The Problem of the Optimal Packing of the Equal Circles for Special Non-Euclidean Metric

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Abstract. The optimal packing problem of equal circles (2-D spheres) in a bounded set P in a two-dimensional metric space is considered. The sphere packing problem is to find an arrangement in which the spheres fill as large proportion of the space as possible. In the case where the space is Euclidean this problem is well known, but the case of non-Euclidean metrics is studied much worse. However there are some applied problems, which lead us to use other special non-Euclidean metrics. For instance such statements appear in the logistics when we need to locate a given number of commercial facilities and to maximize the overall service area. Notice, that we consider the optimal packing problem in the case, where P is a multiply-connected domain. The special algorithm based on optical-geometric approach is suggested and implemented. The results of numerical experiment are presented and discussed.

Keywords: Optimal packing problem · Equal circles · Non-Euclidean space · Multiply-connected domain · Numerical algorithm · Computational experiment

Introduction

The optimal circle packing problem [1] is one of the classical problems of combinatorial geometry. It is of interest both from a theoretical point of view and in connection with a wide variety of applications.

The circle packing problem has a long history, for example, one of the famous statements relating to the packing of spheres in three-dimensional Euclidean space is called "Kepler conjecture" and was formulated more than 400 years ago. It says that no arrangement of equally sized spheres filling space has a greater average density than that of the cubic close packing (face-centered cubic) and hexagonal close packing arrangements. An introduction to its history can be found in [2].

D.I. Ignatov et al. (Eds.): AIST 2016, CCIS 661, pp. 58–68, 2017.
DOI: 10.1007/978-3-319-52920-2_6

In the literature there are a lot of formulations of the problem: from the different variants of the knapsack problem [3] to considering it in the abstract n-dimensional space [4].

Note that nearly always (except for some special cases) the packing problem is NP-complete. So the problem of constructing efficient numerical algorithms is extremely urgent [5].

Apparently, the most popular problems is the problem of the 2-D optimal circle packing of equal radius to a closed set with smooth or piecewise smooth boundary (circles, squares, rectangles, etc.). For example, in the papers [2,6,7] authors deal with the problem to maximize the radius associated with the n circles when the container is the unit square. The number of circles is from 1 to 200. In the case where the number of packing elements is small (up to 36, inclusive), the problem is solved analytically. In other words, it is proved that the constructed packing is the best.

In papers [8–10] the problem of packing identical circles of unit radius in the circle is considered. The results for number of packing elements up to 81 are obtained. Birgin and Gentil [11] consider the problem of packing identical circles of unit radius in a variety of containers (circles, squares, rectangles, equilateral triangles and strips of fixed height) to minimize the size of the latter.

The linear model for the approximate solution of the problem of packing of the maximum number of identical circles to the closed bounded set is suggested in [12]. The problem of packing of various circles of a given radius in order to maximize the number (or weight), or to minimize the waste is considered in [13,14].

Lopez and Beasley present a heuristic algorithm based on the formulation space search method to solve the packing problem for equal [15] and unequal [16] circles. Finally, a substantial and original class of packing problems where circles may be placed either inside or outside other circles, the whole set being packed in a rectangle is considered in [17].

Completing the survey part of the article, we note that the authors, who deal with the packing problem, were not limited by the case when elements are circles. Thus, in [18] authors consider the packing of rectangles (with the possibility of rotation) into triangles.

The survey of publications could be continued because there are hundreds of notable publications. The vast majority of books and articles devoted to the study of the problem of packing in Euclidean space. This is not accidental because such is the most natural formulation. However, sometimes the problems arise in applications, where in order to define the distance between two points it is necessary to use another metrics. For instance such statements appear in the logistics when we need to locate a given number of commercial facilities and to maximize the overall service area [19].

Coxeter [20] and Boroczky [21] deal with congruent circles packing problem for multidimensional spaces of constant curvature (elliptic and hyperbolic) and assess the maximum packing density. Besides above, this problem was studied in a series of papers by Szirmai. In [22,23] he presents a method that determines the data and the density of the optimal ball and horoball packings Coxeter tiling (Coxeter honeycomb) in the hyperbolic 3-, 4- and 5-spaces and based on the projective interpretation of the hyperbolic geometry. The goal of [24] is to extend the problem of finding the densest geodesic ball (or sphere) packing for the other 3-dimensional homogeneous geometries (Thurston geometries).

In this paper we consider the circle packing problem in a bounded set with piecewise smooth boundary in a special metric, which, generally speaking, is not an Euclidean except one particular case. The container is not required to be convex, and even simply connected. We present a numerical algorithm for solving this problem and perform computational experiment, the results of which show the effectiveness of the suggested approach.

1 Formulation

Let X is a metric space, C_i, $i = 1, ..., n$ are congruent circles with centers in $s_i = (x_i, y_i)$, P is closed multiply-connected set.

$$P = \mathrm{cl}\left(D \setminus \bigcup_{k=1}^{m} B_k\right) \subset X \subseteq \mathbb{R}^2.$$

Here $D \subset X$ is the bounded set, $B_k \subset D, k = 1, ..., m$ are compact sets with non-empty interior.

It is necessary to find vector $s = (s_1, ..., s_n) \in \mathbb{R}^{2n}$, which provides the packing of the given number of circles with maximum radius R in P.

The distance between the points of the space X is determined as follows:

$$\rho(a, b) = \min_{G \in G(a,b)} \int\limits_G \frac{dG}{f(x, y)}, \tag{1}$$

where $G(a, b)$ is the set of all continuous curves, which belong X and connect the points a and b, $0 < \alpha \leq f(x, y) \leq \beta$ is continuous function defined instantaneous speed of movement at every point of P. In other words, the shortest route between two points is a curve, that requires to spend the least time.

It is easy to make sure that all the metric axioms are satisfied. In the particular case when $f(x, y) \equiv 1$ we have a Euclidean metric in the two-dimensional space and the shortest route is a straight line.

Thus, we formulate the following problem:

$$R \rightarrow \max \tag{2}$$

$$\rho(s_i, s_j) \geq 2R, \ \forall i = \overline{1, n-1}, \forall j = \overline{i+1, n} \tag{3}$$

$$\rho(s_i, \partial P) \geq R, \ \forall i = \overline{1, n} \tag{4}$$

$$s_i \in P, \ \forall i = \overline{1, n} \tag{5}$$

Here ∂P is the boundary of the set P, $\rho(s_i, \partial P)$ is the distance from a point to a closed set.

The objective, Eq. (2), maximizes the radius associated with the circles. Equation (3) ensure that no circles overlap each other. Equations (4)–(5) are the constraints which ensure that every circle is fully inside the container.

For any vector s, satisfying the conditions (3)–(5) define the sets

$$P_i = \{p \in P : \rho(p, s_i) \leq \rho(p, s_j), \forall j = 1, ..., n, i \neq j\}\,. \tag{6}$$

In the literature, such sets are called Dirichlet cells [25] for points s_i on the set P. It's obvious that $P = \bigcup_{i=1}^{n} P_i$.

The solution of the problem above reduces to the solution of the following sequence of subproblems:

1. For every set P_i find the point $\bar{s}_i \in P_i$ that $\rho(\bar{s}_i, \partial P_i) = \max_{p \in P_i} \rho(p, \partial P_i)$.
2. Find the guaranteed value of the radius satisfying the constraints (3)–(5): $R = \min_{i=1,...,n} \rho(\bar{s}_i, \partial P_i)$.
3. For the new vector $\bar{s} = (\bar{s}_1, ..., \bar{s}_n)$ redefine sets P_i according to formula (6).

The steps 1–3 are carried out while the coordinates of $\bar{s}$ are changed.

2 Solution Method

To solve the described subproblems authors suggest methods based on the physical principles of Fermat and Huygens, which are used in geometric optics. The first principle says that the light in its movement chooses the route that requires to spend a minimum of time. The second one states that each point reached by the light wave, becomes a secondary light source.

Thus, in order to solve of the first subproblems we should carried out the construction of the light wave front, started from the border ∂P_i for each P_i to the time when the front degenerate into a point. It's coordinates are the required solution $\bar{s}_i, i = 1, ..., n$. To solve the third subproblems it's required to simultaneously initiate the light waves from the points $\bar{s}_i$, $i = 1, ..., n$, and to find such points of P, which are simultaneously reached by two or more waves. We presented two algorithms in [26,27].

Let's go back to the first subproblem and suggest an algorithm for it.

Algorithm "BorderWaveInside-BWI"

1. Boundary of the considered set is approximated by the closed polygonal line with nodes at the points $A_i, i = \overline{0, m}$. A_i are called the initial points.
2. For each pair of points A_i, A_{i+1} we construct line segment $A_i A_{i+1}$. Then we construct line segments $A_i B_i'$ and $A_{i+1} B_i''$ which are perpendicular to $A_i A_{i+1}$. The length of the line segments are $f(A_i)\Delta t$ and $f(A_{i+1})\Delta t$ respectively. Let $\mathbf{B}$ is a set of all these line segments. It is easy to see that the amount of new points is twice more than initial one.
3. If there is a pair of line segments $VW \in \mathbf{B}$ and $YZ \in \mathbf{B}$ that $W = Z$, then all initial points between V and Y are eliminated.

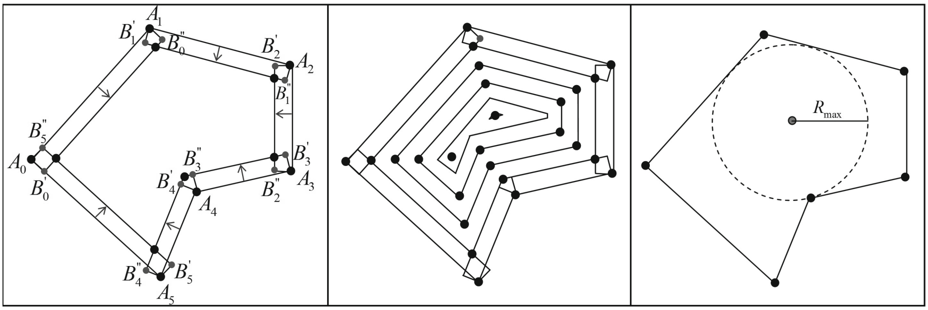

Fig. 1. Several iterations of BWI-algorithm

4. We construct straight lines $B_i'B_i''$, $i = \overline{0, m-1}$, by the points B_i' and B_i''.
5. The points of intersection of $B_i'B_i''$ and $B_{i+1}'B_{i+1}''$, $i = \overline{0, m-2}$, form a set of the new front points.
6. If there is a pair of crossing line segments $VW \in \mathbf{B}$ and $YZ \in \mathbf{B}$, then all initial points between V and Y are eliminated. The point of intersection becomes a point of the new front.
7. If the constructed front is nonclosed line, then the solution is the "middle" of the line, namely the point the distance from which to the ends of line is the same. If the constructed front consists of one point, then this point is the solution. Otherwise, built front is taken as the initial and Go to Step No 1.

Note that after finding of the initial wave front the outer part of considered set becomes to be impassable. So, the perpendicular is directed to inside part.

The steps 3 and 6 provide the correct construction of the front when the "dovetail" problem arises.

Figure 1 illustrates BWI-algorithm. Left part shows the process of the first front constructing, in the middle there is a moment of the front splitting, on the right there is a packed circle of maximum radius.

In the case when the set P_i is not simply connected, in other words, it contains impenetrable for the light wave barriers, in order to solve the first subproblem we need an additional algorithm. The algorithm allow to construct light wave fronts, propagating from the boundary of barrier in the outer area (*BorderWaveOutside-BWO*). This algorithm differs from BWI only by the perpendicular direction.

Thus the algorithm for multiply connected set is follow.

Algorithm (BorderWaveInside Multiply-connected set-BWI-MCS)

1. By the algorithm BWI we construct the fronts of the light wave, which is started from the border of the considered set. By the algorithm BWO we construct the fronts of the light waves, which are started from barriers' borders. The algorithms work until the first contact BWI-wave with one of BWO-waves. Constructed fronts are saved.
2. The set of points, which are not reached neither of the waves, is divided into the maximal simply connected subsets S_j (segments), which are saved in the list of segments S. Segments obtained in the previous iteration are removed.

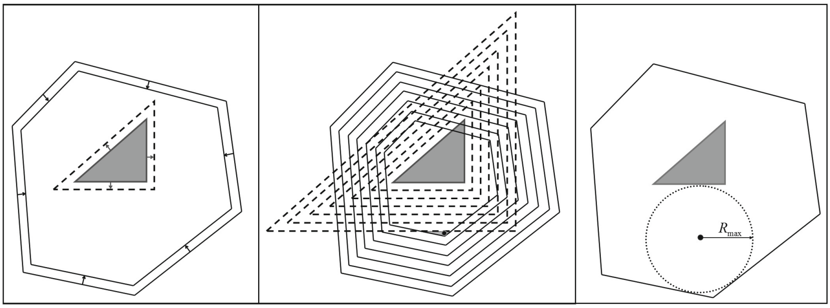

Fig. 2. Several iterations of *BWI-MCS*-algorithm

3. One iteration of algorithms *BWI* and *BWO*, since the stored fronts, is performed. The process 2–3 continues while the set of points, which are not reached by any of the waves, is not empty.
4. All elements of the list of segments are analyzed: If the segment contains a single point, it is a potential solution. If the segment is open curve, its "middle" is a potential solution. In other cases, we find a potential solution by using an algorithm *BWI*, because the segment now is simply connected set.
   All received potential solutions q_j are added to the list Q.
5. The list of potential solutions Q is analyzed: If it contains one point, then this point is the desired solution. Otherwise, for each point $q \in Q$ by using the algorithm proposed in [28], the value of $r_j = \rho(q_j, \partial X_i)$ is calculated, where ∂X_i are sets the boundaries of the P_i. The desired solution is a point $q^* \in Q$, for which $r^* = \max_j r_j$.

Figure 2 illustrates *BWI-MCS*-algorithm. Left part shows the process of the first front constructing, in the middle there is the last iteration, on the right there is a packed circle of maximum radius. The solid line shows the wave front started from the boundary of set. The dashed line shows the wave front initiated from the border of barrier.

Now we are able to present the general algorithm for the problem (2)–(5).

Algorithm of Equal Circles Packing – AECP-MCS

1. Randomly generate an initial solution $s = (s_1, ..., s_n)$, which satisfies the constraint (5). The radius R is assumed to be zero.
2. The set P is divided into subsets P_i, $i = 1, ..., n$, according to the definition (6) by the authors' algorithm proposed in [28].
3. For each P_i, $i = 1, ..., n$, we solve the subproblem 1 by *BWI-MCS* algorithm. As a result, for each P_i, $i = 1, ..., n$, we find the coordinates of the packed circle center $\bar{s}_i$ and its maximum radius r_i.
4. Calculate $R = \min_{i=1,...,n} r_i$.
   Steps 2–4 are repeated until the R increases, then the current vector $\bar{s}$ is saved as an approximation to a global maximum of the problem.

5. The counter of an initial solution generations *Iter* is incremented. If *Iter* becomes equal some preassigned value, then the algorithm is terminated. Otherwise, go to step 1.

3 Numerical Experiment

Example 1. This example illustrates how the given in the previous section algorithms work in the case of the Euclidean metric $f(x, y) \equiv 1$. We solve the equal circle packing problem in unit square. The number of circles is given and we maximize the radius. The results are presented in Table 1. Note, that the Best of known results were obtained from [29].

Considering Table 1 it is clear that our AECP-MCS algorithm produces low percentage deviations (less then 0.1%). In the case when $n \geq 50$ the deviations are retained, but the calculation time is significantly increased. So we can say that AECP-MCS algorithm allows to solve the equal circle packing problem for Euclidian metric, but it is not highly effective. It's advantages will be shown in next examples.

Table 1. AECP-MCS results for Euclidean metric

n	Best of known		AECP-MCS		Deviation	
	Radius (R)	Density (d)	Radius (R)	Density (d)	ΔR	Δd
1	0,50000000	0,78539816	0,50000000	0,78539816	0,00000000	0,00000000
2	0,29289322	0,53901208	0,29289140	0,53900539	0,00000182	0,00000669
3	0,25433310	0,60964481	0,25433090	0,60963429	0,00000220	0,00001052
4	0,25000000	0,78539816	0,25000000	0,78539816	0,00000000	0,00000000
5	0,20710678	0,67376511	0,20710390	0,67374635	0,00000288	0,00001875
6	0,18768060	0,66395691	0,18767851	0,66394208	0,00000210	0,00001483
7	0,17445763	0,66931083	0,17445600	0,66929832	0,00000163	0,00001251
8	0,17054069	0,73096383	0,17053746	0,73093617	0,00000323	0,00002766
9	0,16666667	0,78539816	0,16662136	0,78497118	0,00004531	0,00042698
10	0,14820432	0,69003579	0,14819925	0,68998856	0,00000507	0,00004723
11	0,14239924	0,70074158	0,14239800	0,70072940	0,00000124	0,00001218
12	0,13995884	0,73846822	0,13992800	0,73814277	0,00003084	0,00032545
15	0,12716655	0,76205601	0,12694119	0,75935742	0,00022536	0,00269859
16	0,12500000	0,78539816	0,12500000	0,78539816	0,00000000	0,00000000
2000	0,01172594	0,86392312	0,01151634	0,83331402	0,00020960	0,03060910
3000	0,00967451	0,88212297	0,009243172	0,80521747	0,00043134	0,07690550

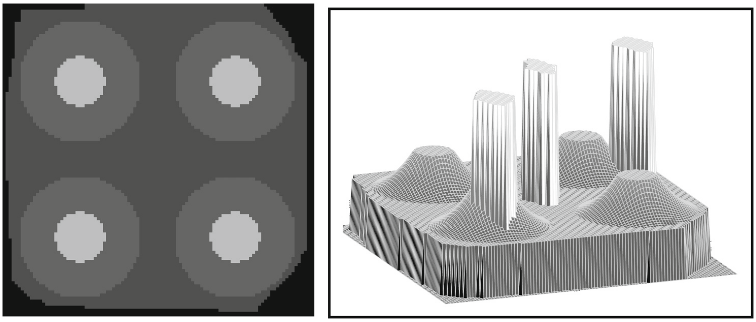

Fig. 3. Level curves (left) and 3-D view with barriers (right) of $f(x,y)$

Example 2. The metric is define by Eq. (1) where $f(x,y)$ has following form:

$$a_1(x,y) = \frac{(x-2.5)^2+(y-2.5)^2}{1+(x-2.5)^2+(y-2.5)^2}, f_1(x,y) = \begin{cases} 0, a_1(x,y) \geq 0.8 \\ a_1(x,y) \end{cases}$$
$$a_2(x,y) = \frac{(x-2.5)^2+(y-7.5)^2}{1+(x-2.5)^2+(y-7.5)^2}, f_2(x,y) = \begin{cases} 0, a_2(x,y) \geq 0.8 \\ a_2(x,y) \end{cases}$$
$$a_2(x,y) = \frac{(x-7.5)^2+(y-2.5)^2}{1+(x-7.5)^2+(y-2.5)^2}, f_3(x,y) = \begin{cases} 0, a_3(x,y) \geq 0.8 \\ a_3(x,y) \end{cases}$$
$$a_3(x,y) = \frac{(x-7.5)^2+(y-7.5)^2}{1+(x-7.5)^2+(y-7.5)^2}, f_4(x,y) = \begin{cases} 0, a_4(x,y) \geq 0.8 \\ a_4(x,y) \end{cases}$$

$$F(x,y) = f_1(x,y) + f_2(x,y) + f_3(x,y) + f_4(x,y)$$
$$f(x,y) = \begin{cases} 0.4, 0 < F(x,y) \leq 0.4 \\ F(x,y) \\ 0.8, F(x,y) = 0 \end{cases}$$

Figure 3 shows level curves of function $f(x,y)$ and location of barriers which is superimposed on the 3-D view of $f(x,y)$.

The metrics like described above arise in infrastructure logistics when we want to locate some objects in the highlands. Here speed of movement depends on the angle of ascent or descent. Therefore the wave fronts are strongly distorted.

The computational results are presented in Table 2. Here $R_{\max}$ is the radius, $d_{\max}(C)$ is the density of package, $d(B)$ is the density of barriers, t is the computing time.

Figure 4 shows the solutions associated with Table 2 for $n = 1, 2, 4, 8, 16, 32$ when the container is multi-connected set in the case where the form of the wave fronts is unknown.

With respect to Fig. 4 it is clear that AECP-MCS algorithm gives acceptable results even for quite complicated metric. Note that, as in the previous example, in the given metric presented on Fig. 4 “circles” have the same radius.

Table 2. AECP-MCS results for multiply-connected set

n	$R_{\max}$	$d_{\max}(C)$	$d(B)$	t
1	52,09715102	0,220326330383	0,0542817532	12,3
2	50,39798735	0,395009064733	0,0542817532	105,2
4	29,65098366	0,440226085102	0,0542817532	151,4
8	25,35214222	0,623120400981	0,0542817532	315,3
16	17,21536804	0,592833528847	0,0542817532	2088,7
32	12,88156661	0,665244747787	0,0542817532	3150,9

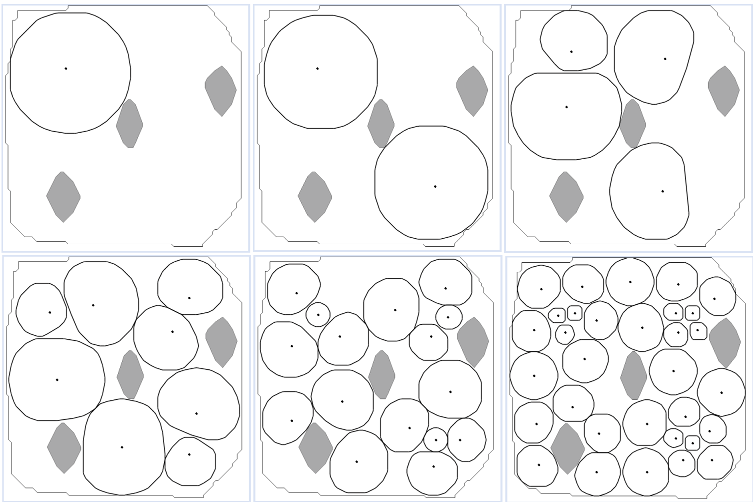

Fig. 4. AECP-MCS results for $n = 1, 2, 4, 8, 16, 32$

4 Conclusion

The presented algorithm is a modification of the Lloyd algorithm (widely known in machine learning and data mining community as k-means). The difference between the traditional k-means method and proposed approach is that the distance of each object to the centroids is not Euclidean.

We suppose that the scope of the clustering and classification problems may be significantly expanded by using special non-Euclidean metrics. So, authors will try to apply it to solve the problem of machine learning and data mining on the completely new aspect.

In conclusion we note that the further development of proposed approach involves consideration of the 3-D packing problem [30].

Acknowledgements. The reported study was particulary funded by RFBR according to the research projects No. 14-07-00222 and No. 16-06-00464.

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