A Good-λ Lemma, Two Weight $T1$ Theorems Without Weak Boundedness, and a Two Weight Accretive Global Tb Theorem

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This paper is dedicated to Dick Wheeden on the occasion of his retirement from Rutgers University, and for all of his fundamental contributions to the theory of weighted inequalities, in particular for the beautiful paper of Hunt, Muckenhoupt and Wheeden that started it all back in 1973.

Abstract Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$, let T^α be a standard α-fractional Calderón-Zygmund operator on $\mathbb{R}^n$ with $0 \leq \alpha < n$, and assume as side conditions the $\mathcal{A}_2^\alpha$ conditions, punctured A_2^α conditions, and certain α*-energy conditions*. Then the weak boundedness property associated with the operator T^α and the weight pair (σ, ω), is 'good-λ' controlled by the testing conditions and the Muckenhoupt and energy conditions. As a consequence, assuming the side conditions, we can eliminate the weak boundedness property from Theorem 1 of Sawyer et al. (A two weight fractional singular integral theorem with side conditions, energy and k-energy dispersed. arXiv:1603.04332v2) to obtain that T^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the testing conditions hold for T^α and its dual. As a corollary we give a simple derivation of a two weight accretive global Tb theorem from a related $T1$ theorem. The role of two different parameterizations of the family of dyadic grids, by scale and by translation, is highlighted in simultaneously exploiting both goodness and NTV surgery with families of grids that are *common* to both measures.

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1 Introduction

The theory of weighted norm inequalities burst into the general mathematical consciousness with the celebrated theorem of Hunt et al. [5] that extended boundedness of the Hilbert transform to measures more general than Lebesgue's, namely showing that H was bounded on the weighted space $L^2(\mathbb{R}^n; w)$ if and only if the A_2 condition of Muckenhoupt,

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{w(x)}dx\right) \lesssim 1\ ,$$

holds when taken uniformly over all cubes Q in $\mathbb{R}^n$. The ensuing thread of investigation culminated in the theorem of Coifman and Fefferman [3] that characterized those nonnegative weights w on $\mathbb{R}^n$ for which all of the 'nicest' of the $L^2(\mathbb{R}^n)$ bounded singular integrals T above are bounded on weighted spaces $L^2(\mathbb{R}^n; w)$, and does so in terms of the above A_2 condition of Muckenhoupt.

Attention then turned to the corresponding two weight inequalities for singular integrals, which turned out to be considerably more complicated. For example, Cotlar and Sadosky gave a beautiful function theoretic characterization of the weight pairs (σ, ω) for which H is bounded from $L^2(\mathbb{R}; \sigma)$ to $L^2(\mathbb{R}; \omega)$, namely a two-weight extension of the Helson-Szegö theorem, which illuminated a deep connection between two quite different function theoretic conditions, but failed to shed much light on when either of them held.[1] On the other hand, the two weight inequality for positive fractional integrals, Poisson integrals and maximal functions were characterized using testing conditions by one of us in [24] (see also [6] for the Poisson inequality with 'holes') and [23], but relying in a very strong way on the positivity of the kernel, something the Hilbert kernel lacks. In a groundbreaking series of papers including [16, 18] and [19], Nazarov, Treil and Volberg used weighted Haar decompositions with random grids, introduced their 'pivotal' condition, and proved that the Hilbert transform is bounded from $L^2(\mathbb{R}; \sigma)$ to $L^2(\mathbb{R}; \omega)$ if and only if a variant of the A_2 condition 'on steroids' held, and the norm inequality and its dual held when tested locally over indicators of cubes—but **only** under the side assumption that their pivotal conditions held.

The last dozen years have seen a resurgence in the investigation of two weight inequalities for singular integrals, beginning with the aforementioned work of NTV, and due in part to applications of the two weight $T1$ theorem in operator theory, such as in [14], where embedding measures are characterized for model spaces K_θ, where θ is an inner function on the disk, and where norms of composition operators are characterized that map K_θ into Hardy and Bergman spaces. A $T1$ theorem could also have implications for a number of problems that are higher dimensional analogues of those connected to the Hilbert transform (rank

[1]However, the testing conditions in Theorem 1 are subject to the same criticism due to the highly unstable nature of singular integrals acting on measures.

one perturbations [20, 32]; products of two densely defined Toeplitz operators; subspaces of the Hardy space invariant under the inverse shift operator [15, 32]; orthogonal polynomials [21, 22, 33]; and quasiconformal theory [1, 2, 8, 11]), and we refer the reader to [28] for more detail on these applications.

Following the groundbreaking work of Nazarov, Treil and Volberg, two of us, Sawyer and Uriarte-Tuero, together with Lacey in [12], showed that the pivotal conditions were not necessary in general, and introduced instead a necessary 'energy' condition as a substitute, along with a hybrid merging of these two conditions that was shown to be sufficient for use as a side condition. The resurgence was then capped along the way with a resolution—involving the work of Nazarov, Treil and Volberg in [19], the authors and M. Lacey in the two part paper [9, 13] and T. Hytönen in [6]—of the two weight Hilbert transform conjecture of Nazarov, Treil and Volberg [32]:

Theorem 1 *The Hilbert transform is bounded from $L^2(\mathbb{R};\sigma)$ to $L^2(\mathbb{R};\omega)$, i.e.*

$$\|H(f\sigma)\|_{L^2(\mathbb{R};\omega)} \lesssim \|f\|_{L^2(\mathbb{R};\sigma)}, \quad f \in L^2(\mathbb{R};\sigma), \tag{1}$$

if and only if the two weight A_2 condition with holes holds,

$$\frac{|Q|_\sigma}{|Q|}\left(\frac{1}{|Q|}\int_{\mathbb{R}\setminus Q}\mathbf{s}_Q^2 d\omega(x)\right)+\left(\frac{1}{|Q|}\int_{\mathbb{R}\setminus Q}\mathbf{s}_Q^2 d\sigma(x)\right)\frac{|Q|_\omega}{|Q|}\lesssim 1,$$

uniformly over all cubes Q, and the two testing conditions hold,

$$\left\|\mathbf{1}_Q H(\mathbf{1}_Q\sigma)\right\|_{L^2(\mathbb{R};\omega)} \lesssim \|\mathbf{1}_Q\|_{L^2(\mathbb{R};\sigma)} = \sqrt{|Q|_\sigma},$$
$$\left\|\mathbf{1}_Q H^*(\mathbf{1}_Q\omega)\right\|_{L^2(\mathbb{R};\sigma)} \lesssim \|\mathbf{1}_Q\|_{L^2(\mathbb{R};\omega)} = \sqrt{|Q|_\omega},$$

uniformly over all cubes Q.

Here $Hf(x)=\int_{\mathbb{R}}\frac{f(y)}{y-x}dy$ is the Hilbert transform on the real line $\mathbb{R}$, and σ and ω are locally finite positive Borel measures on $\mathbb{R}$. The two weight A_2 condition with holes is also a testing condition in disguise, in particular it follows from

$$\left\|H(\mathbf{s}_Q\sigma)\right\|_{L^2(\mathbb{R};\omega)} \lesssim \|\mathbf{s}_Q\|_{L^2(\mathbb{R};\sigma)},$$

tested over all 'indicators with tails' $\mathbf{s}_Q(x)=\frac{\ell(Q)}{\ell(Q)+|x-c_Q|}$ of intervals Q in $\mathbb{R}$. Below we discuss the precise interpretation of the above inequalities involving the singular integral H.

At this juncture, attention naturally turned to the analogous two weight inequalities for *higher dimensional* singular integrals, as well as α-fractional singular integrals such as the Cauchy transform in the plane. A variety of results were obtained, e.g. [10, 14, 26] and [27], in which a $T1$ theorem was proved under certain side conditions that implied the energy conditions. However, in [28], the authors have recently shown that the energy conditions are *not* in general necessary for elliptic singular integrals.

The aforementioned higher dimensional results require refinements of the various one-dimensional conditions associated with the norm inequalities, namely the A_2 conditions, the testing conditions, the weak boundedness property and energy conditions. The purpose of this paper is to prove in higher dimensions that the weak boundedness constant $\mathcal{WBP}_{T^\alpha}(\sigma,\omega)$ that is associated with an α-fractional singular integral T^α and a weight pair (σ,ω) in $\mathbb{R}^n$, is 'good-λ' controlled by the usual testing conditions $\mathfrak{T}_{T^\alpha}(\sigma,\omega)$, $\mathfrak{T}^*_{T^\alpha}(\sigma,\omega)$ and two side conditions on weight pairs, namely the Muckenhoupt conditions $\mathfrak{A}_2^\alpha(\sigma,\omega)$ and the energy conditions $\mathcal{E}_\alpha^{\text{strong}}(\sigma,\omega)$, $\mathcal{E}_\alpha^{\text{strong},*}(\sigma,\omega)$: more precisely, for every $0<\lambda<\frac{1}{2}$, we have the Good-λ Lemma:

$$\mathcal{WBP}_{T^\alpha}(\sigma,\omega)\le C_\alpha\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}^*_{T^\alpha}+\mathcal{E}_\alpha^{\text{strong}}+\mathcal{E}_\alpha^{\text{strong},*}+\sqrt[4]{\lambda}\mathfrak{N}_{T^\alpha}\right).$$

The first instance of this type of conclusion appears in Lacey and Wick in [10])—see Remark 1 in Sect. 2.1 below.

Applications of the Good-λ Lemma are then given to obtain both $T1$ and Tb theorems for two weights. We now turn to a description of the higher dimensional conditions appearing in the above display. As the Good-λ Lemma, along with its corollaries, hold in the more general setting of quasicubes, we describe them first. But the reader interested only in cubes can safely ignore this largely cosmetic generalization (but crucial for our 'measure on a curve' $T1$ theorem in [26]) by simply deleting the prefix 'quasi' wherever it appears.

1.1 Quasicubes

We begin by recalling the notion of quasicube used in [27]—a special case of the classical notion used in quasiconformal theory.

Definition 1 We say that a homeomorphism $\Omega:\mathbb{R}^n\to\mathbb{R}^n$ is a globally biLipschitz map if

$$\|\Omega\|_{Lip}\equiv\sup_{x,y\in\mathbb{R}^n}\frac{\|\Omega(x)-\Omega(y)\|}{\|x-y\|}<\infty, \tag{2}$$

and $\left\|\Omega^{-1}\right\|_{Lip}<\infty$.

Notation 1 *We define $\mathcal{P}^n$ to be the collection of half open, half closed cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. A half open, half closed cube Q in $\mathbb{R}^n$ has the form $Q=Q(c,\ell)\equiv\prod_{k=1}^n\left[c_k-\frac{\ell}{2},c_k+\frac{\ell}{2}\right)$ for some $\ell>0$ and $c=(c_1,\dots,c_n)\in\mathbb{R}^n$. The cube $Q(c,\ell)$ is described as having center c and sidelength ℓ.*

Definition 2 Suppose that $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map.

(1) If E is a measurable subset of $\mathbb{R}^n$, we define $\Omega E \equiv \{\Omega(x) : x \in E\}$ to be the image of E under the homeomorphism Ω.

 (a) In the special case that $E = Q$ is a cube in $\mathbb{R}^n$, we will refer to ΩQ as a quasicube (or Ω-quasicube if Ω is not clear from the context).
 (b) We define the center $c_{\Omega Q} = c(\Omega Q)$ of the quasicube ΩQ to be the point Ωc_Q where $c_Q = c(Q)$ is the center of Q.
 (c) We define the side length $\ell(\Omega Q)$ of the quasicube ΩQ to be the sidelength $\ell(Q)$ of the cube Q.
 (d) For $r > 0$ we define the 'dilation' $r\Omega Q$ of a quasicube ΩQ to be $\Omega r Q$ where rQ is the usual 'dilation' of a cube in $\mathbb{R}^n$ that is concentric with Q and having side length $r\ell(Q)$.

(2) If $\mathcal{K}$ is a collection of cubes in $\mathbb{R}^n$, we define $\Omega\mathcal{K} \equiv \{\Omega Q : Q \in \mathcal{K}\}$ to be the collection of quasicubes ΩQ as Q ranges over $\mathcal{K}$.
(3) If $\mathcal{F}$ is a grid of cubes in $\mathbb{R}^n$, we define the inherited quasigrid structure on $\Omega\mathcal{F}$ by declaring that ΩQ is a child of $\Omega Q'$ in $\Omega\mathcal{F}$ if Q is a child of Q' in the grid $\mathcal{F}$.

Note that if ΩQ is a quasicube, then $|\Omega Q|^{\frac{1}{n}} \approx |Q|^{\frac{1}{n}} = \ell(Q) = \ell(\Omega Q)$. For a quasicube $J = \Omega Q$, we will generally use the expression $|J|^{\frac{1}{n}}$ in the various estimates arising in the proofs below, but will often use $\ell(J)$ when defining collections of quasicubes. Moreover, there are constants R_{big} and R_{small} such that we have the comparability containments

$$Q + \Omega x_Q \subset R_{big}\Omega Q \text{ and } R_{small}\Omega Q \subset Q + \Omega x_Q \,.$$

Example 1 Quasicubes can be wildly shaped, as illustrated by the standard example of a logarithmic spiral in the plane $f_\varepsilon(z) = z|z|^{2\varepsilon i} = ze^{i\varepsilon \ln(z\bar{z})}$. Indeed, $f_\varepsilon : \mathbb{C} \to \mathbb{C}$ is a globally biLipschitz map with Lipschitz constant $1 + C\varepsilon$ since $f_\varepsilon^{-1}(w) = w|w|^{-2\varepsilon i}$ and

$$\nabla f_\varepsilon = \left(\frac{\partial f_\varepsilon}{\partial z}, \frac{\partial f_\varepsilon}{\partial \bar{z}}\right) = \left(|z|^{2\varepsilon i} + i\varepsilon |z|^{2\varepsilon i}, i\varepsilon \frac{z}{\bar{z}} |z|^{2\varepsilon i}\right).$$

On the other hand, f_ε behaves wildly at the origin since the image of the closed unit interval on the real line under f_ε is an infinite logarithmic spiral.

1.2 Standard Fractional Singular Integrals and the Norm Inequality

Let $0 \le \alpha < n$. We define a standard α-fractional CZ kernel $K^\alpha(x, y)$ to be a real-valued function defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following fractional size and smoothness conditions of order $1+\delta$ for some $\delta>0$: For $x\neq y$,

$$|K^{\alpha}(x,y)|\leq C_{CZ}|x-y|^{\alpha-n} \text{ and } |\nabla K^{\alpha}(x,y)|\leq C_{CZ}|x-y|^{\alpha-n-1},$$

$$\left|\nabla K^{\alpha}(x,y)-\nabla K^{\alpha}\left(x',y\right)\right|\leq C_{CZ}\left(\frac{|x-x'|}{|x-y|}\right)^{\delta}|x-y|^{\alpha-n-1}, \quad \frac{|x-x'|}{|x-y|}\leq\frac{1}{2}, \tag{3}$$

and the last inequality also holds for the adjoint kernel in which x and y are interchanged. We note that a more general definition of kernel has only order of smoothness $\delta>0$, rather than $1+\delta$, but the use of the Monotonicity and Energy Lemmas in arguments below, which involve first order Taylor approximations to the kernel functions $K^{\alpha}(\cdot,y)$, requires order of smoothness more than 1 to handle remainder terms.

1.2.1 Defining the Norm Inequality

We now turn to a precise definition of the weighted norm inequality

$$\|T^{\alpha}_{\sigma}f\|_{L^2(\omega)}\leq\mathfrak{N}_{T^{\alpha}_{\sigma}}\|f\|_{L^2(\sigma)}, \quad f\in L^2(\sigma). \tag{4}$$

For this we introduce a family $\left\{\eta^{\alpha}_{\delta,R}\right\}_{0<\delta<R<\infty}$ of nonnegative functions on $[0,\infty)$ so that the truncated kernels $K^{\alpha}_{\delta,R}(x,y)=\eta^{\alpha}_{\delta,R}(|x-y|)K^{\alpha}(x,y)$ are bounded with compact support for fixed x or y. Then the truncated operators

$$T^{\alpha}_{\sigma,\delta,R}f(x)\equiv\int_{\mathbb{R}^n}K^{\alpha}_{\delta,R}(x,y)f(y)\,d\sigma(y), \quad x\in\mathbb{R}^n,$$

are pointwise well-defined, and we will refer to the pair $\left(K^{\alpha},\left\{\eta^{\alpha}_{\delta,R}\right\}_{0<\delta<R<\infty}\right)$ as an α-fractional singular integral operator, which we typically denote by T^{α}, suppressing the dependence on the truncations.

Definition 3 We say that an α-fractional singular integral operator $T^{\alpha}=\left(K^{\alpha},\left\{\eta^{\alpha}_{\delta,R}\right\}_{0<\delta<R<\infty}\right)$ satisfies the norm inequality (4) provided

$$\left\|T^{\alpha}_{\sigma,\delta,R}f\right\|_{L^2(\omega)}\leq\mathfrak{N}_{T^{\alpha}_{\sigma}}\|f\|_{L^2(\sigma)}, \quad f\in L^2(\sigma), 0<\delta<R<\infty.$$

It turns out that, in the presence of Muckenhoupt conditions, the norm inequality (4) is essentially independent of the choice of truncations used, and we now explain this in some detail. A *smooth truncation* of T^{α} has kernel $\eta_{\delta,R}(|x-y|)K^{\alpha}(x,y)$ for a smooth function $\eta_{\delta,R}$ compactly supported in (δ,R), $0<\delta<R<\infty$, and satisfying standard CZ estimates. A typical example of an

α-fractional transform is the α-fractional *Riesz* vector of operators

$$\mathbf{R}^{\alpha,n} = \left\{R_\ell^{\alpha,n} : 1 \le \ell \le n\right\}, \quad 0 \le \alpha < n.$$

The Riesz transforms $R_\ell^{n,\alpha}$ are convolution fractional singular integrals $R_\ell^{n,\alpha} f \equiv K_\ell^{n,\alpha} * f$ with odd kernel defined by

$$K_\ell^{\alpha,n}(w) \equiv \frac{w^\ell}{|w|^{n+1-\alpha}} \equiv \frac{\Omega_\ell(w)}{|w|^{n-\alpha}}, \quad w = \left(w^1, \ldots, w^n\right).$$

However, in dealing with energy considerations, and in particular in the Monotonicity Lemma below where first order Taylor approximations are made on the truncated kernels, it is necessary to use the *tangent line truncation* of the Riesz transform $R_\ell^{\alpha,n}$ whose kernel is defined to be $\Omega_\ell(w)\,\psi_{\delta,R}^\alpha(|w|)$ where $\psi_{\delta,R}^\alpha$ is continuously differentiable on an interval $(0,S)$ with $0 < \delta < R < S$, and where $\psi_{\delta,R}^\alpha(r) = r^{\alpha-n}$ if $\delta \le r \le R$, and has constant derivative on both $(0,\delta)$ and (R,S) where $\psi_{\delta,R}^\alpha(S) = 0$. Here S is uniquely determined by R and α. Finally we set $\psi_{\delta,R}^\alpha(S) = 0$ as well, so that the kernel vanishes on the diagonal and common point masses do not 'see' each other. Note also that the tangent line extension of a $C^{1,\delta}$ function on the line is again $C^{1,\delta}$ with no increase in the $C^{1,\delta}$ norm.

It was shown in the one dimensional case with no common point masses in [13], that boundedness of the Hilbert transform H with one set of appropriate truncations together with the A_2^α condition without holes, is equivalent to boundedness of H with any other set of appropriate truncations, and this was extended to $\mathbf{R}^{\alpha,n}$ and more general operators in higher dimensions, permitting common point masses as well. Thus we are free to use the tangent line truncations throughout the proofs of our results.

1.3 Quasicube Testing Conditions

The following 'dual' quasicube testing conditions are necessary for the boundedness of T^α from $L^2(\sigma)$ to $L^2(\omega)$,

$$\mathfrak{T}_{T^\alpha}^2 \equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{1}{|Q|_\sigma} \int_Q \left|T^\alpha\left(\mathbf{1}_Q\sigma\right)\right|^2 \omega < \infty,$$

$$\left(\mathfrak{T}_{T^\alpha}^*\right)^2 \equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{1}{|Q|_\omega} \int_Q \left|(T^\alpha)^*\left(\mathbf{1}_Q\omega\right)\right|^2 \sigma < \infty,$$

and where we interpret the right sides as holding uniformly over all tangent line truncations of T^α. Equally necessary are the following 'full' testing conditions where the integrations are taken over the entire space $\mathbb{R}^n$:

$$\mathfrak{FT}^2_{T^\alpha} \equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{1}{|Q|_\sigma} \int_{\mathbb{R}^n} \left|T^\alpha\left(\mathbf{1}_Q\sigma\right)\right|^2 \omega < \infty,$$

$$\left(\mathfrak{FT}^*_{T^\alpha}\right)^2 \equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{1}{|Q|_\omega} \int_{\mathbb{R}^n} \left|(T^\alpha)^*\left(\mathbf{1}_Q\omega\right)\right|^2 \sigma < \infty,$$

1.4 Quasiweak Boundedness and Indicator/Touching Property

The quasiweak boundedness property for T^α with constant C is given by

$$\left|\int_Q T^\alpha\left(1_{Q'}\sigma\right) d\omega\right| \leq \mathcal{WBP}_{T^\alpha} \sqrt{|Q|_\omega\, |Q'|_\sigma}, \tag{5}$$

$$\text{for all quasicubes } Q, Q' \text{ with } \frac{1}{C} \leq \frac{\ell(Q)}{\ell(Q')} \leq C,$$

$$\text{and either } Q \subset 3Q' \setminus Q' \text{ or } Q' \subset 3Q \setminus Q,$$

and where we interpret the left side above as holding uniformly over all tangent line trucations of T^α. This condition is used in our $T1$ theorem with an energy side condition in [27], but will be removed in our $T1$ theorem with an energy side condition obtained here as a corollary of the Good-λ Lemma.

We say that two quasicubes Q and Q' in $\Omega\mathcal{P}^n$ are *touching quasicubes* if the intersection of their closures is nonempty and contained in the boundary of the larger quasicube. Finally, let $\mathfrak{I}_{T^\alpha} = \mathfrak{I}_{T^\alpha}(\sigma,\omega)$ be the best constant in the *indicator/touching* inequality for the bilinear form corresponding to T

$$\left|\mathcal{T}^\alpha\left(\mathbf{1}_Q, \mathbf{1}_{Q'}\right)\right| \leq \mathfrak{I}_{T^\alpha}(\sigma,\omega)\, \|\mathbf{1}_Q\|_{L^2(\sigma)}\, \|\mathbf{1}_{Q'}\|_{L^2(\omega)}\ , \tag{6}$$

$$\text{for all touching quasicubes } Q, Q' \in \mathcal{P}^n,$$

$$\text{with } \frac{1}{C} \leq \frac{\ell(Q)}{\ell(Q')} \leq C,$$

$$\text{and either } Q \subset 3Q' \setminus Q' \text{ or } Q' \subset 3Q \setminus Q.$$

1.5 Poisson Integrals and $\mathcal{A}_2^\alpha$

Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$, and suppose Q is an Ω-quasicube in $\mathbb{R}^n$. Recall that $|Q|^{\frac{1}{n}} \approx \ell(Q)$ for a quasicube Q. The two α-fractional

Poisson integrals of μ on a quasicube Q are given by:

$$\mathrm{P}^{\alpha}(Q,\mu) \equiv \int_{\mathbb{R}^n} \frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}} + |x - x_Q|\right)^{n+1-\alpha}} d\mu(x),$$

$$\mathcal{P}^{\alpha}(Q,\mu) \equiv \int_{\mathbb{R}^n} \left(\frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}} + |x - x_Q|\right)^{2}}\right)^{n-\alpha} d\mu(x),$$

where we emphasize that $|x - x_Q|$ denotes Euclidean distance between x and x_Q and $|Q|$ denotes the Lebesgue measure of the quasicube Q. We refer to P^{α} as the *standard* Poisson integral and to $\mathcal{P}^{\alpha}$ as the *reproducing* Poisson integral.

We say that the pair K, K' in $\mathcal{P}^n$ are *neighbours* if K and K' live in a common dyadic grid and both $K \subset 3K' \setminus K'$ and $K' \subset 3K \setminus K$, and we denote by $\mathcal{N}^n$ the set of pairs (K, K') in $\mathcal{P}^n \times \mathcal{P}^n$ that are neighbours. Let

$$\Omega\mathcal{N}^n = \left\{\left(\Omega K, \Omega K'\right) : \left(K, K'\right) \in \mathcal{N}^n\right\}$$

be the corresponding collection of neighbour pairs of quasicubes. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$, and suppose $0 \le \alpha < n$. Then we define the classical *offset* A_2^{α} *constants* by

$$A_2^{\alpha}(\sigma,\omega) \equiv \sup_{(Q,Q') \in \Omega\mathcal{N}^n} \frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q'|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}}. \tag{7}$$

Since the cubes in $\mathcal{P}^n$ are products of half open, half closed intervals $[a, b)$, the neighbouring quasicubes $(Q, Q') \in \Omega\mathcal{N}^n$ are disjoint, and any common point masses of σ and ω do not simultaneously appear in each factor.

We now define the *one-tailed* $\mathcal{A}_2^{\alpha}$ constant using $\mathcal{P}^{\alpha}$. The energy constants $\mathcal{E}_{\alpha}^{\text{strong}}$ introduced below will use the standard Poisson integral P^{α}.

Definition 4 The one-tailed constants $\mathcal{A}_2^{\alpha}$ and $\mathcal{A}_2^{\alpha,*}$ for the weight pair (σ, ω) are given by

$$\mathcal{A}_2^{\alpha} \equiv \sup_{Q \in \Omega\mathcal{P}^n} \mathcal{P}^{\alpha}\left(Q, \mathbf{1}_{Q^c}\sigma\right) \frac{|Q|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}} < \infty,$$

$$\mathcal{A}_2^{\alpha,*} \equiv \sup_{Q \in \Omega\mathcal{P}^n} \mathcal{P}^{\alpha}\left(Q, \mathbf{1}_{Q^c}\omega\right) \frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}} < \infty.$$

Note that these definitions are the analogues of the corresponding conditions with 'holes' introduced by Hytönen [6] in dimension $n = 1$—the supports of the measures $\mathbf{1}_{Q^c}\sigma$ and $\mathbf{1}_{Q}\omega$ in the definition of $\mathcal{A}_2^{\alpha}$ are disjoint, and so the common point masses of σ and ω do not appear simultaneously in each factor. Note also that, unlike in [29], where common point masses were not permitted, we can no longer assert the equivalence of $\mathcal{A}_2^\alpha$ with holes taken over *quasicubes* with $\mathcal{A}_2^\alpha$ with holes taken over *cubes*.

1.5.1 Punctured A_2^α Conditions

The *classical* A_2^α characteristic $\sup_{Q\in\Omega\mathcal{Q}^n} \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}$ fails to be finite when the measures σ and ω have a common point mass—simply let Q in the sup above shrink to a common mass point. But there is a substitute that is quite similar in character that is motivated by the fact that for large quasicubes Q, the sup above is problematic only if just *one* of the measures is *mostly* a point mass when restricted to Q.

Given an at most countable set $\mathfrak{P} = \{p_k\}_{k=1}^\infty$ in $\mathbb{R}^n$, a quasicube $Q \in \Omega\mathcal{P}^n$, and a locally finite positive Borel measure μ, define as in [27],

$$\mu(Q, \mathfrak{P}) \equiv |Q|_\mu - \sup\{\mu(p_k) : p_k \in Q \cap \mathfrak{P}\},$$

where the supremum is actually achieved since $\sum_{p_k\in Q\cap\mathfrak{P}} \mu(p_k) < \infty$ as μ is locally finite. The quantity $\mu(Q, \mathfrak{P})$ is simply the $\widetilde{\mu}$ measure of Q where $\widetilde{\mu}$ is the measure μ with its largest point mass from $\mathfrak{P}$ in Q removed. Given a locally finite measure pair (σ, ω), let $\mathfrak{P}_{(\sigma,\omega)} = \{p_k\}_{k=1}^\infty$ be the at most countable set of common point masses of σ and ω. Then the weighted norm inequality (4) typically implies finiteness of the following *punctured* Muckenhoupt conditions (see [27]):

$$A_2^{\alpha,\text{punct}}(\sigma, \omega) \equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{\omega\left(Q, \mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}},$$

$$A_2^{\alpha,*,\text{punct}}(\sigma, \omega) \equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} \frac{\sigma\left(Q, \mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}.$$

Now we turn to the definition of a quasiHaar basis of $L^2(\mu)$.

1.6 A Weighted QuasiHaar Basis

We will use a construction of a quasiHaar basis in $\mathbb{R}^n$ that is adapted to a measure μ (c.f. [18] for the nonquasi case). Given a dyadic quasicube $Q \in \Omega\mathcal{D}$, where $\mathcal{D}$ is a dyadic grid of cubes from $\mathcal{P}^n$, let $\triangle_Q^\mu$ denote orthogonal projection onto the finite dimensional subspace $L_Q^2(\mu)$ of $L^2(\mu)$ that consists of linear combinations of the indicators of the children $\mathfrak{C}(Q)$ of Q that have μ-mean zero over Q:

$$L_Q^2(\mu) \equiv \left\{ f = \sum_{Q'\in\mathfrak{C}(Q)} a_{Q'} \mathbf{1}_{Q'} : a_{Q'} \in \mathbb{R}, \int_Q f d\mu = 0 \right\}.$$

Then we have the important telescoping property for dyadic quasicubes $Q_1 \subset Q_2$ that arises from the martingale differences associated with the projections $\triangle_Q^\mu$:

$$\mathbf{1}_{Q_0}(x)\left(\sum_{Q\in[Q_1,Q_2]}\triangle_Q^\mu f(x)\right) = \mathbf{1}_{Q_0}(x)\left(\mathbb{E}_{Q_0}^\mu f - \mathbb{E}_{Q_2}^\mu f\right), \quad Q_0 \in \mathfrak{C}(Q_1), f \in L^2(\mu). \tag{8}$$

We will at times find it convenient to use a fixed orthonormal basis $\left\{h_Q^{\mu,a}\right\}_{a\in\Gamma_n}$ of $L_Q^2(\mu)$ where $\Gamma_n \equiv \{0,1\}^n \setminus \{\mathbf{1}\}$ is a convenient index set with $\mathbf{1} = (1,1,\ldots,1)$. Then $\left\{h_Q^{\mu,a}\right\}_{a\in\Gamma_n \text{ and } Q\in\Omega\mathcal{D}}$ is an orthonormal basis for $L^2(\mu)$, with the understanding that we add the constant function $\mathbf{1}$ if μ is a finite measure. In particular we have for an infinite measure

$$\|f\|_{L^2(\mu)}^2 = \sum_{Q\in\Omega\mathcal{D}}\left\|\triangle_Q^\mu f\right\|_{L^2(\mu)}^2 = \sum_{Q\in\Omega\mathcal{D}}\sum_{a\in\Gamma_n}\left|\widehat{f}(Q)\right|^2, \quad \left|\widehat{f}(Q)\right|^2 \equiv \sum_{a\in\Gamma_n}\left|\left\langle f, h_Q^{\mu,a}\right\rangle_\mu\right|^2,$$

where the measure is suppressed in the notation $\widehat{f}$. Indeed, this follows from (8) and Lebesgue's differentiation theorem for quasicubes. We also record the following useful estimate. If I' is any of the 2^n $\Omega\mathcal{D}$-children of I, and $a \in \Gamma_n$, then

$$\left|\mathbb{E}_{I'}^\mu h_I^{\mu,a}\right| \le \sqrt{\mathbb{E}_{I'}^\mu\left(h_I^{\mu,a}\right)^2} \le \frac{1}{\sqrt{|I'|_\mu}}. \tag{9}$$

1.7 The Strong Quasienergy Conditions

Given a dyadic quasicube $K \in \Omega\mathcal{D}$ and a positive measure μ we define the quasiHaar projection $\mathsf{P}_K^\mu \equiv \sum_{J\in\Omega\mathcal{D}:\ J\subset K}\triangle_J^\mu$ on K by

$$\mathsf{P}_K^\mu f = \sum_{J\in\Omega\mathcal{D}:\ J\subset K}\sum_{a\in\Gamma_n}\left\langle f, h_J^{\mu,a}\right\rangle_\mu h_J^{\mu,a} \text{ so that } \left\|\mathsf{P}_K^\mu f\right\|_{L^2(\mu)}^2 = \sum_{J\in\Omega\mathcal{D}:\ J\subset K}\sum_{a\in\Gamma_n}\left|\left\langle f, h_J^{\mu,a}\right\rangle_\mu\right|^2,$$

and where a quasiHaar basis $\left\{h_J^{\mu,a}\right\}_{a\in\Gamma_n \text{ and } J\in\mathcal{D}\Omega}$ adapted to the measure μ was defined in the subsection on a weighted quasiHaar basis above.

Now we define various notions for quasicubes which are inherited from the same notions for cubes. The main objective here is to use the familiar notation that one uses for cubes, but now extended to Ω-quasicubes. We have already introduced the notions of quasigrids $\Omega\mathcal{D}$, and center, sidelength and dyadic associated to quasicubes $Q \in \Omega\mathcal{D}$, as well as quasiHaar functions, and we will continue to extend to quasicubes the additional familiar notions related to cubes as we come across them. We begin with the notion of *deeply embedded*. Fix a quasigrid $\Omega\mathcal{D}$. We say that a dyadic quasicube J is $(\mathbf{r},\varepsilon)$-*deeply embedded* in a (not necessarily dyadic) quasicube K, which we write as $J \Subset_{\mathbf{r},\varepsilon} K$, when $J \subset K$ and both

$$\ell(J) \leq 2^{-\mathbf{r}}\ell(K), \tag{10}$$
$$\operatorname{qdist}(J,\partial K) \geq \frac{1}{2}\ell(J)^{\varepsilon}\,\ell(K)^{1-\varepsilon},$$

where we define the quasidistance $\operatorname{qdist}(E,F)$ between two sets E and F to be the Euclidean distance $\operatorname{dist}\left(\Omega^{-1}E,\Omega^{-1}F\right)$ between the preimages $\Omega^{-1}E$ and $\Omega^{-1}F$ of E and F under the map Ω, and where we recall that $\ell(J) \approx |J|^{\frac{1}{n}}$. For the most part we will consider $J \Subset_{\mathbf{r},\varepsilon} K$ when J and K belong to a common quasigrid $\Omega\mathcal{D}$, but an exception is made when defining the strong energy constants below.

Recall that in dimension $n = 1$, and for $\alpha = 0$, the energy condition constant was defined by

$$\mathcal{E}^2 \equiv \sup_{I=\dot{\cup} I_r} \frac{1}{|I|_{\sigma}} \sum_{r=1}^{\infty} \left(\frac{\mathrm{P}^{\alpha}(I_r,\mathbf{1}_I\sigma)}{|I_r|}\right)^2 \left\|\mathsf{P}^{\omega}_{I_r}\mathbf{x}\right\|^2_{L^2(\omega)} ,$$

where I, I_r and J are intervals in the real line. The extension to higher dimensions we use here is that of 'strong quasienergy condition' defined in [27] and recalled below.

We define a quasicube K (not necessarily in $\Omega\mathcal{D}$) to be an *alternate* $\Omega\mathcal{D}$-quasicube if it is a union of 2^n $\Omega\mathcal{D}$-quasicubes K' with side length $\ell(K') = \frac{1}{2}\ell(K)$ (such quasicubes were called shifted in [29], but that terminology conflicts with the more familiar notion of shifted quasigrid). Thus for any $\Omega\mathcal{D}$-quasicube L there are exactly 2^n alternate $\Omega\mathcal{D}$-quasicubes of twice the side length that contain L, and one of them is of course the $\Omega\mathcal{D}$-parent of L. We denote the collection of alternate $\Omega\mathcal{D}$-quasicubes by $\mathcal{A}\Omega\mathcal{D}$.

The extension of the energy conditions to higher dimensions in [29] used the collection

$$\mathcal{M}_{\mathbf{r},\varepsilon-\mathrm{deep}}(K) \equiv \{\text{maximal dyadic } J \Subset_{\mathbf{r},\varepsilon} K\}$$

of *maximal* $(\mathbf{r},\varepsilon)$-deeply embedded dyadic subquasicubes of a quasicube K (a subquasicube J of K is a *dyadic* subquasicube of K if $J \in \Omega\mathcal{D}$ when $\Omega\mathcal{D}$ is a dyadic quasigrid containing K). This collection of dyadic subquasicubes of K is of course a pairwise disjoint decomposition of K. We also defined there a refinement and extension of the collection $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$ for certain K and each $\ell \geq 1$. For an alternate quasicube $K \in \mathcal{A}\Omega\mathcal{D}$, define $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)$ to consist of the *maximal* $\mathbf{r}$-deeply embedded $\Omega\mathcal{D}$-dyadic subquasicubes J of K. (In the special case that K itself belongs to $\Omega\mathcal{D}$, then $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K) = \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$.) Then in [29] for $\ell \geq 1$ we defined the refinement

$$\mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K) \equiv \Big\{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}\left(\pi^{\ell}K'\right) \text{ for some } K' \in \mathfrak{C}_{\Omega\mathcal{D}}(K) :$$
$$J \subset L \text{ for some } L \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)\Big\},$$

where $\mathfrak{C}_{\Omega\mathcal{D}}(K)$ is the obvious extension to alternate quasicubes of the set of $\Omega\mathcal{D}$-dyadic children. Thus $\mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)$ is the union, over all quasichildren K' of K, of those quasicubes in $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}\left(\pi^{\ell}K'\right)$ that happen to be contained in some $L \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)$. We then define the *strong* quasienergy condition as follows.

Definition 5 Let $0 \leq \alpha < n$ and fix 'goodness' parameters $(\mathbf{r},\varepsilon)$. Suppose σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$. Then the *strong* quasienergy constant $\mathcal{E}^{\mathrm{strong}}_{\alpha}$ is defined by

$$\begin{aligned}\left(\mathcal{E}^{\mathrm{strong}}_{\alpha}\right)^2 &\equiv \sup_{I=\dot{\cup}I_r} \frac{1}{|I|_{\sigma}} \sum_{r=1}^{\infty} \sum_{J\in\mathcal{M}_{\mathbf{r},\varepsilon-\mathrm{deep}}(I_r)} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_I\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\omega}_J \mathbf{x}\right\|^2_{L^2(\omega)} \\ &+ \sup_{\Omega\mathcal{D}} \sup_{I\in\mathcal{A}\Omega\mathcal{D}} \sup_{\ell\geq 0} \frac{1}{|I|_{\sigma}} \sum_{J\in\mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(I)} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_I\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\omega}_J \mathbf{x}\right\|^2_{L^2(\omega)} .\end{aligned}$$

Similarly we have a dual version of $\mathcal{E}^{\mathrm{strong}}_{\alpha}$ denoted $\mathcal{E}^{\mathrm{strong},*}_{\alpha}$, and both depend on $\mathbf{r}$ and ε as well as on n and α. An important point in this definition is that the quasicube I in the second line is permitted to lie *outside* the quasigrid $\Omega\mathcal{D}$, but only as an alternate dyadic quasicube $I \in \mathcal{A}\Omega\mathcal{D}$. In the setting of quasicubes we continue to use the linear function $\mathbf{x}$ in the final factor $\left\|\mathsf{P}^{\omega}_J \mathbf{x}\right\|^2_{L^2(\omega)}$ of each line, and not the pushforward of $\mathbf{x}$ by Ω. The reason of course is that this condition is used to capture the first order information in the Taylor expansion of a singular kernel.

2 The Good-λ Lemma

The basic new result of this paper is the following 'Good-λ Lemma' whose utility will become evident when we pursue its corollaries below. Set *fraktur* A^{α}_2 to be the sum of the four A^{α}_2 conditions:

$$\mathfrak{A}^{\alpha}_2 = \mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,*}_2 + A^{\alpha,\mathrm{punct}}_2 + A^{\alpha,*,\mathrm{punct}}_2 .$$

Lemma 1 (The Good-λ Lemma) *Suppose that T^{α} is a standard α-fractional singular integral in $\mathbb{R}^n$, and that σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$. For every $\lambda \in \left(0,\frac{1}{2}\right)$, we have*

$$\begin{aligned}&\mathcal{WBP}_{T^{\alpha}}(\sigma,\omega) &(11)\\ &\leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_2(\sigma,\omega)} + \left(\mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^*_{T^{\alpha}}\right)(\sigma,\omega) + \left(\mathcal{E}^{\mathrm{strong}}_{\alpha} + \mathcal{E}^{\mathrm{strong},*}_{\alpha}\right)(\sigma,\omega) + \sqrt[4]{\lambda}\,\mathfrak{N}_{T^{\alpha}}(\sigma,\omega)\right).\end{aligned}$$

Thus the effect of the Good-λ Lemma is to 'good-λ replace' the quasiweak boundedness property with just the usual testing conditions in the presence of the side conditions of Muckenhoupt and energy on the weight pair. However, in dimension $n = 1$ a much stronger inequality can be proved (see e.g. [19] and [12]):

$$\mathcal{WBP}_{T^\alpha} \leq C_\alpha \left(\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* \right).$$

2.1 Corollaries

Now we come to the corollaries of the Good-λ Lemma. We first remove the hypothesis of the quasiweak boundedness property from the conclusion of part (1) of Theorem 1 in [27].

Remark 1 In [10], Lacey and Wick have removed the weak boundedness property from their $T1$ theorem by using NTV surgery with two independent grids, one for each function f and g in $\langle T_\sigma^\alpha f, g \rangle$, in the course of their argument. The use of independent grids for each of f and g greatly simplifies the NTV surgery, but does not accommodate our control of functional energy by Muckenhoupt and energy conditions.

Theorem 2 *Suppose $0 \leq \alpha < n$, that T^α is a standard α-fractional singular integral operator on $\mathbb{R}^n$, and that ω and σ are locally finite positive Borel measures on $\mathbb{R}^n$. Set $T_\sigma^\alpha f = T^\alpha (f\sigma)$ for any smooth truncation of T_σ^α. Let $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ be a globally biLipschitz map. Then the operator T_σ^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.*

$$\|T_\sigma^\alpha f\|_{L^2(\omega)} \leq \mathfrak{N}_{T_\sigma^\alpha} \|f\|_{L^2(\sigma)},$$

uniformly in smooth truncations of T^α, and moreover

$$\mathfrak{N}_{T_\sigma^\alpha} \leq C_\alpha \left(\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} \right),$$

provided that the two dual $\mathcal{A}_2^\alpha$ conditions and the two dual punctured Muckenhoupt conditions all hold, and the two dual quasitesting conditions for T^α hold, and provided that the two dual strong quasienergy conditions hold uniformly over all dyadic quasigrids $\Omega\mathcal{D} \subset \Omega\mathcal{P}^n$, i.e. $\mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} < \infty$, and where the goodness parameters $\mathbf{r}$ and ε implicit in the definition of the collections $\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(K)$ and $\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep},\Omega\mathcal{D}}^\ell(K)$ appearing in the strong energy conditions, are fixed sufficiently large and small respectively depending only on n and α.

Proof Let $T_{\delta,R}^\alpha$ be a tangent line approximation to T^α as introduced above. Then $\mathfrak{N}_{T_{\delta,R}^\alpha} < \infty$, indeed $\mathfrak{N}_{T_{\delta,R}^\alpha} \leq C_{n,\alpha,\delta,R} \sqrt{\mathfrak{A}_2^\alpha}$ by an easy argument, and by part (1) of

Theorem 1 in [27] applied to the α-fractional singular integral $T^{\alpha}_{\delta,R}$ we have

$$\mathfrak{N}_{T^{\alpha}_{\delta,R}} \leq C_{\alpha}\left(\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}_{\delta,R}} + \mathfrak{T}^{*}_{T^{\alpha}_{\delta,R}} + \mathcal{E}^{\text{strong}}_{\alpha} + \mathcal{E}^{\text{strong},*}_{\alpha} + \mathcal{WBP}_{T^{\alpha}_{\delta,R}}\right),$$

with C_{α} independent of δ and R. We obtain from the Good-λ Lemma applied to $T^{\alpha}_{\sigma,\delta,R}$ in place of T^{α},

$$\mathcal{WBP}_{T^{\alpha}_{\delta,R}} \leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}_{\delta,R}} + \mathfrak{T}^{*}_{T^{\alpha}_{\delta,R}} + \mathcal{E}^{\text{strong}}_{\alpha} + \mathcal{E}^{\text{strong},*}_{\alpha} + \sqrt[4]{\lambda}\,\mathfrak{N}_{T^{\alpha}_{\delta,R}}\right),$$

and then combining inequalities gives

$$\mathfrak{N}_{T^{\alpha}_{\delta,R}} \leq C'_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}_{\delta,R}} + \mathfrak{T}^{*}_{T^{\alpha}_{\delta,R}} + \mathcal{E}^{\text{strong}}_{\alpha} + \mathcal{E}^{\text{strong},*}_{\alpha} + \sqrt[4]{\lambda}\,\mathfrak{N}_{T^{\alpha}_{\delta,R}}\right),$$

with C'_{α} independent of δ and R. Since $\mathfrak{N}_{T^{\alpha}_{\delta,R}} < \infty$, we can absorb the term $C'_{\alpha}\sqrt[4]{\lambda}\,\mathfrak{N}_{T^{\alpha}_{\delta,R}}$ on the right hand side above into the left hand side for $\lambda > 0$ sufficiently small. Since $T^{\alpha}_{\delta,R}$ is an arbitrary tangent line approximation to T^{α}, the proof of Theorem 2 is complete. □

The first case of the following $T1$ theorem was proved in [26], and the second case is a corollary of Theorem 2 above and Theorem 2 in [27].

Theorem 3 *Suppose* $0 \leq \alpha < n$, *that* T^{α} *is a standard* α*-fractional singular integral operator on* $\mathbb{R}^n$, *and that* ω *and* σ *are locally finite positive Borel measures on* $\mathbb{R}^n$. *Set* $T^{\alpha}_{\sigma}f = T^{\alpha}(f\sigma)$ *for any smooth truncation of* T^{α}_{σ}. *Let* $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ *be a globally biLipschitz map. Then*

$$\mathfrak{N}_{T^{\alpha}_{\sigma}} \approx \sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^{*}_{T^{\alpha}}\ ,$$

in the following two cases:

(1) when T^{α} is a strongly elliptic standard α-fractional singular integral operator on $\mathbb{R}^n$, and one of the weights σ or ω is supported on a compact $C^{1,\delta}$ curve in $\mathbb{R}^n$,
(2) when $T^{\alpha} = \mathbf{R}^{\alpha}$ is the vector of α-fractional Riesz transforms, and both weights σ and ω are k-energy dispersed where $0 \leq k \leq n-1$ satisfies

$$\begin{cases} n-k < \alpha < n,\ \alpha \neq n-1 & \text{if } \ 1 \leq k \leq n-2 \\ 0 \leq \alpha < n,\ \alpha \neq 1, n-1 & \text{if } \quad k = n-1 \end{cases}.$$

There is a further corollary that can be easily obtained, namely a **two weight** accretive global Tb theorem whenever a two weight $T1$ theorem holds for strictly comparable weight pairs. We say that two weight pairs (σ,ω) and $(\widetilde{\sigma},\widetilde{\omega})$ are *strictly comparable* if $\widetilde{\sigma} = h_1\sigma$ and $\widetilde{\omega} = h_2\omega$ where each h_i is a function bounded between two positive constants. The simple proof of the following accretive global

Tb theorem uses only the *statement* of a related $T1$ theorem. We say that a complex-valued function b is *accretive* on $\mathbb{R}^n$ if

$$0 < c_b \leq \operatorname{Re} b(x) \leq |b(x)| \leq C_b < \infty, \quad x \in \mathbb{R}^n .$$

Theorem 4 *Suppose* $0 \leq \alpha < n$, *that* T^α *is a standard* α*-fractional singular integral operator on* $\mathbb{R}^n$, *and that* ω *and* σ *are locally finite positive Borel measures on* $\mathbb{R}^n$ *for which we have the 'T1 theorem' for strictly comparable weight pairs, i.e.*

$$\mathfrak{N}_{T^\alpha_\sigma}(\widetilde{\sigma},\widetilde{\omega}) \approx \sqrt{\mathfrak{A}^\alpha_2(\widetilde{\sigma},\widetilde{\omega})} + \mathfrak{T}_{T^\alpha}(\widetilde{\sigma},\widetilde{\omega}) + \mathfrak{T}^*_{T^\alpha}(\widetilde{\sigma},\widetilde{\omega}), \tag{12}$$

whenever (σ,ω) *and* $(\widetilde{\sigma},\widetilde{\omega})$ *are strictly comparable. Finally, let* b *and* b^* *be two accretive functions on* $\mathbb{R}^n$. *Then the best constant* $\mathfrak{N}_{T^\alpha_\sigma} = \mathfrak{N}_{T^\alpha_\sigma}(\sigma,\omega)$ *in the two weight norm inequality*

$$\|T^\alpha_\sigma f\|_{L^2(\omega)} \leq \mathfrak{N}_{T^\alpha_\sigma} \|f\|_{L^2(\sigma)} ,$$

taken uniformly over tangent line truncations of T^α, *satisfies*

$$\mathfrak{N}_{T^\alpha_\sigma} \approx \sqrt{\mathfrak{A}^\alpha_2} + \mathfrak{T}^b_{T^\alpha} + \mathfrak{T}^{b^*,*}_{T^\alpha}, \tag{13}$$

where the two dual b-testing conditions for T^α *are given by*

$$\int_Q \left|T^\alpha_\sigma(\mathbf{1}_Q b)\right|^2 d\omega \leq \mathfrak{T}^b_{T^\alpha} |Q|_\sigma , \quad \text{for all cubes } Q,$$

$$\int_Q \left|T^{\alpha,*}_\omega(\mathbf{1}_Q b^*)\right|^2 d\sigma \leq \mathfrak{T}^{b^*,*}_{T^\alpha} |Q|_\omega , \quad \text{for all cubes } Q,$$

and where we interpret the left sides above as holding uniformly over all tangent line truncations of T^α.

Note that Theorem 4 applies in particular to both cases (1) and (2) of Theorem 3.

Proof We first note that since the kernel K^α is real-valued,

$$\int_Q \left|T^\alpha_\sigma(\mathbf{1}_Q \operatorname{Re} b)\right|^2 d\omega = \int_Q \left|\operatorname{Re} T^\alpha_\sigma(\mathbf{1}_Q b)\right|^2 d\omega \leq \int_Q \left|T^\alpha_\sigma(\mathbf{1}_Q b)\right|^2 d\omega \leq \mathfrak{T}^b_{T^\alpha} |Q|_\sigma ,$$

$$\int_Q \left|T^{\alpha,*}_\omega(\mathbf{1}_Q \operatorname{Re} b^*)\right|^2 d\sigma = \int_Q \left|\operatorname{Re} T^{\alpha,*}_\omega(\mathbf{1}_Q b^*)\right|^2 d\sigma \leq \int_Q \left|T^{\alpha,*}_\omega(\mathbf{1}_Q b^*)\right|^2 d\sigma \leq \mathfrak{T}^{b^*,*}_{T^\alpha} |Q|_\omega ,$$

and if we now define measures

$$\widetilde{\omega} \equiv \left(\operatorname{Re} b^*\right)\omega \text{ and } \widetilde{\sigma} \equiv (\operatorname{Re} b)\,\sigma ,$$

we see that the operator T^α and the weight pair $(\widetilde{\sigma},\widetilde{\omega})$ satisfy (12). But it follows that $\mathfrak{T}_{T^\alpha}(\widetilde{\sigma},\widetilde{\omega}) \leq \mathfrak{T}^b_{T^\alpha}(\sigma,\omega)$ and $\mathfrak{T}^*_{T^\alpha}(\widetilde{\sigma},\widetilde{\omega}) \leq \mathfrak{T}^{b^*,*}_{T^\alpha}(\sigma,\omega)$, and since the Muckenhoupt A_2 conditions are clearly comparable for strictly comparable weight pairs, we have the equivalence

$$\mathfrak{N}_{T^\alpha_\sigma}(\widetilde{\sigma},\widetilde{\omega}) \approx \sqrt{\mathfrak{A}^\alpha_2(\sigma,\omega)} + \mathfrak{T}^b_{T^\alpha}(\sigma,\omega) + \mathfrak{T}^{b^*,*}_{T^\alpha}(\sigma,\omega).$$

Finally, since $0 < c \leq \operatorname{Re} b, \operatorname{Re} b^* \leq C$, we see that $\mathfrak{N}_{T^\alpha_\sigma}(\widetilde{\sigma},\widetilde{\omega}) \approx \mathfrak{N}_{T^\alpha_\sigma}(\sigma,\omega)$, and this completes the proof of (13). □

Note that the presence of a (b,b^*)-variant of the weak boundedness property here would complicate matters, since in general,

$$\operatorname{Re}\int_Q T^\alpha\left(1_{Q'}b\sigma\right)b^* d\omega \neq \int_Q T^\alpha\left(1_{Q'}\operatorname{Re} b\sigma\right)\operatorname{Re} b^* d\omega.$$

To remind the reader of the versatility of even a *global Tb* theorem, we reproduce a proof of the boundedness of the Cauchy integral on $C^{1,\delta}$ curves.

2.1.1 Boundedness of the Cauchy Integral on $C^{1,\delta}$ Curves

Here we point out how the above *Tb* theorem can apply to obtain the boundedness of the Cauchy integral on $C^{1,\delta}$ curves in the plane (which can be obtained in many other easy ways as well, see e.g. [31, Sect. 4 of Chap. VII]). Recall that the problem reduces to boundedness on $L^2(\mathbb{R})$ of the singular integral operator C_A with kernel

$$K_A(x,y) \equiv \frac{1}{x-y+i(A(x)-A(y))},$$

where the curve has graph $\{x+iA(x) : x\in\mathbb{R}\}$. Now $b(x)\equiv 1+iA'(x)$ is accretive and we have the b-testing condition

$$\int_I |C_A(\mathbf{1}_I b)(x)|^2\,dx \leq \mathfrak{T}^b_H |I|,$$

and its dual. Indeed, if $I=[\alpha,\beta]$, then

$$\begin{aligned}
C_A(\mathbf{1}_I b)(x) &= \int_\alpha^\beta \frac{1+iA'(y)}{x-y+i(A(x)-A(y))}dy \\
&= -\log(x-y+i(A(x)-A(y)))\,|_\alpha^\beta \\
&= \log\left(\frac{x-\alpha+i(A(x)-A(\alpha))}{x-\beta+i(A(x)-A(\beta))}\right),
\end{aligned}$$

gives

$$\left|C_A\left(\mathbf{1}_I b\right)(x)\right|^2 \approx \ln \frac{x-\alpha}{\beta-x}, \qquad x \in I = [\alpha, \beta],$$

and it follows that

$$\begin{aligned}\int_I \left|C_A\left(\mathbf{1}_I b\right)(x)\right|^2 dx &\approx \int_I \left|\ln \frac{x-\alpha}{\beta-x}\right|^2 dx \approx \int_0^{\beta-\alpha} \left|\ln \frac{x}{\beta-\alpha}\right|^2 dx \\ &= (\beta-\alpha)\int_0^1 |\ln w|^2 dw = C|I|.\end{aligned}$$

Since the kernel K_A is $C^{1,\delta}$, the *Tb* theorem above applies with $T = C_A$ and $\sigma = \omega = dx$ Lebesgue measure, to show that C_A is bounded on $L^2(\mathbb{R})$. Of course this proof just misses the case of Lipschitz curves since our two weight *Tb* theorem does not apply to kernels that fail to be $C^{1,\delta}$.

3 Proof of the Good-λ Lemma

We will prove the Good-λ Lemma by first replacing the quasiweak boundedness constant on the left hand side of (11) with the indicator/touching constant introduced in (6) above. To control the indicator/touching constant, we will need to tweak the usual good/bad technology of NTV a bit in the following subsection.

3.1 Good/Bad Technology

First we recall the good/bad cube technology of Nazarov, Treil and Volberg [32] as in [25], but with a small simplification introduced in the real line by Hytönen in [6]. This simplification does not impact the validity of the arguments in [30], but will facilitate the use of NTV surgery in later subsections.

Following [6], we momentarily fix a large positive integer $M \in \mathbb{N}$, and consider the tiling of $\mathbb{R}^n$ by the family of cubes $\mathbb{D}_M \equiv \left\{I_\alpha^M\right\}_{\alpha \in Z^n}$ having side length 2^{-M} and given by $I_\alpha^M \equiv I_0^M + 2^{-M}\alpha$ where $I_0^M = \left[0, 2^{-M}\right)^n$. A *dyadic grid* $\mathcal{D}$ built on $\mathbb{D}_M$ is defined to be a family of cubes $\mathcal{D}$ satisfying:

(1) Each $I \in \mathcal{D}$ has side length $2^{-\ell}$ for some $\ell \in \mathbb{Z}$ with $\ell \leq M$, and I is a union of $2^{n(M-\ell)}$ cubes from the tiling $\mathbb{D}_M$,
(2) For $\ell \leq M$, the collection $\mathcal{D}_\ell$ of cubes in $\mathcal{D}$ having side length $2^{-\ell}$ forms a pairwise disjoint decomposition of the space $\mathbb{R}^n$,
(3) Given $I \in \mathcal{D}_i$ and $J \in \mathcal{D}_j$ with $j \leq i \leq M$, it is the case that either $I \cap J = \emptyset$ or $I \subset J$.

We now momentarily fix a *negative* integer $N \in -\mathbb{N}$, and restrict the above grids to cubes of side length at most 2^{-N}:

$$\mathcal{D}^N \equiv \left\{ I \in \mathcal{D} : \text{side length of } I \text{ is at most } 2^{-N} \right\}.$$

We refer to such grids $\mathcal{D}^N$ as a (truncated) dyadic grid $\mathcal{D}$ built on $\mathbb{D}_M$ of size 2^{-N}. There are now two traditional means of constructing probability measures on collections of such dyadic grids.

Construction #1: Consider first the special case of dimension $n = 1$. Then for any

$$\beta = \{\beta_i\}_{i \in_M^N} \in \omega_M^N \equiv \{0, 1\}^{\mathbb{Z}_M^N},$$

where $\mathbb{Z}_M^N \equiv \{\ell \in \mathbb{Z} : N \le \ell \le M\}$, define the dyadic grid $\mathcal{D}_\beta$ built on $\mathbb{D}_M$ of size 2^{-N} by

$$\mathcal{D}_\beta = \left\{ 2^{-\ell} \left([0, 1) + k + \sum_{i:\ \ell < i \le M} 2^{-i+\ell} \beta_i \right) \right\}_{N \le \ell \le M,\, k \in \mathbb{Z}}.$$

Place the uniform probability measure ρ_M^N on the finite index space $\omega_M^N = \{0, 1\}^{\mathbb{Z}_M^N}$, namely that which charges each $\beta \in \omega_M^N$ equally. This construction is then extended to Euclidean space $\mathbb{R}^n$ by taking products in the usual way and using the product index space $\Omega_M^N \equiv \left(\omega_M^N\right)^n$ and the uniform product probability measure $\mu_M^N = \rho_M^N \times \ldots \times \rho_M^N$.

Construction #2: Momentarily fix a (truncated) dyadic grid $\mathcal{D}$ built on $\mathbb{D}_M$ of size 2^{-N}. For any

$$\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma_M^N \equiv \left\{ 2^{-M} \mathbb{Z}_+^n : |\gamma_i| < 2^{-N} \right\},$$

where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, define the dyadic grid $\mathcal{D}^\gamma$ built on $\mathbb{D}_M$ of size 2^{-N} by

$$\mathcal{D}^\gamma \equiv \mathcal{D} + \gamma.$$

Place the uniform probability measure ν_M^N on the finite index set Γ_M^N, namely that which charges each multiindex γ in Γ_M^N equally.

The two probability spaces $\left(\left\{ \mathcal{D}_\beta \right\}_{\beta \in \Omega_M^N}, \mu_M^N \right)$ and $\left(\{\mathcal{D}^\gamma\}_{\gamma \in \Gamma_M^N}, \nu_M^N \right)$ are isomorphic since both collections $\left\{ \mathcal{D}_\beta \right\}_{\beta \in \Omega_M^N}$ and $\{\mathcal{D}^\gamma\}_{\gamma \in \Gamma_M^N}$ describe the set $\boldsymbol{A}_M^N$ of **all** (truncated) dyadic grids $\mathcal{D}^\gamma$ built on $\mathbb{D}_M$ of size 2^{-N}, and since both measures μ_M^N and ν_M^N are the uniform measure on this space. Indeed, it suffices to verify this in the case $n = 1$. The first construction may be thought of as being *parameterized by scales*—each component β_i in $\beta = \{\beta_i\}_{i \in_M^N} \in \omega_M^N$ amounting to a choice of the two possible tilings at level i that respect the choice of tiling at the level below—and since any grid in $\boldsymbol{A}_M^N$ is determined by a choice of scales , we see that $\{\mathcal{D}_\beta\}_{\beta\in\Omega_M^N} = \boldsymbol{A}_M^N$. The second construction may be thought of as being *parameterized by translation*—each $\gamma \in \Gamma_M^N$ amounting to a choice of translation of the grid $\mathcal{D}$ fixed in construction #2–and since any grid in $\boldsymbol{A}_M^N$ is determined by any of the cubes at the top level, i.e. with side length 2^{-N}, we see that $\{\mathcal{D}^\gamma\}_{\gamma\in\Gamma_M^N} = \boldsymbol{A}_M^N$ as well, since every cube at the top level in $\boldsymbol{A}_M^N$ has the form $Q+\gamma$ for some $\gamma \in \Gamma_M^N$ and $Q \in \mathcal{D}$ at the top level in $\boldsymbol{A}_M^N$ (i.e. every cube at the top level in $\boldsymbol{A}_M^N$ is a union of small cubes in $\mathbb{D}_M$, and so must be a translate of some $Q \in \mathcal{D}$ by an amount 2^{-M} times an element of $\mathbb{Z}_+^n$). Note also that in all dimensions, $\#\Omega_M^N = \#\Gamma_M^N = 2^{n(M-N)}$. We will use $\mathbb{E}_{\Omega_M^N}$ to denote expectation with respect to this common probability measure on $\boldsymbol{A}_M^N$.

The usual NTV probabilistic reduction to 'good' cubes will be implemented below for each positive integer M and each negative integer N assuming that the functions f and g are supported in a large cube L with $\int_L f d\sigma = 0 = \int_L g d\omega$, and moreover assuming that $-N$ is sufficiently large compared to $\ell(L)$ that the small probability estimates claimed below hold ($-N > \ell(L) + \mathbf{r}$ will work where $\mathbf{r}$ is the goodness constant), and finally assuming that f and g are constant on each cube Q in the tiling $\mathbb{D}_M$. Recall that we can always reduce to the case $\int_L f d\sigma = 0 = \int_L g d\omega$ by simply subtracting off averages and controlling the resulting error terms by the testing conditions (see e.g. [32]).

Notation 2 *For purposes of notation and clarity, we often suppress all reference to M and N in our families of grids, and in the notations Ω and Γ for the parameter sets, and we will use $\mathbb{P}_\Omega$ and $\mathbb{E}_\Omega$ to denote probability and expectation, and instead proceed as if all grids considered are unrestricted. The careful reader can supply the modifications necessary to handle the assumptions made above on the grids $\mathcal{D}$ and the functions f and g regarding M and N. In fact, we will exploit the integers M and N explicitly in the subsubsections on NTV surgery below.*

In the case of one independent family of grids, as is the case here, the main result is the following *conditional* probability estimate: for every $I \in \mathcal{P}^n$,

$$\mathbb{P}_\Omega\{\mathcal{D} : I \text{ is a } \textit{bad} \text{ cube in } \mathcal{D} \mid I \in \mathcal{D}\} \leq C2^{-\varepsilon\mathbf{r}}. \tag{14}$$

Provided we obtain estimates independent of M and N, this will be sufficient for our proof—this follows the procedure with *two* independent grids initiated by Hytönen for the Hilbert transform inequality in [6]. The key point of introducing the two different parameterizations above of the same probability space, is that construction #1 is well-adapted to the reduction to good cubes in a *single* independent family of grids, as used in the proof of the main theorem in [30], which is in turn needed below, while construction #2 facilitates the use of NTV surgery below when combined with the construction of Q-good *grids,* to which we next turn.

3.1.1 Q-Good Quasicubes and Q-Good Quasigrids

We first introduce these notions for usual cubes, and later pass to quasicubes. Let $Q \in \mathcal{P}^n$ be an arbitrary cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. For technical reasons associated to our application below, we also want to consider the 'siblings' of Q, i.e. the 'triadic children' of $3Q$.

Definition 6 We say that a cube $I \in \mathcal{P}^n$ is *Q-good* if either $\ell(I) > 2^{-\rho}\ell(Q)$, or for every sibling Q' of Q, we have

$$\operatorname{dist}\left(I, \partial Q'\right) \geq \frac{1}{2}\ell(I)^{\varepsilon}\,\ell\left(Q'\right)^{1-\varepsilon}$$

when $\ell(I) \leq 2^{-\rho}\ell(Q)$. We say $I \in \mathcal{P}^n$ is *Q-bad* if I is **not** Q-good.

Note that for a fixed cube $Q \in \mathcal{P}^n$, we do **not** have a conditional probability estimate $\mathbb{P}_{\Omega}\{\mathcal{D} : I \in \mathcal{D} \text{ and } I \text{ is } Q\text{-bad}\} \leq C2^{-\varepsilon\mathbf{r}}$ since the property of a cube I being Q-bad is independent of which grids $\mathcal{D}$ it belongs to. To rectify this complication we will introduce below a *second independent* family of grids—but this second family will also be used to simultaneously Haar-decompose both $f \in L^2(\sigma)$ and $g \in L^2(\omega)$.[2]

We next wish to capture the idea of a grid $\mathcal{D}$ being '*Q-good*' with respect to this fixed cube Q, and the idea will be to require that Q is I-good for all sufficiently larger cubes I in the grid $\mathcal{D}$. Here we *will* obtain a 'goodness' estimate in Lemma 2 below.

Definition 7 Let $\mathbf{r}$ and ε be goodness constants as in [25]. For $Q \in \mathcal{P}^n$ we declare a grid $\mathcal{D}$ to be *Q-good* if for every sibling Q' of Q and for every $I \in \mathcal{D}$ with $\ell(I) \geq 2^{\mathbf{r}}\ell(Q)$, the following holds: the distance from the cube Q' to the boundary of the cube I satisfies the 'deeply embedded' inequality,

$$\operatorname{dist}\left(Q', \partial I\right) \geq \frac{1}{2}\ell\left(Q'\right)^{\varepsilon}\ell(I)^{1-\varepsilon}.$$

We say the grid $\mathcal{D}$ is *Q-bad* if it is not Q-good.

Note that Q is fixed in this definition and it is easy to see, using the translation parameterization in construction #2 above, that the collection of grids $\mathcal{D}$ that are Q-bad occur with small probability. Indeed, if $I \supset Q$ has side length at least $2^{\mathbf{r}}$ times that of Q, then the translates of I satisfy $Q \Subset_{\mathbf{r}} I$ with probability near 1.

Lemma 2 *Fix a cube $Q \in \mathcal{P}^n$. Then $\mathbb{P}_{\Omega}\{\mathcal{D} : \mathcal{D} \text{ is } Q\text{-bad}\} \leq C2^{-\varepsilon\mathbf{r}}$.*

---

[2]Traditionally, two independent grids are applied to f and g separately, something we *avoid* since the treatment of functional energy in the arguments of [27, 30] (which we use here) relies on using a *common* grid for f and g.

The following is our tweaking of the good/bad technology of NTV [32]. Fix a cube $Q \in \mathcal{P}^n$ and let $\mathcal{D}$ be randomly selected. Define linear operators (depending on the grid $\mathcal{D}$),

$$\mathsf{P}^{\sigma}_{Q;\mathrm{good}} f \equiv \begin{cases} \sum_{I\in\mathcal{D}:\ I \text{ is } \mathbf{r}\text{-good in } \mathcal{D}} \triangle^{\sigma}_I f & \text{if } \mathcal{D} \text{ is } Q\text{-good} \\ 0 & \text{if } \mathcal{D} \text{ is } Q\text{-bad} \end{cases},$$

$$\mathsf{P}^{\sigma}_{Q;\mathrm{bad}} f \equiv f - \mathsf{P}^{\sigma}_{Q;\mathrm{good}} f ,$$

and likewise for $\mathsf{P}^{\omega}_{Q;\mathrm{good}} g$ and $\mathsf{P}^{\omega}_{Q;\mathrm{bad}} g$.

Proposition 1 *Fix a cube $Q \in \mathcal{P}^n$. Then we have the estimates*

$$\mathbb{E}_{\Omega} \left\| \mathsf{P}^{\sigma}_{Q;\mathrm{bad}} f \right\|_{L^2(\sigma)} \leq C 2^{-\frac{\varepsilon \mathbf{r}}{2}} \|f\|_{L^2(\sigma)} ,$$

$$\mathbb{E}_{\Omega} \left\| \mathsf{P}^{\omega}_{Q;\mathrm{bad}} g \right\|_{L^2(\omega)} \leq C 2^{-\frac{\varepsilon \mathbf{r}}{2}} \|g\|_{L^2(\omega)} .$$

Proof We have from (14) and Lemma 2 that

$$\begin{aligned}\mathbb{E}_{\Omega} \left\| \mathsf{P}^{\sigma}_{\mathrm{bad}} f \right\|^2_{L^2(\sigma)} &= \mathbb{E}_{\Omega} \left(\mathbf{1}_{\{\mathcal{D} \text{ is } Q\text{-good}\}} \sum_{I\in\mathcal{D} \text{ is bad}} \left\| \triangle^{\sigma}_I f \right\|^2_{L^2(\sigma)} \right) + \mathbb{E}_{\Omega} \left(\mathbf{1}_{\{\mathcal{D} \text{ is } Q\text{-bad}\}} \sum_{I\in\mathcal{D}} \left\| \triangle^{\sigma}_I f \right\|^2_{L^2(\sigma)} \right) \\ &\leq C 2^{-\varepsilon \mathbf{r}} \sum_{I\in\mathcal{D}} \left\| \triangle^{\sigma}_I f \right\|^2_{L^2(\sigma)} + \mathbb{E}_{\Omega} \left(\mathbf{1}_{\{\mathcal{D} \text{ is } Q\text{-bad}\}} \right) \sum_{I\in\mathcal{D}} \left\| \triangle^{\sigma}_I f \right\|^2_{L^2(\sigma)} \lesssim C 2^{-\varepsilon \mathbf{r}} \|f\|^2_{L^2(\sigma)} .\end{aligned}$$

□

From this we conclude that there is an absolute choice of $\mathbf{r}$ depending on $0 < \varepsilon < 1$ so that the following holds. Let $T : L^2(\sigma) \to L^2(\omega)$ be a bounded linear operator, and let $Q \in \mathcal{P}^n$ be a fixed cube. We then have

$$\|T\|_{L^2(\sigma)\to L^2(\omega)} \leq 2 \sup_{\|f\|_{L^2(\sigma)}=1} \sup_{\|g\|_{L^2(\omega)}=1} \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{good}} f, \mathsf{P}^{\omega}_{Q;\mathrm{good}} g \right\rangle_{\omega} \right| . \tag{15}$$

Indeed, we can choose $f \in L^2(\sigma)$ of norm one, and $g \in L^2(\omega)$ of norm one so that

$$\begin{aligned}\|T\|_{L^2(\sigma)\to L^2(\omega)} &= \langle Tf, g\rangle_{\omega} \\ &\leq \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{good}} f, \mathsf{P}^{\omega}_{Q;\mathrm{good}} g \right\rangle_{\omega} \right| + \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{bad}} f, \mathsf{P}^{\omega}_{Q;\mathrm{good}} g \right\rangle_{\omega} \right| \\ &\quad + \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{good}} f, \mathsf{P}^{\omega}_{Q;\mathrm{bad}} g \right\rangle_{\omega} \right| + \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{bad}} f, \mathsf{P}^{\omega}_{Q;\mathrm{bad}} g \right\rangle_{\omega} \right| \\ &\leq \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{good}} f, \mathsf{P}^{\omega}_{Q;\mathrm{good}} g \right\rangle_{\omega} \right| + 3C \cdot 2^{-\frac{\mathbf{r}\varepsilon}{16}} \|T\|_{L^2(\sigma)\to L^2(\omega)} ,\end{aligned}$$

And this proves (15) for $\mathbf{r}$ sufficiently large depending on $\varepsilon > 0$.

Clearly, all of this extends automatically to the quasiworld.

Implication: *Given a quasicube $Q \in \Omega\mathcal{P}^n$, it suffices to consider only Q-good quasigrids and Q-good quasicubes in these quasigrids, and to prove an estimate for $\|T_\sigma\|_{L^2(\sigma)\to L^2(\omega)}$ that is independent of these assumptions.*

3.2 Control of the Indicator/Touching Property

Recall the indicator/touching constant $\mathfrak{I}_{T^\alpha}$ defined in (6) above. Here we will prove that

$$\mathfrak{I}_{T^\alpha} \leq C_\alpha \left(\frac{1}{\lambda} \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} + \sqrt[4]{\lambda}\mathfrak{N}_{T^\alpha} \right), \tag{16}$$

from which it easily follows that we have the same inequality for the weak boundedness property constant $\mathcal{WBP}_{T^\alpha}$ defined in (5) above,

$$\mathcal{WBP}_{T^\alpha} \leq C_\alpha \left(\frac{1}{\lambda} \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} + \sqrt[4]{\lambda}\mathfrak{N}_{T^\alpha} \right). \tag{17}$$

Indeed an elementary argument shows that $\mathcal{WBP}_{T^\alpha} \lesssim \mathfrak{I}_{T^\alpha} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha}$. For the proof of (16) we assume the reader is already familiar with the proof of the main theorem in [30] or [27], and we now review the parts of this proof that are pertinent here.

We first recall the basic setup in [30]. Let $\Omega\mathcal{D}^\sigma = \Omega\mathcal{D}^\omega$ be a quasigrid on $\mathbb{R}^n$, and let $\left\{h_I^{\sigma,a}\right\}_{I\in\Omega\mathcal{D}^\sigma,\ a\in\Gamma_n}$ and $\left\{h_J^{\omega,b}\right\}_{J\in\Omega\mathcal{D}^\omega,\ b\in\Gamma_n}$ be corresponding quasiHaar bases, so that $f \in L^2(\sigma)$ and $g \in L^2(\omega)$ can be written $f = f_{\text{good}} + f_{\text{bad}}$ and $g = g_{\text{good}} + g_{\text{bad}}$ where

$$f = \sum_{I\in\Omega\mathcal{D}^\sigma} \triangle_I^\sigma f \text{ and } g = \sum_{J\in\Omega\mathcal{D}^\omega} \triangle_J^\omega g\ ,$$

$$f_{\text{good}} = \sum_{I\in\Omega\mathcal{D}^\sigma_{\text{good}}} \triangle_I^\sigma f \text{ and } g_{\text{good}} = \sum_{J\in\Omega\mathcal{D}^\omega_{\text{good}}} \triangle_J^\omega g\ ,$$

and where $\Omega\mathcal{D}^\sigma_{\text{good}} = \Omega\mathcal{D}^\omega_{\text{good}}$ is the $(\mathbf{r},\varepsilon)$-good subgrid, and where the quasiHaar projections $\triangle_I^\sigma f_{\text{good}}$ and $\triangle_J^\omega g_{\text{good}}$ *vanish* if the quasicubes I and J are not good in $\Omega\mathcal{D}^\sigma = \Omega\mathcal{D}^\omega$. Note that we use a *single* independent family of grids $\Omega\mathcal{D}^\sigma = \Omega\mathcal{D}^\omega$ and only include the different superscripts σ and ω to emphasize which measure the grid is being used with in a given situation.

Remark 2 In [27] and [30], the quasiHaar projections $\triangle_I^\sigma f_{\text{good}}$ and $\triangle_J^\omega g_{\text{good}}$ are required to vanish if the quasicubes I and J are not $\boldsymbol{\tau}$-good in $\Omega\mathcal{D}^\sigma = \Omega\mathcal{D}^\omega$, where a quasicube I is $\boldsymbol{\tau}$-good in a quasigrid $\Omega\mathcal{D}$ if I together with its children and its ancestors up to order $\boldsymbol{\tau}$ are all good. This more restrictive condition doesn't affect what is done here.

For future reference note that the argument in [30] applies just as well to the smaller projections $\mathsf{P}^{\sigma}_{Q;\text{good}}f$ and $\mathsf{P}^{\omega}_{Q;\text{good}}g$ in place of f_{good} and g_{good} respectively. We fix $f = f_{\text{good}}$ and $g = g_{\text{good}}$. For now we continue to work with general functions f and g and the projections f_{good} and g_{good}, but keeping in mind that in order to prove (16), we will later specialize to the cases of indicator functions $f = \mathbf{1}_Q$ and $g = \mathbf{1}_R$, and we will then also include the restriction to Q-good grids $\Omega\mathcal{D}_{Q;\text{good}}$ and projections $\mathsf{P}^{\sigma}_{Q;\text{good}}f$ and $\mathsf{P}^{\omega}_{Q;\text{good}}g$ for a fixed quasicube Q - the quasicube Q in the projection $\mathsf{P}^{\sigma}_{Q;\text{good}}f$ is chosen to coincide with the quasicube Q in the indicator $\mathbf{1}_Q$ in order to achieve the three critical reductions in Sect. 3.2.1 below. Continuing with [27, 30], we then proved there the bilinear inequality

$$|\mathcal{T}^{\alpha}(f,g)| = \left| \sum_{I \in \Omega\mathcal{D}^{\sigma}_{\text{good}} \text{ and } J \in \Omega\mathcal{D}^{\omega}_{\text{good}}} \mathcal{T}^{\alpha}\left(\Delta^{\sigma}_{I} f, \Delta^{\omega}_{J} g\right) \right| \tag{18}$$
$$\leq C_{\alpha}\left(\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^{*}_{T^{\alpha}} + \mathcal{E}^{\text{strong}}_{\alpha} + \mathcal{E}^{\text{strong},*}_{\alpha} + \mathcal{WBP}_{T^{\alpha}}\right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)},$$

uniformly over grids $\mathcal{D}$, and we now discuss the salient features of this proof for us.

As in [27, 30] let

$$\mathcal{NTV}_{\alpha} \equiv \sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^{*}_{T^{\alpha}} + \mathcal{WBP}_{T^{\alpha}},$$
$$\mathfrak{A}^{\alpha}_{2} \equiv \mathcal{A}^{\alpha}_{2} + \mathcal{A}^{\alpha,*}_{2} + A^{\alpha,\text{punct}}_{2} + A^{\alpha,*,\text{punct}}_{2},$$

and recall the following brief schematic diagram of the decompositions involved in the proof given in [30], with bounds in $\Box$:

$$\boxed{\begin{array}{lllllll}
\langle T^{\alpha}_{\sigma} f, g\rangle_{\omega} & & & & & & \\
\downarrow & & & & & & \\
\mathsf{B}_{\Subset_{\rho}}(f,g) & + & \mathsf{B}_{_{\rho}\Supset}(f,g) & + & \mathsf{B}_{\cap}(f,g) & + & \mathsf{B}_{\diagup}(f,g) \\
\downarrow & & \boxed{\text{dual}} & & \boxed{\mathcal{NTV}_{\alpha}} & & \boxed{\mathcal{NTV}_{\alpha}} \\
\downarrow & & & & & & \\
\mathsf{T}_{\text{diagonal}}(f,g) & + & \mathsf{T}_{\text{far below}}(f,g) & + & \mathsf{T}_{\text{far above}}(f,g) & + & \mathsf{T}_{\text{disjoint}}(f,g) \\
\downarrow & & \downarrow & & \boxed{\emptyset} & & \boxed{\emptyset} \\
\downarrow & & \downarrow & & & & \\
\mathsf{B}^{A}_{\Subset_{\rho}}(f,g) & & \mathsf{T}^{1}_{\text{far below}}(f,g) & + & \mathsf{T}^{2}_{\text{far below}}(f,g) & & \\
\downarrow & & \boxed{\mathcal{NTV}_{\alpha} + \mathcal{E}^{\text{strong}}_{\alpha}} & & \boxed{\mathcal{NTV}_{\alpha}} & & \\
\downarrow & & & & & & \\
\mathsf{B}^{A}_{stop}(f,g) & + & \mathsf{B}^{A}_{paraproduct}(f,g) & + & \mathsf{B}^{A}_{neighbour}(f,g) & & \\
\boxed{\mathcal{E}^{\text{strong}}_{\alpha} + \sqrt{A^{\alpha}_{2}}} & & \boxed{\mathfrak{T}_{T^{\alpha}}} & & \boxed{\sqrt{A^{\alpha}_{2}}} & &
\end{array}}$$

With reference to this diagram, we now make a sweeping and crucial claim.

The **only** two places in our proof of the main theorem in [30] where the **weak boundedness property** $\mathcal{WBP}_{T^\alpha}$ is used, is

(1) in proving the estimates for terms A_1 and A_2 involving $\left\langle T^\alpha_\sigma\left(\triangle^\sigma_I f\right), \triangle^\omega_J g\right\rangle_\omega$ that arise in estimating the form $\mathsf{B}_{\diagup}(f,g)$ at the top right of the schematic diagram, and
(2) and in the estimates for the inner products $\left\langle T^\alpha_\sigma\left(\triangle^\sigma_I f\right), \triangle^\omega_J g\right\rangle_\omega$ in the form $\mathsf{T}^2_{\text{far below}}(f,g)$ for which I are J are close in both scale and position,
(3) and *even then* in these two cases, only for certain child quasicubes I_θ and $J_{\theta'}$ when they *touch*, i.e. their interiors are disjoint but their closures intersect (even in just a point). In all other instances where $\mathcal{NTV}_\alpha$ appears in the schematic diagram, the weak boundedness property is *not* used.

In order to make the application of the quasiweak boundedness property in these arguments clear, we reproduce the relevant portions of the arguments from [30] that deal with the forms $\mathsf{B}_{\diagup}(f,g)$ and $\mathsf{T}^2_{\text{far below}}(f,g)$. Recall also that the parameters $\boldsymbol{\rho}, \boldsymbol{\tau}, \mathbf{r}$ in [30, Definition 12 on p. 40] were fixed to satisfy

$$\boldsymbol{\tau} > \mathbf{r} \text{ and } \boldsymbol{\rho} > \boldsymbol{\tau} + \mathbf{r}\,.$$

1: Here is the beginning of the proof of (6.1) on page 28 dealing with $\mathsf{B}_{\diagup}(f,g)$ in the statement of Lemma 9 in [30].

Extract from pages 28 and 29 of [30]:

Note that in (6.1) we have used the parameter $\boldsymbol{\rho}$ in the exponent rather than $\mathbf{r}$, and this is possible because the arguments we use here only require that there are finitely many levels of scale separating I and J. To handle this term we first decompose it into

$$\left\{ \sum_{\substack{(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega:\ J\subset 3I \\ 2^{-\boldsymbol{\rho}}\ell(I)\leq\ell(J)\leq 2^{\boldsymbol{\rho}}\ell(I)}} + \sum_{\substack{(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega:\ I\subset 3J \\ 2^{-\boldsymbol{\rho}}\ell(I)\leq\ell(J)\leq 2^{\boldsymbol{\rho}}\ell(I)}} + \sum_{\substack{(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega \\ 2^{-\boldsymbol{\rho}}\ell(I)\leq\ell(J)\leq 2^{\boldsymbol{\rho}}\ell(I) \\ J\not\subset 3I \text{ and } I\not\subset 3J}} \right\} \left|\left\langle T^\alpha_\sigma\left(\triangle^\sigma_I f\right), \triangle^\omega_J g\right\rangle_\omega\right|$$
$$\equiv A_1 + A_2 + A_3.$$

The proof of the bound for term A_3 is similar to that of the bound for the left side of (6.2), and so we will defer the bound for A_3 until after (6.2) has been proved.

We now consider term A_1 as term A_2 is symmetric. To handle this term we will write the quasiHaar functions h^σ_I and h^ω_J as linear combinations of the indicators of the children of their supporting quasicubes, denoted I_θ and $J_{\theta'}$ respectively. Then we use the quasitesting condition on I_θ and $J_{\theta'}$ when they overlap, i.e. their interiors intersect; we use the quasiweak boundedness property on I_θ and $J_{\theta'}$ when they touch, i.e. their interiors are disjoint but their closures intersect (even in just a point); and finally we use the A^α_2 condition when I_θ and $J_{\theta'}$ are separated, i.e. their closures are disjoint. We will suppose initially that the side length of J is at most *the side length I, i.e.* $\ell(J) \leq \ell(I)$, *the proof for* $J = \pi I$ *being similar but for one point mentioned below. So suppose that* I_θ *is a child of* I *and that* $J_{\theta'}$ *is a child of* J. *If* $J_{\theta'} \subset I_\theta$ *we have from (9) that,*

$$\begin{aligned}
\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\,\triangle_I^\sigma f\right),\mathbf{1}_{J_{\theta'}}\,\triangle_J^\omega g\right\rangle_\omega\right| &\lesssim \sup_{a,a'\in\Gamma_n}\frac{\left|\left\langle f,h_I^{\sigma,a}\right\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}}\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right),\mathbf{1}_{J_{\theta'}}\right\rangle_\omega\right|\frac{\left|\left\langle g,h_J^{\omega,a'}\right\rangle_\omega\right|}{\sqrt{|J_{\theta'}|_\omega}}\\
&\lesssim \sup_{a,a'\in\Gamma_n}\frac{\left|\left\langle f,h_I^{\sigma,a}\right\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}}\left(\int_{J_{\theta'}}\left|T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right)\right|^2 d\omega\right)^{\frac{1}{2}}\left|\left\langle g,h_J^{\omega,a'}\right\rangle_\omega\right|\\
&\lesssim \sup_{a,a'\in\Gamma_n}\frac{\left|\left\langle f,h_I^{\sigma,a}\right\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}}\mathfrak{T}_{T_\alpha}\,|I_\theta|_\sigma^{\frac{1}{2}}\left|\left\langle g,h_J^{\omega,a'}\right\rangle_\omega\right|\\
&\lesssim \sup_{a,a'\in\Gamma_n}\mathfrak{T}_{T_\alpha}\left|\left\langle f,h_I^{\sigma,a}\right\rangle_\sigma\right|\left|\left\langle g,h_J^{\omega,a'}\right\rangle_\omega\right|.
\end{aligned}$$

The point referred to above is that when $J = \pi I$ *we write* $\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right),\mathbf{1}_{J_{\theta'}}\right\rangle_\omega = \left\langle\mathbf{1}_{I_\theta},T_\omega^{\alpha,*}\left(\mathbf{1}_{J_{\theta'}}\right)\right\rangle_\sigma$ *and get the dual quasitesting constant* $\mathfrak{T}_{T_\alpha}^*$. *If* $J_{\theta'}$ *and* I_θ *touch, then* $\ell(J_{\theta'}) \leq \ell(I_\theta)$ *and we have* $J_{\theta'} \subset 3I_\theta \setminus I_\theta$, *and so*

$$\begin{aligned}
\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\,\triangle_I^\sigma f\right),\mathbf{1}_{J_{\theta'}}\,\triangle_J^\omega g\right\rangle_\omega\right| &\lesssim \sup_{a,a'\in\Gamma_n}\frac{\left|\left\langle f,h_I^{\sigma,a}\right\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}}\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right),\mathbf{1}_{J_{\theta'}}\right\rangle_\omega\right|\frac{\left|\left\langle g,h_J^{\omega,a'}\right\rangle_\omega\right|}{\sqrt{|J_{\theta'}|_\omega}} \qquad (19)\\
&\lesssim \sup_{a,a'\in\Gamma_n}\frac{\left|\left\langle f,h_I^{\sigma,a}\right\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}}\mathcal{WBP}_{T^\alpha}\sqrt{|I_\theta|_\sigma\,|J_{\theta'}|_\omega}\frac{\left|\left\langle g,h_J^{\omega,a'}\right\rangle_\omega\right|}{\sqrt{|J_{\theta'}|_\omega}}\\
&= \sup_{a,a'\in\Gamma_n}\mathcal{WBP}_{T^\alpha}\left|\left\langle f,h_I^{\sigma,a}\right\rangle_\sigma\right|\left|\left\langle g,h_J^{\omega,a'}\right\rangle_\omega\right|.
\end{aligned}$$

The only place where the quasiweak boundedness property $\mathcal{WBP}_{T^\alpha}$ was used above was in the second line of the display (19) when we invoked

$$\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right),\mathbf{1}_{J_{\theta'}}\right\rangle_\omega\right| \leq \mathcal{WBP}_{T^\alpha}\sqrt{|I_\theta|_\sigma\,|J_{\theta'}|_\omega}$$

for quasicubes $I_\theta \in \mathfrak{C}(I)$ and $J_{\theta'} \in \mathfrak{C}(J)$ that touch.

2: Here is the beginning of the proof on page 41 that controls the form $\mathsf{T}_{\text{far below}}(f,g)$ in [30].

Extract from page 41 of [30]:

The far below term $T_{\text{far below}}(f,g)$ *is bounded using the Intertwining Proposition and the control of functional energy condition by the energy condition given in the* *next two sections. Indeed, assuming these two results, we have from $\tau < \rho$ that*

$$\begin{aligned}
\mathsf{T}_{\text{far below}}(f,g) &= \sum_{\substack{A,B\in\mathcal{A}\\ B\subsetneqq A}} \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega \\
&= \sum_{B\in\mathcal{A}} \sum_{A\in\mathcal{A}:\ B\subsetneqq A} \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\mathbf{fi}-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega \\
&= \sum_{B\in\mathcal{A}} \sum_{A\in\mathcal{A}:\ B\subsetneqq A} \sum_{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\tau-\text{shift}}} \left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega \\
&\quad - \sum_{B\in\mathcal{A}} \sum_{A\in\mathcal{A}:\ B\subsetneqq A} \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\mathbf{fi}-\text{shift}}\\ J\not\Subset_{\rho,\varepsilon} I}} \left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega \\
&= \mathsf{T}_{\text{far below}}^1(f,g) - \mathsf{T}_{\text{far below}}^2(f,g).
\end{aligned}$$

Now $\mathsf{T}_{\text{far below}}^2(f,g)$ is bounded by NTV_α by Lemma 9 since J is good if $\triangle_J^\omega g \neq 0$.

The only place where the quasiweak boundedness property $\mathcal{WBP}_{T^\alpha}$ was used above[3] was in bounding the inner products $\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega$ by Lemma 9 of [30] when in addition I and J were close in both scale and position, and this reduces to the previous extract from pages 28 and 29 of [30] treated above.

Thus we may split the sum in (18) as follows:

$$\begin{aligned}
\mathcal{T}^\alpha(f,g) &= \sum_{I\in\Omega\mathcal{D}_{\text{good}}^\sigma \text{ and } J\in\Omega\mathcal{D}_{\text{good}}^\omega} \mathcal{T}^\alpha\left(\triangle_I^\sigma f,\triangle_J^\omega g\right) \\
&= \left\{ \sum_{\substack{(I,J)\in\Omega\mathcal{D}_{\text{good}}^\sigma\times\Omega\mathcal{D}_{\text{good}}^\omega:\ J\subset 3I\\ 2^{-\rho}\ell(I)\le\ell(J)\le 2^\rho\ell(I)}} + \sum_{\substack{(I,J)\in\Omega\mathcal{D}_{\text{good}}^\sigma\times\Omega\mathcal{D}_{\text{good}}^\omega:\ J\subset 3I\\ 2^{-\rho}\ell(I)\le\ell(J)\le 2^\rho\ell(I)}} \right\} \mathcal{T}^\alpha\left(\triangle_I^\sigma f,\triangle_J^\omega g\right) \\
&\quad + \mathcal{R}^\alpha(f,g) \\
&\equiv \left\{A_1(f,g)+A_2(f,g)\right\} + \mathcal{R}^\alpha(f,g),
\end{aligned}$$

where we are including in the terms $A_1(f,g)+A_2(f,g)$ the corresponding inner products from the form $\mathsf{T}_{\text{far below}}^2(f,g)$ to which Lemma 9 of [30] was applied. Then

[3] On page 41 of [30], there was a typo in that $J \Subset_{\mathbf{j},\varepsilon} I$ appeared in the fourth line of the display instead of $J \not\Subset_{\rho,\varepsilon} I$ as corrected here.

the remainder form $\mathcal{R}^{\alpha}(f,g)$ satisfies the estimate

$$\left|\mathcal{R}^{\alpha}(f,g)\right| \leq C_{\alpha}\left(\sqrt{\mathfrak{A}_2^{\alpha}}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}_{T^{\alpha}}^{*}+\mathcal{E}_{\alpha}^{\text{strong}}+\mathcal{E}_{\alpha}^{\text{strong},*}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}. \tag{20}$$

The key point here is that the quasiweak boundedness constant $\mathcal{WBP}_{T^{\alpha}}$ does **not** appear on the right hand side of this estimate, and this is because the arguments in [30] that are used to bound $\mathcal{R}^{\alpha}(f,g)$ do not use the quasiweak boundedness property at all, as a patient reader can verify. This constitutes the deepest part of our argument to prove (16).

We now turn to the 'good-λ' argument that will substitute for the use of the quasiweak boundedness property in (18) in order to prove (16). First we observe that the constant C in (6) can be taken to be 2^{ρ}, and then an application of the inequality

$$\left|\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{I_{\theta}}\right),\mathbf{1}_{J_{\theta'}}\right\rangle_{\omega}\right| \leq \mathfrak{I}_{T^{\alpha}}\sqrt{\left|I_{\theta}\right|_{\sigma}\left|J_{\theta'}\right|_{\omega}},$$

to the display in (19) above, shows that

$$\begin{aligned}
\left|\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{I_{\theta}}\triangle_I^{\sigma}f\right),\mathbf{1}_{J_{\theta'}}\triangle_J^{\omega}g\right\rangle_{\omega}\right| &\lesssim \sup_{a,a'\in\Gamma_n}\frac{\left|\left\langle f,h_I^{\sigma,a}\right\rangle_{\sigma}\right|}{\sqrt{\left|I_{\theta}\right|_{\sigma}}}\left|\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{I_{\theta}}\right),\mathbf{1}_{J_{\theta'}}\right\rangle_{\omega}\right|\frac{\left|\left\langle g,h_J^{\omega,a'}\right\rangle_{\omega}\right|}{\sqrt{\left|J_{\theta'}\right|_{\omega}}}\\
&\lesssim \sup_{a,a'\in\Gamma_n}\frac{\left|\left\langle f,h_I^{\sigma,a}\right\rangle_{\sigma}\right|}{\sqrt{\left|I_{\theta}\right|_{\sigma}}}\mathfrak{I}_{T^{\alpha}}\sqrt{\left|I_{\theta}\right|_{\sigma}\left|J_{\theta'}\right|_{\omega}}\frac{\left|\left\langle g,h_J^{\omega,a'}\right\rangle_{\omega}\right|}{\sqrt{\left|J_{\theta'}\right|_{\omega}}}\\
&= \sup_{a,a'\in\Gamma_n}\mathfrak{I}_{T^{\alpha}}\left|\left\langle f,h_I^{\sigma,a}\right\rangle_{\sigma}\right|\left|\left\langle g,h_J^{\omega,a'}\right\rangle_{\omega}\right|.
\end{aligned}$$

From this we obtain the following *crude* estimate valid for any $f\in L^2(\sigma)$ and $g\in L^2(\omega)$:

$$\left|A_1(f,g)+A_2(f,g)\right| \leq C_{\alpha}\left(\sqrt{\mathfrak{A}_2^{\alpha}}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}_{T^{\alpha}}^{*}+\mathfrak{I}_{T^{\alpha}}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}. \tag{21}$$

Definition 8 We say that two quasicubes K and L have *η-comparable* side lengths, or simply that $\ell(K)$ and $\ell(L)$ are *η-comparable*, if

$$2^{-\eta}\ell(K)\leq\ell(L)\leq 2^{\eta}\ell(K).$$

Furthermore, we say that K and L are *η-close* if they have *η-comparable* side lengths, and if they belong to a common quasigrid $\Omega\mathcal{D}$ and are touching quasicubes that satisfy either $K\subset 3L$ or $L\subset 3K$.

Now consider the special indicator case $f=\mathbf{1}_Q$ and $g=\mathbf{1}_R$ where Q and R are *ρ-close* in some $\Omega\mathcal{D}$. For this case we will be able to do much better than (21). In fact, for each $0 < \lambda < \frac{1}{2}$ we claim that the following 'good-λ' inequality holds:

$$\left|A_1\left(\mathbf{1}_Q,\mathbf{1}_R\right)\right|+\left|A_2\left(\mathbf{1}_Q,\mathbf{1}_R\right)\right| \leq C_\alpha\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}_{T^\alpha}^*+\sqrt[4]{\lambda}\mathfrak{N}_{T^\alpha}\right)\left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}\left\|\mathbf{1}_R\right\|_{L^2(\omega)}. \tag{22}$$

With (22) proved, we can use it and (20) to complete the proof of the estimate for the indicator/touching property (16) by taking expectations $\mathbb{E}_\Omega$ as usual:

$$\begin{aligned}
&\mathbb{E}_\Omega\left|\sum_{I\in\Omega\mathcal{D}^\sigma \text{ and } J\in\Omega\mathcal{D}^\omega}\mathcal{T}_\sigma^\alpha\left(\triangle_I^\sigma\mathbf{1}_Q,\triangle_J^\omega\mathbf{1}_R\right)\right| \\
&\leq \mathbb{E}_\Omega\left(|A_1|+|A_2|\right)+\mathbb{E}_\Omega\left|\mathcal{R}^\alpha\left(\mathbf{1}_Q,\mathbf{1}_R\right)\right| \\
&\leq C_\alpha\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}_{T^\alpha}^*+\sqrt{\lambda}\mathfrak{N}_{T^\alpha}\right)\left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}\left\|\mathbf{1}_R\right\|_{L^2(\omega)} \\
&\quad+C_\alpha\left(\sqrt{\mathfrak{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}_{T^\alpha}^*+\mathcal{E}_\alpha^{\text{strong}}+\mathcal{E}_\alpha^{\text{strong},*}\right)\left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}\left\|\mathbf{1}_R\right\|_{L^2(\omega)} \\
&\leq C_\alpha\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}_{T^\alpha}^*+\mathcal{E}_\alpha^{\text{strong}}+\mathcal{E}_\alpha^{\text{strong},*}+\sqrt[4]{\lambda}\mathfrak{N}_{T^\alpha}\right)\left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}\left\|\mathbf{1}_R\right\|_{L^2(\omega)},
\end{aligned}$$

which gives (16) upon taking the supremum over such Q and R to get

$$\mathfrak{I}_{T^\alpha} \leq C_\alpha'\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}_{T^\alpha}^*+\mathcal{E}_\alpha^{\text{strong}}+\mathcal{E}_\alpha^{\text{strong},*}+\sqrt[4]{\lambda}\mathfrak{N}_{T^\alpha}\right).$$

Notation 3 *The remainder of this paper is devoted to proving (22) for touching and $\boldsymbol{\rho}$-close quasicubes Q and R. To simplify notation and geometric constructions, we consider only the case of* ordinary *cubes in $\mathcal{P}^n$, and note that the extension to the quasiworld is then routine.*

To prove the claim (22) we use the *parameterization by translation* introduced above. Essentially this approach was used in the averaging technique employed in [23], which in turn was borrowed from Fefferman and Stein [4], later refined in [6], and further refined here in this paper. It suffices to prove that

$$\begin{aligned}
\left|\mathcal{T}^\alpha\left(\left(\mathbf{1}_Q\right)_{\text{good}},\left(\mathbf{1}_R\right)_{\text{good}}\right)\right| &\leq C_\alpha\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}_{T^\alpha}^*+\mathcal{E}_\alpha^{\text{strong}}+\mathcal{E}_\alpha^{\text{strong},*}+\sqrt{\lambda}\mathfrak{N}_{T^\alpha}\right) \\
&\quad\times\left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}\left\|\mathbf{1}_R\right\|_{L^2(\omega)},
\end{aligned}$$

for all $Q,R\in\mathcal{P}^n$ that are $\boldsymbol{\rho}$-close, uniformly over Q-good grids, and where

$$\mathcal{T}^\alpha\left(\left(\mathbf{1}_Q\right)_{\text{good}},\left(\mathbf{1}_R\right)_{\text{good}}\right)=\sum_{I\in\mathcal{D}_{Q;\text{good}}^\sigma \text{ and } J\in\mathcal{D}_{Q;\text{good}}^\omega}\mathcal{T}^\alpha\left(\triangle_I^\sigma\mathbf{1}_Q,\triangle_J^\omega\mathbf{1}_R\right).$$

The grids $\mathcal{D}^{\sigma}_{Q;\mathrm{good}} = \mathcal{D}^{\omega}_{Q;\mathrm{good}}$ are those arising in the projections $\mathsf{P}^{\sigma}_{Q;\mathrm{good}}f$ and $\mathsf{P}^{\omega}_{Q;\mathrm{good}}g$ above. Moreover, due to the key observation above regarding where the weak boundedness property arises in the proof of the main theorem in [30], it suffices to prove

$$\mathbb{E}_{\Omega}\left\{\sum_{\substack{(I,J)\in\mathcal{D}^{\sigma}_{Q;\mathrm{good}}\times\mathcal{D}^{\omega}_{Q;\mathrm{good}}:\ J\subset 3I\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)}}+\sum_{\substack{(I,J)\in\mathcal{D}^{\sigma}_{Q;\mathrm{good}}\times\mathcal{D}^{\omega}_{Q;\mathrm{good}}:\ I\subset 3J\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)}}\right\}\left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right),\triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}\right|$$
$$\leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}^{*}_{T^{\alpha}}+\sqrt[4]{\lambda}\mathfrak{N}_{T^{\alpha}}\right)\left\|\mathbf{1}_{Q}\right\|_{L^{2}(\sigma)}\left\|\mathbf{1}_{R}\right\|_{L^{2}(\omega)},$$

under the assumption that we sum over only Q-good cubes I and J that belong to Q-good grids in the above sums, and where we recall that we may realize the underlying probability space as translations of any fixed grid, say the standard dyadic grid. Note that R is contained in $3Q$, and this accounts for our inclusion of siblings in Definition 7 above.

By symmetry it suffices to prove for all $0<\lambda<\frac{1}{2}$ that

$$\mathbb{E}_{\Omega}\sum_{\substack{(I,J)\in\mathcal{D}^{\sigma}_{Q;\mathrm{good}}\times\mathcal{D}^{\omega}_{Q;\mathrm{good}}:\ J\subset 3I\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)\\ I\text{ and }J\text{ touch}}}\left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right),\triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}\right| \tag{23}$$
$$\leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}^{*}_{T^{\alpha}}+\sqrt[4]{\lambda}\mathfrak{I}_{T^{\alpha}}\right)\left\|\mathbf{1}_{Q}\right\|_{L^{2}(\sigma)}\left\|\mathbf{1}_{R}\right\|_{L^{2}(\omega)},$$

for all cubes $Q,R\in\mathcal{P}^{n}$ that are $\boldsymbol{\rho}$-close (we are including the testing conditions here because we are including children I_{θ} and $J_{\theta'}$ in the display (19) that coincide as well).

3.2.1 Three Critical Reductions

Now we make three critical reductions that permit the application of NTV surgery, and lie at the core of the much better estimate (22).

(1) We must have that I 'cuts across the boundary' of Q, i.e. $|I\cap Q|>0$ and $|I\cap Q^{c}|>0$ (or else $\triangle^{\sigma}_{I}\mathbf{1}_{Q}=0$),
(2) We must have that J 'cuts across the boundary' of R, i.e. $|J\cap R|>0$ and $|J\cap R^{c}|>0$ (or else $\triangle^{\omega}_{J}\mathbf{1}_{R}=0$),
(3) By the assumed 'Q-goodness' in Definition 7, together with reductions (1) and (2) above, we *cannot* have either $\ell(I)\geq 2^{\mathbf{r}}\ell(Q)$ or $\ell(J)\geq 2^{\mathbf{r}}\ell(R)$.

From these reductions, we are left to prove

$$\mathbb{E}_{\Omega} \sum_{\substack{(I,J)\in\mathcal{D}^{\sigma}_{Q;\mathrm{good}}\times\mathcal{D}^{\omega}_{Q;\mathrm{good}}:\ J\subset 3I \\ I \text{ and } J \text{ are } \boldsymbol{\rho}\text{-close} \\ \ell(I)<2^{\mathbf{r}}\ell(Q) \text{ and } \ell(J)<2^{\mathbf{r}}\ell(R)}} \left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right), \triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}\right| \tag{24}$$

$$\leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^{*}_{T^{\alpha}} + \sqrt[4]{\lambda}\mathfrak{N}_{T^{\alpha}}\right)\|\mathbf{1}_{Q}\|_{L^{\sigma f 2}(\sigma)}\|\mathbf{1}_{R}\|_{L^{2}(\omega)},$$

for all $\boldsymbol{\rho}$-close $Q, R \in \mathcal{P}^{n}$.

The *small* pairs of cubes (I, J), i.e. those with both $\ell(I) < 2^{-\mathbf{r}}\ell(Q)$ and $\ell(J) < 2^{\mathbf{r}}\ell(R)$, pose a difficulty and our next task is to further reduce matters to proving the more restricted estimate:

$$\mathbb{E}_{\Omega} \sum_{\substack{(I,J)\in\mathcal{D}^{\sigma}_{Q;\mathrm{good}}\times\mathcal{D}^{\omega}_{Q;\mathrm{good}}:\ J\subset 3I \\ I \text{ and } J \text{ are } \boldsymbol{\rho}\text{-close} \\ \ell(I) \text{ and } \ell(Q) \text{ are } r\text{-comparable} \\ \ell(J) \text{ and } \ell(R) \text{ are } r\text{-comparable}}} \left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right), \triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}\right| \tag{25}$$

$$\leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^{*}_{T^{\alpha}} + \sqrt[4]{\lambda}\mathfrak{N}_{T^{\alpha}}\right)\|\mathbf{1}_{Q}\|_{L^{2}(\sigma)}\|\mathbf{1}_{R}\|_{L^{2}(\omega)},$$

for all $Q, R \in \mathcal{P}^{n}$ that are $\boldsymbol{\rho}$-close. The difference between (25) and (24) is that in (25), we do *not* permit small pairs of (I, J), i.e. those with $\ell(I) < 2^{-\mathbf{r}}\ell(Q)$ or $\ell(J) < 2^{-\mathbf{r}}\ell(RQ)$.

3.2.2 Elimination of Small Pairs

To eliminate the small pairs from (24), we apply for a second time our proof from [30] as outlined above, but this time to each inner product $\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right), \triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}$ appearing in the sum in (24) inside the expectation $\mathbb{E}_{\Omega}$. In other words, for fixed I, J, Q and R, we take $f = \triangle^{\sigma}_{I}\mathbf{1}_{Q}$ and $g = \triangle^{\omega}_{J}\mathbf{1}_{R}$, and we obtain that

$$\mathbb{E}_{\Omega}\mathbb{E}_{\Omega'}\left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right), \triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}\right|$$

$$\leq C_{\alpha}\left(\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^{*}_{T^{\alpha}} + + \mathcal{E}^{\mathrm{strong}}_{\alpha} + \mathcal{E}^{\mathrm{strong},*}_{\alpha} + 2^{-\varepsilon\mathbf{r}}\mathfrak{N}_{T^{\alpha}}\right)\left\|\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right\|_{L^{2}(\sigma)}\left\|\triangle^{\sigma}_{J}\mathbf{1}_{R}\right\|_{L^{2}(\omega)}$$

$$+\mathbb{E}_{\Omega}\mathbb{E}_{\Omega'} \sum_{\substack{(K,L)\in\mathcal{D}'_{Q;\mathrm{good}}\times\mathcal{D}'_{Q;\mathrm{good}}:\ L\subset 3K \\ K \text{ and } L \text{ are } \mathbf{j}\text{-close} \\ \ell(K)<2^{\mathbf{r}}\ell(I) \text{ and } \ell(L)<2^{\mathbf{r}}\ell(JR)}} \left|\left\langle T^{\alpha}_{\sigma}\,\triangle^{\sigma}_{K}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right), \triangle^{\omega}_{L}\left(\triangle^{\omega}_{J}\mathbf{1}_{R}\right)\right\rangle_{\omega}\right|,$$

where here the expectation $\mathbb{E}_{\Omega'}$ is taken to be independent of $\mathbb{E}_{\Omega}$.

But now we may further assume that the pair of grids $(\mathcal{D}, \mathcal{D}')$, for which $(I, J) \in \mathcal{D} \times \mathcal{D}$ and $(K, L) \in \mathcal{D}' \times \mathcal{D}'$, are *mutually good*.[4] Thus we cannot have $\ell(K) < 2^{-\boldsymbol{\rho}} \ell(I)$ because K is I-good, and this eliminates the inclusion of small pairs (K, L), i.e. those with $\ell(K) < 2^{-\boldsymbol{\rho}} \ell(I)$. Note that the term $2^{-\varepsilon \mathbf{r}} \mathfrak{N}_{T^\alpha}$ arises from the bad Haar projections $\triangle_K^\sigma$ and $\triangle_L^\omega$ of $\triangle_I^\sigma \mathbf{1}_Q$ and $\triangle_J^\omega \mathbf{1}_R$ respectively. Finally, we note that $f = \triangle_I^\sigma \mathbf{1}_Q$ is constant on the children of I and that $\left\| \triangle_I^\sigma \mathbf{1}_Q \right\|_{L^2(\sigma)}^2 = \sum_{I' \in \mathfrak{C}(I)} \int_{I'} \left| \mathbb{E}_{I'}^\sigma \mathbf{1}_Q - \mathbb{E}_I^\sigma \mathbf{1}_Q \right|^2 d\sigma$. Thus it suffices to prove the following estimate,

$$\mathbb{E}_{\Omega'} \sum_{\substack{(K,L) \in \mathcal{D}'_{Q;\text{good}} \times \mathcal{D}'_{Q;\text{good}} :\ L \subset 3K \\ K \text{ and } L \text{ are } \boldsymbol{\rho}\text{-close} \\ \ell(K) \text{ and } \ell(I) \text{ are } \mathbf{r}\text{-comparable} \\ \ell(L) \text{ and } \ell(J) \text{ are } \mathbf{r}\text{-comparable}}} \sum_{\substack{I' \in \mathfrak{C}(I) \\ J' \in \mathfrak{C}(J)}} \left| \left\langle T_\sigma^\alpha \triangle_K^\sigma \left(\left[\mathbb{E}_{I'}^\sigma \triangle_I^\sigma \mathbf{1}_Q \right] \mathbf{1}_{I'} \right), \triangle_L^\omega \left(\left[\mathbb{E}_{J'}^\omega \triangle_J^\omega \mathbf{1}_R \right] \mathbf{1}_{J'} \right) \right\rangle_\omega \right|$$

$$\leq C_\alpha \left(\frac{1}{\lambda} \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \sqrt[4]{\lambda} \mathfrak{N}_{T^\alpha} \right) \sum_{\substack{I' \in \mathfrak{C}(I) \\ J' \in \mathfrak{C}(J)}} \left| \mathbb{E}_{I'}^\sigma \triangle_I^\sigma \mathbf{1}_Q \right| \left| \mathbb{E}_{J'}^\omega \triangle_J^\omega \mathbf{1}_R \right| \left\| \mathbf{1}_{I'} \right\|_{L^2(\sigma)} \left\| \mathbf{1}_{J'} \right\|_{L^2(\omega)},$$

which we can write simply as

$$\mathbb{E}_{\Omega'} \sum_{\substack{(K,L) \in \mathcal{D}'_{Q;\text{good}} \times \mathcal{D}'_{Q;\text{good}} :\ L \subset 3K \\ K \text{ and } L \text{ are } \boldsymbol{\rho}\text{-close} \\ \ell(K) \text{ and } \ell(I') \text{ are } \mathbf{r}\text{-comparable} \\ \ell(L) \text{ and } \ell(J') \text{ are } \mathbf{r}\text{-comparable}}} \left| \left\langle T_\sigma^\alpha \triangle_K^\sigma \left(\mathbf{1}_{I'} \right), \triangle_L^\omega \left(\mathbf{1}_{J'} \right) \right\rangle_\omega \right|$$

$$\leq C_\alpha \left(\frac{1}{\lambda} \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \sqrt[4]{\lambda} \mathfrak{N}_{T^\alpha} \right) \left\| \mathbf{1}_{I'} \right\|_{L^2(\sigma)} \left\| \mathbf{1}_{J'} \right\|_{L^2(\omega)}$$

for each $I' \in \mathfrak{C}(I)$ and $J' \in \mathfrak{C}(J)$. Now relabel I' and J' as Q and R respectively (and then also K and L as I and J respectively) to obtain (25).

3.2.3 NTV Surgery

Now in order to prove (25), we invoke the technique of NTV surgery as used in [7, 17] and [10]. Given $0 < \lambda < \frac{1}{2}$, define

$$J_\lambda \equiv \{ x \in J : \text{dist}(x, \partial J) > \lambda \ell(J) \}.$$

Then we write

$$\begin{aligned} \left| \left\langle T_\sigma^\alpha \left(\triangle_I^\sigma \mathbf{1}_Q \right), \triangle_J^\omega \mathbf{1}_R \right\rangle_\omega \right| &\leq \left| \left\langle T_\sigma^\alpha \left(\triangle_I^\sigma \mathbf{1}_Q \right), \mathbf{1}_{J_\lambda} \triangle_J^\omega \mathbf{1}_R \right\rangle_\omega \right| + \left| \left\langle T_\sigma^\alpha \left(\triangle_I^\sigma \mathbf{1}_Q \right), \mathbf{1}_{J \setminus J_\lambda} \triangle_J^\omega \mathbf{1}_R \right\rangle_\omega \right| \\ &\equiv A_1 + A_2. \end{aligned}$$

[4]Both I and J belong to the common grid $\mathcal{D}$, while K and L belong to the independent common grid $\mathcal{D}'$—in contrast to the traditional use of two independent grids where $I \in \mathcal{D}$ and $J \in \mathcal{D}'$.

Now we use first the fact that I and J_λ are separated by a distance at least $\lambda\ell(J) > 0$ in order to bound the first term A_1 by

$$\begin{aligned} A_1 &= \left|\left\langle T_\sigma^\alpha \left(\mathbf{1}_{I'} \triangle_I^\sigma \mathbf{1}_Q\right), \mathbf{1}_{J_\lambda} \triangle_J^\omega \mathbf{1}_R\right\rangle_\omega\right| \\ &\lesssim \frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}\left\|\triangle_I^\sigma \mathbf{1}_Q\right\|_{L^2(\sigma)}\left\|\triangle_J^\omega \mathbf{1}_R\right\|_{L^2(\omega)} \leq \frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}\left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}\left\|\mathbf{1}_R\right\|_{L^2(\omega)} . \end{aligned} \tag{26}$$

We further dominate the square of the second term A_2 by

$$\begin{aligned} A_2^2 &= \left|\left\langle T_\sigma^\alpha \left(\triangle_I^\sigma \mathbf{1}_Q\right), \mathbf{1}_{J\setminus J_\lambda} \triangle_J^\omega \mathbf{1}_R\right\rangle_\omega\right|^2 \\ &= \left|\left\langle T_\sigma^\alpha \left(\sum_{I'\in\mathfrak{C}(I)} \mathbf{1}_{I'} \triangle_I^\sigma \mathbf{1}_Q\right), \mathbf{1}_{J\setminus J_\delta} \sum_{J'\in\mathfrak{C}(J)} \mathbf{1}_{J'} \triangle_J^\omega \mathbf{1}_R\right\rangle_\omega\right|^2 \\ &\lesssim \sum_{I'\in\mathfrak{C}(I)}\sum_{J'\in\mathfrak{C}(J)} \left|\left\langle T_\sigma^\alpha \left(\mathbf{1}_{I'} \triangle_I^\sigma \mathbf{1}_Q\right), \mathbf{1}_{J'\setminus J_\lambda} \triangle_J^\omega \mathbf{1}_R\right\rangle_\omega\right|^2 \\ &\lesssim \sum_{I'\in\mathfrak{C}(I)}\sum_{J'\in\mathfrak{C}(J)} \mathfrak{N}_{T^\alpha}^2 \left\|\mathbf{1}_{I'} \triangle_I^\sigma \mathbf{1}_Q\right\|_{L^2(\sigma)}^2 \left\|\mathbf{1}_{J'\setminus J_\lambda} \triangle_J^\omega \mathbf{1}_R\right\|_{L^2(\omega)}^2 \\ &\lesssim \mathfrak{N}_{T^\alpha}^2 \left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}^2 \sum_{J'\in\mathfrak{C}(J)} \left\|\mathbf{1}_{J'\setminus J_\lambda} \triangle_J^\omega \mathbf{1}_R\right\|_{L^2(\omega)}^2 = \mathfrak{N}_{T^\alpha}^2 \left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}^2 \int_{J'\setminus J_\lambda} \left|\triangle_J^\omega \mathbf{1}_R\right|^2 d\omega . \end{aligned} \tag{27}$$

Then we note the fact that, using the *translation parameterization* of Ω indexed by $\gamma \in \Gamma$, we have

$$\mathbb{E}_\Omega \left|R \cap \left[(J+\gamma)' \setminus (J+\gamma)_\lambda\right]\right|_\omega \leq C_\alpha \lambda \left|R\right|_\omega , \tag{28}$$

which follows upon taking the average over certain translates $\mathcal{D}_0 + \gamma$ where $\mathcal{D}_0$ is a fixed grid containing J. This is of course equivalent to taking instead the average over the same translates $\omega + \gamma$ of the measure ω, and it is in this latter form that (28) is evident.

Now we will apply (28), together with an argument to resolve the difficulty associated with the appearance of J in *both* $J' \setminus J_\lambda$ and $\triangle_J^\omega \mathbf{1}_R$, to obtain the following key estimate for every $0 < \lambda < \frac{1}{2}$:

$$\mathbb{E}_\Omega \int_{J'\setminus J_\lambda} \left|\triangle_J^\omega \mathbf{1}_R\right|^2 d\omega \leq C_\alpha \sqrt{\lambda} \left|R\right|_\omega , \tag{29}$$

for the expected value of the final integral on the right hand side of (27). With (29) and (26) in hand, we will obtain that

$$\begin{aligned} &\mathbb{E}_\Omega \left|\left\langle T_\sigma^\alpha \left(\triangle_I^\sigma \mathbf{1}_Q\right), \triangle_J^\omega \mathbf{1}_R\right\rangle_\omega\right|^2 \\ &\lesssim \mathbb{E}_\Omega \left|\left\langle T_\sigma^\alpha \left(\triangle_I^\sigma \mathbf{1}_Q\right), \mathbf{1}_{J_\lambda} \triangle_J^\omega \mathbf{1}_R\right\rangle_\omega\right|^2 + \mathbb{E}_\Omega \sum_{I'\in\mathfrak{C}(I)}\sum_{J'\in\mathfrak{C}(J)} \left|\left\langle T_\sigma^\alpha \left(\mathbf{1}_{I'} \triangle_I^\sigma \mathbf{1}_Q\right), \mathbf{1}_{J'\setminus J_\lambda} \triangle_J^\omega \mathbf{1}_R\right\rangle_\omega\right|^2 \end{aligned}$$

$$\leq C_\alpha^2 \frac{1}{\lambda^2} \mathfrak{A}_2^\alpha \left\| \mathbf{1}_Q \right\|_{L^2(\sigma)}^2 \left\| \mathbf{1}_R \right\|_{L^2(\omega)}^2 + \mathbb{E}_\Omega \sum_{I' \in \mathfrak{C}(I)} \sum_{J' \in \mathfrak{C}(J)} \mathfrak{N}_{T^\alpha}^2 \left\| \mathbf{1}_{I'} \bigtriangleup_I^\sigma \mathbf{1}_Q \right\|_{L^2(\sigma)}^2 \left\| \mathbf{1}_{J' \setminus J_\lambda} \bigtriangleup_J^\omega \mathbf{1}_R \right\|_{L^2(\omega)}^2$$

$$\leq C_\alpha^2 \frac{1}{\lambda^2} \mathfrak{A}_2^\alpha \left\| \mathbf{1}_Q \right\|_{L^2(\sigma)}^2 \left\| \mathbf{1}_R \right\|_{L^2(\omega)}^2 + \sqrt{\lambda} \mathfrak{N}_{T^\alpha}^2 \left\| \mathbf{1}_Q \right\|_{L^2(\sigma)}^2 \left\| \mathbf{1}_R \right\|_{L^2(\omega)}^2 ,$$

as required. Thus the proof of (16), and hence also that of the Good-λ Lemma, will be complete once we have proved the estimate (29), to which we now turn.

Remark 3 In the third line above we have used the norm inequality $\left| \langle T_\sigma^\alpha f, g \rangle_\omega \right| \leq \mathfrak{N}_{T^\alpha} \| f \|_{L^2(\sigma)} \| g \|_{L^2(\omega)}$ with $f = \mathbf{1}_{I'} \bigtriangleup_I^\sigma \mathbf{1}_Q$ and $g = \mathbf{1}_{J' \setminus J_\lambda} \bigtriangleup_J^\omega \mathbf{1}_R$, and where g is a constant multiple of an indicator of a 'rectangle' $J' \setminus J_\lambda$. This prevents us from using the smaller bound $\lambda \mathfrak{I}_{T^\alpha}^2$ in place of $\lambda \mathfrak{N}_{T^\alpha}^2$.

In order to illuminate the main ideas in the proof of (29), we first prove the simplest case of dimension $n = 1$. So let

$$J \setminus J_\lambda = J_\lambda^{\text{left}} \cup J_\lambda^{\text{right}},$$

where $J_\lambda^{\text{left}} = J_- \setminus J_\lambda$ and $J_\lambda^{\text{right}} = J_+ \setminus J_\lambda$, and write

$$\mathbb{E}_\Omega \int_{J' \setminus J_\lambda} \left| \bigtriangleup_J^\omega \mathbf{1}_R \right|^2 d\omega = \mathbb{E}_\Omega \int_{J_\lambda^{\text{left}}} \left| \bigtriangleup_J^\omega \mathbf{1}_R \right|^2 d\omega + \mathbb{E}_\Omega \int_{J_\lambda^{\text{right}}} \left| \bigtriangleup_J^\omega \mathbf{1}_R \right|^2 d\omega = \textit{Left} + \textit{Right}. \tag{30}$$

Now we recall the parameterization of the expectation by translations $\gamma \in \Gamma_M^N$ of step size 2^{-M}, and let $\eta = \lambda 2^M$ where λ is the side length of the interval $J' \setminus J_\lambda$. Then, by using the 'average of an average' principle, we can rewrite the expectation in terms of the larger step size $\eta 2^{-M}$. We continue to use γ to denote the new step size $\eta 2^{-M}$. Then we further decompose the expectation *Left* in (30) as

$$\begin{aligned}
\textit{Left} &= \mathbb{E}_\Omega \int_{J_\lambda^{\text{left}}} \left| \bigtriangleup_J^\omega \mathbf{1}_R \right|^2 d\omega = \mathbb{E}_\Omega \int_{(J+\gamma)_\lambda^{\text{left}}} \left| \bigtriangleup_{J+\gamma}^\omega \mathbf{1}_R \right|^2 d\omega \\
&= \mathbb{E}_\Omega \mathbf{1}_{\left\{\gamma : (J+\gamma)_\lambda^{\text{left}} \subset R\right\}} \int_{(J+\gamma)_\lambda^{\text{left}}} \left| \bigtriangleup_{J+\gamma}^\omega \mathbf{1}_R \right|^2 d\omega \\
&\quad + \mathbb{E}_\Omega \mathbf{1}_{\left\{\gamma : (J+\gamma)_\lambda^{\text{left}} \text{ lies to the left of } R\right\}} \int_{(J+\gamma)_\lambda^{\text{left}}} \left| \bigtriangleup_{J+\gamma}^\omega \mathbf{1}_R \right|^2 d\omega \\
&\equiv A_3 + A_4 ,
\end{aligned}$$

where because of our change of step size, we have that $\left\{ (J+\gamma)_\lambda^{\text{left}} \right\}_\gamma$ is a pairwise disjoint covering of the top interval containing J that has side length 2^{-N} (see the beginning of Sect. 3.1 above).

For term A_3 we use the elementary estimate

$$\left|\triangle^{\omega}_{J+\gamma}\mathbf{1}_R\right| = \left|\mathbb{E}_{(J+\gamma)_-}\mathbf{1}_R - \mathbb{E}_{(J+\gamma)}\mathbf{1}_R\right| \leq 1$$

together with the estimate in (28), to obtain

$$\begin{aligned} A_3 &= \mathbb{E}_{\Omega}\mathbf{1}_{\{\gamma:(J+\gamma)^{\mathrm{left}}_{\lambda}\subset R\}}\int_{(J+\gamma)^{\mathrm{left}}_{\lambda}}\left|\triangle^{\omega}_{J+\gamma}\mathbf{1}_R\right|^2 d\omega \\ &\leq \mathbb{E}_{\Omega}\left|R\cap (J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega} \leq C_{\alpha}\lambda\,|R|_{\omega}\ . \end{aligned}$$

For term A_4 we proceed as follows. We suppose that $(J+\gamma)^{\mathrm{left}}_{\lambda}$ lies to the left of R, since the case when $(J+\gamma)^{\mathrm{right}}_{\lambda}$ lies to the right of R is similar. We have

$$\begin{aligned} \int_{(J+\gamma)^{\mathrm{left}}_{\lambda}}\left|\triangle^{\omega}_{J+\gamma}\mathbf{1}_R\right|^2 d\omega &= \int_{(J+\gamma)^{\mathrm{left}}_{\lambda}}\left|\mathbb{E}_{(J+\gamma)_-}\mathbf{1}_R - \mathbb{E}_{(J+\gamma)}\mathbf{1}_R\right|^2 d\omega \\ &= \int_{(J+\gamma)^{\mathrm{left}}_{\lambda}}\left|\frac{|R\cap (J+\gamma)_-|_{\omega}}{|(J+\gamma)_-|_{\omega}} - \frac{|R\cap (J+\gamma)|_{\omega}}{|J+\gamma|_{\omega}}\right|^2 d\omega \\ &\leq 2\left|(J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega}\left(\frac{|R\cap (J+\gamma)_-|_{\omega}}{|(J+\gamma)_-|_{\omega}}\right)^2 \\ &\quad +2\left|(J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega}\left(\frac{|R\cap (J+\gamma)|_{\omega}}{|J+\gamma|_{\omega}}\right)^2 . \end{aligned}$$

We now estimate the sum of the first terms above since the sum of the second terms can be estimated with the same argument.

For the sum of the first terms we write

$$\begin{aligned} &\sum_{\gamma:\ (J+\gamma)^{\mathrm{left}}_{\lambda}\text{ is left of }R}\left|(J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega}\left(\frac{|R\cap (J+\gamma)_-|_{\omega}}{|(J+\gamma)_-|_{\omega}}\right)^2 \\ &\leq \left(\sum_{\gamma:\ (J+\gamma)^{\mathrm{left}}_{\lambda}\text{ is left of }R}\frac{\left|(J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega}}{|(J+\gamma)_-|_{\omega}}\frac{|R\cap (J+\gamma)_-|_{\omega}}{|(J+\gamma)_-|_{\omega}}\right)|R|_{\omega}\ , \end{aligned}$$

and let $J+\gamma_1$ be the leftmost translate of J such that

$$\frac{\left|(J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega}}{|(J+\gamma)_-|_{\omega}}\frac{|R\cap (J+\gamma)_-|_{\omega}}{|(J+\gamma)_-|_{\omega}} > \delta, \tag{31}$$

where $\delta > 0$ will be chosen later to be $\sqrt{\lambda}$. We suppose the translations γ are ordered to be increasing. Note that we have both

$$1 \geq \frac{|R \cap (J+\gamma_1)_-|_\omega}{|(J+\gamma_1)_-|_\omega} > \delta$$

and

$$(J+\gamma)_\lambda^{\text{left}} \subset (J+\gamma_1)_- \ ,$$
$$\text{if both } \gamma > \gamma_1 \text{ and } (J+\gamma)_\lambda^{\text{left}} \text{ is left of } R.$$

Thus we compute that

$$\mathbb{E}_\Omega \int_{(J+\gamma)_\lambda^{\text{left}}} \left|\triangle_{J+\gamma}^\omega \mathbf{1}_R\right|^2 d\omega = \frac{1}{\Lambda}\left\{\sum_{\gamma<\gamma_1} + \sum_{\gamma>\gamma_1:\ (J+\gamma)_\lambda^{\text{left}} \text{ is left of } R}\right\} \tag{32}$$

$$\int_{(J+\gamma)_\lambda^{\text{left}}} \left|\triangle_{J+\gamma}^\omega \mathbf{1}_R\right|^2 d\omega \tag{33}$$

$$\leq \frac{1}{\Lambda}\sum_{\gamma<\gamma_1} \frac{\left|(J+\gamma)_\lambda^{\text{left}}\right|_\omega |R\cap (J+\gamma)_-|_\omega}{|(J+\gamma)_-|_\omega^2} |R|_\omega + \frac{1}{\Lambda}\sum_{\gamma>\gamma_1:\ (J+\gamma)_\lambda^{\text{left}} \text{ is left of } R} \left|(J+\gamma)_\lambda^{\text{left}}\right|_\omega$$

$$\leq \frac{1}{\Lambda}\delta\,\#\{\gamma<\gamma_1\}\,|R|_\omega + \frac{1}{\Lambda}|(J+\gamma_1)_-|_\omega \leq \delta\,|R|_\omega + \frac{1}{\Lambda}\frac{1}{\delta}\,|R\cap(J+\gamma_1)_-|_\omega$$

$$\leq \left(\delta + \frac{\lambda}{\delta}\right)|R|_\omega = 2\sqrt{\lambda}\,|R|_\omega\,,$$

if we choose $\delta = \sqrt{\lambda}$. This completes the proof of (29) in dimension $n = 1$.

3.2.4 Higher Dimensions

In the case of $n > 1$ dimensions we decompose the 'corner-like' pieces $J' \setminus J_\lambda$ for each child $J' \in \mathfrak{C}(J)$ into faces $S + \gamma$ of width λ (when $n = 1$ there are only two such faces $S + \gamma$, namely the intervals $(J+\gamma)_\lambda^{\text{left}}$ and $(J+\gamma)_\lambda^{\text{right}}$). Then we apply the above argument for $(J+\gamma)_\lambda^{\text{left}}$ to $S + \gamma$ for each face S of width λ in $J' \setminus J_\lambda$, but using only translations perpendicular to the face S, and finally apply the 'average of an average' principle, to obtain (29). We illustrate the proof in the case $n = 2$ since the general case $n \geq 2$ is no different.

For a square K in the plane, let K_- denote the lower left child of K. Now fix squares J and R in the plane with $\boldsymbol{\rho}$-comparable side lengths and such that $J \subset 3R$. For $\gamma \in \mathcal{H}_\lambda$, where $\mathcal{H}_\lambda$ is the set of *horizontal* translations γ of step size λ with

$|\gamma| \leq C\ell(R)$, denote by $(J+\gamma)_\lambda^{\text{lower left}}$ the L-shaped 'corner'

$$(J+\gamma)_\lambda^{\text{lower left}} \equiv (J+\gamma)_- \setminus J_\lambda \ ,$$

and by $(J+\gamma)_\lambda^{\text{left}}$ the *vertical* portion of the L-shaped set $(J+\gamma)_\lambda^{\text{lower left}}$ (this is one of the faces $S+\gamma$ introduced above). We will show that

$$\frac{1}{\#\mathcal{H}_\lambda} \sum_{\gamma \in \mathcal{H}_\lambda} \int_{(J+\gamma)_\lambda^{\text{left}}} \left| \triangle_{J+\gamma}^{\omega} \mathbf{1}_R \right|^2 d\omega \lesssim \sqrt{\lambda}, \tag{34}$$

where $\#\mathcal{H}_\lambda \approx \frac{C\ell(R)}{\lambda}$, and then by the 'average of an average' principle we obtain (29). To prove (34) we will apply the one-dimensional argument from the previous subsubsection, but with modifications to accommodate the fact that $(J+\gamma)_\lambda^{\text{left}}$ can now spill out over the top of R as well as to the left of R (recall that in the one-dimensional setting, $(J+\gamma)_\lambda^{\text{left}}$ occurred to the left of the interval R if it was not contained in R). As in dimension $n=1$, let $J+\gamma_1$ be the leftmost horizontal translate of J such that

$$\frac{\left|(J+\gamma)_\lambda^{\text{left}}\right|_\omega}{|(J+\gamma)_-|_\omega} \frac{|R \cap (J+\gamma)_-|_\omega}{|(J+\gamma)_-|_\omega} > \delta, \tag{35}$$

so that we have

$$1 \geq \frac{|R \cap (J+\gamma_1)_-|_\omega}{|(J+\gamma_1)_-|_\omega} > \delta.$$

Then with notation analogous to the case $n=1$ we have a similar calculation to that in (33):

$$\begin{aligned}
&\frac{1}{\Lambda}\left\{\sum_{\gamma<\gamma_1} + \sum_{\gamma>\gamma_1:\ (J+\gamma)_\lambda^{\text{left}} \subset (J+\gamma_1)_-}\right\} \int_{(J+\gamma)_\lambda^{\text{left}}} \left|\triangle_{J+\gamma}^{\omega}\mathbf{1}_R\right|^2 d\omega \\
&\leq \frac{1}{\Lambda}\sum_{\gamma<\gamma_1} \frac{\left|(J+\gamma)_\lambda^{\text{left}}\right|_\omega |R\cap (J+\gamma)_-|_\omega}{|(J+\gamma)_-|_\omega^2} |R|_\omega + \frac{1}{\Lambda}\sum_{\gamma>\gamma_1:\ (J+\gamma)_\lambda^{\text{left}} \subset (J+\gamma_1)_-} \left|(J+\gamma)_\lambda^{\text{left}}\right|_\omega \\
&\leq \frac{1}{\Lambda}\delta\,\#\{\gamma<\gamma_1\}\,|R|_\omega + \frac{1}{\Lambda}\,|(J+\gamma_1)_-|_\omega \leq \delta\,|R|_\omega + \frac{1}{\Lambda}\frac{1}{\delta}\,|R\cap(J+\gamma_1)_-|_\omega \\
&\leq \left(\delta + \frac{\lambda}{\delta}\right)|R|_\omega = 2\sqrt{\lambda}\,|R|_\omega\,,
\end{aligned}$$

if we choose $\delta = \sqrt{\lambda}$. Thus we have so far successfully estimated the sum over translations γ that satisfy either $\gamma < \gamma_1$ or $(J+\gamma)_\lambda^{\text{left}} \subset (J+\gamma_1)_-$.

Now we simply repeat the last step considering only the remaining horizontal translations. Since the side lengths of J and R are comparable, there are at most a fixed number of such steps left, and adding up the results, and using the 'average of an average' principle, then gives

$$\mathbb{E}_{\Omega}\int_{(J+\gamma)_{\lambda}^{\mathrm{left}}}\left|\triangle_{J+\gamma}^{\omega}\mathbf{1}_{R}\right|^{2}d\omega\leq C_{\alpha}\sqrt{\lambda}.$$

This completes the proof of (29) in the case of dimension $n=2$, and as mentioned earlier, the above two-dimensional argument easily adapts to the case $n\geq3$.

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Appendix

We assume notation as above. Define the bilinear form

$$\mathcal{B}(f,g)\equiv\left\langle T_{\sigma}^{\alpha}f,g\right\rangle_{\omega}\ ,\quad f\in L^{2}(\sigma),g\in L^{2}(\omega),$$

restricted to functions f and g of compact support and mean zero. For each dyadic grid $\mathcal{D}$ we then have

$$\mathcal{B}(f,g)=\sum_{I,J\in\mathcal{D}}\left\langle T_{\sigma}^{\alpha}\ \triangle_{I}^{\sigma}f,\triangle_{J}^{\omega}g\right\rangle_{\omega}\ .$$

Now define the bilinear forms

$$\mathcal{C}_{\mathcal{D}}(f,g)=\sum_{I,J\in\mathcal{D}:\ I\text{ and }J\text{ are }\mathbf{r}\text{-close}}\left\langle T_{\sigma}^{\alpha}\ \triangle_{I}^{\sigma}f,\triangle_{J}^{\omega}g\right\rangle_{\omega}\ ,\quad f\in L^{2}(\sigma),g\in L^{2}(\omega).$$

Thus the form $\mathcal{C}_{\mathcal{D}}(f,g)$ sums over those pairs of cubes in the grid $\mathcal{D}$ that are close in both scale and position, these being the only pairs where the need for a weak boundedness property traditionally arises. We also consider the subbilinear form

$$\mathcal{S}_{\mathcal{D}}(f,g)=\sum_{I,J\in\mathcal{D}:\ I\text{ and }J\text{ are }\mathbf{r}\text{-close}}\left|\left\langle T_{\sigma}^{\alpha}\ \triangle_{I}^{\sigma}f,\triangle_{J}^{\omega}g\right\rangle_{\omega}\right|\ ,\quad f\in L^{2}(\sigma),g\in L^{2}(\omega),$$

which dominates $\mathcal{C}_{\mathcal{D}}(f,g)$, i.e. $|\mathcal{C}_{\mathcal{D}}(f,g)|\leq\mathcal{S}_{\mathcal{D}}(f,g)$ for all $f\in L^{2}(\sigma),g\in L^{2}(\omega)$. The main results above can be organized into the following two part theorem.

Theorem 5 *With notation as above, we have:*

(1) *For f and g of compact support and mean zero,*

$$\begin{aligned}&\mathbb{E}_\Omega \left|\mathcal{B}(f,g) - \mathcal{C}_\mathcal{D}(f,g)\right| \\ &\le C_\alpha \left(\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} + 2^{-\varepsilon \mathbf{r}}\mathfrak{N}_{T^\alpha}\right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)} \\ &\qquad + C_\alpha \mathbb{E}_\Omega \mathcal{S}_\mathcal{D}(f,g).\end{aligned}$$

(2) *For f and g of compact support and mean zero, and for* $0 < \lambda < \frac{1}{2}$,

$$\mathbb{E}_\Omega \mathcal{S}_\mathcal{D}(f,g) \le C_\alpha \left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \sqrt[4]{\lambda}\,\mathfrak{N}_{T^\alpha}\right) \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}.$$

The reason for emphasizing the two estimates in this way, is that a different proof strategy might produce a *different* bound for $\mathbb{E}_\Omega \left|\mathcal{B}(f,g) - \mathcal{C}_\mathcal{D}(f,g)\right|$, which can then be combined with the bound for $\mathbb{E}_\Omega \mathcal{S}_\mathcal{D}(f,g)$ to control $\left|\mathcal{B}(f,g)\right|$. Note also that the term $C_\alpha \mathbb{E}_\Omega \mathcal{S}_\mathcal{D}(f,g)$ is included in part (1) of the theorem, to allow for some of the inner products in the definition of $\mathcal{C}_\mathcal{D}(f,g)$ to be added back into the form $\mathcal{B}(f,g) - \mathcal{C}_\mathcal{D}(f,g)$ during the course of the proof of estimate (1). Indeed, this was done when controlling the form $\mathsf{T}^2_{\text{far below}}(f,g)$ above.

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