Chapter 1
Introduction

An odd candidate for controversy in mathematics education is the Mean Value Theorem. It is easily, if imprecisely, described to the man in the street: Draw a smooth curve and choose two points A and B on the curve. Somewhere on the arc of the curve between these two points is another point C at which the line tangent to the curve will be parallel to the line connecting A and B. Figures 1.1 and 1.2 illustrate this nicely.

With such pictures in mind, the method of finding such a point C and the reason for the truth of the Theorem suggest themselves. C is the point (or any of the points) on the curve farthest from the line AB as possible and is found by sliding the line AB upward (in Fig. 1.1) or leftward (in Fig. 1.2) or in whatever appropriate direction — without rotating AB at all — until it exits the curve. The last point (or any of the last points if there are more than one) at which AB had contact with the curve is this farthest point C. Why is the tangent at C parallel to AB? Referring to Fig. 1.3, we see that as we move along the curve from A to C, the tangents are always steeper than AB but they are becoming less steep. And as we move from C to B, the tangents continue to grow less steep and none are as steep as AB. The level of steepness, or *slope*, of the tangent at C must lie between these two trends and thus bears the same level of steepness as AB itself, i.e., the tangent is parallel to AB.

The argument given is not mathematically rigorous, but it can be made so first by offering precise definitions of the terms involved ("curve", "tangent") and isolating the conditions under which the Theorem is to hold, and then appealing to more advanced theory to justify the individual steps. This will be done in the course of this book. Here let me just indicate that some conditions must be met. The hyperbola of Fig. 1.4 demonstrates an obvious condition: the Mean Value Theorem presupposes A and B to lie on the same branch of the curve in question. If they don't, there are no points lying on the arc of the curve connecting them simply because there is no such arc. One could try to find a point maximally distant from AB elsewhere on the curve, but again no such point exists. Indeed, on the hyperbola, if A and B lie on different branches, there is no point C at which the tangent is parallel to AB. There

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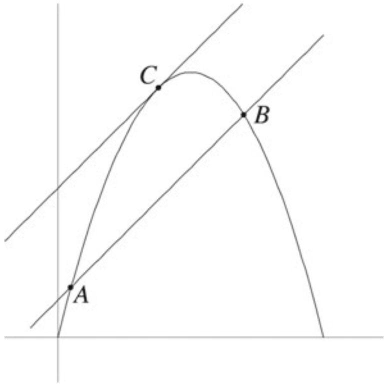


Fig. 1.1 Mean Value Theorem; Example 1

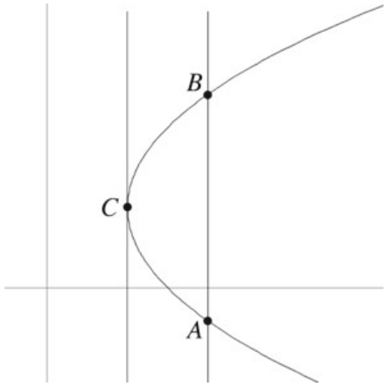


Fig. 1.2 Mean Value Theorem; Example 2

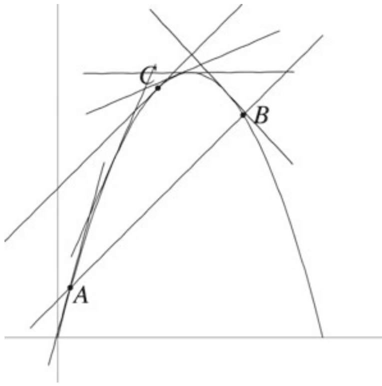


Fig. 1.3 Changing slopes

is a simple reason for this: the slope of AB lies strictly between the slopes of the two asymptotes, while the slopes of the tangent lines are all strictly outside this range.

Also, the Theorem can fail if the curve fails to have tangents everywhere, as in Fig. 1.5. Here, depending on how one defines "tangent", the curve possesses infinitely

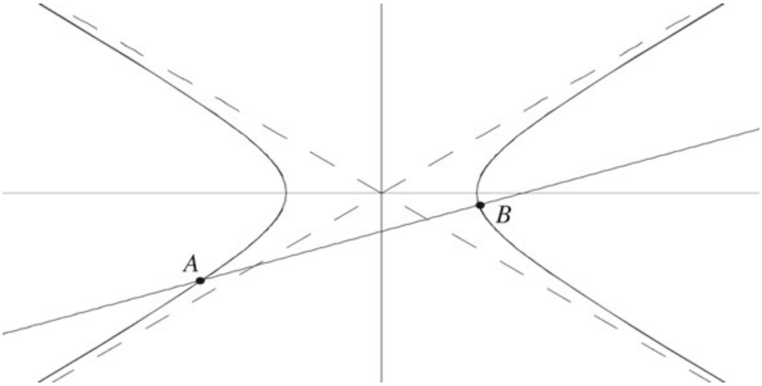
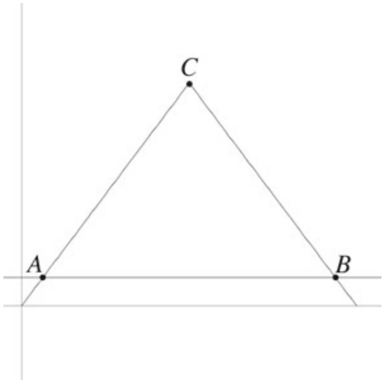


Fig. 1.4 Hyperbola

Fig. 1.5 Corner



many, two, or no tangents at the point C farthest from the line AB. The grammar school definition of a tangent to a curve being a line meeting the curve at a single point without crossing the curve allows for infinitely many tangents through C, one of which is indeed parallel to AB. The kinematic definition of the tangent at C as being the line a particle travelling along the curve would follow if at C, instead of following the curve, the particle would continue in the direction it was going when it arrived at C, yields two tangents — the lines extending past C the segments AC and BC, respectively. And the definition of tangent used in the Calculus as the line passing through C with slope given by the derivative of the curve at that point yields no tangent because the curve is not differentiable at the point C.

The curve of Fig. 1.6 has a cusp rather than a corner. The Mean Value Theorem fails for the same reason. C has infinitely many tangents according to the grammar school definition, one according to the kinematic definition (namely, the vertical line), and none according to the definition used in the Calculus.

These counterexamples dictate the conditions a curve must satisfy for the Mean Value Theorem to apply. It must be "connected" or "continuous", i.e., it must consist

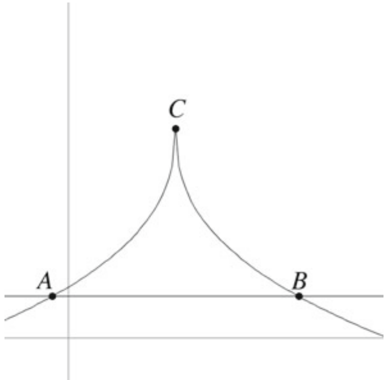


Fig. 1.6 Cusp

of a single branch. And it must be "smooth", i.e., it must have no sudden changes of direction as given by the corner and cusp of the last two figures.

The sequel divides into three chapters. The first (Chap. 2) provides precise formalisations of the concepts of curve, tangent, continuity, and smoothness, and finishes with a rigorous proof of the Mean Value Theorem as geometrically described above. It is a leisurely presentation that seems to take its time getting to the point. However, much of it is referred to later in the book, occasionally explicitly and occasionally implicitly as absorbed background for the discussion of issues that are later raised. I beg the reader's indulgence in what may at first appear excessive background information.

The Chap. 3 is devoted to the Mean Value Theorem itself, in its more familiar analytic form. This includes the Theorem and related results, some applications, and the history of attempts to prove it. In this I stick to the Theorem as it is usually presented in a single-variable Calculus course and do not attempt to discuss generalisations except where such directly impinge on the discussion of the standard version of the Theorem.

Finally, the last chapter discusses the educational question: Is the Mean Value Theorem itself or one of several proposed replacements more appropriate for the introductory course in the Calculus? I present and criticise some of the points that have been raised in the debate, but offer no final solution. My purpose is not to dictate the contents of the Calculus course, but to provide enough information for a more informed discussion than has thus far taken place.

Ostensibly the issues of the present book will only interest those members of departmental curriculum committees deciding on what must be covered in the basic Calculus course, or, more broadly, those who teach Calculus. I think, however, the issues might be of broader interest, say to the student who is taking such a course. And I suppose such a book could have been written for him or her, particularly as the topic comes from such a course. The temptations, however, to pepper the exposition with examples assuming some computational facility and to approach the subject from a higher point of view have played havoc with this good intention. The student who attempts to read the present work while first learning the Calculus might find some of the material a bit heavy going: the present book discusses the *theory* of the Calculus, not the Calculus *per se*, and although the main prerequisite is the sort of mathematical maturity that can be acquired by a reasonably, but not overly, rigorous course in the Calculus, it does presuppose some knowledge of the Calculus as well.

In this book I include many citations from the literature, so many that the book might be viewed as an absurdly heavily annotated source book. I have included excerpts from original mathematics, as well as comments from various historians and other expositors. In doing this I have tried to be as true to the originals as possible, short of including facsimile reproductions of the pages. Where I have taken liberties is mostly footnoted. Exceptions are that: lacking the old "long s", I have used only the standard "s" in reproducing older material; I omit references to the literature of the form "[ABC]" or "[1]" as these are not relevant; and, in place of indentation, paragraphs in quoted material are separated by extra space between them. I am fairly confident in the overall correctness of those passages I've translated. Occasionally I have used an English idiom in place of the literal translation of idioms, and I have been a bit loose when it comes to the tenses of verbs.

The decision to include the mathematical sources needs no justification. Citing historians and other expositors I justify on several counts. Aside from acknowledging the sources of some of my information, I do this in part to introduce whatever students may be reading this book to some of the experts. In part I do this when I have not checked the primary sources myself because I don't read the language in question (Sanskrit, Latin), the original was unavailable to me, or the point was too tangential to check myself. Sometimes I find the historians or expositors to express matters so well it would be impossible to improve upon them and I feel I must quote them; and sometimes I cannot resist quoting someone so that I can spout my disagreement.

There is one by-product of quoting original sources; the notation shifts from author to author. When discussing a particular paper, after quoting a passage, I may stick to the author's notation if it is clear enough, or I may translate the material into more (currently) standard notation. My hope is that this will not confuse the reader.