10

Two and Three-Dimensional Systems

Separation of variables; degeneracy theorem; group of invariance of the two-dimensional isotropic oscillator.

10.1 Consider the Hamiltonian of a two-dimensional anisotropic harmonic oscillator:

$$H = \Big(\frac{p_1^2}{2m} + \frac{1}{2}m\omega_1^2\,q_1^2\Big) + \Big(\frac{p_2^2}{2m} + \frac{1}{2}m\omega_2^2\,q_2^2\Big)\,; \qquad \omega_1 \neq \omega_2\,.$$

a) Exploit the fact that the Schrödinger eigenvalue equation can be solved by separating the variables and find a complete set of eigenfunctions of H and the corresponding eigenvalues.

b) Assume that $\omega_1/\omega_2 = 3/4$. Find the first two degenerate energy levels. What can one say about the degeneracy of energy levels when the ratio between ω_1 and ω_2 is not a rational number?

c) Write the eigenfunctions of the Hamiltonian in the case $\omega_2 = 0$.

Consider now a particle of mass m in two dimensions subject to the potential:

$$V(q_1,\,q_2) = m\omega^2\big(q_1^2 - q_1 q_2 + q_2^2\big)\,.$$

d) Say whether the problem of finding the eigenvalues of the Hamiltonian $H = (p_1^2 + p_2^2)/2m + V(q_1,\,q_2)$ can be solved by the method of separation of variables.

10.2 A particle of mass m in two dimensions is constrained inside a square whose edge is $2a$: $|x| \le a,\ |y| \le a$.

a) Write the Schrödinger equation, separate the variables and find a complete set of eigenfunctions of the Hamiltonian.

b) Find the energy levels of the system and say whether there is degeneracy.

c) Say whether there exist operators (i.e. transformations) that commute with the Hamiltonian but do not commute among themselves. In the affirmative case, give one or more examples.

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Assume now that within the square the potential:

$$V_a(x,y) = V_{0a}\cos(\pi x/2a)\,\cos(\pi y/2a)$$

is present.

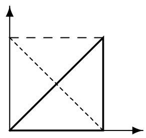
d) Is it still possible to separate the variables in the Schrödinger equation? Do degenerate energy levels exist?

e) Say whether it is possible to guarantee the existence of degenerate energy levels if, instead, the potential is

$$V_b(x,y) = V_{0b}\sin(\pi x/a)\,\sin(\pi y/a)\;.$$

Is any relationship between the eigenfunctions of the Hamiltonian $\psi_E(x,y)$ and $\psi_E(y,x)$ expected? (Namely, are they equal? are they different? ...)

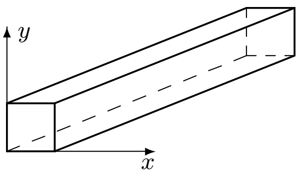
10.3 A particle of mass m in two dimensions is constrained inside the triangle whose vertices have the coordinates $(x=0,\ y=0)$; $(x=a,\ y=0)$; $(x=a,\ y=a)$ (a half of the square with edge a).



a) Find eigenvectors and eigenvalues of the energy.

b) For the same system and exploiting the results of the previous question, find a complete set of eigenvectors of the operator that implements the reflection through the straight line $x+y=a$ (the dotted line in the figure).

10.4 A particle of mass m in three dimensions is confined within an infinite rectilinear guide with a cross section that is a square of edge a.



a) Find eigenfunctions and eigenvalues of the Hamiltonian. What is the minimum energy (threshold energy) the particle must have in order to propagate along the guide?

Consider the wavefunctions:

$$\psi_1(x,y,z) = A\,\sin(2\pi x/a)\,\sin(\pi y/a)\,e^{i k_1 z}$$
$$\psi_2(x,y,z) = B\,\sin(\pi x/a)\,\sin(\pi y/a)\,e^{i k_2 z}\;.$$

b) Determine the normalization coefficients A and B in such a way that the integral of the densities $\rho_{1,2}$ over a slice of the guide of unit volume equals 1 (“one particle per unit volume” normalization).

c) Calculate the probability current densities:

$$\vec{\jmath}_{1,2}\,(x,y,z) = \frac{\hbar}{m}\,\Im m\left(\psi^*_{1,2}(x,y,z)\,\nabla\,\psi_{1,2}(x,y,z)\right)$$

for the states represented by the wavefunctions ψ_1 and ψ_2 normalized as above, and verify that $\operatorname{div}\vec{\jmath}_{1,2}\,(x,y,z)=0$.

d) Say for which values of k_2 the probability current associated to the state represented by $\psi(x,y,z,\,t)$ with $\psi(x,y,z,\,0)=\psi_1(x,y,z)+\psi_2(x,y,z)$, is divergenceless.

10.5 A particle is subject to the potential $V=V(q_1^2+q_2^2,\,q_3)$.

a) Show that the Hamiltonian $H_0=\vec{p}^{\,2}/2m+V$ commutes with the angular momentum operator $L_z=q_1p_2-q_2p_1$.

b) Use the degeneracy theorem to show that there exist degenerate energy levels.

c) Say whether and how the degeneracy is removed if the system is on a platform rotating around the z axis with constant angular velocity ω.

10.6 The Hamiltonian of a two-dimensional isotropic harmonic oscillator of mass m and angular frequency ω is

$$H=\frac{1}{2m}\big(p_1^2+p_2^2\big)+\frac{1}{2}m\omega^2\big(q_1^2+q_2^2\big)=H_1(q_1,\,p_1)+H_2(q_2,\,p_2)\,.$$

a) Exploit the separation of variables ($H=H_1+H_2$) and find the eigenvalues of H and their degeneracies.

b) Write the eigenfunctions of the Hamiltonian in the Schrödinger representation, in the basis in which both H_1 and H_2 are diagonal.

c) Is the degeneracy found in a) in agreement with the result established in Problem 10.5? Find the maximum and the minimum of the eigenvalues m' of $L_3=q_1p_2-q_2p_1$ within each energy level. Do all its possible values ranging between $m'_{\rm max}$ and $m'_{\rm min}$ occur?

d) For each of the first three energy levels, say which eigenvalues of L_3 do occur and explicitly write the wavefunctions relative to the states $|\,E,m'\,\rangle$ (simultaneous eigenstates of H and L_3).

10.7 This problem is devoted to establish a priori the degeneracies of the two-dimensional isotropic harmonic oscillator found in Problem 10.6. Set

$$\eta_a=\frac{1}{\sqrt{2m\omega\,\hbar}}(p_a-\mathrm{i}\,m\omega\,q_a)\,,\qquad\qquad a=1,\,2\,.$$

a) Write the Hamiltonian H of the two-dimensional oscillator in terms of the operators η_a and $\eta_a^\dagger$ and the commutation rules $[\eta_a\,,\,\eta_b^\dagger\,]$, $\quad a,\,b=1,2$.

b) Show that the four operators $\eta_a^\dagger\,\eta_b$ commute with the Hamiltonian H.

Consider the operators:

$$j_1=\frac{1}{2}(\eta_1^\dagger\,\eta_2+\eta_2^\dagger\,\eta_1)\,,\quad j_2=\frac{1}{2\mathrm{i}}(\eta_1^\dagger\,\eta_2-\eta_2^\dagger\,\eta_1)\,,\quad j_3=\frac{1}{2}(\eta_1^\dagger\,\eta_1-\eta_2^\dagger\,\eta_2)\,.$$

c) Show that the operators j_a have the same commutation rules as the angular momentum (divided by $\hbar$). Write j_2 and j_3 in terms of the q's and p's and show that the 'angular momentum operators' j_a have both integer and half-integer eigenvalues.

Setting $h_0 = H/\hbar\,\omega - 1$, the identity

$$\boldsymbol{j}^2 \equiv j_1^2 + j_2^2 + j_3^2 = \frac{h_0}{2} \times \Big(\frac{h_0}{2} + 1\Big)$$

holds (it may be verified using the commutation rules).

d) Exploit the theory of angular momentum (all the properties of the angular momentum follow uniquely from the commutation relations) and the above identity to find the eigenvalues of H and the relative degeneracies. Say which eigenvalues of L_3 do occur in each energy level.

10.8 In Problem 10.7 the energy levels of a two-dimensional isotropic harmonic oscillator and their degeneracies have been found starting from the commutation rules of the three constants of motion j_1, j_2, j_3 'given from the outside'. We now want to establish both the existence and the form of such constants of motion starting from the invariance group of the Hamiltonian. Adopting the notation of Problem 10.7 one has:

$$H = \hbar\,\omega\,\Big(\sum_{a=1}^{2} \eta_a^\dagger\,\eta_a + 1\Big)\,; \quad [\eta_a\,,\,\eta_b\,] = 0\,, \quad [\eta_a\,,\,\eta_b^\dagger\,] = \delta_{a\,b}\,; \quad a,\,b = 1,\,2\;.$$

Consider the linear transformation:

$$\tilde{\eta}_a = \sum\nolimits_b u_{ab}\,\eta_b\;. \tag{1}$$

a) Show that (1) is an invariance transformation both for the Hamiltonian and for the commutation rules if and only if u is a unitary 2×2 matrix.

We shall consider only the transformations that fulfill $\det u = 1$.

b) Show that all the unitary 2×2 matrices, whose determinant is 1, may be written as:

$$\begin{pmatrix} z_1 & z_2 \\ -z_2^* & z_1^* \end{pmatrix}, \qquad |z_1|^2 + |z_2|^2 = 1\,, \qquad z_1,\, z_2 \in \mathbb{C}\;.$$

They, therefore, form a continuous, 3 parameter group – the group $SU(2)$.

The transformation (1) in a neighborhood of the identity takes the form:

$$\tilde{\eta}_a = \eta_a + \mathrm{i}\,\epsilon \sum\nolimits_b g_{ab}\,\eta_b\,, \qquad u \approx \mathbb{1} + \mathrm{i}\,\epsilon\, g\,, \qquad \epsilon \ll 1\;. \tag{2}$$

c) Show that the matrix g is Hermitian and traceless.

Thanks to the von Neumann theorem, for any transformation (1) there exists a unitary operator that implements it:

$$\tilde{\eta}_a = U(u)\,\eta_a\,U^{-1}(u)\;.$$

d) Let $U(g,\,\epsilon) = \mathrm{e}^{-\mathrm{i}\,\epsilon\,G_g}$ be the unitary operator that implements the infinitesimal transformation (2) (the operators $G_g = G_g^\dagger$ are the generators of the group). Compare $\tilde{\eta}_a = U(g,\,\epsilon)\,\eta_a\,U^{-1}(g,\,\epsilon)$, expanded to the first order in ϵ, and (2) and show that $[G_g\,,\,\eta_a] = -\sum_b g_{ab}\,\eta_b$. Find the expression for G_g and show that $[G_g\,,\,H] = 0$.

e) Show that any traceless Hermitian 2×2 matrix g may be written in the form (the factor $\frac{1}{2}$ is there only for the sake of convenience):

$$g = a_1\,\tfrac{1}{2}\sigma_1 + a_2\,\tfrac{1}{2}\sigma_2 + a_3\,\tfrac{1}{2}\sigma_3\,, \qquad a_i \in \mathbb{R}$$

where σ_i are the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

f) Write the expressions for G_g in the three particular cases when only one of the a_i equals 1 and the other two are vanishing: compare the generators G_1, G_2, G_3 so obtained with the operators j_1, j_2, j_3 of Problem 10.7. Show that $[G_{g'}\,,\,G_{g''}] = \sum_{ab}\eta_a^\dagger\,[g'\,,\,g'']_{ab}\,\eta_b$ and make use of the commutation relations of the Pauli matrices:

$$[\tfrac{1}{2}\sigma_a\,,\,\tfrac{1}{2}\sigma_b] = \mathrm{i}\,\epsilon_{abc}\,\tfrac{1}{2}\sigma_c$$

to find the commutation rules of the generators: $[G_a\,,\,G_b] = \mathrm{i}\,\epsilon_{abc}\,G_c$.

Solutions

10.1

a) The Schrödinger equation is

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,y)}{\partial x^2}+\frac{m}{2}\,\omega_1^2\,x^2\,\psi(x,y)-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,y)}{\partial x^2}+\frac{m}{2}\,\omega_2^2\,y^2\,\psi(x,y)$$
$$=\;E\,\psi(x,y)$$

and one is after separate variables solutions $\psi(x,y)=\psi_1(x)\times\psi_2(y)$:

$$\Big[-\frac{\hbar^2}{2m}\psi_1''(x)+\frac{m}{2}\,\omega_1^2\,x^2\,\psi_1(x)\Big]\psi_2(y)$$
$$+\Big[-\frac{\hbar^2}{2m}\psi_2''(y)+\frac{m}{2}\,\omega_2^2\,y^2\,\psi_2(y)\Big]\psi_1(x)=\;E\,\psi_1(x)\times\psi_2(y)\;.$$

Dividing both sides of the equation by $\psi_1(x)\times\psi_2(y)$, one has:

$$\Big(-\frac{\hbar^2}{2m}\frac{\psi_1''(x)}{\psi_1(x)}+\frac{m}{2}\,\omega_1^2\,x^2\Big)+\Big(-\frac{\hbar^2}{2m}\frac{\psi_2''(y)}{\psi_2(y)}+\frac{m}{2}\,\omega_2^2\,y^2\Big)=E\;.$$

The left hand side is the sum of a term depending only on x and a term depending only on y: in order that their sum be a constant, each of them must be a constant:

$$-\frac{\hbar^2}{2m}\frac{\psi_1''(x)}{\psi_1(x)}+\frac{m}{2}\,\omega_1^2\,x^2=E_1\,,\quad -\frac{\hbar^2}{2m}\frac{\psi_2''(y)}{\psi_2(y)}+\frac{m}{2}\,\omega_2^2\,y^2=E_2$$
$$E_1+E_2=E\;.$$

Multiplying the two equations respectively by $\psi_1(x)$ and $\psi_2(x)$ one obtains the eigenvalue equations relative to two independent one-dimensional harmonic oscillators. Therefore:

$$E_1\equiv E_{n_1}=\hbar\,\omega_1\,(n_1+\tfrac{1}{2})\,,\quad E_2\equiv E_{n_2}=\hbar\,\omega_2\,(n_2+\tfrac{1}{2})\quad\Rightarrow$$
$$E\equiv E_{n_1n_2}=\hbar\,\omega_1\,(n_1+\tfrac{1}{2})+\hbar\,\omega_2\,(n_2+\tfrac{1}{2})$$
$$\psi_{n_1n_2}(x,y)=\psi_{n_1}^{(1)}(x)\times\psi_{n_2}^{(2)}(y)$$

where $\psi^{(1)}_{n_1}(x)$ and $\psi^{(2)}_{n_2}(y)$ are the eigenfunctions of the Hamiltonians of one-dimensional oscillators of angular frequencies ω_1 and ω_2 . As both $\psi^{(1)}_{n_1}$ and $\psi^{(2)}_{n_2}$ are a complete set in $L^2(\mathbb{R})$, $\psi_{n_1 n_2}(x,y) = \psi^{(1)}_{n_1}(x) \times \psi^{(2)}_{n_2}(y)$ is a complete set in the space of functions $L^2(x,y)$ and, as a consequence, the eigenvalues $E_{n_1 n_2}$ are all the eigenvalues of H . The eigenvectors of H corresponding to the wavefunctions $\psi^{(1)}_{n_1}(x) \times \psi^{(2)}_{n_2}(y)$ are usually denoted by $|\, n_1,\, n_2\,\rangle$ or also $|\, n_1\,\rangle\,|\, n_2\,\rangle$.

b) Putting $\omega_1 = 3\,\omega$, $\omega_2 = 4\,\omega$, the energy levels write:

$$E_{n_1 n_2} = \hbar\,\omega\,(3n_1 + 4n_2 + \tfrac{7}{2}) = \hbar\,\omega\,(N + \tfrac{7}{2})\,.$$

The first degenerate level is that with $N = 12$: $n_1 = 0,\, n_2 = 3$; $n_1 = 4,\, n_2 = 0$ and the following is that with $N = 15$: $n_1 = 1,\, n_2 = 3$; $n_1 = 5,\, n_2 = 0$.

If the ratio of ω_1 and ω_2 is not rational, all the levels are nondegenerate: it appears that the commensurability of the frequencies is a necessary and sufficient condition for the occurrence of degeneracy, much as in classical mechanics so it is for the closure of trajectories (Lissajous curves).

c) If $\omega_2 = 0$, the particle is a free particle along the y axis and the eigenfunctions of the Hamiltonian are:

$$\psi_{n,k}(x,y) = \psi^{(1)}_n(x)\,\mathrm{e}^{\mathrm{i}\,k\,y}\,.$$

d) The potential is a positive definite quadratic form:

$$V(q_1,q_2) = \frac{1}{2} m\omega^2 \,(\,q_1 \quad q_2\,) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

and it is known (see Problem 1.8) that it can be brought to canonical form by means of a real orthogonal transformation, namely by introducing the normal coordinates:

$$\tilde q_1 = \frac{1}{\sqrt{2}}(q_1 + q_2)\,, \qquad \tilde q_2 = \frac{1}{\sqrt{2}}(q_1 - q_2)$$

and correspondingly:

$$\tilde p_1 = \frac{1}{\sqrt{2}}(p_1 + p_2)\,, \qquad \tilde p_2 = \frac{1}{\sqrt{2}}(p_1 - p_2)\,.$$

One has:

$$H = \frac{\tilde p_1^2}{2m} + \frac{1}{2} m\omega_1^2\, \tilde q_1^2 + \frac{\tilde p_2^2}{2m} + \frac{1}{2} m\omega_2^2\, \tilde q_2^2\,; \qquad \omega_1^2 = \omega^2, \quad \omega_2^2 = 3\,\omega^2$$

and, since the transformation $q \to \tilde q,\ p \to \tilde p$ is canonical, H expressed in terms of the variables $\tilde q,\, \tilde p$ is the Hamiltonian of a two-dimensional oscillator.

10.2

a) The Schrödinger eigenvalue equation:

$$-\frac{\hbar^2}{2m}\Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big)\,\psi_E(x,y) = E\,\psi_E(x,y)\,, \qquad |x| \le a,\ |y| \le a$$

can be solved by the method of separation of variables because both the following conditions are fulfilled: *i)* $H = H_1(q_1,\, p_1) + H_2(q_2,\, p_2)$ and, as a consequence, the Schrödinger equation possesses solutions with the form $\psi_E(x,y) = \psi_{E_1}(x)\,\psi_{E_2}(y)$, $E_1 + E_2 = E$; *ii)* the boundary conditions $\big(\psi_E(x,y) = 0$ on the edges of the square$\big)$ give rise to separate boundary conditions for $\psi_{E_1}(x)$ and $\psi_{E_2}(y)$: $\psi_{E_1}(a) = \psi_{E_1}(-a) = 0$; $\psi_{E_2}(a) = \psi_{E_2}(-a) = 0$ (see Problems 6.9 and 6.10). The equations for $\psi_{E_1}(x)$ and $\psi_{E_2}(y)$ are those for a particle in the segments $|x| \le a$ and $|y| \le a$:

$$-\frac{\hbar^2}{2m}\,\psi''_{E_1}(x) = E_1\,\psi_{E_1}(x)\,, \qquad -\frac{\hbar^2}{2m}\,\psi''_{E_2}(y) = E_2\,\psi_{E_2}(y)$$

that, with the given boundary conditions, have the (nonnormalized) solutions:

$$\psi_{n_1}(x) = \begin{cases} \cos(n_1\pi\, x/2a) & n_1 > 0 \text{ odd} \\ \sin(n_1\pi\, x/2a) & n_1 > 0 \text{ even} \end{cases}$$

$$\psi_{n_2}(y) = \begin{cases} \cos(n_2\pi\, y/2a) & n_2 > 0 \text{ odd} \\ \sin(n_2\pi\, y/2a) & n_2 > 0 \text{ even} \end{cases}$$

and $\psi_{n_1,n_2}(x,y) = \psi_{n_1}(x)\,\psi_{n_2}(y)$ for any pair of positive integers $n_1,\, n_2$ gives rise to a complete set of eigenfunctions of the Hamiltonian H.

b) The energy levels are:

$$E_{n_1,n_2} = E_{n_2} + E_{n_2} = \frac{\hbar^2\pi^2}{8ma^2}\,\big(n_1^2 + n_2^2\big)$$

and, when $n_1 \ne n_2$, they are twice degenerate.

c) As there are degenerate energy levels, there must exist operators that commute with the Hamiltonian H, but do not commute with one another. Indeed H (with the given boundary conditions) exhibits all the symmetries of the square, therefore it commutes with the rotations by an angle that is an integer multiple of $\pi/2$, with the inversions $x \to -x,\ y \to y$ and $x \to x,\ y \to -y$ and therefore also with the exchange $x \leftrightarrow y$, that can be obtained as the product of the rotation by $\pi/2$ and the inversion of the x axis: not all of these transformations, and therefore the operators associated with them, commute with one another (the group of the square is non-Abelian), and this fact guarantees the existence of degenerate levels. All the $\psi_{n_1,n_2}(x,y)$ are simultaneous eigenfunctions of H and of the inversions; if $n_1 = n_2$ they also are eigenfunctions of the operator that exchanges x with y, and therefore of all the other transformations. On the contrary, if $n_1 \ne n_2$, $\psi_{n_1,n_2}(y,x) \ne \psi_{n_1,n_2}(x,y)$ but they have the same energy: in fact $\psi_{n_1,n_2}(y,x) = \psi_{n_2,n_1}(x,y)$.

d) The Schrödinger equation is no longer a separable differential equation: it is still true that, by means of elementary trigonometry formulae, $V_a(x,y)$ can be expressed as the sum of two terms respectively dependent on $x+y$ and $x-y$, but the boundary conditions cannot be expressed in terms of the latter variables.

The potential $V_a(x,y)$ still has all the symmetries of the square: $V_a(x,y) = V_a(-x,y) = V_a(y,x)$, so there exists degenerate levels, even if it is not possible to relate the degeneracy of levels in presence of the potential with that in the absence of $V_a(x,y)$ (for example, it is no longer possible to state that the second and the fourth level have degeneracy 2).

e) The potential $V_b(x,y)$ is no longer invariant under inversions, while still it is so under the exchange $x \leftrightarrow y$ and the rotation by π that commute with each other: as a consequence, it is no longer possible to establish the existence of degenerate levels. In the absence of degeneracy, the eigenstates of H must be eigenstates of the exchange operator $x \leftrightarrow y$ as well, so $\psi_E(x,y) = \pm\psi_E(y,x)$. For any degenerate level that might exist it is still possible to find simultaneous eigenstates of the energy and the exchange operator.

10.3

a) The eigenfunctions of the energy are those of the particle in the square of edge a: $0 \le x \le a,\ 0 \le y \le a$, vanishing at $x = y$ (see Problems 6.9, 6.10) or, equivalently, that are odd under the exchange $x \leftrightarrow y$: the eigenfunctions of the energy for a particle in the square of edge a are:

$$\psi_{n_1,n_2}(x,y) = \alpha\,\sin(n_1\pi\,x/a)\,\sin(n_2\pi\,y/a) + \beta\,\sin(n_2\pi\,x/a)\,\sin(n_1\pi\,y/a)$$

with $n_1 = 1,\,2,\,\cdots,\ \ n_2 = 1,\,2,\,\cdots$ and those that vanish at $x = y$ are:

$$\psi_{n_1,n_2}(x,y) = \sin(n_1\pi\,x/a)\,\sin(n_2\pi\,y/a) - \sin(n_2\pi\,x/a)\,\sin(n_1\pi\,y/a)$$

with $n_1 \ne n_2$. The eigenvalues of the energy are:

$$E_{n_1,n_2} = \frac{\hbar^2\pi^2}{2ma^2}\left(n_1^2 + n_2^2\right), \qquad n_1 \ne n_2\ .$$

b) The Hamiltonian (as well as the boundary conditions) is invariant under the reflection with respect to the straight line $x + y = a$ and, possessing only nondegenerate eigenvalues, its eigenfunctions also are eigenfunctions of this reflection: indeed, if $x \to a - y,\ y \to a - x,\ \ \psi_{n_1,n_2}(x,y) \to (-)^{n_1+n_2}\psi_{n_1,n_2}(x,y)$.

10.4

a) With the boundary condition $\psi_E(x,y,z) = 0$ on the surface of the wave guide, the Schrödinger equation is a separable variables one; taken the Cartesian axes as in the text, the eigenfunctions of the Hamiltonian are:

$$\psi_{n_1,n_2;\,k}(x,y,z) = \sin\Big(n_1\pi\,\frac{x}{a}\Big)\,\sin\Big(n_2\pi\,\frac{y}{a}\Big)\,e^{\,\mathrm{i}\,k\,z}, \qquad n_1,n_2 = 1,\,2,\,\cdots$$

and the eigenvalues:

$$E_{n_1,n_2}(k) = \frac{\hbar^2\pi^2}{2ma^2}\left(n_1^2 + n_2^2\right) + \frac{\hbar^2k^2}{2m} \qquad \big(\,k \in \mathbb{R}\,\big)\ .$$

As k is real, $E > E_{\rm t} = E_{n_1=1,n_2=1}(k=0) = \hbar^2\pi^2/ma^2$; $E_{\rm t}$ is, for particles, the analogue of the cutoff frequency for electromagnetic waves within a wave guide, namely the minimum frequency that can give rise to propagation along the guide.

b) A piece of the guide with unitary volume has length $1/a^2$, therefore, as $|\psi_1|^2$ and $|\psi_2|^2$ do not depend on z,

$$1 = \frac{1}{a^2}\,A^2\int_0^a\!\!\int_0^a \sin^2(2\pi\,x/a)\,\sin^2(\pi\,y/a)\,{\rm d}x\,{\rm d}y = \frac{A^2}{4} \quad\Rightarrow\quad A^2 = 4$$

and likewise $B = 2$.

c) In both cases the components x and y of $\vec{\jmath}$ vanish, whereas:

$$\jmath_{1z} = 4\,\frac{\hbar k_1}{m}\sin^2(2\pi\,x/a)\,\sin^2(\pi\,y/a)\,;\quad \jmath_{2z} = 4\,\frac{\hbar k_2}{m}\sin^2(\pi\,x/a)\,\sin^2(\pi y/a)$$

and obviously in both cases ${\rm div}\,\vec{\jmath} = \partial\jmath_z/\partial z = 0$.

d) The probability current is divergenceless if the state of wavefunction $\psi(x,y,z,\,t)$ is stationary: indeed, from the continuity equation:

$$\begin{aligned}{\rm div}\,\vec{\jmath} &= -\frac{\partial\rho(x,y,z,\,t)}{\partial t} = -\frac{\partial}{\partial t}\,\big|\psi_1\,{\rm e}^{-{\rm i}\,E_1 t/\hbar} + \psi_2\,{\rm e}^{-{\rm i}\,E_2 t/\hbar}\big|^2\\ &= -{\rm i}\frac{E_1-E_2}{\hbar}\,\Big(\psi_1^*\,\psi_2\,{\rm e}^{{\rm i}\,(E_1-E_2)t} - \psi_2^*\,\psi_1\,{\rm e}^{-{\rm i}\,(E_1-E_2)t}\Big)\end{aligned}$$

and, since $\psi_1^*\,\psi_2\,{\rm e}^{{\rm i}\,(E_1-E_2)t} - \psi_2^*\,\psi_1\,{\rm e}^{-{\rm i}\,(E_1-E_2)t}$ is not identically zero, ${\rm div}\,\vec{\jmath} = 0 \Leftrightarrow E_1 = E_2$, namely the state of wavefunction $\psi(x,y,z,\,t)$ must be an eigenstate of the energy. As a consequence $E_{2,1}(k_1) = E_{1,1}(k_2)$, i.e.

$$5\times\frac{\hbar^2\pi^2}{2ma^2} + \frac{\hbar^2 k_1^2}{2m} = 2\times\frac{\hbar^2\pi^2}{2ma^2} + \frac{\hbar^2 k_2^2}{2m} \quad\Rightarrow\quad k_2 = \pm\sqrt{k_1^2 + 3\,\frac{\pi^2}{a^2}}\;\cdot$$

10.5

a) Both the kinetic energy and the potential are invariant under rotations around the z axis, therefore they commute with L_z.

b) The Hamiltonian H commutes also with the operator I_x, the inversion with respect to the plane $x=0$ (as a matter of fact, thanks to the invariance under rotations, H commutes also with the inversion with respect to any plane containing the z axis), but I_x and L_z do not commute, so there must exist degenerate levels. Since $I_x\,L_z\,I_x^{-1} = -L_z$, if one considers the simultaneous eigenstates of H and L_z: $|\,E,m'\,\rangle$, one has $I_x\,|\,E,m'\,\rangle \propto |\,E,-m'\,\rangle$, and, as a consequence, all the energy levels with $m' \neq 0$ are at least twice degenerate. This result holds true whatever the potential, provided it is invariant under rotations around some axis and depends only on the q's: the invariance under reflections follows from these assumptions.

c) In the rotating frame the Hamiltonian is (see Problem 9.10):

$$H = \frac{\vec{p}^{\,2}}{2m} + V(q_1^2 + q_2^2,\, q_3) - \omega\, L_z \equiv H_0 - \omega\, L_z$$

that still commutes with L_z, but does no longer commute with the inversions (in the present case $V - \omega\, L_z$ no longer depends only on the q's), so the existence of degenerate levels cannot be guaranteed: indeed the states $|\, E_0, m' \,\rangle$ and $|\, E_0, -m' \,\rangle$ (eigenstates of H_0 and L_z, therefore of H and L_z) respectively have energies $E_0 \mp m'\, \hbar\, \omega$.

10.6

a) As in Problem 10.1 the eigenvalue equation can be split up into the two equations:

$$H_1 \,|\, E_1 \,\rangle = E_1 \,|\, E_1 \,\rangle, \qquad H_2 \,|\, E_2 \,\rangle = E_2 \,|\, E_2 \,\rangle; \qquad E = E_1 + E_2$$

that are the eigenvalue equations for two one-dimensional independent oscillators. Then:

$$E_1 = \big(n_1 + \tfrac{1}{2}\big)\, \hbar\, \omega\,, \quad E_2 = \big(n_2 + \tfrac{1}{2}\big)\, \hbar\, \omega$$

$$E = E_1 + E_2 \equiv E_{n_1, n_2} = (n_1 + n_2 + 1)\, \hbar\, \omega\ .$$

The degeneracy of the n-th level ($n \equiv n_1 + n_2 = 0,\, 1,\, \cdots$) is the number of ways in which $n_1 + n_2 = n$, namely $n + 1$.

b) In the basis in which H_1 and H_2 are diagonal, the eigenfunctions of H are the product of the eigenfunctions of H_1 and H_2, given in the text of Problem 5.14:

$$|E_{n_1, n_2}\rangle \xrightarrow{\mathrm{SR}} \sqrt{\frac{m\omega/\pi\, \hbar}{2^n\, n_1!\, n_2!}}\, H_{n_1}\big(\sqrt{m\omega/\hbar}\, x\big)\, H_{n_2}\big(\sqrt{m\omega/\hbar}\, y\big)\, \mathrm{e}^{-(m\, \omega/2\, \hbar)\, (x^2 + y^2)}$$

where $H_n(\xi)$ are the Hermite polynomials.

c) The degeneracy found in a) obviously does not disagree with the results of Problem 10.5, but, from the third energy level on ($n \geq 2$), it is greater than that imposed by rotation and reflection invariance. Evidently there must exist further operators that commute with H but do not commute with one another: in Problem 10.7 we shall find these operators and in Problem 10.8 we shall see that their existence and form is determined by the invariance properties of the Hamiltonian.
As $H_k(\xi)$ is a polynomial of degree k, $H_{n_1}\big(\sqrt{m\omega/\hbar}\, x\big) \times H_{n_2}\big(\sqrt{m\omega/\hbar}\, y\big)$ contains $x^{n_1} y^{n_2} = (x^2 + y^2)^{n/2}\, (\cos\phi)^{n_1}\, (\sin\phi)^{n_2}$ and therefore $\mathrm{e}^{\pm \mathrm{i}\, n\, \phi}$, so that $m'_{\mathrm{max}} = n$, $m'_{\mathrm{min}} = -n$. Certainly not all the values of m' ranging from n to $-n$ are possible: they would be $2n + 1 > n + 1$; only those with a definite parity, the same parity of n, are allowed: indeed $H_{n_1} \times H_{n_2}$ has parity $(-1)^{(n_1 + n_2)} = (-1)^n$. Note that the number of integers between n and $-n$ with the same parity as n exactly is $n + 1$, i.e. the degeneracy of the level: actually, in the next problem we will see that the $n + 1$ states of the n-th level are precisely those with $L_3 = n,\, n - 2,\, \cdots,\, -n$.

d) The ground state ($n_1 = n_2 = 0$) is nondegenerate, so it must have $m' = 0$; the nonnormalized wavefunction is

$$|\,E_0 = \hbar\,\omega,\, m' = 0\,\rangle \ \xrightarrow{\text{SR}}\ \mathrm{e}^{-(x^2+y^2)/2a^2}, \qquad a = \sqrt{\hbar/m\omega}$$

that indeed does not depend on the angle ϕ. The first excited level has degeneracy 2, so, thanks to what has been found in c), $m' = \pm 1$; as

$$|\,n_1 = 1,\, n_2 = 0\,\rangle \ \xrightarrow{\text{SR}}\ x\,\mathrm{e}^{-(x^2+y^2)/2a^2}$$

$$|\,n_1 = 0,\, n_2 = 1\,\rangle \ \xrightarrow{\text{SR}}\ y\,\mathrm{e}^{-(x^2+y^2)/2a^2}$$

one has:

$$|\,E_1 = 2\hbar\,\omega,\, m' = \pm 1\,\rangle \ \xrightarrow{\text{SR}}\ (x \pm \mathrm{i}\,y)\,\mathrm{e}^{-(x^2+y^2)/2a^2}.$$

The eigenspace corresponding to the second excited level has dimension 3: thanks to what has been found in c), in it L_3 must have the eigenvalues $m' = \pm 2$ both nondegenerate, otherwise – due to (see Problem 10.5) $I_x\,|\,E, m'\,\rangle = |\,E, -m'\,\rangle$ – the degeneracy of the level would be 4. Then the third eigenvalue must be $m' = 0$. One has (the states are equally normalized):

$$|\,n_1 = 2,\, n_2 = 0\,\rangle \ \xrightarrow{\text{SR}}\ (x^2 - \tfrac{1}{2}a^2)\,\mathrm{e}^{-(x^2+y^2)/2a^2}$$

$$|\,n_1 = 1,\, n_2 = 1\,\rangle \ \xrightarrow{\text{SR}}\ \sqrt{2}\,x\,y\,\mathrm{e}^{-(x^2+y^2)/2a^2}$$

$$|\,n_1 = 0,\, n_2 = 2\,\rangle \ \xrightarrow{\text{SR}}\ (y^2 - \tfrac{1}{2}a^2)\,\mathrm{e}^{-(x^2+y^2)/2a^2}$$

and, since the states with $m' = \pm 2$ have wavefunctions proportional to $(x \pm \mathrm{i}\,y)^2 = x^2 - y^2 \pm 2\mathrm{i}\,x\,y$ (see Problem 8.6), one has:

$$|\,E_2 = 3\hbar\,\omega,\, m' = \pm 2\,\rangle \ \xrightarrow{\text{SR}}\ (x \pm \mathrm{i}\,y)^2\,\mathrm{e}^{-(x^2+y^2)/2a^2}$$

namely (we shall omit $n_1 =,\ n_2 =$)

$$|\,E_2,\, m' = \pm 2\,\rangle = |\,2,\,0\,\rangle - |\,0,\,2\,\rangle \pm \sqrt{2}\,\mathrm{i}\,|\,1,\,1\,\rangle$$

and therefore, by orthogonality:

$$|\,E_2,\, m' = 0\,\rangle = |\,2,\,0\,\rangle + |\,0,\,2\,\rangle \ \xrightarrow{\text{SR}}\ (x^2 + y^2 - a^2)\,\mathrm{e}^{-(x^2+y^2)/2a^2}.$$

10.7

a) The operators $\eta_{1,2}^\dagger$, $\eta_{1,2}$ respectively are the "raising and lowering operators" for the independent one-dimensional oscillators 1 and 2. Therefore:

$$H = \hbar\,\omega\big(\eta_1^\dagger\,\eta_1 + \eta_2^\dagger\,\eta_2 + 1\big); \qquad [\eta_a\,,\,\eta_b\,] = 0\,, \qquad [\eta_a\,,\,\eta_b^\dagger\,] = \delta_{a\,b}\,.$$

b) By direct use of the commutation rules one can verify that the operators $\eta_a^\dagger\,\eta_b$ commute with the Hamiltonian H. Otherwise observe that (notation as in Problem 10.6) $\eta_1^\dagger\,\eta_2\,|\,n_1,\,n_2\,\rangle \propto |\,n_1\!+\!1,\,n_2\!-\!1\,\rangle$ and that $E_{n_1,n_2} = E_{n_1+1,n_2-1}$.

c) $[j_1\,,\,j_2] = -\dfrac{1}{4\,\mathrm{i}}\,[\eta_1^\dagger\,\eta_2\,,\,\eta_2^\dagger\,\eta_1] + \dfrac{1}{4\,\mathrm{i}}\,[\eta_2^\dagger\,\eta_1\,,\,\eta_1^\dagger\,\eta_2] = \dfrac{1}{2\mathrm{i}}\,[\eta_2^\dagger\,\eta_1\,,\,\eta_1^\dagger\,\eta_2]$

$$= \frac{1}{2\mathrm{i}}\Big(\eta_2^\dagger\,[\eta_1\,,\,\eta_1^\dagger\,\eta_2] + [\eta_2^\dagger\,,\,\eta_1^\dagger\,\eta_2]\,\eta_1\Big) = \frac{1}{2\mathrm{i}}\big(\eta_2^\dagger\eta_2 - \eta_1^\dagger\eta_1\big) = \mathrm{i}\,j_3$$

and likewise for the other commutators. In conclusion $[j_a\,,\,j_b] = \mathrm{i}\,\epsilon_{abc} j_c$. With the notation of Problem 10.6 one has:

$$j_2 = \frac{1}{2\,\hbar}\,L_3 = \frac{1}{2\,\hbar}\,(q_1 p_2 - q_2 p_1)\,, \qquad j_3 = \frac{1}{2\,\hbar\,\omega}\,(H_1 - H_2)\,.$$

Since (see Problem 10.6) L_3 has both even and odd eigenvalues, j_2 (and therefore any j_a) has both integer and half-integer eigenvalues: the occurrence of both kinds of eigenvalues is not forbidden as it is instead, for a given system, in the case of the angular momentum.

d) From the theory of angular momentum we know that $\boldsymbol{j}^2$ has the eigenvalues $j(j+1)$ with $j = \frac{1}{2}, 1, \frac{3}{2}, \cdots$ therefore, thanks to the identity given in the text, H has the eigenvalues $(2j+1)\hbar\,\omega = (n+1)\hbar\,\omega$ with $n \equiv 2j = 0, 1, 2, \cdots$. Furthermore, always from the theory of angular momentum, one has that the number of independent states with a given j is $2j+1 = n+1$ and the eigenvalues taken by $L_3/\hbar = 2j_2$ are those between $-2j$ and $+2j$, namely the integers between $-n$ and $+n$ with the same parity as n.

The classification of the states by means of $\boldsymbol{j}^2$ and j_2 is the same as the classifications in terms of n and m (the eigenvalue of L_3), whereas the classification by means of $\boldsymbol{j}^2$ and j_3 is the same as the classifications in terms of $n = n_1 + n_2$ and $n_1 - n_2$, namely in terms of n_1, n_2.

10.8

a) By η with no index we shall denote the pair (η_1, η_2), so $H = \hbar\,\omega\,(\eta^\dagger\eta + 1)$. One has:

$$\tilde\eta_a^\dagger = \sum\nolimits_b u_{ab}^*\,\eta_b^\dagger = \sum\nolimits_b \eta_b^\dagger\,u_{ba}^\dagger\,; \qquad \tilde\eta = u\,\eta\,, \quad \tilde\eta^\dagger = \eta^\dagger\,u^\dagger$$

$$\tilde\eta^\dagger\,\tilde\eta = \eta^\dagger\,u^\dagger\,u\,\eta = \eta^\dagger\,\eta \quad \Leftrightarrow \quad u^\dagger\,u = \mathbb{1}\,.$$

$$[\tilde\eta_a\,,\,\tilde\eta_b^\dagger\,] = \sum\nolimits_{cd} u_{ac}\,[\eta_c\,,\,\eta_d^\dagger\,]\,u_{db}^\dagger = \sum\nolimits_c u_{ac}\,u_{cb}^\dagger = \delta_{ab}\,.$$

b) Any matrix of the given form obviously is unitary and its determinant equals 1. Conversely:

$$\begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix}\begin{pmatrix} z_1^* & z_3^* \\ z_2^* & z_4^* \end{pmatrix} = \mathbb{1} \quad \Leftrightarrow$$

$$\begin{cases} |z_1|^2 + |z_2|^2 = 1 & \\ z_1\,z_3^* = -z_2 z_4^* & \Rightarrow \quad z_3 = -\alpha\,z_2^*, \quad z_4 = \alpha\,z_1^* \\ |z_3|^2 + |z_4|^2 = 1 & \Rightarrow \quad |\alpha| = 1 \end{cases}$$

and, if the determinant must equal 1, then $\alpha = 1$. Unitary 2×2 matrices form a group – the U(2) group – and those with determinant equal to 1 form a subgroup, the group $SU(2)$ (special unitary group).

c) The condition that, to the first order in ϵ, the matrix $\mathbb{1}+\mathrm{i}\,\epsilon\,g$ be unitary is

$$(\mathbb{1}+\mathrm{i}\,\epsilon\,g)(\mathbb{1}-\mathrm{i}\,\epsilon\,g^\dagger)=\mathbb{1}+O(\epsilon^2)\quad\Rightarrow\quad g=g^\dagger$$
$$\det(\mathbb{1}+\mathrm{i}\,\epsilon\,g)=1+\mathrm{i}\,\epsilon\,(g_{11}+g_{22})+O(\epsilon^2)\quad\Rightarrow\quad \mathrm{Tr}\;g=0\,.$$

d) To the first order in ϵ:

$$U(g,\,\epsilon)\,\eta_a\,U^{-1}(g,\,\epsilon)\approx(\mathbb{1}-\mathrm{i}\,\epsilon\,G_g)\,\eta_a\,(\mathbb{1}+\mathrm{i}\,\epsilon\,G_g)\approx\eta_a-\mathrm{i}\,\epsilon\,[G_g\,,\,\eta_a]$$

and by comparison with (2) in the text, it follows that $[G_g\,,\,\eta]=-g\,\eta$. Therefore, by analogy with $[\eta^\dagger\eta,\,\eta]=-\eta$, one has:

$$G_g=\eta^\dagger g\,\eta\,:\quad \sum\nolimits_{ab}[\eta_a^\dagger\,g_{ab}\,\eta_b\,,\,\eta_c]=\sum\nolimits_{ab}[\eta_a^\dagger\,,\,\eta_c]\;g_{ab}\,\eta_b=-\sum\nolimits_b g_{cb}\,\eta_b\,.$$

Since the operators G_g are generators of invariance transformations for the Hamiltonian H: $U(u)\,H\,U^{-1}(u)=H$, they commute with H itself.

e) Any 2×2 matrix can be written as a linear combination of the three Pauli matrices and of the identity:

$$\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}=\frac{1}{2}\,(\alpha+\delta)\,\mathbb{1}+\frac{1}{2}\,(\beta+\gamma)\,\sigma_1+\frac{\mathrm{i}}{2}\,(\beta-\gamma)\,\sigma_2+\frac{1}{2}\,(\alpha-\delta)\,\sigma_3\,.$$

The Hermitian matrices are linear combinations with *real* coefficients (the Pauli matrices are Hermitian), and those with vanishing trace ($\alpha+\delta=0$) are combinations only of the three Pauli matrices.

f) If $a_1=1$, $a_2=a_3=0$ one has $g=\frac{1}{2}\,\sigma_1$; therefore:

$$G_1=\frac{1}{2}\,\eta^\dagger\,\sigma_1\,\eta=\frac{1}{2}\,(\eta_1^\dagger\,\eta_2+\eta_2^\dagger\,\eta_1)\equiv j_1$$

and likewise:

$$G_2=\frac{1}{2}\,\eta^\dagger\,\sigma_2\,\eta=\frac{1}{2\mathrm{i}}\,(\eta_1^\dagger\,\eta_2-\eta_2^\dagger\,\eta_1)\equiv j_2\,,$$
$$G_3=\frac{1}{2}\,\eta^\dagger\,\sigma_3\,\eta=\frac{1}{2}\,(\eta_1^\dagger\,\eta_1-\eta_2^\dagger\,\eta_2)\equiv j_3\,.$$

$$\begin{aligned}[G_{g'}\,,\,G_{g''}\,]&=\sum_{abcd}[\,\eta_a^\dagger\,g'_{ab}\,\eta_b\,,\,\eta_c^\dagger\,g''_{cd}\,\eta_d\,]\\ &=\sum_{abcd}\Big(\eta_a^\dagger\,g'_{ab}\,[\eta_b\,,\,\eta_c^\dagger]\,g''_{cd}\,\eta_d+\eta_c^\dagger\,g''_{cd}\,[\eta_a^\dagger\,,\,\eta_d]\,g'_{ab}\,\eta_b\Big)\\ &=\sum_{ab}\eta_a^\dagger\,[g'\,,\,g''\,]_{ab}\,\eta_b\end{aligned}$$

therefore (and likewise for the others):

$$[G_1\,,\,G_2\,]=\eta^\dagger\,[\tfrac{1}{2}\sigma_1\,,\,\tfrac{1}{2}\sigma_2]\,\eta=\frac{\mathrm{i}}{2}\,\eta^\dagger\,\sigma_3\,\eta=\mathrm{i}\,G_3$$

then, thanks to the identity of the G_a's with the j_a's, we have established again the commutation rules between the j_a's found in Problem 10.7.