Relation of Semi-Classical Orthogonal Polynomials to General Schlesinger Systems via Twistor Theory

Hironobu Kimura

Abstract We study the relation between semi-classical orthogonal polynomials and nonlinear differential equations coming from the isomonodromic deformation of linear system of differential equations on $\mathbb{P}^1$. There are many works establishing this kind of relations between the Painlevé equations and semi-orthogonal polynomials with the weight functions taking from the integrands for hypergeometric, Kummer, Bessel, Hermite, Airy integrals. Some extension of these results is obtained for the semi-classical orthogonal polynomials with the weight functions coming from the general hypergeometric integrals on the Grassmannian $G_{2,N}$. To establish the desired relations, we make use of the Atiyah-Ward Ansatz construction of particular solutions for the 2×2 Schlesinger system and its degenerated ones.

Keywords Isomonodromic deformation • Semi-classical orthogonal polynomial • Twistor theory

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1 Introduction

In this note we discuss a relation of semi-classical orthogonal polynomials to the nonlinear systems of partial differential equations obtained from the theory of isomonodromic deformation of linear differential equations on the projective line $\mathbb{P}^1$.

First we explain our motivation. Let $w(t)$ be a positive weight function on some subset $I \subset \mathbb{R}$ and let $(f, g) = \int_I f(t)g(t)w(t)dt$ be the inner product for polynomials with respect to the measure $w(t)dt$. By the process of orthogonalization we have a series of monic orthogonal polynomials $p_n(t)$ of degree $n \in \mathbb{Z}_{\geq 0}$. One of the

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important features of orthogonal polynomials is the three-term recurrence relations

$$tp_n(t) = p_{n+1}(t) + \alpha_n p_n(t) + \beta_n p_{n-1}(t).$$

It is important to know the coefficients α_n, β_n. These quantities can be expressed using the determinants

$$D_n = \det\left(\int_I t^{j+k} w(t)dt\right)_{j,k=0}^{n-1}$$

of the Hankel matrix whose (i,j) entry is the $i+j$ th moment of $w(t)$. It is known that β_n can be expressed as

$$\beta_n = \frac{D_{n-1}D_{n+1}}{D_n^2}$$

and α_n is also computable in terms of $\{D_n\}$. For the classical orthogonal polynomials, namely, Jacobi, Laguerre and Hermite polynomials, we take $w(t) = t^\alpha(1-t)^\beta$ on $[0, 1]$, $t^\alpha e^{-t}$ on $[0,\infty)$ and e^{-t^2} on $(-\infty,\infty)$ as the weight function, respectively. Evidently, we impose the condition $\alpha, \beta > -1$ so that the inner product can be defined for polynomials. In these cases, D_n are constants depending on the parameters α, β contained in the weight function. It should be noted here that, these weight functions are integrands of Beta, Gamma and Gaussian integrals, respectively:

$$B(\alpha+1,\beta+1) = \int_0^1 t^\alpha(1-t)^\beta dt,\ \Gamma(\alpha+1) = \int_0^\infty t^\alpha e^{-t}dt,\ \sqrt{\pi} = \int_{-\infty}^\infty e^{-t^2}dt.$$

Several semi-classical orthogonal polynomials are defined using the weight functions $w(t,x)$ depending on a parameter x. In these cases the Hankel determinants D_n depend on x, so we denote them as $D_n(x)$. A numerous works are devoted to clarify how $D_n(x)$ are related to the Painlevé equations P2,...,P6. For example, Dai and Zhang [3] considered the semi-classical orthogonal polynomials attached to the weight function $w(t,x) = t^\alpha(1-t)^\beta(t-x)^\gamma$ and showed that the function

$$H_n(x) := x(x-1)\frac{d}{dx}\log D_n(x) + d_1 x + d_2$$

with

$$d_1 = -n(n+\alpha+\beta+\gamma) - \frac{(\alpha+\beta)^2}{4},$$

$$d_2 = -\frac{1}{4}\left[2n(n+\alpha+\beta+\gamma) + \beta(\alpha+\beta) - \gamma(\alpha-\beta)\right]$$

satisfies the Okamoto's σ-form equation for the sixth Painlevé equation P6. This result indicates that $D_n(x)$ is the τ-function for some particular solution of P6.

Similar connection of semi-classical orthogonal polynomials of other type to the Painlevé equations was also discussed [1, 2, 4, 9]. The form of weight functions and the related Painlevé equations are listed in the following table.

$w(t,x)$	Painlevé	Special function
$t^{\alpha}(1-t)^{\beta}(t-x)^{\gamma}$	P6	Gauss
$t^{\alpha}(1-t)^{\beta}e^{-x/t}$	P5	Kummer
$(1+t)^{\alpha}(1-t)^{\beta}e^{-xt}$	”	”
$\lvert t-x\rvert^{\alpha}e^{-t^2}$	P4	Hermite-Weber
$t^{\alpha}e^{-t^2+xt}$	”	”
$\lvert t^{2\alpha+1}\rvert e^{-t^4+xt^2}$	”	”
$t^{\alpha}e^{-t-x/t}$	P3	Bessel
$e^{t^3/3+xt}$	P2	Airy

In the third column of the above table, we listed the name of special functions, where the weight function, or rather the measure $w(t,x)dt$, is essentially the integrand of the integral representation of the corresponding special function.

It is natural to ask if it is possible to extend the above story by taking an appropriate class of weight functions and a class of nonlinear differential equations. Here we take, as a weight function w, the integrand of integral representation of the general hypergeometric function (GHGF) on the Grassmannian manifold $G_{2,N}$ consisting of 2-dimensional subspaces in $\mathbb{C}^N$, see [6]. As is explained in Sect. 2, GHGF is defined as a Radon transform of a character of the universal covering group of a maximal abelian subgroup $H_\lambda \subset \mathrm{GL}_N(\mathbb{C})$ indexed by a partition λ of N. We observe that the Beta, Gamma and Gaussian integral are regarded as GHGF on $G_{2,3}$ for the partitions $\lambda = (1,1,1),(2,1),(3)$, respectively, and the special functions listed above, namely, Gauss, Kummer, Bessel, Hermite-Weber and Airy are GHGF on $G_{2,4}$ for the partitions $\lambda = (1,1,1,1),(2,1,1),(2,2),(3,1)$ and (4), respectively.

The nonlinear differential equations which we consider are those obtained from the isomonodromic deformation of systems of linear differential equations on $\mathbb{P}^1$ with regular and irregular singular points for 2×2 unknowns, so the nonlinear equations are equivalent to the Garnier system [5] and the systems of its confluent type. We call these systems as general Schlesinger system (GSS). The twistor theoretic approach is used to describe the isomonodromic deformation, where the deformation parameters live in the subspace $Z_\lambda$of $\mathrm{Mat}_{2,N}(\mathbb{C})$ whose quotient space $\mathrm{GL}_2(\mathbb{C}) \setminus Z_\lambda$ is a Zariski open subset of $G_{2,N}$, and the infinitesimal action of the group H_λ on Z_λ plays an important role.

A connection of the Hankel determinants to GSS is a consequence of the result due entirely to Shah and Woodhouse [10] on the construction of particular solutions, so called Ward ansatz solutions, for GSS.

This note is of expository nature and serves as a remark on the recognition of a possible extension of the connection between the theory of semi-classical orthogonal polynomials and nonlinear systems coming from isomonodromic deformation.

This note is organized as follows. In Sect. 2, we recall the definition of general hypergeometric functions (GHGF) on the Grassmannian manifold $G_{2,N}$. Then we review the twistor theoretic treatment of isomonodromic deformation in Sect. 3 following [10] and [8]. In Sect. 4, we explain the construction of Ward ansatz solution of the generalized anti-self-dual Yang-Mills equation (GYM) and of the related GSS in terms of general hypergeometric functions, which say that the determinant of the Hankel matrix, whose entries are moments of integrand of general hypergeometric integral on the Grassmannian, describes a particular solution of GSS. This establishes an extension of the results on the relation of semi-classical orthogonal polynomial theory to Painlevé equations.

2 Hypergeometric Function on $G_{2,N}$

Maximal Abelian Subgroup

We shall recall the definition of general hypergeometric functions (GHGF). Let N be a positive integer and $\lambda = (n_1, \dots, n_\ell)$ be a partition of N. For λ, we associate a maximal abelian subgroup of complex general linear group $\mathrm{GL}_N(\mathbb{C})$ defined by

$$H_\lambda := J(n_1) \times \cdots \times J(n_\ell),$$

where $J(n) \subset \mathrm{GL}_n(\mathbb{C})$ is the abelian subgroup obtained as a centralizer of the shift matrix $\Lambda = (\delta_{i+1,j})_{0\le i,j<n}$ and is called the Jordan group of size n. Explicitly we have

$$J(n) = \left\{h = h_0 I + h_1 \Lambda + \cdots + h_{n-1}\Lambda^{n-1} \mid h_0 \neq 0\right\} \subset \mathrm{GL}_n(\mathbb{C}),$$

from which we can know the isomorphism $J(n) \simeq \mathbb{C}[X]/(X^n)$ as multiplicative groups, where $\mathbb{C}[X]$ is the ring of polynomials in X and (X^n) is the ideal generated by X^n. The Lie algebras for H_λ and $J(n)$ will be denoted by $\mathfrak{h}_\lambda$ and $\mathfrak{j}(n)$, respectively.

Character

Let $\tilde{H}_\lambda$ be the universal covering group of H_λ and consider a character of $\tilde{H}_\lambda$, namely a group homomorphism $\chi : \tilde{H}_\lambda \to \mathbb{C}^\times$. Explicit description is as follows. Let $\theta_m(x)$ $(m \ge 0)$ be the functions of $x = (x_0, x_1, \dots)$ defined by

$$\sum_{0\le m<\infty} \theta_m(x) T^m = \log(x_0 + x_1 T + x_2 T^2 + \cdots). \tag{1}$$

$$= \log x_0 + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m} \left(\frac{x_1}{x_0} T + \frac{x_2}{x_0} T^2 + \cdots\right)^m. \tag{2}$$

Then we see that $\theta_0(x) = \log x_0$ and

$$\theta_1(x) = \frac{x_1}{x_0}$$
$$\theta_2(x) = \frac{x_2}{x_0} - \frac{1}{2}\left(\frac{x_1}{x_0}\right)^2$$
$$\theta_3(x) = \frac{x_3}{x_0} - \left(\frac{x_1}{x_0}\right)\left(\frac{x_2}{x_0}\right) + \frac{1}{3}\left(\frac{x_1}{x_0}\right)^3$$
$$\vdots$$

Since the correspondence $\tilde{J}(n) \to \mathfrak{j}(n)$, given by $h \mapsto (\theta_0(h), \theta_1(h), \dots, \theta_{n-1}(h))$, defines the identification of $\tilde{J}(n)$ with its Lie algebra $\mathfrak{j}(n)$, exponentiating a character of $\mathfrak{j}(n)$ and using this correspondence, we have a character $\chi_n : \tilde{J}(n) \to \mathbb{C}^\times$ as $\chi_n(h; \alpha) = \exp\left(\sum_{0 \le i < n} \alpha_i \theta_i(h)\right)$ with a weight $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_{n-1}) \in \mathbb{C}^n$. Since $\tilde{H}_\lambda$ is a direct product of $\tilde{J}(n_k)$, the characters χ of $\tilde{H}_\lambda$ are given by

$$\chi(h; \alpha) = \prod_{1 \le k \le \ell} \chi_{n_k}(h^{(k)}, \alpha^{(k)}) = \prod_{1 \le k \le \ell} \exp\left(\sum_{0 \le i < n_k} \alpha_i^{(k)} \theta_i(h^{(k)})\right),$$

for $h = (h^{(1)}, \dots, h^{(\ell)}) \in \tilde{H}_\lambda,\ h^{(k)} \in \tilde{J}(n_k)$. Here $\alpha = (\alpha^{(1)}, \dots, \alpha^{(\ell)}) \in \mathbb{C}^N$, $\alpha^{(k)} = (\alpha_0^{(k)}, \dots, a_{n_k-1}^{(k)}) \in \mathbb{C}^{n_k}$ is a weight.

General Hypergeometric Function

The general hypergeometric function (GHGF) is defined as a Radon transform of the characters χ of $\tilde{H}_\lambda$ as follows. Let Z_λ be the open subset of $\mathrm{Mat}_{2,N}(\mathbb{C})$ consisting of matrices $z = (z^{(1)}, \dots, z^{(\ell)}), z^{(k)} = (z_0^{(k)}, \dots, z_{n_k-1}^{(k)}) \in \mathrm{Mat}_{2,n_k}(\mathbb{C})$ satisfying

$$\det(z_0^{(k)}, z_1^{(k)}) \neq 0,\ (\text{if } n_k \geq 2)$$
$$\det(z_0^{(k)}, z_0^{(l)}) \neq 0,\ (k \neq l).$$

Definition 2.1 Assume that the weight of a character χ of $\tilde{H}_\lambda$ satisfies the condition

$$\sum_{1 \le k \le \ell} \alpha_0^{(k)} = -2. \tag{3}$$

Then the general hypergeometric function of type λ is defined by

$$F(z,\alpha) = \int_C \chi(\vec{t}z,\alpha)dt \qquad (z \in Z_\lambda) \tag{4}$$

where $\vec{t} = (1,t)$, $\vec{t}z = (\vec{t}z_0^{(1)},\ldots,\vec{t}z_{n_1-1}^{(1)},\ldots,\vec{t}z_0^{(\ell)},\ldots,\vec{t}z_{n_\ell-1}^{(\ell)})$ and C is a one dimensional cycle in $\mathbb{C}$ of the homology group defined by the integrand. We do not enter in detailed explanation for the homology group.

On the space Z_λ, the groups $\mathrm{GL}_2(\mathbb{C})$ and H_λ act from left and right, respectively, by the matrix multiplication $\mathrm{GL}_2(\mathbb{C}) \times Z_\lambda \times H_\lambda \ni (g,z,h) \mapsto gzh \in Z_\lambda$. Then we know the following results.

Proposition 2.2 *$F(z,\alpha)$ satisfies*

$$F(gz,\alpha) = (\det g)^{-1}F(z,\alpha) \quad (g \in \mathrm{GL}_2(\mathbb{C})),$$

$$F(zh,\alpha) = \chi(h,\alpha)F(z,\alpha) \quad (h \in \tilde{H}_\lambda) \tag{5}$$

$$(\partial_{0i}\partial_{1j} - \partial_{1i}\partial_{0j})F(z,\alpha) = 0 \quad (\forall i,j) \tag{6}$$

Roughly speaking, the last equation (6) comes from the fact that GHGF is defined as a Radon transform of a function on $\tilde{H}_\lambda$.

Relation to the Classical Special Functions

We explain how the integral representation for the classical special functions is obtained as GHGF on the Grassmannian manifold $G_{2,4}$. We list up the following data:

1. the character of $\tilde{H}_\lambda$,
2. GHGF of type λ,
3. a subspace X_λ of Z_λ which is a realization of $\mathrm{GL}_2(\mathbb{C})\backslash Z_\lambda/H_\lambda$,
4. restriction of GHGF to X_λ with a normalization of parameters.

Gauss HGF($\boldsymbol{\lambda = (1,1,1,1)}$)

1. $\chi(h) = h_1^{\alpha_1}\cdots h_4^{\alpha_4}$ with $\alpha_1+\alpha_2+\alpha_3+\alpha_4 = -2$,
2. $F(z,\alpha) = \int_C (z_{01}+z_{11}t)^{\alpha_1}\cdots(z_{04}+z_{14}t)^{\alpha_4}dt$,
3. $X_{(1,1,1,1)} = \left\{ \mathbf{x} = \begin{pmatrix} 1 & 0 & 1 & -x \\ 0 & 1 & -1 & 1 \end{pmatrix} \mid x \neq 0,1 \right\}$,
4. $F(\mathbf{x},\alpha) = \int_C t^{\alpha_2}(1-t)^{\alpha_3}(t-x)^{\alpha_4}dt$.

Kummer's Confluent HGF($\boldsymbol{\lambda} = (2, 1, 1)$)

1. $\chi(h) = h_1^{\alpha_1} \exp(\alpha_2 \frac{h_2}{h_1}) h_3^{\alpha_3} h_4^{\alpha_4}$ with $\alpha_1 + \alpha_3 + \alpha_4 = -2,\ \alpha_2 = -1$
2. $F(z, \alpha) = \int_C (z_{01} + z_{11} t)^{\alpha_1} \exp\left(\alpha_2 \frac{z_{02} + z_{12} t}{z_{01} + z_{11} t}\right) \prod_{i=3,4} (z_{0i} + z_{1i} t)^{\alpha_i} dt,$
3. $X_{(2,1,1)} = \left\{ \mathbf{x} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & x & 1 & -1 \end{pmatrix} \mid x \neq 0 \right\},$
4. $F(\mathbf{x}, \alpha) = \int_C e^{-xt} t^{\alpha_3} (1 - t)^{\alpha_4} dt.$

Hermite-Weber($\boldsymbol{\lambda} = (3, 1)$)

1. $\chi(h) = h_1^{\alpha_1} \exp\left(\alpha_2 \frac{h_2}{h_1} + \alpha_3 (\frac{h_3}{h_1} - \frac{1}{2} (\frac{h_2}{h_1})^2)\right) h_4^{\alpha_4}$ with $\alpha_1 + \alpha_4 = -2,\ \alpha_2 = 0,\ \alpha_3 = 1,$
2. $F(z, \alpha) = \int_C (z_{01} + z_{11} t)^{\alpha_1} \exp\left(\frac{z_{03} + z_{13} t}{z_{01} + z_{11} t} - \frac{1}{2} (\frac{z_{02} + z_{12} t}{z_{01} + z_{11} t})^2\right) (z_{04} + z_{14} t)^{\alpha_4} dt,$
3. $X_{(3,1)} = \left\{ \mathbf{x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & x & 1 \end{pmatrix} \mid x \in \mathbb{C} \right\},$
4. $F(\mathbf{x}, \alpha) = \int_C \exp(xt - \frac{1}{2} t^2) t^{\alpha_4} dt.$

Bessel($\boldsymbol{\lambda} = (2, 2)$)

1. $\chi(h) = h_1^{\alpha_1} \exp\left(\alpha_2 \frac{h_2}{h_1}\right) h_3^{\alpha_3} \exp(\alpha_4 \frac{h_4}{h_3})$ with $\alpha_1 + \alpha_3 = -2,\ \alpha_2 = 1,\ \alpha_3 = 1,$
2. $F(z, \alpha) = \int_C (z_{01} + z_{11} t)^{\alpha_1} \exp\left(\frac{z_{02} + z_{12} t}{z_{01} + z_{11} t}\right) (z_{03} + z_{13} t)^{\alpha_3} \exp\left(\frac{z_{04} + z_{14} t}{z_{03} + z_{13} t}\right) dt,$
3. $X_{(2,2)} = \left\{ \mathbf{x} = \begin{pmatrix} 1 & 0 & 0 & -x \\ 0 & 1 & 1 & 0 \end{pmatrix} \mid x \neq 0 \right\},$
4. $F(\mathbf{x}, \alpha) = \int_C \exp(t - \frac{x}{t}) t^{\alpha_3} dt.$

Airy($\boldsymbol{\lambda} = (4)$)

1. $\chi(h) = h_1^{\alpha_1} \exp\left(\alpha_2 \frac{h_2}{h_1} + \alpha_3 (\frac{h_3}{h_1} - \frac{1}{2} (\frac{h_2}{h_1})^2) + \alpha_4 (\frac{h_4}{h_1} - \left(\frac{h_2}{h_1}\right)\left(\frac{h_3}{h_1}\right) + \frac{1}{3}\left(\frac{h_2}{h_1}\right)^3)\right)$ with $\alpha_1 = -2,\ \alpha_2 = \alpha_3 = 0,\ \alpha_4 = 1,$
2. $F(z, \alpha) = \int_C (z_{01} + z_{11} t)^{\alpha_1} \exp\left(\frac{z_{04} + z_{14} t}{z_{01} + z_{11} t} - \frac{z_{02} + z_{12} t}{z_{01} + z_{11} t} \cdot \frac{z_{03} + z_{13} t}{z_{01} + z_{11} t} + \frac{1}{3} (\frac{z_{02} + z_{12} t}{z_{01} + z_{11} t})^3\right) dt,$
3. $X_{(4)} = \left\{ \mathbf{x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & x \end{pmatrix} \mid x \in \mathbb{C} \right\},$
4. $F(\mathbf{x}, \alpha) = \int_C \exp(xt + \frac{1}{3} t^3) dt.$

3 General Schlesinger System and the Result

Schlesinger System

Consider a family of linear differential equations on $\mathbb{P}^1$

$$\frac{dy}{d\zeta} = \left(\sum_{j=1}^{N-1} \frac{A_j(x)}{\zeta - x_j}\right) y, \quad A_j(x) \in sl_2(\mathbb{C}) \tag{7}$$

where $A_j(x)$ depends holomorphically on x in some open subset of $(\mathbb{P}^1)^{N-1}$. Assume that $A_N := -\sum_{j=1}^{N-1} A_j(x) = \mathrm{diag}(a, -a)$.

Definition 3.1 The Eq. (7) gives an isomonodromic family if there exists a fundamental system of solutions $Y(\zeta, x)$ such that the associated monodromy representation is invariant under the variation of x_js.

Assume here that, for each j, 2 eigenvalues of A_j do not differ by an integer. Then we know the following result.

Proposition 3.2 *The family of Eqs. (7) gives an isomonodromic family with a fundamental system of solutions $Y(\zeta, x)$ which has the form $Y(\zeta, x) = (\sum_{m=0}^{\infty} Y_m \zeta^{-m})\mathrm{diag}(\zeta^a, \zeta^{-a})$ at $\zeta = \infty$ with $Y_0 = I_2$, if and only if (7) together with*

$$\frac{\partial y}{\partial x_j} = -\frac{A_j(x)}{\zeta - x_j} \quad (1 \leq j < N)$$

form an integrable system. This condition can be written as the Schlesinger system:

$$dA_j = \sum_{i(\neq j)} [A_i, A_j] d \log(x_i - x_j) \qquad (1 \leq j < N). \tag{8}$$

General Schlesinger System

For a given partition $\lambda = (n_1, \ldots, n_\ell)$ of N, let us consider the system of linear differential equations of the form

$$\frac{dy}{d\zeta} = \left(\sum_{k=1}^{\ell} \sum_{j=0}^{n_k - 1} A_j^{(k)}(z) \frac{d\theta_j(\vec{\zeta} z^{(k)})}{d\zeta}\right) y \tag{9}$$

where θ_j are those defined in (1), $z = (z^{(1)}, \dots, z^{(\ell)})$ varies in some open subset $U \subset Z_\lambda$ and $A_j^{(k)}(z) \in sl_2(\mathbb{C})$ depends holomorphically on $z \in U$ and satisfy $\sum_{k=0}^{\ell} A_0^{(k)}(z) = 0$. The equation has the singular points $\{x_1, \dots, x_\ell\}$, where $x_k = -\frac{z_{00}^{(k)}}{z_{10}^{(k)}}$ $(1 \le k \le \ell)$. Since $\frac{d\theta_j(\vec{\zeta} z^{(k)})}{d\zeta}$ has a pole of order n_k, the Eq. (9) has in general an irregular singular point at x_k when $n_k \ge 2$. Roughly speaking, the family (9) gives an isomonodromic family if there is a fundamental system of solutions such that the associated monodromy representation is independent of z and the connection matrices among canonical solutions at irregular singular points(including Stokes matrices) are also independent of z. We refer Sect. 4 of [10] for the detailed explanation for it. The nonlinear system for $A_j^{(k)}(z)$ governing the isomonodromic deformation is called the general Schlesinger system (GSS). We remark that when the partition of N is $\lambda = (1, \dots, 1)$, the GSS coincides with the Schlesinger system. Note also that the Schlesinger system is known to be completely integrable, but the integrability of GSS is checked in a particular case [7].

Result

Here we present a particular solution of GSS given in terms of the Hankel determinants of moments associated with the general hypergeometric function of type λ.

As above, take a partition $\lambda = (n_1, \dots, n_\ell)$ of N, a character $\chi(h, \alpha)$ of $\tilde{H}_\lambda$. For any fixed $n_0 \in \mathbb{Z}$, consider moments of $\chi(\vec{tz}, \alpha)$:

$$\phi_n(z) = \int_C t^{n+n_0} \chi(\vec{tz}, \alpha) dt, \tag{10}$$

where C is a cycle and the Hankel determinant $\tau_m^p(z) = \det(\phi_{i+j+p-m+1}(z))_{i,j=0}^{m-1}$, where m is a size of matrix and p denotes the index of moment which is arrayed in the main anti-diagonal entries. For example when $p = 0$,

$$\tau_m^0(z) = \begin{vmatrix} \phi_{1-m} & \phi_{2-m} & \cdots & & \phi_0 \\ \phi_{2-m} & & & \phi_0 & \\ \vdots & & \iddots & & \vdots \\ & \phi_0 & & & \phi_{m-2} \\ \phi_0 & & \cdots & \phi_{m-2} & \phi_{m-1} \end{vmatrix},$$

where ϕ_0 is arrayed along the main anti-diagonal line.

Put

$$f_0(z) = \frac{(-1)^m}{\tau_m^0} \begin{pmatrix} \tau_m^1 & \tau_{m+1}^0 \\ \tau_{m-1}^0 & \tau_m^{-1} \end{pmatrix} \tag{11}$$

We know that $f_0(z)$ belongs to $\mathrm{SL}_2(\mathbb{C})$ by virtue of Sylvester's formula of determinants.

Theorem 3.3 *For any positive integer m, we have an isomonodromic family (9) of the form*

$$A_j^{(k)}(z) = -\frac{1}{2} \begin{pmatrix} \alpha_j^{(k)} & \\ & -\alpha_j^{(k)} \end{pmatrix} + \sum_{0 \le i} z_{1i}^{(k)} \frac{\partial f_0}{\partial z_{1,i+j}^{(k)}} \cdot f_0^{-1} \quad (1 \le k \le \ell, 0 \le j < n_k)$$

and hence a particular solution to GSS expressed in terms of Hankel determinants associated with GHGF of type λ.

This theorem is just a rephrase of the result due to Shah and Woodhouse [10]. It relies on the description of isomonodromic deformation via twistor theory and on the construction of particular solution of generalized anti-self-dual Yang-Mills equation on $G_{2,N}$ using Ward ansatz. It is not yet known that τ_m^0 is so called the τ-function for the isomonodromy problem. In the following sections, we explain how the above solution can be obtained.

4 Twistor Theory and Isomonodromic Deformation

In this section, we explain how the Schlesinger system and its confluent type systems can be obtained from the twistor theoretic point of view following [8, 10].

Generalized Yang-Mills Equation

Let $Z = \{z \in \mathrm{Mat}_{2,N}(\mathbb{C}) \mid rk\, z = 2\}$ with the coordinates $z = (z_{ij})_{0 \le i \le 1, 1 \le j \le N}$. Let $U \subset Z$ be an open set and consider a holomorphic connection D on the trivial bundle $U \times \mathbb{C}^2$ with connection matrices in $sl_2(\mathbb{C})$. Then D can be written as

$$D = d + \sum_{i,j} \Phi_{ij}(z) dz_{ij} = \sum_{ij} D_{ij} dz_{ij},$$

where

$$D_{ij} = \frac{\partial}{\partial z_{ij}} + \Phi_{ij}(z), \quad \Phi_{ij}(z) \in sl_2(\mathbb{C}).$$

Definition 4.1 A holomorphic connection D is called the generalized (anti-self-dual) Yang-Mills (GYM) connection, if we have

$$[\zeta D_{0j} - D_{1j}, \zeta D_{0k} - D_{1k}] = 0 \quad (j \neq k, \zeta \in \mathbb{C}). \tag{12}$$

Equivalently, we have the nonlinear equations for $\Phi_{ij}(z)$:

$$[D_{0j}, D_{0k}] = 0, \quad [D_{1j}, D_{1k}] = 0, \quad [D_{0j}, D_{1k}] + [D_{1j}, D_{0k}] = 0, \tag{13}$$

$1 \leq j,k \leq N$, which we call the generalized Yang-Mills equation.

Ward-Penrose Transform

The important feature in treating GYM equation is to encode its solutions in terms of some class of vector bundles on the twistor space $\mathbb{P}^{N-1}$. Consider a double fibration

$$\begin{array}{ccc} & \mathbb{P}\mathcal{C} & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}^{N-1} & & Z \end{array}$$

where $\mathbb{P}\mathcal{C} := \{([t_0,t_1],z) \in \mathbb{P}^1 \times Z \mid (t_0,t_1) \neq (0,0)\}$ with the homogeneous coordinates (t_0,t_1) of $\mathbb{P}^1$ and the maps π_1 and π_2 are defined by

$$\pi_1([t_0,t_1],z) = [t_0\vec{z}_0 + t_1\vec{z}_1], \quad z = \begin{pmatrix} \vec{z}_0 \\ \vec{z}_1 \end{pmatrix}$$

$$\pi_2([t_0,t_1],z) = z.$$

$\mathbb{P}\mathcal{C}$ is called the correspondence space and $\mathbb{P}^{N-1}$ the twistor space. The above double fibration gives the correspondence from Z to the twistor space $\mathbb{P}^{N-1}$ by

$$Z \ni z \mapsto \hat{z} := \pi_1(\pi_2^{-1}(z)) = \{[t_0\vec{z}_0 + t_1\vec{z}_1] \mid (t_0,t_1) \neq (0,0)\} \subset \mathbb{P}^{N-1}.$$

Here $\hat{z}$ is a projective line in $\mathbb{P}^{N-1}$ joining two points $[\vec{z}_0]$, $[\vec{z}_1]$ and is called the twistor line determined by z. On the other hand, it gives a correspondence from $\mathbb{P}^{N-1}$ to Z by

$$\mathbb{P}^{N-1} \ni p = [x] \mapsto \tilde{p} := \pi_2(\pi_1^{-1}([x])) = \{z \in Z \mid \vec{z}_0 \wedge \vec{z}_1 \wedge x = 0\},$$

where $\tilde{p}$ is a plane in Z of $\dim = N-1$ called the twistor surface.

It is known that a connection D on $U \times \mathbb{C}^2$ is GYM if and only if $D|_{\tilde{p}}$ is integrable for $\forall p \in \hat{U} = \pi_1(\pi_2^{-1}(U)) \subset \mathbb{P}^{N-1}$.

The Ward-Penrose transform is the correspondence between the following two sets

$$\left\{\begin{array}{l}\text{holomorphic } \mathrm{SL}_2(\mathbb{C})\text{-vector bundles}\\ E \to \hat{U}, \text{trivial on twistor lines } \hat{q}(q \in U)\end{array}\right\} \leftrightarrow \left\{\begin{array}{l}\text{solutions } D \text{ of}\\ \text{GYM on } U\end{array}\right\}$$

We call an element of the left-hand side a twistor bundle. We shall explain the Ward-Penrose transform from a twistor bundle to a solution of GYM.

Let $E \to \hat{U}$ be a $\mathrm{SL}_2(\mathbb{C})$-twistor bundle, and $\pi_1^* E$ be the lift of E to $\pi_1^{-1}(\hat{U}) = \mathbb{P}^1 \times U$. If $F \in \mathrm{SL}_2(\mathbb{C})$ is a patching function for E, then that for $\pi_1^* E$ is $F^* = F(\vec{z}_0 + \zeta \vec{z}_1)$, where $\zeta = t_1/t_0$ is the affine coordinates of $\mathbb{P}^1$. Let $\{V, \tilde{V}\}$ be an open covering of $\pi_1^{-1}(\hat{U})$ defined by

$$V = \{|\zeta| < r\} \times U, \ \tilde{V} = \{|\zeta| > \tilde{r}\} \times U, \quad \tilde{r} < r.$$

We may assume that F^* is defined on the intersection $V \cap \tilde{V}$. Since E is trivial on twistor lines, there is a Birkhoff decomposition $F^* = \tilde{f}^{-1} \cdot f$, where $f, \tilde{f} \in \mathrm{SL}_2(\mathbb{C})$ are holomorphic respectively on V and $\tilde{V}$ such that $\tilde{f}(\infty, z) = 1_2$. Combining this with the fact that F^* is a lift of F, we have $(\zeta \partial_{0j} - \partial_{1j})F^* = (\zeta \partial_{0j} - \partial_{1j})(\tilde{f}^{-1} \cdot f) = 0$ for any j. It implies

$$\zeta \partial_{0j} f \cdot f^{-1} - \partial_{1j} f \cdot f^{-1} = \zeta \partial_{0j} \tilde{f} \cdot \tilde{f}^{-1} - \partial_{1j} \tilde{f} \cdot \tilde{f}^{-1}. \tag{14}$$

The left-hand side and the right-hand side are defined on V and $\tilde{V}$, respectively. By Liouville Theorem, both sides define a polynomial function in ζ with a simple pole at $\zeta = \infty$. From $\tilde{f}(\infty, z) = 1_2$, we see that $\zeta = \infty$ is not a pole of both sides, and hence (14) defines a $sl_2(\mathbb{C})$ valued function depending only on z, which we denote as $\Phi_{1j}(z) \in sl_2(\mathbb{C})$. Then we have

$$[\zeta(\partial_{0j} + 0) - (\partial_{1j} + \Phi_{1j}(z))]f = 0, \quad [\zeta(\partial_{0j} + 0) - (\partial_{1j} + \Phi_{1j}(z))]\tilde{f} = 0.$$

This implies that the connection $\nabla = d + \sum_j \Phi_{1j}(z) dz_{1j}$ is a solution to GYM equation.

Note that, if $f = f_0(z) + f_1(z)\zeta + \cdots$, then Φ_{1j} can be determined only from f_0;

$$\Phi_{1j}(z) = -\partial_{1j} f_0 \cdot f_0^{-1} \quad (1 \le j \le N). \tag{15}$$

Isomonodromic Deformation

First we introduce some notation. Let $x = (x_1, \ldots, x_N)$ be the homogeneous coordinates of $\mathbb{P}^{N-1}$. We denote by $[x]$ (or sometimes by x) a point of $\mathbb{P}^{N-1}$. For $\xi \in \mathfrak{h}_\lambda$, define a vector field X_ξ on $\mathbb{P}^{N-1}$ and Y_ξ on $\mathbb{P}\mathcal{C}$ by

$$X_\xi g := \frac{d}{ds} g([x \exp s\xi])|_{s=0}, \quad Y_\xi h := \frac{d}{ds} h(([t_0, t_1], z \exp s\xi))|_{s=0}.$$

Sometimes we use also the notation $x = (x^{(1)}, \dots, x^{(\ell)})$, $x^{(k)} = (x_0^{(k)}, \dots, x_{n_k-1}^{(k)}) \in \mathbb{C}^{n_k}$ for the homogeneous coordinates of $\mathbb{P}^{N-1}$.

Definition 4.2 $\mathrm{SL}_2(\mathbb{C})$-twistor bundle $E \to \hat{U}$ is said to be symmetric with respect to H_λ, if the infinitesimal action of H_λ on $\hat{U} \subset \mathbb{P}^{N-1}$ can be lifted to E, in other terms, if there is a Lie derivation $\mathcal{L}_\xi$ for any $\xi \in \mathfrak{h}_\lambda$, which acts on local sections of E, such that $\xi \mapsto \mathcal{L}_\xi$ is a Lie algebra homomorphism.

Note that the Lie derivation $\mathcal{L}_\xi$ can be written locally as $\mathcal{L}_\xi = X_\xi + B_\xi(x)$ with $B_\xi(x) \in sl_2(\mathbb{C})$. Since X_ξ spans a tangent space $T_x\mathbb{P}^{N-1}$at any points x such that $x_0^{(k)} \neq 0$ for any k, symmetry of the twistor bundle implies the integrable connection ∇ on $\hat{U}$ whose connection form can be written locally as

$$\sum_{k=1}^{\ell} \sum_{j=0}^{n_k-1} B_j^{(k)}(x) d\theta_j(x^{(k)}), \tag{16}$$

$B_j^{(k)} \in sl_2(\mathbb{C})$ is B_ξ for $\xi = E_j^{(k)} := 0 \oplus \cdots \oplus (\Lambda^{(k)})^j \oplus \cdots \oplus 0) \in \mathfrak{h}_\lambda$, where $\Lambda^{(k)} = (\delta_{i+1,j}) \in \mathfrak{j}(n_k)$ is the shift matrix of size n_k.

We want to get an isomonodromic family of linear differential equations on $\mathbb{P}^1$ by restricting the connection ∇ on twistor lines, or ∇^* on the lines $\mathbb{P}^1 \times \{z\} \subset \mathbb{P}\mathcal{C}$. To get a such family, we trivialize the twistor bundle, which is symmetric with respect to H_λ, on twistor lines. Let ∇^* be the lift of ∇ to the pullback bundle $\pi_1^* E \to \pi_1^{-1}(\hat{U}) \subset \mathbb{P}\mathcal{C}$ whose connection form on $V, \tilde{V}$ are of the forms

$$\omega = \sum_{k=1}^{\ell} \sum_{j=0}^{n_k-1} B_j^{(k)}(\zeta, z) d\theta_j(\vec{\zeta} z^{(k)}), \quad \tilde{\omega} = \sum_{k=1}^{\ell} \sum_{j=0}^{n_k-1} \tilde{B}_j^{(k)}(\zeta, z) d\theta_j(\vec{\zeta} z^{(k)})$$

and let F^* be the patching function for the bundle $\pi_1^* E$ on $V \cap \tilde{V}$. Then it is known that the symmetry is assured if F^* satisfies the condition

$$Y_\xi F^* = F^* B_\xi - \tilde{B}_\xi F^* \qquad (\forall \xi \in \mathfrak{h}_\lambda), \tag{17}$$

where B_ξ and $\tilde{B}_\xi$ are holomorphic on V and $\tilde{V}$ respectively which comes from the local form of $\mathcal{L}_\xi$ on V and $\tilde{V}$. Since $\pi_1^* E$ is trivial on $\mathbb{P}^1 \times \{z\}$ for any $z \in U$, we can find $f, \tilde{f} \in \mathrm{SL}_2(\mathbb{C})$, holomorphic respectively on V and $\tilde{V}$, such that $F^* = \tilde{f}^{-1} f$ with $\tilde{f}(\infty, z) = 1_2$. Then the integrable connection ∇^* on the product bundle:

$$\nabla^* = d + \Omega := d + f\omega f^{-1} - df \cdot f^{-1} = d + \tilde{f}\tilde{\omega}\tilde{f}^{-1} - d\tilde{f} \cdot \tilde{f}^{-1} \tag{18}$$

gives an isomonodromic deformation and the integrability condition $(\nabla^*)^2 = 0$ is a nonlinear system of differential equations. This nonlinear equation is the general Schlesinger system(GSS) of type λ. We want to express Ω more explicitly. Put $\Theta_\xi(z) := fB_\xi f^{-1} - Y_\xi f \cdot f^{-1} = \tilde{f}\tilde{B}_\xi \tilde{f}^{-1} - Y_\xi \tilde{f} \cdot \tilde{f}^{-1}$. Then we can show the following.

Proposition 4.3 *The connection form* Ω *for the connection* ∇^* *is given by*

$$\Omega = -\sum_{k=1}^{\ell}\sum_{j=0}^{n_k-1} A_j^{(k)}(z)d\theta_j(\vec{\zeta}z^{(k)}) + \Phi, \tag{19}$$

where $A_j^{(k)}(z) := \Theta_{E_j^{(k)}}(z) - i_{Y_{E_j^{(k)}}}\Phi$, $i_{Y_{E_j^{(k)}}}$ *being the interior product with respect to the vector field* $Y_{E_j^{(k)}}$, *and* $\Phi := \sum \Phi_{1j}(z)dz_{1j}$ *is a solution of GYM given by (15) corresponding to the twistor bundle* E.

The integrability of the connection $\nabla^* = d + \Omega$ describes the isomonodromic deformation of the linear differential equation (9).

5 Ward Ansatz Solution of GYM

We explain the construction of particular solutions of GSS of type λ following the description of [10], which gives Theorem 3.3.

Ward Ansatz Solution

Let $U \subset Z$ be an open set as in the previous section, and put $\hat{U} = \pi_1(\pi_2^{-1}(U))$. We set the following Ansatz:

(i) $\mathrm{SL}_2(\mathbb{C})$-twistor bundle E on $\hat{U}$ corresponds to a solution of GYM equation,
(ii) the patching function F^* of π_1^*E has the form

$$F^* = \begin{pmatrix} \zeta^m & \phi(\zeta, z) \\ & \zeta^{-m} \end{pmatrix} \text{ on } V \cap \tilde{V}. \tag{20}$$

Let $\phi(\zeta, z) = \sum_{n=-\infty}^{\infty} \phi_n(z)\zeta^{-n}$ be the Laurent expansion with respect to ζ. Then we follow the process of Ward-Penrose transform explained above. We can construct the Birkhoff decomposition

$$F^* = \tilde{f}^{-1} \cdot f \tag{21}$$

uniquely under the condition $\tilde{f}(\infty, z) = 1_2$ using linear algebra. Especially the constant term f_0 in the Taylor expansion of f at $\zeta = 0$ can be given by (11). Hence if we can find the twistor bundle E such that the lifted patching function F^* satisfies the ansatz (i), (ii) and the condition (17) of symmetry with respect to H_λ, we can get a particular solution of GSS by the process of previous section.

First we consider the condition for $\phi(\zeta, z)$ so that F^* has the form (20). Since F^* is a lift of a transition F of E, we have $(\zeta\partial_{0j} - \partial_{1j})F^* = 0$, equivalently, $(\zeta\partial_{0j} - \partial_{1j})\phi = 0$ for any j. Substituting the Laurent expansion of $\phi(\zeta, z)$ with respect to ζ, we get

$$\partial_{0j}\phi_n = \partial_{1j}\phi_{n-1} \quad (1 \le j \le N, n \in \mathbb{Z}). \tag{22}$$

Notice that (22) implies

$$(\partial_{0j}\partial_{1k} - \partial_{0k}\partial_{1j})\phi_n = 0 \quad (j \neq k, n \in \mathbb{Z}), \tag{23}$$

which are just the Eqs. (6) used in characterizing the image of Radon transform.

Particular Solutions for GSS

In the construction of Ward ansatz solution described in the previous subsection, let us determine $\phi_n(z)$ so that the resulting twistor bundle becomes symmetric with respect to H_λ and as a result, it gives a particular solution to GSS of type λ.

Let $\chi : \tilde{H}_\lambda \to \mathbb{C}^\times$ be a character with a weight $\alpha \in \mathbb{C}^N$. Take a fixed $n_0 \in \mathbb{Z}$ and define

$$\phi_n(z) = \int_C t^{n+n_0}\chi(\vec{t}z, \alpha)dt, \tag{24}$$

where C is a cycle of the homology group associated with $\chi(\vec{t}z, \alpha)$. This choice of C assure the exchange of differentiation with respect to z and integration. We can check easily that $\phi_n(z)$ satisfies contiguous relation (22) and $\phi_n(zh) = \phi_n(z)\chi(h, \alpha)$. The last identity implies

$$Y_\xi\phi_n(z) = \langle \xi, \alpha \rangle\, \phi_n(z) \quad (\xi = \sum_{j,k} \xi_j^{(k)} E_j^{(k)} \in \mathfrak{h}_\lambda)$$

where $\langle \xi, \alpha \rangle = \sum_{j,k} \xi_j^{(k)}\alpha_j^{(k)}$. It follows that F^*, given by (20) and (24), satisfies the condition (17) for the symmetry of the twistor bundle with

$$B_\xi(z) = -\tilde{B}_\xi(z) = \frac{1}{2}\begin{pmatrix} \langle \xi, \alpha \rangle & \\ & -\langle \xi, \alpha \rangle \end{pmatrix}. \tag{25}$$

Finally, putting in $\Theta_\xi(z)$ the expression (25) for $\tilde{B}_\xi(z)$ and $\Phi = -\sum \partial_{1j}f_0 \cdot f_0^{-1}$ with f_0 given by (11) with (24), Proposition 4.3 produces a particular solution of GSS in terms of GHGF given in Theorem 3.3.

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