48, 1.96, 2.27, 2.61, 3.18, 3.8, 4.01, 4.85, 5.10, 5.26])

# Plot, size of dots, labels, ranges of values, initial values

plt.scatter(x, y)
plt.xlabel("x")

# set the x-axis label

plt.ylabel("y")

# set the y axis label

plt.grid(True)

# Evaluation

N = len(x)
X = sum(x)
XX = sum(x**2)
Y = sum(y)
XY = sum(x*y)
```

```
DENOM = N*XX-X**2
DETA = Y*XX-X*XY
DETL = N*XY-X*Y

a = DETA/DENOM
lambda = DETL/DENOM

d = y - a - lambda*x

DD = sum(d**2)

Da = math.sqrt((DD*XX)/((N-2)*DENOM))
Dlambda = math.sqrt((N*DD)/((N-2)*DENOM))

# Results

print("Value of a:", a)
print("Value of lambda:", lambda)
print("Standard error in a:", Da)
print("Standard error in λ:", Dlambda)

# Plot least-squares line

xx = np.linspace(min(x), max(x), 200)
yy = a + b * xx
plt.plot(xx, yy, '-')

plt.show()
```

The plot shown below is produced.
The values of the parameters are:

```
Value of a: 0.970909090909
Value of λ: 2.25090909091
Standard error in a: 0.08445667109784327
Standard error in λ: 0.07137891491854917
```



From the relation $y = 0.97 + 2.25x$ we find that for $x = 1.5$ it is $y = 4.35$.

Example 11.5 [R]

Solve the problem of Example 11.1 using R.

i	x_i	y_i
1	0.0	0.92
2	0.2	1.48
3	0.4	1.96
4	0.6	2.27
5	0.8	2.61
6	1.0	3.18
7	1.2	3.80
8	1.4	4.01
9	1.6	4.85
10	1.8	5.10
11	2.0	5.26

Define vectors x and y:

```
> x <- c(0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2)
> y <- c(0.92, 1.48, 1.96, 2.27, 2.61, 3.18, 3.80, 4.01, 4.85, 5.10, 5.26)
```

Plot $y(x)$:

```
> plot(x, y, pch=20, cex=0.5, xlab="x", ylab="y", xlim=c(0, 2), ylim=c(0, 6))
```

Find least-squares best-fit straight line:

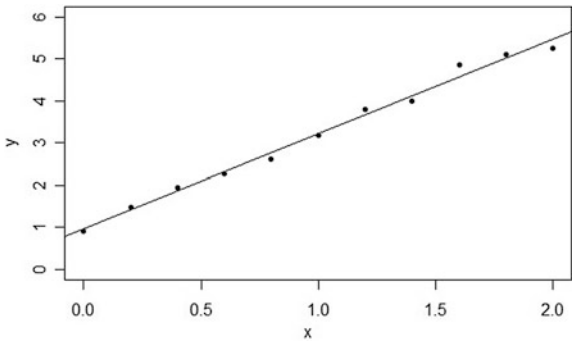
```
> fit <- lm(y~x)
> fit
```

Read intercept and slope of line:

```
Call:
> lm(formula = y ~ x)

Coefficients:
 (Intercept)       x
0.9709    2.2509
```

Plot least-squares best-fit straight line:

```
> abline(fit)
```



The equation of the line is $y = 0.9709 + 2.2509\,x$. For $x = 1.5$, it is $y = 0.9709 + 2.2509 \times 1.5 = 4.347$.

Example 11.6

In this example we will demonstrate the role of the errors in the magnitudes evaluated by the method of least squares. For this reason, the experimental points were chosen to have a high dispersion, corresponding to large measurement errors.

Using the method of least squares, fit a straight line to the points:

i	1	2	3	4	5	6	7
x_i	1	2	4	6	8	9	10
y_i	0.2	0.8	0.4	1	0.7	1.2	0.8

Find the value of y for $x = 5$.

It is: $n = 7$, $\bar{x} = 5.714$ and $\bar{y} = 0.7286$.

$$[x] = 40 \quad [y] = 5.1 \quad [xy] = 33.8 \quad [x^2] = 302 \quad [d^2] = 0.399$$
$$a = 0.3661 \quad \lambda = 0.0634$$

and the required straight line is $y = 0.366 + 0.0634\,x$.

Also, $\sigma_y = \sqrt{\frac{0.399}{5}} = 0.282 \quad \delta a = 0.166 \quad \delta\lambda = 0.025$.

The errors in the y values are given by the relation

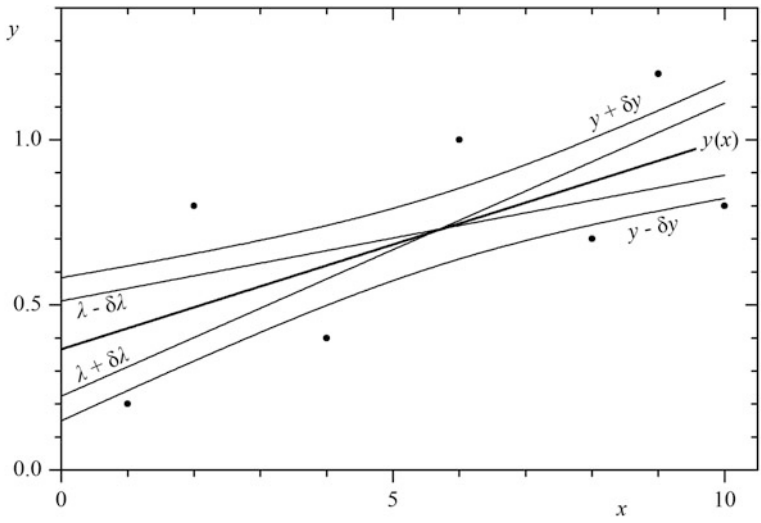
$$\delta y = \frac{\sigma_y}{\sqrt{N}} \sqrt{1 + \frac{N^2}{N[x^2] - [x]^2}(x - \bar{x})^2} = 0.1068\sqrt{1 + 0.0953(x - 5.714)^2}$$

or $\delta y = \sqrt{0.0469 - 0.01242\,x + 0.001087\,x^2}$

From the equation $y = 0.366 + 0.0634\,x$ we find that for $x = 5$ it is $y = 0.68$. The error in y is $\delta y = 0.11$. Therefore, for $x = 5$ it is $y = 0.68 \pm 0.11$.

In the figure below were drawn:

1. The experimental points and the least-squares straight line $y(x)$.
2. The straight lines passing through the center K: (5.71, 0.73) of the line $y(x)$ and having slopes $\lambda \pm \delta\lambda$, i.e. 0.0634 ± 0.0253. The equations of these straight lines are $y_1 = 0.512 + 0.0381\,x$ and $y_2 = 0.224 + 0.0887\,x$.
3. The curves $y(x) \pm \delta y$ or
$$y = 0.366 + 0.0634x \pm \sqrt{0.0469 - 0.01242\,x + 0.001087\,x^2}$$
which mark off the region of y values lying between $y \pm \delta y$.



Example 11.7 [E]

Using Excel® and the method of least squares, fit a straight line to the points:

i	1	2	3	4	5	6	7
x_i	1	2	4	6	8	9	10
y_i	0.2	0.8	0.4	1	0.7	1.2	0.8

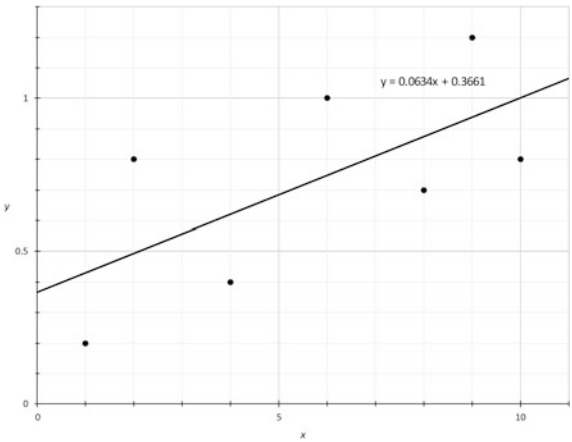
Find the value of y for $x = 5$.

We place the data x_i and y_i in columns A and B, respectively. We highlight columns A and B and open the **Insert** window. We open the **Recommended Charts** window and choose **Scatter**.

A scatter chart is produced. We double-click on a point and in the **Format Data Series** window that opens we select **Series Options > Marker Options** for **Fill** we select **Solid Fill** and color **Black**. Also, in **Border** we select **Solid Line**, color **Black**, **Width** 0.75 pt, **Dash Type** continuous line.

We double-click on the straight line and select **Line**, **Solid Line**, color **Black**, **Width** 1.5 pt, **Dash Type** continuous line. In **Trendline Options** we choose **Linear**, **Forecast**, **Forward** 1.0 period, **Backward** 1.0 period. We also tick the box **Display Equation on Chart**.

The graph produced is shown below.



The coefficients of the equation of line are:

$$a = 0.36615 \pm 0.21651 \quad \text{and} \quad \lambda = 0.06342 \pm 0.03296$$

The equation of the line is $y = 0.36615 + 0.06342x$.
For x = 5 it is $y(5)$ = 0.6833.

Example 11.8 [O]

Using Origin® and the method of least squares, fit a straight line to the points:

i	1	2	3	4	5	6	7
x_i	1	2	4	6	8	9	10
y_i	0.2	0.8	0.4	1	0.7	1.2	0.8

Find the value of y for $x = 5$.

We place the data x_i and y_i in columns A and B, respectively. We highlight columns A and B and open the **Plot** window. We select **Symbol** and then **Scatter**. A scatter plot is produced.

In the Graph1 window, and then

Analysis > **Fitting** > **Linear Fit** > **Open Dialog**...

In the **Linear Fit** window we select **Fitted Curves Plot** and set **X Data Type**, **Margin** [%] to 10. This will ensure that the straight line fitted, when plotted in the graph, will extend by 10% to the left and the right of the experimental points at the two ends. We also open the **Find X/Y** window and tick the **Find Y from X** box. Pressing **OK** produces the best fit straight line in the graph.

The program also returns the results:

$$\textbf{Intercept}\ (= a) = 0.36615 \pm 0.21651 \quad \text{and} \quad \textbf{Slope}\ (= \lambda) = 0.06342 \pm 0.03296.$$

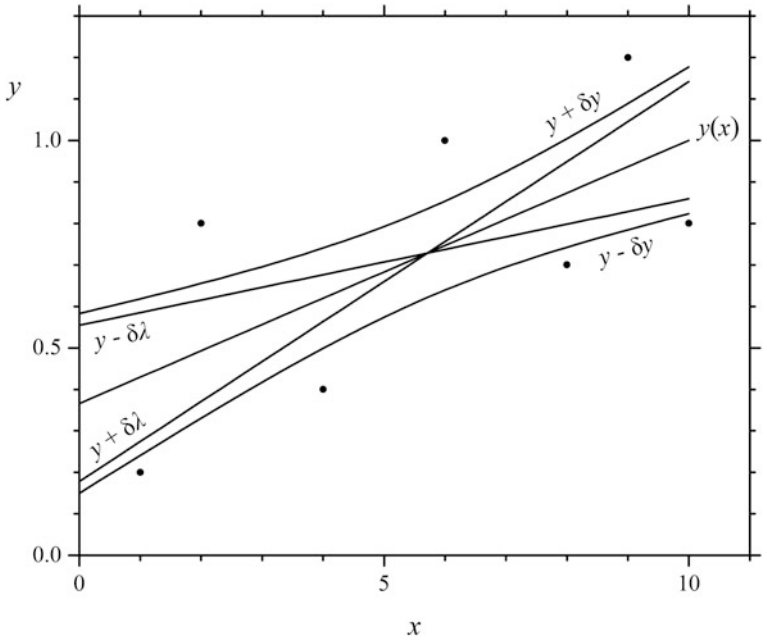
In **Book1** page **FitLinearFindYFromX1**, by typing x = 5 in the x column, we get $y(5) = 0.6833$.

The equations of the two straight lines passing through the center of the points K:(5.714, 0.7286) and having slopes $\lambda \pm \delta\lambda$, i.e. 0.06342 ± 0.03296, are

$$y_1 = 0.5546 + 0.03046\,x \quad \text{and} \quad y_2 = 0.1779 + 0.09638\,x.$$

In column D we enter the values of x from 0 to 10 in steps of 1/3. We highlight column D and open the **Column** window, where we **Set As X** column D. For these values of x, we evaluate y in column E, y_1 in column F and y_2 in column G.

Using the expression for the error δy in y found in Example 11.5, we evaluate δy in column I and the values of $y - \delta y$ and $y + \delta y$ in columns J and K respectively. We highlight columns D, E, F, G, J and K. We open the **Plot** window and select **Line > Line**. The plot shown below is produced. The experimental points were added to this plot by right clicking on the number (**1**) appearing in the top left corner, selecting **Layer Contents** and including column B in the contents of the graph shown on the right. This is done by selecting **B[Y1]** from the table in the left and using the arrow to include it in the table on the right. The final result is shown in the figure below.



Example 11.9 [P]

Using Python and the method of least squares, fit a straight line to the points:

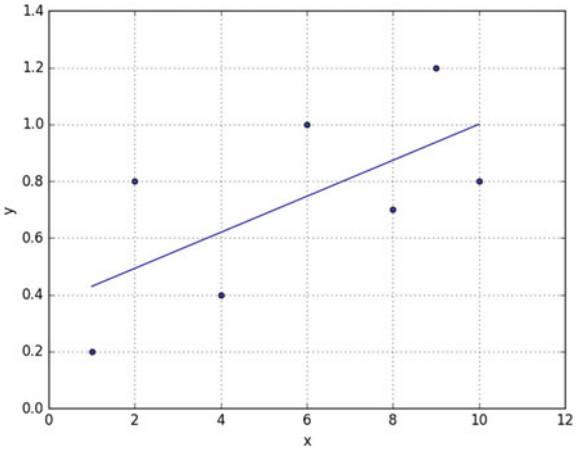
i	1	2	3	4	5	6	7
x_i	1	2	4	6	8	9	10
y_i	0.2	0.8	0.4	1	0.7	1.2	0.8

Find the value of y for $x = 5$.

```
# Enter the values of x, y:

x = np.array([1, 2, 4, 6, 8, 9, 10])
y = np.array([0.2, 0.8, 0.4, 1, 0.7, 1.2, 0.8])
```

The rest of the program is identical to that of Example 11.4
The plot shown below is produced.
The values of the parameters are:

```
Value of a: 0.366147859922
Value of λ: 0.0634241245136
Standard error in a: 0.21650834831061155
Standard error in λ: 0.03296250399459645
```



From the relation $y = 0.366 + 0.0634\,x$ we find that for $x = 5$ it is $y = 0.683$.

Example 11.10 [R]

Using R and the method of least squares, fit a straight line to the points:

i	1	2	3	4	5	6	7
x_i	1	2	4	6	8	9	10
y_i	0.2	0.8	0.4	1	0.7	1.2	0.8

Find the value of y for $x = 5$.

We form the vectors x and y. We plot the scatter plot $y(x)$.

```
> x <- c(1, 2, 4, 6, 8, 9, 10)
> y <- c(0.2, 0.8, 0.4, 1, 0.7, 1.2, 0.8)
> plot(x, y, pch=20, xlab="x", ylab="y", xlim=c(0, 10), ylim=c(0, 1))
```

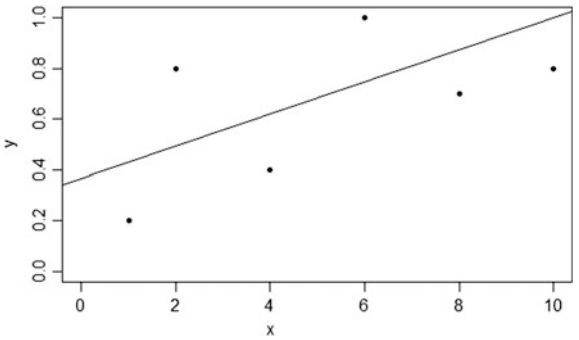
We fit a least-squares straight line to the data:

```
> fit <- lm(y~x)
> fit

Call:
lm(formula = y ~ x)

Coefficients:
(Intercept)            x
    0.36615      0.06342
```

We plot the straight line:

```
> abline(fit)
```

The equation of the line is $y = 0.36615 + 0.06342x$. For $x = 5$ it is $y(5) = 0.6833$.



11.3.1.1 Least Squares Using Weighted Measurements

If the results of the measurements are weighted, with weight w_i for the measurement $(x_i,\ y_i)$, then the results are modified as follows: The normal equations are

$$\alpha[w] + \lambda[wx] = [wy] \quad \text{and} \quad \alpha[wx] + \lambda[wx^2] = [wxy] \tag{11.41}$$

from which it follows that

$$\alpha = \frac{[wy][wx^2] - [wx][wxy]}{[w][wx^2] - [wx]^2} \tag{11.42}$$

$$\lambda = \frac{[w][wxy] - [wx][wy]}{[w][wx^2] - [wx]^2} \tag{11.43}$$

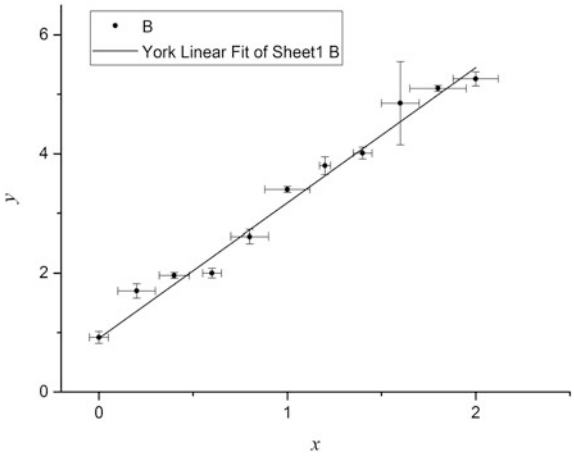
$$\delta\,\alpha = \sigma_\alpha = \sqrt{\frac{[wd^2]}{(N-2)}\frac{[wx^2]}{[w][wx^2] - [wx]^2}} \tag{11.44}$$

$$\delta\,\lambda = \sigma_\lambda = \sqrt{\frac{[wd^2]}{(N-2)}\frac{[w]}{[w][wx^2] - [wx]^2}}. \tag{11.45}$$

Example 11.11 [O]

Fit a straight line to the set of values $(x_i,\ y_i)$ with their errors given in the table below, applying the method of least squares and taking into account the errors.

	col(A)	col(B)	col(C)	col(D)
i	x_i	y_i	δx_i	δy_i
1	0	0.92	0.05	0.1
2	0.2	1.7	0.1	0.12
3	0.4	1.96	0.08	0.05
4	0.6	2	0.05	0.08
5	0.8	2.61	0.1	0.12
6	1	3.4	0.12	0.05
7	1.2	3.8	0.03	0.15
8	1.4	4.01	0.05	0.1
9	1.6	4.85	0.1	0.7
10	1.8	5.1	0.15	0.05
11	2	5.26	0.12	0.12



We place the data of columns x_i, y_i and their errors δx_i and δy_i, in columns A, B, C and D, respectively. We highlight column A. Then

Analysis > Fitting > Fit Linear with Errors > Open Dialog. . .

In the window that opens, we open **Input, Input Data**. In **Range 1** we enter for **X** column **A(X)**, for **Y** column **B(Y)**, for **Y Error** column **D(Y)** and for **X Error** column **C(Y)**. Then press **OK**. The program returns the results

$$\textbf{Intercept}(=a) = 0.90289 \pm 0.10616 \quad \text{and} \quad \textbf{Slope}\,(=\lambda) = 2.27214 \pm 0.10816.$$

The graph shown on the previous page is also produced.

The least squares method used is that of York, which uses weights for each point, based on the errors δx_i and δy_i of the measurements.

Example 11.12 [P]

Using Python, fit a least-squares straight line to the set of values $(x_i,\ y_i)$ of Example 11.11 [O], taking as weights of the points the inverses of the squares of the errors δy_i.

The weights will be taken to be $w_i = 1/(\delta y_i)^2$. The weight vector will therefore be:

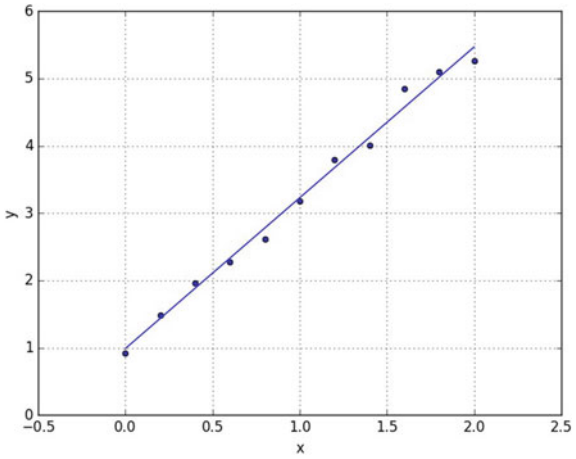
```
w = np.array([100, 69.4, 400, 156.3, 69.4, 400, 44.4, 100, 2, 400, 69.4])
```

```
# Program:

from __future__ import division
import math
import numpy as np
import matplotlib.pyplot as plt

# Enter the values of x, y and their corresponding weights w:

x = np.array([0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2])
y = np.array
([0.92, 1.48, 1.96, 2.27, 2.61, 3.18, 3.8, 4.01, 4.85, 5.1, 5.26])
w = np.array([100, 69.4, 400, 156.3, 69.4, 400, 44.4, 100, 2, 400, 69.4])

# Plot, size of dots, labels, ranges of values, initial values

plt.scatter(x, y)
plt.xlabel("x")                    # set the x-axis label
plt.ylabel("y")                     # set the y-axis label
plt.grid(True)

# Evaluation

N = len(x)

W = sum(w)
WX = sum(w*x)
WXX = sum(w*x**2)
```

```
WY = sum(w*y)
WXY = sum(w*x*y)

DENOM = W*WXX-(WX)**2
DETA = WY*WXX-WX*WXY
DETL = W*WXY-WX*WY

a = DETA/DENOM
lambda = DETL/DENOM

d = y - a - lambda*x

WDD = sum(w*d**2)

Da = math.sqrt((WDD*WXX)/((N-2)*DENOM))
Dlambda = math.sqrt((WDD*W)/((N-2)*DENOM))

# Results

print("Value of a:", a)
print("Value of b:", lambda)
print("Standard error in a:", Da)
print("Standard error in b:", Dlambda)

# Plot least-squares line

xx = np.linspace(min(x), max(x), 200)
yy = a + lambda * xx
plt.plot(xx, yy, '-')

plt.show()
```

The plot shown in the next page is produced.
The numerical values of the parameters are:

```
Value of a: 0.986465071722
Value of λ: 2.24054597889
Standard error in a: 0.05624562965433959
Standard error in λ: 0.04879527127536099
```



Example 11.13 [R]

Fit a straight line to a given set of values (x_i, y_i), applying the method of least squares and taking into account the errors in y.

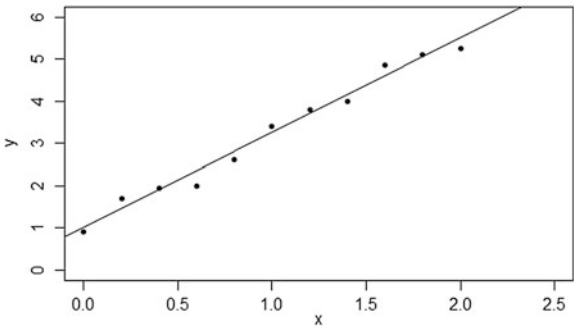
i	x_i	y_i	δy_i
1	0	0.92	0.1
2	0.2	1.7	0.12
3	0.4	1.96	0.05
4	0.6	2	0.08
5	0.8	2.61	0.12
6	1	3.4	0.05
7	1.2	3.8	0.15
8	1.4	4.01	0.1
9	1.6	4.85	0.7
10	1.8	5.1	0.05
11	2	5.26	0.12

This is the same problem as in Example 11.9 but taking into account only the errors in y.

```
# We form the vectors for x, y and the errors in y:
x = c(0, 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8, 2)
y = c(0.92, 1.7, 1.96, 2, 2.61, 3.4, 3.8, 4.01, 4.85, 5.1, 5.26)
erry = c(0.1, 0.12, 0.05, 0.08, 0.12, 0.05, 0.15, 0.1, 0.7, 0.05, 0.12)
```

```
# The weights are taken to be w_i = 1/(δ_yi)^2:
weights = (1/erry^2)
weights
[1] 100.000000  69.444444 400.000000 156.250000  69.444444 400.000000
44.444444 100.000000   2.040816 400.000000
[11]  69.444444

# The scatter plot of the points is created:
plot(x, y, pch=20, xlab="x", ylab="y", xlim=c(0, 2.5), ylim=c(0, 6))
fit <- lm(y~x, weights = weights)
fit

Call:
 lm(formula = y ~ x, weights = weights)

Coefficients:
 (Intercept)        x
    1.012     2.249

# The best straight line is drawn:
abline(fit)
```

The equation of the straight line found is: $y = 1.012 + 2.249x$.



11.3.2 Polynomial

In many cases, the relationship between x and y is not linear. In general, the relationship may be thought of as being expresses by a polynomial of the form [1]:

$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n. \tag{11.46}$$

The determination of the $n+1$ unknown coefficients is achieved by the minimization, with respect to these coefficients, of the quantity

$$S \equiv \sum_{i=1}^{N} (y_i - a_0 - a_1 x_i - a_2 x_i^2 - \ldots - a_n x_i^n)^2, \tag{11.47}$$

where $(x_i, y_i)(i = 1, 2, \ldots, N)$ are the results of the N measurements we have performed. In general, it must be $n < N - 1$.

Differentiating Eq. (11.47) with respect to the coefficients a_k and equating to zero, we have the normal equations:

$$\begin{aligned}
&a_0 N + a_1[x] + a_2[x^2] + \ldots + a_n[x^n] = [y] \\
&a_0[x] + a_1[x^2] + a_2[x^3] + \ldots + a_n[x^{n+1}] = [xy] \\
&a_0[x^2] + a_1[x^3] + a_2[x^4] + \ldots + a_n[x^{n+2}] = [x^2 y] \\
&\ldots\ldots\ldots \\
&a_0[x^n] + a_1[x^{n+1}] + a_2[x^{n+2}] + \ldots + a_n[x^{2n}] = [x^n y]
\end{aligned} \tag{11.48}$$

From these equations, the coefficients a_k may be found.

11.3.2.1 Parabola

For the case

$$y = a_0 + a_1 x + a_2 x^2 \tag{11.49}$$

we have the normal equations

$$\begin{aligned}
&a_0 N + a_1[x] + a_2[x^2] = [y] \\
&a_0[x] + a_1[x^2] + a_2[x^3] = [xy] \\
&a_0[x^2] + a_1[x^3] + a_2[x^4] = [x^2 y]
\end{aligned} \tag{11.50}$$

Applying Cramer's rule, we have for a_0, a_1 and a_2:

$$\frac{a_0}{\begin{vmatrix} [y] & [x] & [x^2] \\ [xy] & [x^2] & [x^3] \\ [x^2 y] & [x^3] & [x^4] \end{vmatrix}} = \frac{a_1}{\begin{vmatrix} N & [y] & [x^2] \\ [x] & [xy] & [x^3] \\ [x^2] & [x^2 y] & [x^4] \end{vmatrix}} = \frac{a_2}{\begin{vmatrix} N & [x] & [y] \\ [x] & [x^2] & [xy] \\ [x^2] & [x^3] & [x^2 y] \end{vmatrix}} = \frac{1}{\begin{vmatrix} N & [x] & [x^2] \\ [x] & [x^2] & [x^3] \\ [x^2] & [x^3] & [x^4] \end{vmatrix}}. \tag{11.51}$$

The errors in the coefficients $\delta a_0, \delta a_1$ and δa_2 are given by the relations:

$$\frac{(\delta a_0)^2}{\begin{vmatrix} [x^2] & [x^3] \\ [x^3] & [x^4] \end{vmatrix}} = \frac{(\delta a_1)^2}{\begin{vmatrix} N & [x^2] \\ [x^2] & [x^4] \end{vmatrix}} = \frac{(\delta a_2)^2}{\begin{vmatrix} N & [x] \\ [x] & [x^2] \end{vmatrix}} = \frac{\sigma_y^2}{\begin{vmatrix} N & [x] & [x^2] \\ [x] & [x^2] & [x^3] \\ [x^2] & [x^3] & [x^4] \end{vmatrix}}, \tag{11.52}$$

where

$$\sigma_y^2 = \frac{[d^2]}{N-3}, \quad [d^2] \equiv \sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2. \tag{11.53}$$

Example 11.14

Using the method of least squares, fit a parabolic curve to the measurements given in the first three columns of the table below.

i	t (s)	y (m)	t^2 (s^2)	t^3 (s^3)	t^4 (s^4)	ty (s m)	t^2y (s^2 m)	y_{th} (m)	d (m)	d^2 (m^2)
1	0	1	0	0	0	0	0	2.32	1.32	1.74
2	1	8	1	1	1	8	8	6.28	−1.72	2.96
3	2	20	4	8	16	40	80	19.77	−0.23	0.05
4	3	45	9	27	81	135	405	42.80	−2.20	4.84
5	4	70	16	64	256	280	1120	75.36	5.36	28.73
6	5	120	25	125	625	600	3000	117.46	−2.54	6.45
N = 6	15 = [t]	264 = [y]	55 = [t^2]	225 = [t^3]	979 = [t^4]	1063 = [ty]	4613 = [t^2y]			44.78 = [d^2]

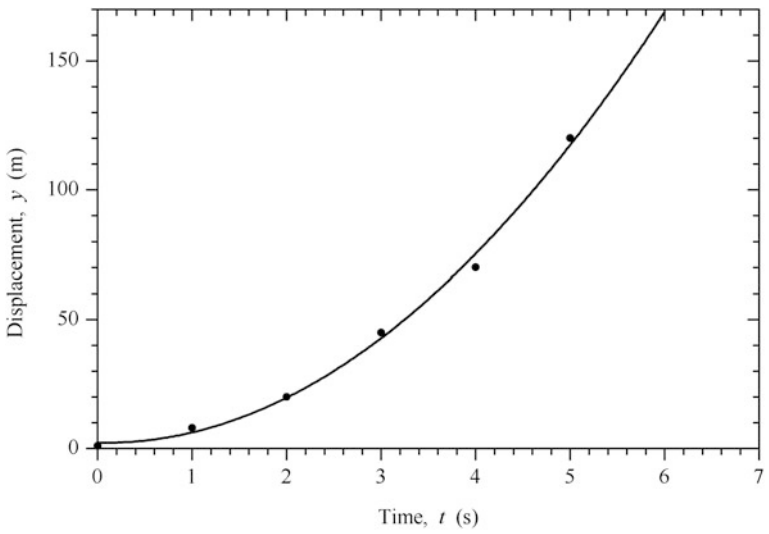
We will fit the curve $y = a_0 + a_1 t + a_2 t^2$ to the experimental results. From Eq. (11.51), in S.I. units,

$$\frac{a_0}{\begin{vmatrix} 264 & 15 & 55 \\ 1063 & 55 & 225 \\ 4613 & 225 & 979 \end{vmatrix}} = \frac{a_1}{\begin{vmatrix} 6 & 264 & 55 \\ 15 & 1063 & 225 \\ 55 & 4613 & 979 \end{vmatrix}} = \frac{a_2}{\begin{vmatrix} 6 & 15 & 264 \\ 15 & 55 & 1063 \\ 55 & 225 & 4613 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{vmatrix}}$$

we find $\dfrac{a_0}{9100} = \dfrac{a_1}{-3178} = \dfrac{a_2}{18690} = \dfrac{1}{3920}$

and $a_0 = 2.32$ m, $a_1 = -0.811$ m/s, $a_2 = 4.768$ m/s^2.

The curve is $y = 2.32 - 0.811t + 4.768t^2$ (in m, when the time t is expressed in s). The experimental points and the least-squares curve are shown in the figure below.



The errors in the parameters are found using Eqs. (11.52) and (11.53)

$$\frac{(\delta a_0)^2}{\begin{vmatrix} 55 & 225 \\ 225 & 979 \end{vmatrix}} = \frac{(\delta a_1)^2}{\begin{vmatrix} 6 & 55 \\ 55 & 979 \end{vmatrix}} = \frac{(\delta a_2)^2}{\begin{vmatrix} 6 & 15 \\ 15 & 55 \end{vmatrix}} = \frac{\sigma_y^2}{\begin{vmatrix} 6 & 15 & 55 \\ 15 & 55 & 225 \\ 55 & 225 & 979 \end{vmatrix}}$$

where $[d^2] \equiv \sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} (y_i - a_0 - a_1 x_i - a_2 x_i^2)^2 = 44.78$ m^2 and $\sigma_y^2 = \frac{[d^2]}{N-3} = \frac{44.78}{3} = 14.9$ m^2 or $\sigma_y = 3.86$ m.

Therefore, $\frac{(\delta a_0)^2}{3220} = \frac{(\delta a_1)^2}{2849} = \frac{(\delta a_2)^2}{105} = \frac{14.9}{3920} = 0.003\,80$ and

$$\delta a_0 = 3.5 \text{ m}, \quad \delta a_1 = 3.3 \text{ m/s}, \quad \delta a_2 = 0.63 \text{ m/s}^2$$

or

$$a_0 = 2.3 \pm 3.5 \text{ m}, \quad a_1 = -0.8 \pm 3.3 \text{ m/s}, \quad a_2 = 4.77 \pm 0.63 \text{ m/s}^2.$$

We notice that the presence of points at large values of t makes the fractional errors in a_0 and a_1 large, since a_0 and a_1 are important at low values of t. Of course, we must not forget that the values of the parameters we found depend on each other. If, in other words, we suppose a different value for one of the parameters, the optimum values of the other two will have to be modified.

Example 11.15 [E]

Using Excel®, fit a parabola to the data of Example 11.14.

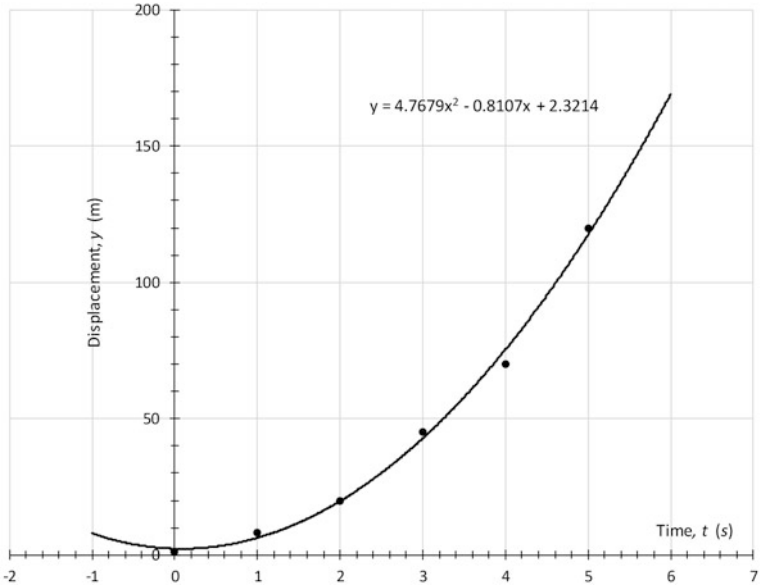
We enter the t and y values in columns A and B respectively. We highlight columns A and B. Opening the **Insert** window, we select **Scatter Plot**. We double-click on a point and change the color of the points to black and their size to 0.75 pt.

Pressing the ⊞ key at the top right hand corner of the graph, we click on **Trendline**, **More Options**. In **Format Trendline**, **Trendline Options** we select **Polynomial**, **Order** 2. We also select **Forecast**, **Forward** 1 period **Backward** 1 period and **Display Equation on chart**. We delete the straight line present in the graph.

The graph of the best fit parabola is produced, which, suitably formatted, looks like the figure shown here.

The equation of the parabola is found to be:

$$y = 2.3214 - 0.8107t + 4.7679t^2.$$



Example 11.16 [O]

Using Origin®, fit a parabola to the data of Example 11.14.

We place the data of columns t and y in columns A and B, respectively. We highlight both columns by left-clicking on the label of column A and then, holding the **Shift** key down, left-clicking on the label of column B. Then

Analysis > **Fitting** > **Polynomial Fit** > **Open Dialog**...

In the window that opens, we select: **Input Data, Range 1**, **X** select column A, **Y** select column B, **Polynomial Order, 2**. Press **Fit**. The program fits the parabola $y = A + Bx + Cx^2$ to the experimental results, where $x = t$. It is given that

$$A(= a_0) = 2.32143 \pm 3.50266,\ B(= a_1) = -0.81071 \pm 3.2947 \text{ and}$$
$$C(= a_2) = 4.76786 \pm 0.63251.$$

The equation of the parabola is:

$$y = 2.3214 - 0.8107t + 4.7679t^2$$

These results agree with those of Example 11.3.

Example 11.17 [P]

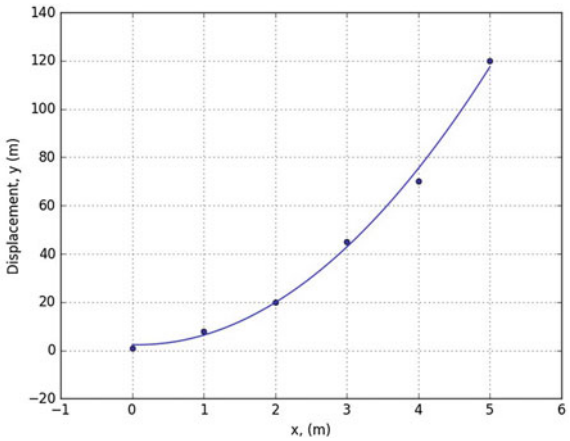
Using Python, fit a parabola to the data of Example 11.14.

```
import math
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt

# Enter the values of x and the corresponding y:

x = np.array([0, 1, 2, 3, 4, 5])
y = np.array([1, 8, 20, 45, 70, 120])

# Plot, size of dots, labels, ranges of values, initial values

plt.scatter(x, y)
plt.xlabel("x, (m)")                # set the x-axis label
plt.ylabel("Displacement, y (m)")# set the y-axis label
plt.grid(True)
```

```
# Evaluation
N = len(x)

X = sum(x)
XX = sum(x**2)
XXX = sum(x**3)
XXXX = sum(x**4)
Y = sum(y)
XY = sum(x*y)
XXY = sum(x**2*y)

DENOM = N*(XX*XXXX-XXX*XXX) - X*(X*XXXX-XX*XXX) + XX*(X*XXX-XX*XX)
DET0 = Y*(XX*XXXX-XXX*XXX) - X*(XY*XXXX-XXX*XXY) + XX*(XY*XXX-XX*XXY)
DET1 = N*(XY*XXXX-XXX*XXY) - Y*(X*XXXX-XX*XXX) + XX*(X*XXY-XX*XY)
DET2 = N*(XX*XXY-XXX*XY) - X*(X*XXY-XX*XY) + Y*(X*XXX-XX*XX)

a0 = DET0/DENOM
a1 = DET1/DENOM
a2 = DET2/DENOM

d = y - a0 - a1*x - a2*x**2
S = math.sqrt(sum(d**2)/(N-3))

Da0 = S*math.sqrt(abs((XX*XXXX-XXX*XXX)/DENOM))
Da1 = S*math.sqrt(abs((N*XXXX-XX*XX)/DENOM))
Da2 = S*math.sqrt(abs((N*XX-X*X)/DENOM))

# Results

print("Value of a0:", a0)
print("Value of a1:", a1)
print("Value of a2:", a2)
print("Standard error in a0:", Da0)
print("Standard error in a1:", Da1)
print("Standard error in a2:", Da2)

# Plot least-squares line

xx = np.linspace(min(x), max(x), 200)
yy = a0 + a1*xx + a2*xx**2
plt.plot(xx, yy, '-')

plt.show()
```

The plot shown is produced.
The values of the parameters are:

```
Value of a0: 2.32142857143
Value of a1: -0.810714285714
Value of a2: 4.76785714286
Standard error in a0: 3.5026593395561
Standard error in a1: 3.2947023804146847
Standard error in a2: 0.632505948991947
```

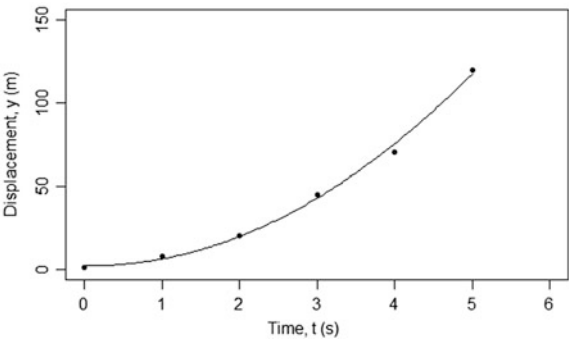


Example 11.18 [R]

Using R, fit a parabola to the data of Example 11.14.

```
# The data vectors

tdata = c(0, 1, 2, 3, 4, 5)
ydata = c(1, 8, 20, 45, 70, 120)

# Plot, size of dots, labels, ranges of values, initial values

plot(tdata, ydata, pch=20, xlab="Time, t (s)", ylab="Displacement, y (m)",
xlim=c(0, 6), ylim=c(0, 150))

# Fit least-squares line

A=2
B=-10
C=5
fit = nls(ydata~A+B*tdata+C*tdata^2, start=list(A=A, B=B, C=C))
```

```
summary(fit)
 Formula: ydata ~ A + B * tdata + C * tdata^2
 Parameters:
  Estimate Std. Error t value Pr(>|t|)
A  2.3214    3.5027  0.663  0.55486
B -0.8107    3.2947 -0.246  0.82151
C  4.7679    0.6325  7.538  0.00484 **
 Residual standard error: 3.865 on 3 degrees of freedom
 Number of iterations to convergence: 1
 Achieved convergence tolerance: 1.47e-07

# Plot least-squares line

new = data.frame(tdata = seq(min(tdata),max(tdata), len=200))
lines(new$tdata, predict(fit, newdata=new))

# Sum of squared residuals

sum(resid(fit)^2)
 [1] 44.80714

# Parameter confidence intervals

confint(fit)

     2.5%   97.5%
A -8.825597 13.468454
B -11.295928  9.674499
C  2.754941  6.780773
```



11.3.2.1.1 Errors in the Values Read from a Least-Squares Parabola

It is not clear how the errors in a_0, a_1 and a_2 combine to give the error δy in y for a given x. To get a *reasonable estimate* of δy, we work as follows.

We assume that a_0 is known and define the new variable

$$Y = \frac{y - a_0}{x}. \tag{11.54}$$

Then,

$$Y = a_1' + a_2' x. \tag{11.55}$$

By the method of least squares we find

$$a_1' = \frac{[Y][x^2] - [x][xY]}{N[x^2] - [x]^2} \quad a_2' = \frac{N[xY] - [x][Y]}{N[x^2] - [x]^2}. \tag{11.56}$$

If

$$\sigma_Y = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}(Y_i - a_1' - a_2' x_i)^2} \quad \text{or} \quad \sigma_Y = \sqrt{\frac{[D^2]}{N-2}}, \tag{11.57}$$

with

$$D_i \equiv Y_i - a_1' - a_2' x_i, \tag{11.58}$$

the errors in a_1' and a_2' are

$$\delta a_1' = \sigma_{a_1'} = \sigma_Y \sqrt{\frac{[x^2]}{N[x^2] - [x]^2}} \qquad \delta a_2' = \sigma_{a_2'} = \sigma_Y \sqrt{\frac{N}{N[x^2] - [x]^2}}. \tag{11.59}$$

According to Eq. (11.33), the error in Y at the point x is

$$\delta Y = \frac{\sigma_Y}{\sqrt{N}} \sqrt{1 + \frac{N^2}{N[x^2] - [x]^2}(x - \bar{x})^2}, \quad \text{where} \quad \bar{x} = \frac{[x]}{N}. \tag{11.60}$$

Since it is $y = a_0 + xY$, the error in y is given by
$\delta y(x) = \sigma_y(x) = \sqrt{(\delta a_0)^2 + x^2(\delta Y)^2}$ or

$$\delta y(x) = \sigma_y(x) = \sqrt{(\delta a_0)^2 + \frac{\sigma_Y^2}{N}x^2 + \frac{N\sigma_Y^2}{N[x^2] - [x]^2}x^2(x - \bar{x})^2}. \tag{11.61}$$

It should be clear what this error expresses. Considering the arguments used above, $\delta y(x)$ gives a measure of the dispersion of the values of y derived from measurements at the point x. The error in the actual value of $y(x)$ determined from the least squares parabola, which is the result of many measurements, is in fact much smaller. To use an analogy, $\delta y(x)$ corresponds to what we called *error of a single observation* in the case of a series of measurements of y at the same value of x. The error in the reading of $y(x)$ from the least squares curve would correspond to the *error of the mean value*. It is not clear how one may determine the last magnitude from a series of measurements $y(x)$ at various values of x. We will suggest below some ideas to address this problem.

An example will help illustrate the theory presented above.

Example 11.19

Using the method of least squares, fit a parabola $y(x)$ to the points:

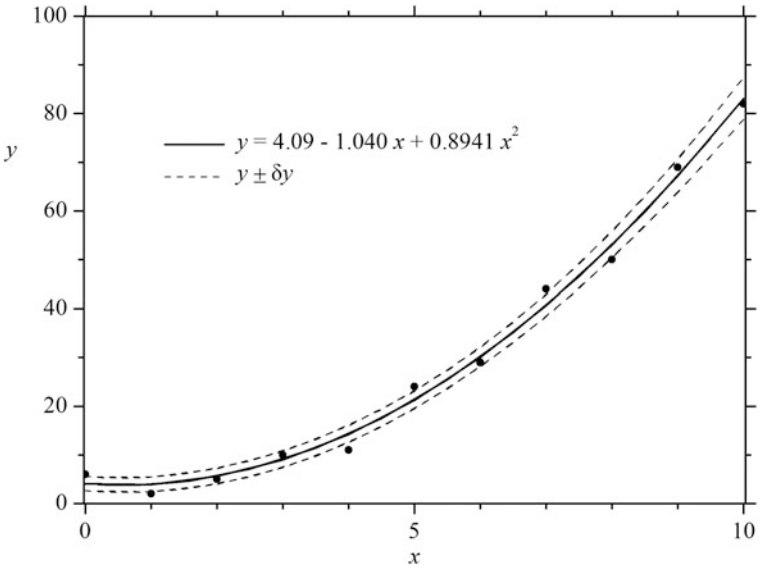
i	0	1	2	3	4	5	6	7	8	9	10
x_i	0	1	2	3	4	5	6	7	8	9	10
y_i	6	2	5	10	11	24	29	44	50	69	82

Find the value of y and its standard deviation from the mean for $x = 6$.

We evaluate the sums

$$N = 11,\ [x] = 55,\ [x^2] = 385,\ [x^3] = 3025,\ [x^4] = 25333,\ [y] = 332,$$
$$[xy] = 2529,\ [x^2y] = 21{,}077,$$

which we use in order to fit the least-squares parabola $y = 4.09 - 1.040x + 0.8941x^2$ to the given points. The points and the parabola are shown in the figure that follows.



We also find the deviations d_i of the points from the parabola

i	0	1	2	3	4	5	6	7	8	9	10
d_i	1.91	−1.95	−0.59	0.98	−3.24	2.77	−1.04	3.38	−2.99	1.85	−1.10

and use them to find $[d^2] = 52.94$, $\sigma_y = 2.57$ and $\delta a_0 = 1.45$.

We now define the variable

$$Y_i = \frac{y_i - 4.09}{x_i}$$

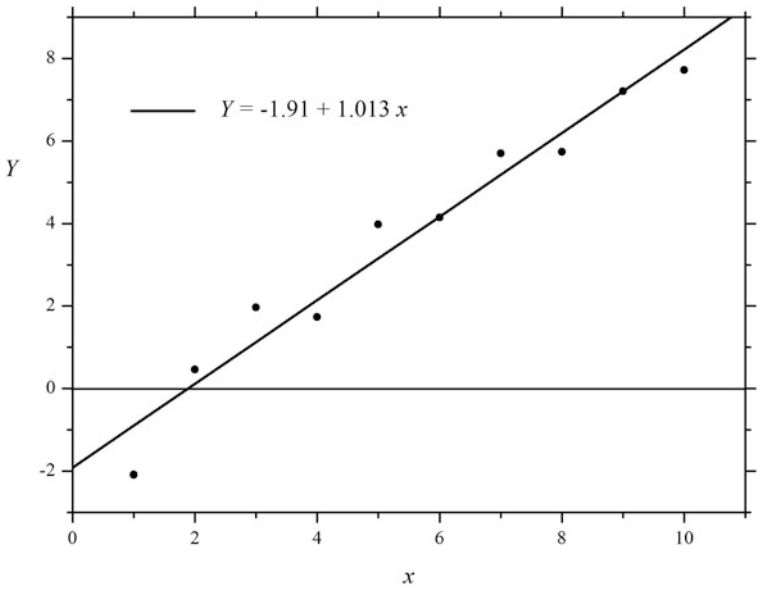
and find its values at the points x_i.

i	0	1	2	3	4	5	6	7	8	9	10
x_i	0	1	2	3	4	5	6	7	8	9	10
Y_i	–	−2.09	0.46	1.97	1.73	3.98	4.15	5.70	5.74	7.21	7.72
D_i	–	−1.20	0.34	0.84	−0.42	0.83	−0.02	0.52	−0.46	0.00	−0.43

Using the sums $N = 10$, $[x] = 55$, $[x^2] = 385$, $[Y] = 36.64$ and $[xY] = 285.1$, we find the values of $a_1' = -1.91$ and $a_2' = 1.013$, from which we get the straight line fitted to the points Y_i by least squares

$$Y = -1.91 + 1.013x.$$

The points $Y_i(x_i)$ and the straight line $Y = -1.91 + 1.013x$ are shown in the figure that follows.



The errors in a'_1 and a'_2 can be found. Evaluating $D_i \equiv Y_i - a'_1 - a'_2 x_i$ for each value of x_i, we find

$$[D^2] = 3.77 \quad \text{and} \quad \sigma_Y = \sqrt{\frac{[D^2]}{N-2}} = \sqrt{\frac{3.77}{8}} = 0.686.$$

The errors in the coefficients are $\delta a'_1 = \sigma_{a'_1} = 0.468$ and $\delta a'_2 = \sigma_{a'_2} = 0.0754$.

We may now use Eq. (11.61) to find the error in y:

$$\delta y(x) = \sigma_y(x) = \sqrt{(\delta a_0)^2 + \frac{\sigma_Y^2}{N}x^2 + \frac{N\sigma_Y^2}{N[x^2] - [x]^2}x^2(x-\bar{x})^2},$$

where, here, $\delta a_0 = 1.45$ and $\bar{x} = 5.5$. It follows that

$$\delta y(x) = \sigma_y(x) = \sqrt{2.10 + 0.0471x^2 + 0.005704x^2(x-5.5)^2}$$

or, finally,

$$\delta y(x) = \sigma_y(x) = \sqrt{2.10 + 0.2196x^2 - 0.06274x^3 + 0.005704x^4}.$$

For the values of x considered, the values of the error δy range from $\sigma_y(0) = 1.45$ to $\sigma_y(10) = 4.28$.

At $x = 6$ it is $\sigma_y(6) = 1.96 = 2.0$.

The value of y at $x = 6$ is, therefore, $y(6) = 30.0 \pm 2.0$. This agrees with the value of 29 given in the initial table. As mentioned above, the error δy gives the spread of measurements of y at the point x and not the accuracy with which y is known as a result of the series of measurements made and plotted in the first figure.

11.3.3 Other Curves

The least squares method is also applied in the same way for other kinds of curves. Usually, the normal equations are difficult or impossible to solve. In most cases, approximate numerical methods have to be used for the determination of the parameters.

The solution is easily found in the cases of curves of the form

$$y(x) = a\,A(x) + b\,B(x) + \ldots + m\,M(x) \tag{11.62}$$

where $A(x), B(x), \ldots, M(x)$ are known functions of x with all their parameters known. The normal equations for the determination of $a, b, \ldots, m$ are

$$\begin{aligned} &a[A^2] + b[AB] + \ldots + m[AM] = [yA] \\ &a[BA] + b[B^2] + \ldots + m[BM] = [yB] \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &a[MA] + b[MB] + \ldots + m[M^2] = [yM] \end{aligned} \tag{11.63}$$

where $[AB] \equiv \sum_{i=1}^{N} A(x_i)B(x_i)$, $[yA] \equiv \sum_{i=1}^{N} y_i A(x_i)$ etc. These may be solved for $a, b, \ldots, m$.

Example 11.20

A simple harmonic oscillator with negligible damping has angular frequency $\omega = 1$ rad/s, known with great accuracy. If its motion is described by a relation of the form

$$y(t) = A\,\sin\omega t + B\,\cos\omega t,$$

find, using the method of least squares, the constants A and B, in terms of the co-ordinates $(t_i,\ y_i)$ from N measurements.

The method of least squares requires the minimization of the quantity

$$S \equiv \sum_{i=1}^{N} (y_i - A\ \sin \omega t_i - B\ \cos \omega t_i)^2.$$

With partial differentiation with respect to A and B, we find

$$\frac{\partial S}{\partial A} = -2 \sum_{i=1}^{N} (y_i - A\ \sin \omega t_i - B\ \cos \omega t_i) \sin \omega t_i = 0$$
$$\frac{\partial S}{\partial B} = -2 \sum_{i=1}^{N} (y_i - A\ \sin \omega t_i - B\ \cos \omega t_i) \cos \omega t_i = 0$$

from which the normal equations

$$[y\ \sin \omega t] - A[\sin^2 \omega t] - B[\sin \omega t\ \cos \omega t] = 0$$
$$[y\ \cos \omega t] - A[\sin \omega t\ \cos \omega t] - B[\cos^2 \omega t] = 0$$

are obtained, where

$$[y\ \sin \omega t] \equiv \sum_{i=1}^{N} y_i \sin \omega t_i, \quad [\sin \omega t\ \cos \omega t] \equiv \sum_{i=1}^{N} \sin \omega t_i \cos \omega t_i,$$
$$[\sin^2 \omega t] \equiv \sum_{i=1}^{N} \sin^2 \omega t_i$$

etc. From these, the parameters A and B are found to be

$$A = \frac{[\sin^2 \omega t][y\ \cos \omega t] - [y\ \sin \omega t][\sin \omega t\ \cos \omega t]}{[\ \sin^2 \omega t][\cos^2 \omega t] - [\sin \omega t\ \cos \omega t]^2}$$
$$B = \frac{[\cos^2 \omega t][y\ \sin \omega t] - [y\ \cos \omega t][\sin \omega t\ \cos \omega t]}{[\sin^2 \omega t][\cos^2 \omega t] - [\sin \omega t\ \cos \omega t]^2}.$$

These are functions of time.

11.3.4 The Reduction of Non-linear Relations to Linear

In certain cases, when the method of least squares is difficult or impossible to apply to a non-linear relation which is considered to apply between the variables, this relation may be transformed to a linear, relating new variables which are suitably defined.

For example, if we have measurements $[t,\ R(t)]$ of the variation of the activity $R(t)$ of the radioactive sample with time t and we wish to fit to them a relation of the form

$$R(t) = R_0 \mathrm{e}^{-\lambda t}, \tag{11.64}$$

we may have a linear relation between the variables

$$x = t, \quad \text{and} \quad y = \ln R(t), \tag{11.65}$$

as it is obviously true that

$$\ln R(t) = \ln R_0 - \lambda t \tag{11.66}$$

and

$$y = \ln R_0 - \lambda x. \tag{11.67}$$

The method of least squares may be applied to relation (11.67) for the determination of R_0 and λ. Of course, the method will give results which are not exactly equal to those we would have obtained by applying the method to the relation of Eq. (11.64). It is obvious that the transformation of the variables changes the relative importance of the measurements. The transformation $y = \ln R$, for example, increases the importance of the small values of R. This will be demonstrated in the example that follows. This somewhat arbitrary use of the method of least squares to the linearized relation is often the only solution we have. Other relations which may be linearized with the suitable change of variables will be examined in Chap. 12 (Sect. 12.4).

Example 11.21

$N = 6$ measurements gave the results (x_i, y_i) of the table below:

i	1	2	3	4	5	6
x_i	1	2	3	4	5	6
y_i	0.8	1.3	1.9	1.9	2.4	2.7

Apply the method of least squares in order to fit to these results a curve, first using the relation $y = \alpha\sqrt{x}$ and then the linearized relation $\ln y = \ln \alpha + \frac{1}{2}\ln x$.

Method 1

Using $y = \alpha\sqrt{x}$, the deviations of the experimental points are $d_i = y_i - \alpha\sqrt{x_i}$.

The magnitude to be minimized is $S \equiv \sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} \left(y_i - \alpha\sqrt{x_i}\right)^2$.

From $\dfrac{\partial S}{\partial \alpha} = -2\sum_{i=1}^{N} (y_i - \alpha\sqrt{x_i})\sqrt{x_i}$, we get, for $\dfrac{\partial S}{\partial \alpha} = 0$,

$\sum_{i=1}^{N} \left(\sqrt{x_i}y_i - \alpha x_i\right) = [\sqrt{x}y] - \alpha[x] = 0.$

This gives the value

$$\alpha = \frac{[\sqrt{x}y]}{[x]}.$$

We form the table

i	x_i	y_i	$\sqrt{x_i}$	$\sqrt{x_i}y_i$
1	1	0.8	1	0.8
2	2	1.3	1.414	1.838
3	3	1.9	1.732	3.291
4	4	1.9	2	3.8
5	5	2.4	2.236	5.366
6	6	2.7	2.449	6.612
Sums:	21 = $[x]$		11.831 = $[\sqrt{x}]$	21.707 = $[\sqrt{x}y]$

From the sums of which we get

$$\alpha = \frac{[\sqrt{x}y]}{[x]} = \frac{21.707}{21} = 1.033 \quad \alpha = 1.033.$$

Method 2

Linearizing the relation $y = \alpha\sqrt{x}$, we get $\ln y = \ln\alpha + \frac{1}{2}\ln x$.

Defining $d_i = \ln y_i - \ln\alpha - \dfrac{1}{2}\ln x_i$ and $S \equiv \sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} (\ln y_i - \ln\alpha - \frac{1}{2}\ln x_i)^2$ and demanding that $\dfrac{\partial S}{\partial \alpha} = 0$ or $-\dfrac{2}{\alpha}\sum_{i=1}^{N} (\ln y_i - \ln\alpha - \dfrac{1}{2}\ln x_i) = 0$, we obtain the equation $[\ln y] - N\ln\alpha - \frac{1}{2}[\ln x] = 0$ from which $\ln\alpha = \frac{1}{N}\left([\ln y] - \frac{1}{2}[\ln x]\right)$ or $\alpha = \exp\left\{\frac{1}{N}\left([\ln y] - \frac{1}{2}[\ln x]\right)\right\}$.

This may also be written as $\alpha = \left\{\dfrac{y_1 y_2 \ldots y_N}{\sqrt{x_1 x_2 \ldots x_N}}\right\}^{1/N}$ or $\alpha = (\alpha_1\alpha_2\ldots\alpha_N)^{1/N}$, where $\alpha_i = \dfrac{y_i}{\sqrt{x_i}}$.

The values of the table give

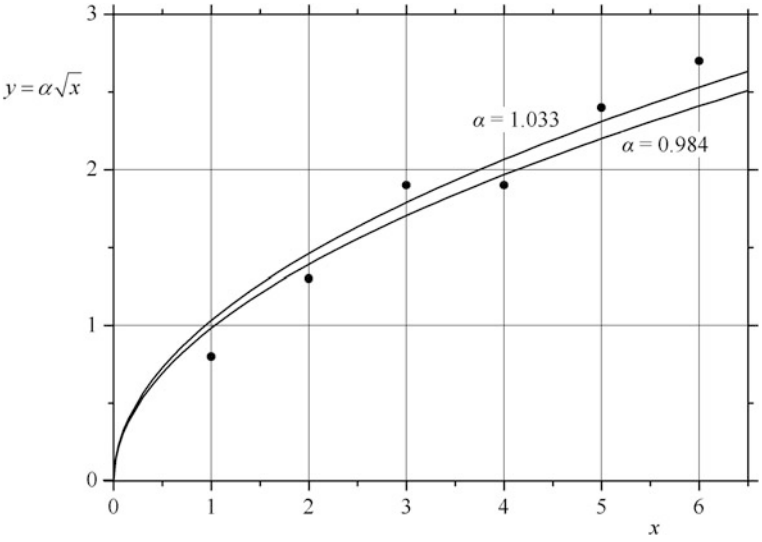
$$\prod_i x_i = 720 \quad \prod_i \sqrt{x_i} = 26.83 \quad \prod_i y_i = 24.329,$$

from which we get

$$\alpha = \left\{ \frac{y_1 y_2 \ldots y_N}{\sqrt{x_1 x_2 \ldots x_N}} \right\}^{1/N} = \left\{ \frac{24.329}{26.83} \right\}^{1/6} = 0.9068^{1/6} = 0.984$$

or, finally $\alpha = 0.984$.

The two methods give the slightly different values $\alpha = 1.033$ and $\alpha = 0.984$, respectively. The curves for these two values of α are drawn in the figure below. The difference is clearly visible.

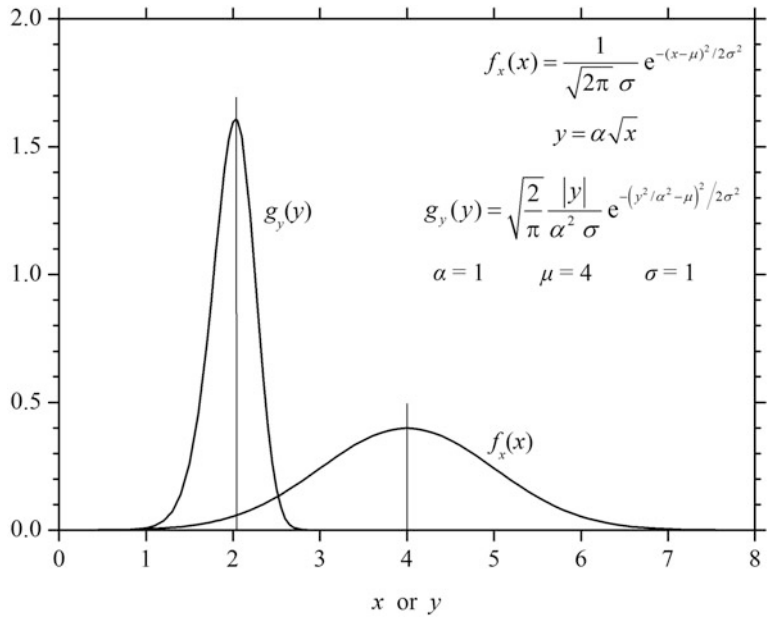


What is the effect of a transformation of variables on their probability densities? Assume that the variable x has a probability density $f_x(x)$. Let this variable be changed into $y = y(x)$. We want to determine the probability density $g_y(y)$ of y. If to an interval $\mathrm{d}x$ there corresponds an interval $\mathrm{d}y$ and equating the probability $g_y(y)\mathrm{d}y$ of a result in the region between y and $y + \mathrm{d}y$ with that for a result in the corresponding region between x and $x + \mathrm{d}x$, i.e. $f_x(x)\mathrm{d}x$, we have $g_y(y)\mathrm{d}y = f_x(x)\mathrm{d}x$, from which we finally get

$$g_y(y) = \frac{f_x(x)}{|\mathrm{d}y/\mathrm{d}x|}, \tag{11.68}$$

where we have taken the absolute value of the derivative since the probability density must be positive. This relation is true for a relation $y = y(x)$ describing a one-to-one correspondence between the variables x and y [2].

As an example, if it is $f_x(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$ and $y = \alpha\sqrt{x}$, the probability density of y is $g_y(y) = \sqrt{\frac{2}{\pi}} \frac{|y|}{\alpha^2\sigma} e^{-\left(y^2/\alpha^2-\mu\right)^2/2\sigma^2}$. These two probability densities are drawn in the figure that follows.



The differences in the two distributions are visible. Apart from the shift on the axis, the Gaussian $f_x(x)$ is changed into the asymmetrical function $g_y(y)$. The two methods, therefore, are bound to give different results.

Example 11.22 [O]

Using Origin® fit a parabola to the data of Example 11.21.

We place the data of columns x and y in columns A and B, respectively. We highlight both columns by left-clicking on the label of column A and then, holding the **Shift** key down, left-clicking on the label of column B. Then,

Analysis > Fitting > Nonlinear Curve Fit > Open Dialog...

In the window that opens, we select: **Settings: Function Selection**, **Category: Power**, **Function: Power 1**. The function Power 1 is $y = A|x - x_c|^p$. We wish to fit

the function $y = a\sqrt{x}$, so we must set $x_c = 0$ and $p = 1/2$. To do this, we open **Parameters**. For x_c we tick the box **Fixed** and enter the value 0. For p we tick the box **Fixed** and enter the value 0.5 (not 1/2). Then we press **Fit**.

The result returned is: $A(= a) = 1.03379 \pm 0.03866$, so that it is $y = 1.034\sqrt{x}$. This agrees well with the results of Example 11.16.

11.4 The Choice of the Optimum Function Fitted to a Set of Experimental Results

The method of least squares gives us the best coefficients for the function we chose to fit to a series of experimental results. What it does not give us is the best function to be used. It can, however, tell us which of the various functions we have tried has a better fit to the experimental results.

Let the two functions which were used, α and β, have n_α and n_β parameters respectively (2 for a straight line, 3 for a parabola etc.). For the N values x_i, which are assumed as known with absolute accuracy, the two functions give, respectively, the values $y_{\alpha,i}$ and $y_{\beta,i}$. We evaluate the magnitudes

$$\Phi_\alpha \equiv \frac{1}{N - n_\alpha} \sum_{i=1}^{N} (y_i - y_{\alpha,i})^2 \quad \text{and} \quad \Phi_\beta \equiv \frac{1}{N - n_\beta} \sum_{i=1}^{N} (y_i - y_{\beta,i})^2. \tag{11.69}$$

It is proved that the function with the smaller value of Φ gives the best fit to the experimental results.

11.5 The Fractional Absolute Deviation of the Experimental Values from the Values of the Curve

Assume that a curve has been fitted to the scatter data of an experiment, such as the ones shown in Fig. 11.3, passing between the experimental points, either using the least squares method or by applying smoothing to the data (see next section). If we read the value of y as given by this curve for a particular x, what is a measure of dispersion for this y value? If, as is the case for a straight line or a simple curve used in the method of least squares, the standard deviation in the y values is given by a formula, then there is no problem. In most cases, however, this is not possible. The results obtained in Chap. 4 do not apply here, as we do not have many measurements of a physical quantity under the same experimental conditions but many measurements performed at different values of the independent variable x. We may obtain *a measure* for the scatter of the points about the smoothed curve by working as described below.

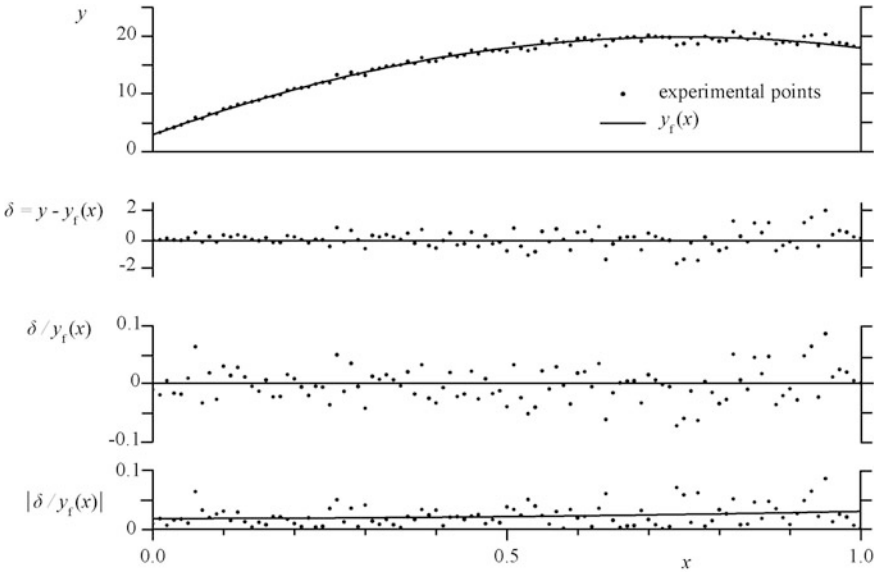


Fig. 11.3 The curve $y_f(x)$ is fitted to the experimental results. The deviation $\delta = y - y_f(x)$, the fractional deviation $\delta/y_f(x)$ and the absolute fractional deviation $|\delta/y_f(x)|$ of the experimental points from the values given by the graph are evaluated for all the experimental points. A parabola is fitted by the method of least squares to the points $|\delta/y_f(x)|$ (line in the lower part of the figure)

We wish to evaluate an estimate for the fractional absolute deviation of the experimental values from the curve,

$$\frac{\delta(x)}{y_f(x)} = \frac{y_{\text{exper.}}(x) - y_f(x)}{y_f(x)}, \tag{11.70}$$

as a function of x. Here, $y_{\text{exper.}}(x)$ is the value of y at x, expected from an experimental measurement and $y_f(x)$ is the value given by the curve at x. The steps of the procedure followed are shown in Fig. 11.3.

Using the experimental results y_i and the curve fitted to them, $y_f(x)$, we evaluate the deviation $\delta = y - y_f(x)$, the fractional deviation $\delta/y_f(x)$ and the absolute fractional deviation $|\delta/y_f(x)|$ for each experimental point. Figure 11.4 shows the experimental points, y_i, the best curve fitted to them, $y_f(x)$, and the deviations of the experimental points from the curve, δy, as functions of x.

The values of δy evaluated by the method described above, simply gives a measure of the dispersion of the experimental points about the fitted curve. It does not give the error in a value of y read off the curve. What was found above is the equivalent of the standard deviation of the measurements about their mean. We need the equivalent of the standard deviation of the mean, which may also be considered to be the error in y. A suggestion on how an estimate for such a magnitude may be obtained will be given below.

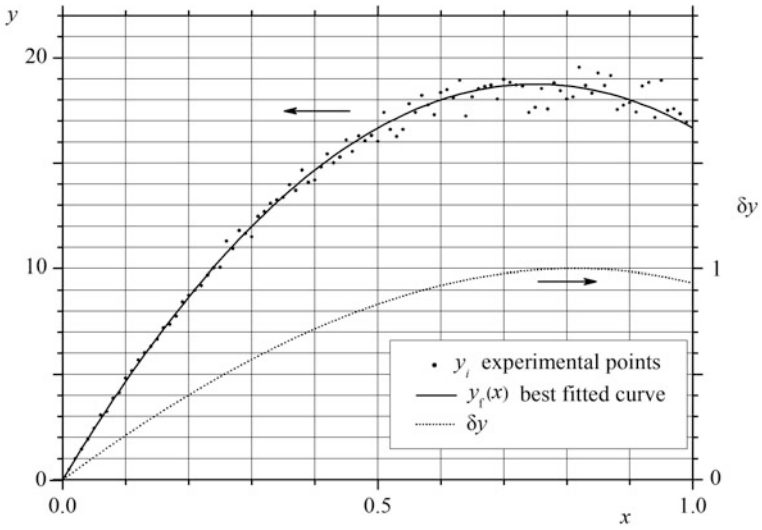


Fig. 11.4 The experimental points, y_i, the best curve fitted to them, $y_f(x)$, and the deviations of the experimental points from the curve, δy, as functions of x

11.6 Smoothing

It is often impossible to fit a simple curve to the experimental values by the method of least squares or otherwise. Usually, the reason is that a curve which would agree sufficiently with the experimental points does not have the form of a polynomial or other simple functions. An example is given in Fig. 11.5. There is a very rich library of specialized functions used in particular branches of science, e.g. in optical, dielectric or gamma-ray spectroscopy. Even so, in many cases, curves appear which do not have a known or simple structure enabling the fitting to them of a curve of known mathematical form. In some cases we can settle for a curve through the points which is smooth enough so that we can read the result of a possible measurement at any value of the independent variable. This is achieved with a procedure called *smoothing*.

Figure 11.5 shows the momentum spectrum of electrons emitted by the radioisotope ^{137}Cs. The details do not concern us here, but in essence the points represent a histogram of the momenta of the electrons emitted, each point representing the electrons counted in a narrow interval of values of the momentum. The dispersion of the points is due to the statistical fluctuations in the numbers of the electrons counted. This dispersion is made more obvious in Fig. 11.6, in which consecutive points have been joined by straight lines. It is clear that it would not be easy to apply to the whole curve the method of least squares without destroying the fine structure of the spectrum in the region of the two narrow peaks at the large values of momentum.

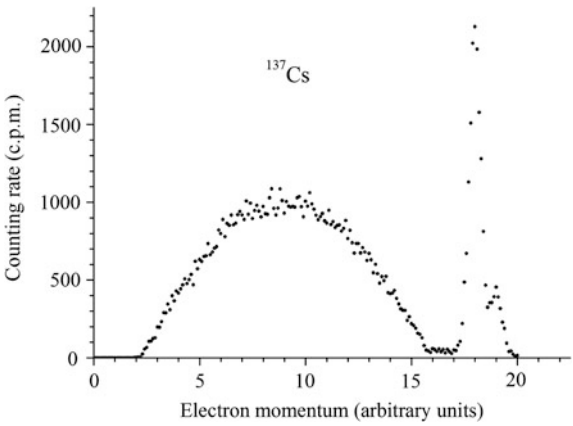


Fig. 11.5 The momentum spectrum of the electrons emitted from the radioisotope ^{137}Cs, as recorded by a multichannel analyzer. Apart from the continuous spectrum which is due to the β emission from the nucleus, two narrow peaks are also observed, of monoenergetic electrons due to internal conversion: a large one with electrons from the K shell and a smaller one with electrons from the L shell

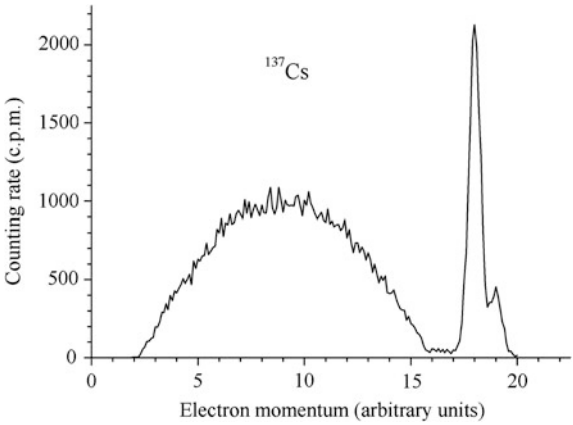


Fig. 11.6 The spectrum of Fig. 11.5. Consecutive points have been joined with straight lines

The smoothing of a curve is achieved by applying the method of least squares or some procedure of averaging on parts of the curve separately. Using this method is much simpler when the experimental points are at the same distance between them along the axis of the independent variable (x), as is the case in Fig. 11.5. A new value of y is calculated for every value of x, by fitting a curve to only $2N+1$ consecutive points, with central point that at which the new value of y is being evaluated. The values of $2N+1$ are 3, 5, 7 etc. and the curve fitted to these points is a simple polynomial of the second or not very much higher degree. There are various equations for the application of the method, for points which are mutually equidistant or not, or others which take into account, for example, that at the edges the points available are not sufficient for the calculations. Relative weights may also be attributed to each point, depending on its distance from the central point. It is possible, of course, to apply the same procedure two or more times in succession. This must be avoided if the number of points involved is too large, as it will lead to

an over-smoothing of the curve, in which points at very large distances from the central point affect its value.

Various smoothing methods are available in data analysis computer programs. The simplest method is that of averaging the y-values of $2N+1$ points symmetrically situated about each experimental point in succession [e.g. $y'_n = (y_{n-2}+y_{n-1}+y_n+y_{n+1}+y_{n+2})/5$ for the nth point of the data]. Here, y'_n replaces y_n in the smoothed curve. A better and more popular method is known as the *Savitzki-Golay method*. This uses, for each experimental point, $2N+1$ points symmetrically situated about the central one and fits a least-squares polynomial to them. The value of y at the central point is then evaluated using the resulting polynomial. Obviously, the value of N must be decided taking into account the total number of points available. The degree of the polynomial must not be too high, otherwise the effectiveness of the method is reduced. In the limit, if the degree of the polynomial is equal to N, there will be no change in the re-calculated y-values of the points! The method was applied to the data of Fig. 11.5, with $2N+1=7$ points and a polynomial of the second degree (parabola). The differences between the curves of Figs. 11.6 and 11.7 are obvious.

Smoothing must be applied with great caution and only when it would offer an improvement to a scatter plot or to a table of data. It is useful to remember that smoothing is equivalent to 'filtering' the curve by a filter that cuts off the high frequencies. In other words, the process removes the high-frequency variations from the curve. In the final analysis, this is equivalent to the diminishing of the power of discrimination of the experimental method used. In Fig. 11.7, this is demonstrated by the broadening of the two narrow peaks. Greater smoothing might possibly make invisible the small peak at the higher values of momentum. The fine structure in the data, which may be of great physical importance, could be made to disappear by an excessive use of smoothing.

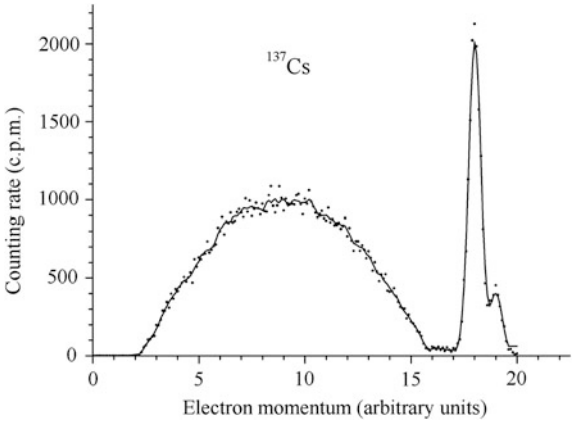


Fig. 11.7 The spectrum of Fig. 11.5, after smoothing by the method of Savitzki-Golay, using $2N+1=7$ points and a polynomial of the second degree. The scatter points are also shown for comparison

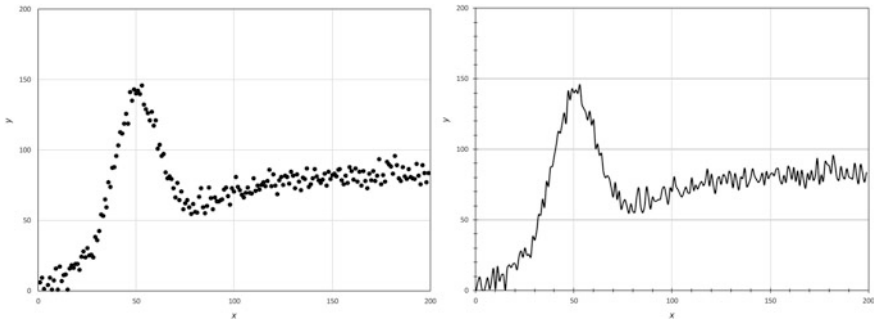
Example 11.23 [E]

A number of experimental results are given, which are shown in a scatter plot below. Use Excel® to transform the data to a smoothed curve.

We enter the values of x in column A and those of y in column B. We highlight the two columns and, through **Insert**, we produce a scatter plot of the experimental points. This is shown in the left hand side figure below.

While in the chart page, we right-click on one of the points and, in the window that opens, we select **Format Data Series**. The **Format Data Series** task pane opens. Click the **Fill and Line** icon. Select **Solid Line** and then check the **Smoothed Line** box. Click **OK**. A line appears in the plot, joining the points.

Right-click on the line and, in the window that opens, select **Change Series Chart Type**. Select **Line** plot and **Scatter With Smooth Lines**. The dots will disappear. After some formatting, the graph looks like the right-hand figure shown below.

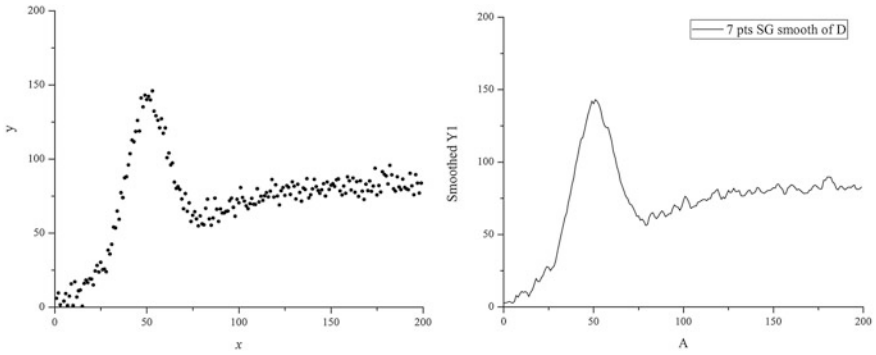


Strictly speaking, what Excel® does here is not smoothing. It just joins the dots with straight lines and rounds off the corners.

Example 11.24 [O]

A number of experimental results are given, which are shown in a scatter plot below. Using Origin Origin®, perform a 7-point parabola Savitzki-Golay smoothing operation on these data and show the result.

We import the data (x, y) and place them in columns A and D, respectively. The scatter plot of the data is shown in the figure on the left.



We select column D by left clicking on its label. Then,

Analysis > **Signal Processing** > **Smooth** > **Open Dialog**

In the window that opens, we select:

Method: **Savitzki**−**Golay**, **Points of Window**: **7**, **Polynomial Order**: **2**.

Press **OK**. The smoothed data appear in a new column. Give the instructions:

Plot > **Line** > **Line**.

The smoothed data are plotted as shown above, in the figure on the right.

Example 11.25 [P]

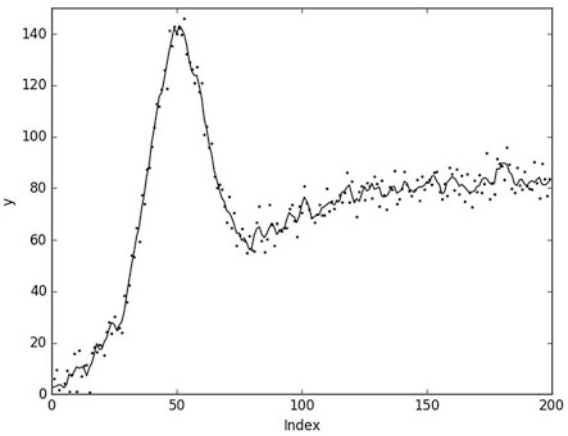
A number of experimental results are given, which are shown in a scatter plot below. Use Python to transform the data to a smoothed curve.

First the data vector y is entered (we omit this operation for brevity), and we create a vector x, containing a series of integers from 0 to the length of y. We calculate a corresponding vector of smoothed data using the `savgol_filter` function from the `scipy.signal` sub-package. The function accepts three parameters: the original data, the number of window points and the polynomial order; like in the previous example, we use a 7-point window and a 2nd degree polynomial for the Savitzky-Golay smoothing operation.

```
import numpy as np
import matplotlib.pyplot as plt
from scipy.signal import savgol_filter

y = np.array([-1.43019, 6.04592, 9.58303, ... 83.65553])
x = np.arange(0, len(y))
plt.scatter(x, y, s=2, color="black")
```

```
plt.xlim(0, 200)
plt.ylim(0, 150)
plt.xlabel("Index")
plt.ylabel("y")
ysmooth = savgol_filter(y, 7, 2)
plt.plot(x, ysmooth, '-', color="blue")
plt.show()
```

The following figure is produced, combining the scatter plot of the original data and the smoothed curve.



Example 11.26 [R]

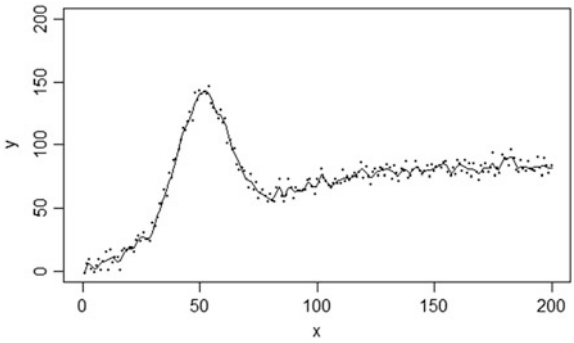
A number of experimental results are given, which are shown in a scatter plot below. Use R to transform the data to a smoothed curve.

The points are the same as in the two previous examples. We will achieve smoothing by the use of a cubic spline.

```
# The data vectors are entered:
> y <- c(-1.43019, 6.04592, 9.58303, 1.54254, ...
     ... 75.90812, 89.37784, 83.67599, 77.00079, 83.65553)
> x <- seq(1, length(y), len = 201)

# The scatter plot is drawn:
> s02 <- smooth.spline(y, spar = 0.2)
> plot(y, pch = 20, cex = 0.5, xlab = "x", ylab = "y", xlim=c
(0, 200), ylim=c(0, 200), col.main = 2)
>
```

```
# The smoothed curve is drawn:
> lines(predict(s02, x))
```

The results are shown in the figure below.



11.7 The Error in a Value Read off a Smoothed Curve

We need an estimate for the standard deviation or error of a point on a curve which was obtained by the method of least squares or by smoothing of data. This is possible in the case of a curve obtained by smoothing using the simple averaging procedure. For example, if the smoothing of a curve is done by finding the average

$$y'_n = (y_{n-k} + y_{n-k+1} + \ldots + y_n + \ldots + y_{n+K-1} + y_{n+k})/(2k+1), \qquad (11.71)$$

then we may consider the $2k+1 = N$ measurements at slightly different values of x, as measurements performed under approximately the same conditions and evaluate their mean, $\bar{y}$, standard deviation s_y and standard deviation of their mean $\sigma_{\bar{y}}$.

An example is shown in Fig. 11.8. The noisy original signal is shown in (a). In three different cases, smoothing is performed by averaging 9, 17 and 25 adjacent points [figures (b), (c) and (d), respectively]. In each case, estimates of the standard deviations s_y and $\sigma_{\bar{y}}$ are evaluated. It is seen that s_y, as expected, tends to stabilize at some value, while $\sigma_{\bar{y}}$ decreases, as the number of points averaged (N) increases. This is as expected, since it is $\sigma_{\bar{y}} = s_y/\sqrt{N-1}$.

The question which arises concerns the optimum number of points to be used in the smoothing of the curve and, therefore, in the evaluation of $\sigma_{\bar{y}}$. No quantitative criterion exists, so we are obliged to make a subjective judgment, trying to minimize $\sigma_{\bar{y}}$ as much as possible (using a large N) while not deforming the curve too

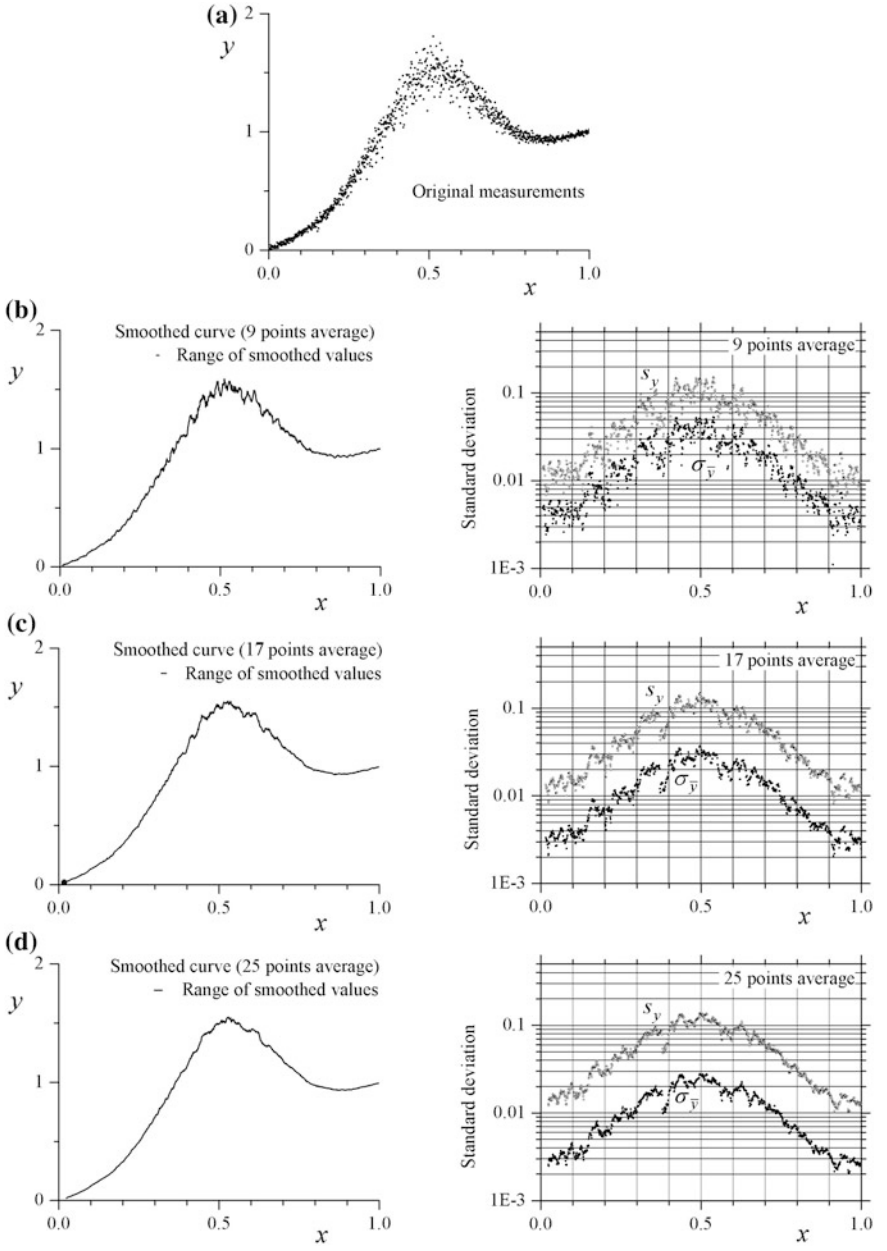


Fig. 11.8 The smoothing of a curve consisting of 1001 experimental points, **a**, by taking the averages of various numbers of points (9, 17 and 25 here) and the evaluation of the corresponding estimates for the standard deviation of the points, s_y, and of their mean $\sigma_{\bar{y}}$ (**b**, **c** and **d**)

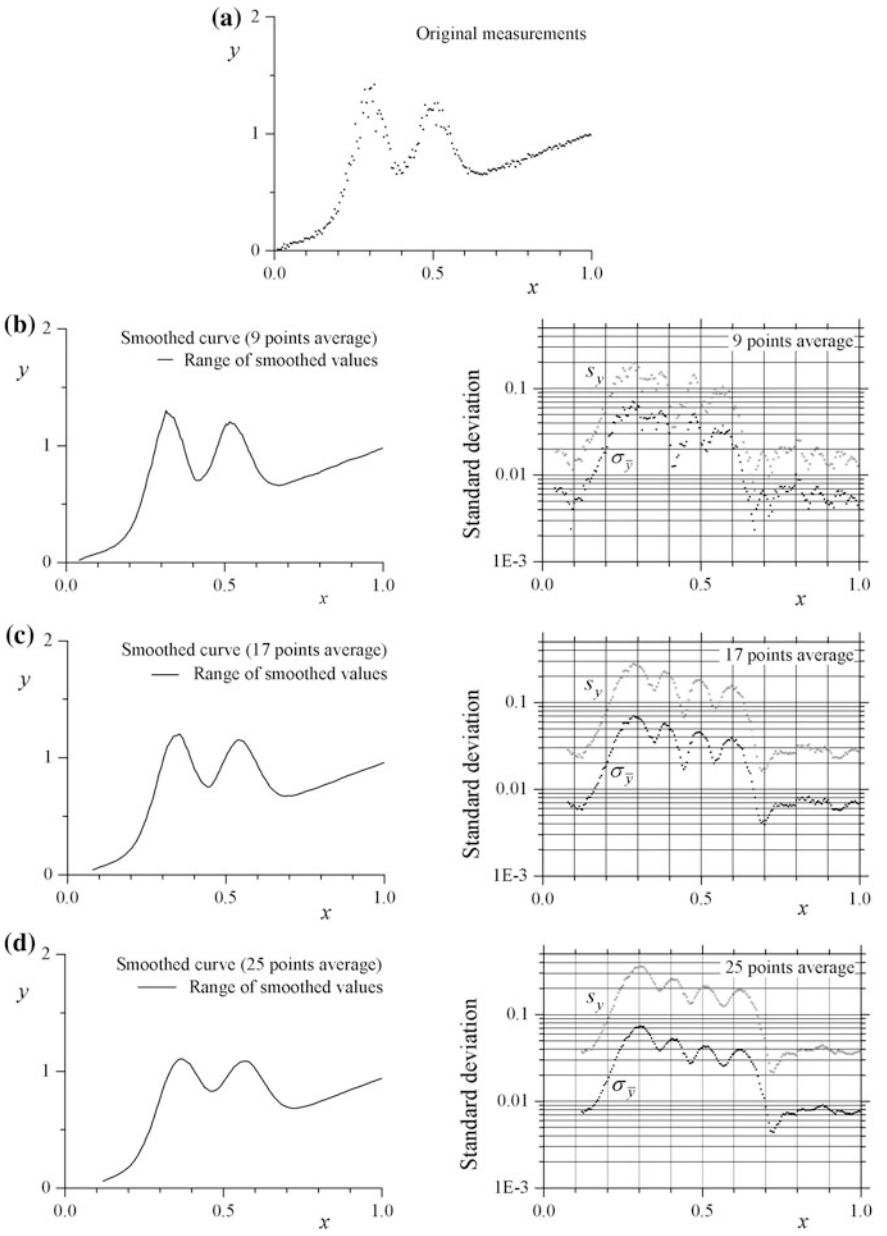


Fig. 11.9 The smoothing of a curve consisting of 201 experimental points, **a**, by taking the averages of various numbers of points (9, 17 and 25 here) and the evaluation of the corresponding estimates for the standard deviation of the points, s_y, and of their mean $\sigma_{\bar{y}}$ [**b**, **c** and **d**]

much (by using a small N). In the example of Fig. 11.8, the series of measurements consists of 1001 results. Given the variation of the signal, an averaging using 25 points does not seem unreasonable. It represents 1/40th of the whole range, and it is seen that the signal does not change significantly over this range. Figure 11.9 shows another case. There are only 201 points and in the smoothing, the 17 or 25 points used cover a significant part of the whole range of values (the ranges are shown in the graphs by small horizontal lines). As a result, over the range of the smoothing, the signal varies significantly and, consequently, s_y and $\sigma_{\bar{y}}$ increase with increasing N. The loss of detail in the smoothed curves is also obvious. It is seen, that in this case, using more than 9 points in the averaging for the smoothing does not offer any advantage.

Ideally, there should be a strict mathematical method for finding the error ($\sigma_{\bar{y}}$) at any point of a series of measurements such as that of Fig. 11.8a. This estimate would depend on the values at all the measurements. Such a method, however, is not available. We are thus forced to use the somewhat arbitrary method described above, based on smoothing. In all cases, we should consider the results obtained using the method described above as giving an order of magnitude estimate for the error in y as a function of the independent variable, x.

11.8 The Regression Line and the Coefficient of Correlation

In Chap. 6, Sect. 6.2.3, we found the mean value and the standard deviation of a function $Q = Q(x, y)$ of two variables $x,\ y$. Making use of those results in the case of N pairs of values $(x_i, y_i)(i = 1, 2, \ldots, N)$, if we expand the function $Q = Q(x, y)$ in a Taylor series in the region of the point $(\bar{x}, \bar{y})$, where $\bar{x}$ and $\bar{y}$ are the means of x and y, we have

$$Q(x, y) = Q(\bar{x}, \bar{y}) + \left(\frac{\partial Q}{\partial x}\right)_{\bar{x},\bar{y}} (x - \bar{x}) + \left(\frac{\partial Q}{\partial y}\right)_{\bar{x},\bar{y}} (y - \bar{y}) + \ldots \tag{11.72}$$

and find for the mean of the function, approximately,

$$\bar{Q} = Q(\bar{x}, \bar{y}). \tag{11.73}$$

The standard deviation of Q is found from the relation

$$\sigma_Q^2 = \frac{1}{N} \sum_i \left[\left(\frac{\partial Q}{\partial x}\right)_{\bar{x},\bar{y}} (x_i - \bar{x}) + \left(\frac{\partial Q}{\partial y}\right)_{\bar{x},\bar{y}} (y_i - \bar{y}) \right]^2, \tag{11.74}$$

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2_{\bar{x},\bar{y}} \frac{1}{N}\sum_i (x_i - \bar{x})^2 + \left(\frac{\partial Q}{\partial y}\right)^2_{\bar{x},\bar{y}} \frac{1}{N}\sum_i (y_i - \bar{y})^2 + 2\left(\frac{\partial Q}{\partial x}\right)_{\bar{x},\bar{y}} \left(\frac{\partial Q}{\partial y}\right)_{\bar{x},\bar{y}} \frac{1}{N}\sum_i (x_i - \bar{x})(y_i - \bar{y}) \tag{11.75}$$

This expression may be written in the form

$$\sigma_Q^2 = \left(\frac{\partial Q}{\partial x}\right)^2_{\bar{x},\bar{y}} \sigma_x^2 + \left(\frac{\partial Q}{\partial y}\right)^2_{\bar{x},\bar{y}} \sigma_y^2 + 2\left(\frac{\partial Q}{\partial x}\right)_{\bar{x},\bar{y}} \left(\frac{\partial Q}{\partial y}\right)_{\bar{x},\bar{y}} \sigma_{xy} \tag{11.76}$$

where σ_x and σ_y are the standard deviations of x and y, and

$$\sigma_{xy} \equiv \frac{1}{N}\sum_i (x_i - \bar{x})(y_i - \bar{y}) \tag{11.77}$$

is the *covariance* of x and y. This is a property of the sample of the measurements. The best estimate for the covariance of the parent population is

$$\hat{\sigma}_{xy} = \frac{N}{N-1}\sigma_{xy} = \frac{1}{N-1}\sum_i (x_i - \bar{x})(y_i - \bar{y}). \tag{11.78}$$

Equation (11.76) gives the standard deviation of Q whether x and y are independent of each other or not. If they are independent of each other, their covariance tends to zero as the number of measurements tends to infinity.

In the case of fitting a straight line to the points (x_i, y_i) using the method of least squares, we have found that the equation of the line may be written in the form

$$\alpha + \lambda\frac{[x]}{N} = \frac{[y]}{N}, \tag{11.79}$$

i.e. that the straight line passes through the point $\mathrm{K} : (\bar{x} = [x]/N,\ \bar{y} = [y]/N)$, which we called center of the line. If we define the variables

$$X \equiv x - \bar{x} \quad \text{and} \quad Y \equiv y - \bar{y}, \tag{11.80}$$

the equation of the straight line is

$$Y = \lambda X \tag{11.81}$$

and, according to the method of least squares, Eq. (11.35), it will be

$$\lambda = \frac{[XY]}{[XX]}. \tag{11.82}$$

Therefore, the straight line is given by the equation

$$y - \bar{y} = \frac{[XY]}{[XX]}(x - \bar{x}). \tag{11.83}$$

Using the relations $[XX] = N\sigma_x^2$, $[YY] = N\sigma_y^2$ and $[XY] = N\sigma_{xy}$, and defining the *(Pearson) coefficient of linear correlation*

$$r \equiv \frac{[XY]}{\sqrt{[XX][YY]}}, \tag{11.84}$$

we may write Eq. (11.83) as

$$\frac{y - \bar{y}}{\sigma_y} = r\frac{x - \bar{x}}{\sigma_x}. \tag{11.85}$$

This straight line is called *regression line of y on x*.

The correlation coefficient is written in the forms

$$r = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}, \tag{11.86}$$

where the sums are performed over all the values $i = 1, 2, \ldots, N$.

The coefficient of correlation r is a measure of how well the points $(x_i,\ y_i)$ are described by the regression line. It may take values

$$-1 \leq r \leq 1. \tag{11.87}$$

If the coefficient of correlation r is near the values ± 1, the points are near a straight line. If it has values near 0, the points are not correlated and there is no line that could be fitted to them satisfactorily. Let us note that, if all the points lie on the straight line $y = \alpha + \lambda x$, then it is $y_i = \alpha + \lambda x_i$ for every i and, also, $\bar{y} = \alpha + \lambda \bar{x}$. Subtracting, we find that $y_i - \bar{y} = \lambda(x_i - \bar{x})$ for every point. Therefore, Eq. (11.86) gives

$$r = \frac{\lambda \sum (x_i - \bar{x})^2}{\sqrt{\sum (x_i - \bar{x})^2 \lambda^2 \sum (x_i - \bar{x})^2}} = \frac{\lambda}{|\lambda|} = \pm 1. \tag{11.88}$$

The conclusion is: if all the points lie exactly on a straight line, then $r = \pm 1$ and the sign is that of the line's slope.

At the other end, if x and y are not correlated to each other, then the sum $\sum (x_i - \bar{x})(y_i - \bar{y})$ tends to zero as the number of points increases, since the terms are equally probable to be positive or negative. For a finite number of uncorrelated measurements, the coefficient of correlation r has values near 0.

Table 11.1 The probability $P\{|r| \geq r_0\}$ for the absolute value of the coefficient of linear correlation $|r|$ of a number N of points (x_i, y_i) to be greater than or equal to some value r_0 due to a coincidence and not due to the correlation of the variables x and y with each other

N	$P\{\|r\| \geq r_0\}$										
	r_0										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
4	1	0.90	0.87	0.81	0.74	0.67	0.59	0.51	0.41	0.29	0
6	1	0.85	0.70	0.56	0.43	0.31	0.21	0.12	0.056	0.014	0
8	1	0.81	0.63	0.47	0.33	0.21	0.12	0.053	0.017	0.002	0
10	1	0.78	0.58	0.40	0.25	0.14	0.067	0.024	0.005		0
12	1	0.76	0.53	0.34	0.20	0.098	0.039	0.011	0.002		0
14	1	0.73	0.49	0.30	0.16	0.069	0.023	0.005	0.001		0
16	1	0.71	0.46	0.26	0.12	0.049	0.014	0.003			0
18	1	0.69	0.43	0.23	0.10	0.035	0.008	0.001			0
20	1	0.67	0.40	0.20	0.081	0.025	0.005	0.001			0
25	1	0.63	0.34	0.15	0.048	0.011	0.002				0
30	1	0.60	0.29	0.11	0.029	0.005					0
35	1	0.57	0.25	0.080	0.017	0.002					0
40	1	0.54	0.22	0.060	0.011	0.001					0
45	1	0.51	0.19	0.045	0.006						0
50	1	0.49	0.16	0.034	0.004						0

When no value is given, the probability is less than 0.0005

Having fitted a straight line to a group of experimental points, it would be very useful to know whether the two variables are not correlated with each other and that the curve fit is simply the result of a coincidence. Given in Table 11.1 is, for a given number of measurements N, the probability $P\{|r| \geq r_0\}$ for the value of the correlation coefficient to be greater than or equal to some value r_0 due to a coincidence and not because of the correlation of the variables x and y with each other.

For example, for $N = 10$ points, a coefficient of linear correlation greater than or equal to $r_0 = 0.5$ has a probability of 0.14 (or 14%) to be due to a coincidence and not to a correlation of x and y with each other. For the same number of points, the value $r_0 = 0.8$ has a probability of 0.005 (or 0.5%) to be due to a coincidence.

Example 11.27

Find the coefficient of linear correlation for the points $(x_i,\ y_i)$ of Example 11.1 and the probability for the linear relationship between x and y to be due to a coincidence.

In Example 11.1 we found $\bar{x} = 1.00$ and $\bar{y} = 3.22$.

For the evaluation of $r = \dfrac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}$ we complete the table below:

i	x_i	y_i	$x_i - \bar{x}$	$y_i - \bar{y}$	$(x_i - \bar{x})^2$	$(y_i - \bar{y})^2$	$(x_i - \bar{x})(y_i - \bar{y})$
1	0.0	0.92	−1.0	−2.3	1.00	5.2900	2.300
2	0.2	1.48	−0.8	−1.74	0.64	3.0276	1.392
3	0.4	1.96	−0.6	−1.26	0.36	1.5876	0.756
4	0.6	2.27	−0.4	−0.95	0.16	0.9025	0.380
5	0.8	2.61	−0.2	−0.61	0.04	0.3721	0.122
6	1.0	3.18	0	−0.04	0	0.0016	0
7	1.2	3.80	0.2	0.58	0.04	0.3364	0.116
8	1.4	4.01	0.4	0.79	0.16	0.6241	0.316
9	1.6	4.85	0.6	1.63	0.36	2.6569	0.978
10	1.8	5.10	0.8	1.88	0.64	3.5344	1.504
11	2.0	5.26	1.0	2.04	1.00	4.1616	2.040
Sums					4.40	22.5	9.90

Therefore, $r = \dfrac{9.90}{\sqrt{4.40 \times 22.5}} = 0.995$.

The probability that this value of the coefficient of linear correlation is due to a coincidence is extremely small, as seen in Table 11.1.

Example 11.28 [E]

Using Excel®, evaluate the correlation coefficient for the data of Example 11.27.

Copy the x values into column A and the y values into column B. Highlight an empty cell. From **Formulas > More Functions, Statistical**, select **Correl**. This opens the correlation window. Fill **Array1** by right-clicking on cell A3 and dragging the cursor to A13. Similarly, fill **Array2** with the values in the cells B3 to B13. Pressing **OK** returns the value for the coefficient of correlation as $r = 0.99551$.

Example 11.29 [O]

Using Origin®, evaluate the correlation coefficient for the data of Example 11.27.

We place the data of columns x and y of the table of Example 11.27 in columns A and B, respectively. We highlight both columns by left-clicking on the label of column A and then, holding the **Shift** key down, left-clicking on the label of column B. Then

Statistics > Descriptive Statistics > Correlation Coefficient > Open Dialog...

In the window that opens, we select: **Correlation Types: Pearson**. Press **OK**.

The result returned is: AB or BA Pearson Correlation Coefficient = 0.99551. This is the same result as the one found in Example 11.27.

Example 11.30 [P]

Evaluate the correlation coefficient for the data of Example 11.27.

The scipy.stats subpackage includes the function pearsonr to calculate the Pearson correlation coefficient. We first enter the data into two vectors, and then invoke the function: it returns the value of r and the two-tailed p-value for testing non-correlation.

```
import numpy as np
from scipy.stats.stats import pearsonr

x = np.array([0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0])
y = np.array
([0.92, 1.48, 1.96, 2.27, 2.61, 3.18, 3.80, 4.01, 4.85, 5.10, 5.26])
pearsonr(x, y)
```

The result is r = 0.99551, with a *p*-value of 1.5913E−10.

Example 11.31 [R]

Evaluate the correlation coefficient for the data of Example 11.27.

Enter the data vectors:

```
> x <- c(0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0)
> y <- c(0.92, 1.48, 1.96, 2.27, 2.61, 3.18, 3.80, 4.01, 4.85, 5.10, 5.26)
```

Calculate the Pearson correlation coefficient:

```
> cor(x, y, method = "pearson")
[1] 0.9955053
```

The result, $r = 0.99551$, is in agreement with those of the two previous Examples.

11.9 The Use of the Method of Least Squares in the Solution of a System of Overdetermined Linear Equations

The method of least squares was used by Legendre in order to find the optimum solutions of systems of linear equations, in those cases when the number of equations is larger than the number of unknowns and the equations are not all satisfied by a certain set of values of the unknowns.

11.9.1 Equations in Two Variables

Let the linear equations involve two variables, x and y. Given are $N > 2$ equations

$$a_i x + b_i y = h_i \quad (i = 1, 2, \ldots, N) \tag{11.89}$$

where a_i, b_i and h_i are unknown constants. The problem is overdetermined, in the sense that there exist more equations than needed for the unique determination of the unknowns x and y. The equations are said to form an *overdetermined system of equations*.

To find the *most probable values* of x and y, the method of least squares is used as follows:

Defining the 'error' of the ith equation as

$$e_i \equiv a_i x + b_i y - h_i, \tag{11.90}$$

we find the values of x and y which minimize the sum

$$S \equiv \sum_{i=1}^{N} e_i^2 = \sum_{i=1}^{N} (a_i x + b_i y - h_i)^2. \tag{11.91}$$

Differentiating S partially with respect to x and with respect to y and equating both to zero, we obtain the normal equations

$$[a^2]x + [ab]y = [ah] \tag{11.92}$$

$$[ab]x + [b^2]y = [bh] \tag{11.93}$$

the solutions of which are given by the relations

$$\frac{x}{\begin{vmatrix} [ah] & [ab] \\ [bh] & [b^2] \end{vmatrix}} = \frac{y}{\begin{vmatrix} [a^2] & [ah] \\ [ab] & [bh] \end{vmatrix}} = \frac{1}{\begin{vmatrix} [a^2] & [ab] \\ [ab] & [b^2] \end{vmatrix}}. \tag{11.94}$$

In cases where weights are attributed to the equations, with the ith equation having a weight equal to w_i, the normal equations are

$$[wa^2]x + [wab]y = [wah] \tag{11.95}$$

$$[wab]x + [wb^2]y = [wbh] \tag{11.96}$$

and the solutions are suitably readjusted.

In order to find the errors in x and y, we define the residuals

$$d_i \equiv a_i x + b_i y - h_i, \tag{11.97}$$

and their standard deviation,

$$\sigma = \sqrt{\frac{[d^2]}{N-2}}. \tag{11.98}$$

The errors δx and δy in x and y are given by the relations:

$$\frac{(\delta x)^2}{[b^2]} = \frac{(\delta y)^2}{[a^2]} = \frac{\sigma^2}{\begin{vmatrix} [a^2] & [ab] \\ [ab] & [b^2] \end{vmatrix}}. \tag{11.99}$$

Example 11.32

Find the most probable solutions of the equations

$$x+y=5.3 \qquad 2x-y=0.8 \qquad x-y=-0.6 \qquad 3x+2y=11.2$$

and their errors.

We construct the following table:

i	a_i	b_i	h_i	a_i^2	b_i^2	$a_i b_i$	$a_i h_i$	$b_i h_i$
1	1	1	5.3	1	1	1	5.3	5.3
2	2	–1	0.8	4	1	–2	1.6	–0.8
3	1	–1	–0.6	1	1	–1	–0.6	0
4	3	2	11.2	9	4	6	33.6	22.4
Sums	7	1	16.7	15	7	4	39.9	27.5

The normal Eq. (11.94) give $\begin{matrix} 15x+4y=39.9 \\ 4x+7y=27.5 \end{matrix}$ the solutions of which are $x=1.90 \quad y=2.84$. These are the most probable solutions of the equations given.

The residuals of the four equations are, respectively:

$$d_1 = x+y-5.3 = -0.56 \quad d_2 = 2x-y-0.8 = 0.16$$
$$d_3 = x-y+0.6 = -0.34 \quad d_4 = 3x+2y-11.2 = 0.18$$

Therefore, $[d^2] = 0.487$ and $\sigma = \sqrt{\frac{[d^2]}{N-2}} = \sqrt{\frac{0.487}{2}} = 0.494$.

$$\frac{(\delta x)^2}{7} = \frac{(\delta y)^2}{15} = \frac{\sigma^2}{\begin{vmatrix} 15 & 4 \\ 4 & 7 \end{vmatrix}} = \frac{0.244}{89} = 0.002742$$

and, finally, $\delta x = 0.139 \quad \delta y = 0.203$.

The most probable values of x and y are, therefore, $x = 1.90 \pm 0.14$, $y = 2.84 \pm 0.20$.

Example 11.33 [E]

Using Excel®, find the most probable solutions of the equations

$$x+y=5.3 \quad 2x-y=0.8 \quad x-y=-0.6 \quad 3x+2y=11.2$$

as well as their errors.

We enter the 4 coefficients a_i in column A (cells A3 to A6), the 4 coefficients b_i in column B (cells B3 to B6) and the h_i's in column C (cells C3 to C6).

We evaluate [aa] = **sumsq(A3:A6)** = 15 in cell B9.
We evaluate [bb] = **sumsq(B3:B6)** = 7 in cell D9.
We evaluate [ab] = **sumproduct(A3:A6;B3:B6)** = 4 in cell F9.
We evaluate [ah] = **sumproduct(A3:A6;C3:C6)** = 39.9 in cell B11.
We evaluate [bh] = **sumproduct(B3:B6;C3:C6)** = 27.5 in cell D11.

Using Eq. (11.94), we find: x = 1.9022 and y = 2.8416.

We calculate the values of d_i in column E: In cell E3 type **1.9022*A3 + 2.8416*B3 − C3** and press **ENTER**. This puts d_1 in cell E1. We fill down to cell E6.

In cell B13 we calculate **sumsq(E3:E6)**.

Using Eq. (11.99), we find δx = 0.1384 and δy = 0.2026.

The most probable values of x and y are, therefore, $x = 1.90 \pm 0.14$, $y = 2.84 \pm 0.20$.

Example 11.34 [O]

Using Origin®, find the most probable solutions of the equations

$$x+y=5.3 \quad 2x-y=0.8 \quad x-y=-0.6 \quad 3x+2y=11.2$$

as well as their errors.

We enter the 4 coefficients a_i in column A (cells A1 to A4), the 4 coefficients b_i in column B (cells B1 to B4) and the h_i's in column C (cells C1 to C4).

Using **Column > Set Column Values…** we evaluate a^2, b^2, ab, ah and bh in columns D, E, F, G and H, respectively.

In each column we highlight those cells containing data and, using the **Σ** operator, we evaluate:

[aa] = 15 in cell M2 [bb] = 7 in cell M3 [ab] = 4 in cell M4
[ah] = 39.9 in cell M5 [bh] = 27.5 in cell M6.

Using Eq. (11.94), we find: x = 1.9022 and y = 2.8416.

We calculate the values of d_i^2 in column J: We highlight column J and, using **Column > Set Column Values…**, we evaluate (**1.9022*col(A) + 2.8416*col(B) – col(C))^2** in column J. Summing these values, we find [dd] = 0.48708. This value gives σ = 0.4935.

Using Eq. (11.99), we find δx = 0.1384 and δy = 0.2026.

The most probable values of x and y are, therefore, $x = 1.90 \pm 0.14$, $y = 2.84 \pm 0.20$.

Example 11.35 [P]

Using Python, find the most probable solutions of the equations

$$x + y = 5.3 \quad 2x - y = 0.8 \quad x - y = -0.6 \quad 3x + 2y = 11.2$$

as well as their errors.

```
from __future__ import division
import math
import numpy as np

# Enter the values of the coefficients a, b and h:
a = np.array([1, 2, 1, 3])
b = np.array([1, -1, -1, 2])
h = np.array([5.3, 0.8, -0.6, 11.2])

# Evaluation
AA = sum(a**2)
BB = sum(b**2)
AB = sum(a*b)
AH = sum(a*h)
BH = sum(b*h)

DENOM = AA*BB-AB*AB
DETX = AH*BB-AB*BH
DETY = BH*AA-AB*AH

x = DETX/DENOM
y = DETY/DENOM

d = x*a + y*b - h
S = math.sqrt(sum(d**2)/2)

DX = S*math.sqrt(BB/DENOM)
DY = S*math.sqrt(AA/DENOM)

# Results
print("Value of x:", x)
print("Value of y:", y)
print("Standard error in x:", DX)
print("Standard error in y:", DY)
```

```
Value of x: 1.90224719101
Value of y: 2.84157303371
Standard error in x: 0.1384007891490002
Standard error in y: 0.2025980103399658
```

Example 11.36 [R]

Using R, find the most probable solutions of the equations

$$x+y=5.3 \quad 2x-y=0.8 \quad x-y=-0.6 \quad 3x+2y=11.2$$

as well as their errors.

```
# Enter data vectors:

> a <- c(1, 2, 1, 3)
> b <- c(1, -1, -1, 2)
> h <- c(5.3, 0.8, -0.6, 11.2)

# Calculate sums of products:

> AA = sum(a^2)
> BB = sum(b^2)
> AB = sum(a*b)
> AH = sum(a*h)
> BH = sum(b*h)

# Calculate determinants:

> DENOM = AA*BB-AB*AB
> DETX = AH*BB-AB*BH
> DETY = BH*AA-AB*AH

# Find x and y:

> x = DETX/DENOM
> x
[1] 1.902247
> y = DETY/DENOM
> y
[1] 2.841573

# Calculate σ:

> d = 1.902247*a+2.841573*b-h
> S = sqrt(sum(d^2)/2)
> S
[1] 0.493497
```

```
# Calculate errors in x and y:

> DX = S*sqrt(BB/DENOM)
> DX
[1] 0.1384008
> DY = S*sqrt(AA/DENOM)
> DY
[1] 0.202598
```

The final results are: $x = 1.90 \pm 0.14$, $y = 2.84 \pm 0.20$.

11.9.2 Equations in Three Variables

Let the linear equations involve three variables, x, y and z. Given are $N > 3$ equations

$$a_i x + b_i y + c_i z = h_i \quad (i = 1, 2, \ldots, N) \tag{11.100}$$

where a_i, b_i, c_i and h_i are unknown constants. The problem is overdetermined, in the sense that there exist more equations than needed for the unique determination of the unknowns x, y and z. The equations are said to form an overdetermined system of equations.

To find the *most probable values* of x, y and z, we work as in Sect. 11.9.1. The equations derived are just presented here:

$$a_i x + b_i y + c_i z = h_i \quad (i = 1, 2, \ldots, N) \tag{11.101}$$

$$S \equiv \sum_{i=1}^{N} (a_i x + b_i y + c_i z - h_i)^2 \tag{11.102}$$

The normal equations are

$$[a^2]x + [ab]y + [ac]z = [ah] \tag{11.103}$$

$$[ab]x + [b^2]y + [bc]z = [bh] \tag{11.104}$$

$$[ac]x + [bc]y + [c^2]z = [ch] \tag{11.105}$$

and their solutions,

$$\frac{x}{\begin{vmatrix} [ah] & [ab] & [ac] \\ [bh] & [b^2] & [bc] \\ [ch] & [bc] & [c^2] \end{vmatrix}} = \frac{y}{\begin{vmatrix} [a^2] & [ah] & [ac] \\ [ab] & [bh] & [bc] \\ [ac] & [ch] & [c^2] \end{vmatrix}} = \frac{z}{\begin{vmatrix} [a^2] & [ab] & [ah] \\ [ab] & [b^2] & [bh] \\ [ac] & [bc] & [ch] \end{vmatrix}} = \frac{1}{\begin{vmatrix} [a^2] & [ab] & [ac] \\ [ab] & [b^2] & [bc] \\ [ac] & [bc] & [c^2] \end{vmatrix}}. \tag{11.106}$$

In cases where weights are attributed to the equations, with the ith equation having a weight equal to w_i, the normal equations are

$$[wa^2]x + [wab]y + [wac]z = [wah] \tag{11.107}$$

$$[wab]x + [wb^2]y + [wbc]z = [wbh] \tag{11.108}$$

$$[wac]x + [wbc]y + [wc^2]z = [wch] \tag{11.109}$$

and the solutions are suitably readjusted.

To find the errors δx, δy and δz in the variables x, y and z, we define

$$d_i \equiv a_i x + b_i y + c_i z - h_i \tag{11.110}$$

and

$$\sigma = \sqrt{\frac{[d^2]}{N-3}}, \tag{11.111}$$

in which case we have the relations

$$\frac{(\delta x)^2}{\begin{vmatrix} [b^2] & [bc] \\ [bc] & [c^2] \end{vmatrix}} = \frac{(\delta y)^2}{\begin{vmatrix} [a^2] & [ac] \\ [ac] & [c^2] \end{vmatrix}} = \frac{(\delta z)^2}{\begin{vmatrix} [a^2] & [ab] \\ [ab] & [b^2] \end{vmatrix}} = \frac{\sigma^2}{\begin{vmatrix} [a^2] & [ab] & [ac] \\ [ab] & [b^2] & [bc] \\ [ac] & [bc] & [c^2] \end{vmatrix}}. \tag{11.112}$$

Programs
Excel
Ch. 11. *Excel—Least Squares—Overdetermined Equations—2 Variables*
Ch. 11. *Excel—Least Squares—Overdetermined Equations—3 Variables*
Ch. 11. *Excel—Least Squares—Smoothing—Adjacent Averaging*
Ch. 11. *Excel—Least Squares Fit—Straight Line*
Ch. 11. *Excel—Least Squares Fit—Straight Line—Weighted Points*
Ch. 11. *Excel—Least Squares Fit—Straight Line Through Origin*
Ch. 11. *Excel—Least Squares Fit—Straight Line Through Origin—Weighted Points*

(continued)

(continued)

Programs
Ch. 11. *Excel—Least Squares Fit—Parabola*
Ch. 11. *Excel—Least Squares Fit—Cubic*
Ch. 11. *Excel—Least Squares Fit—Curve of 4th Degree*
Ch. 11. *Excel—Least Squares Fit—Curve of 5th Degree*
Ch. 11. *Excel—Least Squares Fit—Curve of 6th Degree*
Ch. 11. *Excel—Least Squares Fit—Exponential*
Origin
Ch. 11. *Origin—Least Squares—Overdetermined Equations—2 Variables*
Ch. 11. *Origin—Least Squares—Overdetermined Equations—3 Variables*
Ch. 11. *Origin—Least Squares—Smoothing—Adjacent Averaging and Savitzki-Golay*
Ch. 11. *Origin—Least Squares Fit—Straight Line*
Ch. 11. *Origin—Least Squares Fit—Straight Line—Weighted Points*
Ch. 11. *Origin—Least Squares Fit—Straight Line Through Origin*
Ch. 11. *Origin—Least Squares Fit—Straight Line Through Origin—Weighted Points*
Ch. 11. *Origin—Least Squares Fit—Parabola*
Ch. 11. *Origin—Least Squares Fit—Cubic*
Ch. 11. *Origin—Least Squares Fit—Curve of 4th Degree*
Ch. 11. *Origin—Least Squares Fit—Curve of 5th Degree*
Ch. 11. *Origin—Least Squares Fit—Curve of 6th Degree*
Ch. 11. *Origin—Least Squares Fit—Power*
Ch. 11. *Origin—Least Squares Fit—Exponential*
Ch. 11. *Origin—Least Squares Fit—Gaussian*
Ch. 11. *Origin—Least Squares Fit—Poisson*
Python
Ch. 11. *Python—Least Squares—Overdetermined Equations—2 Variables*
Ch. 11. *Python—Least Squares—Overdetermined Equations—3 Variables*
Ch. 11. *Python—Least Squares—Smoothing—Savitzki-Golay*
Ch. 11. *Python—Least Squares Fit—Straight Line*
Ch. 11. *Python—Least Squares Fit—Straight Line—Weighted Points*
Ch. 11. *Python—Least Squares Fit—Straight Line Through Origin*
Ch. 11. *Python—Least Squares Fit—Straight Line Through Origin—Weighted Points*
Ch. 11. *Python—Least Squares Fit—Parabola*
Ch. 11. *Python—Least Squares Fit—Cubic*
Ch. 11. *Python—Least Squares Fit—Curve of 4th Degree*
Ch. 11. *Python—Least Squares Fit—Curve of 5th Degree*
Ch. 11. *Python—Least Squares Fit—Curve of 6th Degree*
Ch. 11. *Python—Least Squares Fit—Exponential*
R
Ch. 11. *R—Least Squares—Overdetermined Equations—2 Variables*
Ch. 11. *R—Least Squares—Overdetermined Equations—3 Variables*
Ch. 11. *R—Least Squares—Smoothing—Cubic Spline*
Ch. 11. *R—Least Squares Fit—Straight Line*
Ch. 11. *R—Least Squares Fit—Straight Line—Weighted Points*
Ch. 11. *R—Least Squares Fit—Straight Line Through Origin*
Ch. 11. *R—Least Squares Fit—Straight Line Through Origin—Weighted Points*
Ch. 11. *R—Least Squares Fit—Parabola*
Ch. 11. *R—Least Squares Fit—Cubic*
Ch. 11. *R—Least Squares Fit—Curve of 4th Degree*

(continued)

(continued)

Programs
Ch. 11. *R—Least Squares Fit—Curve of 5th Degree*
Ch. 11. *R—Least Squares Fit—Curve of 6th Degree*
Ch. 11. *R—Least Squares Fit—Exponential*

Problems

The reader is reminded of the fact that most scientific hand-held calculators have the possibility of evaluating the quantities mentioned in this book. The n pairs of values x, y are entered using the key $\Sigma+$ *(and* $\Sigma-$ *for correcting erroneous entries). When entering the data is complete, the calculator's memories contain the quantities n,* $[x]$, $[y]$, $[x^2]$, $[y^2]$, $[xy]$, $\bar{x}$, $\bar{y}$, s_x, $\sigma_{\bar{x}}$, s_y, $\sigma_{\bar{y}}$, *which may be used to evaluate magnitudes such as the ones mentioned so far in this book. Some scientific calculators also return the parameters of the regression line,* α, λ *and r. Of course, statistics calculators offer even more.*

11.1 **[E.O.P.R.]** Given the experimental results

x_i	0	1	2	3	4	5	6	7	8	9
y_i	3.8	11.3	18.5	24.5	31.1	37.7	45.8	52.7	60.5	66.2

(a) In a figure draw the straight line that you consider to fit better to these points.
(b) Find the least-squares straight line $y = a + \lambda x$ fitted to the points. Draw this line in the figure drawn in (a).
(c) Find the coefficient of correlation r of the least-squares line.
(d) What are the errors in the values of a and λ ?

11.2 **[E.O.P.R.]** Measurements of y as a function of x gave the results

x_i	0.8	2.2	3.6	4.8	6.2	7.8	9.0
y_i	8.0	6.8	6.1	5.2	4.4	4.0	2.8

(a) Find the parameters $a \pm \delta a$ and $\lambda \pm \delta \lambda$ of the straight line $y = a + \lambda x$ fitted to these data using the method of least squares. Assuming that a and λ are correlated to a negligible degree, so that from the relation $y = (a \pm \delta a) + (\lambda \pm \delta \lambda)x$ the error in y to be given by $\delta y = \sqrt{(\delta a)^2 + x^2(\delta \lambda)^2}$, find:
(b) the value of y and its error δy for $x = 5$ and
(c) for which value of x (and its error, δx) y is equal to 0.

11.3 To the pairs of experimental values $(x_i, y_i) \quad (i = 1, 2, \ldots, N)$ we wish to fit a straight line $y = a + \lambda x$.

(a) Assuming that a is known with great accuracy, show that the method of least squares gives $\lambda = \frac{[xy]-a[x]}{[x^2]}$.

(b) Assuming that λ is known with great accuracy, show that the method of least squares gives $a = \frac{1}{N}([y] - \lambda[x])$.

11.4 **[E.O.P.R.]** Using the method of least squares, fit a parabolic curve to the experimental points

x_i	0	0.1	0.2	0.3	0.4	0.5	0.6
y_i	2.8	4.2	8.4	16.0	27.5	41.9	59.3

11.5 If in Example 11.4 the function was $y = (A \sin \omega t + B \cos \omega t)\mathrm{e}^{-\kappa t}$, where ω and κ are known, what does the method of least squares give for the parameters A and B?

11.6 A sample contains two radioisotopes, whose decay constants, λ_1 and λ_2, are known with great accuracy. If N_{01} and N_{02} are the initial numbers of nuclei of the two isotopes, the total activity of the sample at time t is $R = R_1 + R_2$,

$$R(t) = \lambda_1 N_{01}\mathrm{e}^{-\lambda_1 t} + \lambda_2 N_{02}\mathrm{e}^{-\lambda_2 t}.$$

From N measurements (t_i, R_i), find N_{01} and N_{02} by the method of least squares.

(*Suggestion:* For convenience, use the notation $x_i \equiv \lambda_1 \mathrm{e}^{-\lambda_1 t_i}$ and $y_i \equiv \lambda_2 \mathrm{e}^{-\lambda_2 t_i}$. The values of x_i and y_i are known for every value t_i.)

11.7 **[E.O.P.R.]** The viscosity of water, η (in units of centipoise) varies with the temperature in the following way, as determined by measurements:

t (°C)	10	20	30	40	50	60	70
η	1.308	1.005	0.801	0.656	0.549	0.469	0.406

Assume that a relation of the form $\eta = A\mathrm{e}^{\lambda/T}$ holds, where $T(\mathrm{K}) = t(^\circ\mathrm{C}) + 273.15$ is the absolute temperature. Using $x = 1/T$ as variable and the methods of curve fitting, determine A and λ. Find also the errors in these parameters.

11.8 In an experiment for the determination of the radius of the Earth, R, by measuring the acceleration of gravity as a function of height H above the surface of the Earth, the results were as follows:

H (m)	0	500	1000	1500	2000	2500	3000
$g(\mathrm{m/s^2})$	9.8070	9.8051	9.8044	9.8020	9.8015	9.7990	9.7976

The theoretical relation for the acceleration of gravity as a function of height is $g = \frac{g_0}{(1+H/R)^2}$. From this we have $\frac{1}{\sqrt{g}} = \frac{1}{\sqrt{g_0}} + \frac{1}{R\sqrt{g_0}}H$. Putting $x = H$, $y = 1/\sqrt{g}$, $a = 1/\sqrt{g_0}$ and $\lambda = \frac{1}{R\sqrt{g_0}}$, it follows that $y = a + \lambda x$.

Using the method of least squares determine $a \pm \delta a$ and $\lambda \pm \delta\lambda$, and then $g_0 \pm \delta g_0$ and $R \pm \delta R$.

This method, which would not give accurate results, assumes that g may be measured with sufficient accuracy and that its variation is due solely to the change in height.

11.9 The rolling resistance F for a vehicle was measured at various speeds υ and found to be

υ_i (m/s)	1	2	3	4	5
F_i (N)	15	20	30	40	60

Assuming a relation of the form $F = a + \lambda\upsilon^2$ and using as variable $x = \upsilon^2$, find the least-squares straight line for $F(x)$ and from it find the coefficients a and λ.

[E.O.P.R.] Using non-linear curve fitting, find the parabola of the form $F = a + \lambda\upsilon^2$ that gives the best fit to the points.

11.10 The activity $R(t)$ of a radon sample is initially equal to R_0. The variation of the ratio $R(t)/R_0$ in measurements which were made at intervals of one day each from the other is:

t_i (d)	0	1	2	3	4	5	6	7	8
$R(t)/R_0$	1	0.835	0.695	0.580	0.485	0.405	0.335	0.280	0.235

Assuming that it is $R(t)/R_0 = \mathrm{e}^{-\lambda t}$ and, therefore, $\ln[R(t)/R_0] = -\lambda t$, find the value of λ applying the method of least squares to the last relation.

[E.O.P.R.] Using non-linear curve fitting, find the curve $R(t)/R_0 = \mathrm{e}^{-\lambda t}$ that gives the best fit to the points.

11.11 **[E.O.P.R.]** Measurements of y as a function of x gave the following results:

x_i	2	6	8	12	16	18	22	28
y_i	2	4	8	8	10	14	16	18

(a) Using the method of least squares, find the straight line $y(x)$, when x is considered to be the independent variable.

(b) Using the method of least squares, find the straight line $x(y)$, when y is considered to be the independent variable.

(c) Draw both lines in a graph.
(d) Show that both lines pass through the point $(\bar{x}, \bar{y})$.

11.12 **[E.O.P.R.]** The main measurements of the speed of light performed between 1900 and 1956 are given in the table below.

#	Researcher	t	c (km/s)	#	Researcher	t	c (km/s)
1	Rosa, Dorsey	1906	299 781	10	Houston	1950	299 775
2	Mercier	1923	299 782	11	Bol, Hansen	1950	299 789.3
3	Michelson	1926	299 796	12	Aslakson	1951	299 794.2
4	Karolus, Mittelstaedt	1928	299 778	13	Rank, Ruth, Ven der Sluis	1952	299 776
5	Michelson, Pease, Pearson	1932	299 774	14	Froome	1952	299 792.6
				15	Florman	1954	299 795.1
6	Huettel	1940	299 768	16	Rank, Shearer, Wiggins	1954	299 789.8
7	Anderson	1941	299 776				
8	Bergstrand	1950	299 792.7	17	Edge	1956	299 792.9
9	Essen	1950	299 792.5				

Using the method of least squares, fit a straight line of the form $c = a + \lambda(t - 1956)$ to the measurements, where t is the year each measurement was performed. Investigate the possibility that the results support the hypothesis that the speed of light varies with time.

11.13 From the equations

$$3x + 2y = 5.8 \quad x - 4y = 1.8 \quad 4x - 6y = 3.8$$

find the most probable values of x and y.

11.14 Find the most probable values of x, y and z, as these are determined from the equations:

$$x + 2y + 3z = 12.1 \quad 2x - 2y + 3z = 3.2 \quad x + 6y - 6z = 15.1$$
$$3x + 2y = 14.9.$$

11.15 Find the most probable values of x and y, and their errors, as these are determined by applying the method of least squares to the equations

$$x + 2y = 31.8 \quad x - 4y = -4.8 \quad x - 2y = 3.6 \quad 2x + 6y = 67.2.$$

References

1. For a complete analysis of the problem of fitting polynomial curves to experimental points, see, for example, F.B. Hildebrand, *Introduction to Numerical Analysis*, 2nd edn. (McGraw-Hill Inc., New York, 1974). Chap. 7
2. For the general expression, see, for example, Athanasios Papoulis, *Probability, Random Variables and Stochastic Processes*, 3rd edn. (McGraw-Hill Co. Inc., New York, 1991). Chap. 5, Sect. 5-2

Chapter 12
Graphs

12.1 Introduction

The graphical representation of experimental results may serve one or more of the following purposes:

1. To show the relationship between two quantities, bringing out characteristics which would not be obvious in a table of numerical values.
2. The curve of the graph may be used in the evaluation of the slope or the intercept with one of the axes, especially when the relationship between the two magnitudes is linear. Important physical quantities and natural constants are usually determined by this method.
3. To investigate the form of the relation connecting two variables (linear, exponential etc.) which then may be expressed in the form of a mathematical equation for greater accuracy.
4. For the verification or not of a theoretical relation between two magnitudes, by comparison of the theoretical curve with the experimental results.
5. To determine the calibration curve of an instrument or of a process or, in general, to enable finding the value of one of the variables corresponding to a certain value of the other. Even if the result is present in the table of results, reading the required value off the best curve drawn between the experimental points gives a better value, as it is based on more than one measurement.

Having used so many graphs in the previous chapters of the book, it is certainly unnecessary to try to put forward arguments in favor of using graphs in the presentation of information. The numerical values of a table are obviously useful, but they do not transmit the same amount of information as a graph does. Figure 12.1 demonstrates the truth of this statement.

In this chapter we will present the main characteristics of graphs and the criteria on the basis of which these are selected for the best presentation of the data. We will only examine cases in which the results of measurements we have at our

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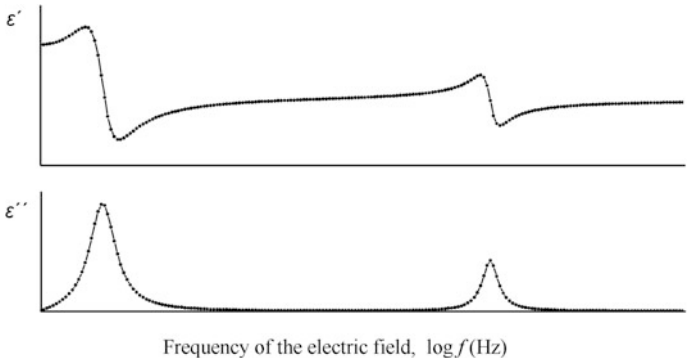


Fig. 12.1 Real (ε') and imaginary part (ε'') of the dielectric constant of a material, as functions of the frequency. The frequency scale is logarithmic and common to both quantities

disposal are N pairs of values $(x,\ y)$ of a dependent variable y as a function of an independent variable x. We wish to exhibit the relationship between the two magnitudes in the best way.

12.2 The Axes

The first things that are drawn in a graph are the axes. The choice of these will determine the kind of graph that will result, the range of values that will be covered as well as the kind of curve that will be obtained.

The magnitude considered to be the independent variable is usually recorded along the axis of the abscissae (x-axis), while the magnitude considered to be the dependent variable is recorded along the ordinate axis (y-axis). Although the distinction is not always possible, in general, if in an experiment we set the values of one of the variables (e.g. the potential difference across the ends of a resistance), then this variable is considered to be the independent variable and the result (e.g. the current through the resistance) is the dependent variable.

12.2.1 Linear Scales

In the simplest and most common graphs, we use linear scales for both x and y. Examples of axes with linear scales have been drawn in Fig. 12.2. Although the axes drawn are horizontal, ordinate axes may be drawn in the same manner.

The numerical values for a reasonable number of points are given on the axis, in such a way that intermediate values of the variable would be easy to find. This is done without overloading the axis with numbers which may cause confusion without

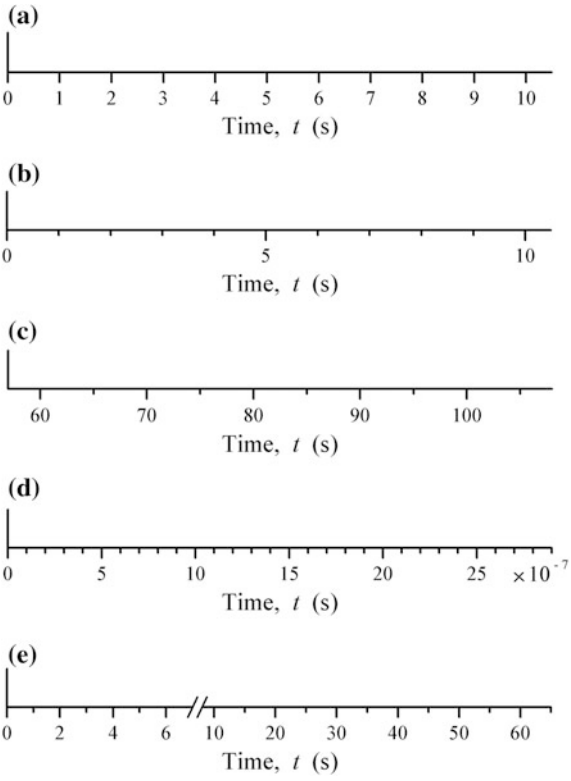


Fig. 12.2 Examples of axes with linear scales

giving any useful information. Having drawn the axis and the numerical scale corresponding to it, the name of the magnitude to which the axis corresponds is written near it and (usually in parentheses) the units used, e.g. time, t (s), height, h (m), or, simply, $\lambda I^2/m$ (S.I.).

Figure 12.2 shows some common kinds of linear axes:

(a) The simplest kind, with a number at every division.
(b) With numerical indications every 5 units and 4 subdivisions between them.
(c) The scale does not start at 0. In this way, the region of interest or the region in which experimental results exist may be shown in greater detail.
(d) The numerical indications have a common multiplying factor, e.g. $\times 10^{-7}$. The range covered by the axis in this case is between 0 and 29×10^{-7} s.
(e) Two different scales are used on the axis. This may be necessary in order to bring out the details in a region of values (here, between 0 and 6 s). The 'break' in the axis must be marked clearly.

For convenience, every subdivision of the graph paper used (usually with mm subdivisions) corresponds to 1, 2 or 5 units or to the corresponding multiples of a power of 10, depending on the range of values to be covered. This is done to

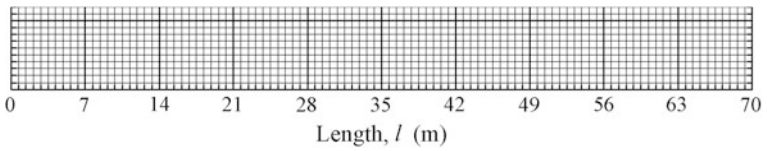


Fig. 12.3 The choice of a scale (1 cm = 7 units) which makes the reading of values off the graph difficult

facilitate marking the points on the graph, as well as in reading numerical values from it. A scale on which one cm corresponded to, say, 7 units, would be difficult to use (Fig. 12.3).

Subdivisions of the units in 1/2, 1/4, 1/8 etc. are actually used in scales in inches, as these are the subdivisions usually found on inch rulers. If years are used as units, they are usually subdivided in 12 months. Angles and times may be subdivided in 60 min and 60 s.

It is preferable to avoid using multi-digit numbers (e.g. 0.000 01, 0.000 02, or 10 000, 20 000 etc.). This is easily achieved by the suitable choice of units (e.g. mA or μA instead of A) or the use of a power of 10 as a common multiplying factor $\left(\times 10^{3}, \times 10^{-6}\,\text{etc.}\right)$. Some examples of linear scales are given in Figs. 12.4 and 12.5. Figure 12.5 shows the advantage of the suitable choice of the region of values in the scales, for the best presentation of the measurements.

12.2.2 Logarithmic Scales

Quite often, when a large range of values has to be covered without losing the details in the region of small values, we use as variable not the magnitude itself. e.g. x, but its logarithm $\log x$ (we usually use the common (decimal) logarithm $\log x$, while the use of the natural logarithm, $\ln x$, is rarer). For convenience in the

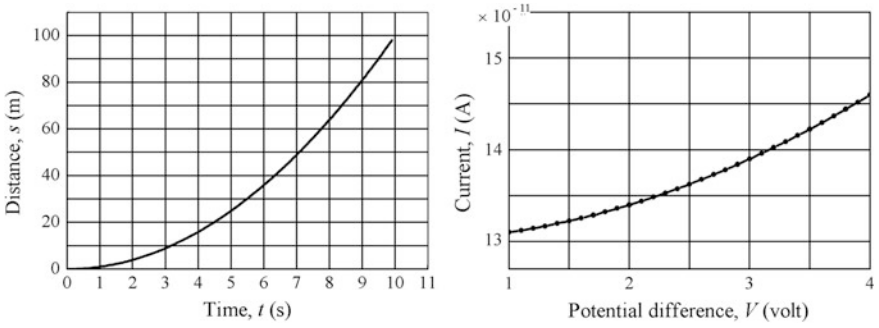


Fig. 12.4 Examples of use of various kinds of linear axes

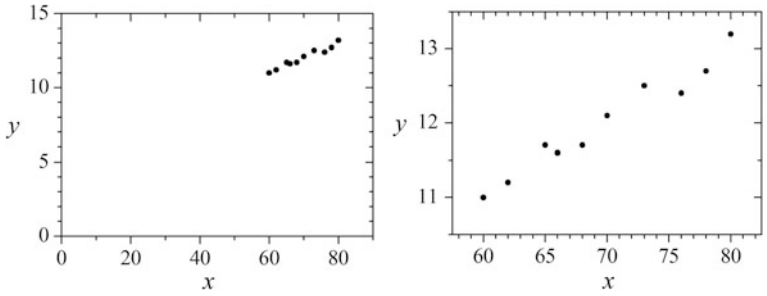


Fig. 12.5 The choice of the suitable range of values in the scales

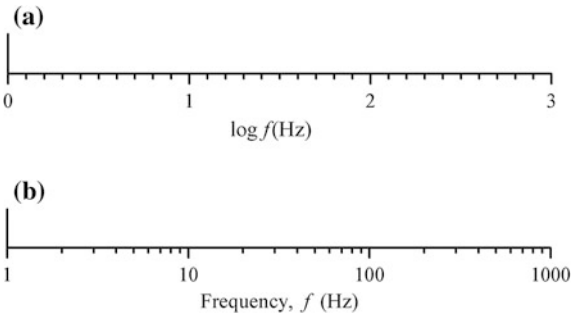


Fig. 12.6 A logarithmic scale of three decades (or three cycles)

reading of values off the graph, the subdivisions of the scale as well as the numerical values marked on the axis are those of x and not those of log x. This is demonstrated in Fig. 12.6a, where an axis was drawn with the logarithm of f, log f, taking values from 1 to 3, while in Fig. 12.6b the same axis is drawn, on which now the subdivisions and values marked are those of f, which takes values between 1 and 1000, corresponding to the values of log f between 0 and 3. On the axis of Fig. 12.6b the subdivisions corresponding to the values of 2, 3, … 9, 20, 30, … 90, 200, 300, … 900 are also drawn. It should be noted that, although the logarithm is a dimensionless magnitude, the numerical value of the logarithm of a physical quantity depends on the units used in expressing this quantity. For this reason, it is necessary to state the units used, in parentheses, immediately after the symbol for the physical quantity, e.g. log f(Hz). Figure 12.7 shows four of the many kinds of semi-log (one linear and one logarithmic axis) and log-log (both axes logarithmic) graph paper available. Naturally, on the logarithmic graph paper available, there are given, whenever possible, more subdivisions than shown in our figures. For example, the interval between 1 and 2 is often subdivided into tenths, etc.

The choice of the scales to be used in any particular case depends mainly on the relationship expected to exist between the plotted variables. Thus, semi-log graph paper is used, apart from the case we have already mentioned in which we wish to

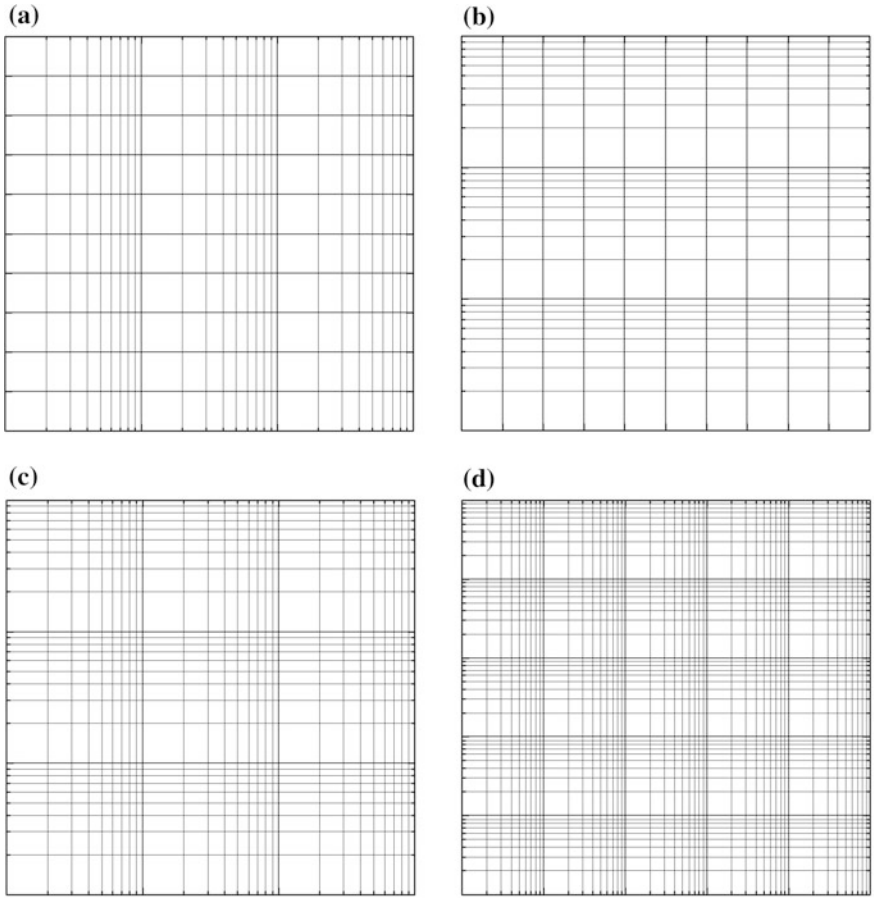


Fig. 12.7 **a**, **b** Three-decade semi-log graph paper, **c** log-log paper with 3 × 3 decades, **d** log-log paper with 5 × 5 decades

cover a large range of values, also when the relation between two variables is exponential, as, for example, when it is

$$y = A\,\mathrm{e}^{-\kappa x}. \tag{12.1}$$

Then, since it is

$$\ln y = \ln A - \kappa x, \tag{12.2}$$

plotting the natural logarithm of y as a function of x, will result in a linear graph. The same is true if the common logarithm of y is plotted as a function of x (since it is $\log y = \ln y / \ln 10$). A linear relation is desirable, as it is easier to draw a straight line between the points $(x, \log y)$ than it is to draw the corresponding exponential curve.

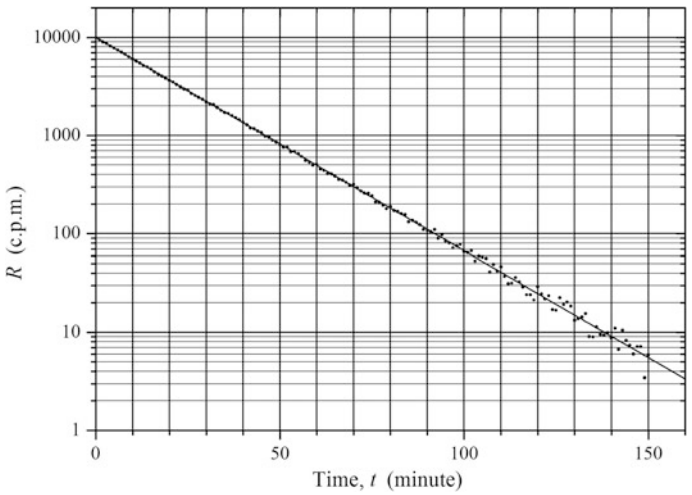


Fig. 12.8 The variation with time t of the activity R of a radioactive sample. The scale of R is logarithmic but the corresponding *values* of R are shown

Using $\log y$ as variable, we succeed in depicting the large values of y as well as the small ones. A classic example is the case of the decrease of the activity of a radioactive sample with time, presented in Fig. 8.2 which we reproduce here (Fig. 12.8).

The mathematical relation between the activity R and the time t is:

$$R(t) = R_0 \mathrm{e}^{-\lambda t}. \tag{12.3}$$

Plotting $\log R$ as a function of time, we have the straight line shown in the figure.

For relationships of the form $y = Ax^n$, the use of logarithmic scales for both variables leads to a linear relation:

$$\log y = \log A + n \ \log x. \tag{12.4}$$

For example, the Child-Langmuir law for the anode current I_a passing through a vacuum tube diode is found theoretically to be of the form

$$I_a = K V_a^{3/2}, \tag{12.5}$$

where V_a is the anode potential and K is a constant which depends on the geometry of the diode. Figure 12.9 shows the experimental results of I_a as a function of V_a, plotted using logarithmic scales. The linear relation

$$\log I_a = \log K + \frac{3}{2} \log V_a \tag{12.6}$$

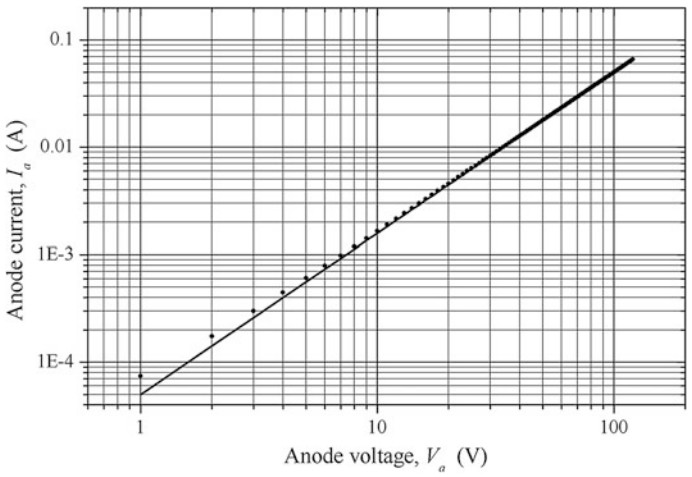


Fig. 12.9 Anode current as a function of anode potential for a vacuum tube diode

is seen to apply for large values of I_a and V_a, with the slope of the straight line actually being equal to 3/2. The deviation from linearity at low currents is due to the fact that current passes through the diode even when no potential difference is applied between cathode and anode. The relation is, in practice,

$$I_a = K(V_0 + V_a)^{3/2}, \tag{12.7}$$

with $V_0 = 0.3$ V for the diode of Fig. 12.9. If we plot I_a as a function of $V_a + 0.3$ V, using logarithmic scales, we will have the linear relation holding over all the range of values.

Another example is shown in Fig. 12.10, where the periods T of the planets were plotted as a function of the planets' distances from the Sun (the semi-major axes of their elliptic orbits), a, using logarithmic scales. The unit for the distance is the astronomic unit (1 ua = mean distance of Sun-Earth) and the unit for time is the year. As a result, the point for Earth is (1, 1). The resulting linear relation verifies Kepler's third law, $T^2 \propto a^3$ (the straight line has a slope of 3/2).

A variety of combinations of scales can be used to bring out a certain characteristic of a graph. Examples are shown in Fig. 12.11.

12.2.3 Polar Diagrams

It is often the case that the dependence of one magnitude on another is angular, $r(\theta)$. In these cases, it is useful to draw the relation using a polar diagram, such as that shown in Fig. 12.12. The independent variable is plotted as an angle on a circle

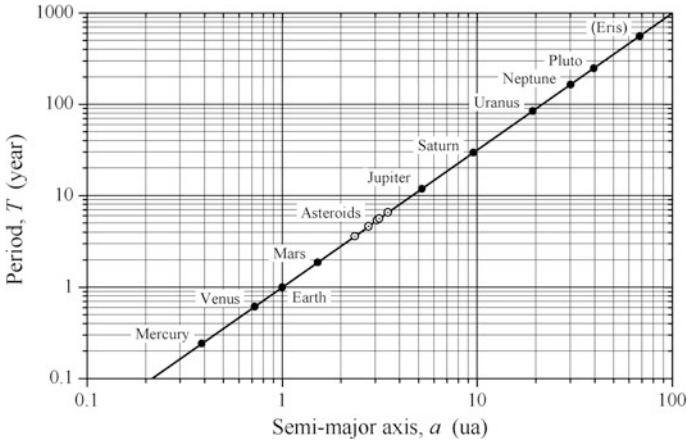


Fig. 12.10 Kepler's third law. The point for the newly discovered dwarf planet Eris should not be taken as verifying the law, as it is certain that the law was actually used in order to evaluate its period from the knowledge of its orbit

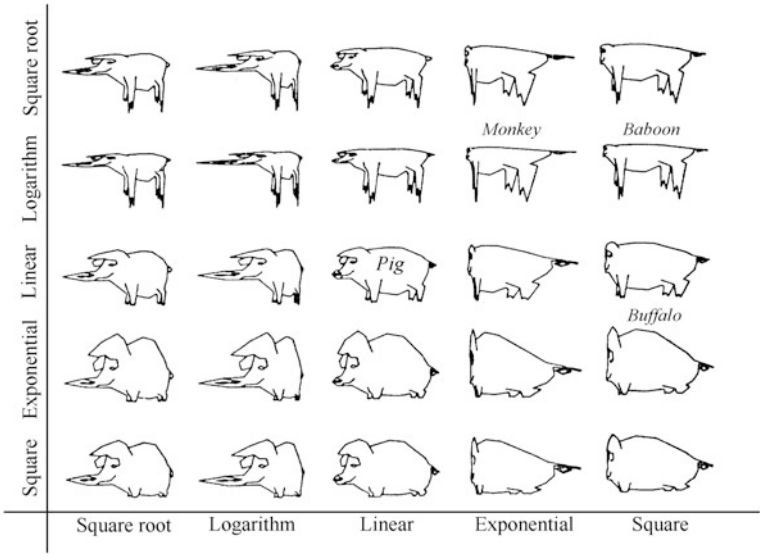


Fig. 12.11 The effect of different kinds of scales on a graph or mapping a pig (with acknowledgements to unknown author)

and the dependent variable is given by the corresponding distance from the center of the diagram.

An example of a polar plot is shown in Fig. 12.13, where the relative luminosity of an electric bulb has been drawn as a function of the direction, as this is given by the angle θ it forms with the axis of the bulb.

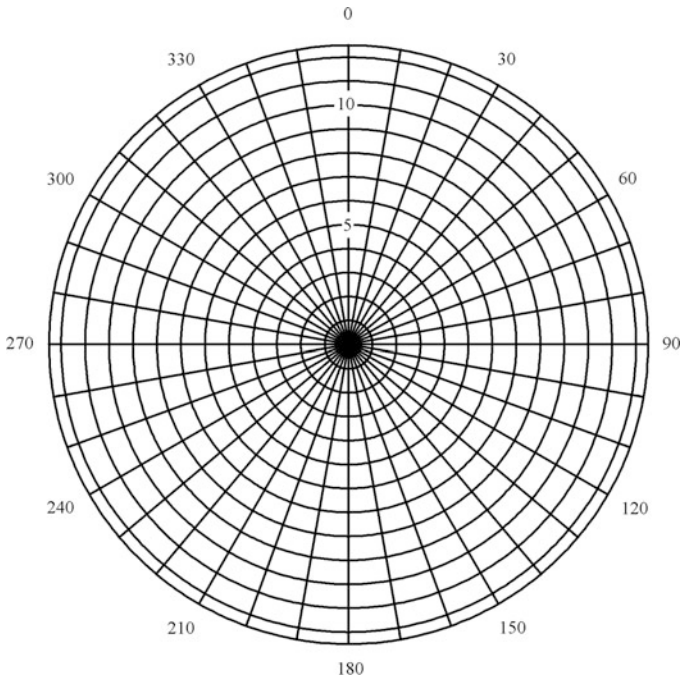


Fig. 12.12 The scales of a polar diagram

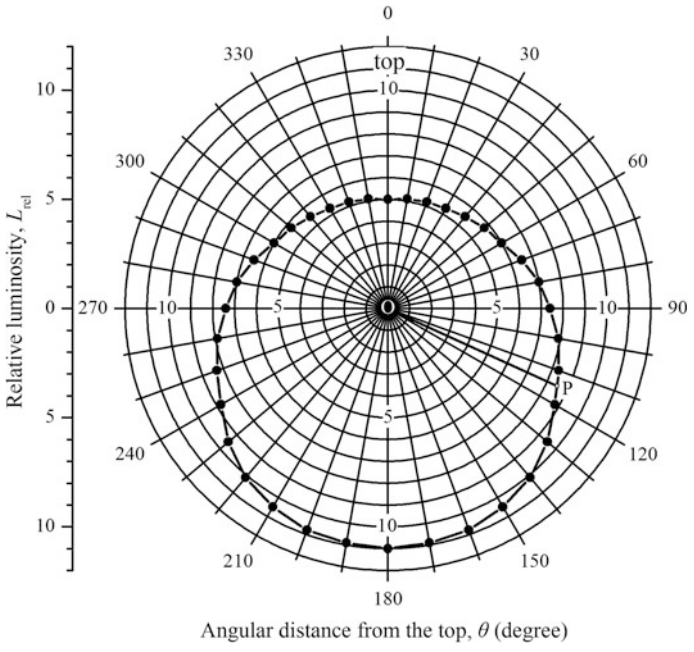


Fig. 12.13 A polar diagram of the relative luminosity of an electric bulb as a function of direction, as this is given by the angle θ it forms with the axis of the bulb. Point P has co-ordinates $(\theta = 115^{o},\ L_{rel} = 8.5)$

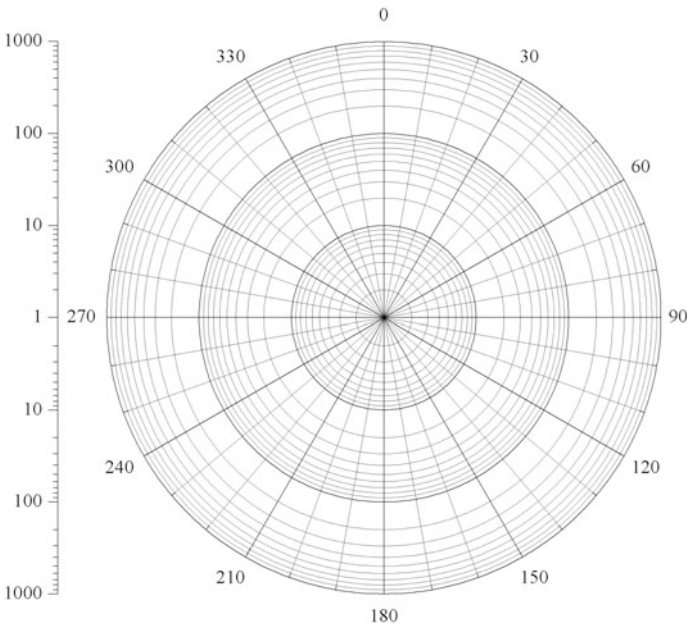


Fig. 12.14 Polar logarithmic graph paper

To give an example, point P shows that the relative luminosity of the bulb in a direction forming an angle of $\theta = 115^{\circ}$ with respect to the upward vertical is $L_{\text{rel}} = 8.5$.

Polar diagrams which have a logarithmic radial scale are sometimes used, in order to cover a wide range of values without losing the details at low values. An example of graph paper used in such cases, with 3 cycles of logarithmic scale, is shown in Fig. 12.14.

12.2.4 Other Matters Relating to the Axes and the Scales of Graphs

It is sometimes desirable to give in a graph a second scale, which has some mathematical relationship with the main scale and which gives additional information. For example, let us examine the Arrhenius equation

$$\tau = \tau_0 \, \mathrm{e}^{E/kT}, \tag{12.8}$$

where τ_0, E and k are constants and T is the absolute temperature. Then, since it is

$$\ln \tau = \ln \tau_0 + \left(\frac{E}{k}\right) \frac{1}{T}, \tag{12.9}$$

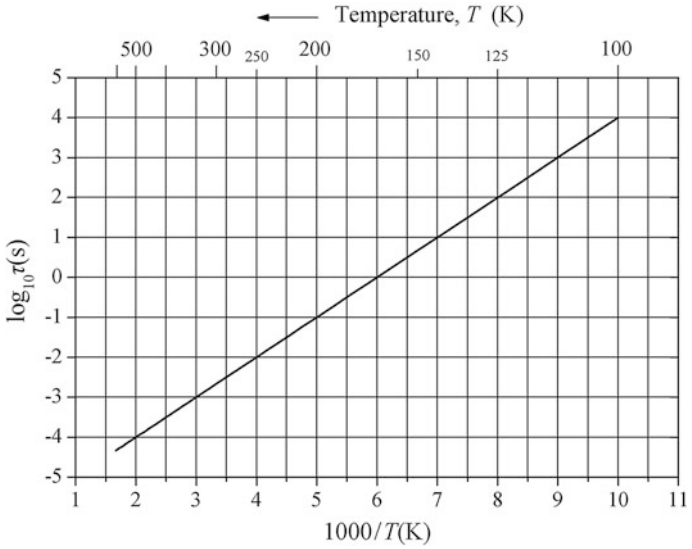


Fig. 12.15 The use of an additional auxiliary scale. Apart from the basic scale for $1/T$, the scale for the temperature values T is also given along the upper axis

if we plot log τ or ln τ as a function of $1/T$ we will have a straight line (Fig. 12.15). Apart from the main scale of $1/T$, it is useful to also have a scale for T, from which we can read directly the values of the absolute temperature. In Fig. 12.15 the scale of the temperature T is given along the top axis. This scale is not, of course, linear.

One more example of a graph with double axes which are mathematically related is shown in Fig. 12.16. Figure 12.17 shows an example where the second axis gives some specific values of the independent variable.

In some cases, when using linear scales, the values of the dependent variable may be too small in a certain region to be clearly visible. In such cases we re-draw the region of interest in a different scale. Next to this re-drawn section we give the factor by which the values had to be multiplied in order to reach their new level (e.g. $\times 10$, $\times 100$ etc.). Such an example is shown in Fig. 12.18, in which the second peak was re-drawn at a more sensitive scale in order to make visible details of the peak. As indicated in the figure, the values were re-drawn having been multiplied by 10. We must be careful here, as some researchers mark on the graph the factor by which the scale numbers must be multiplied in order to read a value off the graph.

If we wish to compare two different quantities, we may plot both in the same graph (Fig. 12.19). One axis (usually the x-axis) is common. In general, the other two axes are drawn, if they are different, on the left-hand-side and on the right-hand-side. As has been done in Fig. 12.19, the correspondence of curves and scales is shown by arrows. Alternatively, we might write next to the curves which one gives y_1 and which y_2.

More than one series of measurements may be presented in one graph, using different symbols for the points of each. In Fig. 12.20 the values of the specific heat

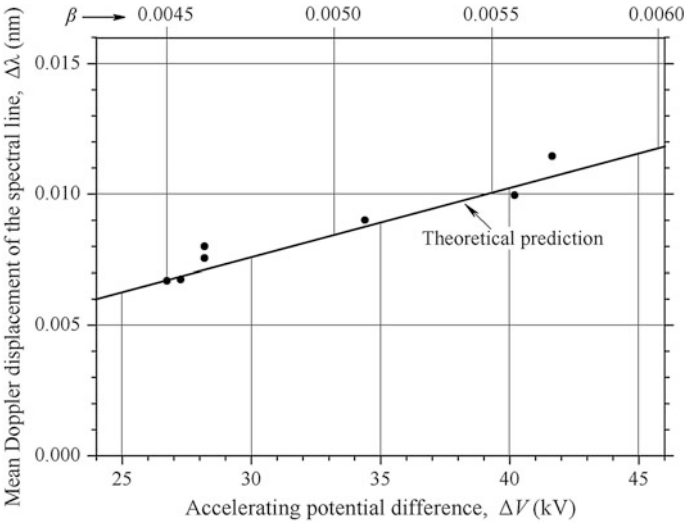


Fig. 12.16 The use of an additional auxiliary axis. β is the ratio of the particle's speed to the speed of light in vacuum, c

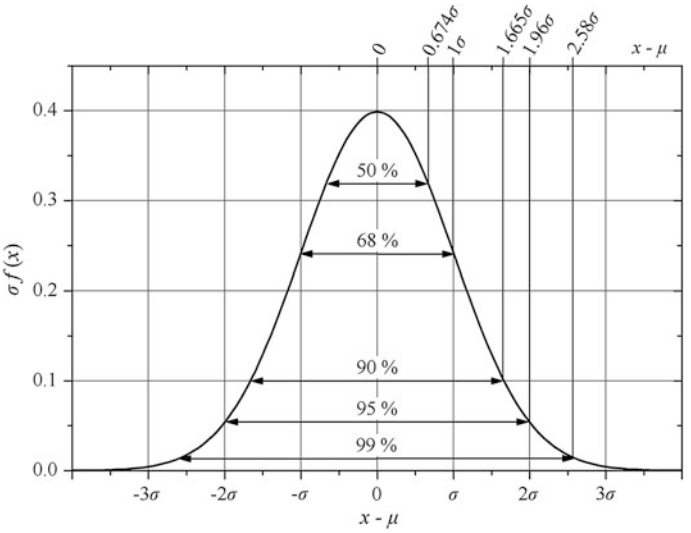


Fig. 12.17 The use of an additional auxiliary axis which gives some specific values of the independent variable

of four solids were drawn as functions of the reduced temperature T/θ_D, where θ_D is the Debye temperature of each solid. The symbol used for each solid is given in the text box on the graph. In this particular case when using as variable the reduced

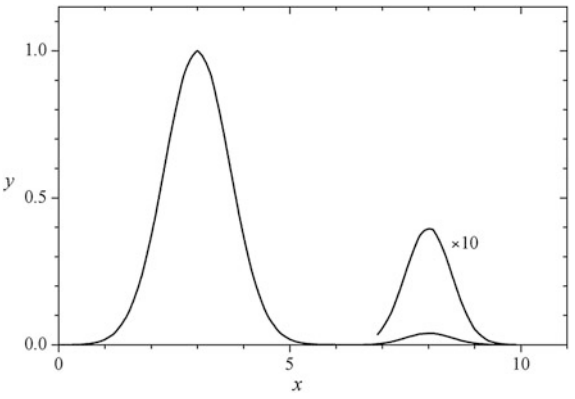


Fig. 12.18 Drawing a part of the graph in a different scale, to highlight possible details. The multiplying factor given should be the factor by which the values were multiplied in order to get the curve shown

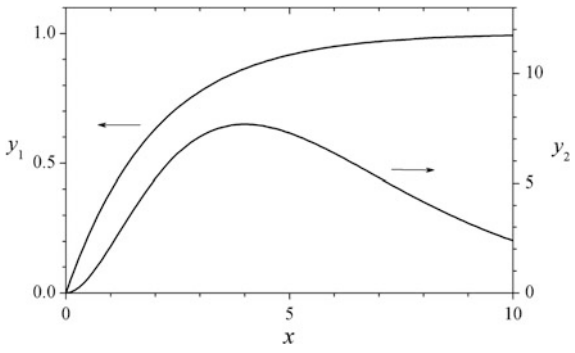


Fig. 12.19 Plotting, on the same graph, two different physical magnitudes, y_1 and y_2, having a common scale for the independent variable (x). The *arrows* point towards the scale to be used with each curve

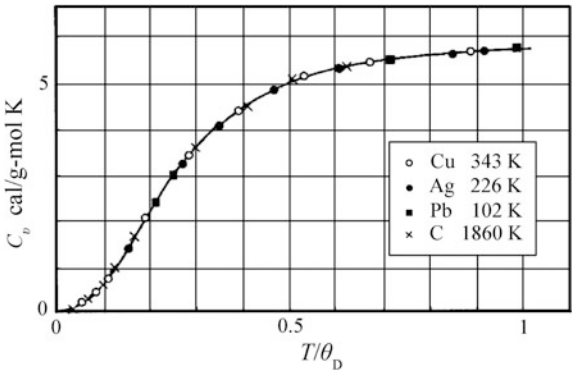


Fig. 12.20 The drawing of four series of measurements in one graph. The points give the results of measurements of the specific heat of four materials as a function of the absolute temperature T divided by the Debye temperature, θ_D, characteristic of each material. The distinction of the series is achieved by using different symbols for the points of each one of them. The values of the Debye temperature for each material are given in the box in the figure. The continuous curve shows the theoretical prediction of the Debye theory for the specific heats of solids. When C_v is plotted as a function of the reduced temperature T/θ_D, a curve results which is universal for all solids

temperature T/θ_D for each solid, where T is the absolute temperature and θ_D a temperature characteristic of the material (the Debye temperature) the values for all the materials fall on a common, universal, theoretical curve, which is derived in Debye's theory of the specific heats of solids. This curve has been drawn in the figure. In most cases, however, in which this drawing technique is used, different curves are obtained for different materials or for different experimental conditions, if these vary from one series of measurements to another.

12.2.5 Legends of the Figure, Labels of the Axes and the Units of Physical Magnitudes

As a rule, the figure should convey as much information as possible by itself, without the reader having to resort to the text. The contents of the graph, together with its legend, should be enough to enable the reader to understand what the figure describes. The legend of a figure should be informative without being of excessive size, a rule which was probably not followed in Fig. 12.20. Of course, for the details, the text is indispensable.

Each axis of a graph should be clearly labeled with the magnitude it records, its symbol and its units. If the name is too long to be written along the axis, a suitable symbol may be used, which should be clearly explained in the legend. Examples are the following:

Increasing number of the measurement, i
Distance, s (m)
Speed, υ (m/s)
Electric current, I (μA)
Temperature gradient, $\mathrm{d}T/\mathrm{d}x$ (K/m)
Thermal conductivity, κ (W/m·K)
Thermal conductivity/Electrical conductivity, κ/σ (W·Ω/K)
Differential cross-section, $\mathrm{d}\sigma/\mathrm{d}\,\Omega$ (mbarn/sterad)
or just symbols: E/k (K) $2Nh^2d/\kappa r_0$ (J·K/(s m^4)) or $2Nh^2d/\kappa r_0$ (S.I.)

There are different opinions regarding the way in which the units should be given. It is of course a rule that the units are written in upright letters (m, s, K, km, W, V,...), as opposed to the variables, which are symbolized by italics $(x, l, m, I, E, V, dx/dt, \ldots)$. There is also the point of view that fraction slashes (solidi) should not be used in stating the units. It is suggested that m s^{-1} should be written instead of m/s, N m^2 kg^{-2} instead of N m^2 kg^2 etc. The reason for this is to avoid confusion when many fraction slashes appear without it being clear which quantity is divided each time. For example, by the expression N m/s/K/kg it is not clear that it is meant N m/(s/K/kg) or (N m/s)/(K/kg) or N (m/s/K)/kg or something else. The use of fraction slashes is accepted, however, if such uncertainties are

avoided by the suitable use of parentheses, as, for example, in the case of W/(K/m). The problem also disappears if we take care for only one slash to appear, as, for example, in the case of W m/K. The S.I. system of units has already been presented in Chap. 5.

According to the Symbols Committee of the Royal Society [1], the symbols and their units should be written, in tables and graphs, as dimensionless numbers:

$$\text{Magnitude}/\text{Units}.$$

For example, a column of a table or the axis of a graph giving the temperature in degrees Kelvin must be labeled as T/K. Thus, the number 400 in a table or a graph means $T/\text{K} = 400$, or $T = 400$ K. This symbolism, however, presents a problem when we are dealing with complex quantities. For example, if (God forbid!) the magnitude

$$(\text{Stefan-Boltzmann's constant}) \times (\text{electric conductivity})$$

should arise in some theoretical work and we express the Stefan-Boltzmann constant in terms of universal constants, together with their units, we would have the expression

$$(\pi^2/60)k^4\hbar^{-3}c^2\sigma_\eta/\text{W m}^{-3}\ \text{K}^{-4}\,\Omega^{-1}$$

which might cause some confusion.

Different scientific journals have adopted different notations regarding the presentation of units. In this book, for tables and graphs, we prefer to write the physical magnitude and its units separately, with the units given in parentheses, as in the examples given above.

12.3 The Points

A point (x, y) is marked in a graph using a symbol, such as the ones shown in Fig. 12.21.

The symbol must be large enough to be clearly visible. The same symbol is used for all the points corresponding to the same series of measurements, unless we wish to separate a point from the others due to some reason. If there are more than one series of measurements and their points are not sufficiently separated in the graph, we use different symbols for the different series of measurements (see, for example, Fig. 12.20).

If the errors $\delta x = \sigma_{\bar{x}}$ and $\delta y = \sigma_{\bar{y}}$ in the values of x and y, respectively, are known, these are marked on the graph, as shown in Fig. 12.22. For the point on the left in Fig. 12.22, only the value of y has an error. If the error in y is $\delta y = \sigma_{\bar{y}}$, then we draw a vertical line which stretches from the point (x, y) up and down to the points $(x, y \pm \delta y)$. For the point on the right in the figure, for which there are errors both in x and in y, the same procedure is followed for the error in x. Drawing the errors in the graph is important as it affects the form of the curve we will draw in order to describe the mathematical relationship between x and y, which results from the experimental values. Two examples of graphs drawn with errors in the measurements (x, y) are shown in Fig. 12.23.

Fig. 12.21 Some of the symbols commonly used for denoting points in graphs

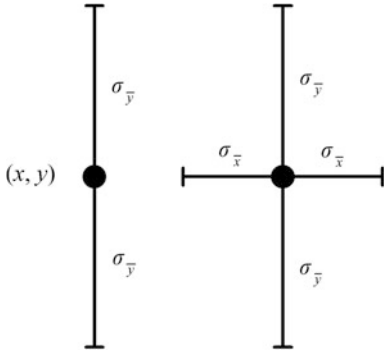


Fig. 12.22 Denoting the errors $\delta x = \sigma_{\bar{x}}$ and $\delta y = \sigma_{\bar{y}}$ in the values of x and y in a graph. In the case of the point on the *left*, there is an error only in y, while in the case of the point on the *right* there are errors in both x and y

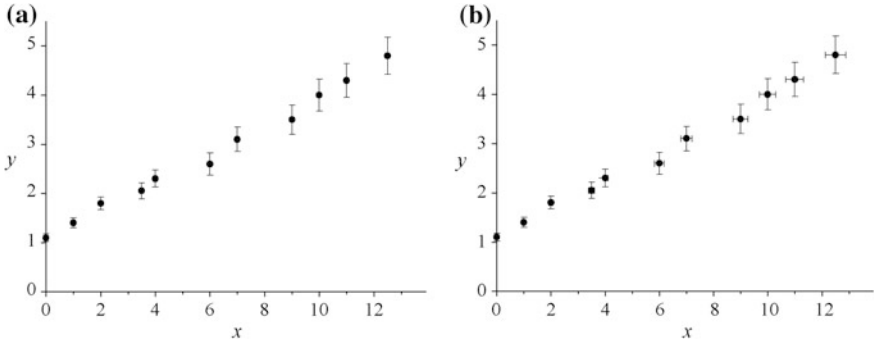


Fig. 12.23 Two graphs on which the errors in the values of x and y are indicated. **a** Errors exist only for the values of y, **b** errors exist for the values of both x and y

12.4 The Curve

Having chosen the axes and marked all the experimental points $(x_i,\ y_i)$ with their errors in the diagram, we wish to draw the best curve describing the relation $y(x)$. The tern '*best curve*' does not have a unique interpretation. Experience has taught us that mathematical simplicity is one criterion. This, however, cannot mean that a straight line is always preferable to a parabola and so on. The existence of a theoretical prediction for the particular experiment is usually the best way out of the difficulty. Even then, however, the possibility exists that the theoretical curve is not followed exactly by the experimental points due to errors. For example, a straight line may not pass through the origin, although theory predicts that it should do so. Summarizing, we may say that the main factors which determine our choice of the mathematical expression best describing a series of measurements are:

1. The relation 'suggested' by the positions of the points, taking into account their errors.
2. A theoretically predicted mathematical relation.
3. The simplicity of the mathematical relation.
4. Our experience from similar cases.

We will examine below some issues related to this procedure.

At the start, it should be stressed that the curve does not necessarily have to pass through all the experimental points (nor even through any of them). This is a frequent mistake. Two examples of this erroneous practice are shown in Fig. 12.24. Although the picture presented in the figure in not always impossible to be true, in the cases when something like this happens, it should be adequately documented. An example, from outside the physical sciences, is the daily variation of a stock exchange index. In this case, values between the points have no meaning and the connection of adjacent points with straight lines is justified, in order that the succession of values should be clear. In Physics or Chemistry, the recording of the values of a particular property of the elements as a function of their atomic number, for example, would justify a graph such as that of Fig. 12.24a. The straight lines help us follow the succession of points. In general, if the independent variable is quantized, a plot such as the one in Fig. 12.24a is usually justified.

In general, the curve adjusted to a series of measurements must be as smooth as possible. This can be seen as another application of Occam's razor. Deviation from a straight line or a smooth curve is justified only if there are an adequate number of reliable experimental points in the region of the deviation, taking into account the magnitude of the errors. Whenever possible, the curve should pass between the limits of the errors (see, for example, Fig. 12.25), always remembering that deviations from the real value by one or even more standard deviations are not rare. On the other hand, we should bear in mind that a point differing by more than about two standard deviations from the curve should be investigated, in order to decide whether the point is acceptable or it should be rejected.

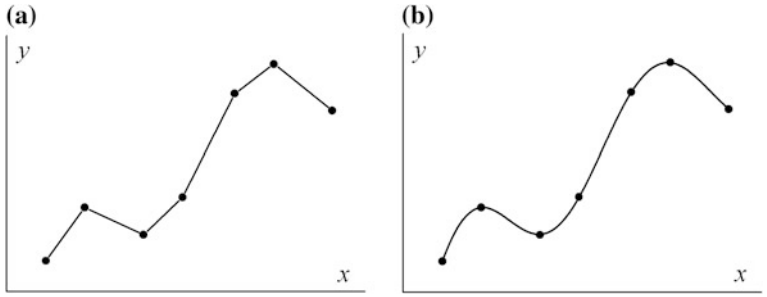


Fig. 12.24 Two examples of the wrong plotting of the curve between the experimental points, when the independent variable is continuous

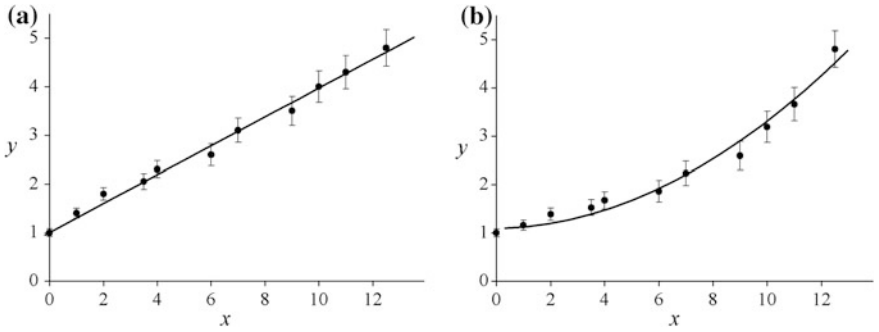


Fig. 12.25 The adjustment of a curve to a number of experimental points. **a** We see that, taking into account the errors in the values of y, a straight line adequately describes the relation $y(x)$. **b** In this example, a second degree curve is necessary for the description of $y(x)$

In Fig. 12.25a, a straight line expresses the relation $y(x)$ to a satisfactory degree. In Fig. 12.25b, however, given the systematic behavior of the points as well as the magnitudes of the errors in y, a straight line is not satisfactory and a curve of the second degree (parabola) is required.

Theory suggests that the best curve through the experimental points should cross, on average, only 2 out of 5 error bars. To demonstrate this, we apply the results of Example 9.5 to the points of graph in Fig. 12.23b. We draw ellipses with semi-axes equal to $\sigma_{\bar{x}}$ and $\sigma_{\bar{y}}$, and centers at the corresponding points $(\bar{x}, \bar{y})$ (Fig. 9.26a). The probability of the real point (x, y_0) lying within the corresponding ellipse is 39%. With this in mind, we expect the straight line drawn between the 11 experimental points (Fig. 9.26b) to pass through 4 of the ellipses. In fact, it cuts 8. This is evidence that the straight line fit is a good one. We also found that 87% of the ellipses with semi-axes equal to $2\sigma_{\bar{x}}$ and $2\sigma_{\bar{y}}$ should be intersected by the straight line. Here, all $2\sigma_{\bar{x}} - 2\sigma_{\bar{y}}$ ellipses are crossed by the line (Fig. 12.26).

In programs used in personal computers nowadays, there are large libraries of mathematical functions which may be fitted to the experimental results. Apart from

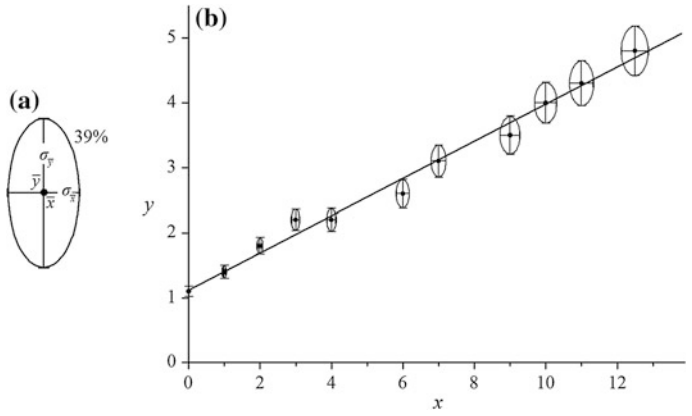


Fig. 12.26 **a** The ellipse with center an experimental point and semi-axes equal to the uncertainties $\delta x = \sigma_{\bar{x}}$ and $\delta y = \sigma_{\bar{y}}$ in $\bar{x}$ and $\bar{y}$ respectively. The real point corresponding to x and y has a 39% probability to lie within this ellipse. **b** The straight line fitted to the experimental points is expected to cut 39% or approximately 4 of the 11 ellipses such as that of **a** corresponding to the experimental points. Instead it cuts 8, a fact that must suggest that the linear fit is a very good one

the common mathematical functions such as polynomials of various degrees, exponential functions, trigonometric functions etc. and their combinations, there are specialized functions which are used in specific branches of science, such as Statistics, optical, dielectric and gamma-ray spectroscopy etc. The fast and expedient fitting of the best curve to the experimental points is, therefore, a matter of experience, mainly in the use of the suitable programs. Of course, the thing that cannot be done efficiently by any other method is the preliminary examination of the point 'by eye' in order to check whether there is a problem with some of the points, which has to be resolved before a curve is fitted.

Example 12.1 [E]

Graph plotting with Excel®

i	t_i (s)	z_i (m)	δt_i (s)	δz_i (m)
1	0	0	0.1	0
2	5	12	0.2	0.5
3	10	55	0.3	4
4	15	100	0.25	8
5	20	200	0.3	10
6	25	305	0.5	15
7	30	380	0.5	15
8	35	430	0.6	20
9	40	485	0.7	25
10	45	490	0.7	30

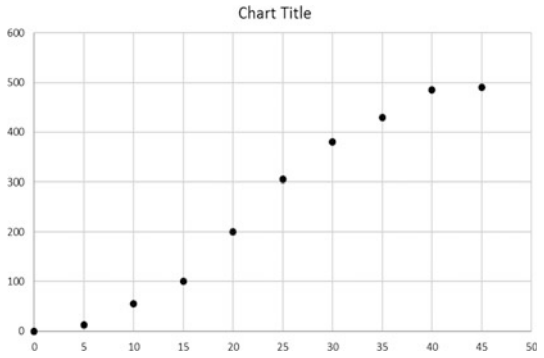
A table is given, containing the pairs of experimental results t_i and z_i of a relation $z(t)$ between the position of a particle and time, together with their respective errors, δt_i and δz_i.

Create a scatter plot of $z(t)$. Format the graph. Find the best polynomial curve $z(t)$ that fits the experimental points.

Differentiate the curve found, $z(t)$, to obtain the velocity of the particle, $\upsilon(t)$. Differentiate this curve, $\upsilon(t)$, to obtain the acceleration of the particle, $a(t)$. Plot the curves $\upsilon(t)$ and $a(t)$ in the graph of $z(t)$.

We enter the data t_i, $z(t)$, δt_i and δz_i in columns A, B, C and D, respectively. Highlight columns A and B. Open the **Insert** window and from **Charts** select **Insert Scatter (X, Y) or Bubble Chart**. A graph now appears, which is shown in the figure below. We will format this graph.

Pressing the ⊞ key that appears when we click on the top right hand side of the graph's box opens the **Chart Elements** dialog box. We choose



Error Bars > More Options > Format Error Bars > Error Bar Options

In **Vertical Bar Options**, we select **Direction: Both**, **End Style: Cap**, **Error Amount: Custom**. Then,

Specify Values > Positive Error Values and **Negative Error Values** type **= Sheet1!$D$4:$D$13**

Press **OK**. The vertical error bars appear in the figure at each point.

Ticking on a horizontal error bar opens a dialog box in which we select **Direction: Both**, **End Style: Cap**, **Error Amount: Custom**. Then,

Specify Values > Positive Error Values and **Negative Error Values** type **= Sheet1!$C$4:$C$13**

Press **OK**. The horizontal error bars also appear in the figure at each point.

We will fit a polynomial curve to the points.

Pressing the ⊞ key opens the **Chart Elements** dialog box. We choose

Error Bars > More Options > Trent Line Options > Polynomial: Order 4 and **Display Equation on chart**

The equation of the curve fitted is:

$$z(t) = 3.3741 - 4.355\,t + 1.0543\,t^2 - 0.0177\,t^3 + 4 \times 10^{-5}\,t^4 \text{ (in m for } t \text{ in s)}.$$

Double-clicking on the curve opens the **Format Trendline**, **Trendline Options** window. For **Line**, we choose: **Solid Line**, **Color: Black**, **Width: 1 pt**, **Dash Line: solid line**.

We will now plot the curves for $v(t)$ and $a(t)$ in the graph.

Differentiating with respect to time, we have the speed

$$v(t) = -4.355 + 2.1086\,t - 0.0531\,t^2 + 1.6 \times 10^{-4}\,t^3 \text{ (in m/s for } t \text{ in s)}.$$

Differentiating with respect to time, we have the acceleration

$$a(t) = 2.1086 - 0.1062\,t + 4.8 \times 10^{-4}\,t^2 \left(\text{in m/s}^2 \text{for } t \text{ in s}\right).$$

We label an empty column, say E(Y) as **v (m/s)**. In cell E4 we type

(−4.355 + 2.1086 * A1 − 0.0531 * A1^2 + 0.00016 * A1^3) * 10

and press **ENTER**. The factor of 10 is used since we are going to plot $z(t)$, $v(t)$ and $a(t)$ in the same graph, using a common axis, and we want to do this using comparable numbers, for easier readability. We **Fill Down** from E4 to E13.These cells now contain the values of $10v(t)$.

We label an empty column, say F(Y) as **a (m/s^2)**. In cell F4 we type

(2.1086 − 0.1062 * A1 + 0.00048 * A1^2) * 100

and press **ENTER**. The factor 100 serves the same purpose as the factor 10 for the velocity, as explained above. We **Fill Down** from F4 to F13. These cells now contain the values of $100a(t)$.

We highlight cells A4 to A13 and E4 to E13. In **Insert**, **Charts** we chose the option smooth line plot without points. This produces a graph of $v(t)$.

We highlight cells A4 to A13 and F4 to F13. In **Insert**, **Charts** we chose the option smooth line plot without points. This produces a graph of $a(t)$.

We **Cut** the graph $v(t)$ and **Paste** it on the graph $a(t)$. We then **Cut** the graph $z(t)$ and **Paste** it on the graph of $a(t)$ and $v(t)$.

We format the graph containing $z(t)$, $v(t)$ and $a(t)$ by changing the colors to black etc., as described below. In order to write something on the plot, we open **Insert** and insert a **Text Box**. We write the text in the box and then move it to the appropriate position.

We write z, $10v$ and $100a$ near the corresponding curves, in order to identify them.

We click and select the numbers on the X-axis opening **Chart Tools**. In **Format** we open Text Fill and select **black**. We do the same with the Y-scale.

We click anywhere in the area of the numbers of the X-axis. This opens **Chart Tools**. **Open** the **Format** window. In the top left corner of the screen select **Horizontal (Value) Axis**. This opens the **Format Axis** window for the horizontal axis. In **Line** select **Solid Line**, **Color** black, **Width** 0.75 pt. In the same window,

click the icon showing a histogram. Open **Axis Options** and select **Bounds**, **Minimum** 0 and **Maximum** 50. For **Units**, we select **Major** 5 and **Minor** 1. Open **Tick Marks** and select **Major Type**, **Outside** and **Minor Type**, **Outside**.

Repeat the same procedure for the **Vertical (Value) Axis**. In this case select **Bounds**, **Minimum** −200 and **Maximum** 600. For **Units**, we select **Major** 100 and **Minor** 20.

We click anywhere in the plot area. This opens **Chart Tools**. **Open** the **Format** window. In the top left corner of the screen select **Horizontal (Value) Axis Major Gridlines**. This opens the **Format Axis** window for the horizontal major gridlines. Select **Solid Line**, **Color** black and **Width** 0.75 pt. We now open the **Horizontal (Value) Axis Minor Gridlines**. This opens the **Format Axis** window for the horizontal minor gridlines. Select **Solid Line**, **Color** gray and **Width** 0.5 pt.

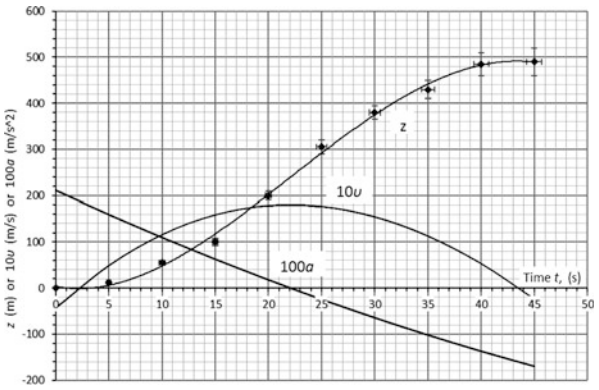
We repeat the same procedure for the vertical gridlines.

To white out the background of lettering so that they are easy to read, we double-click on the area of the X-axis numbers. This opens the **Format Axis** window. Clicking on the first icon, we select **Fill**, **Pattern Fill**, **Foreground** white and **Background** white.

To white out the background of text in the plot area, we right-click on the text. In the window that opens we select **Format Object**. Clicking on the first icon, we select **Fill**, **Pattern Fill**, **Foreground** white and **Background** white.

To remove the border line around the plot, we right-click on the plot area and open the **Format Chart Area** window. We click on the first icon in **Chart Options**. We select **No Line**.

The final result is shown in the figure below.



Example 12.2 [O]

Graph plotting with Origin®

A table is given, containing the pairs of experimental results t_i and z_i of a relation $z(t)$ between the position of a particle and time, together with their respective errors, δt_i and δz_i.

i	t_i (s)	z_i (m)	δt_i (s)	δz_i (m)
1	0	0	0.1	0
2	5	12	0.2	0.5
3	10	55	0.3	4
4	15	100	0.25	8
5	20	200	0.3	10
6	25	305	0.5	15
7	30	380	0.5	15
8	35	430	0.6	20
9	40	485	0.7	25
10	45	490	0.7	30

Create a scatter plot of $z(t)$. Format the graph. Find the best polynomial curve $z(t)$ that fits the experimental points.

Differentiate the curve found, $z(t)$, to obtain the velocity of the particle, $v(t)$. Differentiate this curve, $v(t)$, to obtain the acceleration of the particle, $a(t)$. Plot the curves $v(t)$ and $a(t)$ in the graph of $z(t)$.

We enter the data t_i, $z(t)$, δt_i and δz_i in columns A(X), B(Y), C(Y) and D(Y), respectively. Right-click on C(Y). Then

Set As > X Error

The label of the column now becomes C(xEr±). This indicates that the column contains the errors in the values of X, i.e. in t.

Right-click on D(Y). Then

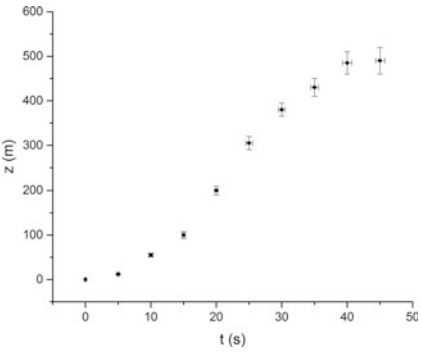
Set As > Y Error

The label of the column now becomes D(yEr±). This indicates that the column contains the errors in the values of Y, i.e. in z.

Highlight columns A, B, C and D. Then

Plot > Symbol > Scatter

The plot shown in the figure below appears.



We will format this graph.

We will fit a polynomial curve to the points. While in the plot environment, we follow the path

Analysis > Fitting > Nonlinear Curve Fit > Open Dialog…

In the window that opens, we select

Settings > Function Selection > Category set to **Polynomial > Function** set to **Poly4**

This will fit a polynomial of the fourth degree to the experimental points. Press **Fit**. The fitted curve appears on the graph. The equation of the curve fitted is:

$$z(t) = 3.34 - 4.312\,t + 1.0437\,t^2 - 0.01756\,t^3 + 3.946 \times 10^{-5}\,t^4 \ (\text{in m for } t \text{ in s}).$$

Differentiating with respect to time, we have the speed

$$v(t) = -4.312\ + 2.0874\,t - 0.05268\,t^2 + 1.3784 \times 10^{-4}\,t^3 \ (\text{in m/s for } t \text{ in s}).$$

Differentiating with respect to time, we have the acceleration

$$a(t) = 2.0874 - 0.10536\,t + 4.135 \times 10^{-4}\,t^2 \ \left(\text{in m/s}^2 \text{for } t \text{ in s}\right).$$

We will now plot the curves for $v(t)$ and $a(t)$ in the graph.

In the data sheet (Book1) we highlight an empty column, say E(Y), which we label as **v (m/s)**. Then

Column > Set Column Values

and in the dialog box that opens we type

−4.312 + 2.0874 * col(A) − 0.05268 * (col(A))^2 + 0.00013784 * (col(A))^3

Pressing **OK** fills column E with the values of $v(t)$.

In the data sheet (Book1) we highlight an empty column, say F(Y), which we label as **a (m/s)**. Then

Column > Set Column Values

and in the dialog box that opens we type

2.0874 − 0.10536 * col(A) + 0.0004135 * (col(A))^2

Pressing **OK** fills column F with the values of $a(t)$.

We will plot these values in the graph of $z(t)$. Before we do so, however, we want to bring the values to be plotted in the same range as those of $z(t)$. To achieve this, we fill column G with the values of $10v(t)$ and column H with the values of $100a(t)$.

Returning to the graph environment (**Window, Graph1**), we right-click on the number at the top left hand side of the page and then we click on **Layer Contents…** . In the window that opens, we highlight the line corresponding to column G by clicking on any point of the line. Then, pressing the arrow ⟶, we include column G in the list on the right, which shows the columns plotted in the graph. We repeat for column H. Then press **Plot Setup…** . The graph appears, now containing the scatter plots of $10v(t)$ and $100a(t)$.

We will change the plots of $10v(t)$ and $100a(t)$ from scatter to line plots. To do this we double-click on one of the points, opening the **Plot Details—Plot Properties** and in the **Plot Type** select **Line**. We do this for both the plots $10v(t)$ and $100a(t)$.

We double-click on each of the three curves in turn and change the line width to 1.5 and the color to black.

To change the ranges of the scales, we double-click on one of the axes. In the window that opens we set the **Horizontal** scale **From** 0 **To** 50 and the **Vertical** scale **From** −200 **To** 600.

In the same window, we open **Tick Labels**. For the **Left** and **Bottom** axes we tick the **Show** box.

In the same window, we open **Grids**. For **Vertical**, **Major Grid Lines**, tick the **Show** box and select **Color** to be **Black**, **Style** to be **Solid** and **Thickness** to be 0.5. For **Minor Grid Lines**, we tick the **Show** box and select **Color** to be **Black**, **Style** to be **Solid** and **Thickness** to be 0.3. In Additional Lines tick Y = 0. For **Horizontal**, **Major Grid Lines**, we tick the **Show** box and select **Color** to be **Black**, **Style** to be **Solid** and **Thickness** to be 0.5. For **Minor Grid Lines**, we tick the **Show** box and select **Color** to be **Black**, **Style** to be **Solid** and **Thickness** to be 0.3. In **Additional Lines** tick Y = 0.

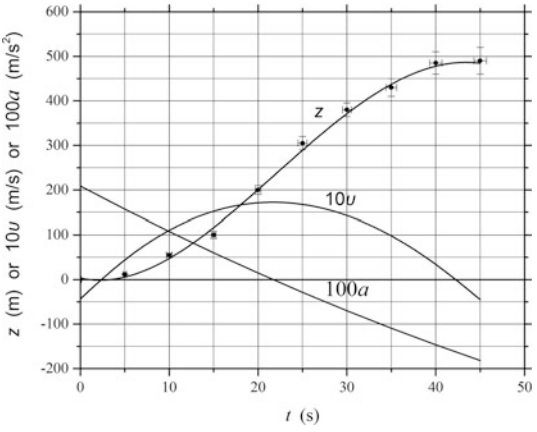
In the same window, we open **Lines and Ticks**.

For **Bottom**, we tick the **Show Line and Ticks** box and select: for **Line** we tick the **Show** box, **Color** to be **Black**, **Thickness 1.5** and **Axis Position Bottom**. For **Major Ticks** we select **Out**. For **Minor Ticks** we select **Out**. For **Top** we tick the **Show Line and Ticks** box and the **Use Same Options for Bottom and Top** box. For **Left** we use same settings as **Bottom**. For **Right** we tick the **Show Line and Ticks** box and the **Use Same Options for Left and Right** box. Press **OK**.

We double click on each of the three lines in turn and set for each **Line**: **Connect** to be **Straight**, **Style** to be **Solid**, **Width** to be **1** and **Color** to be **Black**.

We double click on the X label and write ***t* (s)**. We double click on the Y axis and write ***z* (m) or 10*v* (m/s) or 100*a* (m/s^2)** . Identify the three curves by writing ***z***, **10*v*** and **100*a*** near the corresponding curve.

The final result is shown in the figure below.



Example 12.3 [P]

Graph plotting with Python

A table is given, containing the pairs of experimental results t_i and z_i of a relation $z(t)$ between the position of a particle and time, together with their respective errors, δt_i and δz_i.

Create a scatter plot of $z(t)$. Format the graph. Find the best polynomial curve $z(t)$ that fits the experimental points.

Differentiate the curve found, $z(t)$, to obtain the velocity of the particle, $v(t)$. Differentiate this curve, $v(t)$, to obtain the acceleration of the particle, $a(t)$. Plot the curves $v(t)$ and $a(t)$ in the graph of $z(t)$.

i	t_i (s)	z_i (m)	δt_i (s)	δz_i (m)
1	0	0	0.1	0
2	5	12	0.2	0.5
3	10	55	0.3	4
4	15	100	0.25	8
5	20	200	0.3	10
6	25	305	0.5	15
7	30	380	0.5	15
8	35	430	0.6	20
9	40	485	0.7	25
10	45	490	0.7	30

As usual, we will import the numpy and matplotlib modules, and then entering the experimental data into vectors t and z, and the corresponding error values into vectors errt and errz.

```
import numpy as np
import matplotlib.pyplot as plt
t = np.array([0, 5, 10, 15, 20, 25, 30, 35, 40, 45])
z = np.array([0, 12, 55, 100, 200, 305, 380, 430, 485, 490])
errt = np.array([0.1, 0.2, 0.3, 0.25, 0.3, 0.5, 0.5, 0.6, 0.7, 0.7])
errz = np.array([0, 0.5, 4, 8, 10, 15, 15, 20, 25, 30])
```

We use the errorbar function of matplotlib to produce a scatter plot with error bars:

```
plt.errorbar(t, z, xerr=errt, yerr=errz, fmt='o', color='b')
plt.xlim(0, 50)
plt.ylim(-200, 600)
plt.xlabel("t (s)")
plt.ylabel("z (m) or 10v (m/s) or 100a (m/s^2)"
plt.grid(True)
```

The fmt = 'o' option indicates that the experimental points will be drawn as small circles on the scatter plot. Other available markers include 's' for a square, '*' for a star, 'D' for a diamond, and '1', '2', '3', '4' for a triangle (down-, up-, left- or right-oriented respectively). A complete list of markers can be found on http://matplotlib.org/api/markers_api.html. These markers can also be used with the plot and scatter commands. The color parameter sets the colour of the points: 'b' is shorthand for blue, 'r' for red, 'g' for green, 'k' for black etc. A complete list of colours available for use in matplotlib graphs can be found on http://matplotlib.org/api/colors_api.html.

We then use the least-squares method to fit a fourth degree polynomial to the experimental data, as follows:

```
fit = np.polyfit(t, z, 4)
p = np.poly1d(fit)
```

In order to draw this polynomial as a curve, we create a series of 200 points between min(t) and max(t) using the linspace command from numpy, and then use the plot command. The third parameter ('-') to the plot command indicates that the points should be linked (to form a smooth curve).

```
xp = np.linspace(min(t), max(t), 200)
plt.plot(xp, p(xp), '-', color="red")
```

The numpy polynomial object supports the deriv function that calculates derivatives of the polynomial. We store the velocity (first derivative) and acceleration (second derivative) polynomials in objects v and a as follows:

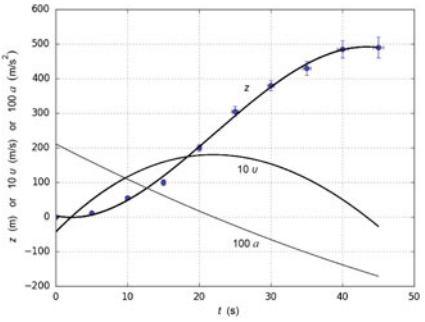
```
v = p.deriv(1)
a = p.deriv(2)
```

Using the same method as above, we can plot $10v$ and $100a$ on the graph, using the following commands:

```
plt.plot(xp, 10*v(xp), '-', color="blue")
plt.plot(xp, 100*a(xp), '-', color="black")
```

Our graph is ready: to see it on the screen, or export it as an image file, we issue the show() command:

```
plt.show()
```



Example 12.4 [R]

Graph plotting with R

A table is given, containing the pairs of experimental results t_i and z_i of a relation $z(t)$ between the position of a particle and time, together with their respective errors, δt_i and δz_i.

i	t_i (s)	z_i (m)	δt_i (s)	δz_i (m)
1	0	0	0.1	0
2	5	12	0.2	0.5
3	10	55	0.3	4
4	15	100	0.25	8
5	20	200	0.3	10
6	25	305	0.5	15
7	30	380	0.5	15
8	35	430	0.6	20
9	40	485	0.7	25
10	45	490	0.7	30

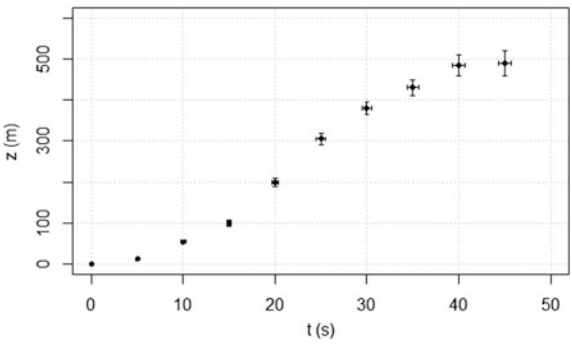
Create a scatter plot of $z(t)$. Format the graph. Find the best polynomial curve $z(t)$ that fits the experimental points.

Differentiate the curve found, $z(t)$, to obtain the velocity of the particle, $v(t)$. Differentiate this curve, $v(t)$, to obtain the acceleration of the particle, $a(t)$. Plot the curves $v(t)$ and $a(t)$ in the graph of $z(t)$.

We first create the scatter plot of the experimental points. We enter the vectors for t, z and their errors:

```
# t and z vectors
t <- c(0, 5, 10, 15, 20, 25, 30, 35, 40, 45)
z <- c(0, 12, 55, 100, 200, 305, 380, 430, 485, 490)
```

```
# errors in t and z
errt <- c(0.1, 0.2, 0.3, 0.25, 0.3, 0.5, 0.5, 0.6, 0.7, 0.7)
errz <- c(0, 0.5, 4, 8, 10, 15, 15, 20, 25, 30)

#scatter plot of data with x and y axes labels, lengths and grid
plot(t, z, pch=20, xlab="t (s)", ylab="z (m)", xlim=c(0, 50),
ylim=c(0, 600), grid())

# add t error bars
arrows(t-errt, z, t+errt, z, length=0.02, angle=90, code=3)

# add z error bars
arrows(t, z-errz, t, z+errz, length=0.02, angle=90, code=3)
# ENTER returns the scatter plot
```



We fit to the points a least-squares polynomial of the fourth degree:

```
# Least-squares curve fit
> nls(z~a0+a1*t+a2*t^2+a3*t^3+a4*t^4)
Nonlinear regression model
 model: z~a0+a1*t+a2*t^2+a3*t^3+a4*t^4
  data: parent.frame()
     a0      a1      a2      a3      a4
3.374e+00 -4.355e+00  1.054e+00 -1.774e-02  3.986e-05
 residual sum-of-squares: 751.5
Number of iterations to convergence: 1
Achieved convergence tolerance: 5.72e-07
```

The equation of the curve fitted is:

$$z(t) = 3.374 - 4.355\,t + 1.054\,t^2 - 0.01774\,t^3 + 3.986 \times 10^{-5}\,t^4 \,(\text{in m for } t \text{ in s}).$$

Differentiating with respect to time, we have the speed

$$\upsilon(t) = -4.355 + 2.108\,t - 0.05322\,t^2 + 1.5944 \times 10^{-4}\,t^3 \,(\text{in m/s for } t \text{ in s}).$$

Differentiating with respect to time, we have the acceleration

$$a(t) = 2.108 - 0.10644\,t + 4.7832 \times 10^{-4}\,t^2 \left(\text{in m/s}^2 \text{for } t \text{ in s}\right).$$

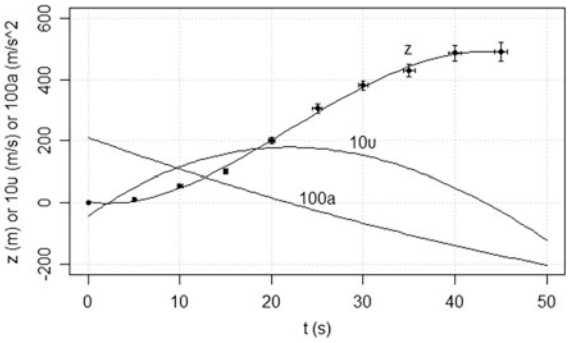
We will now plot the curves for $\upsilon(t)$ and $a(t)$ in the graph. We first re-plot the curve $z(t)$ with the z-axis taking values between -200 and 600. To enable easy reading, we will plot 10υ and $100a$ on the graph. We also change the Y-axis label to 'z (m) or 10υ (m/s) or $100a$ (m/s^2)'.

```
# Plot the curve z(t):
curve(3.374-4.355*x+1.054*x^2-0.01774*x^3
+3.986e-05*x^4, from=0, to=50, add=T)
# add the curves for 10υ and 100a
curve(-43.55+21.08*x-0.5322*x^2+0.001594*x^3, from=0, to=50, add=T)
curve(210.8-10.644*x+0.04783*x^2, from=0, to=50, add=T)
```

Label the curve for z and the two curves for 10υ and 100a:

```
# add labels to the curves
text(35, 500, "z", cex=1)
text(30, 200, "10υ", cex=1)
text(25, 20, "100a", cex=1)
```

The final graph is shown below



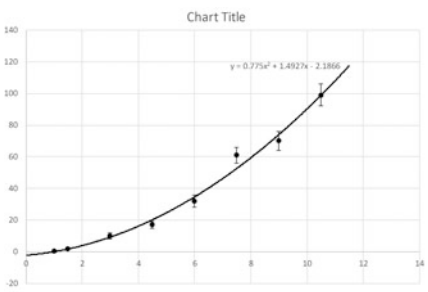
Example 12.5 [E]

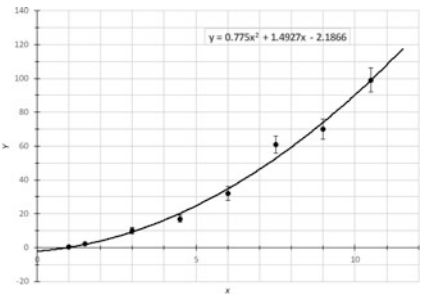
Eight results of measurements of the quantity $Y(x)$ are given in the table below. Using Excel©, create a graph showing the measurements and the best parabolic curve between them.

i	1	2	3	4	5	6	7	8
x	1.0	1.5	3.0	4.5	6.0	7.5	9.0	10.5
$Y \pm \delta Y$	0.5 ± 0.3	2.1 ± 0.5	10 ± 2	17 ± 2	32 ± 4	61 ± 5	70 ± 6	99 ± 7

We enter x, Y and δY in columns A, B and C, respectively. To plot $Y(x)$, we highlight cells A1–A8 and B1–B8 and open the **Insert** window. We choose scatter plot. We left-click at the top right corner of the plot and open ⊞. We open **Error Bars**, **More Options**, **Format Error Bars**. Double-click at a point near and to the left or the right of a point of the plot, in order to open the **Horizontal Errors** window. In **Format Error Bars > Error Bar Options** we open ◇. We select **Line > No Line**. We double-click at a point near and above or below a point of the plot, we open the **Vertical Errors** window. In **Format Error Bars > Error Bar Options** we open ◇. We select **Line > Solid Line**, **Color** black and **Width** 0.75 pt. We open ▮▮ **Vertical Error Bar**. We select **Direction Both**, **End Style Cap**, **Error Amount Custom**. Clicking **Specify Value** will open the window **Custom Error Bars**. In both **Positive Error Value** and **Negative Error Value** type **= Sheet1!$C$1:$C$8**. We press **OK**.

We will now fit the best parabola to the experimental points. We press ⊞, **Trendline**, **More Options**. In ◇, **Line**, we select **Solid Line**, **Color** black, **Width** 1.5 pt, **Dash Type** continuous line. We open ▮▮ and select **Polynomial Order** 2. Also, select **Forecast Forward** 1 period and **Backward** 1 period. Finally, click the box Display Equation on Chart. Left-click on the straight line present in the plot and delete it. The graph shown on the left below is produced.





We will format this graph.

We delete the **Chart Title** text box. We click anywhere in the area of the numbers of the X-axis. This opens **Chart Tools**. **Open** the **Format** window. In the top left corner of the screen select **Horizontal (Value) Axis**. This opens the **Format**

Axis window for the horizontal axis. In **Line** select **Solid Line**, **Color** black, **Width** 0.75 pt, **Dash Type** continuous. In the same window, click the icon . Open **Axis Options** and select **Bounds**, **Minimum** 0 and **Maximum** 12. For **Units**, we select **Major** 5 and **Minor** 1. Open **Tick Marks** and select **Major Type**, **Outside** and **Minor Type**, **Outside**.

Repeat the same procedure for the **Vertical (Value) Axis**. In this case select **Bounds**, **Minimum** –20 and **Maximum** 140. For **Units**, we select **Major** 20 and **Minor** 10.

We click anywhere in the plot area. This opens **Chart Tools**. **Open** the **Format** window. In the top left corner of the screen select **Horizontal (Value) Axis Major Gridlines**. This opens the **Format Axis** window for the horizontal major gridlines. Select , **Solid Line**, **Color** gray and **Width** 0.5 pt. We now open the **Horizontal (Value) Axis Minor Gridlines**. This opens the **Format Axis** window for the horizontal minor gridlines. Select **Solid Line**, **Color** gray and **Width** 0.5 pt.

We repeat the same procedure for the vertical gridlines.

To insert axis titles we open , **Axis Titles**. For the X-Axis title we type x. For the Y-Axis title we type Y. In **Format Axis Title > Text Options > Text Fill** select Solid fill color black. Repeat for the Y-Axis, typing y.

We left-click on the equation of the parabola and change the lettering to **Bold** and **Size** 11 pts.

The final result is that shown in the right-hand-side figure above.

Example 12.6 [O]

Eight results of measurements of the quantity $Y(x)$ are given in the table below. Using Origin©, create a graph showing the measurements and the best parabolic curve between them.

i	1	2	3	4	5	6	7	8
x	1.0	1.5	3.0	4.5	6.0	7.5	9.0	10.5
$Y \pm \delta Y$	0.5 ± 0.3	2.1 ± 0.5	10 ± 2	17 ± 2	32 ± 4	61 ± 5	70 ± 6	99 ± 7

We enter the data x, Y and δY in columns A, B and C respectively. Highlight column C. Right-click on C. Then

Set As > Y Error

Highlight columns A, B and C. Then

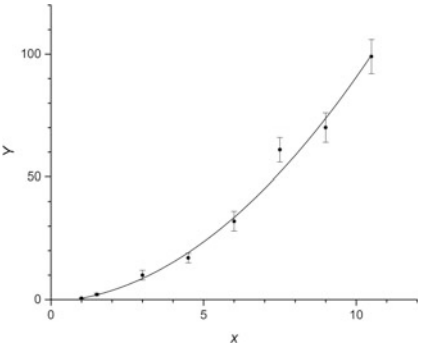
Plot > Symbol > Scatter

A plot is produced. We will fit a parabola to the points.

Analysis > Fitting > Polynomial Fit > Open Dialog…

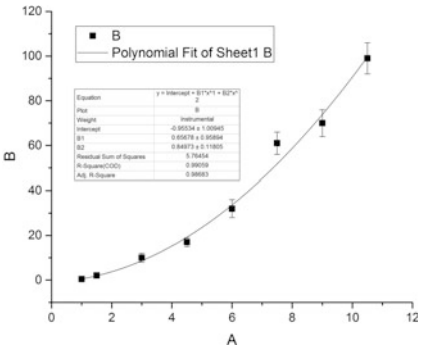
Select **Polynomial** of **Order 2**. On pressing **OK** the graph shown in the figure on the left below is produced. It is seen that the parabola $Y = (-0.95534 \pm 1.00945) + (0.65678 \pm 0.95894)x + (0.84973 \pm 0.11805)x^2$

We will improve the appearance of the graph.



1. We delete the two text boxes given in the figure.
2. We change the thickness of the line by double-clicking on it and changing **Width** from 0.5 to 1. We change the color of the line from red to black.
3. We double-click on a point and change the shape and size of the points from square and 9 pts to circular and 5 points.
4. We change the labels of the axes by double-clicking on them and writing x in place of A and Y in place of B. We use the default font of Arial 22 pts.
5. We change the **Horizontal** x axis by double-clicking on it. We set the **Scale** from 0 to 12, **Major Ticks Value** 5 and **Minor Ticks Count** 4.
6. We change the **Vertical** Y axis by double-clicking on it. We set the **Scale** from 0 to 120, **Major Ticks Value** 50 and **Minor Ticks Count** 4.
7. We may, if we wish, write the equation of the best fit parabola on the graph.

The result is shown in the figure on the right above.



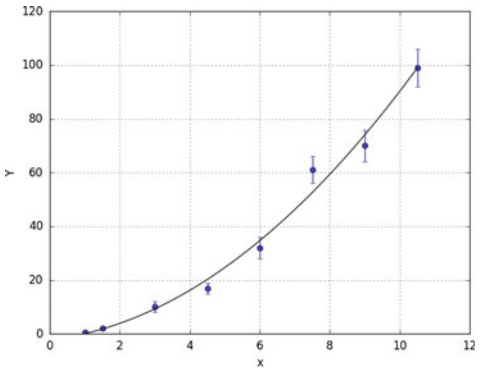
Example 12.7 [P]

Eight results of measurements of the quantity $Y(x)$ are given in the table below. Using Python, create a graph showing the measurements and the best parabolic curve between them.

i	1	2	3	4	5	6	7	8
x	1.0	1.5	3.0	4.5	6.0	7.5	9.0	10.5
$Y \pm \delta Y$	0.5 ± 0.3	2.1 ± 0.5	10 ± 2	17 ± 2	32 ± 4	61 ± 5	70 ± 6	99 ± 7

```
import numpy as np
import matplotlib.pyplot as plt

# Enter the values of x and Y:
x = np.array([1, 1.5, 3, 4.5, 6, 7.5, 9, 10.5])
Y = np.array([0.5, 2.1, 10, 17, 32, 61, 70, 99])

# Enter the errors in Y:
errx = np.array([0, 0, 0, 0, 0, 0, 0, 0])
errY = np.array([0.3, 0.5, 2, 2, 4, 5, 6, 7])

# Production of the scatter plot of data with linear axes and grid:
plt.errorbar(x, Y, xerr=errx, yerr=errY, fmt='o', color='b')
plt.xlim(0, 12)          # set the x-axis range of values
plt.ylim(0, 120)         # set the Y-axis range of values
plt.xlabel("x")         # set the x-axis label
plt.ylabel("Y")         # set the Y-axis label
plt.grid(True)

# Least-squares curve fit (parabola)
fit = np.polyfit(x, Y, 2)
p = np.poly1d(fit)

xp = np.linspace(min(x), max(x), 200)

# Plot result:
plt.plot(xp, p(xp), '-', color="black")

plt.show()
```

The plot produced is shown here.



Example 12.8 [R]

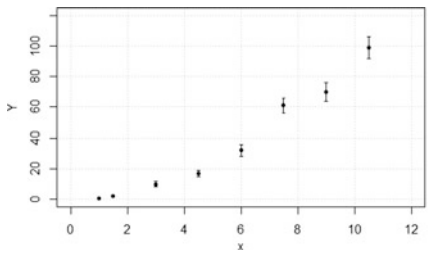
Eight results of measurements of the quantity $Y(x)$ are given in the table below. Using R, create a graph showing the measurements and the best parabolic curve between them.

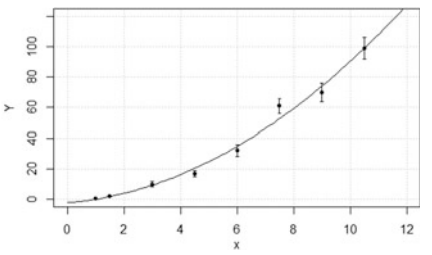
i	1	2	3	4	5	6	7	8
x	1.0	1.5	3.0	4.5	6.0	7.5	9.0	10.5
$Y \pm \delta Y$	0.5 ± 0.3	2.1 ± 0.5	10 ± 2	17 ± 2	32 ± 4	61 ± 5	70 ± 6	99 ± 7

We first create the scatter plot of the experimental points. We enter the vectors for x, Y and δY:

```
# x, Y and errY vectors:
x <- c(1.0, 1.5, 3.0, 4.5, 6.0, 7.5, 9.0, 10.5)
Y <- c(0.5, 2.1, 10, 17, 32, 61, 70, 99)
errY <- c(0.3, 0.5, 2, 2, 4, 5, 6, 7)

#scatter plot of data with x and Y axes labels, lengths and grid:
plot(x, Y, pch=20, xlab="x", ylab="Y", xlim=c(0, 12), ylim=c(0, 120), grid
())

# add Y error bars
arrows(x, Y-errY, x, Y+errY, length=0.02, angle=90, code=3)
# ENTER returns the scatter plot
```

The scatter plot is shown below on the left.





We fit to the points a least-squares parabola:

```
# Least-squares curve fit
> nls(Y~a0+a1*x+a2*x^2)
```

We add the least-squares curve to the graph:

```
# plot least-squares curve
curve(-2.187+1.493*x+0.775*x^2, from=0, to=12, add=T)
```

The final graph is shown above on the right.

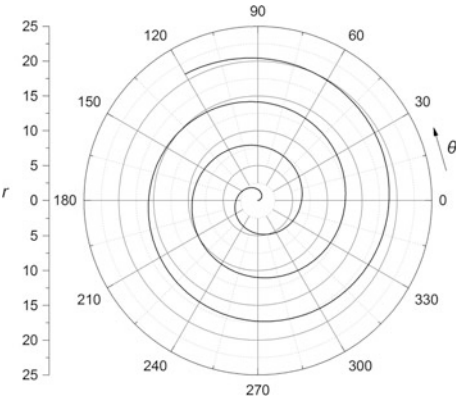
Example 12.9 [O]

Plot 3 turns of the Archimedean spiral, $r = \theta$, in polar coordinates.

We will give the angle θ in degrees but the program will evaluate r with θ in radians. For 3 turns, we therefore need values of θ between 0 and $3 \times 360^{o} = 1080^{o}$. We will use $0 \leq \theta \leq 1200^{o}$.

To plot the graph, we act as follows:

Values of column A. Highlight column A



Column > Set Column Values

and enter **(i − 1)** for i between 1 and 1201. Press **OK**.

Values of column B. Highlight column B

Column > Set Column Values

and enter **col(A) * 2 * pi/360** for i between 1 and 1201. Press **OK**.

Highlight both columns A and B. Then,

Plot > Specialized > Plot theta(X) r(Y)

The polar plot of $r = \theta$ appears.

We will improve the appearance of the graph.

We delete the text box containing B by clicking on it and pressing **Delete**.

We increase the thickness of the line in the graph by double-clicking on them and changing **Width** from 0.5 to 1.

On the r scale, we delete B. Using the **Text Tool T** we write r in italics and size 22.

For the θ scale, using the **Arrow Tool** we draw an arrow in the region between 0 and 30 degrees. If we wish, we may change the size, shape or color of the arrow by double-clicking on it and opening the appropriate window. Changing the font to Arial Greek we write θ next to the arrow.

We export the figure by using

File > Export Graphs > Open Dialog…

naming the file and selecting the directory in which we wish to save it.

The ease with which we can fit a straight line to the points for which a linear relation $y(x)$ holds, makes it desirable to reduce other mathematical expressions to linear. We have already seen that an exponential relation

$$R(t) = R_0\, \mathrm{e}^{-\lambda t} \tag{12.10}$$

may be reduced to a linear by the transformation

$$y \equiv \ln R, \quad a \equiv \ln R_o, \quad x \equiv t, \tag{12.11}$$

when we have

$$y = a - \lambda x \tag{12.12}$$

which is a linear relation. We may then draw, even by hand, a straight line between the points (x, y). The intercept of the line with the y-axis gives a and so also $R_0 = \mathrm{e}^a$, while the slope of the line gives $-\lambda$. [It should be noted that this presupposes the use of natural logarithms in the plot, as required by Eq. (12.11). The determination of λ from the slope of the straight line when common logarithms are used will be described below.]

Many common mathematical relations may be reduced to linear by a suitable change of variables. The more common of these cases are given in Table 12.1.

Table 12.1 Examples of variable transformations that reduce non-linear relations to linear

#	Non-linear relation	New variables		Resulting linear relation
		x	y	
1	$s = s_0 + \frac{1}{2}\gamma t^2$	t^2	s	$y = s_0 + \frac{1}{2}\gamma x$
2	$s = \upsilon_0 t + \frac{1}{2}\gamma t^2$	t	$\frac{s}{t}$	$y = \upsilon_0 + \frac{1}{2}\gamma x$
3	$V = -\frac{k}{r}$	$\frac{1}{r}$	V	$y = -kx$
4	$Y = \frac{1}{a+bX}$	X	$\frac{1}{Y}$	$y = a + bx$
5	$F = \frac{k}{r^2}$	$\frac{1}{r^2}$	F	$y = kx$
6	$\upsilon = k r^n$	$\log\, r$	$\log\, \upsilon$	$y = \log k + nx$
7	$N = N_0 e^{-\lambda t}$	t	$\ln N$	$y = \ln N_0 - \lambda x$
		t	$\log N$	$y = \log N_0 - (\lambda\, \log e)\, x$
8	$Y = ab^{cX}$	X	$\log Y$	$y = \log a + (c\, \log b)x$
9	$Y = ab^{cX} + d$	X	$\log(Y - d)$	$y = \log a + (c\, \log b)x$
10	$Y = aX^b$	$\log X$	$\log Y$	$y = \log a + bx$
11	$Y = aX^b + d$	$\log X$	$\log(Y - d)$	$y = \log a + bx$
12	$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}$	$\frac{1}{s'}$	$\frac{1}{s}$	$y = \frac{1}{f} - x$

Example 12.10 [E]

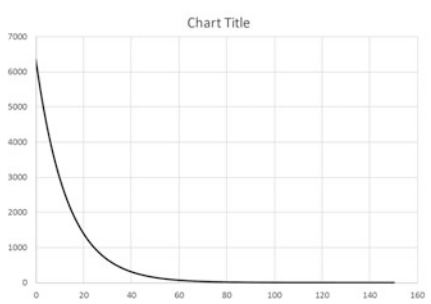
The activity of an isotope at $t = 0$ is $R_0 = 6350$ cpm, ignoring errors. The mean lifetime of the isotope is $1/\lambda = 13.2$ min. Plot a graph of the expected activity $R(t)$ of the isotope up to $t = 150$ min. Use a logarithmic scale for R.

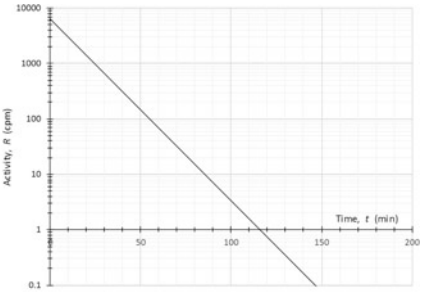
We will use the relation $R(t) = R_0 e^{-\lambda t}$. We will evaluate $R(t)$ for $0 \leq t \leq 150$ min in steps of 1 min. The values of t will be stored in column A, while the corresponding values of $R(t)$ will be stored in column B.

Evaluation of t values: Type 0 in cell A1. In cell A2 type = A1 + 1. After **ENTER**, we fill down from cell A2 to cell A151. Cells A1 to A151 now contain the numbers 0 to 150.

In cell B1 we type = 6350 * exp(−A1/13.2). After **ENTER**, we fill down from B1 to B151. Cells B1 to B151 now contain the values of $R(t)$ for $0 \leq t \leq 150$ min in steps of 1 min.

Highlight columns A and B and through **Insert** select the **Scatter** smooth line plot. Pressing **ENTER** results in the graph shown on the left, below.





We delete the text box for the **Chart Title**. We double-click on the Y-Axis and open the **Format Axis** window. In **Axis Options**, we click the **Logarithmic Base 10** box.

We format the plot in the way described in previous examples so that, finally, we have the graph shown on the right above.

Example 12.11 [O]

The activity of an isotope at $t = 0$ is $R_0 = 6350$ cpm, ignoring errors. The mean lifetime of the isotope is $1/\lambda = 13.2$ min. Plot a graph of the expected activity $R(t)$ of the isotope up to $t = 150$ min. Use a logarithmic scale for R.

We will use the relation $R(t) = R_0 e^{-\lambda t}$. We will evaluate $R(t)$ for $0 \leq t \leq 150$ min in steps of 1 min. The values of t will be stored in column A, while the corresponding values of $R(t)$ will be stored in column B.

Evaluation of t values: Highlight column A. Then

Column > Set Column Values

In the window that opens we type **i − 1** for i from 1 to 151. We press **OK**. The values of t in min are entered in column A.

Evaluation of R values: Highlight column B. Then,

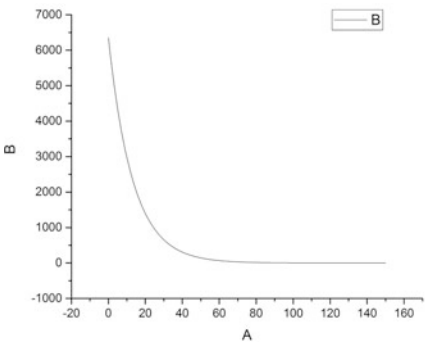
Column > Set Column Values

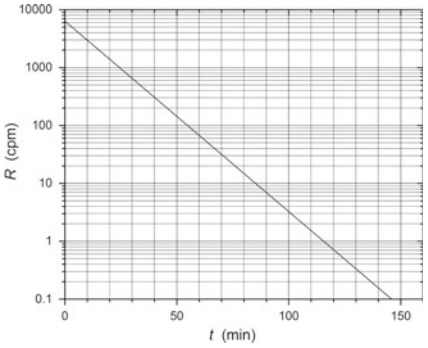
In the window that opens we type 6350 * exp(−col(A)/13.2) for i from 1 to 151. We press **OK**. The values of $R(t)$ in cpm are entered in column B.

We now plot $R(t)$. Highlight columns A and B. Then

Plot > Symbol > Scatter

The graph shown in the figure on the left below appears.





We will improve the appearance of the graph.

1. We delete the two text box given in the figure.
2. We change the thickness of the line by double-clicking on it and changing **Width** from 0.5 to 1.
3. We change the labels of the axes by double-clicking on them and writing ***t* (min)** in place of A and ***R* (c.p.m.)** in place of B. We use the default font of Arial 22 pts.
4. We change the **Horizontal** t axis by double-clicking on it. We set the **Scale** from 0 to 160, **Major Ticks Value** 50 and **Minor Ticks Count** 4.
5. We change the **Vertical** R axis by double-clicking on it. We set the **Scale** from 0.1 to 10 000, Type **Log10**. **Major Ticks** and **Minor Ticks** are set by the program to 1 and 8 respectively.
6. It is of great assistance in reading values off the graph to have the grid lines drawn. This is particularly true when one or both the axes are logarithmic. We will now draw the grid lines:

   i. We double click on the t axis, thus opening the **X Axis** window. Then, having opened **Horizontal, Grids**, set
   Major Grid Lines: Tick **Show**, **Color Black**, **Style Solid**, **Thickness 0.5**
   Minor Grid Lines: Tick **Show**, **Color Black**, **Style Solid**, **Thickness 0.3.**
   Then,
   Vertical, Grids, set the same as for **Horizontal, Grids**.

ii. We open the window for **Line and Ticks**.

Bottom: Tick **Show Line and Ticks**.

Line: Tick **Show**, **Color Black**, **Thickness 1.5**.
Major Ticks: Style Out.
Minor Ticks: Style Out.

Top: Tick **Show Line and Ticks**.

Line: Tick **Show**, **Color Black**, **Thickness 1.5**.
Major Ticks: Style In.
Minor Ticks: Style In.

Left: Same as Bottom.
Right: Same as Top.

Press **OK**.
The result is shown in the figure on the right above.

Example 12.12 [P]

The activity of an isotope at $t = 0$ is $R_0 = 6350$ cpm, ignoring errors. The mean lifetime of the isotope is $1/\lambda = 13.2$ min. Plot a graph of the expected activity $R(t)$ of the isotope up to $t = 150$ min. Use a logarithmic scale for R.

We will use the relation $R(t) = R_0 e^{-\lambda t}$. We will evaluate $R(t)$ for $0 \leq t \leq 150$ min in steps of 1 min. We first plot $R(t)$ with linear scales:

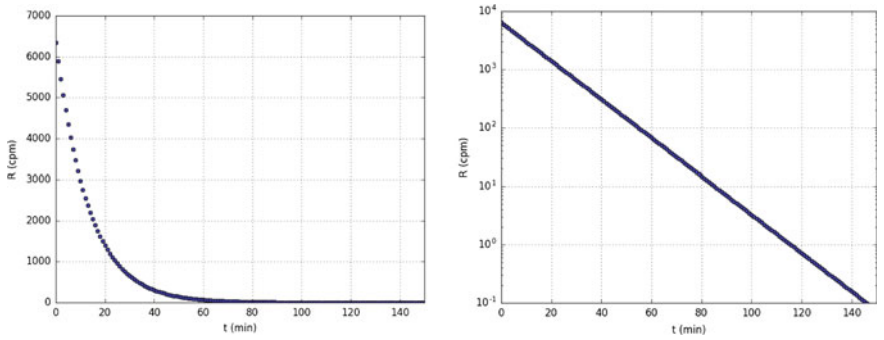
```
from __future__ import division
import numpy as np
import matplotlib.pyplot as plt
t = np.arange(0, 151)
R = 6350 * np.exp(-t/13.2)

plt.scatter(t, R)
plt.xlim(0, 150)
plt.ylim(0, 7000)
plt.xlabel("t (min)")
plt.ylabel("R (cpm)")
plt.grid(True)
plt.show()
```

We then plot R(t) with a logarithmic scale for R (the y axis):

```
plt.scatter(t, R)
plt.yscale('log')
plt.xlim(0, 150)
plt.ylim(0.1, 10000)
plt.xlabel("t (min)")
```

```
plt.ylabel("R (cpm)")
plt.grid(True)
plt.show()
```

The following figure shows the graph with linear scales (left) and the graph with logarithmic R scale (right).

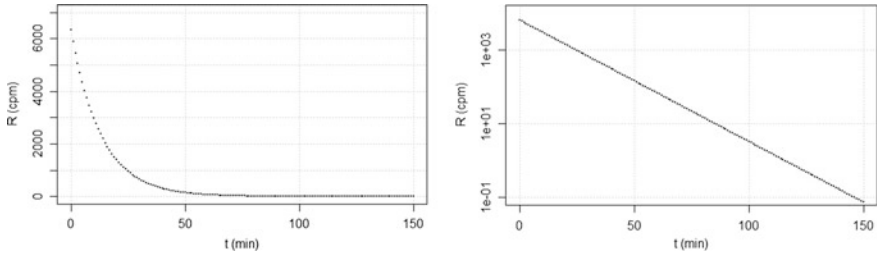


Example 12.13 [R]

The activity of an isotope at $t = 0$ is $R_0 = 6350$ cpm, ignoring errors. The mean lifetime of the isotope is $1/\lambda = 13.2$ min. Plot a graph of the expected activity $R(t)$ of the isotope up to $t = 150$ min. Use a logarithmic scale for R.

We will use the relation $R(t) = R_0 e^{-\lambda t}$. We will evaluate $R(t)$ for $0 \leq t \leq 150$ min in steps of 1 min. We first plot $R(t)$ with linear scales:

```
> # scatter plot with linear scales:
> t <- seq(0, 150, by=1)
> R = 6350*exp(-t/13.2)
> plot(t, R, pch=20, cex=0.5, xlab="t (min)", ylab="R (cpm)",
  xlim=c(0, 150), ylim=c(0, 7000), grid())
```



This plot is shown above, on the left.

We now plot $R(t)$ using a logarithmic scale for R. So that we avoid infinities, we change the lower limit of R to 0.1 cpm.

```
# Scatter plot with logarithmic R scale:
plot(t, R, log="y", pch=20, cex=0.5, xlab="t (min)", ylab="R (cpm)",
xlim=c(0, 150), ylim=c(0.1, 10000), grid())
```

This plot is shown above, on the right.

Example 12.14

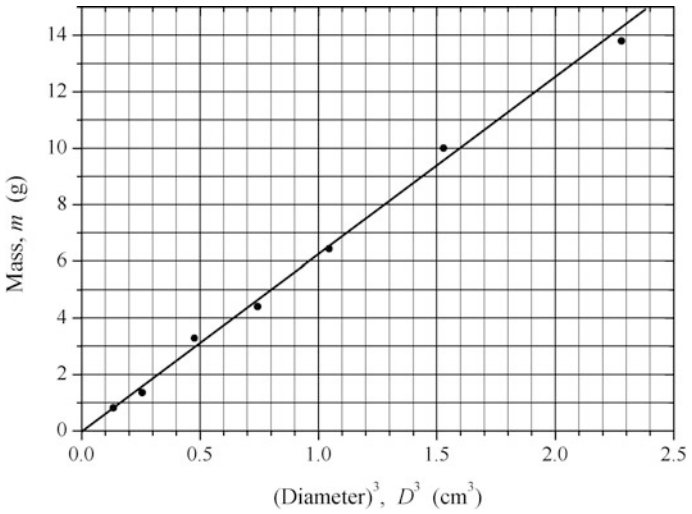
With the aim of determining their density, we measure the diameters of 7 metal spheres, as well as their masses. The results are given in columns 2 and 3 of the table below. Assuming that the 'spheres' are perfectly spherical, find their density using a graphical method.

The condition 'using a graphical method' is obviously imposed because we could determine the density of each sphere directly and then find the mean value and its standard deviation. However, for demonstration purposes, we will use the graphical method.

The relation between the mass m and the diameter D of a sphere is $m = \frac{\pi}{6}\rho D^3$, where ρ is its density. If we use the new variable $x = D^3$, we have the linear relation $m = \frac{\pi}{6}\rho x$. The straight line $m(x)$ passes through the origin and has a slope equal to $\lambda = \frac{\pi}{6}\rho$.

#	D (cm)	m (g)	D^3 (cm^3)
1	0.512	0.824	0.1342
2	0.635	1.362	0.2561
3	0.781	3.285	0.4764
4	0.906	4.401	0.7437
5	1.015	6.428	1.0457
6	1.152	10.853	1.5288
7	1.316	13.806	2.2791

The values of the variable D^3 are given in the table. The mass m is plotted as a function of D^3 in the figure that follows, where the linear relationship between the variables D^3 and m is seen.



The slope of the straight line is found to be approximately equal to $\lambda = 6.30\,\text{g/cm}^3$, from which we find that $\rho = \frac{6}{\pi}\lambda = 12.0\,\text{g/cm}^3$.

A simple method of obtaining an estimate for the error in the density will be presented below. It should be noted that for each point we could also have drawn the errors in both D^3 (as they are evaluated from the accuracy of measuring D) and in the masses m of the spheres.

12.5 The Slope of the Curve

The most accurate method for finding the best straight line or, in general, the best curve which corresponds to a series of experimental points, is the method of least squares which, at least in the case of the straight line, also gives the errors in the values of the two parameters which describe the line. The method has been presented in Chap. 11. Here, we will describe how we may find the slope of a straight line or of a curve at a point by a graphical method.

By the term 'best straight line' we mean a straight line that passes between the experimental points in such a way that, to the degree that this is possible, there are equal numbers of points above and below the line, both in the case of all the points and in as many smaller regions of values of the points as possible. Obviously this is not easy or possible, unless there are a great number of points with most of them not deviating by much more than the others from the general linear trend. Drawing this line requires more than the use of a transparent ruler. It is already becoming clear that this process depends on a number of subjective judgments. The same is true for non-linear curves, with the difference that in their case things are more difficult.

In general, large and sudden changes in the slope of the curve should be avoided, unless this is justified by the systematic behavior of an adequate number of points.

Having drawn what in our judgment is the best curve, we may find for a straight line its intercepts with the axes and its slope, and for a curve its slope at a certain point of interest. In Fig. 12.27a the method of finding the slope of a straight line is described. Having drawn the best straight line between the points, we choose two geometrical points on the line, A and B. These points *must not be* experimental points; otherwise the procedure would virtually cancel itself by giving weight to only two experimental points, ignoring all the others! For more accurate results, points A and B must at as great a distance from each other as possible. Also, for better reading of the co-ordinates of these points but also in order to simplify arithmetic, it is desirable that these points have abscissas (x) which correspond to whole numbers or integers or, in any case, to numbers for which there correspond lines on the graph paper used (for example, in Fig. 12.27a, the abscissas of points A and B are 2 and 12). The same comments apply in case we might choose to start by selecting two points with given values of the dependent variable (y). We read the co-ordinates of the points, A:(x_A, y_A) and B:(x_B, y_B). The slope of the straight line is

$$\lambda \equiv \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\Delta y}{\Delta x} = \frac{y_B - y_A}{x_B - x_A}. \tag{12.13}$$

The need for as large as possible values of the differences $x_B - x_A$ and $y_B - y_A$ is obvious, since, otherwise, the reading errors in the co-ordinates will be significant fractions of the differences $x_B - x_A$ and $y_B - y_A$. It will be mentioned here that the units of the slope are given by the relation:

$$[\text{Units of slope}] = [\text{Units of } y]/[\text{Units of } x]. \tag{12.14}$$

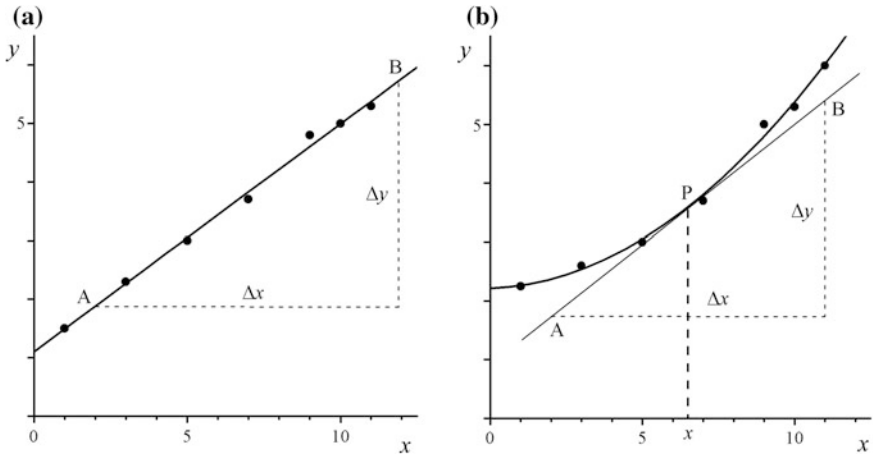


Fig. 12.27 The evaluation of the slope of **a** a straight line and **b** a curve at a point P

It must also be mentioned that the point of view that the slope is equal to $\tan\theta$, where θ is the angle formed by the straight line with the x-axis is wrong. This is true only in those cases that the scales of the variables x and y are such that it is

(units of x per unit of length of the x-axis) = (units of y per unit of length of the y-axis).

It must also be taken into account that the slope has dimensions while $\tan\theta$ is a pure (dimensionless) number.

In the general case of a curve, having drawn the curve, we also draw a tangent AB to the curve at the point P at which we wish to determine the slope of the curve, Fig. 12.27b. The required slope at P, $(\mathrm{d}y/\mathrm{d}x)_{\mathrm{P}}$, is equal to the slope of this tangent and is evaluated as described above.

12.5.1 A Graphical Method of Evaluating the Errors $\delta\alpha$ and $\delta\lambda$ in the Parameters α and λ of the Straight Line $y = \alpha + \lambda x$

A graphical method is described here for the determination of the errors $\delta\alpha$ and $\delta\lambda$ in the parameters α and λ of the straight line $y = \alpha + \lambda x$, which is simpler than the method of least squares. Although subjective judgments are made in applying the method, its results are satisfactory in most cases. It must be borne in mind, however, that the values of the errors found by this method have uncertainties of the order of 50% or even more.

Having drawn the best straight line for the given points and found the values of the parameters α and λ, we determine their errors by the following procedure (see Fig. 12.28):

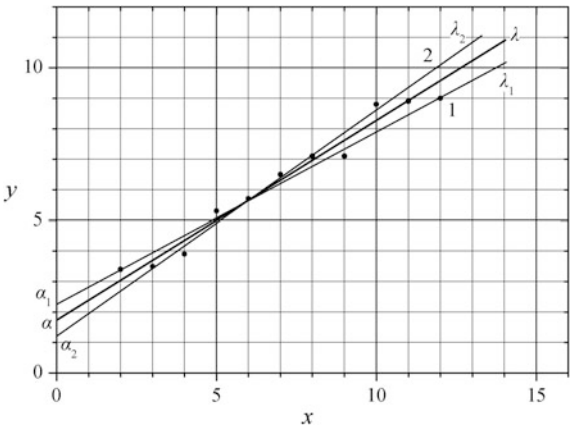


Fig. 12.28 The determination, by a graphical method, of the errors in the slope of a straight line and in the points of intersection of the axes by the line

1. We draw the best straight line, 1, passing between those points which lie *above* the main straight line on the left and *below* the main straight line on the right. The line we draw in this way is given by the equation $y = \alpha_1 + \lambda_1 x$ and from its slope and its intercept with the y-axis we determine the values of λ_1 and α_1, respectively.
2. We draw the best straight line, 2, passing between those points which lie *below* the main straight line on the left and *above* the main straight line on the right. The line we draw in this way is given by the equation $y = \alpha_2 + \lambda_2 x$ and from its slope and its intercept with the y-axis we determine the values of λ_2 and α_2, respectively.

Satisfactory estimates for the errors in α and λ are the values

$$\delta\alpha = \frac{1}{2}(\alpha_1 - \alpha_2) \quad \text{and} \quad \delta\lambda = \frac{1}{2}(\lambda_2 - \lambda_1). \tag{12.15}$$

Example 12.15

Find the values of the parameters α and λ and of their errors $\delta\alpha$ and $\delta\lambda$ for the straight line $y = \alpha + \lambda x$ of Fig. 12.28.

Referring to Fig. 12.28, we choose to find the ordinates of the three lines which correspond to the values $x = 0$ and $x = 13$. These are:

Main line:	$(x = 0,\ y = 1.75)$	$(x = 13,\ y = 10.25)$
Line 1:	$(x = 0,\ y = 2.25)$	$(x = 13,\ y = 9.55)$
Line 2:	$(x = 0,\ y = 1.20)$	$(x = 13,\ y = 10.85)$

We consider these magnitudes to be dimensionless, so that we do not have to give units.

The points of intersection of the y-axis are: $\alpha = 1.75$, $\alpha_1 = 2.25$, $\alpha_2 = 1.20$.

The slopes of the straight lines are:

$$\lambda = \frac{10.25 - 1.75}{13 - 0} = \frac{8.50}{13} = 0.654$$

$$\lambda_1 = \frac{9.55 - 2.25}{13 - 0} = \frac{7.30}{13} = 0.562 \quad \lambda_2 = \frac{10.85 - 1.20}{13 - 0} = \frac{9.65}{13} = 0.742.$$

The errors are

$$\delta\alpha = \frac{1}{2}(2.25 - 1.20) = 0.5 \quad \text{and} \quad \delta\lambda = \frac{1}{2}(0.742 - 0.562) = 0.09.$$

Finally,

$$\alpha = 1.8 \pm 0.5 \quad \text{and} \quad \lambda = 0.65 \pm 0.09.$$

Example 12.16 [E]

A converging lens is used for the formation of the image of a bright object. If the distance of the object from the center of the lens is s, then the distance of the image from the center of the lens is s' and the two distances are connected by the relation $\frac{1}{s}+\frac{1}{s'}=\frac{1}{f}$, where the length f is characteristic of the particular lens used and is called the *focal length* of the lens.

Given in the table that follows are pairs of values of s and s', as these were determined experimentally for a particular lens.

Column A	s (cm)	13	14	15	16	18	20	25
Column B	s' (cm)	43.0	35.0	30.0	27.0	22.5	20.0	16.7

(a) Mark in a graph the points $(s,\ s')$. Assume that the values of s are known with errors of ± 0.5 cm and that the values of s' have fractional errors equal to $\pm 5\%$ and draw the errors $\pm\delta s$ and $\pm\delta s'$ for every point of the graph. Investigate whether there are any points that you think should be neglected as the results of wrong measurements.

(b) Use the theoretical relation between s and s' for the determination of the focal length f of the lens as follows: From the theoretical relation it follows that $s'=f(1+s'/s)$. Make a table in which you record the values of s and s', with their errors, δs and $\delta s'$, and the corresponding values of the variable $x=1+s'/s$ with their errors, $\delta\, x$.

Column A	Column B	Column C	Column D	Column E	Column F
s (cm)	s' (cm)	δs (cm)	$\delta s'$ (cm)	$x=1+\frac{s'}{s}$	δx

Plot the points $(x,\ s')$, with their errors $\pm\delta x$ and $\pm\delta s'$, and the best straight line that passes between these points. From the value of the slope of the straight line determine the focal length of the lens, f.

(a) We enter the values of $s, s', \delta s$ and $\delta s'$ in the columns A, B, C and D, respectively. We highlight columns A and B and, from **Insert**, we select the plot **Scatter** for them.

We delete the **Chart Title** text box. We double-click on a point and change the color of the points to black. We open ⊞, **Error Bars**, **More options**, to open the **Format Error Bars** window. In ▮▮▮, **Horizontal Error Bars**, we select **Direction: Both** and **End Style: Cup**. We also select **Error Amount: Custom** and tick the **Specify Value** box. In the window that opens, we type **= Sheet1!$C$1:$C$7** in both **Positive** and **Negative Error Value**. We repeat for the **Vertical Error Bars**. In the last step, we type **= Sheet1!$D$1:$D$7** in both **Positive** and **Negative Error Value**.

In ⊞, we select **Axis Titles**. We change the **X-Axis Title** to ***s* (cm)** and black color. We change the **Y-Axis Title** to ***s′* (cm)** and black color.

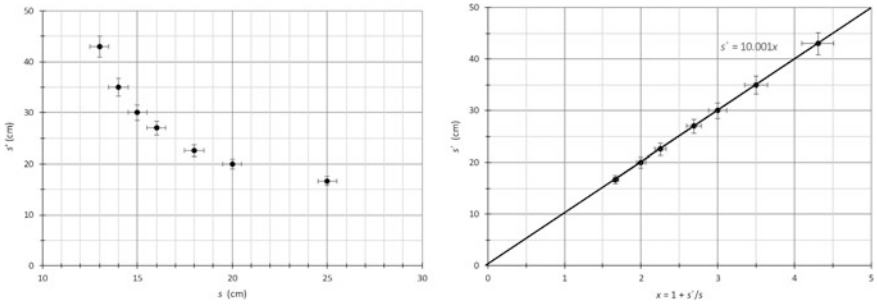
We double-click on the X-Axis and open the **Format Axis** window. In , **Axis Options**, we select **Bounds Minimum** 10, **Maximum** 30, **Units Major** 5, **Minor** 1. In **Tick Marks**, we select **Cross** for both **Major** and **Minor Type**.

For the Y-Axis, in , **Axis Options**, we select **Bounds Minimum** 0, **Maximum** 50, **Units Major** 10, **Minor** 5. In **Tick Marks**, we select **Cross** for both **Major** and **Minor Type**.

In **Format Axis > Text Options > Text Fill**, we select color black for both axes.

In , **Grid Lines**, we also tick **Primary Minor Horizontal** and **Primary Minor Vertical** grid lines.

We finally obtain the graph shown on the left below.



We see that the points lie on a smooth curve and that there do not appear to be any points that should be neglected as the results of wrong measurements.

(b) The theoretical relation between s and s' may be written as $s' = f(1 + s'/s)$. We enter the values of $x = 1 + s'/s$ in column E. In cell E1 we type **= 1 + B1/A1**. After **ENTER**, we fill down from E1 to E7.

We estimate the errors in x from the errors in s and s'. From the theory of the propagation of errors we have

$$\delta x = (x - 1)\sqrt{\left(\frac{\delta s}{s}\right)^2 + \left(\frac{\delta s'}{s'}\right)^2}.$$

In cell F1 we type **= (E1 − 1) * sqrt((C1/A1)^2 + (D1/B1)^2)**. After **ENTER**, we fill down from F1 to F7. Column F now contains the values of δx.

We will plot the points (x, s') with their errors. We copy column B, which contains s', in column G. We highlight columns E and G and through **Insert** we choose a scatter plot for these two variables.

We delete the **Chart Title** text box. We double-click on a point and change the color of the points to black. We open , **Error Bars**, **More options**, to open the **Format Error Bars** window. In , **Horizontal Error Bars**, we select **Direction:**

Both and **End Style: Cup**. We also select **Error Amount: Custom** and tick the **Specify Value** box. In the window that opens, we type **= Sheet1!$F$1:$F$7** in both **Positive** and **Negative Error Value**. We repeat for the **Vertical Error Bars**. In the last step, we type **= Sheet1!$D$1:$D$7** in both **Positive** and **Negative Error Value**.

In ⊞, we select **Axis Titles**. We change the **X-Axis Title** to $x = 1 + s'/s$ and black color. We change the **Y-Axis Title** to s' **(cm)** and black color.

We double-click on the X-Axis and open the **Format Axis** window. In ▮▮▮, **Axis Options**, we select **Bounds Minimum** 0, **Maximum** 5, **Units Major** 1, **Minor** 0.5. In **Tick Marks**, we select **Cross** for both **Major** and **Minor Type**.

For the Y-Axis, in ▮▮▮, **Axis Options**, we select **Bounds Minimum** 0, **Maximum** 50, **Units Major** 10, **Minor** 5. In **Tick Marks**, we select **Cross** for both **Major** and **Minor Type**.

In **Format Axis > Text Options > Text Fill**, we select color black for both axes.

In ⊞, **Grid Lines**, we also tick **Primary Minor Horizontal** and **Primary Minor Vertical** grid lines.

We open ⊞, **Trendline**, **More Options**. We choose **Linear** and **Set Intercept** to (0, 0). We also choose for the equation of the line to be shown in the plot. The final result is shown above by the figure on the right above.

The equation of the line is given as $s' = 10.001x$. This means that the value of the focal length is $f = 10.001$ cm. It is obvious that the error in f as derived from the points of the plot would be about 10 times smaller than that expected from the errors in x and s' of the points.

Example 12.17 [O]

Solve Example 12.16 [E] using Origin®.

We fill columns A and B with the values of s and s' and label them s **(cm)** and s' **(cm)**, respectively. The values of δs and $\delta s'$ are entered in columns C and D. We select column C and then

Column > Set As > X Error
We select column D and then,

Column > Set As > Y Error
We select columns A, B, C and D and

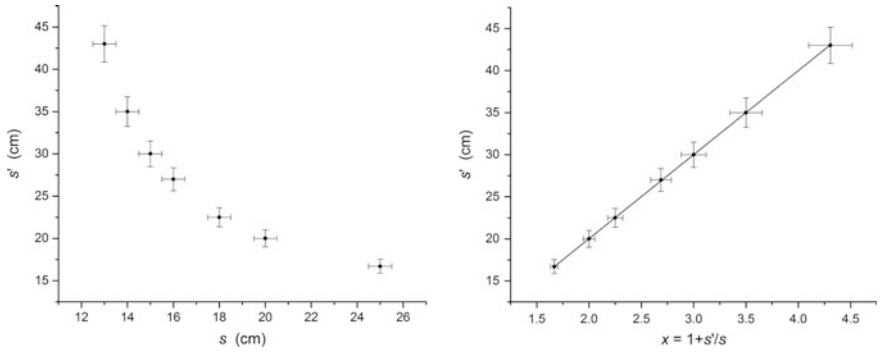
Plot > Symbol > Scatter
A graph appears, showing the points (s, s') with their error bars (see below, left). We will improve the appearance of the graph.

1. We delete the two text box given in the figure.
2. The labels and ranges of the two axes are in an acceptable form. The ranges are 12–26 cm for the s-axis and 15–45 cm for the s'-axis.

We export the graph using

File > Export Graphs > Open Dialog…

Select **Image Type** (say jpg), **File Name** and **Path**. The figure may be imported in the text here. The result is shown in the figure on the left below. It is seen that all the points lie on a smooth curve and no point needs to be rejected.



In column E we enter the values of $x = 1 + s'/s$ as follows:

Column > Set Column Values

and entering **1 + col(B)/col(A)**. Pressing **OK** fills column E with the x values.

We estimate the errors in x from the errors in s and s'. From the theory of the propagation of errors we have

$$\delta x = (x-1)\sqrt{\left(\frac{\delta s}{s}\right)^2 + \left(\frac{\delta s'}{s'}\right)^2}.$$

We highlight column F. Then

Column > Set Column Values

and type **(col(E) − 1) * sqrt((col(C)/col(A))^2 + (col(D)/col(B))^2)** and then **OK**. The errors in x are entered in column F.

We arrange the data in columns in preparation for plotting $s'(x)$.
Column E contains the values of x. We highlight column E and then

Column > Set As > X

Column F contains the values of δx. We highlight column F and then

Column > Set As > X Error

We copy column B in column G. We highlight column G and then

Column > Set As > Y

We copy column D in column H. We highlight column H and then

Column > Set As > Y Error

We highlight columns E, F, G and H. Then

Plot > Symbol > Scatter

A graph appears, showing the points (x, s') with their error bars.
We will improve the appearance of the graph.

1. We delete the text box given in the figure.
2. We double-click on a point and change the shape and size of the points from square and 8 pts to circular and 5 points.
3. We change the labels of the axes by double-clicking on them and writing ***x* = 1 + *s′*/*s*** in place of x and ***s′* (cm)** for the Y axis. We use the default font of Arial 22 pts.
4. The ranges of the axes are satisfactory.

While in the graph of (x, s') we select
Analysis > Fitting > Linear Fit > Open Dialog
Press **OK**. A straight line is drawn between the experimental points.
Double-click on the line and change its color from red to black and its thickness from 0.5 to 1. Press **OK**.
The result is shown in the figure on the right above.
The straight line has a slope of 9.98841 ± 0.02756 cm. The focal length of the lens is, therefore, equal to $f = 9.99 \pm 0.03$ cm.

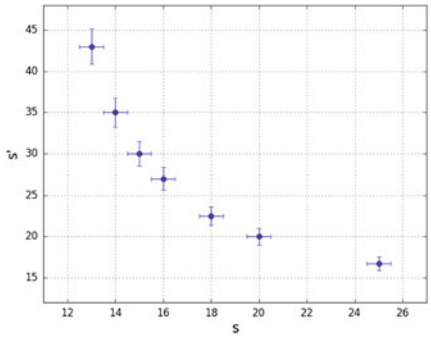
Example 12.18 [P]

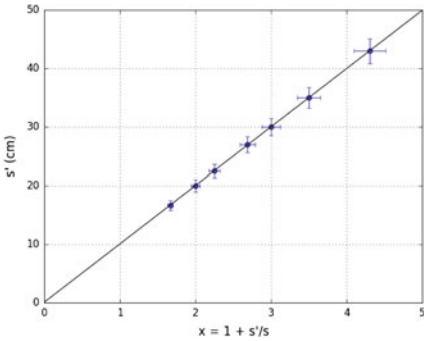
Solve Example 12.16 [E] using Python.

(a) We first plot the graph $s'(s)$.

```
import numpy as np
import matplotlib.pyplot as plt
s1 = np.array([13, 14, 15, 16, 18, 20, 25])
s2 = np.array([43, 35, 30, 27, 22.5, 20, 16.7])
errs1 = np.array([0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5])

errs2 = 0.05*s2

plt.errorbar(s1, s2, xerr=errs1, yerr=errs2, fmt='o', color='b')
plt.xlim(11, 27)
plt.ylim(12,48)
plt.xlabel("s")
plt.ylabel("s'")
plt.grid(True)
plt.show()
```

The resulting scatter plot is shown in the figure below, on the left.





(b) We will now plot $s'(x)$.

We evaluate $x = 1 + s'/s$ and its errors $\delta x = (x-1)\sqrt{\left(\frac{\delta s}{s}\right)^2 + \left(\frac{\delta s'}{s'}\right)^2}$.

```
x = 1 + s2/s1
errx = (x-1)*np.sqrt((errs1/s1)**2 + (errs2/s2)**2)
```

We plot a scatter plot of x versus s2. We also add the best-fit least-squares straight line:

```
plt.errorbar(x, s2, xerr=errx, yerr=errs2, fmt='o', color='b')
plt.xlim(0, 5)
plt.ylim(0, 50)
plt.xlabel("x = 1 + s'/s")
plt.ylabel("s' (cm)")
plt.grid(True)
fit = np.polyfit(x, s2, 1)
p = np.poly1d(fit)
xp = np.linspace(0, 5, 200)
plt.plot(xp, p(xp), '-', color="black")
plt.show()
```

The resulting graph is shown in the figure on the right, above. By looking at the contents of the p object (the polynomial of the straight line fit to the data) we get:

```
In[]: p
Out[]: poly1d([ 9.97003572,  0.0928256 ])
```

Therefore, we calculated intercept = 0.09283 cm and slope = 9.97004 cm. This means that the focal length of the lens is f = 9.97 cm.

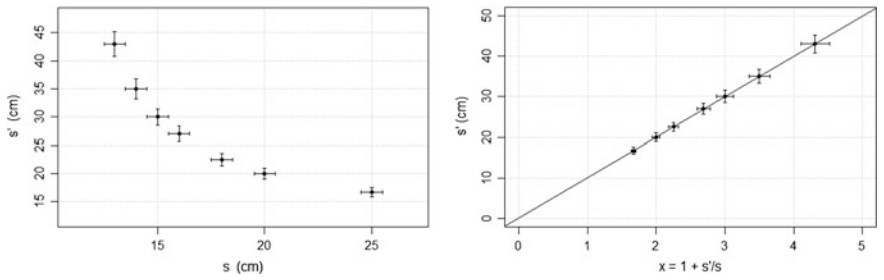
Example 12.19 [R]

Solve Example 12.16 [E] using R.

(a) We first plot the graph $s'(s)$.

```
# data vectors:
s1 <- c(13, 14, 15, 16, 18, 20, 25)
s2 <- c(43, 35, 30, 27, 22.5, 20, 16.7)
errs1 <- c(0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5)
errs2 = 0.05*s2

# scatter plot of data with s1 and s2 axes labels, lengths and grid:
plot(s1, s2, pch=20, xlab="s", ylab="s'", xlim=c(11, 27),
ylim=c(12, 48), grid())

# add s1 error bars
arrows(s1-errs1, s2, s1+errs1, s2, length=0.02, angle=90, code=3)
# add s2 error bars
arrows(s1, s2-errs2, s1, s2+errs2, length=0.02, angle=90, code=3)
```

The resulting scatter plot is shown in the figure below, on the left.



We see that there do not appear to be any points that should be neglected as the results of wrong measurements.

(b) We will now plot $s'(x)$.

We evaluate $x = 1 + s'/s$ and its errors $\delta x = (x-1)\sqrt{\left(\frac{\delta s}{s}\right)^2 + \left(\frac{\delta s'}{s'}\right)^2}$.

```
# evaluate x and its errors:
x = 1+s2/s1
errx = (x-1)*sqrt((errs1/s1)^2+(errs2/s2)^2)

#scatter plot of data with x and s2 axes labels, lengths and grid:
plot(x, s2, pch=20, xlab="x = 1 + s'/s", ylab="s' (cm)", xlim=c(0, 5),
```

```
ylim=c(0, 50), grid())

# add x error bars
arrows(x-errx, s2, x+errx, s2, length=0.02, angle=90, code=3)
# add s2 error bars
arrows(x, s2-errs2, x, s2+errs2, length=0.02, angle=90, code=3)
```

The resulting graph is shown in the figure on the right, above.
We find the best-fit least-squares straight line through the points:

```
# find least-squares best-fit straight line:
fit <- lm(s2~x)
fit

# Plot least-squares best-fit straight line:
abline(fit)
```

The best-fit least-squares straight line is also drawn in the figure.

The least-squares fit gives: intercept = 0.09283 cm and slope = 9.97004 cm. This means that the focal length of the lens is $f = 9.97$ cm. An estimate of the error in f can be made, based on the scatter of the experimental points about the straight line. It appears to be less than 1%.

12.5.2 The Evaluation of Slopes of Straight Lines in Graphs with Logarithmic Scales

We will now discuss the way to evaluate the slopes of straight lines in graphs with one or both the scales being logarithmic.

12.5.2.1 Two Logarithmic Scales

In the case of a straight line in a graph with two logarithmic scales, if the actual values of the logarithms are marked on the axes, the slope is found as described above for a straight line in a graph with linear axes. We must remember that, in this case, the slope, being the ratio of two differences of logarithms, is a dimensionless number. Also, since the common logarithm and the natural logarithm are related by the expressions

$$z = 10^{\log z}, \quad \ln z = (\ln 10)\log z, \quad \ln z = (2.3026\ldots) \times \log z, \tag{12.16}$$

$$z = e^{\ln z}, \quad \log z = (\log e)\ln z, \quad \log z = (0.4343\ldots) \times \ln z, \tag{12.17}$$

i.e. through a multiplying factor, the value of the slope is the same both in the case of two scales of common logarithms and in the case of two scales of natural logarithms.

More frequently, however, although the scales are logarithmic, log x and log y, the numbers x and y are marked on them, as for example in Fig. 12.29. In these cases, if A and B are two geometrical (not experimental) points on the straight line, the slope is given by the ratio

$$\kappa \equiv \frac{\Delta \log y}{\Delta \log x} = \frac{\log y_B - \log y_A}{\log x_B - \log x_A} \quad \text{or} \quad \kappa \equiv \frac{\Delta \ln y}{\Delta \ln x} = \frac{\ln y_B - \ln y_A}{\ln x_B - \ln x_A} \tag{12.18}$$

After we read the values of x and y off the graph, we must substitute their logarithms, common or natural, in Eq. (12.18). The slope κ, being the ratio of two differences of logarithms, is a dimensionless quantity (pure number).

Points A and B must be chosen to be at a distance between them as large as possible, but also at such positions that the reading of the values of x and y may be done with as much accuracy as possible, especially since the scales are not linear. Of course, on the logarithmic paper usually used, there are lines at as many subdivisions as possible, more than in our figures, and the reading of values is easier.

As a numerical example, we evaluate the slope of the straight line of Fig. 12.9, which we reproduce here as Fig. 12.30. We choose points A and B to have $V_a = 1$ and 100 V, respectively. The corresponding values of I_a are 5.0×10^{-5} and 5.0×10^{-2} A. Thus, we have for the slope

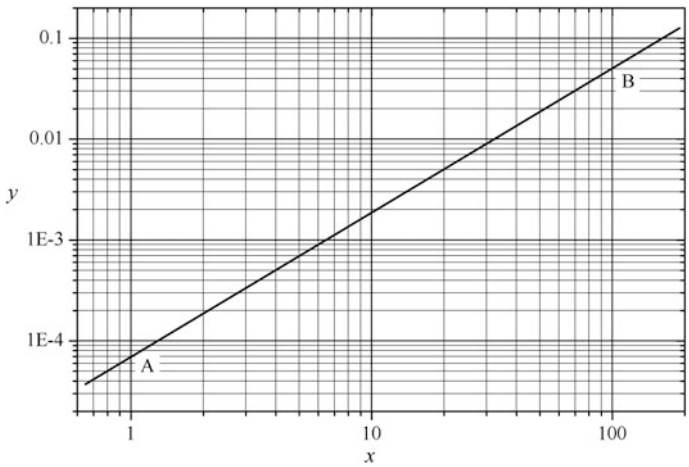


Fig. 12.29 The evaluation of the slope of a straight line in a graph with two logarithmic scales

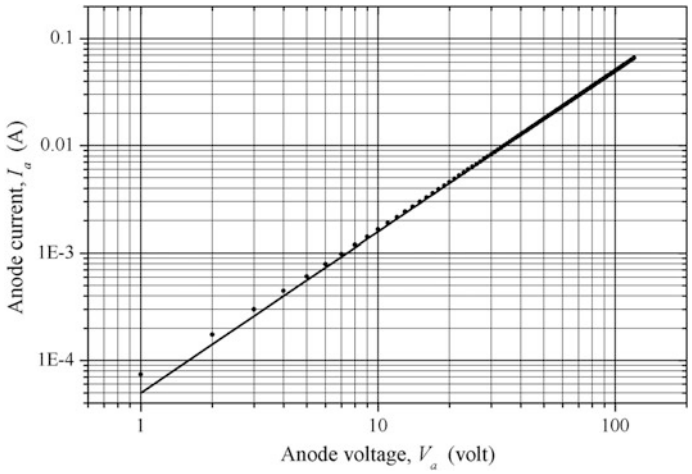


Fig. 12.30 A numerical example of the evaluation of the slope of a straight line in a graph with two logarithmic scales. The notation 1E-3 = 10^{-3} etc. is used for powers of 10

$$\kappa = \frac{\log I_B - \log I_A}{\log V_B - \log V_A} = \frac{\log(5.0 \times 10^{-2}) - \log(5.0 \times 10^{-5})}{\log(100) - \log(1)}$$
$$= \frac{(-1.301) - (-4.301)}{2 - 0} = \frac{3}{2}, \tag{12.19}$$

as predicted by theory.

A second example is given in Fig. 12.31, in which the periods of revolution of the planets around the Sun, T, are plotted as a function of their distances from it, a,

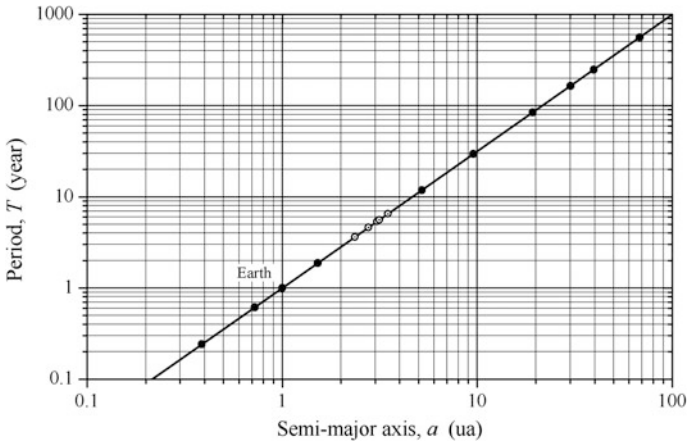


Fig. 12.31 Kepler's third law (Fig. 12.10). A numerical example of the evaluation of the slope of a straight line, in a graph with two logarithmic scales

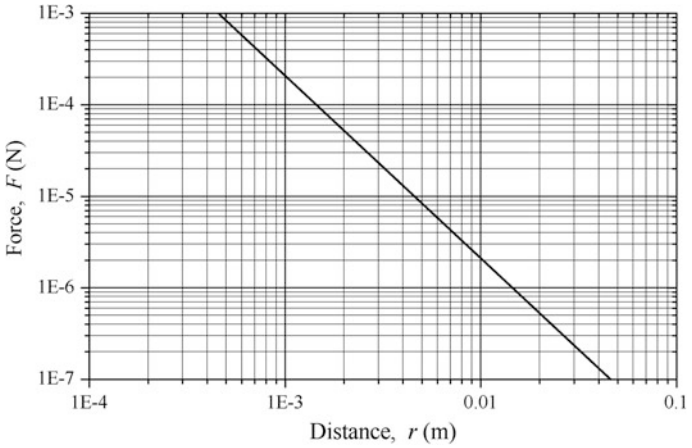


Fig. 12.32 The force F between two point charges as a function of their distance r

using logarithmic scales. The relationship is seen to be linear. For the calculation of the slope we use the points Earth: (1, 1) and B: (100, 1000). The slope of the line is

$$\kappa = \frac{\log T_{\mathrm{B}} - \log T_{\mathrm{Earth}}}{\log a_{\mathrm{B}} - \log a_{\mathrm{Earth}}} = \frac{\log(1000) - \log(1)}{\log(100) - \log(1)} = \frac{3}{2}, \tag{12.20}$$

verifying Kepler's third law, $\mathrm{T} \propto a^{3/2}$.

A third example is shown in Fig. 12.32, in which the force F between two point electric charges is plotted as a function of the distance between them, r. We use the points

$$\mathrm{A}:(4.5 \times 10^{-4}\ \mathrm{m}, \quad 1 \times 10^{-3}\ \mathrm{N}) \quad \text{and} \quad \mathrm{B}:(0.045\ \mathrm{m}, \quad 1 \times 10^{-7}\ \mathrm{N}) \tag{12.21}$$

and find for the slope of the straight line the value

$$\kappa = \frac{\log(1 \times 10^{-7}) - \log(1 \times 10^{-3})}{\log(0.045) - \log(4.5 \times 10^{-4})} = \frac{(-7) - (-3)}{(-1.347) - (-3.347)} = \frac{-4}{2} = -2. \tag{12.22}$$

Therefore, it is

$$\log F = K - 2 \log r \quad \text{or} \quad F = A/r^2, \tag{12.23}$$

as expected from Coulomb's law.

12.5.2.2 *One Linear Scale and One Logarithmic Scale*

We will assume that the logarithmic scale is that of the dependent variable (y), as in Fig. 12.33. If A and B are two (geometrical) points on the straight line, the slope is given by the ratio

$$\kappa \equiv \frac{\Delta \ln y}{\Delta x} = \frac{\ln y_B - \ln y_A}{x_B - x_A} \tag{12.24}$$

and, after we read the values of x and y, we must substitute for the natural logarithms of the y values in Eq. (12.23). This presupposes that the relation connecting x and y is of the form

$$y = A\, e^{\kappa x}. \tag{12.25}$$

This is no limitation, as the relation may be transformed to an exponential of any other base, through the relation $y = A\, a^{(\kappa / \ln a)\, x}$. It should be noted that, since the difference in the logarithms for y_A and y_B is the logarithm of their ratio, which is a dimensionless quantity, the units of the slope κ are the inverse of those of x:

$$[\text{Units of the slope } \kappa] = 1/[\text{Units of } x]. \tag{12.26}$$

Points A and B must be chosen to be at as great a distance from each other as possible and in such positions that they make the reading of the values of y as accurate as possible, also taking into account the fact that the scale is not linear. A good choice of points would be those with $y_A = 10$ and $y_B = 10\ 000$. We start with values of y, since we can choose such values that are easy to read off the graph

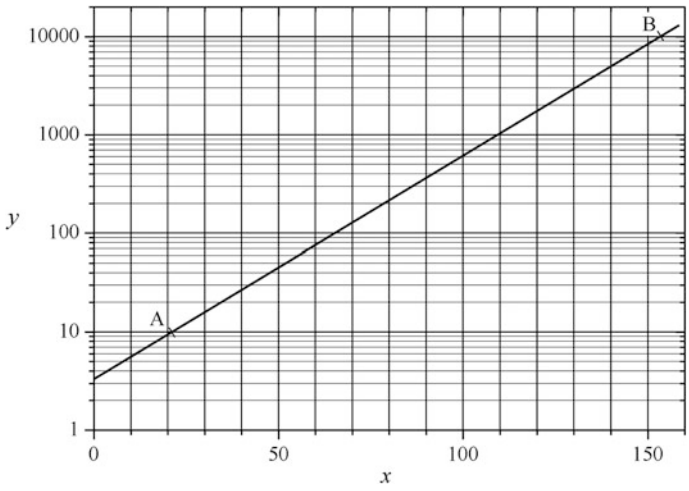


Fig. 12.33 The evaluation of the slope of a straight line in a graph with one linear scale and one logarithmic scale

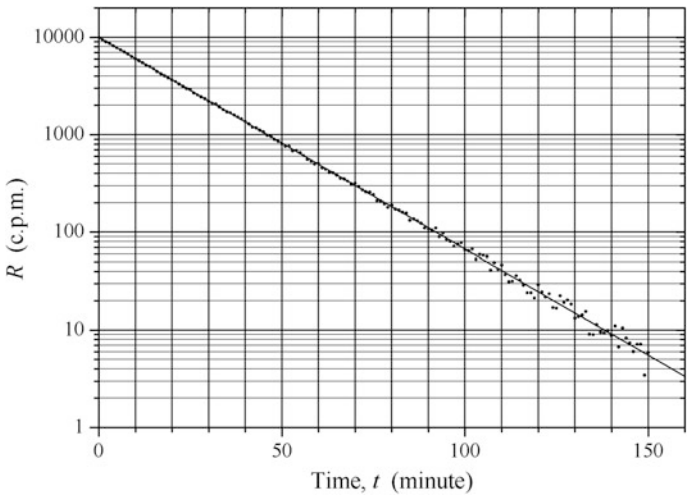


Fig. 12.34 A numerical example of the evaluation of the slope of a straight line in a graph with one linear scale and one logarithmic scale. The graph shows the variation with time of the counting rate of a radioactive sample

than other values of y which do not correspond to a grid line of the logarithmic scale. This would have been the difficulty, had we started by choosing two values of x.

A numerical example is given in Fig. 12.34, in which we reproduce the variation as a function of time of the disintegration counting rate, R, of a radioactive sample of Fig. 8.2.

As points A and B we choose those with $R_A = 10\,000$ and $R_B = 10$ c.p.m. The co-ordinates of these points are A: (0, 10^4 c.p.m.) and B: (138 min, 10 c.p.m.). Then,

$$\kappa = \frac{\ln R_B - \ln R_A}{t_B - t_A} = \frac{\ln(10) - \ln(10^4)}{138 - 0} = \frac{2.303 - 9.210}{138} = -0.0501\ \text{min}^{-1}. \tag{12.27}$$

Alternatively, we may choose as points A and B those with $t = 0$ and $t = 160$ min. The co-ordinates of these points are A: (0, 10^4 c.p.m.) and B: (160 min, 3.3 c.p.m.). The slope of the straight line is:

$$\kappa = \frac{\ln R_B - \ln R_A}{t_B - t_A} = \frac{\ln(3.3) - \ln(10^4)}{160 - 0} = \frac{1.194 - 9.210}{160} = -0.0501\ \text{min}^{-1}. \tag{12.28}$$

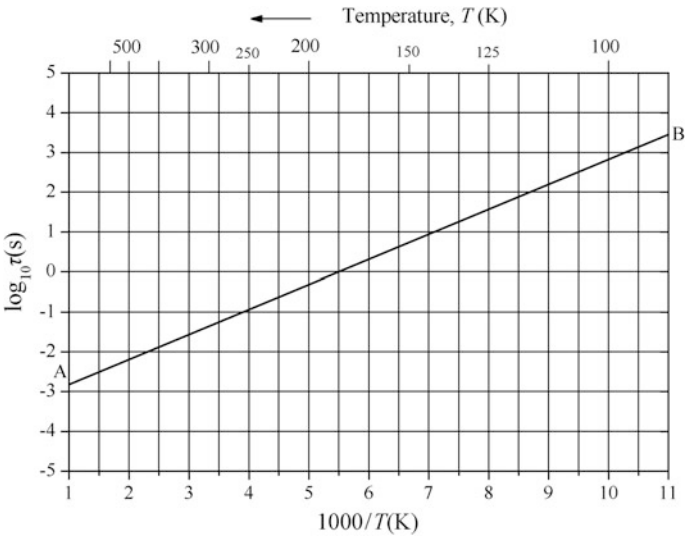


Fig. 12.35 An Arrhenius diagram ($\log_{10}\tau$ – 1/T)

From the law of radioactivity, $R(t) = R_0 e^{-\lambda t}$, we expect

$$\ln R = \ln R_0 - \lambda t \tag{12.29}$$

so that the slope found is $\kappa = -\lambda$. It follows from these measurements that for the particular radioisotope it is $\lambda = 0.050\,\text{min}^{-1}$. The mean life-time of the isotope is $\tau = 1/\lambda = 20\,\text{min}$. The errors in these values could be found either by a graphical method or the method of least squares applied to the straight line $\ln R(t)$. It can be seen that these errors are mostly due to the dispersion of values at low R's.

Another example is given in Fig. 12.35, which shows a series of measurements similar to that of Fig. 12.15. The slope of the straight line may be found from the points

$$\text{A: } (1/T = 1 \times 10^{-3}\ \text{K}^{-1}, \log_{10}\tau(\text{s}) = -2.8)$$
$$\text{and B: } (1/T = 11 \times 10^{-3}\ \text{K}^{-1}, \log_{10}\tau(\text{s}) = 3.5).$$

We change the common logarithms to natural logarithms by multiplying then by 2.3026:

$$\text{A: } (1/T = 1 \times 10^{-3}\ \text{K}^{-1}, \ln\tau(\text{s}) = -6.45)$$
$$\text{and B: } (1/T = 11 \times 10^{-3}\ \text{K}^{-1}, \ln\tau(\text{s}) = 8.06).$$

The slope of the straight line is

$$\kappa = \frac{[\ln\ \tau(\mathrm{s})]_\mathrm{B} - [\ln\ \tau(\mathrm{s})]_\mathrm{A}}{1/T_\mathrm{B} - 1/T_\mathrm{A}} = \frac{8.06 - (-6.45)}{11 \times 10^{-3} - 1 \times 10^{-3}} = \frac{14.51}{10^{-2}} = 1451\,\mathrm{K}. \quad (12.30)$$

Assuming a relation of the form $\tau = \tau_0 \mathrm{e}^{E/kT}$ [Eq. (12.8)], where k is Boltzmann's constant, τ_0 a time constant and E an energy, we find that the straight line

$$\ln\ \tau = \ln\ \tau_0 + \frac{E}{k}\frac{1}{T} \quad (12.31)$$

which results from plotting the values of $\ln\ \tau$ as a function of $1/T$ has a slope of

$$\kappa = \frac{E}{k}. \quad (12.32)$$

Therefore, the experimental values give

$$\kappa = \frac{E}{k} = 1451\,\mathrm{K}. \quad (12.33)$$

As it is $1/k = 11604.5$ K/eV, the value of the energy E is

$$E = \frac{1451}{11604.5} = 0.125\,\mathrm{eV}. \quad (12.34)$$

Given that the accuracy with which values are read off the graph cannot be better than about 2%, we accept that $\pm 2\,\%$ is a reasonable estimate for the fractional error in the value of E. Thus,

$$E = 0.125 \pm 0.003\,\mathrm{eV}, \quad (12.35)$$

unless we have another indication from the dispersion of the experimental points.

12.5.2.3 *The Graphical Resolution of the Activity of a Radioactive Sample into Components*

It is often the case that the activity of a radioactive sample is due to two or more radioisotopes, with different decay constants. In the case of two radioisotopes with comparable activities but having decay constants which are quite different the separation of their activities is possible by a graphical method.

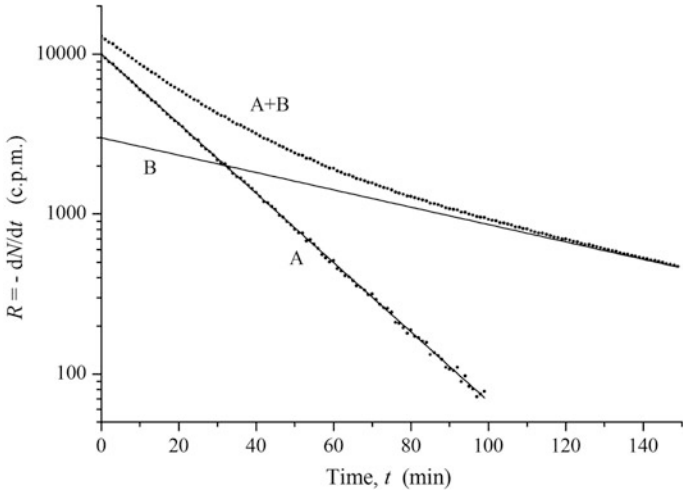


Fig. 12.36 The graphical resolution of the activity of a radioactive sample into two components, A and B, due to two different radioisotopes

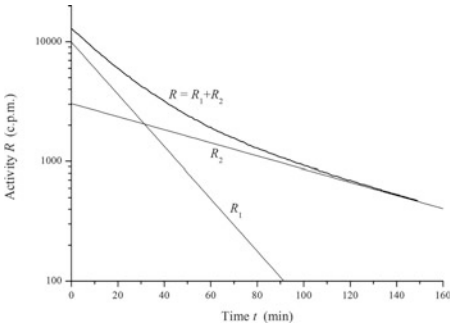
If the total activity of the sample is

$$R = R_{\mathrm{A}} + R_{\mathrm{B}} = R_{0\mathrm{A}}\mathrm{e}^{-\lambda_{\mathrm{A}}t} + R_{0\mathrm{B}}\mathrm{e}^{-\lambda_{\mathrm{B}}t} \tag{12.36}$$

and we have at our disposal a large enough number of measurements at different times t, the curve of the total activity as a function of time will be as the curve A + B of Fig. 12.36. At large values of time, the activity of the shorter-lived of the two radioisotopes is reduced to negligible values compared to the activity of the other radioisotope. If the plot is $\log R - t$, and the background, which is independent of time, has been subtracted, the straight line B may be determined, which gives the activity of radioisotope B. Subtracting this from the total activity, we find the activity of radioisotope A. In this way we may determine the quantities λ_{A}, λ_{B}, $R_{0\mathrm{A}}$ and $R_{0\mathrm{B}}$. In the example of Fig. 12.35, it is found that $\tau_{\mathrm{A}} = 1/\lambda_{\mathrm{A}} = 20$ min, $\tau_{\mathrm{B}} = 1/\lambda_{\mathrm{B}} = 80$ min, $R_{0\mathrm{A}} = 10,000$ c.p.m. and $R_{0\mathrm{B}} = 3000$ c.p.m. Obviously, the graphical method has a limited accuracy. More accurate results may be obtained by numerical methods. The method is very rarely useful for more than two radioisotopes.

Example 12.20 [O]

Measurements of the activity of a radioactive sample, R, are given for $0 \leq t \leq 150$ min (They are the same as those of Example 8.5). Plot $\log R(t)$ and verify that the activity seems to be due to two isotopes with different decay constants. Assume that the activity is given by $R = R_{10}\mathrm{e}^{-t/\tau_1} + R_{20}\mathrm{e}^{-t/\tau_2}$ and find the parameters of the functions.



The values of t are entered in column A and those of R in column B. We plot log$R$ (t). While in the graph environment, we press

Analysis > Fitting > Nonlinear Curve Fit > Open Dialog…

In the window that opens we select

Settings > Function Selection > Origin Basic Functions > Exponential > ExpDec2

The curve fitted is y = A1 * exp(−x/t1) + A2 * exp(−x/t2) + y0. We want to set y0 = 0.

Open **Parameters**. Tick **y0** and set its value to zero. Press **Fit**. The results are:

$$\text{A1} = R_{10} = 9950 \pm 48 \text{ c.p.m.}, \quad \text{t1} = \tau_1 = 19.88 \pm 0.10 \text{ min}$$
$$\text{A2} = R_{20} = 3036 \pm 51 \text{ c.p.m.}, \quad \text{t2} = \tau_2 = 79.43 \pm 0.95 \text{ min}$$

The curve $R = R_{10}\mathrm{e}^{-t/\tau_1} + R_{20}\mathrm{e}^{-t/\tau_2}$ is plotted in the graph. We also plot $R_1 = R_{10}\mathrm{e}^{-t/\tau_1}$ and $R_2 = R_{20}\mathrm{e}^{-t/\tau_2}$.

The graph is shown in the figure presented above.

Is a resolution into three components possible? A radioactive sample contains three radioisotopes, A, B and C, which contribute counting rates $R_\mathrm{A}(t) = 10\,000\ \mathrm{e}^{-t/100}$, $R_\mathrm{B}(t) = 2000\ \mathrm{e}^{-t/300}$ and $R_\mathrm{C}(t) = 300\ \mathrm{e}^{-t/900}$, respectively, in c.p.m. when t is in min. The total counting rate of the sample is $R(t) = R_\mathrm{A}(t) + R_\mathrm{B}(t) + R_\mathrm{C}(t)$. Discuss the possibility of performing a graphical analysis of this curve into its three components. Assume that the counting rates are evaluated using the results of 10-minute measurements Fig. 12.37.

From the figure, it is seen that drawing a tangent to the $R(t)$ curve is not easy. It should be noted that at the lower point of the curve, a 10-min measurement will give the result of $20 \times 10 = 200 \pm 14$ counts, corresponding to a counting rate of 20 ± 1.4 c.p.m. We see that there is an error of 7% in the counting rates at times near 2500 min. This makes drawing a tangent to the curve even more difficult. The best we can do is to draw two straight lines, C1 and C2 as the limiting cases

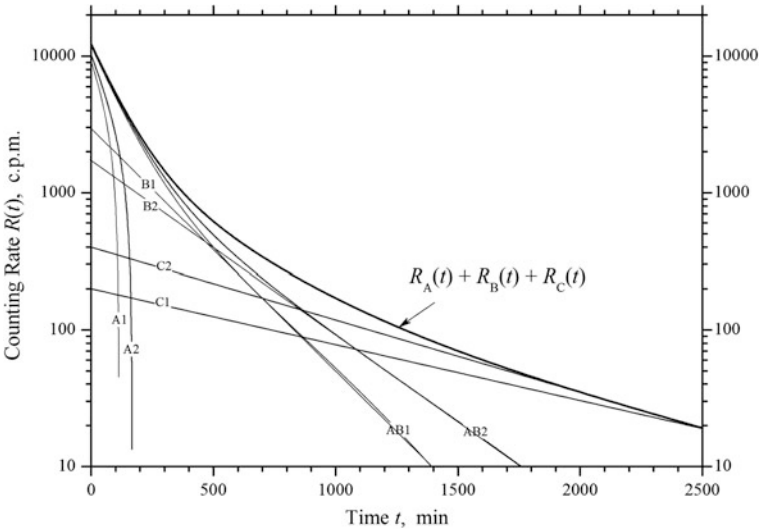


Fig. 12.37 The graphical resolution of the activity of a radioactive sample into three components, A, B and C, due to three different radioisotopes

between which the correct tangent lies. We find that these lines cut the R axis at $R_{C1}(0) = 200$ and $R_{C2}(0) = 400$ c.p.m., respectively, and correspond to lifetimes equal to $\tau_{C1} = 1062\,\text{min}$ and $\tau_{C2} = 820\,\text{min}$. The limiting cases for the counting rate from isotope C are $R_{C1}(t) = 200\,\text{e}^{-t/1062}$ and $R_{C2}(t) = 400\,\text{e}^{-t/820}$ c.p.m.

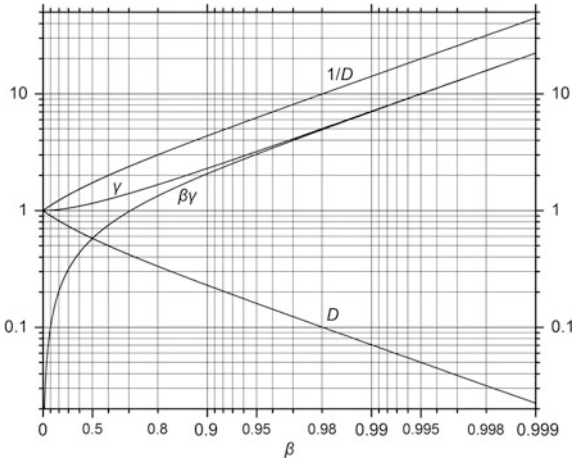
Subtracting the straight lines C1 and C2 from the total $R(t)$, we have the curves AB1 and AB2, which are the limits within which $R_A(t) + R_B(t)$ lies. The tangents to these curves at high times give the straight lines B1 and B2 for isotope B. The equations of these lines are found to be $R_{B1}(t) = 2900\,\text{e}^{-t/245}$ and $R_{B2}(t) = 1700\,\text{e}^{-t/341}$.

Subtracting the sums B1 + C1 and B2 + C2 from the total $R(t)$, we have the curves A1 and A2, which are supposed to be giving the counting rate from isotope A. These curves are far from being straight. The method is seen to fail for the analysis of a curve with counting rates from three isotopes.

Example 12.21 [O]

In the special theory of relativity, we are interested in the quantities $\gamma = \frac{1}{\sqrt{1-\beta^2}}$, $\beta\gamma$, $D = \sqrt{\frac{1-\beta}{1+\beta}}$ and $1/D = \sqrt{\frac{1+\beta}{1-\beta}}$, especially at values of the speed approaching that of light in vacuum, c, since then the reduced speed $\beta = \upsilon/c$ approaches unity and some of these quantities diverge. We wish to find suitable scales that will show the detail in the above quantities at values of β approaching unity.

We plot these quantities on a logarithmic scale from 0.02 to 50, as functions of the variable $1 - \beta$, for values of β increasing from 0 to 0.999. We use a logarithmic scale for $1 - \beta$, but instead of setting the range of this from 0.001 to 1 we invert this scale by setting the range to be from 1 to 0.001. This means that the leftmost point of the X-axis will correspond to the value $\beta = 0$ and the one on the extreme right will correspond to $\beta = 0.999$, increasing from left to right. On the X-axis, instead of giving the values of $1 - \beta$, we give those of β. This means that the values of β are not entered automatically but have to be inserted 'by hand'. The results are shown in the graph that follows.



It is seen that, due to the method used, the scale for β can be made as detailed as we like for $\beta \to 1$, by adding more decades to the logarithmic scale of $1 - \beta$.

Programs
Excel
Ch. 12. *Excel—Column and Label*
Ch. 12. *Excel—Histogram*
Ch. 12. *Excel—Scatter Plot—Linear Scales*
Ch. 12. *Excel—Scatter Plot—Linear Scales—Errors*
Ch. 12. *Excel—Scatter Plot—Linear Scales—Multiple*
Ch. 12. *Excel—Scatter Plot—Linear-Log Scales*
Ch. 12. *Excel—Scatter Plot—Linear-Log Scales—Errors*
Ch. 12. *Excel—Scatter Plot—Linear-Log Scales—Multiple*
Ch. 12. *Excel—Scatter Plot—Log-Linear Scales*
Ch. 12. *Excel—Scatter Plot—Log-Linear Scales—Errors*
Ch. 12. *Excel—Scatter Plot—Log-Linear Scales—Multiple*
Ch. 12. *Excel—Scatter Plot—Log-Log Scales*
Ch. 12. *Excel—Scatter Plot—Log-Log Scales—Errors*
Ch. 12. *Excel—Scatter Plot—Log-Log Scales—Multiple*

(continued)

(continued)

Programs
Origin
Ch. 12. *Origin—Column Plot*
Ch. 12. *Origin—Column and Label*
Ch. 12. *Origin—Histogram*
Ch. 12. *Origin—Scatter Plot—Linear Scales*
Ch. 12. *Origin—Scatter Plot—Linear Scales—Errors*
Ch. 12. *Origin—Scatter Plot—Linear Scales—Multiple*
Ch. 12. *Origin—Scatter Plot—Linear Scales—Weights*
Ch. 12. *Origin—Scatter Plot—Linear Scales—Errors—Weights*
Ch. 12. *Origin—Scatter Plot—Linear Scales—Errors—Weights from Errors*
Ch. 12. *Origin—Scatter Plot—Linear-Log Scales*
Ch. 12. *Origin—Scatter Plot—Linear-Log Scales—Errors*
Ch. 12. *Origin—Scatter Plot—Linear-Log Scales—Multiple*
Ch. 12. *Origin—Scatter Plot—Linear-Log Scales—Weights*
Ch. 12. *Origin—Scatter Plot—Linear-Log Scales—Errors—Weights*
Ch. 12. *Origin—Scatter Plot—Linear-Log Scales—Errors—Weights from Errors*
Ch. 12. *Origin—Scatter Plot—Log-Linear Scales*
Ch. 12. *Origin—Scatter Plot—Log-Linear Scales—Errors*
Ch. 12. *Origin—Scatter Plot—Log-Linear Scales—Multiple*
Ch. 12. *Origin—Scatter Plot—Log-Linear Scales—Weights*
Ch. 12. *Origin—Scatter Plot—Log-Linear Scales—Errors—Weights*
Ch. 12. *Origin—Scatter Plot—Log-Linear Scales—Errors—Weights from Errors*
Ch. 12. *Origin—Scatter Plot—Log-Log Scales*
Ch. 12. *Origin—Scatter Plot—Log-Log Scales—Errors*
Ch. 12. *Origin—Scatter Plot—Log-Log Scales—Multiple*
Ch. 12. *Origin—Scatter Plot—Log-Log Scales—Weights*
Ch. 12. *Origin—Scatter Plot—Log-Log Scales—Errors—Weights*
Ch. 12. *Origin—Scatter Plot—Log-Log Scales—Errors—Weights from Errors*
Ch. 12. *Origin—Polar Diagram*
Ch. 12. *Origin—Polar Diagram—Errors*
Ch. 12. *Origin—Polar Diagram—Weights*
Ch. 12. *Origin—Polar Diagram—Errors—Weights*
Ch. 12. *Origin—Polar Diagram—Errors—Weights from Errors*
Python
Ch. 12. *Python—Histogram*
Ch. 12. *Python—Column and Label*
Ch. 12. *Python—Scatter Plot—Linear Scales*
Ch. 12. *Python—Scatter Plot—Linear Scales—Errors*
Ch. 12. *Python—Scatter Plot—Linear Scales—Multiple*
Ch. 12. *Python—Scatter Plot—Linear-Log Scales*
Ch. 12. *Python—Scatter Plot—Linear-Log Scales—Errors*
Ch. 12. *Python—Scatter Plot—Linear-Log Scales—Multiple*
Ch. 12. *Python—Scatter Plot—Log-Linear Scales*
Ch. 12. *Python—Scatter Plot—Log-Linear Scales—Errors*
Ch. 12. *Python—Scatter Plot—Log-Linear Scales—Multiple*
Ch. 12. *Python—Scatter Plot—Log-Log Scales*
Ch. 12. *Python—Scatter Plot—Log-Log Scales—Errors*
Ch. 12. *Python—Scatter Plot—Log-Log Scales—Multiple*

(continued)

(continued)

Programs
R
Ch. 12. *R—Histogram*
Ch. 12. *R—Column and Label*
Ch. 12. *R—Scatter Plot—Linear Scales*
Ch. 12. *R—Scatter Plot—Linear Scales—Errors*
Ch. 12. *R—Scatter Plot—Linear Scales—Multiple*
Ch. 12. *R—Scatter Plot—Linear-Log Scales*
Ch. 12. *R—Scatter Plot—Linear-Log Scales—Errors*
Ch. 12. *R—Scatter Plot—Linear-Log Scales—Multiple*
Ch. 12. *R—Scatter Plot—Log-Linear Scales*
Ch. 12. *R—Scatter Plot—Log-Linear Scales—Errors*
Ch. 12. *R—Scatter Plot—Log-Linear Scales—Multiple*
Ch. 12. *R—Scatter Plot—Log-Log Scales—Errors*
Ch. 12. *R—Scatter Plot—Log-Log Scales*
Ch. 12. R—*Scatter Plot—Log-Log Scales—Multiple*

Problems

12.1 **[E.O.P.R.]** The position y of a moving body is given as a function of time in the table that follows:

t (s)	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
y (m)	1.1	2.6	4.1	4.4	7.1	7.6	9.1	10.0	10.5

Mark the points (t, y) in a graph.
Make sure that a straight line would adequately describe the relation $y(t)$ and that there are no points that should be rejected as due to measurements with excessive errors.
Assume that a relation of the form $y = \alpha + \lambda t$ holds for the variables y and t.
Complete a table and use the method of least squares in order to determine the parameters α, λ and their errors $\delta\alpha$ and $\delta\lambda$.
Mark the point K:$(\bar{t},\ \bar{y}\,)$ on the graph.
Draw the following straight lines on the graph:

1. the straight line $y = \alpha + \lambda t$, using the values found for α and λ
2. the straight line passing through point K and having a slope equal to $\lambda - \delta\lambda$
3. the straight line passing through point K and having a slope equal to $\lambda + \delta\lambda$.

12.2 **[E.O.P.R.]** Given in the table that follows, as a function of time t, are the measurements of the distance y of a body from the origin, as this moves on the y-axis.

t (s)	1.0	1.5	1.8	2.0	2.2	2.5
y (m)	30	50	68	73	100	110

(a) Mark the points (t, y) on a graph. Suppose that the values of t are known with errors equal to ± 0.2 s and that the values of y have fractional errors equal to $\pm 7\%$ and draw the errors $\pm\delta t$ and $\pm\delta y$ for each point. Investigate whether there are any points that you think should be neglected as the results of wrong measurements. Draw with a pencil the best (smooth) curve passing between the points of the graph (perfection is desirable but unattainable!).

(b) Assume that for y and t the relation $y = y_0 + \frac{1}{2}\gamma t^2$ holds. Construct a table in which you record the values of t, y and the corresponding values of the variable $x = t^2$. Evaluate also and enter in the table the errors δt, δy and δx in these magnitudes.

t (s)	δt (s)	y (m)	δy (m)	$x = t^2$ (s^2)	δx (s^2)

Mark on a graph the points (x, y) and draw their errors $\pm\delta x$ and $\pm\delta y$. Verify that the relation $y(x)$ is linear to a good enough approximation and draw the best in your judgment straight line between the points of the graph. Without marking anything on the graph, find the coordinates of two points of the straight line (geometrical points, *not* experimental) and evaluate its slope. For better results, these points must be at a distance between them which is as large as possible. From the intercept of the line with the y-axis and from its slope, determine the values of y_0 and γ.

[E.O.P.R.] Use the methods of non-linear curve fitting to fit a curve of the form $y = y_0 + \frac{1}{2}\gamma t^2$ to the experimental points.

12.3 **[E.O.P.R.]** Given in the table that follows, as a function of time t, are the measurements of the distance y of a body from the origin, as this moves on the y-axis.

t (s)	1.0	2.0	3.0	4.0	5.0	6.0
y (m)	1.4	4.0	7.2	12.0	17.0	24.0

(a) Mark on a graph the points (t, y). Assume that the values of t are known with great accuracy but in the values of y there are fractional errors equal to $\pm 7\%$ and draw the errors $\pm\delta y$ for all the points. Investigate whether there are any points that you think should be neglected as the results of wrong measurements. Draw with a pencil and by hand the best (smooth) curve passing between the points of the graph.

(b) Assume that the theoretical relation between y and t is the expression $y = \upsilon_0 t + \frac{1}{2}\gamma t^2$. Make a table in which you record the values of t and y and the corresponding values of the variable $z = y/t$.

t (s)	y (m)	$z = y/t$ (m/s)

Plot in a graph the points (t, z). Satisfy yourselves that the function $z(t)$ is linear. Draw the best, in your opinion, straight line that passes between these points. Determine the coordinates of two points of the line (geometrical points, *not* experimental) and evaluate its slope. From the intercept of the line with the z-axis and its slope determine the values of υ_0 and γ.

12.4 **[E.O.P.R.]** A mass m is connected to the free end of a spring which has a constant k, and the other end of which is fixed. Theory predicts that the motion of the mass, if this is displaced from its equilibrium point and let free, is simple harmonic, with a period equal to $T = 2\pi\sqrt{m/k}$.

The table below shows the values of T for various masses m connected to the same spring.

m (kg)	0.1	0.2	0.3	0.4	0.5	0.6	0.7
T (s)	1.00	1.50	1.80	2.05	2.15	2.55	2.70

(a) Mark the points (m, T) on a graph. Assume that the values of m are known with great accuracy and that the values of T have fractional errors equal to $\pm 5\%$ and draw the errors $\pm\delta T$ of each point. Investigate whether there are any points that you think should be neglected as the results of wrong measurements. Draw with a pencil and by hand the best (smooth) curve passing between the points of the graph.

(b) Use the theoretical relation $T = 2\pi\sqrt{m/k}$ for the determination of the constant k as follows: From the relation it follows that $T^2 = (4\pi^2/k)\, m$. Construct a table in which you record the values of m and T and the corresponding values of the variable $y = T^2$.

m (kg)	T (s)	$y = T^2$ (s^2)

Mark in a graph the points $(m, y = T^2)$ and the best, in your opinion, straight line passing between them. Read off the straight line the coordinates of two points (geometrical points, *not* experimental) and evaluate the slope of the straight line. From the value of the slope evaluate the constant k.

Reference

1. The Royal Society, *Quantities, Units, and Symbols*, 1975

Chapter 13
The Written Report of the Results of an Experiment

13.1 Introduction

The results of an experiment may be rendered useless if they are not presented in written form, so that they may be used by other researchers or the person who performed the measurements at a later time, if needed. Regarding the way in which an experimental procedure and its results may be presented, there is no unique recipe which could be followed at all times. The form of the report depends on the purpose for which it is written. In order of increasing 'importance', the following forms may be mentioned:

1. The presentation of an experiment performed in an educational laboratory. The aim here is usually for the student to give the instructor the opportunity to appreciate the experimental work performed and obtain the highest possible grade.
2. The presentation of the results of an experiment to a small circle of people, such as, for example, the members of the group to which a researcher belongs. Based on the report, the research group may possibly make important decisions regarding the future of their work.
3. The writing of a scientific article, either for presentation at a conference or for publication in a scientific journal.
4. The presentation of experimental results in the framework of a diploma thesis or of postgraduate studies.
5. The exposition of experimental results in a monograph, intended for publication so that it becomes available to a wide public.

The way in which the report is written differs, depending on the purpose for which it is intended. There is a rich bibliography concerning the writing of scientific reports, articles and theses [1] as well as the presentation of the results to audiences [2]. Publishers have their own specific rules which must be followed in the writing of books [3] and scientific papers [4]. The rules and the advice for the best

C. Christodoulides and G. Christodoulides, *Analysis and Presentation of Experimental Results*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-53345-2_13

presentation of a subject apply, of course, not only for the presentation of experimental results but for the results of purely theoretical work as well.

In this chapter, we will discuss the simplest problem of the writing of a report for the presentation of the experimental results and of the conclusions drawn from them, in the case of experimental work performed in the framework of an educational laboratory.

13.2 The Preparation of the Writing of the Report While Performing the Experiment

It must be stated from the start that, even in an educational laboratory in which the results of the experiment are usually known beforehand, a strict experimental procedure must be followed. If the purpose of the exercise is for the student to be trained in the proper experimental practice, he or she must behave as if performing an important experiment for the first time.

The writing of the report on the results of an experiment begins with the entrance of the experimenter in the laboratory (or even earlier). The recording of all the information which are relevant to the experiment as well as the actions taken in the laboratory are of fundamental importance and a prerequisite for the writing of a useful report.

The experimenter must record all the information acquired as well as the details of his actions during the experiment. It is recommended that for this purpose a *notebook* is used rather than loose sheets of paper as the latter may be easily lost or not be kept in the right order to describe what was performed in the right time sequence.

What is worth recording is largely a matter of judgment by the experimenter. Obviously the numerical readings of instruments must be written down, but the experimenter must decide which other information is necessary for the interpretation of the results (for example, in some experiments knowing the temperature in the laboratory might be needed but not in others). Since the possibilities are countless, the experimenter must have as complete an understanding of the phenomenon to be studied as possible. As a complete understanding is of course impossible (otherwise there would be no need for the experiment to be performed!) there is obviously a need for very good *preparation* for the experiment with a thorough study of the relevant bibliography before starting the experimental work.

Recording the information in the laboratory notebook must be made having in mind that the analysis of the observations will possibly take place some considerable time later. Facts that the experimenter knows well at the time of the experiment may not be accurately recorded in his memory. This means that the information must be recorded (in legible form!) with clarity and with as much detail as would be needed for the notes to be understood by another person. Drawings,

makeshift and by hand, are extremely useful and indispensable in the description of apparatus or experimental arrangements.

A general but by no means complete enumeration of the useful information that must be recorded during the experiment, could be the following:

1. The title (if there is one) of the experiment to be performed.
2. The names of the instructor and of the co-workers for the particular experiment.
3. The date and time. The time should be recorded whenever a new procedure is started.
4. The aim of the experiment.
5. Environmental conditions which might prove useful.
6. The relevant bibliography used, as well as any additional documents that might be used in the analysis of the results.
7. The procedure of preparation of the experiment.
8. Drawings and a description of the experimental setup used.
9. The results of the measurements. Wherever necessary, the numerical values should be recorded in well designed tables, with clear descriptions of the magnitudes recorded (magnitude, units, common multiplying factor if there is one etc.).
10. The problems met and the questions that arose during the experiment and the actions taken for their resolution. Anything that is considered relevant for the analysis of the results.
11. Ideas the experimenter may have had during the experiment, which, although in a crude form at the time, might be adequately developed later and help in the better understanding of the results.
12. Suggestions for further work that might be done or for improvement of the procedures followed that might help the same or another experimenter in a repetition of the experiment at a later time. Ideas for similar research that might be done.

It is obvious that this list is simply indicative and may differ from experiment to experiment.

One good piece of advice is that if the need arises for the rejection of some results, these should be crossed out in such a way that they remain legible. A thin straight line drawn over them, preferably in a different color, is all that is needed for the purpose. Very frequently, results which were considered useless at some stage of the experiment turn out to be correct. The same is true for a whole experimental procedure which had been followed before being replaced by another which was considered at the time to be a better one.

The writing of the report describing an experiment and analyzing its results commences in the laboratory. During the experiment, the experimenter must compute intermediate results and draw, roughly, graphs, in order to check the development of the experiment and take the necessary measures for the optimum continuation of it, at every stage. The accuracy of the calculations does not have to be high and the graphs may be drawn by hand whenever that is satisfactory. Their value, however, is enormous.

The experimental setup must not be dismantled before the experimenter is satisfied that there is no need for additional measurements to be taken. In scientific research, this happens after the results have been analyzed completely. Very often, during the analysis of the results, it is found that additional measurements have to be made or that some needed information has not been recorded. In an educational laboratory this is usually impossible to do. For this reason, before leaving the laboratory, the experimenter should be satisfied that the experiment has been concluded.

13.3 The Written Report of an Experiment

The written report of an experiment in the framework of an educational laboratory must contain the following parts:

1. *Title.* The title must be brief and comprehensive.
2. *Names.* The names of the people who conducted the experiment should be given, as well as that of the supervisor, if any.
3. *Dates and time.* The date and time on which the experiment was conducted.
4. *Summary.* In about one paragraph, a short description should be given of the work done, the scientific reasons for which this was done and the method followed. The main results obtained may also be mentioned.
5. *Theory.* A *short* summary of the theoretical background of the experiment should be given, without this being an extensive presentation of theory and unnecessary equations. The method which is to be followed for the extraction of the information needed from the experimental results should also be described.
6. *Apparatus used.* A detailed list should be given of the equipment available and used in the experiment. All relevant technical specifications should be mentioned.
7. *Experimental results.* The crude results of the measurements should be presented. Whenever possible, the results should be given in tables, in the order in which these were performed, before being subjected to any arithmetical treatment. Both the experimental arrangement and the procedures followed in taking the results must be described with as much detail as needed. In this part, drawings showing the experimental setup are particularly useful.
8. *Analysis of the results.* The analysis of the results should be presented with clarity, without it being necessary to show all the intermediate results of the arithmetic calculations. On the other hand, results should not appear out of nowhere! In this part, extensive use of tables and graphs should be made.

9. *Conclusions.* The conclusions that follow from the analysis of the measurements should be presented and justified. A short general assessment of the success or not of the whole exercise should also be given.
10. *Bibliography.* A list of the bibliography used should be given (books, journals, theses, manuals etc.). There are many ways of presenting references and bibliography. For the purposes of an educational laboratory, the following method is adequate:

    1. Books:
       H.D. Young and R.A. Freedman, *University Physics with Modern Physics* (Pearson—Addison Wesley, 11th edition, 2004). Chaps. 6, 9.
       If a single reference is made to a book, the relevant chapters or pages are given in the bibliography and this is referred to in the text as (Young and Freeman 2004). Otherwise, multiple references are made in the text: (Young and Freeman 2004, Chap. 6) and elsewhere (Young and Freeman 2004, pp 345–9).
    2. Journal articles:
       C. Christodoulides (2017), 'The construction of a perpetual motion machine of the fourth kind', *Journal of Non-Reproducible Results*, **1**, 1–665.
       The name of the journal is given in full or in the accepted abbreviated form. The title of the article is optional but can be of great help if given. The volume number of the journal is given in bold. The number of the article's first page is then given. The end page might also be useful.
       A. Genius (2003), 'The creation of a mini black hole in the educational laboratory'. *J. Black Hole Res.*, **13**, (7), 123–4.
       Here, the issue number of the journal (7) is also given.
    3. Reference books—tables:
       G.W.C. Kaye and T.H. Laby, *Tables of Physical and Chemical Constants* (Longmans, 14th ed., 1973). Page 165, Table 15.6.
       W.M. Haynes (ed.), *CRC Handbook of Chemistry and Physics* (CRC Press, 95th ed., 2014). Page 983.
       The reference is given in a manner similar to the way a book is referred to (1). The table or graph used should be mentioned with enough detail so that a reader may easily find the information which has been used.
    4. Web pages:
       The NIST reference on Constants, Units and Uncertainty.
       http://physics.nist.gov/cuu/Constants/
       National Physical Laboratory, *Tables of Physical and Chemical Constants*.
       http://www.kayelaby.npl.co.uk/

Some general rules for the write-up of the report are the following:

(a) The tense used is that appropriate to what is described. The experiment was performed in the past, so the past tense must be used in describing it. Phrases are used such as: '*The experimental arrangement used is shown in Figure 1*' or '*The resistance was varied from 0 to 1000 Ω, in steps of 100 Ω*' etc.
(b) The use of impersonal expressions is preferred. For example, instead of the statement '*We decided to investigate further the range of current values between 20 and 30 mA*', it is preferable to write: '*It was judged necessary for the range of current values between 20 and 30 mA to be investigated in greater detail*'.
(c) When referring to decisions that are taken by us or to conclusions we reached during the analysis of the results, the use of first person present tense is allowed and, in some cases, necessary. For example, we may write: '*We conclude that the two results for the electronic charge agree with each other, within experimental error*'. This can also be written as: '*It is concluded that the two results for the electronic charge agree with each other, within experimental error*'.
(d) Bibliographic references should be given in the text with adequate detail. For example, a reference may be given as (Kittel, p. 467). If the same author is mentioned more than once for the same year in the bibliography, the distinction between the two or more books or articles can be achieved by using letters (a, b, …) after the year of publication (e.g. Genius 2003a, b). The details of the book or article (publisher, place and time of publication, edition number, volume) are given in the bibliography table. It must be borne in mind that there are many different ways of giving bibliographic references.

13.4 An Example of a Written Report of an Experiment

An example of a written report of an experiment conducted in the framework of an educational laboratory is given below. The method used is by no means unique but it may be used as a rough guide for similar reports.

Although software is available for evaluating mean values, standard deviations and least-squares parameters, in this exercise these magnitudes will be evaluated 'by hand', in order to understand the principles of the methods.

Figure 13.1 shows the notes taken in the laboratory during an experiment.

EXERCISE 17

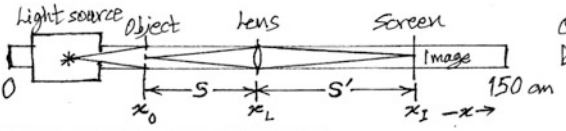
The measurement of the focal length of a lens, its refractive index and the observation of its chromatic aberration

Instr.: M. Faraday Partner: G. Marx 1 April 2015

Equipment: Optical bench, light source. Stands. Lenses. Wratten filters. Slides (object, screen). Spherometer. Ruler

PART A

Setup

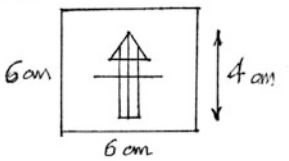


i	x_L (cm)	x_0 (cm)	x_I (cm)	s_i (cm)	s'_i (cm)	h'_i* (cm)
1	50	25	145.2	25	95.2	—**
2	"	22.5	125.1	27.5	75.1	—**
3	"	20	107.3	30	57.3	—
4	"	15	97.4	35	47.4	-5.2
5	100	60	140.5	40	40.5	-4.2
6	"	55	139.3	45	39.3	-3.6
7	"	50	130.8	50	30.8	-2.2
8	"	40	129.0	60	29.0	-1.9
9	"	30	127.1	70	27.1	-1.7
10	"	20	125.8	80	25.8	-1.0

* Negative because the image is inverted.
** For $s = 25$–30 cm h'_i cannot be measured because the image is too big.

Light source: White light

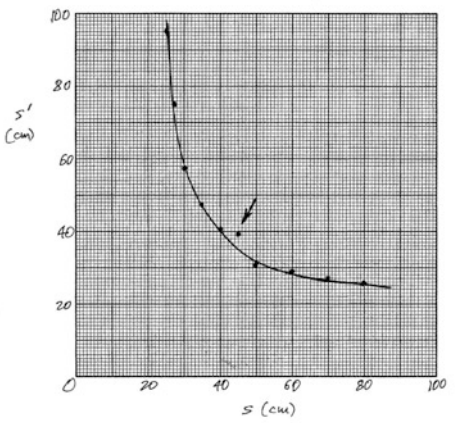
Object:



Scale readings: ± 0.5 mm
Errors in s_i, s'_i: ± 1 mm

Accuracy in focusing (approx.): from ±0.3 cm for s near 40 cm to ±1 cm for $s = 25$ cm.

Plot of $s'(s)$



The measurement for $s = 45$ cm looks wrong.

Repeat the measurement:

$s = 45$ cm $s' = 36.2$ $h' = -3.6$ cm

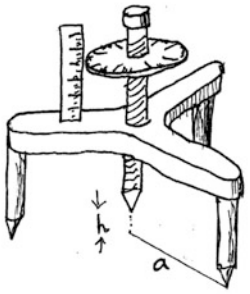
Fig. 13.1 The notes taken in the laboratory during an experiment

PART B

Sherometer

Distance of central pointed end from 3 others: $a = 3$ cm (given by manufacturer)

Accuracy of h readings: ± 0.0005 cm



Radii of lens using the spherometer (both positive)

For R_1

i	h_{1i} cm
1	0.1875
2	0.1820
3	0.1845
4	0.1860
5	0.1825

For R_2

i	h_{2i} cm
1	0.1865
2	0.1820
3	0.1835
4	0.1850
5	0.1825

PART C

1. The image is formed using red or blue light:

Filters used: red: Wratten 24 blue: Wratten 38 A

For $s = 40$ cm
Distance of image with red light $s'_R = 40.5$ cm
" " " " blue " $s'_B = 38.5$ cm (± 1 cm)

2. Chromatic aberration of lens:

Form image of the filament of the light source. Spectral analysis is visible near edges of image.

3. Chromatic aberration of telescope:

Lens has diameter $d \approx 12$ cm and $f \approx 25$ cm.

Holding the lens with one hand and a jeweller's eyepice near the eye, we have a tesescope (astronomical). Magnification $\sim \times 8$.

Chromatic aberration is clearly visible.
Blue and red images slightly different.
Spectral analysis visible near the edges of the image

Fig. 13.1 (continued)

Exercise 17

The Measurement of the Focal Length of a Lens, its Refractive Index and the Observation of its Chromatic Aberration

A. Fresher

Instructor: M. Faraday Partner: G. Marx

Date the experiment was performed: 1 April, 2017

Aim of the Experiment

In this experiment the focal length of a converging lens was measured. From the knowledge of the lens's focal length and the measurement of the radii of its spherical surfaces, the refractive index of the glass the lens is made of was evaluated. Finally, the phenomenon of chromatic aberration in the lens was observed.

Bibliography

C. Christodoulides and G. Christodoulides, *Analysis and Presentation of Experimental Results* (Springer, 2017). To be referred to as APER.

E. Hecht, *Optics* (Addison-Wesley, 3rd ed., 1998).

W.M. Haynes (ed.), *CRC Handbook of Chemistry and Physics* (CRC Press, 95th ed., 2014).

H.D. Young and R.A. Freedman, *University Physics with Modern Physics* (Pearson–Addison Wesley, 11th edition, 2004). To be referred to as Y&F.

Theory

A converging lens may be used for the formation of the real image of an object. If s and s' are the distances from the center of the lens of the object and of the image, respectively, the relation between them is (Y&F, Sect. 34.4)

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'} \tag{1}$$

where f is a length characteristic of the lens, its focal length (Fig. 1a). By measuring the values of s' for a series of values of s, the focal length of the lens may be determined.

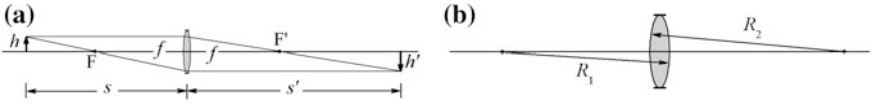


Figure 1 **a** Formation of the real image of an object by a converging lens. **b** The radii of the lens's spherical surfaces

The size of the image is related to the size of the object through the expression

$$\frac{h'}{h} = -\frac{s'}{s}, \tag{2}$$

where the negative sign indicates the fact that the image is inverted relative to the object.

If R_1 and R_2 are the radii of the lens's two spherical surfaces (Fig. 1b), then its focal length is given by the lens maker's equation (Y&F, Sect. 34.4)

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \tag{3}$$

where n is the index of refraction of the lens's glass. With the sign convention used, R_2 must be taken as negative. The equation is valid for thin lenses, otherwise a correction is needed.

Equipment Available

The following equipments was supplied (Fig. 2):

An optical bench of 150 cm length.
A source of white light.
A set of Wratten filters.
Stands which can be placed on the optical bench and can hold various optical elements.
Lenses: The lens to be studied, a hand held magnifier and a jeweler's eyepiece.
A slide with the shape of an arrow drawn in transparent lines on it. To be used as the object.
A screen on which the image may be formed.
A ruler.

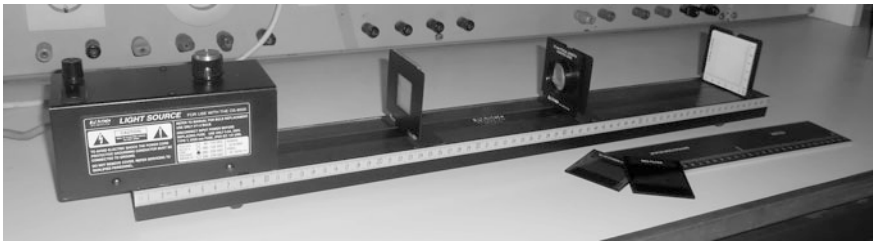


Figure 2 The experimental setup

Experimental Procedure
The experiment was performed in three parts:

Part A. In this part of the experiment the focal length of a converging lens was determined by measuring the distances of the images from the lens for various distances of an object.
Part B. The index of refraction of the glass of the lens was determined by measuring the radii of its surfaces and using the value of its focal length found in Part A.
Part C. Phenomena were observed which are associated with the chromatic aberration of the lens.

Presentation of the Experimental Procedure and Analysis of the Results

Part A. The Determination of the Focal Length of a Converging Lens by Measuring the Distances of the Images from the Lens for Various Distances of the Object
The experimental arrangement used in this part of the exercise is shown in Figs. 2 and 3. The experiment was conducted in a darkened part of the laboratory. On an optical bench a source of white light was placed and stands that could be moved along the bench. The position of each element on the bench could be read on a scale along it, with an accuracy of 1 mm. The length of the optical bench was 150 cm.

A converging lens was placed on a stand approximately at the middle of the bench, at a distance of x_L from the end of the bench on which the source was situated. As object, an opaque slide provided was used, which was transparent along thin lines on it forming an arrow of height $h = 4.0 \pm 0.1$ cm (Fig. 4). The object was placed at x_O. The light from the source illuminated the slide from behind

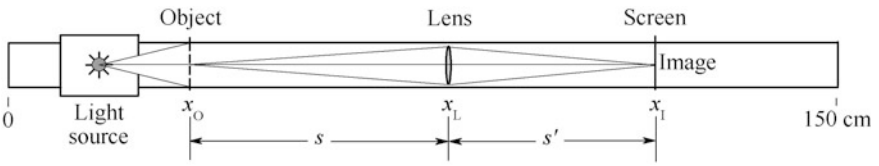


Figure 3 The optical bench used with the light source, the object, the lens and the screen in position

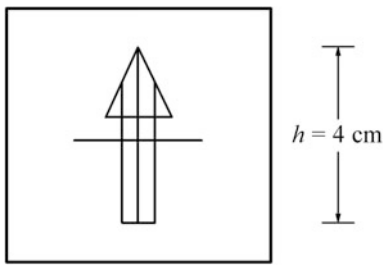


Figure 4 The slide used as object. The thin lines were transparent

Table 1 The measurements for the first part of the exercise

i	x_L (cm)	x_O (cm)	x_I (cm)	s_i (cm)	s_i'(cm)	h_i'(cm)
1	50	25	145.2	25	95.2	–
2	50	22.5	125.1	27.5	75.1	–
3	50	20	107.3	30	57.3	–
4	50	15	97.4	35	47.4	−5.2
5	100	60	140.5	40	40.5	−4.2
6	100	55	139.3	45	39.3	−3.6
7	100	50	130.8	50	30.8	−2.2
8	100	40	129.0	60	29.0	−1.9
9	100	30	127.1	70	27.1	−1.7
10	100	20	125.8	80	25.8	−1.0

so that light rays would emerge from it through the transparent lines. The image of the arrow was formed on a screen placed on the other side of the lens, at x_I. Keeping the lens in a fixed position, the object was placed at various distances s from the center of the lens. At each position the screen was moved along the bench until a clear well focused image of the object was formed on it. Theory suggests that s and s' are related to each other by the expression of Eq. (2). Since the optical bench was only 150 cm long, in order to use values of s and s' in as large a range as possible, the first 4 measurements were performed with the lens at $x_L = 50$ cm and then the lens was moved to $x_L = 100$ cm. The magnitudes read off the scale were x_L, x_O and x_I. The distance of the object from the lens, $s = x_L - x_O$, was chosen to be a round number of centimeters. The distance of the image from the lens, $s' = x_I - x_L$, was determined as described above.

The values of x_L, x_O and x_I, together with the calculated values of s and s' are shown in Table 1. The height of every image formed, h_i', was also measured and is given in the table. The values given are negative, to state the fact that the image was inverted relative to the object. It was not possible to measure the size of the image for the distances of $s = 25$ cm, $s = 27.5$ cm and $s = 30$ cm, as in these cases the image was too big, bigger than the slide serving as screen.

A plot of s' as a function of s drawn in the laboratory (Fig. 5) suggested that point #6 might be the result of a mistake in reading the scale as its deviation from the curve between the points seems rather large. The measurement for $s = 45$ cm was repeated and it was found that $s' = 36.2$ cm. The results, with the new value of measurement #6 are given in columns 2 and 3 of Table 3.

The possibility that result #6 is spurious will be tested by evaluating the focal length f_i of the lens substituting each pair of values of s and s' in Eq. (1). This is done with the aid of Table 2.

From these 10 values it is found (APER, Sect. 4.2) that:

The mean value of the focal distance: $\overline{f} = \frac{198.56}{10} = 19.86$ cm.

The graph drawn in Fig. 5 is $s'(s)$ from Eq. (1) with $f = 19.9$ cm.

The standard deviation of the distribution of f values: $\sigma_f = \sqrt{\frac{2.4328}{10}} = 0.49$ cm.

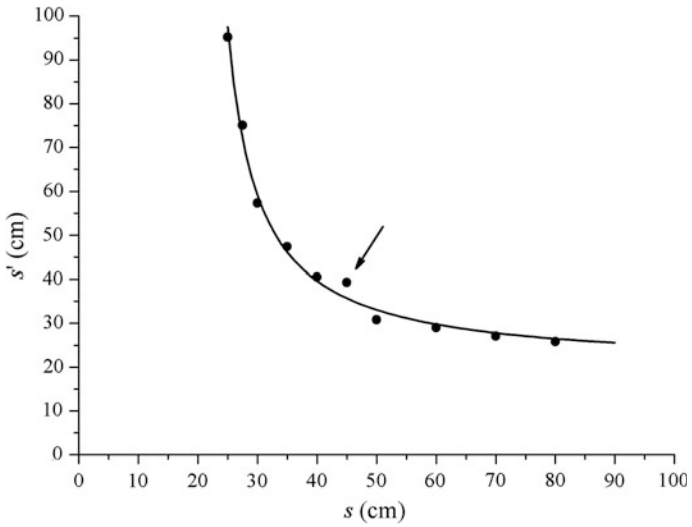


Figure 5 Plot of s' as a function of s. Initial measurements. The measurement for s = 45 cm is suspected to deviate too much from the best curve through the points. The line drawn is $s'(s)$ from Eq. (1) with f = 19.9 cm

Table 2 The initial measurements for the first part of the exercise

i	s_i (cm)	s'_i (cm)	f_i (cm)	$f_i - \bar{f}$ (cm)	$(f_i - \bar{f})^2$ (cm^2)
1	25	95.2	19.80	−0.06	0.0036
2	27.5	75.1	20.13	0.27	0.0729
3	30	57.3	19.69	−0.17	0.0289
4	35	47.4	20.13	0.27	0.0729
5	40	40.5	20.12	0.26	0.0676
6	45	39.3	20.98	1.12	1.2544
7	50	30.8	19.06	−0.80	0.6400
8	60	29.0	19.60	−0.26	0.0676
9	70	27.1	19.54	−0.32	0.1024
10	80	25.8	19.51	−0.35	0.1225
Sums			198.56		2.4328

It is seen that measurement #6 differs from the mean by 1.12 cm or by 2.3 σ_f.

Chauvenet's criterion (APER, Chap. 10) states that, for 10 measurements, a measurement differing from the mean by more than 1.96 σ_f should be rejected. This is the case here, so we reject the initial measurement for $s = 45$ cm is rejected.

Using the new measurement for $s = 45$ cm, we plot $s'(s)$ in Fig. 6.

Table 3 The measurements for the first part of the exercise, with the repeated measurement

i	s_i (cm)	s'_i (cm)	h'_i (cm)	f_i (cm)	$f_i - \overline{f}$ (cm)	$(f_i - \overline{f})^2$ (cm^2)
1	25	95.2	–	19.80	0.04	0.0016
2	27.5	75.1	–	20.13	0.37	0.1369
3	30	57.3	–	19.69	−0.07	0.0049
4	35	47.4	−5.2	20.13	0.37	0.1369
5	40	40.5	−4.2	20.12	0.36	0.1296
6	45	36.2	−3.6	20.06	0.30	0.0900
7	50	30.8	−2.2	19.06	−0.70	0.4900
8	60	29.0	−1.9	19.60	−0.16	0.0256
9	70	27.1	−1.7	19.54	−0.22	0.0484
10	80	25.8	−1.0	19.51	−0.25	0.0625
Sums				197.64		1.1264

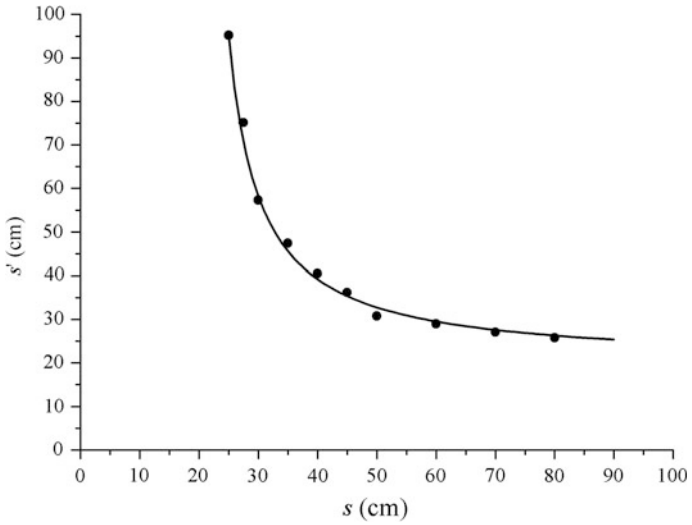


Figure 6 Plot of s' as a function of s. With the measurement for $s = 45$ cm repeated. The line drawn is $s'(s)$ from Eq. (1) with $f = 19.8$ cm

Determination of the focal length of the lens using the equation $1/f = 1/s + 1/s'$ *and the pairs of values of s and* s'.

The focal length of the lens may be evaluated by substituting in Eq. (1) each pair of the values of s and s'. The values of f_i thus found are given in the 5th column of Table 3. From these 10 values of f the mean may be found as well as the standard deviation of the distribution of the values and the standard deviation of the mean.

Mean value of the focal distance: $\overline{f} = \frac{197.64}{10} = 19.76\,\text{cm}$.

The graph drawn in Fig. 6 is $s'(s)$ from Eq. (1) with $f = 19.8\,\text{cm}$.

Table 4 The data used for testing the validity of equation $-h'/h = s'/s$

i	s_i (cm)	s'_i (cm)	$-h'_i$ (cm)	s'_i/s_i	$-h'_i/h$
1	25.0 ± 0.2	95.2 ± 4.8	–	3.81 ± 0.19	–
2	27.5 ± 0.2	75.1 ± 3.8	–	2.73 ± 0.14	–
3	30.0 ± 0.2	57.3 ± 2.9	–	1.91 ± 0.10	–
4	35.0 ± 0.2	47.4 ± 2.4	5.2 ± 0.3	1.35 ± 0.07	1.30 ± 0.10
5	40.0 ± 0.2	40.5 ± 2.0	4.2 ± 0.2	1.01 ± 0.05	1.05 ± 0.07
6	45.0 ± 0.2	36.2 ± 1.8	3.6 ± 0.2	0.80 ± 0.04	0.90 ± 0.07
7	50.0 ± 0.2	30.8 ± 1.5	2.2 ± 0.2	0.62 ± 0.03	0.55 ± 0.06
8	60.0 ± 0.2	29.0 ± 1.5	1.9 ± 0.2	0.48 ± 0.02	0.48 ± 0.06
9	70.0 ± 0.2	27.1 ± 1.4	1.7 ± 0.2	0.39 ± 0.02	0.43 ± 0.05
10	80.0 ± 0.2	25.8 ± 1.3	1.0 ± 0.1	0.32 ± 0.02	0.25 ± 0.03

The standard deviation of the values of f is: $\sigma_f = \sqrt{\frac{1.1264}{10}} = 0.34\,\text{cm}$.

Having rejected one of the initial measurements, the next candidate for rejection is the measurement for $s = 50\,\text{cm}$. It differs from the new mean by 0.7 cm or 2 σ_f. We consider this measurement to be acceptable.

From Table 4 it is found that the standard deviation of the mean $\overline{f}$ is: $\sigma_{\overline{f}} = \sqrt{\frac{1.1264}{10\times 9}} = 0.11\,\text{cm}$.

The final result for the focal length of the lens used is, therefore,

$$\boxed{f = 19.8 \pm 0.1\,\text{cm}}$$

Test of the validity of the equation $-h'/h = s'/s$ for the size of the image

The values of the image size, h' are used in order to check the validity of Eq. (2). In this experiment, the object size was $h = 4.0 \pm 0.1\,\text{cm}$. Due to the uncertainties in determining the positions of the lens and the image, the error in s was of the order of 2 mm. The errors in s' were estimated to be of the order of 5%. The best estimate for the errors in h' is (1 mm + 5%). The first term results from the procedure of measuring the size of the image with a ruler and the second from the limited definition in the image. In Table 4 we give the values of s, s', h', s'/s and h'/h, with their errors.

In Fig. 7, $-h'/h$ is plotted as a function of s'/s. The errors in both variables are also marked. Drawing a straight line passing through the origin and between the points of the graph, it is seen that the linear relation is verified within the accuracy of the measurement, in agreement with Eq. (2).

Determination of the focal length of the lens using the method of least squares on the linearized graph of the equation $1/f = 1/s + 1/s'$.

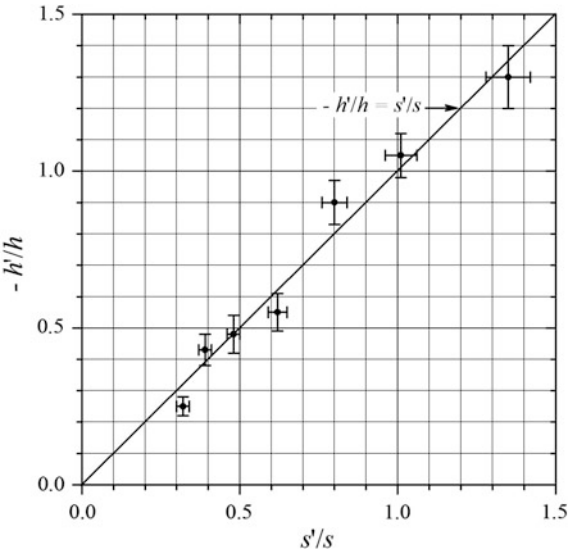


Figure 7 The ratio of the image height to the object height as a function of the ratio s'/s of their respective distances from the lens

In Table 5 are recorded: the values of s and s', the values of $x = 1/s$ and $y = 1/s'$, and of the products needed for the application of the method of least squares.

In terms of x and y, Eq. (1) takes the form $y = 1/f - x$. The graph of $y = 1/s'$ as a function of $x = 1/s$ is plotted in Fig. 8. As expected, the points lie near a straight line which has a slope of about -1. This line was determined using the method of least squares.

Assuming that the best fit to the experimental points is given by the straight line $y = a + \lambda x$, the coefficients are found from (APER, Sect. 11.3)

$$\alpha = \frac{[y][x^2] - [x][xy]}{N[x^2] - [x]^2} = 0.0512\,\text{cm}^{-1} \quad \lambda = \frac{N[xy] - [x][y]}{N[x^2] - [x]^2} = -1.031$$

If $d_i \equiv y_i - \alpha - \lambda x_i$, the errors in α and λ are

$$\delta\alpha = \sigma_\alpha = \sqrt{\frac{1}{N-2} \cdot \frac{[x^2][d^2]}{N[x^2] - [x]^2}} = 0.0012\,\text{cm}^{-1} \quad \text{and}$$

$$\delta\lambda = \sigma_\lambda = \delta\alpha \sqrt{\frac{N}{[x^2]}} = 0.048$$

so, finally, $\alpha = 0.0512 \pm 0.0012$ cm^{-1} and $\lambda = -1.031 \pm 0.048$.

Table 5 The quantities needed in the application of the method of least squares

i	s_i (cm)	s_i' (cm)	$x_i = 1/s_i$ (cm^{-1})	$y_i = 1/s_i'$ (cm^{-1})	x_i^2 (10^{-4} cm^{-2})	$x_i y_i$ (10^{-4} cm^{-2})	d_i (cm^{-1})	d_i^2 (10^{-6} cm^{-2})
1	25	95.2	0.0400	0.0105	16.00	4.20	0.00054	0.292
2	27.5	75.1	0.0364	0.0133	13.22	4.84	−0.00037	0.137
3	30	57.3	0.0333	0.0175	11.11	5.82	0.00063	0.397
4	35	47.4	0.0286	0.0211	8.16	6.03	−0.00061	0.372
5	40	40.5	0.0250	0.0247	6.25	6.17	−0.00073	0.533
6	45	36.2	0.0222	0.0276	4.94	6.14	−0.00298	8.880
7	50	30.8	0.0200	0.0325	4.00	6.49	0.00192	3.686
8	60	29.0	0.0167	0.0345	2.78	5.75	0.00048	0.230
9	70	27.1	0.0143	0.0369	2.04	5.27	0.00044	0.194
10	80	25.8	0.0125	0.0388	1.56	4.84	0.00049	0.240
		Sums	0.2490 = $[x]$	0.2555 = $[y]$	70.07 × 10^{-4}= $[x^2]$	55.56 × 10^{-4}= $[xy]$		14.961 × 10^{-6} = $[d^2]$

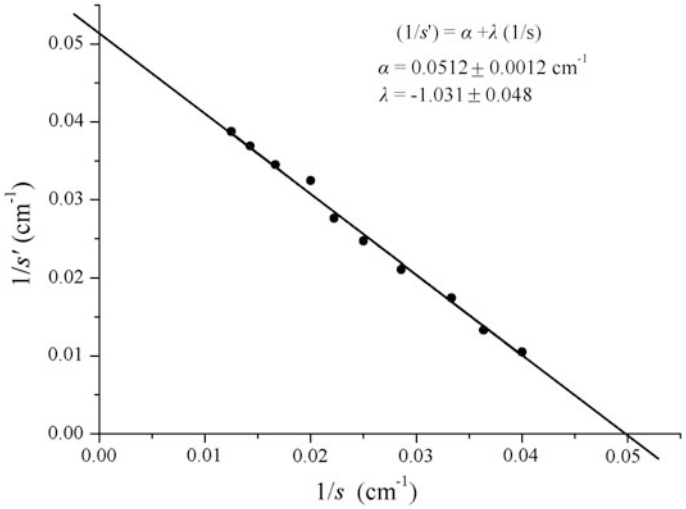


Figure 8 Plot of $1/s'$ as a function of $1/s$

The focal length of the lens is $f = 1/\alpha$ and its error $\delta f = \delta\alpha/\alpha^2$. Substituting,

$$\boxed{f = 19.5 \pm 0.5\,\text{cm}}$$

Part B. The Index of Refraction of the Glass of the Lens from its Radii and its Focal Length

The index of refraction of the glass the lens is made of may be determined from a knowledge of the lens's radii and its focal length.

Measurement of the radii of the lens using a spherometer

A spherometer is shown on Fig. 9. The pointed ends of the three legs of the spherometer, forming an equilateral triangle, are placed on the surface of the lens whose curvature is to be measured. A central leg is then lowered until it makes contact with the surface. The height h of the point of the central leg above the plane of the other three is indicated on the spherometer's scale. One complete turn of the dial shifts the central leg by 0.5 mm. The dial is graduated into subdivisions, making it possible to measure h with greater precision. With the spherometer used, the accuracy with which this is measured is ± 0.0005 cm

If the distance of the three fixed legs of the spherometer from each other is a and the radius of the sphere being measured is R, then, with reference to Fig. 9, it is

$$(\text{OC}) = h + \sqrt{(\text{OA})^2 - a^2} \quad \text{or} \quad R = h + \sqrt{R^2 - a^2}.$$

Squaring, $R^2 - a^2 = R^2 - 2Rh + h^2$, from which it follows that

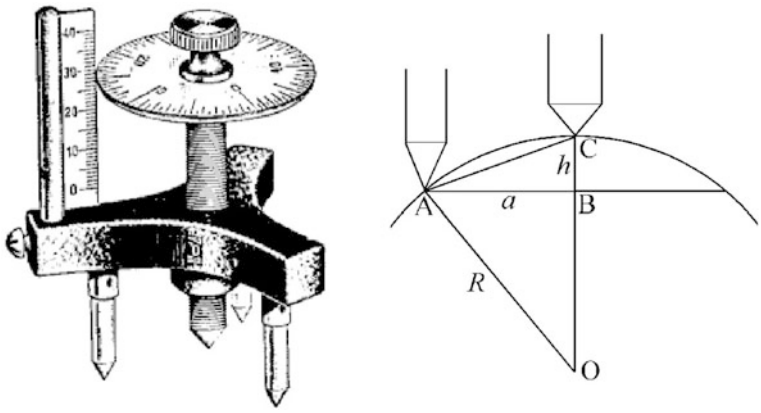


Figure 9 A spherometer and its geometry

$$R = \frac{a^2}{2h} + \frac{h}{2}. \tag{4}$$

The results of the measurements performed on the two surfaces of the lens are given in the second columns of the two following tables.

Measurements for R_1			
i	h_{1i} (cm)	$h_{1i} - \overline{h_1}$ (cm)	$\left(h_{1i} - \overline{h_1}\right)^2$ (10^{-6} cm^2)
1	0.1875	0.0030	9.00
2	0.1820	−0.0025	6.25
3	0.1845	0	0
4	0.1860	0.0015	2.25
5	0.1825	−0.0020	4.00
Sums	0.9225	0	21.50

Measurements for R_2			
i	h_{2i} (cm)	$h_{2i} - \overline{h}_2$ (cm)	$\left(h_{2i} - \overline{h}_2\right)^2$ (10^{-6} cm^2)
1	0.1865	0.0026	6.76
2	0.1820	−0.0019	3.61
3	0.1835	−0.0004	0.16
4	0.1850	0.0011	1.21
5	0.1825	−0.0014	1.96
Sums	0.9195	0	13.70

The first table gives: $\overline{h}_1 = 0.9225/5 = 0.1845\,\text{cm}$ and $\sigma_{\overline{h}_1} = \sqrt{21.50 \times 10^{-6}/5 \times 4} = 0.0010\,\text{cm}$.

The second table gives: $\overline{h}_2 = 0.9195/5 = 0.1839\,\text{cm}$ and $\sigma_{\overline{h}_2} = \sqrt{13.70 \times 10^{-6}/5 \times 4} = 0.0008\,\text{cm}$.

The final results are $h_1 = 0.1845 \pm 0.0010\,\text{cm}$ and $h_2 = 0.1839 \pm 0.0008\,\text{cm}$.

The errors are 0.54% for h_1 and 0.44% for h_2.

The radii corresponding to these values of h are found, given that, according to the manufacturers of the spherometer, $a = 3.000\,\text{cm}$ with an error of the order of 0.05%. The values of R_1 and R_2 are found using Eq. (4): $R_1 = 24.48\,\text{cm}, R_2 = 24.56\,\text{cm}$.

The errors in R_1 and R_2 are given by (APER, Sect. 6.2.3)

$$\delta R = \sqrt{\left(\frac{\partial R}{\partial h}\right)^2 (\delta h)^2 + \left(\frac{\partial R}{\partial a}\right)^2 (\delta a)^2} = \sqrt{\left(\frac{1}{2} - \frac{a^2}{2h^2}\right)^2 (\delta h)^2 + \left(\frac{a}{h}\right)^2 (\delta a)^2}.$$

By substitution it is found that $\delta R_1 = 0.13\,\text{cm}$ and $\delta R_2 = 0.11\,\text{cm}$. The final results for the radii of the lens are

$$R_1 = 24.48 \pm 0.13\,\text{cm} \quad \text{and} \quad R_2 = 24.56 \pm 0.11\,\text{cm}.$$

Determination of the index of refraction of the glass of the lens

The value of the refractive index of the glass of the lens may be found using the value of $f = 19.8 \pm 0.1\,\text{cm}$ for the focal length of the lens found in Part A and $R_1 = 24.48 \pm 0.13\,\text{cm}$ and $R_2 = 24.56 \pm 0.11\,\text{cm}$ in the lens maker's equation (Y&F, Sect. 34.4)

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right), \quad \text{from which it follows that} \quad n = 1 + \frac{R_1 R_2}{f(R_2 - R_1)}$$

(with R_2 negative, here). Substituting, it is found that $n = 1.6150$.

The error in n is found from the last equation and the general formula

$$\delta n = \sqrt{\left(\frac{\partial n}{\partial f}\delta f\right)^2 + \left(\frac{\partial n}{\partial R_1}\delta R_1\right)^2 + \left(\frac{\partial n}{\partial R_2}\delta R_2\right)^2}$$

to be

$$\delta n = (n-1)\sqrt{\left(\frac{\delta f}{f}\right)^2 + \frac{1}{(R_2 - R_1)^2}\left[\left(\frac{R_2}{R_1}\delta R_1\right)^2 + \left(\frac{R_1}{R_2}\delta R_2\right)^2\right]}.$$

Substituting, $\delta n = 0.6150\sqrt{0.00505^2 + 0.00266^2 + 0.00224^2} = 0.0038$.

Since it is

$$\begin{aligned}\delta n &= 0.6150 \times 0.00505 \times \sqrt{1 + 0.4742} \\ &= 0.6150 \times 0.00505 \times \sqrt{1.4742} \times \left\{ \frac{1}{\sqrt{1.4742}} + \frac{\sqrt{1.4742} - 1}{\sqrt{1.4742}} \right\} \\ &= 0.0038 \times (0.82 + 0.18)\end{aligned}$$

it is seen that 82% of the error in n is due to the error in f and 18% to the errors in R_1 and R_2.

Finally,

$$\boxed{n = 1.615 \pm 0.004}$$

is the index of refraction of the glass the lens is made of.

Part C. The Observation of the Chromatic Aberration of a Lens

The image of the object was focused using blue light and then red light, isolated by suitable filters (Fig. 10).

From tables (Hecht 1998, Fig. 3.37) it is found that the index of refraction for light flint glass is $n_B = 1.65$ for blue light and $n_R = 1.62$ for red light, approximately. For a lens with two convex surfaces with radii equal to $R = 24.5$ cm, the focal lengths for the two colors would be $f_B = 18.8$ cm and $f_R = 19.8$ cm. For an object at a distance of $s = 40$ cm from the lens, the images formed by the lens at the two colors would appear at distances $s'_B = 35.5$ cm and $s'_R = 39.2$ cm. The lens used, with $f_W = 19.8 \pm 0.1$ cm measured using white light, would give for $s = 40$ cm an image distance of $s'_W = 39.2$ cm.

Two filters were used in order to obtain blue or red light: A Wratten 38A filter in order to obtain blue light and Wratten 24 filter for red. From tables (*Handbook of Chemistry and Physics,* 2014), it is found that the dominant wavelength transmitted by each of the two filters are: for filter 38A $\lambda_B = 479$ nm and for filter 24 $\lambda_R = 611$ nm. With the object at $s = 40$ cm, the image was formed at a distance of $s'_B = 38.5 \pm 1.0$ cm for blue light and $s'_R = 40.5 \pm 1.0$ cm for red. The focusing was rather difficult and the uncertainties in the image distances might be greater than ± 1 cm. Nevertheless, the phenomenon of the dispersion of light and the chromatic aberration of the lens used were observed, at least qualitatively.

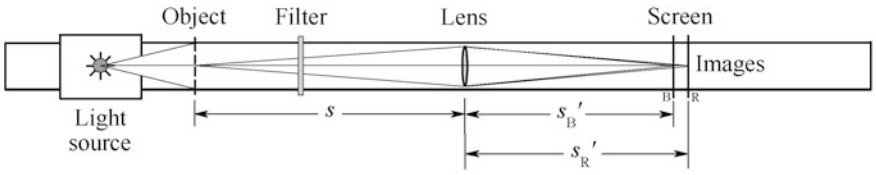


Figure 10 Forming the image of an object using *blue* (B) or *red* (R) light

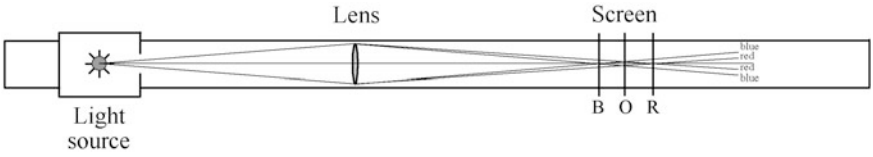


Figure 11 The observation of chromatic aberration by forming the image of the filament of the light source

Observation of chromatic aberration by forming the image of the filament of the light source. With the lens in a given position, an image of the light source's filament was formed on the screen (position O in Fig. 11). The image at position O had the best focusing that could be achieved. Moving the screen towards the light source (point B) the focusing changes, with blue being in focus but not red. The separation of the colors is clearly visible. Moving the screen away from the light source focuses red at the expense of blue. The distribution of colors is now different.

Observation of the chromatic aberration of a telescope. A hand-held stamp collector's lens and a jeweler's magnifier were held so that they formed an astronomical telescope, as shown in Fig. 12. The objective lens had a diameter of 12 cm and a focal length of about 25 cm. While the eyepiece was held in position using one hand, the objective lens, held in the other hand, was moved forward or backwards until focusing was achieved. The telescope had a magnification of 8, approximately. Chromatic aberration was clearly visible as the blue and red colors were separated at the edges of the objects being observed.

Figure 12 A makeshift astronomical telescope

Conclusions

The focal length of the converging lens studied was determined by two methods: By using the relation $1/f = 1/s + 1/s'$ to evaluate f directly and by applying the method of least squares to this relation, suitably linearized. The results were $f = 19.8 \pm 0.1$ cm and $f = 19.5 \pm 0.5$ cm, respectively. The first method is seen to be more accurate.

The refractive index of the glass the lens is made of was determined using the focal length found and the values of the lens's radii which were measured. The result was $n = 1.615 \pm 0.004$. The error in n is due largely to the error in f. This means that the best way to improve the accuracy in n is to lower the error in f.

The dispersion of light, although difficult to observe by focusing the images of an object illuminated by light of different colors, was observed qualitatively. This effect is important in the construction of telescopes, as verified experimentally.

Focusing the image was found to be easier for distances of s and s' near the value of $2f$. For this reason, it is expected that, if more measurements were performed in this region of values, a more accurate value of the lens's focal length f would have been obtained. Better results would be obtained if the determination of the position of the lens (its center) could be determined with more accuracy. The use of monochromatic light would also give a more accurate value of f for particular wavelengths.

Problem

13.1 **[E.O.P.R.]** Perform the analysis of the data in the tables and draw the figures of the report above.

References

1. L. Finkelstein Jr., *Pocket Book of Technical Writing for Engineers and Scientists* (McGraw-Hill, New York, 2000)
2. P. Kenney, *A Handbook of Public Speaking for Scientists and Engineers* (Adam Hilger, Bristol, 1982)
3. Springer, *Manuscript Guidelines*. Prentice-Hall, *Author's Guide*. Both frequently revised
4. American Institute of Physics, *AIP Style Manual*, 4th edn (1990)

Chapter 14
Appendices

14.1 Appendix 1: Least Squares Straight Line $y = \alpha + \lambda x$. The Errors in α and λ

Let us assume that we have performed a large number M of series of measurements, each consisting of N measurements, in which for various values of x_i we have determined the corresponding values of y_i. The N values of x $[x_i, (i = 1, 2, \ldots, N)]$ are kept the same in each series. Thus, for every value of x_i we have M values of y_i. Assume that the errors are significant only in the values of y_i and that these values are mutually independent. We therefore have, for every value of x_i, a distribution, which we assume to be normal, of M values of y_i, with the same standard deviation σ about the real value of y which corresponds to the particular value x_i and is symbolized by Y_i (see Fig. 14.1).

If the real relationship between x and y is

$$y = A + \Lambda x, \tag{14.1}$$

then

$$Y_i = A + \Lambda x_i. \tag{14.2}$$

For each of the M series of N measurements (x_i, y_i) we may find, using the method of least squares, the values of α and λ in the relation $y = \alpha + \lambda x$, which we assume to connect x and y. The mean value of λ over all the M series of measurements is Λ and the standard deviation of λ or the standard error in λ is $\delta\lambda$, where

$$(\delta\lambda)^2 = \frac{1}{M}\sum_{r=1}^{M}(\lambda_r - \Lambda)^2. \tag{14.3}$$

C. Christodoulides and G. Christodoulides, *Analysis and Presentation of Experimental Results*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-53345-2_14

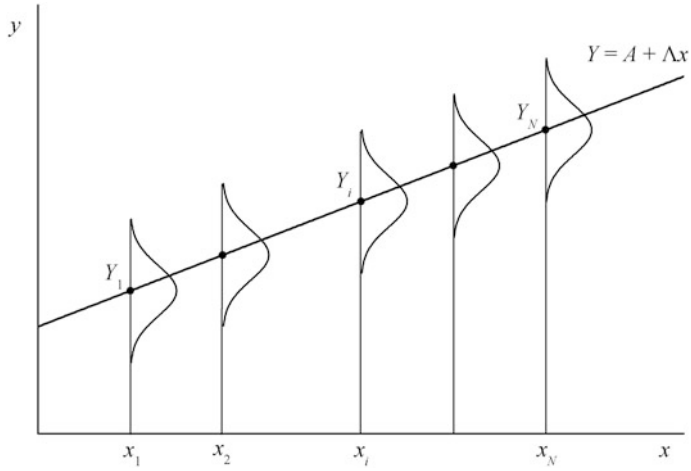


Fig. 14.1 The real linear relation between x and y and the distributions of the results of the measurements y_i about the real values Y_i (of y) which correspond to the values x_i

Similarly, the mean value of the α's is A and the standard deviation of α or the standard error in α is $\delta\alpha$, where

$$(\delta\alpha)^2 = \frac{1}{M}\sum_{r=1}^{M}(\alpha_r - A)^2. \tag{14.4}$$

The quantities $\delta\alpha$ and $\delta\Lambda$ are to be determined.

Naturally, having performed only one experiment (consisting of only one series of N measurements), we have only one value of α and one of λ, which we consider to be the best estimates we have for A and Λ, respectively. Let us define the new variable

$$\chi \equiv x - \bar{x}, \tag{14.5}$$

where

$$\bar{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{14.6}$$

is the mean of the values of x for our series of measurements. It is

$$\sum_{i=1}^{N}\chi_i = \sum_{i=1}^{N}(x_i - \bar{x}) = N\bar{x} - N\bar{x} = 0. \tag{14.7}$$

Also, the quantity defined as

$$D \equiv \sum_{i=1}^{N} \chi_i^2 \tag{14.8}$$

is equal to

$$D = \sum_{i=1}^{N} (x_i - \bar{x})^2 = \sum_{i=1}^{N} x_i^2 - 2\bar{x} \sum_{i=1}^{N} x_i + \bar{x}^2 \sum_{i=1}^{N} 1 = \sum_{i=1}^{N} x_i^2 - N\bar{x}^2. \tag{14.9}$$

The straight line $y = \alpha + \lambda x$ may now be expressed as

$$y = \beta + \lambda \chi, \tag{14.10}$$

where

$$\beta \equiv \alpha + \lambda \bar{x}. \tag{14.11}$$

The best values we have for β and λ are derived from the application of the method of least squares to the values (χ_i, y_i). From Eq. (11.15) we have

$$(\beta - \lambda \bar{x})N + \lambda \sum_{i=1}^{N} (\chi_i + \bar{x}) = \sum_{i=1}^{N} y_i, \quad \beta N + \lambda \sum_{i=1}^{N} \chi_i = \sum_{i=1}^{N} y_i. \tag{14.12}$$

However, $\sum_{i=1}^{N} \chi_i = 0$ and so we have, finally,

$$\beta = \frac{1}{N} \sum_{i=1}^{N} y_i = \bar{y}. \tag{14.13}$$

where $\bar{y}$ is the mean value of the y_i.

From Eq. (11.16),

$$(\beta - \lambda \bar{x}) \sum_{i=1}^{N} (\chi_i + \bar{x}) + \lambda \sum_{i=1}^{N} (\chi_i + \bar{x})^2 = \sum_{i=1}^{N} (\chi_i + \bar{x}) y_i \tag{14.14}$$

Expanding and taking into account that $\sum_{i=1}^{N} \chi_i = 0$, we have

$$N\bar{x}(\beta - \bar{y}) + \lambda \sum_{i=1}^{N} \chi_i^2 = \sum_{i=1}^{N} \chi_i y_i. \tag{14.15}$$

However, $\beta - \bar{y} = 0$ and so

$$\lambda = \frac{1}{D}\sum_{i=1}^{N} \chi_i y_i, \tag{14.16}$$

with D defined by Eq. (14.8).

For each series of measurements, it is

$$\lambda = \frac{\chi_1}{D} y_1 + \frac{\chi_2}{D} y_2 + \ldots + \frac{\chi_N}{D} y_N, \tag{14.17}$$

where the coefficients of y_i are the same for all series. Since the values y_i are mutually independent, we may write for the error in λ,

$$(\delta\lambda)^2 = \left(\frac{\chi_1}{D}\right)^2 (\delta y_1)^2 + \left(\frac{\chi_2}{D}\right)^2 (\delta y_2)^2 + \ldots + \left(\frac{\chi_N}{D}\right)^2 (\delta y_N)^2. \tag{14.18}$$

We have assumed, however, that the standard deviations of the y_i values are the same. Thus,

$$(\delta y_1)^2 = (\delta y_2)^2 = \ldots = \sigma^2 \tag{14.19}$$

and, therefore,

$$(\delta\lambda)^2 = \frac{\sigma^2}{D^2}\sum_{i=1}^{N} \chi_i^2 = \frac{\sigma^2}{D^2} D = \frac{\sigma^2}{D}. \tag{14.20}$$

For β we have $\beta = \frac{1}{N}\sum_{i=1}^{N} y_i$ and so

$$(\delta\beta)^2 = \frac{1}{N^2}(\delta y_1)^2 + \frac{1}{N^2}(\delta y_2)^2 + \ldots + \frac{1}{N^2}(\delta y_N)^2 = N\frac{1}{N^2}\sigma^2 = \frac{\sigma^2}{N}. \tag{14.21}$$

What we want to find is the error $\delta\alpha$ in $\alpha = \beta - \lambda\bar{x}$, which is given by the relation

$$(\delta\alpha)^2 = (\delta\beta)^2 + (\bar{x})^2(\delta\lambda)^2 \tag{14.22}$$

or

$$(\delta\alpha)^2 = \left(\frac{1}{N} + \frac{(\bar{x})^2}{D}\right)\sigma^2. \tag{14.23}$$

We will now evaluate σ. If the real value of β is B, then

$$Y_i = \Lambda\chi_i + B. \tag{14.24}$$

Summing all the Y_i, and since it is $\sum_{i=1}^{N} \chi_i = 0$, we have

$$B = \frac{1}{N}\sum_{i=1}^{N} Y_i. \tag{14.25}$$

Also, multiplying Eq. (14.24) by χ_i and summing, we get

$$\Lambda = \frac{1}{D}\sum_{i=1}^{N} \chi_i Y_i. \tag{14.26}$$

Figure 14.2 shows the various quantities which will be used in the analysis of the problem.

The error in the ith measurement is

$$e_i = y_i - Y_i = y_i - (\Lambda\chi_i - B). \tag{14.27}$$

The least squares straight line gives for x_i the value of $\lambda\chi_i + \beta$ for y. The residual is, therefore,

$$d_i = y_i - (\lambda\chi_i + \beta), \tag{14.28}$$

as shown in Fig. 14.2. The values of e_i are not known but those of d_i are. We define the quantity

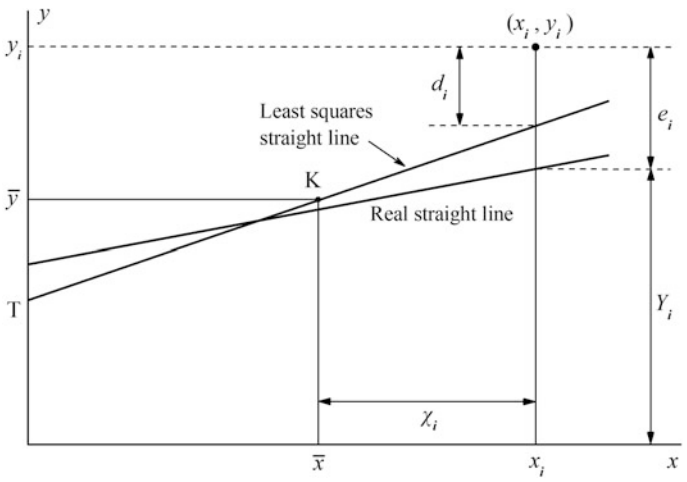


Fig. 14.2 The various quantities used in the evaluation of the errors $\delta\alpha$ and $\delta\lambda$

$$s^2 \equiv \sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} [y_i - (\lambda\, \chi_i + \beta)]^2. \tag{14.29}$$

From Eqs. (14.27) and (14.28) we have

$$d_i = e_i - [(\lambda - \Lambda)\chi_i + (\beta - B)]\,. \tag{14.30}$$

From Eqs. (14.13) and (14.25),

$$\beta - B = \frac{1}{N}\sum_{i=1}^{N} e_i \tag{14.31}$$

and from Eqs. (14.16) and (14.26),

$$\lambda - \Lambda = \frac{1}{D}\sum_{i=1}^{N} \chi_i(y_i - Y_i) = \frac{1}{D}\sum_{i=1}^{N} \chi_i e_i. \tag{14.32}$$

Substituting from Eqs. (14.31) and (14.32) in Eq. (14.30),

$$d_i = e_i - \chi_i \frac{1}{D}\sum_{i=1}^{N} \chi_i e_i - \frac{1}{N}\sum_{i=1}^{N} e_i. \tag{14.33}$$

We verify, in passing, that it is

$$\begin{aligned} \sum_{i=1}^{N} d_i &= \sum_{i=1}^{N} e_i - \frac{1}{D}\left(\sum_{i=1}^{N} \chi_i\right)\left(\sum_{i=1}^{N} \chi_i e_i\right) - \frac{1}{N}\sum_{i=1}^{N}\left(\sum_{i=1}^{N} e_i\right) \\ &= \sum_{i=1}^{N} e_i - \frac{1}{D}(0)\left(\sum_{i=1}^{N} \chi_i e_i\right) - \frac{1}{N}N\left(\sum_{i=1}^{N} e_i\right) = 0. \end{aligned} \tag{14.34}$$

Squaring d_i,

$$\begin{aligned} d_i^2 = e_i^2 &+ \frac{\chi_i^2}{D^2}\left(\sum_{i=1}^{N} \chi_i e_i\right)^2 + \frac{1}{N^2}\left(\sum_{i=1}^{N} e_i\right)^2 \\ &- 2\frac{e_i \chi_i}{D}\sum_{i=1}^{N} \chi_i e_i - 2\frac{e_i}{N}\sum_{i=1}^{N} e_i + 2\frac{\chi_i}{DN}\left(\sum_{i=1}^{N} \chi_i e_i\right)\left(\sum_{i=1}^{N} e_i\right) \end{aligned} \tag{14.35}$$

and summing over all i's

$$\sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} e_i^2 + \frac{1}{D^2}\left(\sum_{i=1}^{N}\chi_i^2\right)\left(\sum_{i=1}^{N}\chi_i e_i\right)^2 + \frac{1}{N^2}\sum_{i=1}^{N}\left(\sum_{i=1}^{N} e_i\right)^2 - \frac{2}{D}\left(\sum_{i=1}^{N}\chi_i e_i\right)\left(\sum_{i=1}^{N}\chi_i e_i\right) - \frac{2}{N}\left(\sum_{i=1}^{N} e_i\right)\left(\sum_{i=1}^{N} e_i\right) + \frac{2}{DN}\left(\sum_{i=1}^{N}\chi_i\right)\left(\sum_{i=1}^{N}\chi_i e_i\right)\left(\sum_{i=1}^{N} e_i\right) \tag{14.36}$$

Since it is $\sum_{i=1}^{N}\chi_i^2 = D$ and $\sum_{i=1}^{N}\chi_i = 0$, the last relation simplifies to

$$\sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} e_i^2 + \frac{1}{D}\left(\sum_{i=1}^{N}\chi_i e_i\right)^2 + \frac{1}{N}\left(\sum_{i=1}^{N} e_i\right)^2 - \frac{2}{D}\left(\sum_{i=1}^{N}\chi_i e_i\right)^2 - \frac{2}{N}\left(\sum_{i=1}^{N} e_i\right)^2 + 0$$

or

$$\sum_{i=1}^{N} d_i^2 = \sum_{i=1}^{N} e_i^2 - \frac{1}{D}\left(\sum_{i=1}^{N}\chi_i e_i\right)^2 - \frac{1}{N}\left(\sum_{i=1}^{N} e_i\right)^2. \tag{14.37}$$

This relation holds for every one of the M series of measurements. Using indices on the quantities of Eq. (14.37), we have, for $r = 1, 2, \ldots, M$, the relations

$$\sum_{i=1}^{N} d_{ri}^2 = \sum_{i=1}^{N} e_{ri}^2 - \frac{1}{D}\left(\sum_{i=1}^{N}\chi_i e_{ri}\right)^2 - \frac{1}{N}\left(\sum_{i=1}^{N} e_{ri}\right)^2, \tag{14.38}$$

where we have left D and χ_i without r indices, since, as mentioned at the start, we used the same values of x_i in the M series of measurements. We will now sum the r Eqs. (14.38). The sums of the various terms are:

$$\begin{aligned}\sum_{r=1}^{M}\sum_{i=1}^{N} d_{ri}^2 &= (d_{11}^2 + d_{12}^2 + \ldots + d_{1N}^2) + (d_{21}^2 + d_{22}^2 + \ldots + d_{2N}^2) \\ &\quad + \ldots + (d_{M1}^2 + d_{M2}^2 + \ldots + d_{MN}^2) \\ &= (d_{11}^2 + d_{21}^2 + \ldots + d_{M1}^2) + (d_{12}^2 + d_{22}^2 + \ldots + d_{M2}^2) \\ &\quad + \ldots + (d_{1N}^2 + d_{2N}^2 + \ldots + d_{MN}^2)\end{aligned} \tag{14.39}$$

$$\sum_{r=1}^{M}\left(\sum_{i=1}^{N} e_{ri}^2\right) = MN\,\sigma^2 \tag{14.40}$$

$$\sum_{r=1}^{M}\left(\sum_{i=1}^{N}\chi_i e_{ri}\right)^2=\sum_{r=1}^{M}\left(\sum_{i=1}^{N}\chi_i^2 e_{ri}^2+2\sum_{\substack{i,j\\ i\neq j}}^{N}\chi_i\chi_j e_{ri}e_{rj}\right) \tag{14.41}$$

For given i and j (and, therefore, also given $\chi_i\,\chi_j$), the sum of the terms $\chi_i\,\chi_j\,e_{ri}\,e_{rj}$ over all values of r tends to zero, since the quantities e_{rj} and e_{ri} are mutually independent. Thus,

$$\sum_{r=1}^{M}\left(\sum_{i=1}^{N}\chi_i e_{ri}\right)^2=\chi_1^2\sum_{r=1}^{M}e_{r1}^2+\chi_2^2\sum_{r=1}^{M}e_{r2}^2+\ldots=(\chi_1^2+\chi_2^2+\ldots)M\sigma^2=DM\sigma^2. \tag{14.42}$$

Also,

$$\begin{aligned}\sum_{r=1}^{M}\left(\sum_{i=1}^{N}e_{ri}\right)^2&=\sum_{r=1}^{M}\left(\sum_{i=1}^{N}e_{ri}^2+2\sum_{\substack{i,j\\ i\neq j}}^{N}e_{ri}e_{rj}\right)=\sum_{r=1}^{M}\left(\sum_{i=1}^{N}e_{ri}^2+2\times 0\right)\\&=\sum_{r=1}^{M}\left(N\sigma^2\right)=MN\sigma^2.\end{aligned} \tag{14.43}$$

Substituting from Eqs. (14.39) to (14.43) in Eq. (14.38), we find that it is

$$\sum_{r=1}^{M}\sum_{i=1}^{N}d_{ri}^2=MN\,\sigma^2-M\,\sigma^2-M\,\sigma^2. \tag{14.44}$$

For the M sums of the $\sum_{i=1}^{N}d_{ri}^2$ we do not have an accurate value. We have an estimate, equal to the sum $\sum_{i=1}^{N}d_i^2$, which is derived from our N measurements. The best estimate we can make is, therefore,

$$\sum_{r=1}^{M}\sum_{i=1}^{N}d_{ri}^2\approx M\sum_{i=1}^{N}d_i^2 \tag{14.45}$$

and Eq. (14.44) finally gives

$$M\sum_{i=1}^{N}d_i^2=MN\,\sigma^2-2M\,\sigma^2 \tag{14.46}$$

or

$$\sigma^2 = \frac{1}{N-2}\sum_{i=1}^{N} d_i^2, \tag{14.47}$$

$$\sigma_y \equiv \sigma = \sqrt{\frac{1}{N-2}\sum_{i=1}^{N}(y_i - \alpha - \lambda x_i)^2}. \tag{14.48}$$

Substituting in Eqs. (14.23) and (14.20), respectively, we have for the errors in α and λ,

$$(\delta\alpha)^2 = \frac{1}{N-2}\left(\frac{1}{N} + \frac{(\bar{x})^2}{D}\right)\sum_{i=1}^{N} d_i^2 \tag{14.49}$$

$$(\delta\lambda)^2 = \frac{1}{D}\frac{1}{N-2}\sum_{i=1}^{N} d_i^2. \tag{14.50}$$

We adopt the notation

$$\sum_{i=1}^{N} x_i \equiv [x] \quad \sum_{i=1}^{N} y_i \equiv [y] \quad \sum_{i=1}^{N} x_i^2 \equiv [x^2] \quad \sum_{i=1}^{N} x_i y_i \equiv [xy]. \tag{14.51}$$

Taking into account the fact that $D = \sum_{i=1}^{N} x_i^2 - N\bar{x}^2 = [x^2] - \frac{1}{N}[x]^2$ and $(\bar{x})^2 = \left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)^2 = \frac{[x]^2}{N^2}$, we have

$$\sigma_y^2 = \frac{[d]}{N-2}, \tag{14.52}$$

$$\delta\alpha = \sigma_y\sqrt{\frac{[x^2]}{N[x^2] - [x]^2}} = \sqrt{\frac{[d]}{N(N-2)}\cdot\frac{[x^2]}{N[x^2] - [x]^2}} \tag{14.53}$$

and

$$\delta\lambda = \sigma_y\sqrt{\frac{N}{N[x^2] - [x]^2}} = \sqrt{\frac{[d]}{N(N-2)}\cdot\frac{N}{N[x^2] - [x]^2}}. \tag{14.54}$$

The question to be answered now is this: Given the relation $y = \alpha + \lambda x$ and the errors in $\delta\alpha$ and $\delta\lambda$, what is the error in y, as this is evaluated for a given value of x? The first thought, which is to write the equation with the errors in its coefficients as

$$y = (\alpha \pm \delta\alpha) + (\lambda \pm \delta\lambda)\, x \tag{14.55}$$

and the error in y as

$$\delta y = \sqrt{(\delta\alpha)^2 + (x\,\delta\lambda)^2}, \tag{14.56}$$

would be wrong, because the magnitudes α and λ are not mutually independent. This can be verified by the fact that

$$\sum_{r=1}^{M} (\lambda_r - \Lambda)(\alpha_r - A) = -\bar{x}(\delta\,\lambda)^2, \tag{14.57}$$

whereas it would have to be equal to zero for mutually independent α and λ. On the other hand, β and λ are mutually independent magnitudes, as testified by Eq. (14.21). The correct equation for the straight line with errors is, therefore,

$$y = (\beta \pm \delta\beta) + (\lambda \pm \delta\lambda)\,(x - \bar{x}), \tag{14.58}$$

from which the error in y is found to be

$$\delta y = \sqrt{(\delta\beta)^2 + [(x - \bar{x})\,\delta\lambda]^2}. \tag{14.59}$$

Since according to Eq. (14.21) and since the best estimate we have for σ is σ_y, it is $\delta\beta = \sigma_y/\sqrt{N}$, the error in y is:

$$\delta\, y = \frac{\sigma_y}{\sqrt{N}} \sqrt{1 + \frac{N^2}{N\,[x^2] - [x]^2}\,(x - \bar{x})^2}. \tag{14.60}$$

The errors which are mutually independent and contribute to the error in y are: the error in the value of $\beta = \bar{y}$ which determines the center K of the straight line (Fig. 14.2) and the error in the slope of the straight line, which determines its orientation as the line is considered to rotate about the center K. The erroneous Eq. (14.56) would mean that the error in y is determined by the error in the ordinate α of the point of intersection T of the y-axis with the least squares straight line and the error $\delta\,\lambda$ in the slope of the line, as this would now be considered to rotate about the point T.

14.2 Appendix 2: Dimensional Analysis

We know that a mathematical equation describing the relationship between physical quantities does not only lead to numerical equality when in place of the symbols for these quantities we place their numerical values. The equation must also be homogeneous regarding the units and in general the *dimensions* of the two sides of the equation. Dimensions are attributed to all physical quantities, starting from those considered to be the fundamental ones. In Mechanics, *fundamental dimensions* are considered to be those of mass (M), length (L) and time (T). It is found that the definitions of other quantities as well as the physical laws can be expressed in terms of these three quantities. All masses, lengths and time intervals are measured by comparison with the prototypes defined for these three fundamental quantities. If we include electromagnetic phenomena, no additional dimension is required when using the c.g.s. system of units (electrostatic and electromagnetic system of units), since both the electric charge and the magnetic force are defined in terms of the three fundamental quantities of Mechanics. In the S.I. system of units the 'electric charge' is introduced via the definition of the electric current and so four fundamental dimensions are required: mass, length, time and electric charge (Q). We should mention that pure numbers (0, 1, π, e etc.) have no dimensions.

We may use the requirement that a mathematical relation between physical quantities should be homogeneous as regards dimensions, in order to extract conclusions regarding this relation, even if we do not know its form. This process is called *dimensional analysis*. Naturally, dimensional analysis is also used in order to check whether a relation between physical quantities is or is not the right one. For example, if a term of a sum has dimensions of L/T, i.e. (length)/(time), then, for the equation to be correct, all the other terms must also have these dimensions. Another very useful application of dimensional analysis is the derivation of functional relations connecting the physical quantities describing a certain phenomenon. The description of this method is the main purpose of this appendix [1].

14.2.1 The Dimensions of Physical Quantities

We denote by $[X]$ the dimensions of a physical quantity X. Thus, according to what has been said for the three fundamental quantities of Mechanics, we have for mass m, length l and time t,

$$[m] = \mathrm{M}, \quad [l] = \mathrm{L}, \quad [t] = \mathrm{T}. \tag{14.61}$$

For consistency of the relations connecting the dimensions of physical quantities, we must accept that for a pure number A it is

$$[A] = 1. \tag{14.62}$$

Thus, for example, we have $[1] = 1, [\sqrt{2}] = 1, [\pi] = 1$ etc. Pure numbers are *dimensionless*.

The dimensions of a derivative can be found if we remember that

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}x}\cdots\frac{\mathrm{d}}{\mathrm{d}x}(n \text{ times}) \cdot y. \tag{14.63}$$

The operator $\dfrac{\mathrm{d}}{\mathrm{d}x}$ means that we take a difference and divide by $\mathrm{d}x$. The result of this is to multiply the dimensions of the denominator by the dimensions of x, i.e. $[x]$. Thus

$$\left[\frac{\mathrm{d}^n y}{\mathrm{d}x^n}\right] = \left[\frac{\mathrm{d}}{\mathrm{d}x}\right]^n [y] = \frac{[y]}{[x]^n}. \tag{14.64}$$

In short, we find the dimensions of a derivative by erasing the operators[1] d or ∂. For example, the dimensions of speed or velocity are

$$[v] = \left[\frac{\mathrm{d}x}{\mathrm{d}t}\right] = \frac{[x]}{[t]} = \frac{\mathrm{L}}{\mathrm{T}} = \mathrm{LT}^{-1}. \tag{14.65}$$

The dimensions of acceleration are

$$[a] = \left[\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right] = \frac{[x]}{[t^2]} = \frac{[x]}{[t]^2} = \mathrm{LT}^{-2}. \tag{14.66}$$

The dimensions of other quantities may be found either using the equations of their definition or through some physical law. For example,

$$\text{the dimensions of force are } [F] = [ma] = \left[m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2}\right] = [m]\frac{[x]}{[t^2]} = \mathrm{M}\frac{\mathrm{L}}{\mathrm{T}^2} = \mathrm{MLT}^{-2} \tag{14.67}$$

$$\text{the dimensions of energy are } [E] = [Fx] = [F][x] = (\mathrm{MLT}^{-2})(\mathrm{L}) = \mathrm{ML}^2\mathrm{T}^{-2} \tag{14.68}$$

and so on. Similarly, the dimensions of the Newtonian constant of gravitation may be found using Newton's law of gravity

$$F = G\frac{m_1 m_2}{r^2}. \tag{14.69}$$

[1]From its definition, the operator ∇ has the dimensions of L^{-1} and the Laplacian ∇^2 those of L^{-2}.

Thus, we have $G = \frac{Fr^2}{m_1 m_2}$ and

$$[G] = \left[\frac{Fr^2}{m_1 m_2}\right] = \frac{[F][r^2]}{[m^2]} = \frac{(\mathrm{MLT}^{-2})(\mathrm{L}^2)}{\mathrm{M}^2} = \mathrm{M}^{-1}\mathrm{L}^3\mathrm{T}^{-2}. \tag{14.70}$$

Continuing in this way we may create a table of the dimensions of all the physical quantities.

14.2.2 The Dimensional Homogeneity of Equations

As we have already mentioned, the mathematical relations connecting physical quantities must be homogeneous regarding dimensions. As an example, we check the relation

$$s = s_0 + v_0 t + \tfrac{1}{2} a\, t^2. \tag{14.71}$$

We have

$$[s] = [s_0] + [v_0 t] + [\tfrac{1}{2} a\, t^2]$$

$$[s] = [s_0] + [v_0][t] + [\frac{1}{2}][a][t^2]$$

$$\mathrm{L} = \mathrm{L} + (\mathrm{LT}^{-1})\,(\mathrm{T}) + (1)\,(\mathrm{LT}^{-2})\,(\mathrm{T}^2) \tag{14.72}$$

or, finally,

$$\mathrm{L} = \mathrm{L} + \mathrm{L} + \mathrm{L} = \mathrm{L}, \tag{14.73}$$

which of course must not be interpreted algebraically but must be considered as stating that 'the length on the left is the sum of the three lengths on the right', i.e. it is equal to a length, as it should. All equations involving physical quantities must pass this test if they are to be correct.

In many equations of Physics there appear functions such as the trigonometric, exponential, logarithmic etc. We have to examine the question of the dimensions of these functions. From the Maclaurin series for these functions,

$$\begin{aligned} &\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 - \ldots \quad \cos z = 1 - \frac{1}{2!}z^2 + \frac{1}{4!}z^4 - \ldots \\ &\mathrm{e}^z = 1 + \frac{1}{1!}z + \frac{1}{2!}z^2 + \frac{1}{3!}z^3 + \ldots, \end{aligned} \tag{14.74}$$

it is obvious that both their arguments and the functions themselves must be dimensionless quantities. Otherwise, it would not be possible for different powers of the arguments to have the same dimensions and have dimensional homogeneity in the equation. The same is true for the logarithmic function. We can prove this in various ways. For example, if it is $y = \ln x$, then $x = e^y$, and, from what we have said about the exponential function, both x and y must be dimensionless quantities. Besides, if the arguments of these functions were not dimensionless numbers, the values of the functions would be different for different systems of units we might use.

Angles, defined as the ratios of two lengths, are also dimensionless. They must, of course, be expressed in radians (rad).

As a consequence, in the expression for the potential energy of a simple harmonic oscillator consisting of a mass m at the free end of a spring with constant k oscillating with an angular frequency ω and an amplitude a,

$$U(t) = \frac{1}{2}ka^2 \sin^2(\omega t + \phi), \tag{14.75}$$

it is immediately obvious that the sine is a pure number and so are ωt and ϕ. If this is not the case, then we have made a mistake in the derivation of the equation. We must, however, be careful in those cases in which some symbols have been replaced by their numerical values. For example, if in the previous equation we substitute $\omega = 1$ rad/s and $k = 10^6$ N/m, we will have the equation $U(t) = \frac{1}{2}10^6 a^2 \sin^2(t + \phi)$, which would appear to us to be dimensionally wrong. If, however, we are informed of the substitutions, we realize that the equation is dimensionally correct and also that we must use S.I. units when substituting for t and a.

14.2.3 The Derivation of Relations Between Physical Quantities Using Dimensional Analysis

We will demonstrate the power of dimensional analysis, as well as its limitations, with certain examples of the derivation of relations between the physical quantities involved in some phenomena.

Example 14.1

A simple harmonic oscillator consists of a mass m fixed at the free end of a spring, whose constant is k. We wish to find the functional relation for the period T of the oscillator.

The first thing we must decide is which are the quantities involved in the determination of the quantity we need to find. We must be very careful, since, if we

include too many irrelevant quantities we will not have a unique relation. On the other hand, if we omit some relevant quantity, we will derive a relation which will be wrong.

The quantities on which the oscillator's period may depend, are its mass, m, the spring's constant k and the amplitude of the oscillation, a. We assume that the oscillators are executed on a smooth horizontal plane, so that gravity does not affect the period. We will investigate this matter later.

We assume that the period of the oscillator is given by a relation of the form

$$T = A a^{\alpha} m^{\beta} k^{\gamma}$$

where A is a numerical coefficient and the exponents α, β, γ are to be determined.

We write the equation of the dimensions of the two sides of the relation. The spring constant, being a force per unit length, has dimensions $[k] = \mathrm{MT}^{-2}$. Therefore

$$[T] = [A][a^{\alpha}][m^{\beta}][k^{\gamma}] = [A][a]^{\alpha}[m]^{\beta}[k]^{\gamma}$$

or

$$\mathrm{T} = (1)(\mathrm{L})^{\alpha}(\mathrm{M})^{\beta}(\mathrm{MT}^{-2})^{\gamma} \quad \mathrm{T} = \mathrm{L}^{\alpha}\mathrm{M}^{\beta+\gamma}\mathrm{T}^{-2\gamma}.$$

Since L, M and T are dimensionally independent of each other, we may equate their exponents, respectively, on the left and on the right. We thus find that, for the assumed relation to be valid, it must be:

$$1 = -2\gamma, \quad 0 = \alpha, \quad 0 = \beta + \gamma$$

from which it follows that

$$\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = -\frac{1}{2}.$$

The period of the oscillator is, therefore,

$$T = A\sqrt{\frac{m}{k}}.$$

The method does not tell us what the value of A is. (From theory we know that $A = 2\pi$.)

The procedure we followed cannot be considered to be a proof of the relation derived. Nor is the derived relation necessarily correct. If someone includes the mass of the Earth M in the relevant variables, the ratio m/M, which is dimensionless, could appear in any of an infinite number of forms, without violating the

dimensional homogeneity of the equation. For example, we could have, among others, the expressions

$$T = A\sqrt{\frac{m}{k}\left(\frac{m}{M}\right)}, \quad T = A\sqrt{\frac{m}{k}\left(1+\frac{m}{M}\right)}, \quad T = A\sqrt{\frac{m}{k}}\left(\frac{m}{M}\right)^{7\pi},$$
$$T = A\sqrt{\frac{m}{k}}\,\mathrm{e}^{m/M}.$$

Some of these equations may appear highly improbable or even ugly but this is no reason for us to reject them (Dirac's advice that 'it is more important to have beauty in one's equations than to have them fit experiment' is not always easy to apply in practice!). In the final analysis, if theory cannot help us, experiment will show us which of these equations is the acceptable one.

If we include the acceleration of gravity in the magnitudes determining T, and we set $T = A\,a^{\alpha}m^{\beta}k^{\gamma}g^{\delta}$, it follows that $[T] = [A][a]^{\alpha}[m]^{\beta}[k]^{\gamma}[g]^{\delta}$ or $\mathrm{T} = (1)(\mathrm{L})^{\alpha}(\mathrm{M})^{\beta}(\mathrm{MT}^{-2})^{\gamma}(\mathrm{LT}^{-2})^{\delta}$. Finally, $\mathrm{T} = \mathrm{L}^{\alpha+\delta}\mathrm{M}^{\beta+\gamma}\mathrm{T}^{-2\gamma-2\delta}$. Equating the exponents of L, M and T, respectively, left and right, we find that, for the assumed relation to be valid, it must be:

$$1 = -2\gamma - 2\delta, \quad 0 = \alpha + \delta, \quad 0 = \beta + \gamma.$$

We now have three equations with four unknowns. We, therefore, have no unique solutions. We may express the exponents β, γ and δ in terms of α:

$$\beta = \frac{1}{2} - \alpha, \quad \gamma = \alpha - \frac{1}{2}, \quad \delta = -\alpha.$$

For each value of α we have a different solution. If by experiment we prove that the period of the oscillator does not depend on the amplitude of the oscillations, then it is $\alpha = 0$ and, therefore, $\beta = \frac{1}{2}, \gamma = -\frac{1}{2}, \delta = 0$. The independence of the period of the oscillation from the amplitude of the oscillations leads to the conclusion that the period does not depend on the acceleration of gravity. All these, of course, with the reservations already expressed.

Example 14.2

The Cepheid variables are stars whose luminosities vary periodically due to the expansion and contraction of their radii. There is a relation between the period of the pulsations of such a star and the absolute luminosity of the star and this makes the Cepheid variables useful in the determination of distances. What is the relation for the period of the variation of the luminosity of the Cepheid variables?

We suspect that the period depends on the radius and the mass of the star. Since, obviously, the gravitational forces must play an important role in the whole process, supplying the restoring forces during the oscillations, we must also include the Newtonian constant of gravitation in the quantities to be taken into account in the dimensional analysis. Let the period of the oscillations be given by a relation of the form

$$T = A\, G^{\alpha} R^{\beta} M^{\gamma}.$$

The dimensions are

$$[T] = [A]\,[G]^{\alpha}[R]^{\beta}[M]^{\gamma} \Rightarrow \mathrm{T} = [\mathrm{M}^{-1}\mathrm{L}^{3}\mathrm{T}^{-2}]^{\alpha}[\mathrm{L}]^{\beta}[\mathrm{M}]^{\gamma} = \mathrm{M}^{-\alpha+\gamma}\mathrm{L}^{3\alpha+\beta}\mathrm{T}^{-2\alpha}.$$

Therefore, $1 = -2\alpha, 0 = 3\alpha + \beta, 0 = -\alpha + \gamma$ and $\alpha = -\frac{1}{2}, \beta = \frac{3}{2}, \gamma = -\frac{1}{2}$.

Thus, $T = A\sqrt{\frac{R^3}{GM}}$. In terms of the star's density ρ, it is $T = A'/\sqrt{G\rho}$ where A' is a new constant. The theoretical relation found by Sterne is $T = \sqrt{6\pi\beta}/\sqrt{G\rho}$, where β is a numerical parameter which depends on the mode of oscillation and on the ratio of the specific heats of the stellar material, i.e. two dimensionless quantities. The agreement with observation is very good.

In those cases, in which the absolute temperature T is involved in the expression assumed, this is always taken as its product kT with Boltzmann's constant, k. This results in the conversion of the temperature to energy, whose dimensions are known.

Example 14.3

Using dimensional analysis, derive Stefan's law, which gives the amount of energy emitted per unit time and per unit area, Φ, from a body at absolute temperature T.

Assuming that, apart from the temperature, there may appear in the relation assumed universal constants related to electromagnetic radiation, i.e. the speed of light c and Planck's constant, h. Boltzmann's constant will be taken together with the absolute temperature in the product kT.

The amount of energy emitted per unit time per unit area has dimensions

$$[\Phi] = [\text{energy}]/([\text{time}][\text{area}]) = (\mathrm{ML}^2\mathrm{T}^{-2})/(\mathrm{T})(\mathrm{L}^2) = \mathrm{MT}^{-3}.$$

We assume that the required relation is of the form

$$\Phi = A\, c^{\alpha} h^{\beta} (kT)^{\gamma}.$$

The dimensions give

$$[\Phi] = [A][c]^{\alpha}[h]^{\beta}[kT]^{\gamma}, \quad \mathrm{MT}^{-3} = (\mathrm{LT}^{-1})^{\alpha}(\mathrm{ML}^2\mathrm{T}^{-1})^{\beta}(\mathrm{ML}^2\mathrm{T}^{-2})^{\gamma},$$

$$1 = \beta + \gamma, \quad 0 = \alpha + 2\beta + 2\gamma, \quad -3 = -\alpha - \beta - 2\gamma$$
$$\text{or} \quad \alpha = -2, \quad \beta = -3, \quad \gamma = 4$$

from which we have

$$\Phi = A\,(kT)^4/c^2h^3.$$

Theory gives $A = (2\pi^5 / 15)$.

Example 14.4

(a) Find, using the method of dimensional analysis, a possible relation giving the period of a mathematical pendulum.
(b) In an experiment, a mass m, having very small dimensions, was tied at the end of a very thin string, thus forming a pendulum. The measurements of the period T of the pendulum as a function of its length l gave the following results, for oscillations of small amplitude:

l (m)	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
T (s)	0.60	0.87	1.15	1.26	1.41	1.52	1.67	1.83	1.90

Use these measurements in order to find the numerical coefficient in the mathematical relation found by dimensional analysis.

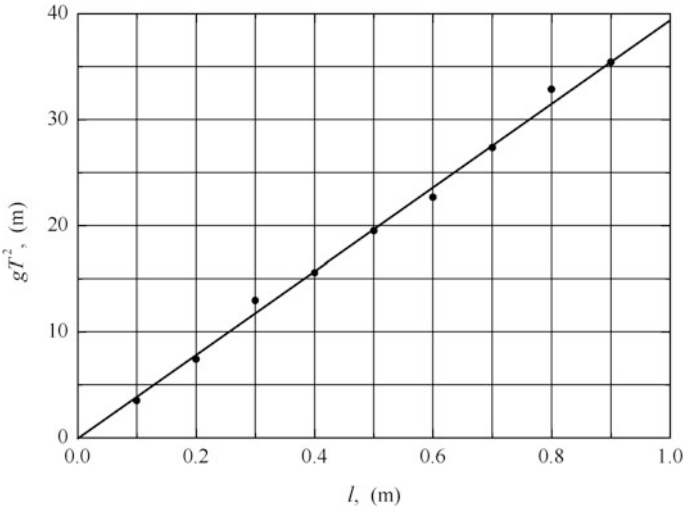
(a) We assume a relation of the form $T = A\,g^{\alpha}m^{\beta}l^{\gamma}$. Substituting the dimensions of the quantities, we have

$$[T] = [A][g]^{\alpha}[m]^{\beta}[l]^{\gamma} \quad \Rightarrow \quad [\mathrm{T}] = [\mathrm{LT}^{-2}]^{\alpha}[\mathrm{M}]^{\beta}[\mathrm{L}]^{\gamma}, \quad \mathrm{T} = \mathrm{L}^{\alpha+\gamma}\mathrm{T}^{-2\alpha}\mathrm{M}^{\beta},$$
$$1 = -2\alpha, \quad 0 = \alpha + \gamma, \quad 0 = \beta, \quad \Rightarrow \quad \alpha = -\frac{1}{2}, \quad \beta = 0, \quad \gamma = \frac{1}{2}.$$

Therefore,

$$T = A\sqrt{\frac{l}{g}}.$$

(b) We will use the experimental results in order to determine the constant A. If the relation found is correct, then plotting gT^2 as a function of l must result in a straight line passing through the origin and having a slope equal to A^2.



We see that this is indeed the case. The point at which the straight line cuts the abscissa line for $l = 1.0$ m has $gT^2 = 39 \pm 1$ m, where the error ± 1 m is an estimate based on the straight lines of maximum and of minimum slope allowed by the experimental points. Therefore, it is $A^2 = 39 \pm 1$. The fractional error in A^2 is 1/39 = 0.026. The fractional error in A will be half this value. Thus, $A^2 = 39 \times (1 \pm 0.026)$ and $A = \sqrt{39} \times (1 \pm 0.013)$. Finally, $A = 6.24 \pm 0.08$. The real value, derived from theory, is $A = 2\pi = 6.283\ldots$

We have here an example in which the combination of dimensional analysis and experiment helped us discover a relation describing the behavior of a physical system.

In fact, if the mass is not a point mass but has some dimensions (i.e. we have a physical and not a mathematical pendulum), the relation for the period of the pendulum is shown by theory to be

$$T = 2\pi\sqrt{\frac{l}{g}\left(1 + \frac{I_C}{Ml^2}\right)},$$

where l is the distance of the body's center of mass from the fixed end of the string, M is the mass of the body and I_C is its moment of inertia with respect to the axis passing through its center of mass and is normal to the plane of oscillations. Dimensional analysis does not give the dimensionless quantity $\left(1 + \frac{I_C}{Ml^2}\right)$. Its existence can, however, be detected by experiment. Plotting the variation of the quantity $\left(\frac{T}{2\pi}\right)^2 \frac{g}{l} - 1$ as a function of l, we will have the curve

$$\left(\frac{T}{2\pi}\right)^2 \frac{g}{l} - 1 = \frac{a^2}{l^2},$$

where a is a length characteristic of the body (its radius of gyration), which is equal to zero for a mathematical pendulum. If at the end of the string we have a solid sphere of radius R, then it is $a = \frac{2}{5}R$.

Plotting the quantity $y = \left(\frac{T}{2\pi}\right)^2 \frac{g}{l} - 1$ as a function of $x = \frac{1}{l^2}$, we will have the straight line $y = \left(\frac{2}{5}R\right)^2 x$, from which we can determine, experimentally, the dependence on R. Plotting y as a function of x in fact brings out the correction (residue) to the result of dimensional analysis. The deviations of y from zero is shown at large values of x, i.e. small values of l. If in the experiment it is $R = 1$ cm, we expect the straight line $y = \frac{4}{25}x$ (with x given in cm^{-2}).

l (cm)	2	3	4	5	6	10
x (cm^{-2})	0.250	0.111	0.0625	0.040	0.0278	0.010
y	0.040	0.018	0.010	0.0064	0.0044	0.0016

It appears that, in order to detect experimentally the presence of the term $\left(1 + \frac{I_C}{Ml^2}\right)$ with measurements at small values of l, an accuracy of the order of 1% is necessary in the measurement of T and l.

If there is a dependence of the period of the pendulum on the amplitude of the oscillations, this will not be disclosed by dimensional analysis. The amplitude of the oscillations, θ_0, is an angle, which is a dimensionless quantity. If it is

$$T = 2\pi\sqrt{\frac{l}{g}}\, f(\theta_0),$$

where $f(\theta_0)$ is a function of the amplitude of the oscillations, the existence of this function may be detected by measuring T for different amplitudes of oscillation. Theory gives

$$f(\theta_0) = 1 + \frac{1}{4}\sin^2\frac{\theta_0}{2} + \frac{9}{64}\sin^4\frac{\theta_0}{2} + \ldots.$$

In this example we demonstrated the capabilities as well as the limitations of the method of dimensional analysis.

Problems

14.1 Using dimensional analysis, derive Kepler's third law, which relates the period of revolution of a planet about the Sun, T, with its mean distance from the Sun, R. The force of gravity (i.e. the constant G) and, therefore, the mass of the Sun, M, must also be taken into account.

14.2 Find, using dimensional analysis, Boyle's law, which relates the pressure P with the absolute temperature T of a quantity of gas with n_0 mol per unit volume. It should be noted that the number of mol, being a pure number, has no dimensions and, therefore, the dimensions of n_0 are those of (volume) $^{-1}$.

14.3 Find, using dimensional analysis, the expression giving the volumetric flow rate Q (= volume per unit time) of a fluid through a pipe. Assume that the flow rate possibly depends on the radius a of the pipe, the density ρ and the viscosity η of the fluid and on the pressure gradient, $\Delta p/\Delta x$, between the ends of the pipe. (The dimensions of η are $[\eta] = \mathrm{ML}^{-1}\mathrm{T}^{-1}$).

14.4 A string having a linear density λ (mass per unit length) is stretched by a tension (force) F. Using dimensional analysis, find the form of the relation giving the speed υ of transverse waves along the string.

If Young's modulus E of the string's material is included in the quantities on which the speed υ depends, does the solution change? (The dimensions of E are [E] = [Force] /[Area]).

14.5 An electric charge Q is uniformly distributed in a sphere of radius R. Using dimensional analysis, find the form of the relation giving the electrostatic energy W of the charge distribution in S.I. units. The dimensions of the electric constant ε_0 involved in electrostatic phenomena are $[\varepsilon_0] = \mathrm{M}^{-1}\mathrm{L}^{-3}\mathrm{T}^2\mathrm{Q}^2$, where Q is the dimension of electric charge.

Would your result change if the charge is not uniformly distributed inside the sphere but is spread on its surface?

14.3 Appendix 3: The Use of Random Numbers in Finding Values *x* of a Variable x Which Are Distributed According to a Given Probability Density Function *f*(*x*)

We will discuss the following problem:

We have random numbers uniformly distributed in the interval [0, 1). How can they be made to correspond to the values x of the variable **x** in such a way that these have a distribution described by a given probability density function $f(x)$?

For example, if we have at our disposal N random numbers between 0 and 1, how can we obtain an equal number of x values which are normally distributed about a certain mean value with a given standard deviation?

Problems of this kind have to be solved in applications of the Monte Carlo method, either in Statistics, as was done in Chap. 3 and elsewhere in this book or in many other applications of the method, e.g. in Physics etc.

Before we present the general theory for the problem, we should mention that the generation of random numbers with a given distribution may be achieved using

various computer software programs. Using Microsoft Excel®, for example, random numbers may be produced which are distributed according to 7 different probability density functions (normal, Poisson, binomial etc.).

14.3.1 The Use of Random Numbers in Finding Values x of a Variable x Which Are Distributed According to a Given Probability Density Function f(x)

We assume that we are given the probability density function $f(x)$ and the corresponding distribution function $F(x)$ of a random variable **x**. These have been drawn in Fig. 14.3, using a common x-axis. We know that, by definition, $f(x)$ takes positive values in the interval $-\infty < x < \infty$ and that $F(x)$ increases monotonically with x, assuming values between 0 and 1. The two functions are related via the expressions

$$f(x) = \frac{\mathrm{d}F}{\mathrm{d}x} \quad \text{and} \quad F(x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t. \tag{14.76}$$

With reference to Fig. 14.3, we divide the range of values of $F(x)$ (0 to 1) into N equal sections, which we denote by the increasing number n, staring with $n = 0$ for

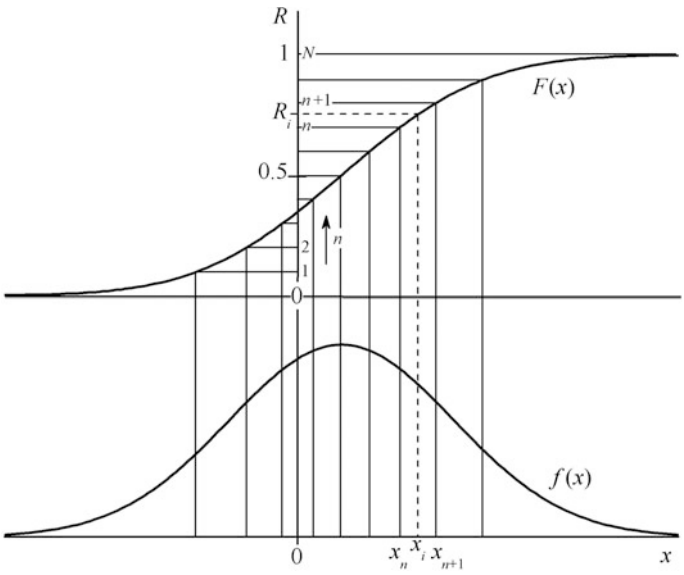


Fig. 14.3 The probability density function $f(x)$ and the distribution function $F(x)$ of the random variable **x**, plotted with a common x-axis. The method of making random numbers R to correspond to values of x is described in the text

$F(x) = 0$. We draw N straight lines parallel to the x-axis and at equal distances between them. From the points of intersection of these with the $F(x)$ curve we draw straight lines normal to the x-axis, which intersect this axis at the N points x_n. From the relation connecting the functions $f(x)$ and $F(x)$, it follows that these lines divide the area under the curve $f(x)$ into N equal parts. The points of intersection of the x-axis define the N intervals $(-\infty, x_1], (x_1, x_2], \ldots, (x_{N-1}, \infty)$. Since to these intervals there correspond equal areas between the x-axis and the curve $f(x)$, a value x of the random variable **x** is equally probable to lie in any one of these intervals.

We see that, using this method of projection, points uniformly distributed on the axis n [or of $F(x)$] are distributed on the x-axis according to the probability density $f(x)$. The greater the number N is, the better the definition of the position of each point on the x-axis. If we have many different random numbers R in the interval [0, 1), putting for each one of them

$$F(x) = R \tag{14.77}$$

and finding the value of x using the method described above, we have an equal number of points on the x-axis which are distributed according to the probability density function $f(x)$.

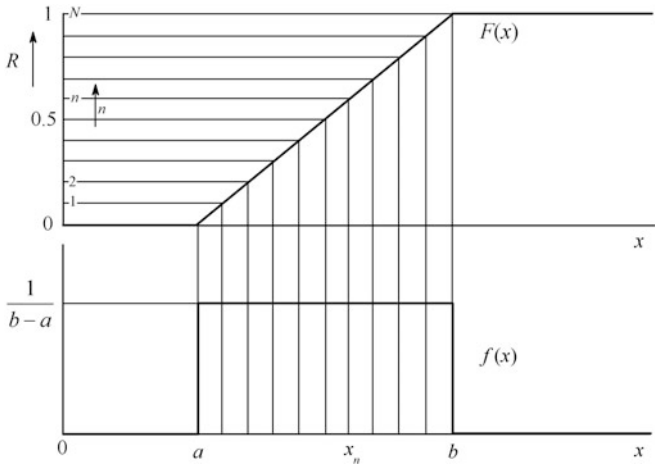
The geometrical procedure followed for finding the value of x for which it is $F(x) = R$, is equivalent to the solution of this equation by the inversion of function $F(x)$, i.e.

$$x(R) = F^{-1}(R). \tag{14.78}$$

For a given random number R in the interval [0, 1), in order to find the corresponding x-value we must solve Eq. (14.77) for x. We will demonstrate the method with some examples. In all examples, the random numbers used are the decimal digits of π (the first 50 000 digits).

Example 14.5

Use 10 000 five-digit random numbers in the interval [0, 1) in order to produce an equal number of values of x, which are uniformly distributed in the interval $a \leq x \leq b$.



It was given that $f(x) = \frac{1}{b-a}$ for $a \leq x \leq b$ and $f(x) = 0$ outside this interval.

Obviously, $f(x)$ is normalized. We find the distribution function

$$F(x) = \int_a^x f(x)\,\mathrm{d}x = \int_a^x \frac{\mathrm{d}x}{b-a} = \frac{x-a}{b-a},$$

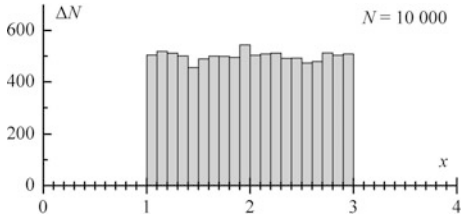
which may be solved for x,

$$x = a + (b-a)F(x).$$

Putting $F(x) = R$, where R is a random number in the interval [0, 1), we have the corresponding value of x,

$$x = a + (b-a)R.$$

We find the values of x which correspond to 10 000 random numbers, for the values $a = 1, b = 3$. The histogram below shows the results using a bin width of 0.1.



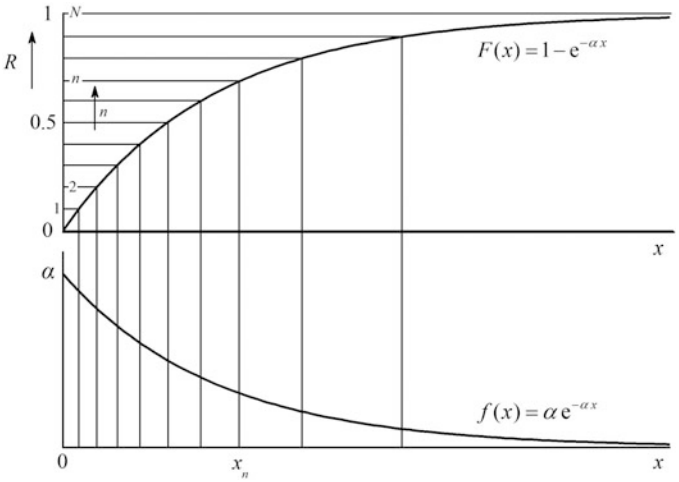
Example 14.6

Use 10 000 five-digit random numbers in the interval [0, 1) in order to produce an equal number of values of x, which are distributed according to the probability density function $f(x) = \alpha\, e^{-\alpha x}$ in the interval $0 \le x < \infty$.

The function $f(x)$ is normalized. We find the distribution function

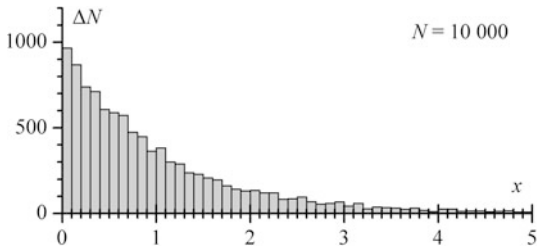
$$F(x) = \int_0^x f(x)\, dx = \int_0^x \alpha\, e^{-\alpha x}\, dx = [-e^{-\alpha x}]_0^x = 1 - e^{-\alpha x}.$$

The relation between $f(x)$ and $F(x)$, as well as the geometrical procedure used in corresponding values of x to the values of the random numbers, are shown in the following figure.



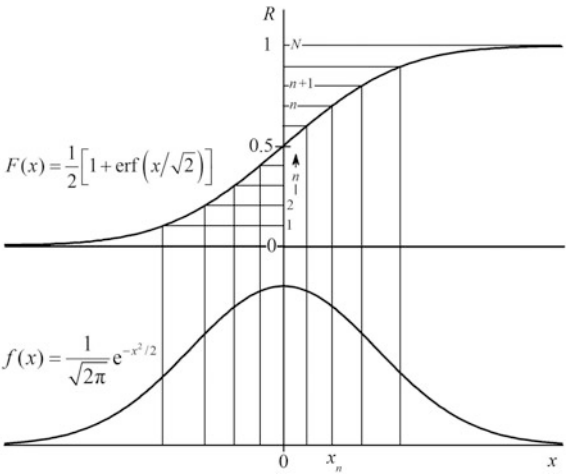
Equating $F(x) = R$ and solving for x, we have $x = -\frac{1}{\alpha} \ln (1 - R)$.

As an example, for the case of $\alpha = 1$, we find the values of x which correspond to 10 000 random numbers. Their histogram is shown in the figure below, with a bin width of 0.1.



Example 14.7

Use 10 000 random numbers in order to produce an equal number of values of x, which are normally distributed about a mean of $\mu_x = 0$ with a standard deviation of $\sigma_x = 1$.



The given probability density function is $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$.

The corresponding distribution function is $F(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt = \frac{1}{2}\left[1 + \mathrm{erf}\left(x/\sqrt{2}\right)\right]$.

If R is a random number in the range $0 \le R < 1$, the corresponding value of **x** is found by solving the equation $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt = R$ for x.

The inversion of $F(x)$ will be done in this example using an approximation. If we define the function $Q(x) \equiv \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-t^2/2}\, dt$, then it is $F(x) = 1 - Q(x)$ and $Q(x) = 1 - F(x) = 1 - R$. Since it is $0 \le R < 1$, we may consider the number $1 - R$ as the random number and set $Q(x) = R$ for convenience. For positive values of x, $Q(x)$ takes values between ½ and 0.

For $Q(x)$ the following approximate method exists [2]:
If it is $Q(x_p) = p$, where $0 < p \le 0.5$, and $t \equiv \sqrt{-2 \ln p}$, then

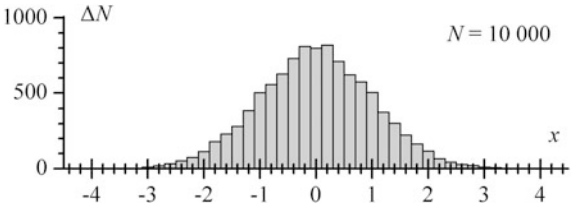
$$x_p = t - \frac{2.515517 + 0.802853\, t + 0.010328\, t^2}{1 + 1.432788\, t + 0.189269\, t^2 + 0.001308\, t^3} + \varepsilon(p)$$

$$\text{where } |\varepsilon(p)| < 4.5 \times 10^{-4}.$$

The method described gives only the positive values of x. We may cover negative values of x as well by using one of the following two methods:

1. We put $p = |R - 0.5|$. We find x_p. If it is $R - 0.5 < 0$, we consider x_p to be negative. If it is $R - 0.5 > 0$, we consider x_p to be positive. This is equivalent to multiplying the value of x_p found by $\frac{R-0.5}{|R-0.5|}$.
2. We put $p = R/2$. We find x_p. We decide whether x_p is to be considered to be positive or negative using another *unbiased* method, e.g. by whether the last digit of R is odd or even.

Using the first method and 10 000 random numbers, we find the corresponding values of x, the histogram of which is given below with a bin width of 0.2.



If we wish the distribution to have a standard deviation σ_x instead of unity, we multiply the values of x, which were found as described above, by the numerical value of σ_x.

If we wish the distribution to have a standard deviation σ_x instead of unity and a mean equal to μ_x instead of 0, having multiplied the values of x by the numerical value of σ_x, we add to the results the numerical value of μ_x.

Example 14.8 [E]

Using Excel©, produce 1000 numbers which are derived from a normal population with $\mu = 5$ and $\sigma = 2$.

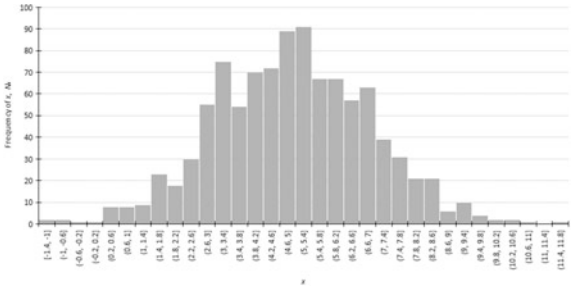
Excel 2016 creates random numbers using the add-in **Data Analysis**, found in **Data** if it is installed.

Highlight cell 14. In **Data Analysis** choose **Random Number Generation**. In the window that opens, set **Number of Variables** 1, **Number of Random Numbers** 1000, **Distribution** Normal, **Mean =** 5, **Standard Deviation =** 2. Pressing OK fills column A with 1000 random numbers, normally distributed, with mean = 5 and standard deviation = 2.

Highlight column A. In **Insert > Recommended Chart** we choose **Histogram**. This produces a histogram of the random numbers.

Excel sets the bin width using the lowest and highest of the random numbers. This has the effect that the bin limits are numbers with many digits. To avoid this, we first sort the numbers in increasing order. In this case, the smallest number is −1.363261491. We change this to −1.4. The largest number is 11.75441697. We change this to 11.8. These changes will not affect the histogram. They will, however simplify the bin widths and limits. Double-clicking on the X-Axis, we open the

Format Axis window. In **Axis Options**, we choose **Bin Width** 0.5. With these changes, the histogram has the form of the figure shown here.



Example 14.9 [O]

Using Origin©, produce 1000 numbers which are derived from a normal population with $\mu = 5$ and $\sigma = 2$.

We highlight the column in which we want the numbers to be entered, say A. Then

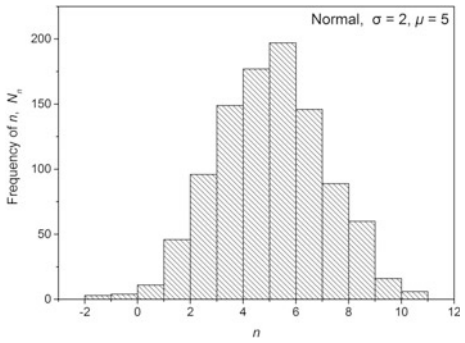
$$\textbf{Column} > \textbf{Set Column Values}$$

In the window that opens, we select:

$$\textbf{Function} > \textbf{Data Generation} > \textbf{normal}(\textbf{nps}[,\ \textbf{seed}])$$

To obtain numbers distributed normally with standard deviation σ and mean μ, we must give the instruction: **normal([*n*])*[σ] + [μ]**, where **[*n*]** is the number of numbers we wish to obtain, **[σ]** is the numerical value of σ and **[μ]** is the numerical value of μ.

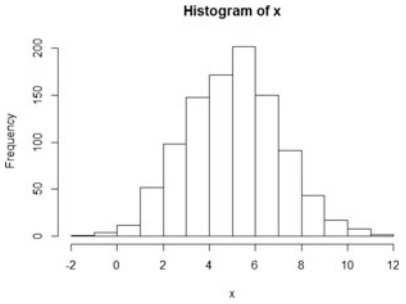
Here, we type the instruction **normal(1000)*2 + 5** and press **OK**. 1000 numbers normally distributed with $\mu = 5$ and $\sigma = 2$ are entered in the selected column. The histogram of these is shown in the figure.



Example 14.10 [R]

Produce 1000 numbers which are derived from a normal population with $\mu = 5$ and $\sigma = 2$.

The function producing random numbers normally distributed is `rnorm(n, μ, σ)`. It produces n random numbers from a parent distribution with a mean μ and standard deviation σ. Using the function for $\mu = 5$ and $\sigma = 2$, produces the numbers whose histogram is shown




```
x <- rnorm(1000, 5, 2)
hist(x)
```

Programs
Excel
Origin
Ch. 14. *Origin—Random Number Generation*
Python
R
Ch. 14. *R—Random Number Generation*

Problems

14.6 Find the relation which transforms random numbers R, distributed uniformly in the range [0, 1), to values of x which are distributed according to the probability density function $f(x) = 1.5 \times 10^{-3}\sqrt{x}$ in the range [0, 100).

14.7 Find the relation which transforms random numbers R, distributed uniformly in the range [0, 1), to values of x which are distributed according to the probability density function $f(x) = \sin x$ in the range $[0, \pi/2)$.

14.8 The probability density function for the displacement of the simple harmonic oscillator is $f(x) = \frac{2/\pi}{\sqrt{a^2 - x^2}}$, for values of x between 0 and a. Find the relation which transforms random numbers R, uniformly distributed in the range [0, 1), to values of x which are distributed according to this density function in the range [0, a).

14.9 **[O]** Produce 1000 numbers which are derived from a Poisson population with $\mu = 5$. Hint: Follow the procedure of Example 14.5 [O], with the final entry in the window that opens being **poisson(n, μ)**.

14.4 Appendix 4: The Values of the Fundamental Physical Constants

The values of the fundamental physical constants given below are the internationally recommended for 2010 by CODATA, The Committee on Data for Science and Technology.

The uncertainty in a value (standard deviation of the mean) is given in parentheses at the end of the numerical value and refers to the corresponding last digits. For example, the value $G = 6.673\ 84(80) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$ should be interpreted to mean $G = (6.673\ 84 \pm 0.000\ 80) \times 10^{-11}\ \mathrm{m^3\ kg^{-1}\ s^{-2}}$.

Most recently revised values of the fundamental physical constants may be found at the web page Fundamental Constants Data Center of the National Institute of Standards and Technology (NIST) of USA: http://physics.nist.gov/cuu/Constants/.

Recommended values of the fundamental physical constants. CODATA 2010

Universal constants			
Quantity	Symbol	Value	Units
Speed of light in vacuum	c	299 792 458 (by definition)	$\mathrm{m\ s^{-1}}$
Magnetic constant	μ_0	$4\pi \times 10^{-7}$ (exact) $12.566\ 370\ 614\ldots \times 10^{-7}$	$\mathrm{N\ A^{-2}}$ $\mathrm{N\ A^{-2}}$
Electric constant	ε_0	$1/\mu_0 c^2$ (exact) $8.854\ 187\ 817\ldots \times 10^{-12}$	$\mathrm{F\ m^{-1}}$ $\mathrm{F\ m^{-1}}$
Newtonian constant of gravitation	G	$6.673\ 84(80) \times 10^{-11}$	$\mathrm{m^3\ kg^{-1}\ s^{-2}}$
Planck constant	h	$6.626\ 069\ 57(29) \times 10^{-34}$	J s
		$4.135\ 667\ 516(91) \times 10^{-15}$	eV s

Electromagnetic constants			
Quantity	Symbol	Value	Units
Elementary charge	e	$1.602\ 176\ 565(35) \times 10^{-19}$	C
Magnetic flux quantum $h/2e$	Φ_0	$2.067\ 833\ 758(46) \times 10^{-15}$	Wb
Conductance quantum $2e^2/h$	G_0	$7.748\ 091\ 7346(25) \times 10^{-5}$	S
Bohr magneton $e\hbar/2m_e$	μ_B	$9.274\ 009\ 68(20) \times 10^{-24}$	J T^{-1}
Nuclear magneton $e\hbar/2m_p$	μ_N	$5.050\ 783\ 53(11) \times 10^{-27}$	J T^{-1}

Atomic and nuclear constants			
Quantity	Symbol	Value	Units
Fine-structure constant $e^2/4\pi\varepsilon_0\hbar c$	α	$7.297\ 352\ 5698(24) \times 10^{-3}$	–
Inverse of fine-structure constant	$1/\alpha$	137.035 999 074(44)	–
Rydberg constant $\alpha^2 m_e c/2h$	R_∞	10 973 731.568 539(55)	m^{-1}
$R_\infty hc$ in eV	$R_\infty hc$	13.605 692 53(30)	eV
Bohr radius $\alpha/4\pi R_\infty = 4\pi\varepsilon_0\hbar^2/m_e e^2$	a_0	$0.529\ 177\ 210\ 92(17) \times 10^{-10}$	m
Hartree energy $e^2/4\pi\varepsilon_0 a_0 = 2R_\infty hc = \alpha^2 m_e c^2$	E_h	$4.359\ 744\ 34(19) \times 10^{-18}$	J
		27.211 385 05(60)	eV
Electron mass	m_e	$9.109\ 382\ 91(40) \times 10^{-31}$	kg
		$5.485\ 799\ 0946(22) \times 10^{-4}$	u
Proton mass	m_p	$1.672\ 621\ 777(74) \times 10^{-27}$	kg
		1.007 276 466 812(90)	u
Neutron mass	m_n	$1.674\ 927\ 351(74) \times 10^{-27}$	kg
		1.008 664 916 00(43)	u
Electron mass energy equivalent	$m_e c^2$	$8.187\ 105\ 06(36) \times 10^{-14}$	J
		0.510 998 928(11)	MeV
Proton mass energy equivalent	$m_p c^2$	$1.503\ 277\ 484(66) \times 10^{-10}$	J
		938.272 046(21)	MeV
Neutron mass energy equivalent	$m_n c^2$	$1.505\ 349\ 631(66) \times 10^{-10}$	J
		939.565 379(21)	MeV
Proton mass/electron mass	m_p/m_e	1836.152 672 45(75)	–
Neutron mass /proton mass	m_n/m_p	1.001 378 419 17(45)	–
Electron charge to mass quotient	$-e/m_e$	$-1.758\ 820\ 088(39) \times 10^{11}$	C kg^{-1}
Proton charge to mass quotient	e/m_p	$9.578\ 833\ 58(21) \times 10^7$	C kg^{-1}
Electron Compton wavelength $h/m_e c$	λ_C	$2.426\ 310\ 2389(16) \times 10^{-12}$	m
Proton Compton wavelength $h/m_p c$	$\lambda_{C,p}$	$1.321\ 409\ 856\ 23(94) \times 10^{-15}$	m
Neutron Compton wavelength $h/m_n c$	$\lambda_{C,n}$	$1.319\ 590\ 9068(11) \times 10^{-15}$	m
Classical electron radius $\alpha^2 a_0$	r_e	$2.817\ 940\ 3267(27) \times 10^{-15}$	m

(continued)

(continued)

Atomic and nuclear constants			
Quantity	Symbol	Value	Units
Electron magnetic moment	μ_e	$-9.284\ 764\ 30(21) \times 10^{-24}$	J T^{-1}
in Bohr magnetons	μ_e/μ_B	$-1.001\ 159\ 652\ 180\ 76(27)$	–
Proton magnetic moment	μ_p	$1.410\ 606\ 743(33) \times 10^{-26}$	J T^{-1}
in Bohr magnetons	μ_p/μ_B	$1.521\ 032\ 210(12) \times 10^{-3}$	–
Neutron magnetic moment	μ_n	$-0.966\ 236\ 47(23) \times 10^{-26}$	J T^{-1}
in Bohr magnetons	μ_n/μ_B	$-1.041\ 875\ 63(25) \times 10^{-3}$	–

Physico-chemical constants			
Quantity	Symbol	Value	Units
Avogadro constant	N_A	$6.022\ 141\ 29(27) \times 10^{23}$	mol^{-1}
Atomic mass constant $m_u = \frac{1}{12} m(^{12}C)$ $= 1\ u = 10^{-3}\ kg\ mol^{-1}/N_A$	m_u	$1.660\ 538\ 921(73) \times 10^{-27}$	kg
Atomic mass constant energy equivalent	$m_u c^2$	$1.492\ 417\ 954(66) \times 10^{-10}$	J
		931.494 061(21)	MeV
Faraday constant $N_A e$	F	96 485.3365(21)	C mol^{-1}
Molar gas constant	R	8.314 4621(75)	J mol^{-1} K^{-1}
Boltzmann constant R/N_A	k	$1.380\ 6488(13) \times 10^{-23}$	J K^{-1}
		$8.617\ 3324(78) \times 10^{-5}$	eV/K^{-1}
Inverse of Boltzmann's constant in K eV^{-1}	$1/k$	11 604.519(11)	K eV^{-1}
Molar volume of ideal gas RT/p (T = 273.15 K, p = 101.325 kPa)	V_m	$22.413\ 968(20) \times 10^{-3}$	m^3 mol^{-1}
Stefan-Boltzmann constant $(\pi^2/60)\, k^4/\hbar^3 c^2$	σ	$5.670\ 373(21) \times 10^{-8}$	W m^2 K^4
Wien wavelength displacement constant $b = \lambda_{max} T = (hc/k)/4.965\ 114\ 231\ldots$	b	$2.897\ 7721(26) \times 10^{-3}$	m K

Values which are internationally accepted:
Standard atmosphere = 101 325 Pa.
Standard acceleration of gravity $g_n = 9.806\ 65\ m/s^2$.

References

1. P.W. Bridgman, *Dimensional Analysis* (Yale University Press, 1931. Reprinted 1963). E. de Isaacson and M. de Isaacson, *Dimensional Methods in Engineering and Physics* (Edward Arnold Publishers, London, 1975). H.L. Langhaar, *Dimensional Analysis and Theory of Models* (R.E. Krieger Publishing Company, 1951. Reprinted 1979)
2. C. Hastings Jr., *Approximations for Digital Computers* (Princeton University Press, Princeton, N.J., 1955). Also in M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions* (Dover Publications, Inc., New York, 1972). Chap. 26, Eq. (26.2.23)

Answers to the Problems

Chapter 1

1.1 (a) $\bar{x} = 23.288$ cm, (b) $\overline{|d|} = 0.019$ cm, (c) $s_x = 0.021$ cm.

1.2 (a) $\bar{T} = 2.19$ s, (b) $\overline{|d_T|} = 0.02$ s (c) $s_T = 0.03$ s.

1.3 (b) $\bar{h} = 179.2$ cm, (c) $s_h = 3.6$ cm

1.4 (a) $\bar{x}' = \bar{x} + 1$, $s'_x = s_x$, (b) $\bar{x}' = 1.1\bar{x}$, $s'_x = 1.1 s_x$.

1.5 (a) $A = 1$, (b) $\bar{x} = \frac{1}{2}$, $\sigma_x = \sqrt{\frac{1}{12} - \frac{1}{2\pi^2}}$.

1.6 (a) $\bar{x} = 1$, (b) $\sigma_x = \sqrt{7 - \frac{32}{\pi^2}}$.

1.7 $\bar{x} = a, \quad \sigma_x = a$.

1.8 (a) $A = \frac{2}{L}, \quad \bar{x} = 0, \quad \sigma_x = \frac{\sqrt{3}}{12} L$, (b) $A = \frac{4}{L}, \quad \bar{x} = 0, \quad \sigma_x = \frac{\sqrt{6}}{24} L$.

1.9 (a) $f(x)\,\mathrm{d}x = \dfrac{\mathrm{d}x}{\pi\sqrt{a^2 - x^2}}$, (b) $g(\upsilon)\,\mathrm{d}\upsilon = \dfrac{\mathrm{d}\upsilon}{\pi\sqrt{a^2\omega^2 - \upsilon^2}}$, (c) $\bar{x} = 0$ and $\bar{\upsilon} = 0$, (d) $\overline{|x|} = \frac{a}{\pi}$, $\overline{|\upsilon|} = \frac{a\omega}{\pi}$, (e) $\sigma_x = \frac{a}{\sqrt{2}}$, $\sigma_\upsilon = \frac{a\omega}{\sqrt{2}}$.

1.10 (a) $A = \frac{1}{\sqrt{\pi a}}$, (c) $P\{x < -a \text{ or } x > a\} = 1 - \mathrm{erf}(1) = 0.157$.

1.11 $A = \sqrt{\frac{hm}{\pi}}$, $\overline{|u|} = \frac{1}{\sqrt{\pi hm}}$.

Chapter 2

2.1 72 kg.

2.2 1%.

2.3 (a) 6 m^3, 6.19 m^3, (b) 0.032, (c) $\frac{\delta a}{a} = 0.020$, $\frac{\delta b}{b} = -0.005$ and $\frac{\delta c}{c} = 0.017$ (d) $\frac{\delta V}{V} = 0.032$ and the two results are identical to 2 significant figures.

2.4 $\frac{\delta Q}{Q} = 2\frac{\delta x}{x} + \frac{\delta y}{y} - 2\frac{\delta z}{z}$

2.5 $\frac{\delta Q}{Q} = 2\frac{\delta x}{x} + \frac{\delta y}{y+2} - 2\frac{\delta z}{z}$

C. Christodoulides and G. Christodoulides, *Analysis and Presentation of Experimental Results*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-53345-2

2.6 $\frac{\delta f}{f} = \frac{\delta s}{s} + \frac{\delta s'}{s'} - \frac{\delta s + \delta s'}{s + s'} = 0.046$

2.7 (a) $\frac{\delta \phi}{\phi} = \frac{\delta p}{p} + 4\frac{\delta r}{r} - \frac{\delta l}{l} - \frac{\delta \eta}{\eta}$, (b) the radius r.

2.8 $\frac{\delta m}{m} = \left(\frac{V^2}{c^2}\right)\frac{\delta V}{V}$.

2.9 $\frac{\delta x}{x} = \frac{\delta a}{a_{\mathrm{m}}} + \frac{t\,\delta \omega}{\tan \omega_{\mathrm{m}} t}$.

Chapter 4

4.1 0.5.

4.2 $\bar{x} = 14.5$, $s_x = 2.9$, $\sigma_{\bar{x}} = 1.0$.

4.4 (a) $\bar{x}' = \bar{x} + a$, (b) $s'_x = s_x$.

4.5 (a) $\bar{x}' = a\bar{x}$, (b) $s'_x = as_x$.

4.6 (a) 2.00 s, (b) 0.09 s, (c) 0.03 s.

4.7 $\bar{x} = 14.5$, $s_x = 2.2$, $\sigma_{\bar{x}} = 0.4$, (a) $\bar{x} = 29.0$, $s_x = 4.4$, $\sigma_{\bar{x}} = 0.8$, (b) $\bar{x} = 14.5$, $s_x = 2.2$, $\sigma_{\bar{x}} = 0.3$.

4.8 $\bar{x} = 128.3$ cm, $s_x = 1.5$ cm, $\sigma_{\bar{x}} = 0.21$ cm.

4.9 $\bar{x} = 10.38$, $s_x = 0.17 \pm 0.04$, $\sigma_{\bar{x}} = 0.060 \pm 0.015$.

4.10 (a) 3, (b) 6, (c) 100.

4.11 At least 200.

4.12 (a) 0.023, (b) 0.046, (c) 0.023, (d) 0.027.

4.13 (a) 4.55%, (b) 31.7%, (c) 50%, (d) 63.4%, (e) 99.73%, (f) 45.5%.

4.14 (a) 0.39%, (b) 2.9%, (c) 1.0%, (d) 2.5%.

4.15 0.67%.

4.16 (a) 0.683, (b) 0.954, (c) 0.997.

Chapter 5

5.1

1 significant figure	2 significant figures	3 significant figures	4 significant figures
0.002	0.0018	0.001 76	0.001 762
8	8.1	8.14	8.144
3×10^5	2.7×10^5	2.68×10^5	2.680×10^5
1×10^5	1.3×10^5	1.27×10^5	1.274×10^5
8×10^{-8}	8.1×10^{-8}	8.13×10^{-8}	8.132×10^{-8}

5.2

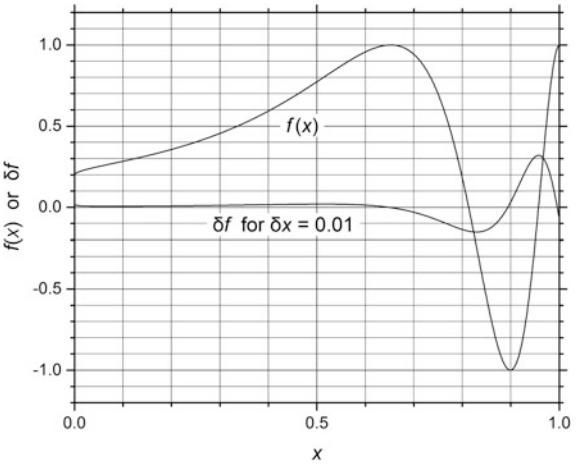
1 significant figure	2 significant figures	3 significant figures	4 significant figures
5×10^4	4.7×10^4	4.74×10^4	4.739×10^4
5×10^{-8}	5.1×10^{-8}	5.13×10^{-8}	5.132×10^{-8}
0.0003	0.000 30	0.000 298	0.000 297 6
3	3.1	3.14	3.142
9×10^5	9.4×10^5	9.38×10^5	9.380×10^5

5.3 523.6 ± 5.4, $0.007\,832 \pm 0.000\,063$, $(4.73 \pm 0.94) \times 10^6$.
5.4 263.58 ± 0.25, 0.0033 ± 0.0026, $(4.6 \pm 1.2) \times 10^3$.
5.5 $823.6 \pm 5.4 \times (1 \pm 0.35)$, $0.007\,832 \pm 0.000\,043 \times (1 \pm 0.24)$, $[4.73 \pm 0.44 \times (1 \pm 0.14)] \times 10^6$.
5.6 5, 10, 40.

Chapter 6

6.1 (a) 1.84 ± 0.27, (b) 17.0 ± 0.5, (c) 11.6 ± 0.7, (d) 0.294 ± 0.009, (e) 0.086 ± 0.005.
6.2 (a) 1.25 ± 0.13 (b) 0.00 ± 0.05, (c) 16.0 ± 2.4, (d) 0.50 ± 0.05.
6.3 $\frac{\delta y}{|y|} = \frac{\delta x}{|x(1+x)|}$.
6.4 (a) $\sin\theta = 0.520 \pm 0.003$, (b) $\cos\theta = 0.855 \pm 0.002$, (c) $\tan^2\theta = 0.370 \pm 0.006$, (d) $\sin^2\theta = 0.270 \pm 0.003$.
6.5 (a) $f = 12.50 \pm 0.10$, (b) $f = (73 \pm 5) \times 10^{-5}$, (c) $f = (4.9 \pm 0.5) \times 10^{-5}$, (d) $f = 4.50 \pm 0.26$, (e) $f = 13.3 \pm 0.9$, (f) $f = 0.487 \pm 0.014$.
6.6 (a) $x + y = 11.8 \pm 0.6$ cm, (b) $x - y = -2.6 \pm 0.6$ cm, (c) $x/y = 0.64 \pm 0.06$, (d) $xy = 33.1 \pm 3.2$ cm^2.
6.7 (a) 2.2%, 1.6%, (b) 0.06, (c) 1%, 16%.
6.8 $Q = 9 \pm 17$. No.
6.9 $x = 1.41 \pm 0.09$ m.
6.10 $x = 0.207 \pm 0.021$ m.
6.11 $x = 0.35 \pm 0.06$ m.
6.12 $x = 0.92 \pm 0.08$ m.
6.13 $\delta y = \sqrt{[\sin(\omega t + \phi)\,\delta A]^2 + [A\cos(\omega t + \phi)\,\delta\phi]^2 + [At\cos(\omega t + \phi)\,\delta\omega]^2}$.
6.14 (a) $B = 60.0^\circ \pm 0.3^\circ$, (b) $a = 5.00 \pm 0.11$ m, $b = 8.66 \pm 0.18$ m.
6.15 $c = 2.5 \pm 0.4$ m.
6.16 $a = 0.85 \pm 0.02$ m, $b = c = 1.10 \pm 0.02$ m.
6.17 $F_x = 6.0 \pm 0.2$ N, $F = 10.4 \pm 0.3$ N.
6.18 $n = 1.2896 \pm 0.0023$.
6.19 $x = 0.71 \pm 0.09$ m.
6.20 $x = 2.00 \pm 0.10$ m
6.21 $\delta V/V = 0.015$, $\rho = 5.62 \pm 0.08$ g/cm^3.

6.22 $f = 4.82 \pm 0.05$ cm.
6.23 $T = 2.007 \pm 0.010$ s.
6.24 5%.
6.25 $G = 37.24 \pm 0.23$ db.
6.26 (a) $Q = 6396 \pm 156$, (b) $Q = 6388 \pm 153$.
6.27 $\bar{x} = 3.46$, $s_x = 0.26$, $\sigma_{\bar{x},\alpha} = 0.0151$, $\sigma_{\bar{x},\beta} = 0.0156$, $\sigma_{\bar{x}} = 0.0150$.
6.28 $\bar{x}_\beta = 5.17$, $s_\beta = 0.08$.
6.29 $n = 1.602\,46 \pm 0.000\,08$.
6.30 $\frac{\delta\phi}{\phi} = \sqrt{\left(\frac{\delta p}{p}\right)^2 + \left(4\frac{\delta r}{r}\right)^2 + \left(\frac{\delta l}{l}\right)^2 + \left(\frac{\delta \eta}{\eta}\right)^2}$.
6.31 $n = 1.788 \pm 0.005$.
6.32



Chapter 7

7.1 $\bar{x} = 2.00$, $np = 2$, $\sigma = 1.155$, $\sqrt{npq} = 1.155$.
7.2

(continued)

(continued)

x	0	1	2	3	4	5	6	7	8	9	10
x	0	1	2	3	4	5	6	7	8	9	10
$P_n(x)$	0.028	0.121	0.233	0.267	0.200	0.103	0.037	0.0090	0.0014	0.00014	0.000006

7.3 (a) $\frac{1}{32}$, $\frac{5}{32}$, $\frac{10}{32}$, $\frac{10}{32}$, $\frac{5}{32}$, $\frac{1}{32}$, (b) 4, 20, 40, 40, 20, 4.
7.4 $n = 18$, $p = \frac{1}{3}$ and $q = \frac{2}{3}$.
7.5 0.040, 0,045.
7.6 For x up to and including 12, $\bar{x} = 3.000$, $\sigma = \sqrt{3} = 1.73$.
7.7 $\bar{x} = 29.2$, $\sigma = 1.42$. The expression $\frac{N}{\sqrt{2\pi}\sigma} e^{-(x-\bar{x})^2/2\sigma^2}$ gives the values n'_r:

x_r	25	26	27	28	29	30	31	32	33	34
n_r	1	5	17	49	85	52	25	11	4	1
n'_r	0.9	5.4	21	49	69	60	32	10.3	2.0	0.9

There is satisfactory agreement between the histogram and the curve found.
7.8 $\bar{x} = 46.0$ mm and $\sigma = 1.58$ mm, $f(x) = 12.4 \exp\left\{-0.200(x - 46.0)^2\right\}$ x in mm.

Chapter 8

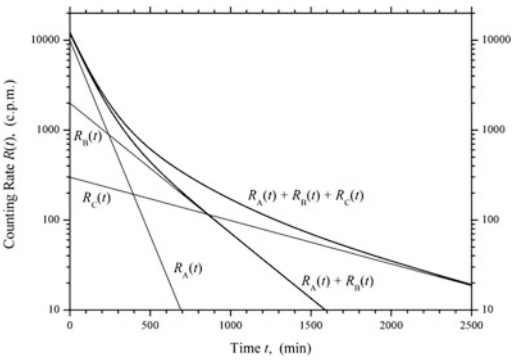
8.1 (a) $R_1 = 98.6 \pm 4.4$ c.p.m., $R_2 = 96.8 \pm 2.2$ c.p.m. (b) 2.
8.2 (a) 1.67, (b) 0.188.
8.3 $\lambda = \ln n_1 - \ln n_2 = 1.65$ min^{-1},

$$\delta\lambda = \sqrt{\left(\frac{\delta n_1}{n_1}\right)^2 + \left(\frac{\delta n_2}{n_2}\right)^2} = \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \approx \frac{1}{7} \approx 0.14 \text{ min}^{-1}.$$

8.4 According to Example 8.4, (a) $\frac{t_{\text{tot}}}{t_{\text{back}}} = \sqrt{\frac{R_{\text{tot}}}{R_{\text{back}}}} = 2$,
(b) $\sigma_{\text{s,min}} = \sqrt{\frac{R_{\text{tot}}}{t}} + \sqrt{\frac{R_{\text{tot}}}{t}} = 1$ c.p.s.
8.5 $r = 1.55 \pm 0.10$.
8.6 The counting rates due to the three isotopes are, in c.p.m. and t in min,

$$R_{\text{A}}(t) = -\frac{\text{d}N_{\text{A}}}{\text{d}t} = 10\,000\ \text{e}^{-t/100}$$
$$R_{\text{B}}(t) = -\frac{\text{d}N_{\text{B}}}{\text{d}t} = 2000\ \text{e}^{-t/300}$$
$$R_{\text{C}}(t) = -\frac{\text{d}N_{\text{C}}}{\text{d}t} = 300\ \text{e}^{-t/900}.$$

In the figure we plot these rates as well as the sums $R_{\text{A}}(t) + R_{\text{B}}(t)$ and $R_{\text{A}}(t) + R_{\text{B}}(t) + R_{\text{C}}(t)$.



Chapter 9

9.1 (a) $x = 5.24 \pm 0.05$, (b) $x = 5.21 \pm 0.05$.
9.2 (a) $x = 8.52 \pm 0.09$, (b) $x = 8.51 \pm 0.09$.
9.3 $\alpha = \sqrt{\frac{N}{2\sum_i (x_i - \bar{x})^2}}$

Chapter 10

10.1 (a) $\bar{x} = 7.5$, $s_x = 2.4$, (b) Yes.
10.2 Yes, the value 252.
10.3 $x = 32.3 \pm 0.2$. $\Pr\left\{\frac{|x-\bar{x}|}{s_x} \geq 3.4\right\} = 0.0007 < \frac{1}{2N}$ and the measurement is rejected.
10.4 $|\bar{x}_2 - \bar{x}_1| = 0.017$, $\sigma_{\bar{x}_2 - \bar{x}_1} = 0.019$. $|\bar{x}_2 - \bar{x}_1| < \sigma_{\bar{x}_2 - \bar{x}_1}$ and the results are mutually compatible.
10.5 $|\bar{x}_2 - \bar{x}_1| = 17$, $\sigma_{\bar{x}_2 - \bar{x}_1} = 7.2$. $|\bar{x}_2 - \bar{x}_1| > \sigma_{\bar{x}_2 - \bar{x}_1}$ and the results are not mutually compatible.

Chapter 11

11.1 (b) $y = 3.92 + 6.96\,x$, (c) $r = 0.999$, (d) $\delta a = 0.36, \delta \lambda = 0.06$.
11.2 (a) $a = 8.25 \pm 0.19$ $\lambda = -0.60 \pm 0.04$, (b) For $x = 5$ it is $y = 5.25 \pm 0.28$, (c) For $y = 0$ it is $x = 13.8 \pm 0.1$.

11.4 $y = 2.85 - 4.84\,x + 165.2\,x^2$.

11.5 $$A = \frac{[\mathrm{e}^{-2\kappa t}\sin^2\omega t][y\,\mathrm{e}^{-\kappa t}\cos\omega t] - [y\,\mathrm{e}^{-\kappa t}\sin\omega t]\,[\mathrm{e}^{-2\kappa t}\sin\omega t\,\cos\omega t]}{[\mathrm{e}^{-2\kappa t}\sin^2\omega t]\,[\mathrm{e}^{-2\kappa t}\cos^2\omega t] - [\mathrm{e}^{-2\kappa t}\sin\omega t\,\cos\omega t]^2},$$

$$B = \frac{[\mathrm{e}^{-2\kappa t}\cos^2\omega t][y\,\mathrm{e}^{-\kappa t}\sin\omega t] - [y^{-\kappa t}\cos\omega t][\mathrm{e}^{-2\kappa t}\sin\omega t\,\cos\omega t]}{[\mathrm{e}^{-2\kappa t}\sin^2\omega t][\mathrm{e}^{-2\kappa t}\cos^2\omega t] - [\mathrm{e}^{-2\kappa t}\sin\omega t\,\cos\omega t]^2},$$

where $[f(t)]$ implies summation over all $f(t)$.

11.6 $$N_{01} = \frac{[Ry][xy] - [Rx][y^2]}{[xy]^2 - [x^2][y^2]}, N_{02} = \frac{[Rx][xy] - [Ry][x^2]}{[xy]^2 - [x^2][y^2]}.$$

11.7 $A = 0.00130 \pm 0.00020$ centipoise, $\lambda = 1953 \pm 45$ K.

11.8 $a = 0.319\,325 \pm 0.000\,005$ s/m$^{1/2}$, $\lambda = (5.00 \pm 0.28)\times 10^{-8}$ s/m$^{3/2}$.

11.9 $a = 12.7 \pm 1.1$ N, $\lambda = 1.84 \pm 0.08$ Ns^2/m^3. [Origin]: $a = 14.8 \pm 1.4$ N, $\lambda = 1.47 \pm 0.10$ Ns^2/m^3.

11.10 $\lambda = 0.18141 \pm 0.00020$ d^{-1}.

11.11 (a) $y = 1.091 + 0.636x$, (b) $x = -1.00 + 1.50y$.

11.12 It is found that $a = 299\,788.1 \pm 3.1$ km/s and $\lambda = 0.22 \pm 0.16$ (km/s) per year. Therefore, $c = 299\,788.1 + 0.22(t - 1956)$ km/s. The possible rate of change of c with time was found to be equal to $\lambda = 0.22 \pm 0.16$ (km/s) per year, which has too high an uncertainty for it to be evidence of such a change. Besides, the mean value of the measurements is $\bar{c} = 299\,785.0$ km/s with a standard deviation of the measurements equal to $s_c = 8.8$ km/s. Over the period of the 50 years examined, the total variation in c, if this did exist, would have been equal to $50\lambda = 11 \pm 8$ km/s, which is comparable to the uncertainty s_c.

11.13 $x = 1.61$, $y = 0.34$.

11.14 $x = 3.0$, $y = 3.0$, $z = 1.0$.

11.15 $x = 17.0 \pm 1.1$, $y = 5.7 \pm 0.2$.

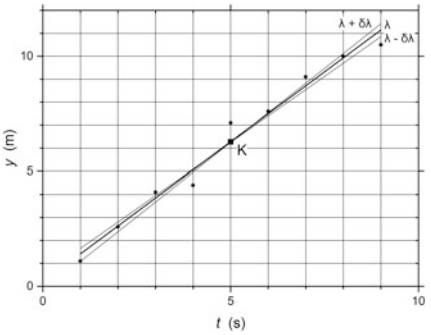
Chapter 12

12.1 $a = 0.19 \pm 0.37$ m, $\lambda = 1.22 \pm 0.07$ m/s, K:($\bar{t} = 5$ s, $\bar{y} = 6.27$ m) see figure.

12.2 $y_0 = 15.0$ m, $\gamma = 31.7$ m/s^2.

12.3 $v_0 = 0.92$ m/s, $\gamma = 1.017$ m/s^2.

12.4 $k = 3.81 \pm 0.16$ N/m.



Chapter 14

14.1 $T = \frac{A}{\sqrt{GM}} R^{3/2}$, A = constant.

14.2 $P = An_0 kT$, A = constant, k = Boltzmann's constant.

14.3 $Q = A\frac{a^4}{\eta}\left(\frac{\Delta p}{\Delta x}\right)$, A = constant.

14.4 $\upsilon = A\sqrt{F/\lambda}$, A = constant. No.

14.5 $W = A\frac{Q^2}{\varepsilon_0 R}$, A = constant. No, but A will be different.

14.6 $x = 100\,R^{2/3}$

14.7 $x = \arcsin R$

14.8 $x = a\,\sin\left(\frac{\pi}{2}R\right)$

List of Programs and Code Samples

(continued)

C. Christodoulides and G. Christodoulides, *Analysis and Presentation of Experimental Results*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-53345-2

(continued)

Ch. 12. *Excel—Scatter Plot—Linear Scales—Errors*
Ch. 12. *Excel—Scatter Plot—Linear Scales—Multiple*
Ch. 12. *Excel—Scatter Plot—Linear-Log Scales*
Ch. 12. *Excel—Scatter Plot—Linear-Log Scales—Errors*
Ch. 12. *Excel—Scatter Plot—Linear-Log Scales—Multiple*
Ch. 12. *Excel—Scatter Plot—Log-Linear Scales*
Ch. 12. *Excel—Scatter Plot—Log-Linear Scales—Errors*
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Origin

Chapter 4 The Statistical Analysis of Experimental Results
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Ch. 07. *Origin—Binomial Distribution—Probability of x successes*
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