Weighted Norm Inequalities of $(1, q)$-Type for Integral and Fractional Maximal Operators

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Dedicated to Richard L. Wheeden

Abstract We study weighted norm inequalities of $(1, q)$- type for $0 < q < 1$,

$$\|\mathbf{G}\nu\|_{L^q(\Omega, d\sigma)} \leq C\,\|\nu\|, \quad \text{for all positive measures } \nu \text{ in } \Omega,$$

along with their weak-type counterparts, where $\|\nu\| = \nu(\Omega)$, and G is an integral operator with nonnegative kernel,

$$\mathbf{G}\nu(x) = \int_\Omega G(x, y)d\nu(y).$$

These problems are motivated by sublinear elliptic equations in a domain $\Omega \subset \mathbb{R}^n$ with non-trivial Green's function $G(x, y)$ associated with the Laplacian, fractional Laplacian, or more general elliptic operator.

We also treat fractional maximal operators M_α $(0 \leq \alpha < n)$ on $\mathbb{R}^n$, and characterize strong- and weak-type $(1, q)$-inequalities for M_α and more general maximal operators, as well as $(1, q)$-Carleson measure inequalities for Poisson integrals.

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1 Introduction

In this paper, we discuss recent results on weighted norm inequalities of $(1, q)$- type in the case $0 < q < 1$,

$$\|\mathbf{G}\nu\|_{L^q(\Omega, d\sigma)} \leq C \, \|\nu\|, \tag{1}$$

for all positive measures ν in Ω, where $\|\nu\| = \nu(\Omega)$, and $\mathbf{G}$ is an integral operator with nonnegative kernel,

$$\mathbf{G}\nu(x) = \int_\Omega G(x, y) d\nu(y).$$

Such problems are motivated by sublinear elliptic equations of the type

$$\begin{cases} -\Delta u = \sigma u^q \quad \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

in the case $0 < q < 1$, where Ω is an open set in $\mathbb{R}^n$ with non-trivial Green's function $G(x, y)$, and $\sigma \geq 0$ is an arbitrary locally integrable function, or locally finite measure in Ω.

The only restrictions imposed on the kernel G are that it is quasi-symmetric and satisfies a weak maximum principle. In particular, $\mathbf{G}$ can be a Green operator associated with the Laplacian, a more general elliptic operator (including the fractional Laplacian), or a convolution operator on $\mathbb{R}^n$ with radially symmetric decreasing kernel $G(x, y) = k(|x - y|)$ (see [1, 12]).

As an example, we consider in detail the one-dimensional case where $\Omega = \mathbb{R}_+$ and $G(x, y) = \min(x, y)$. We deduce explicit characterizations of the corresponding $(1, q)$-weighted norm inequalities, give explicit necessary and sufficient conditions for the existence of weak solutions, and obtain sharp two-sided pointwise estimates of solutions.

We also characterize weak-type counterparts of (1), namely,

$$\|\mathbf{G}\nu\|_{L^{q,\infty}(\Omega, d\sigma)} \leq C \, \|\nu\|. \tag{2}$$

Along with integral operators, we treat fractional maximal operators M_α with $0 \leq \alpha < n$ on $\mathbb{R}^n$, and characterize both strong- and weak-type $(1, q)$-inequalities for M_α, and more general maximal operators. Similar problems for Riesz potentials were studied earlier in [6–8]. Finally, we apply our results to the Poisson kernel to characterize $(1, q)$-Carleson measure inequalities.

2 Integral Operators

2.1 Strong-Type $(1,q)$-Inequality for Integral Operators

Let $\Omega \subseteq \mathbb{R}^n$ be a connected open set. By $\mathscr{M}^+(\Omega)$ we denote the class of all nonnegative locally finite Borel measures in Ω. Let $G\colon \Omega \times \Omega \to [0,+\infty]$ be a nonnegative lower-semicontinuous kernel. We will assume throughout this paper that G is quasi-symmetric, i.e., there exists a constant $a > 0$ such that

$$a^{-1}\, G(x,y) \le G(y,x) \le a\, G(x,y), \quad x,y \in \Omega. \tag{3}$$

If $\nu \in \mathscr{M}^+(\Omega)$, then by $\mathbf{G}\nu$ and $\mathbf{G}^*\nu$ we denote the integral operators (potentials) defined respectively by

$$\mathbf{G}\nu(x) = \int_\Omega G(x,y)\, d\nu(y), \quad \mathbf{G}^*\nu(x) = \int_\Omega G(y,x)\, d\nu(y), \quad x \in \Omega. \tag{4}$$

We say that the kernel G satisfies the *weak maximum principle* if, for any constant $M > 0$, the inequality

$$\mathbf{G}\nu(x) \le M \quad \text{for all } x \in S(\nu)$$

implies

$$\mathbf{G}\nu(x) \le hM \quad \text{for all } x \in \Omega,$$

where $h \ge 1$ is a constant, and $S(\nu) := \operatorname{supp}\nu$. When $h = 1$, we say that $\mathbf{G}\nu$ satisfies the *strong maximum principle*.

It is well-known that Green's kernels associated with many partial differential operators are quasi-symmetric, and satisfy the weak maximum principle (see, e.g., [2, 3, 12]).

The kernel G is said to be *degenerate* with respect to $\sigma \in \mathscr{M}^+(\Omega)$ provided there exists a set $A \subset \Omega$ with $\sigma(A) > 0$ and

$$G(\cdot, y) = 0 \quad d\sigma - \text{a.e. for } y \in A.$$

Otherwise, we will say that G is *non-degenerate* with respect to σ. (This notion was introduced in [19] in the context of (p,q)-inequalities for positive operators $T\colon L^p \to L^q$ in the case $1 < q < p$.)

Let $0 < q < 1$, and let G be a kernel on $\Omega \times \Omega$. For $\sigma \in \mathscr{M}^+(\Omega)$, we consider the problem of the existence of a *positive solution* u to the integral equation

$$u = \mathbf{G}(u^q d\sigma) \quad \text{in } \Omega, \quad 0 < u < +\infty \;\; d\sigma\text{–a.e.}, \quad u \in L^q_{\mathrm{loc}}(\Omega). \tag{5}$$

We call u a positive *supersolution* if

$$u \geq \mathbf{G}(u^q d\sigma) \quad \text{in } \Omega, \quad 0 < u < +\infty \ d\sigma\text{–a.e.}, \quad u \in L^q_{\mathrm{loc}}(\Omega). \tag{6}$$

This is a generalization of the sublinear elliptic problem (see, e.g., [4, 5], and the literature cited there):

$$\begin{cases} -\Delta u = \sigma u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where σ is a nonnegative locally integrable function, or measure, in Ω.

If Ω is a bounded C^2-domain then solutions to (7) can be understood in the "very weak" sense (see, e.g., [13]). For general domains Ω with a nontrivial Green function G associated with the Dirichlet Laplacian Δ in Ω, solutions u are understood as in (5).

Remark 2.1 In this paper, for the sake of simplicity, we sometimes consider positive solutions and supersolutions $u \in L^q(\Omega, d\sigma)$. In other words, we replace the natural local condition $u \in L^q_{\mathrm{loc}}(\Omega, d\sigma)$ with its global counterpart. Notice that the local condition is necessary for solutions (or supersolutions) to be properly defined.

To pass from solutions u which are globally in $L^q(\Omega, d\sigma)$ to all solutions $u \in L^q_{\mathrm{loc}}(\Omega, d\sigma)$ (for instance, very weak solutions to (7)), one can use either a localization method developed in [8] (in the case of Riesz kernels on $\mathbb{R}^n$), or *modified* kernels $\tilde{G}(x,y) = \frac{G(x,y)}{m(x)\, m(y)}$, where the modifier $m(x) = \min\Big(1, G(x, x_0)\Big)$ (with a fixed pole $x_0 \in \Omega$) plays the role of a regularized distance to the boundary $\partial\Omega$. One also needs to consider the corresponding $(1, q)$-inequalities with a weight m (see [16]). See the next section in the one-dimensional case where $\Omega = (0, +\infty)$.

Remark 2.2 Finite energy solutions, for instance, solutions $u \in W^{1,2}_0(\Omega)$ to (7), require the global condition $u \in L^{1+q}(\Omega, d\sigma)$, and are easier to characterize (see [6]).

The following theorem is proved in [16]. (The case where $\Omega = \mathbb{R}^n$ and $\mathbf{G} = (-\Delta)^{-\frac{\alpha}{2}}$ is the Riesz potential of order $\alpha \in (0, n)$ was considered earlier in [8].)

Theorem 2.3 *Let $\sigma \in \mathscr{M}^+(\Omega)$, and $0 < q < 1$. Suppose G is a quasi-symmetric kernel which satisfies the weak maximum principle. Then the following statements are equivalent:*

(1) *There exists a positive constant $\varkappa = \varkappa(\sigma)$ such that*

$$\|\mathbf{G}\nu\|_{L^q(\sigma)} \leq \varkappa \|\nu\| \quad \textit{for all } \nu \in \mathscr{M}^+(\Omega).$$

(2) *There exists a positive supersolution $u \in L^q(\Omega, d\sigma)$ to (6).*
(3) *There exists a positive solution $u \in L^q(\Omega, d\sigma)$ to (5), provided additionally that G is non-degenerate with respect to σ.*

Remark 2.4 The implication (1) $\Rightarrow$ (2) in Theorem 2.3 holds for any nonnegative kernel G, without assuming that it is either quasi-symmetric, or satisfies the weak maximum principle. This is a consequence of Gagliardo's lemma [10, 21]; see details in [16].

Remark 2.5 The implication (3) $\Rightarrow$ (1) generally fails for kernels G which do not satisfy the weak maximum principle (see examples in [16]).

The following corollary of Theorem 2.3 is obtained in [16].

Corollary 2.6 *Under the assumptions of Theorem 2.3, if there exists a positive supersolution $u \in L^q(\Omega, \sigma)$ to (6), then $\mathbf{G}\sigma \in L^{\frac{q}{1-q}}(\Omega, d\sigma)$.*

Conversely, if $\mathbf{G}\sigma \in L^{\frac{q}{1-q},1}(\Omega, d\sigma)$, then there exists a non-trivial supersolution $u \in L^q(\Omega, \sigma)$ to (6) (respectively, a solution u, provided G is non-degenerate with respect to σ).

2.2 The One-Dimensional Case

In this section, we consider positive weak solutions to sublinear ODEs of the type (7) on the semi-axis $\mathbb{R}_+ = (0, +\infty)$. It is instructive to consider the one-dimensional case where elementary characterizations of $(1, q)$-weighed norm inequalities, along with the corresponding existence theorems and explicit global pointwise estimates of solutions are available. Similar results hold for sublinear equations on any interval $(a, b) \subset \mathbb{R}$.

Let $0 < q < 1$, and let $\sigma \in \mathscr{M}^+(\mathbb{R}_+)$. Suppose u is a positive weak solution to the equation

$$- u'' = \sigma u^q \quad \text{on } \mathbb{R}_+, \quad u(0) = 0, \tag{8}$$

such that $\lim_{x\to+\infty} \frac{u(x)}{x} = 0$. This condition at infinity ensures that u does not contain a linear component. Notice that we assume that u is concave and increasing on $[0, +\infty)$, and $\lim_{x\to 0+} u(x) = 0$.

In terms of integral equations, we have $\Omega = \mathbb{R}_+$, and $G(x, y) = \min(x, y)$ is the Green function associated with the Sturm-Liouville operator $\Delta u = u''$ with zero boundary condition at $x = 0$. Thus, (8) is equivalent to the equation

$$u(x) = \mathbf{G}(u^q d\sigma)(x) := \int_0^{+\infty} \min(x, y) u(y)^q d\sigma(y), \quad x > 0, \tag{9}$$

where σ is a locally finite measure on $\mathbb{R}_+$, and

$$\int_0^a y\, u(y)^q d\sigma(y) < +\infty, \quad \int_a^{+\infty} u(y)^q d\sigma(y) < +\infty, \quad \text{for every } a > 0. \tag{10}$$

This "local integrability" condition ensures that the right-hand side of (9) is well defined. Here intervals $(a, +\infty)$ are used in place of balls $B(x, r)$ in $\mathbb{R}^n$.

Notice that

$$u'(x) = \int_x^{+\infty} u(y)^q d\sigma(y), \quad x > 0. \tag{11}$$

Hence, u satisfies the global integrability condition

$$\int_0^{+\infty} u(y)^q d\sigma(y) < +\infty \tag{12}$$

if and only if $u'(0) < +\infty$.

The corresponding $(1, q)$-weighted norm inequality is given by

$$\|\mathbf{G}\nu\|_{L^q(\sigma)} \le \varkappa \|\nu\|, \tag{13}$$

where $\varkappa = \varkappa(\sigma)$ is a positive constant which does not depend on $\nu \in \mathscr{M}^+(\mathbb{R}_+)$. Obviously, (13) is equivalent to

$$\|H_+\nu + H_-\nu\|_{L^q(\sigma)} \le \varkappa \|\nu\| \quad \text{for all } \nu \in \mathscr{M}^+(\mathbb{R}_+), \tag{14}$$

where $H_\pm$ is a pair of Hardy operators,

$$H_+\nu(x) = \int_0^x y\, d\nu(y), \quad H_-\nu(x) = x \int_x^{+\infty} d\nu(y).$$

The following proposition can be deduced from the known results on two-weight Hardy inequalities in the case $p = 1$ and $0 < q < 1$ (see, e.g., [20]). We give here a simple independent proof.

Proposition 2.7 *Let* $\sigma \in \mathscr{M}^+(\mathbb{R}_+)$*, and let* $0 < q < 1$*. Then* (13) *holds if and only if*

$$\varkappa(\sigma)^q = \int_0^{+\infty} x^q d\sigma(x) < +\infty, \tag{15}$$

where $\varkappa(\sigma)$ *is the best constant in* (13)*.*

Proof Clearly,

$$H_+\nu(x) + H_-\nu(x) \le x \|\nu\|, \quad x > 0.$$

Hence,

$$\|H_+\nu + H_-\nu\|_{L^q(\sigma)} \le \Big(\int_0^{+\infty} x^q d\sigma(x)\Big)^{\frac{1}{q}} \|\nu\|,$$

which proves (14), and hence (13), with $\varkappa = \Big(\int_0^{+\infty} x^q d\sigma(x)\Big)^{\frac{1}{q}}$.

Conversely, suppose that (14) holds. Then, for every $a > 0$, and $\nu \in \mathscr{M}^+(\mathbb{R}_+)$,

$$\begin{aligned}
&\Big(\int_0^a x^q d\sigma(x)\Big)\Big(\int_a^{+\infty} d\nu(y)\Big)^q \\
&\le \int_0^a \Big(x \int_x^{+\infty} d\nu(y)\Big)^q d\sigma(x) \\
&\le \int_0^{+\infty} (H_-\nu)^q d\sigma \le \varkappa^q \|\nu\|^q.
\end{aligned}$$

For $\nu = \delta_{x_0}$ with $x_0 > a$, we get

$$\int_0^a x^q d\sigma(x) \le \varkappa^q.$$

Letting $a \to +\infty$, we deduce (15). □

Clearly, the Green kernel $G(x, y) = \min(x, y)$ is symmetric, and satisfies the strong maximum principle. Hence, by Theorem 2.3, Eqs. (8) and (9) have a non-trivial (super)solution $u \in L^q(\mathbb{R}_+, \sigma)$ if and only if (15) holds.

From Proposition 2.7, we deduce that, for "localized" measures $d\sigma_a = \chi_{(a,+\infty)} d\sigma$ $(a > 0)$, we have

$$\varkappa(\sigma_a) = \Big(\int_a^{+\infty} x^q d\sigma(x)\Big)^{\frac{1}{q}}. \tag{16}$$

Using this observation and the localization method developed in [8], we obtain the following existence theorem for general weak solutions to (7), along with sharp pointwise estimates of solutions.

We introduce a new potential

$$\mathbf{K}\sigma(x) := x\Big(\int_x^{+\infty} y^q d\sigma(y)\Big)^{\frac{1}{1-q}}, \quad x > 0. \tag{17}$$

We observe that $\mathbf{K}\sigma$ is a one-dimensional analogue of the potential introduced recently in [8] in the framework of intrinsic Wolff potentials in $\mathbb{R}^n$ (see also [7] in the radial case). Matching upper and lower pointwise bounds of solutions are obtained below by combining $\mathbf{G}\sigma$ with $\mathbf{K}\sigma$.

Theorem 2.8 *Let $\sigma \in \mathscr{M}^+(\mathbb{R}_+)$, and let $0 < q < 1$. Then Eq.* (7)*, or equivalently* (8) *has a nontrivial solution if and only if, for every $a > 0$,*

$$\int_0^a x\, d\sigma(x) + \int_a^{+\infty} x^q d\sigma(x) < +\infty. \tag{18}$$

Moreover, if (18) *holds, then there exists a positive solution u to* (7) *such that*

$$C^{-1}\Big[\Big(\int_0^x y\, d\sigma(y)\Big)^{\frac{1}{1-q}} + \mathbf{K}\sigma(x)\Big] \tag{19}$$

$$\leq u(x) \leq C\Big[\Big(\int_0^x y\, d\sigma(y)\Big)^{\frac{1}{1-q}} + \mathbf{K}\sigma(x)\Big]. \tag{20}$$

The lower bound in (19) *holds for any non-trivial supersolution u.*

Remark 2.9 The lower bound

$$u(x) \geq (1-q)^{\frac{1}{1-q}}\Big[\mathbf{G}\sigma(x)\Big]^{\frac{1}{1-q}}, \quad x > 0, \tag{21}$$

is known for a general kernel G which satisfies the strong maximum principle (see [11], Theorem 3.3; [16]), and the constant $(1-q)^{\frac{1}{1-q}}$ here is sharp. However, the second term on the left-hand side of (19) makes the lower estimate stronger, so that it matches the upper estimate.

Proof The lower bound

$$u(x) \geq (1-q)^{\frac{1}{1-q}}\Big[\int_0^x y\, d\sigma(y)\Big]^{\frac{1}{1-q}}, \quad x > 0, \tag{22}$$

is immediate from (21).

Applying Lemma 4.2 in [8], with the interval $(a, +\infty)$ in place of a ball B, and combining it with (16), for any $a > 0$ we have

$$\int_a^{+\infty} u(y)^q d\sigma(y) \geq c(q)\varkappa(\sigma_a)^{\frac{q}{1-q}} = c(q)\Big[\int_a^{+\infty} y^q d\sigma(y)\Big]^{\frac{1}{1-q}}.$$

Hence,

$$u(x) \geq \mathbf{G}(u^q d\sigma) \geq x\int_x^{+\infty} u(y)^q d\sigma(y) \geq c(q)\, x\Big[\int_x^{+\infty} y^q d\sigma(y)\Big]^{\frac{1}{1-q}}.$$

Combining the preceding estimate with (22), we obtain the lower bound in (19) for any non-trivial supersolution u. This also proves that (18) is necessary for the existence of a non-trivial positive supersolution.

Conversely, suppose that (18) holds. Let

$$v(x) := c\left[\left(\int_0^x y\, d\sigma(y)\right)^{\frac{1}{1-q}} + \mathbf{K}\sigma(x)\right], \quad x > 0, \tag{23}$$

where c is a positive constant. It is not difficult to see that v is a supersolution, so that $v \geq \mathbf{G}(v^q d\sigma)$, if $c = c(q)$ is picked large enough. (See a similar argument in the proof of Theorem 5.1 in [7].)

Also, it is easy to see that $v_0 = c_0(\mathbf{G}\sigma)^{\frac{1}{1-q}}$ is a subsolution, i.e., $v_0 \leq \mathbf{G}(v_0^q d\sigma)$, provided $c_0 > 0$ is a small enough constant. Moreover, we can ensure that $v_0 \leq v$ if $c_0 = c_0(q)$ is picked sufficiently small. (See details in [7] in the case of radially symmetric solutions in $\mathbb{R}^n$.) Hence, there exists a solution which can be constructed by iterations, starting from $u_0 = v_0$, and letting

$$u_{j+1} = \mathbf{G}(u_j^q d\sigma), \quad j = 0, 1, \ldots.$$

Then by induction $u_j \leq u_{j+1} \leq v$, and consequently $u = \lim_{j\to+\infty} u_j$ is a solution to (9) by the Monotone Convergence Theorem. Clearly, $u \leq v$, which proves the upper bound in (19). □

2.3 Weak-Type $(1, q)$-Inequality for Integral Operators

In this section, we characterize weak-type analogues of $(1, q)$-weighted norm inequalities considered above. We will use some elements of potential theory for general positive kernels G, including the notion of *inner capacity*, $\text{cap}(\cdot)$, and the associated *equilibrium* (extremal) measure (see [9]).

Theorem 2.10 *Let $\sigma \in \mathscr{M}^+(\Omega)$, and $0 < q < 1$. Suppose G satisfies the weak maximum principle. Then the following statements are equivalent:*

(1) *There exists a positive constant $\varkappa_w$ such that*

$$\|\mathbf{G}\nu\|_{L^{q,\infty}(\sigma)} \leq \varkappa_w \|\nu\| \quad \textit{for all } \nu \in \mathscr{M}^+(\Omega).$$

(2) *There exists a positive constant c such that*

$$\sigma(K) \leq c\Big(\text{cap}(K)\Big)^q \quad \textit{for all compact sets } K \subset \Omega.$$

(3) $\mathbf{G}\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$.

Proof (1) ⇒ (2) Without loss of generality we may assume that the kernel G is *strictly positive*, that is, $G(x, x) > 0$ for all $x \in \Omega$. Otherwise, we can consider the kernel G on the set $\Omega \setminus A$, where $A := \{x \in \Omega : G(x, x) \neq 0\}$, since A is negligible for the corresponding $(1, q)$-inequality in statement (1). (See details in [16] in the case of the corresponding strong-type inequalities.)

We remark that the kernel G is known to be strictly positive if and only if, for any compact set $K \subset \Omega$, the inner capacity $\mathrm{cap}(K)$ is finite [9]. In this case there exists an equilibrium measure λ on K such that

$$\mathbf{G}\lambda \geq 1 \text{ n.e. on } K, \quad \mathbf{G}\lambda \leq 1 \text{ on } S(\lambda), \quad \|\lambda\| = \mathrm{cap}(K). \tag{24}$$

Here n.e. stands for *nearly everywhere*, which means that the inequality holds on a given set except for a subset of zero capacity [9].

Next, we remark that condition (1) yields that σ is absolutely continuous with respect to capacity, i.e., $\sigma(K) = 0$ if $\mathrm{cap}(K) = 0$. (See a similar argument in [16] in the case of strong-type inequalities.) Consequently, $\mathbf{G}\lambda \geq 1$ $d\sigma$-a.e. on K. Hence, by applying condition (1) with $\nu = \lambda$, we obtain (2).

(2) $\Rightarrow$ (3) We denote by σ_E the restriction of σ to a Borel set $E \subset \Omega$. Without loss of generality we may assume that σ is a finite measure on Ω. Otherwise we can replace σ with σ_F where F is a compact subset of Ω. We then deduce the estimate

$$\|\mathbf{G}\sigma_F\|_{L^{\frac{q}{1-q},\infty}(\sigma_F)} \leq C < \infty,$$

where C does not depend on F, and use the exhaustion of Ω by an increasing sequence of compact subsets $F_n \uparrow \Omega$ to conclude that $\mathbf{G}\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$ by the Monotone Convergence Theorem.

Set $E_t := \{x \in \Omega \colon \mathbf{G}\sigma(x) > t\}$, where $t > 0$. Notice that, for all $x \in (E_t)^c$,

$$\mathbf{G}\sigma_{(E_t)^c}(x) \leq \mathbf{G}\sigma(x) \leq t.$$

The set $(E_t)^c$ is closed, and hence the preceding inequality holds on $S(\sigma_{(E_t)^c})$. It follows by the weak maximum principle that, for all $x \in \Omega$,

$$\mathbf{G}\sigma_{(E_t)^c}(x) \leq \mathbf{G}\sigma(x) \leq h\, t.$$

Hence,

$$\{x \in \Omega \colon \mathbf{G}\sigma(x) > (h+1)t\} \subset \{x \in \Omega \colon \mathbf{G}\sigma_{E_t}(x) > t\}. \tag{25}$$

Denote by $K \subset \Omega$ a compact subset of $\{x \in \Omega \colon \mathbf{G}\sigma_{E_t}(x) > t\}$. By (2), we have

$$\sigma(K) \leq c \left(\mathrm{cap}(K) \right)^q$$

If λ is the equilibrium measure on K, then $\mathbf{G}\lambda \le 1$ on $S(\lambda)$, and $\lambda(K) = \text{cap}(K)$ by (24). Hence by the weak maximum principle $\mathbf{G}\lambda \le h$ on Ω. Using quasi-symmetry of the kernel G and Fubini's theorem, we have

$$\begin{aligned} \text{cap}(K) &= \int_K d\lambda \\ &\le \frac{1}{t}\int_K \mathbf{G}\sigma_{E_t}\, d\lambda \\ &\le \frac{a}{t}\int_{E_t} \mathbf{G}\lambda d\sigma \\ &\le \frac{ah}{t}\sigma(E_t). \end{aligned}$$

This shows that

$$\sigma(K) \le \frac{c(ah)^q}{t^q}\Big(\sigma(E_t)\Big)^q.$$

Taking the supremum over all $K \subset E_t$, we deduce

$$\Big(\sigma(E_t)\Big)^{1-q} \le \frac{c(ah)^q}{t^q}.$$

It follows from the preceding estimate and (25) that, for all $t > 0$,

$$t^{\frac{q}{1-q}}\sigma\Big(\{x \in \Omega\colon \mathbf{G}\sigma(x) > (h+1)t\}\Big) \le t^{\frac{q}{1-q}}\sigma(E_t) \le c^{\frac{1}{1-q}}(ah)^{\frac{q}{1-q}}.$$

Thus, (3) holds.

(3) $\Rightarrow$ (2) By Hölder's inequality for weak L^q spaces, we have

$$\begin{aligned} \|\mathbf{G}\nu\|_{L^{q,\infty}(\sigma)} &= \left\|\frac{\mathbf{G}\nu}{\mathbf{G}\sigma}\mathbf{G}\sigma\right\|_{L^{q,\infty}(\sigma)} \\ &\le \left\|\frac{\mathbf{G}\nu}{\mathbf{G}\sigma}\right\|_{L^{1,\infty}(\sigma)} \|\mathbf{G}\sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)} \\ &\le C\,\|\mathbf{G}\sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)}\,\|\nu\|, \end{aligned}$$

where the final inequality,

$$\left\|\frac{\mathbf{G}\nu}{\mathbf{G}\sigma}\right\|_{L^{1,\infty}(\sigma)} \le C\,\|\nu\|,$$

with a constant $C = C(h, a)$, was obtained in [16], for quasi-symmetric kernels G satisfying the weak maximum principle. □

3 Fractional Maximal Operators

Let $0 \le \alpha < n$, and let $\nu \in \mathscr{M}^+(\mathbb{R}^n)$. The fractional maximal function $M_\alpha \nu$ is defined by

$$M_\alpha \nu(x) := \sup_{Q \ni x} \frac{|Q|_\nu}{|Q|^{1-\frac{\alpha}{n}}}, \quad x \in \mathbb{R}^n, \tag{26}$$

where Q is a cube, $|Q|_\nu := \nu(Q)$, and $|Q|$ is the Lebesgue measure of Q. If $f \in L^1_{\text{loc}}(\mathbb{R}^n, d\mu)$ where $\mu \in \mathscr{M}^+(\mathbb{R}^n)$, we set $M_\alpha(fd\mu) = M_\alpha \nu$ where $d\nu = |f| d\mu$, i.e.,

$$M_\alpha(fd\mu)(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f| \, d\mu, \quad x \in \mathbb{R}^n. \tag{27}$$

For $\sigma \in \mathscr{M}^+(\mathbb{R}^n)$, it was shown in [22] that in the case $0 < q < p$,

$$M_\alpha \colon L^p(dx) \to L^q(d\sigma) \iff M_\alpha \sigma \in L^{\frac{q}{p-q}}(d\sigma), \tag{28}$$

$$M_\alpha \colon L^p(dx) \to L^{q,\infty}(d\sigma) \iff M_\alpha \sigma \in L^{\frac{q}{p-q},\infty}(d\sigma), \tag{29}$$

provided $p > 1$.

More general two-weight maximal inequalities

$$\|M_\alpha(fd\mu)\|_{L^q(\sigma)} \le \varkappa \, \|f\|_{L^p(\mu)}, \quad \text{for all } f \in L^p(\mu), \tag{30}$$

where characterized by E.T. Sawyer [18] in the case $p = q > 1$, R.L. Wheeden [24] in the case $q > p > 1$, and the second author [22] in the case $0 < q < p$ and $p > 1$, along with their weak-type counterparts,

$$\|M_\alpha(fd\mu)\|_{L^{q,\infty}(\sigma)} \le \varkappa_w \, \|f\|_{L^p(\mu)}, \quad \text{for all } f \in L^p(\mu), \tag{31}$$

where $\sigma, \mu \in \mathscr{M}^+(\mathbb{R}^n)$, and $\varkappa, \varkappa_w$ are positive constants which do not depend on f.

However, some of the methods used in [22] for $0 < q < p$ and $p > 1$ are not directly applicable in the case $p = 1$, although there are analogues of these results for real Hardy spaces, i.e., when the norm $\|f\|_{L^p(\mu)}$ on the right-hand side of (30) or (31) is replaced with $\|M_\mu f\|_{L^p(\mu)}$, where

$$M_\mu f(x) := \sup_{Q \ni x} \frac{1}{|Q|_\mu} \int_Q |f| d\mu. \tag{32}$$

We would like to understand similar problems in the case $0 < q < 1$ and $p = 1$, in particular, when $M_\alpha \colon \mathscr{M}^+(\mathbb{R}^n) \to L^q(d\sigma)$, or equivalently, there exists a constant

$\varkappa > 0$ such that the inequality

$$\|M_\alpha \nu\|_{L^q(\sigma)} \le \varkappa \, \|\nu\| \tag{33}$$

holds for all $\nu \in \mathscr{M}^+(\mathbb{R}^n)$.

In the case $\alpha = 0$, Rozin [17] showed that the condition

$$\sigma \in L^{\frac{1}{1-q},1}(\mathbb{R}^n, dx)$$

is sufficient for the Hardy-Littlewood operator $M = M_0 \colon L^1(dx) \to L^q(\sigma)$ to be bounded; moreover, when σ is radially symmetric and decreasing, this is also a necessary condition. We will generalize this result and provide necessary and sufficient conditions for the range $0 \le \alpha < n$. We also obtain analogous results for the weak-type inequality

$$\|M_\alpha \nu\|_{L^{q,\infty}(\sigma)} \le \varkappa_w \, \|\nu\|, \quad \text{for all } \nu \in \mathscr{M}^+(\mathbb{R}^n), \tag{34}$$

where $\varkappa_w$ is a positive constant which does not depend on ν.

We treat more general maximal operators as well, in particular, dyadic maximal operators

$$M_\rho \nu(x) := \sup_{Q \in \mathscr{Q}: Q \ni x} \rho_Q \, |Q|_\nu, \tag{35}$$

where $\mathscr{Q}$ is the family of all dyadic cubes in $\mathbb{R}^n$, and $\{\rho_Q\}_{Q \in \mathscr{Q}}$ is a fixed sequence of nonnegative reals associated with $Q \in \mathscr{Q}$. The corresponding weak-type maximal inequality is given by

$$\|M_\rho \nu\|_{L^{q,\infty}(\sigma)} \le \varkappa_w \, \|\nu\|, \quad \text{for all } \nu \in \mathscr{M}^+(\mathbb{R}^n). \tag{36}$$

3.1 Strong-Type Inequality

Theorem 3.1 *Let $\sigma \in M^+(\mathbb{R}^n)$, $0 < q < 1$, and $0 \le \alpha < n$. The inequality (33) holds if and only if there exists a function $u \not\equiv 0$ such that*

$$u \in L^q(\sigma), \quad \text{and} \quad u \ge M_\alpha(u^q \sigma).$$

Moreover, u can be constructed as follows: $u = \lim_{j\to\infty} u_j$, where $u_0 := (M_\alpha \sigma)^{\frac{1}{1-q}}$, $u_{j+1} \ge u_j$, and

$$u_{j+1} := M_\alpha(u_j^q \sigma), \quad j = 0, 1, \ldots. \tag{37}$$

In particular, $u \ge (M_\alpha \sigma)^{\frac{1}{1-q}}$.

Proof ($\Rightarrow$) We let $u_0 := (M_\alpha \sigma)^{\frac{1}{1-q}}$. Notice that, for all $x \in Q$, we have $u_0(x) \geq \left(\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\right)^{\frac{1}{1-q}}$. Hence,

$$u_1(x) := M_\alpha(u_0^q d\sigma)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q u_0^q d\sigma \geq \sup_{Q \ni x} \left(\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\right)^{\frac{1}{1-q}} = u_0(x).$$

By induction, we see that

$$u_{j+1} := M_\alpha(u_j^q d\sigma) \geq M_\alpha(u_{j-1}^q d\sigma) = u_j, \quad j = 1, 2, \ldots.$$

Let $u = \lim u_j$. By (33), we have

$$\begin{aligned} \|u_{j+1}\|_{L^q(\sigma)} &= \|M_\alpha(u_j^q \sigma)\|_{L^q(\sigma)} \\ &\leq \varkappa \|u_j\|_{L^q(\sigma)}^q \\ &\leq \varkappa \|u_{j+1}\|_{L^q(\sigma)}^q. \end{aligned}$$

From this we deduce that $\|u_{j+1}\|_{L^q(\sigma)} \leq \varkappa^{\frac{1}{1-q}}$ for $j = 0, 1, \ldots$. Since the norms $\|u_j\|_{L^q(\sigma)}^q$ are uniformly bounded, by the Monotone Convergence Theorem, we have for $u := \lim_{j\to\infty} u_j$ that $u \in L^q(\sigma)$. Note that by construction $u = M_\alpha(u^q d\sigma)$.

($\Leftarrow$) We can assume here that $M_\alpha \nu$ is defined, for $\nu \in \mathscr{M}^+(\mathbb{R}^n)$, as the centered fractional maximal function,

$$M_\alpha \nu(x) := \sup_{r>0} \frac{\nu(B(x,r))}{|B(x,r)|^{1-\frac{\alpha}{n}}},$$

since it is equivalent to its uncentered analogue used above. Suppose that there exists $u \in L^q(\sigma)$ ($u \not\equiv 0$) such that $u \geq M_\alpha(u^q d\sigma)$. Set $\omega := u^q d\sigma$. Let $\nu \in \mathscr{M}^+(\mathbb{R}^n)$.

We note that we have

$$\begin{aligned} \frac{M_\alpha \nu(x)}{M_\alpha \omega(x)} &= \frac{\sup_{r>0} \frac{|B(x,r)|_\nu}{|B(x,r)|^{1-\frac{\alpha}{n}}}}{\sup_{\rho>0} \frac{|B(x,\rho)|_\omega}{|B(x,\rho)|^{1-\frac{\alpha}{n}}}} \\ &\leq \sup_{r>0} \frac{|B(x,r)|_\nu}{|B(x,r)|_\omega} \\ &=: M_\omega \nu(x). \end{aligned}$$

Thus,

$$\begin{aligned}\|M_\alpha \nu\|_{L^q(\sigma)} &= \left\| \frac{M_\alpha \nu}{M_\alpha \omega} \right\|_{L^q((M_\alpha \omega)^q d\sigma)} \\ &\leq \left\| \frac{M_\alpha \nu}{M_\alpha \omega} \right\|_{L^q(d\omega)} \\ &\leq \|M_\omega \nu\|_{L^q(d\omega)} \\ &\leq C\, \|M_\omega \nu\|_{L^{1,\infty}(\omega)} \leq C\|\nu\|,\end{aligned}$$

by Jensen's inequality and the $(1, 1)$-weak-type maximal function inequality for $M_\omega \nu$. This establishes (33). □

3.2 Weak-Type Inequality

For $0 \leq \alpha < n$, we define the *Hausdorff content* on a set $E \subset \mathbb{R}^n$ to be

$$H^{n-\alpha}(E) := \inf \left\{ \sum_{i=1}^{\infty} r_i^{n-\alpha} : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \right\} \tag{38}$$

where the collection of balls $\{B(x_i, r_i)\}$ forms a countable covering of E (see [1, 15]).

Theorem 3.2 *Let $\sigma \in M^+(\mathbb{R}^n)$, $0 < q < 1$, and $0 \leq \alpha < n$. Then the following conditions are equivalent:*

(1) *There exists a positive constant $\varkappa_w$ such that*

$$\|M_\alpha \nu\|_{L^{q,\infty}(\sigma)} \leq \varkappa_w \|\nu\| \quad \text{for all } \nu \in \mathscr{M}(\mathbb{R}^n).$$

(2) *There exists a positive constant $C > 0$ such that*

$$\sigma(E) \leq C\,(H^{n-\alpha}(E))^q \quad \text{for all Borel sets } E \subset \mathbb{R}^n.$$

(3) $M_\alpha \sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$.

Remark 3.3 In the case $\alpha = 0$ each of the conditions (1)–(3) is equivalent to $\sigma \in L^{\frac{1}{1-q},\infty}(dx)$.

Proof (1) $\Rightarrow$ (2) Let $K \subset E$ be a compact set in $\mathbb{R}^n$ such that $H^{n-\alpha}(K) > 0$. It follows from Frostman's theorem (see the proof of Theorem 5.1.12 in [1]) that there exists a measure ν supported on K such that $\nu(K) \leq H^{n-\alpha}(K)$, and, for every $x \in K$ there exists a cube Q such that $x \in Q$ and $|Q|_\nu \geq c\,|Q|^{1-\frac{\alpha}{n}}$, where c depends only on n and α. Hence,

$$M_\alpha \nu(x) \geq \sup_{Q \ni x} \frac{|Q|_\nu}{|Q|^{1-\frac{\alpha}{n}}} \geq c \quad \text{for all } x \in K,$$

where c depends only on n and α. Consequently,

$$c^q \, \sigma(K) \leq \|M_\alpha \nu\|^q_{L^{q,\infty}(\sigma)} \leq \varkappa_w^q \Big(H^{n-\alpha}(K)\Big)^q.$$

If $H^{n-\alpha}(E) = 0$, then $H^{n-\alpha}(K) = 0$ for every compact set $K \subset E$, and consequently $\sigma(E) = 0$. Otherwise,

$$\sigma(K) \leq \varkappa_w^q \Big(H^{n-\alpha}(K)\Big)^q \leq \varkappa_w^q \Big(H^{n-\alpha}(K)\Big)^q,$$

for every compact set $K \subset E$, which proves (2) with $C = c^{-q} \varkappa_w^q$.

(2) $\Rightarrow$ (3) Let $E_t := \{x : M_\alpha \sigma(x) > t\}$, where $t > 0$. Let $K \subset E_t$ be a compact set. Then for each $x \in K$ there exists $Q_x \ni x$ such that

$$\frac{\sigma(Q_x)}{|Q_x|^{1-\frac{\alpha}{n}}} > t.$$

Now consider the collection $\{Q_x\}_{x \in K}$, which forms a cover of K. By the Besicovitch covering lemma, we can find a subcover $\{Q_i\}_{i \in I}$, where I is a countable index set, such that $K \subset \bigcup_{i \in I} Q_i$ and $x \in K$ is contained in at most b_n sets in $\{Q_i\}$. By (2), we have

$$\sigma(K) \leq [H^{n-\alpha}(K)]^q,$$

and by the definition of the Hausdorff content we have

$$H^{n-\alpha}(K) \leq \sum |Q_i|^{1-\alpha/n}.$$

Since $\{Q_i\}$ have bounded overlap, we have

$$\sum_{i \in I} \sigma(Q_i) \leq b_n \sigma(K).$$

Thus,

$$\sigma(K) \leq \left(b_n \frac{\sigma(K)}{t}\right)^q,$$

which shows that

$$t^{\frac{q}{1-q}}\sigma(K) \le (b_n)^{\frac{1}{1-q}} < +\infty.$$

Taking the supremum over all $K \subset E_t$ in the preceding inequality, we deduce $M_\alpha\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$.

(3) $\Rightarrow$ (1). We can assume again that M_α is the centered fractional maximal function, since it is equivalent to the uncentered version. Suppose that $M_\alpha\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$. Let $\nu \in \mathscr{M}(\mathbb{R}^n)$. Then, as in the case of the strong-type inequality,

$$\begin{aligned}\frac{M_\alpha\nu(x)}{M_\alpha\sigma(x)} &= \frac{\sup_{r>0}\frac{|B(x,r)|_\nu}{|B(x,r)|^{1-\frac{\alpha}{n}}}}{\sup_{\rho>0}\frac{|B(x,\rho)|_\sigma}{|B(x,\rho)|^{1-\frac{\alpha}{n}}}}\\ &\le \sup_{r>0}\frac{|B(x,r)|_\nu}{|B(x,r)|_\sigma} =: M_\sigma\nu(x).\end{aligned}$$

Thus, by Hölder's inequality for weak L^p-spaces,

$$\begin{aligned}\|M_\alpha\nu\|_{L^{q,\infty}(\sigma)} &\le \|(M_\alpha\sigma)\,(M_\sigma\nu)\|_{L^{q,\infty}(\sigma)}\\ &\le \|M_\alpha\sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)}\,\|M_\sigma\nu\|_{L^{1,\infty}(\sigma)}\\ &\le c\|M_\alpha\sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)}\,\|\nu\|,\end{aligned}$$

where in the last line we have used the $(1, 1)$-weak-type maximal function inequality for the centered maximal function $M_\sigma\nu$. □

We now characterize weak-type $(1, q)$-inequalities (36) for the generalized dyadic maximal operator M_ρ defined by (35). The corresponding (p, q)-inequalities in the case $0 < q < p$ and $p > 1$ were characterized in [22]. The results obtained in [22] for weak-type inequalities remain valid in the case $p = 1$, but some elements of the proofs must be modified as indicated below.

Theorem 3.4 *Let $\sigma \in \mathscr{M}^+(\mathbb{R}^n)$, $0 < q < 1$, and $0 \le \alpha < n$. Then the following conditions are equivalent:*

(1) *There exists a positive constant $\varkappa_w$ such that* (36) *holds.*
(2) $M_\rho\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$.

Proof (2) $\Rightarrow$ (1) The proof of this implication is similar to the case of fractional maximal operators. Let $\nu \in \mathscr{M}(\mathbb{R}^n)$. Denoting by $Q, P \in \mathscr{Q}$ dyadic cubes in $\mathbb{R}^n$, we estimate

$$\frac{M_\rho \nu(x)}{M_\rho \sigma(x)} = \frac{\sup_{Q \ni x}(\rho_Q \, |Q|_\nu)}{\sup_{P \ni x}(\rho_P \, |P|_\sigma)} \le \sup_{Q \ni x} \frac{|Q|_\nu}{|Q|_\sigma} =: M_\sigma \nu(x).$$

By Hölder's inequality for weak L^p-spaces,

$$\begin{aligned} \|M_\rho \nu\|_{L^{q,\infty}(\sigma)} &\le \|(M_\rho \sigma)\,(M_\sigma \nu)\|_{L^{q,\infty}(\sigma)} \\ &\le \|M_\rho \sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)} \, \|M_\sigma \nu\|_{L^{1,\infty}(\sigma)} \\ &\le c \|M_\rho \sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)} \, \|\nu\|, \end{aligned}$$

by the $(1,1)$-weak-type maximal function inequality for the dyadic maximal function M_σ.

(1) $\Rightarrow$ (2) We set $f = \sup_Q(\lambda_Q \chi_Q)$ and $d\nu = f \, d\sigma$, where $\{\lambda_Q\}_{Q \in \mathcal{Q}}$ is a finite sequence of non-negative reals. Then obviously

$$M_\rho \nu(x) \ge \sup_Q(\lambda_Q \rho_Q \chi_Q), \quad \text{and} \quad \|\nu\| \le \sum_Q \lambda_Q \, |Q|_\sigma.$$

By (1), for all $\{\lambda_Q\}_{Q \in \mathcal{Q}}$,

$$\| \sup_Q(\lambda_Q \rho_Q \chi_Q)\|_{L^{q,\infty}(\sigma)} \le \varkappa_v \sum_Q \lambda_Q \, |Q|_\sigma.$$

Hence, by Theorem 1.1 and Remark 1.2 in [22], it follows that (2) holds. □

4 Carleson Measures for Poisson Integrals

In this section we treat $(1,q)$-Carleson measure inequalities for Poisson integrals with respect to Carleson measures $\sigma \in \mathcal{M}^+(\mathbb{R}^{n+1}_+)$ in the upper half-space $\mathbb{R}^{n+1}_+ = \{(x,y) : x \in \mathbb{R}^n, y > 0\}$. The corresponding weak-type (p,q)-inequalities for all $0 < q < p$ as well as strong-type (p,q)-inequalities for $0 < q < p$ and $p > 1$, were characterized in [23]. Here we consider strong-type inequalities of the type

$$\|\mathbf{P}\nu\|_{L^q(\mathbb{R}^{n+1}_+, \sigma)} \le \varkappa \, \|\nu\|_{\mathcal{M}^+(\mathbb{R}^n)}, \quad \text{for all } \nu \in \mathcal{M}^+(\mathbb{R}^n), \tag{39}$$

for some constant $\varkappa > 0$, where $\mathbf{P}\nu$ is the Poisson integral of $\nu \in \mathscr{M}^+(\mathbb{R}^n)$ defined by

$$\mathbf{P}\nu(x,y) := \int_{\mathbb{R}^n} P(x-t,y)d\nu(t), \quad (x,y) \in \mathbb{R}^{n+1}_+.$$

Here $P(x, y)$ denotes the Poisson kernel associated with $\mathbb{R}^{n+1}_+$.

By $\mathbf{P}^*\mu$ we denote the formal adjoint (balayage) operator defined, for $\mu \in \mathscr{M}^+(\mathbb{R}^{n+1}_+)$, by

$$\mathbf{P}^*\mu(t) := \int_{\mathbb{R}^{n+1}_+} P(x-t,y)d\mu(x,y), \quad t \in \mathbb{R}^n.$$

We will also need the symmetrized potential defined, for $\mu \in \mathscr{M}^+(\mathbb{R}^{n+1}_+)$, by

$$\mathbf{PP}^*\mu(x,y) := \mathbf{P}\Big[(\mathbf{P}^*\mu)dt\Big] = \int_{\mathbb{R}^{n+1}_+} P(x-\tilde{x}, y+\tilde{y})d\mu(\tilde{x},\tilde{y}), \quad (x,y) \in \mathbb{R}^{n+1}_+.$$

As we will demonstrate below, the kernel of $\mathbf{PP}^*\mu$ satisfies the weak maximum principle with constant $h = 2^{n+1}$.

Theorem 4.1 *Let $\sigma \in \mathscr{M}^+(\mathbb{R}^{n+1}_+)$, and let $0 < q < 1$. Then inequality (39) holds if and only if there exists a function $u > 0$ such that*

$$u \in L^q(\mathbb{R}^{n+1}_+, \sigma), \quad \textit{and} \quad u \geq \mathbf{PP}^*(u^q\sigma) \quad \text{in } \mathbb{R}^{n+1}_+.$$

Moreover, if (39) holds, then a positive solution $u = \mathbf{PP}^*(u^q\sigma)$ such that $u \in L^q(\mathbb{R}^{n+1}_+, \sigma)$ can be constructed as follows: $u = \lim_{j\to\infty} u_j$, where

$$u_{j+1} := \mathbf{PP}^*(u_j^q\sigma), \quad j = 0, 1, \ldots, \quad u_0 := c_0(\mathbf{PP}^*\sigma)^{\frac{1}{1-q}}, \tag{40}$$

for a small enough constant $c_0 > 0$ (depending only on q and n), which ensures that $u_{j+1} \geq u_j$. In particular, $u \geq c_0\,(\mathbf{PP}^*\sigma)^{\frac{1}{1-q}}$.

Proof We first prove that (39) holds if and only if

$$\|\mathbf{PP}^*\mu\|_{L^q(\mathbb{R}^{n+1}_+,\sigma)} \leq \varkappa\, \|\mu\|_{\mathscr{M}^+(\mathbb{R}^{n+1}_+)}, \quad \text{for all } \mu \in \mathscr{M}^+(\mathbb{R}^{n+1}_+). \tag{41}$$

Indeed, letting $\nu = \mathbf{P}^*\mu$ in (39) yields (41) with the same embedding constant $\varkappa$.

Conversely, suppose that (41) holds. Then by Maurey's factorization theorem (see [14]), there exists $F \in L^1(\mathbb{R}^{n+1}_+, \sigma)$ such that $F > 0$ $d\sigma$-a.e., and

$$\|F\|_{L^1(\mathbb{R}^{n+1}_+,\sigma)} \leq 1, \qquad \sup_{(x,y)\in\mathbb{R}^{n+1}_+} \mathbf{PP}^*(F^{1-\frac{1}{q}}d\sigma)(x,y) \leq \varkappa. \tag{42}$$

By letting $y \downarrow 0$ in (42) and using the Monotone Convergence Theorem, we deduce

$$\sup_{x\in\mathbb{R}^n} \mathbf{P}^*(F^{1-\frac{1}{q}} d\sigma)(x) \le \varkappa. \tag{43}$$

Hence, by Jensen's inequality and (43), for any $\nu \in \mathscr{M}^+(\mathbb{R}^n)$, we have

$$\|\mathbf{P}\nu\|_{L^q(\mathbb{R}^{n+1}_+,\sigma)} \le \|\mathbf{P}\nu\|_{L^1(\mathbb{R}^{n+1}_+, F^{1-\frac{1}{q}} d\sigma)} = \|\mathbf{P}^*(F^{1-\frac{1}{q}} d\sigma)\|_{L^1(\mathbb{R}^n, d\nu)} \le \varkappa\, \|\nu\|_{\mathscr{M}^+(\mathbb{R}^n)}.$$

We next show that the kernel of $\mathbf{PP}^*$ satisfies the weak maximum principle with constant $h = 2^{n+1}$. Indeed, suppose $\mu \in \mathscr{M}^+(\mathbb{R}^{n+1}_+)$, and

$$\mathbf{PP}^*\mu(\tilde{x}, \tilde{y}) \le M, \quad \text{for all } (\tilde{x}, \tilde{y}) \in S(\mu).$$

Without loss of generality we may assume that $S(\mu) \Subset \mathbb{R}^{n+1}_+$ is a compact set. For $t \in \mathbb{R}^n$, let $(x_0, y_0) \in S(\mu)$ be a point such that

$$|(t,0) - (x_0, y_0)| = \text{dist}\Big((t,0), S(\mu)\Big).$$

Then by the triangle inequality, for any $(\tilde{x}, \tilde{y}) \in S(\mu)$,

$$|(x_0, y_0) - (\tilde{x}, -\tilde{y})| \le |(x_0, y_0) - (t,0)| + |(t,0) - (\tilde{x}, -\tilde{y})| \le 2|(t,0) - (\tilde{x}, \tilde{y})|.$$

Hence,

$$\sqrt{|t-\tilde{x}|^2 + \tilde{y}^2} \ge \frac{1}{2} \sqrt{\Big[|x_0 - \tilde{x}|^2 + (y_0 + \tilde{y})^2\Big]}.$$

It follows that, for all $t \in \mathbb{R}^n$ and $(\tilde{x}, \tilde{y}) \in S(\mu)$, we have

$$P(t - \tilde{x}, \tilde{y}) \le 2^{n+1} P(x_0 - \tilde{x}, y_0 + \tilde{y}).$$

Consequently, for all $t \in \mathbb{R}^n$,

$$\mathbf{P}^*\mu(t) \le 2^{n+1}\mathbf{PP}^*\mu(x_0, y_0) \le 2^{n+1} M.$$

Applying the Poisson integral $\mathbf{P}[dt]$ to both sides of the preceding inequality, we obtain

$$\mathbf{PP}^*\mu(x,y) \le 2^{n+1} M \quad \text{for all } (x,y) \in \mathbb{R}^{n+1}_+.$$

This proves that the weak maximum principle holds for $\mathbf{PP}^*$ with $h = 2^{n+1}$. It follows from Theorem 2.3 that (39) holds if and only if there exists a non-trivial $u \in$

$L^q(\mathbb{R}^{n+1}_+, \sigma)$ such that $u \geq \mathbf{PP}^*(u^q d\sigma)$. Moreover, a positive solution $u = \mathbf{PP}^*(u^q \sigma)$ can be constructed as in the statement of Theorem 4.1 (see details in [16]). □

Corollary 4.2 *Under the assumptions of Theorem 4.1, inequality (39) holds if and only if there exists a function $\phi \in L^1(\mathbb{R}^n)$, $\phi > 0$ a.e., such that*

$$\phi \geq \mathbf{P}^*\Big[(\mathbf{P}\phi)^q d\sigma\Big] \quad a.e. \quad in\ \mathbb{R}^n.$$

Moreover, if (39) holds, then there exists a positive solution $\phi \in L^1(\mathbb{R}^n)$ to the equation $\phi = \mathbf{P}^*\Big[(\mathbf{P}\phi)^q d\sigma\Big]$.

Proof If (39) holds then by Theorem 4.1 there exists $u = \mathbf{PP}^*(u^q d\sigma)$ such that $u > 0$ and $u \in L^q(\mathbb{R}^{n+1}_+, \sigma)$. Setting $\phi = \mathbf{P}^*(u^q d\sigma)$, we see that

$$\mathbf{P}\phi = \mathbf{PP}^*(u^q d\sigma) = u,$$

so that $\phi = \mathbf{P}^*[(\mathbf{P}\phi)^q d\sigma]$, and consequently

$$\|\phi\|_{L^1(\mathbb{R}^n)} = \|u\|^q_{L^q(\mathbb{R}^{n+1}_+, \sigma)} = \int_{\mathbb{R}^n} u(x, y) dx < \infty.$$

Conversely, if there exists $\phi > 0$, $\phi \in L^1(\mathbb{R}^n)$ such that $\phi \geq \mathbf{P}^*\Big[(\mathbf{P}\phi)^q d\sigma\Big]$, then letting $u = \mathbf{P}\phi$, we see that u is a positive harmonic function in $\mathbb{R}^{n+1}_+$ so that

$$u(x, y) = \mathbf{P}\phi(x, y) \geq \mathbf{PP}^*(u^q d\sigma)(x, y), \quad (x, y) \in \mathbb{R}^{n+1}_+.$$

Notice that the kernel $P(x - \tilde{x}, y + \tilde{y})$ of the operator $\mathbf{PP}^*$ has the property

$$\int_{\mathbb{R}^n} P(x - \tilde{x}, y + \tilde{y}) dx = 1, \quad y > 0, \ (\tilde{x}, \tilde{y}) \in \mathbb{R}^{n+1}_+,$$

and consequently, for all $y > 0$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+1}_+} P(x - \tilde{x}, y + \tilde{y}) u(\tilde{x}, \tilde{y})^q d\sigma(\tilde{x}, \tilde{y})\, dx = \int_{\mathbb{R}^{n+1}_+} u(\tilde{x}, \tilde{y})^q d\sigma(\tilde{x}, \tilde{y}),$$

Hence,

$$\|u\|^q_{L^q(\mathbb{R}^{n+1}_+, \sigma)} = \int_{\mathbb{R}^n} \Big[\mathbf{PP}^*(u^q d\sigma)\Big](x, y)\, dx \leq \int_{\mathbb{R}^n} u(x, y) dx = \|\phi\|_{L^1(\mathbb{R}^n)} < \infty.$$

Thus, inequality (39) holds by Theorem 4.1. □

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