Intrinsic Difference Quotients

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Abstract An alternative characterizations of intrinsic Lipschitz functions within Carnot groups through the boundedness of appropriately defined difference quotients is provided. It is also shown how intrinsic difference quotients along horizontal directions are naturally related with the intrinsic derivatives, introduced e.g. in Franchi et al. (Comm Anal Geom 11(5):909–944, 2003) and Ambrosio et al. (J Geom Anal 16:187–232, 2006) and used to characterize intrinsic real valued C^1 functions inside Heisenberg groups. Finally the question of the equivalence of the two conditions: (1) boundedness of horizontal intrinsic difference quotients and (2) intrinsic Lipschitz continuity is addressed in a few cases.

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1 Introduction

The aim of this paper is to contribute to the theory of *intrinsic Lipschitz functions* within Carnot groups.

We provide here an alternative characterizations of intrinsic Lipschitz functions through the boundedness of appropriately defined difference quotients. We show also how intrinsic difference quotients are strictly related with the intrinsic derivatives, introduced in [3, 15] and used by Serra Cassano et al. to characterize intrinsic real valued C^1 functions inside Heisenberg groups. Finally in the last section we attach the related question when the boundedness of only horizontal intrinsic difference quotients yields intrinsic Lipschitz continuity.

For a first description of Carnot groups we refer to the beginning of next section and to the literature there indicated. We anticipate here that we identify a Carnot group $\mathbb{G}$ with $\mathbb{R}^n$ endowed with a non commutative polynomial group

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operation denoted as $\cdot$ (see (1) and (2)). Moreover (non commutative) Carnot groups, endowed with their natural Carnot-Carathéodory distance (see Definition 2.1) are not Riemannian manifolds being also non Riemannian at any scale [26].

In the last years, there has been a general attempt aimed to carry on geometric analysis in non-Euclidean structures, and, in particular, to develop a good notion of rectifiable sets in sub-Riemannian metric structures and, specifically, in Carnot groups. For different notions of rectifiability in these general settings see [1, 2, 4, 14, 18, 23–25] and the references therein.

We recall that in Euclidean spaces, rectifiable sets are obtained, up to a negligible subset, by 'gluing up' countable families of graphs of C^1 or of Lipschitz functions. Hence, understanding the objects that, within Carnot groups, naturally take the role of C^1 or of Lipschitz functions seems to be preliminary in order to develop a satisfactory theory of *intrinsic* rectifiable sets. It has been clear for a long time that considering Euclidean notions, even in the simplest Carnot groups i.e. the Heisenberg groups, may be both too general and too restrictive (see [22] for a striking example). More intrinsic definitions are necessary.

Observe that, the adjective "intrinsic" is meant to emphasize the role played by the algebra of the group, in particular by its horizontal layer, and by group translations and dilations. In other words, "intrinsic" notions or properties in $\mathbb{G}$ are those depending only on the structure of its Lie algebra $\mathfrak{g}$. In particular, an intrinsic geometric property, such as e.g. being an intrinsic graph, or an intrinsic regular graph, or an intrinsic Lipschitz graph, must be invariant under group translations and group dilations. By this we mean that, after a translation or a dilation, they keep being graphs or regular graphs or Lipschitz graphs.

The notion of graph *within* Carnot groups is somehow more delicate than in Euclidean spaces, since Carnot groups in general are not cartesian products of subgroups (unlike Euclidean spaces). A notion of *intrinsic graph* fitting the structure of the group $\mathbb{G}$ is needed.

An intrinsic graph inside $\mathbb{G}$ is associated with a decomposition of the ambient group $\mathbb{G}$ as a product $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ of two homogeneous *complementary subgroups* $\mathbb{M}$, $\mathbb{H}$ (Definition 2.2) and the idea is the following one: let $\mathbb{M}$, $\mathbb{H}$ be complementary homogeneous subgroups of a group $\mathbb{G}$, then the intrinsic (left) graph of $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ is the set

$$\text{graph}\,(f) = \{g \cdot f(g) : g \in \mathcal{A}\}.$$

Intrinsic graphs appeared naturally in [5, 17, 19] in relation with the study of non critical level sets of differentiable functions from $\mathbb{G}$ to $\mathbb{R}^k$. Indeed, implicit function theorems for groups [14, 15, 18] can be rephrased stating precisely that non critical level sets are always, locally, intrinsic graphs.

What are then appropriate *intrinsic* notions of Lipschitz functions or of differentiable functions when dealing with functions acting between complementary subgroups?

Both these notions were originally given in a somewhat indirect way as intrinsic geometric properties of the graphs of the functions in question. Precisely, a function acting between complementary subgroups is an intrinsic Lipschitz function when its graph does not intersect appropriately defined cones (see Definitions 3.2 and 3.3). Analogously, a function is an intrinsically differentiable function when its graph admits an appropriately defined tangent homogeneous subgroup at each point (see [5, 20, 27]).

Both these notions, are invariant under group translations and dilations, hence they are intrinsic and seem to be the right ones to be considered inside groups (see e.g [24]).

On the other hand, in the Euclidean setting, the most common and elementary definition of Lipschitz function is through the boundedness of the difference quotients of the function itself and the natural definition of a differentiable function is through existence and continuity of its partial derivatives.

As anticipated before, we introduce here an analogous definition of *intrinsic difference quotients* (see Definition 3.7). These intrinsic difference quotients, though their form may be algebraically complicated, can be explicitly computed given the group $\mathbb{G}$ and the couple of complementary subgroups $\mathbb{M}$ and $\mathbb{H}$. Moreover it is easy to characterize intrinsic Lipschitz functions as intrinsic functions with bounded intrinsic difference quotients (see Proposition 3.11).

The problem of characterizing intrinsic differentiable or intrinsic C^1 graphs in terms of intrinsic differentiability properties of their underlying functions, is definitely much more complicated. The available results are up to now limited to the case of hypersurfaces inside Heisenberg groups that is to the case of 'real valued' functions inside Heisenberg groups. By this we mean precisely that $\mathbb{G}$ is an Heisenberg group and that the target space $\mathbb{H}$, in the decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$, is 1-dimensional and horizontal.

Moreover the actual form of the intrinsic derivatives (in many significant cases they are first order non linear differential operators) was obtained in the above mentioned cases, in a rather indirect way through the use of Dini theorem. We observe here as, in perfect analogy with Euclidean calculus, intrinsic derivatives of functions acting between complementary subgroups of $\mathbb{G}$ can be obtained as *limits of intrinsic difference quotients along horizontal directions* (when these limits exist). So we provide an explicit way of computing the form of intrinsic derivatives, given the group $\mathbb{G}$ and the couple of complementary subgroups $\mathbb{M}$ and $\mathbb{H}$.

Finally we observe that it is not clear when informations on boundedness or continuity of intrinsic derivatives of $f : \mathbb{M} \to \mathbb{H}$ are sufficient to yield that the graph of f is intrinsic Lipschitz or intrinsic differentiable in $\mathbb{G}$. Related to this is the fact that in many significant instances the homogeneous subgroup $\mathbb{M}$, though a stratified group, is not a Carnot group. The validity of an intrinsic Lipschitz continuity result, such as in Theorem 3.21, that does not have up to now a corresponding result in term of continuity or boundedness of intrinsic derivatives, might suggest that also in this case such a result might hold true.

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2 Notations and Definitions

2.1 Carnot Groups

We recall here only the notions about Carnot groups that will be used in the following of this paper. For general accounts on Carnot groups, see e.g. [10–12, 21].

A *graded group* of step κ is a connected, simply connected Lie group $\mathbb{G}$ whose finite dimensional Lie algebra $\mathfrak{g}$ is the direct sum of k subspaces $\mathfrak{g}_i$, $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\kappa$, such that

$$\left[\mathfrak{g}_i, \mathfrak{g}_j\right] \subset \mathfrak{g}_{i+j}, \quad \text{for } 1 \le i, j \le \kappa,$$

where $\mathfrak{g}_i = 0$ for $i > \kappa$. We denote as n the dimension of $\mathfrak{g}$ and as n_j the dimension of $\mathfrak{g}_j$, for $1 \le j \le \kappa$.

A *Carnot group* $\mathbb{G}$ of step κ is a graded group of step κ, where $\mathfrak{g}_1$ generates all of $\mathfrak{g}$. That is $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$, for $i = 1, \dots, \kappa$.

Let $X_1, \dots, X_n$ be a base for $\mathfrak{g}$ such that $X_1, \dots, X_{m_1}$ is a base for $\mathfrak{g}_1$ and, for $1 < j \le \kappa$, $X_{m_{j-1}+1}, \dots, X_{m_j}$ is a base for $\mathfrak{g}_j$. Here we have $m_0 = 0$ and $m_j - m_{j-1} = n_j$, for $1 \le j \le \kappa$.

Because the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a one to one diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$, any $p \in \mathbb{G}$ can be written, in a unique way, as $p = \exp(p_1 X_1 + \cdots + p_n X_n)$ and we identify p with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, i.e. $\mathbb{R}^n$ endowed with the product $\cdot$. The identity of $\mathbb{G}$ is denoted as $0 = (0, \dots, 0)$.

If $\mathbb{G}$ is a graded group, for all $\lambda > 0$, the *(non isotropic) dilations* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ are automorphisms of $\mathbb{G}$ defined as

$$\delta_\lambda(p_1, \dots, p_n) = (\lambda^{\alpha_1} p_1, \lambda^{\alpha_2} p_2, \dots, \lambda^{\alpha_n} p_n),$$

where $\alpha_i = j$, if $m_{j-1} < i \le m_j$. We denote the product of p and $q \in \mathbb{G}$ as $p \cdot q$ (or sometimes as pq). The explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula. It has the form

$$p \cdot q = p + q + \mathcal{Q}(p, q), \qquad \text{for all } p, q \in \mathbb{R}^n, \tag{1}$$

where $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Each $\mathcal{Q}_i$ is a homogeneous polynomial of degree α_i with respect to the intrinsic dilations of $\mathbb{G}$. That is

$$\mathcal{Q}_i(\delta_\lambda p, \delta_\lambda q) = \lambda^{\alpha_i} \mathcal{Q}_i(p, q), \quad \text{for all } p, q \in \mathbb{G} \text{ and } \lambda > 0. \tag{2}$$

We collect now further properties of $\mathcal{Q}$ following from Campbell-Hausdorff formula. First of all $\mathcal{Q}$ is antisymmetric, that is

$$\mathcal{Q}_i(p, q) = -\mathcal{Q}_i(-q, -p), \qquad \text{for all } p, q \in \mathbb{G}.$$

Each $\mathcal{Q}_i(p,q)$ depends only on a section of the components of p and q. Precisely

$$\begin{aligned} \mathcal{Q}_1(p,q) &= \ldots = \mathcal{Q}_{m_1}(p,q) = 0 \\ \mathcal{Q}_j(p,q) &= \mathcal{Q}_j(p_1,\ldots,p_{m_{i-1}},q_1,\ldots,q_{m_{i-1}}), \end{aligned} \tag{3}$$

if $m_{i-1} < j \le m_i$ and $2 \le i$. By Proposition 2.2.22 (4) in [10], for $m_1 < i \le n$ we can write

$$\mathcal{Q}_i(p,q) = \sum_{k,h} \mathcal{R}^i_{k,h}(p,q)(p_k q_h - p_h q_k), \tag{4}$$

where the functions $\mathcal{R}^i_{k,h}$ are polynomials, homogenous of degree $\alpha_i - \alpha_k - \alpha_h$ with respect to group dilations, and the sum is extended to all h,k such that $\alpha_h + \alpha_k \le \alpha_i$. From (4) it follows in particular that

$$\mathcal{Q}_i(p,0) = \mathcal{Q}_i(0,q) = 0 \quad \text{and} \quad \mathcal{Q}_i(p,p) = \mathcal{Q}_i(p,-p) = 0. \tag{5}$$

Finally, it is useful to think $\mathbb{G} = \mathbb{G}^1 \oplus \mathbb{G}^2 \oplus \cdots \oplus \mathbb{G}^\kappa$, where $\mathbb{G}^i = \exp(\mathfrak{g}_i) = \mathbb{R}^{n_i}$ is the ith layer of $\mathbb{G}$ and to write $p \in \mathbb{G}$ as $(p^1,\ldots,p^\kappa)$, with $p^i \in \mathbb{G}^i$. $\mathbb{G}^1$ is denoted as the *horizontal layer* of $\mathbb{G}$.

Accordingly we also denote $\mathcal{Q} = (\mathcal{Q}^1,\ldots,\mathcal{Q}^\kappa)$ where $\mathcal{Q}^1 \equiv 0$ and for $2 \le i \le \kappa$ each $\mathcal{Q}^i$ is a vector valued polynomial homogeneous of degree i with respect to the intrinsic dilations of $\mathbb{G}$. With this notation (1) becomes

$$p \cdot q = \left(p^1 + q^1, p^2 + q^2 + \mathcal{Q}^2(p,q), \ldots, p^\kappa + q^\kappa + \mathcal{Q}^\kappa(p,q)\right), \qquad \text{for all } p,q \in \mathbb{G}. \tag{6}$$

An homogeneous norm in $\mathbb{G}$ is a function $\|\cdot\| : \mathbb{G} \to \mathbb{R}^+$ such that for all $p,q \in \mathbb{G}$ and for all $\lambda \ge 0$

$$\|p \cdot q\| \le \|p\| + \|q\|, \qquad \|\delta_\lambda p\| = \lambda \|p\|.$$

Homogeneous norms exist. A convenient one (see [16, Theorem 5.1]) is

$$\|p\| := \max_{j=1,\ldots,\kappa} \{\varepsilon_j \left\|p^j\right\|^{1/j}_{\mathbb{R}^{n_j}}\}, \qquad \text{for all } p = (p^1,\ldots,p^\kappa) \in \mathbb{G}, \tag{7}$$

where $\varepsilon_1 = 1$, and $\varepsilon_2, \ldots \varepsilon_\kappa \in (0,1]$ are suitable positive constants depending on $\mathbb{G}$.

Definition 2.1 An absolutely continuous curve $\gamma : [0,T] \to \mathbb{G}$ is a *sub-unit curve* if there exist measurable real functions $c_1(s),\ldots,c_{m_1}(s)$, $s \in [0,T]$ such that $\sum_j c_j^2 \le 1$ and

$$\dot\gamma(s) = \sum_{j=1}^{m_1} c_j(s) X_j(\gamma(s)), \qquad \text{for a.e. } s \in [0,T].$$

If $p, q \in \mathbb{G}$, we define their *Carnot-Carathéodory distance* as

$$d_c(p,q) := \inf\{T > 0 : \text{there exists a sub-unit curve } \gamma \text{ with } \gamma(0) = p,\ \gamma(T) = q\}\,.$$

By Chow's Theorem, the set of sub-unit curves joining p and q is not empty, furthermore d_c is a distance on $\mathbb{G}$ that induces the Euclidean topology (see Chap. 19 in [10]).

More generally, given any homogeneous norm $\|\cdot\|$, it is possible to define a distance in $\mathbb{G}$ as

$$d(p,q) = d(q^{-1} \cdot p, 0) = \left\|q^{-1} \cdot p\right\|, \qquad \text{for all } p, q \in \mathbb{G}. \tag{8}$$

The distance d in (8) is comparable with the Carnot-Carathéodory distance of $\mathbb{G}$ and

$$d(g \cdot p, g \cdot q) = d(p,q) \quad , \quad d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p,q) \tag{9}$$

for all $p, q, g \in \mathbb{G}$ and all $\lambda > 0$.

2.2 Complementary Subgroups and Graphs

From now on $\mathbb{G}$ will always be a Carnot group, identified with $\mathbb{R}^n$ through exponential coordinates.

Definition 2.2 A *homogeneous subgroup* of $\mathbb{G}$ (see [28, 5.2.4]) is a Lie subgroup $\mathbb{H}$ such that $\delta_\lambda g \in \mathbb{H}$, for all $g \in \mathbb{H}$ and for all $\lambda \geq 0$. Homogeneous subgroups are linear subspaces of $\mathbb{G} \equiv \mathbb{R}^n$.

Two homogeneous subgroups $\mathbb{M}, \mathbb{H}$ of $\mathbb{G}$ are *complementary subgroups* in $\mathbb{G}$, if $\mathbb{M} \cap \mathbb{H} = \{0\}$ and if for all $g \in \mathbb{G}$, there are $m \in \mathbb{M}$ and $h \in \mathbb{H}$ such that $g = m \cdot h$. If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$ we say that $\mathbb{G}$ is the *product of* $\mathbb{M}$ *and* $\mathbb{H}$ and we denote this as

$$\mathbb{G} = \mathbb{M} \cdot \mathbb{H}.$$

If $\mathbb{M}, \mathbb{H}$ are complementary subgroups of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ then they are also complementary linear subspaces of $\mathbb{R}^n$ and we denote this as $\mathbb{G} = \mathbb{M} \oplus \mathbb{H}$. If one of them is a normal subgroup then $\mathbb{G}$ is said to be the *semi-direct product* of $\mathbb{M}$ and $\mathbb{H}$. If both $\mathbb{M}$ and $\mathbb{H}$ are normal subgroups then $\mathbb{G}$ is said to be the *direct product* of $\mathbb{M}$ and $\mathbb{H}$.

Remark 2.3 If $\mathbb{M}$ is an homogeneous subgroup of $\mathbb{G}$ then also $\mathbb{M}$ is a stratified group, but it is not necessarily a Carnot group. If $\mathbb{M}, \mathbb{H}$ are complementary subgroups of $\mathbb{G}$ then $\mathbb{G}^i = \mathbb{M}^i \oplus \mathbb{H}^i$, for $i = 1, \ldots, \kappa$.

Example 2.4 Complementary subgroups always exist in any Carnot group $\mathbb{G}$. Indeed, choose any horizontal homogeneous subgroup $\mathbb{H} = \mathbb{H}^1 \subset \mathbb{G}^1$ and a subgroup $\mathbb{M} = \mathbb{M}^1 \oplus \cdots \oplus \mathbb{M}^\kappa$ such that: $\mathbb{H} \oplus \mathbb{M}^1 = \mathbb{G}^1$, and $\mathbb{G}^j = \mathbb{M}^j$ for all $2 \le j \le \kappa$. Then it is easy to check that $\mathbb{M}$ and $\mathbb{H}$ are complementary subgroups in $\mathbb{G}$ and that the product $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ is semidirect because $\mathbb{M}$ is a normal subgroup.

Given two complementary subgroups $\mathbb{M}, \mathbb{H}$ of $\mathbb{G}$, then for any $g \in \mathbb{G}$ the elements $m \in \mathbb{M}$ and $h \in \mathbb{H}$ such that $g = mh$ are unique because $\mathbb{M} \cap \mathbb{H} = \{0\}$. These elements are denoted as *components* of g along $\mathbb{M}$ and $\mathbb{H}$ or as *projections* of g on $\mathbb{M}$ and $\mathbb{H}$.

Proposition 2.5 *If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$ there is $c_0 = c_0(\mathbb{M}, \mathbb{H}) > 0$ such that for all $g = mh$*

$$c_0 \left(\|m\| + \|h\|\right) \le \|g\| \le \|m\| + \|h\| . \tag{10}$$

From now on, we will keep the following convention: when $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, $\mathbb{M}$ will always be the first 'factor' and $\mathbb{H}$ the second one, hence $g_\mathbb{M} \in \mathbb{M}$ and $g_\mathbb{H} \in \mathbb{H}$ are the unique elements such that

$$g = g_\mathbb{M} g_\mathbb{H}. \tag{11}$$

We stress that this notation is ambiguous because $g_\mathbb{M}$ and $g_\mathbb{H}$ depend on both the complementary subgroups $\mathbb{M}$ and $\mathbb{H}$ and also on the order under which they are taken.

The projection maps $\mathbf{P}_\mathbb{M} : \mathbb{G} \to \mathbb{M}$ and $\mathbf{P}_\mathbb{H} : \mathbb{G} \to \mathbb{H}$ are defined as

$$\mathbf{P}_\mathbb{M}(g) := g_\mathbb{M}, \qquad \mathbf{P}_\mathbb{H}(g) := g_\mathbb{H} \tag{12}$$

Proposition 2.6 *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups of $\mathbb{G}$, then the projection maps $\mathbf{P}_\mathbb{M} : \mathbb{G} \to \mathbb{M}$ and $\mathbf{P}_\mathbb{H} : \mathbb{G} \to \mathbb{H}$ defined in (12) are polynomial maps. More precisely, if κ is the step of $\mathbb{G}$, there are 2κ matrices $A^1, \dots, A^\kappa, B^1, \dots, B^\kappa$, depending on $\mathbb{M}$ and $\mathbb{H}$, such that*

(i) *A^j and B^j are (n_j, n_j)-matrices, for all $1 \le j \le \kappa$,*

and, with the notations of (1),

(ii) $\mathbf{P}_\mathbb{M} g = \big(A^1 g^1, A^2(g^2 - \mathcal{Q}^2(A^1 g^1, B^1 g^1)), \dots, A^\kappa(g^\kappa - \mathcal{Q}^\kappa(A^1 g^1, \dots, B^{\kappa-1} g^{\kappa-1}))\big)$;
(iii) $\mathbf{P}_\mathbb{H} g = \big(B^1 g^1, B^2(g^2 - \mathcal{Q}^2(A^1 g^1, B^1 g^1)), \dots, B^\kappa(g^\kappa - \mathcal{Q}^\kappa(A^1 g^1, \dots, B^{\kappa-1} g^{\kappa-1}))\big)$;
(iv) *A^j is the identity on $\mathbb{M}^j$, and B^j is the identity on $\mathbb{H}^j$, for $1 \le j \le \kappa$.*

Recall that n_j is the dimension of the layer $\mathfrak{g}_j$.

Definition 2.7 Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$. We say that a set $S \subset \mathbb{G}$ is a *(left)* $\mathbb{H}$*-graph* (or a *left graph in direction* $\mathbb{H}$) if S intersects each left coset of $\mathbb{H}$ in one point, at most.

If $\mathcal{A} \subset \mathbb{G}$ parametrizes the left cosets of $\mathbb{H}$—in particular if $\mathcal{A}$ itself intersect each left coset of $\mathbb{H}$ at most one time—and if S is an $\mathbb{H}$-graph, then there is a unique function $f : \mathcal{E} \subset \mathcal{A} \to \mathbb{H}$ such that *S is the graph of f*, that is

$$S = \text{graph}\,(f) := \{\xi \cdot f(\xi) : \xi \in \mathcal{E}\}.$$

Conversely, for any $\psi : \mathcal{D} \subset \mathcal{A} \to \mathbb{H}$ the set graph (ψ) is an $\mathbb{H}$-graph.

One has an important special case when $\mathbb{H}$ admits a complementary subgroup $\mathbb{M}$. Indeed, in this case, $\mathbb{M}$ naturally parametrizes the left cosets of $\mathbb{H}$ and we have that

$$S \text{ is a } \mathbb{H}\text{-graph if and only if } S = \text{graph}\,(f)$$

for $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$. By uniqueness of the components along $\mathbb{M}$ and $\mathbb{H}$, if $S =$ graph (f) then f is uniquely determined among all functions from $\mathbb{M}$ to $\mathbb{H}$.

If a set $S \subset \mathbb{G}$ is an intrinsic graph then it keeps being an intrinsic graph after left translations or group dilations.

Proposition 2.8 *Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$. If S is a $\mathbb{H}$-graph then, for all $\lambda > 0$ and for all $q \in \mathbb{G}$, $\delta_\lambda S$ and $q \cdot S$ are $\mathbb{H}$-graphs.*

If, in particular, $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, if $S =$ graph (f) with $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$, then

$$\begin{aligned} &\textit{For all } \lambda > 0, \delta_\lambda S = \text{graph}\,(f_\lambda), \textit{with} \\ &f_\lambda : \delta_\lambda \mathcal{E} \subset \mathbb{M} \to \mathbb{H} \textit{ and} \\ &f_\lambda(m) = \delta_\lambda f(\delta_{1/\lambda} m), \textit{ for } m \in \delta_\lambda \mathcal{E}. \end{aligned} \tag{13}$$

$$\begin{aligned} &\textit{For any } q \in \mathbb{G}, \quad q \cdot S = \text{graph}\,(f_q), \textit{ where} \\ &f_q : \mathcal{E}_q \subset \mathbb{M} \to \mathbb{H}, \quad \mathcal{E}_q = \{m : \mathbf{P}_\mathbb{M}(q^{-1} \cdot m) \in \mathcal{E}\} \textit{ and} \\ &f_q(m) = \big(\mathbf{P}_\mathbb{H}(q^{-1} \cdot m)\big)^{-1} \cdot f\big(\mathbf{P}_\mathbb{M}(q^{-1} \cdot m)\big), \textit{ for all } m \in \mathcal{E}_q. \end{aligned} \tag{14}$$

Remark 2.9 The algebraic expression of f_q in (14) is more explicit when $\mathbb{G}$ is a semi-direct product of $\mathbb{M}, \mathbb{H}$. Precisely

(i) If $\mathbb{M}$ is normal in $\mathbb{G}$ then $f_q(m) = q_\mathbb{H} f\big((q^{-1}m)_\mathbb{M}\big)$, for $m \in \mathcal{E}_q = q\mathcal{E}(q_\mathbb{H})^{-1}$.
(ii) If $\mathbb{H}$ is normal in $\mathbb{G}$ then $f_q(m) = (q^{-1}m)_\mathbb{H}^{-1} f(q_\mathbb{M}^{-1} m)$, for $m \in \mathcal{E}_q = q_\mathbb{M}\mathcal{E}$.

If both $\mathbb{M}$ and $\mathbb{H}$ are normal in $\mathbb{G}$—that is if $\mathbb{G}$ is a direct product of $\mathbb{M}$ and $\mathbb{H}$—then we get the well known Euclidean formula

(iii) $f_q(m) = q_{\mathbb{H}} f(q_{\mathbb{M}}^{-1} m)$, for $m \in \mathcal{E}_q = q_{\mathbb{M}}\mathcal{E}$.

See also [5, Proposition 3.6].

3 Intrinsic Lipschitz Functions

3.1 General Definitions

As anticipated in the introduction, *intrinsic Lipschitz functions* in $\mathbb{G}$ are functions, acting between complementary subgroups of $\mathbb{G}$, with graphs non intersecting naturally defined cones. Hence, the notion of *intrinsic Lipschitz graph* respects strictly the geometry of the ambient group $\mathbb{G}$. Intrinsic Lipschitz functions appeared for the first time in [14] and were studied, more diffusely, in [13, 18, 19, 30].

We begin with two definitions of intrinsic (closed) cones.

Definition 3.1 Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$, $q \in \mathbb{G}$. Then, the *cones* $X(q, \mathbb{H}, \alpha)$ with *axis* $\mathbb{H}$*, vertex* q*, opening* α, $0 \le \alpha \le 1$ are defined as

$$X(q, \mathbb{H}, \alpha) = q \cdot X(0, \mathbb{H}, \alpha), \text{ where } X(0, \mathbb{H}, \alpha) = \left\{p : \operatorname{dist}(p, \mathbb{H}) \le \alpha \, \|p\| \right\}.$$

Notice that Definition 3.1 does not require that $\mathbb{H}$ is a complemented subgroup.

Frequently, while working with functions acting between complementary subgroups, it will be convenient to consider also the following family of cones.

Definition 3.2 If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, $q \in \mathbb{G}$ and $\beta \ge 0$, the cones $C_{\mathbb{M},\mathbb{H}}(q, \beta)$, with base $\mathbb{M}$, axis $\mathbb{H}$, vertex q, opening β are defined as

$$C_{\mathbb{M},\mathbb{H}}(q, \beta) = q \cdot C_{\mathbb{M},\mathbb{H}}(0, \beta), \text{ where } C_{\mathbb{M},\mathbb{H}}(0, \beta) = \{p : \|p_{\mathbb{M}}\| \le \beta \, \|p_{\mathbb{H}}\|\}.$$

Observe that

$$\mathbb{H} = X(0, \mathbb{H}, 0) = C_{\mathbb{M},\mathbb{H}}(0, 0), \qquad \mathbb{G} = X(0, \mathbb{H}, 1) = \overline{\cup_{\beta>0} C_{\mathbb{M},\mathbb{H}}(0, \beta)}.$$

Moreover, the cones $C_{\mathbb{M},\mathbb{H}}(q, \beta)$ are equivalent with the cones $X(q, \mathbb{H}, \alpha)$ that is: for any $\alpha \in (0, 1)$ there is $\beta \ge 1$, depending on α, $\mathbb{M}$ and $\mathbb{H}$, such that

$$C_{\mathbb{M},\mathbb{H}}(q, 1/\beta) \subset X(q, \mathbb{H}, \alpha) \subset C_{\mathbb{M},\mathbb{H}}(q, \beta), \tag{15}$$

Now we introduce the basic definition of this paragraph.

Definition 3.3

(i) Let $\mathbb{H}$ be an homogeneous subgroup, not necessarily complemented in $\mathbb{G}$. We say that an $\mathbb{H}$-graph S is an *intrinsic Lipschitz $\mathbb{H}$-graph* if there is $\alpha \in (0,1)$ such that,

$$S \cap X(p, \mathbb{H}, \alpha) = \{p\}, \qquad \text{for all } p \in S.$$

(ii) If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, we say that $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is *intrinsic Lipschitz* in $\mathcal{E}$ when graph (f) is an *intrinsic Lipschitz $\mathbb{H}$-graph*.

(iii) We say that $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is *intrinsic L-Lipschitz* in $\mathcal{E}$ if there is $L > 0$ such that

$$C_{\mathbb{M},\mathbb{H}}(p, 1/L) \cap \operatorname{graph}(f) = \{p\}, \quad \text{for all } p \in \operatorname{graph}(f). \tag{16}$$

The Lipschitz constant of f in $\mathcal{E}$ is the infimum of the $L > 0$ such that (16) holds.

It follows immediately from (15) that f is intrinsic Lipschitz in $\mathcal{E}$ if and only if it is intrinsic L-Lipschitz for an appropriate constant L, depending on α, f and $\mathbb{M}$.

Because of Proposition 2.8 and Definition 3.2 left translations of intrinsic Lipschitz $\mathbb{H}$-graphs, or of intrinsic L-Lipschitz functions, are intrinsic Lipschitz $\mathbb{H}$-graphs, or intrinsic L-Lipschitz functions. We state these facts in the following theorem.

Theorem 3.4 *If $\mathbb{G}$ is a Carnot group, then for all $q \in \mathbb{G}$,*

(i) *$S \subset \mathbb{G}$ is an intrinsic Lipschitz $\mathbb{H}$-graph $\implies$ $q{\cdot}S$ is an intrinsic Lipschitz $\mathbb{H}$-graph;*
(ii) *$f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is intrinsic L-Lipschitz, $\implies$ $f_q : \mathcal{E}_q \subset \mathbb{M} \to \mathbb{H}$ is intrinsic L-Lipschitz.*

The geometric definition of intrinsic Lipschitz graphs has equivalent algebraic forms (see also [5, 17, 19]).

Proposition 3.5 *Let $\mathbb{M}, \mathbb{H}$ be complementary subgroups in $\mathbb{G}$, $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ and $L > 0$. Then* (i) *to* (iii) *are equivalent.*

(i) *f is intrinsic L-Lipschitz in $\mathcal{E}$.*
(ii) $\left\|\mathbf{P}_{\mathbb{H}}(\bar{q}^{-1}q)\right\| \le L \left\|\mathbf{P}_{\mathbb{M}}(\bar{q}^{-1}q)\right\|$, *for all* $q, \bar{q} \in \operatorname{graph}(f)$.
(iii) $\left\|f_{\bar{q}^{-1}}(m)\right\| \le L \|m\|$, *for all* $\bar{q} \in \operatorname{graph}(f)$ *and* $m \in \mathcal{E}_{\bar{q}^{-1}}$.

Remark 3.6 If $\mathbb{G}$ is the semi-direct product of $\mathbb{M}$ and $\mathbb{H}$, (ii) of Proposition 3.5 takes a more explicit form. Indeed, from Remark 2.9, we get

(i) If $\mathbb{M}$ is normal in $\mathbb{G}$ then f is intrinsic L-Lipschitz if and only if

$$\left\| f(\bar{m})^{-1} f(m) \right\| \leq L \left\| f(\bar{m})^{-1} \bar{m}^{-1} m f(\bar{m}) \right\|, \qquad \text{for all } m, \bar{m} \in \mathcal{E}.$$

(ii) If $\mathbb{H}$ is normal in $\mathbb{G}$ then f is intrinsic L-Lipschitz if and only if

$$\left\| m^{-1} \bar{m} f(\bar{m})^{-1} \bar{m}^{-1} m f\big(m\big) \right\| \leq L \left\| \bar{m}^{-1} m \right\|, \qquad \text{for all } m, \bar{m} \in \mathcal{E}.$$

(iii) If $\mathbb{G}$ is a direct product of $\mathbb{M}$ and $\mathbb{H}$ we get the well known expression for Lipschitz functions

$$\left\| f(\bar{m})^{-1} f(m) \right\| \leq L \left\| \bar{m}^{-1} m \right\|, \qquad \text{for all } m, \bar{m} \in \mathcal{E}.$$

Hence in this case intrinsic Lipschitz functions are the same as the usual metric Lipschitz functions from $(\mathbb{M}, d_\infty)$ to $(\mathbb{H}, d_\infty)$.

3.2 Intrinsic Difference Quotients

A different new characterization of intrinsic Lipschitz functions can be given in terms of boundedness of appropriately defined *intrinsic difference quotients*. Let us begin with this notion. In the spirit of the previous paragraphs, first we propose the definition in the particular case of a function vanishing in the origin of the group and then we get the general definition extending the particular case in a translation invariant way.

Let $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ and $Y \in \mathfrak{m}$. Assume $0 \in \mathcal{E}$ and $f(0) = 0$. In this case the *difference quotients* $\Delta_Y f(0; t)$ of f (from $0 \in \mathcal{E}$ in direction Y) are defined as

$$\Delta_Y f(0; t) := \delta_{1/t} f(\delta_t \exp Y)$$

for all $t > 0$ such that $\delta_t \exp Y \in \mathcal{E}$. Then we extend this definition to any $m \in \mathcal{E}$. Let $q := m \cdot f(m) \in \operatorname{graph}(f)$, then $f_{q^{-1}}$ vanishes in $0 \in \mathcal{E}_{q^{-1}}$ and we define

$$\Delta_Y f(m; t) := \Delta_Y f_{q^{-1}}(0; t) = \delta_{1/t} f_{q^{-1}}(\delta_t \exp Y) \tag{17}$$

once more for all $t > 0$ such that $\delta_t \exp Y \in \mathcal{E}_{q^{-1}}$.

To make the previous definition less implicit, i.e. given directly on the function f and not on its translated $f_{q^{-1}}$, we consider the following steps making also more transparent the underlying geometry of the construction.

- Let $f : \mathbb{M} \to \mathbb{H}$. Fix $m \in \mathbb{M}$ and $Y \in \mathfrak{m}$. Then consider the line from $q_m := m \cdot f(m)$

$$s \mapsto q_m \cdot \delta_s \exp Y \qquad \text{for } 0 \leq s$$

and its projection on $\mathbb{M}$

$$s \mapsto \mathbf{P}_{\mathbb{M}}\left(q_m \cdot \delta_s \exp Y\right) \qquad \text{for } 0 \leq s.$$

Let

$$\Delta_{Y,t} q_m := \mathbf{P}_{\mathbb{M}}\left(q_m \cdot \delta_t \exp Y\right) = m \cdot \mathbf{P}_{\mathbb{M}}\left(f(m) \cdot \delta_t \exp Y\right).$$

- Consider the projection on $\mathbb{H}$ of the difference between the two points on graph (f): $\Delta_{Y,t} q_m \cdot f\left(\Delta_{Y,t} q_m\right)$ and $q_m := m \cdot f(m)$,

$$\mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \Delta_{Y,t} q_m \cdot f\left(\Delta_{Y,t} q_m\right)\right) = \mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \Delta_{Y,t} q_m\right) \cdot f\left(\Delta_{Y,t} q_m\right)$$

- Finally the *intrinsic difference quotient* of f from m in direction Y is

$$\Delta_Y f(m;t) := \delta_{1/t}\left(\mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \Delta_{Y,t} q_m \cdot f\left(\Delta_{Y,t} q_m\right)\right)\right). \tag{18}$$

The previous definition of $\Delta_Y f(m;t)$ can be given a different expression.

$$\begin{aligned}
&\mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \Delta_{Y,t} q_m \cdot f\left(\Delta_{Y,t} q_m\right)\right) \\
&= \mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \mathbf{P}_{\mathbb{M}}\left(q_m \cdot \delta_t \exp Y\right) \cdot f\left(\Delta_{Y,t} q_m\right)\right) \\
&= \mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \mathbf{P}_{\mathbb{M}}\left(q_m \cdot \delta_t \exp Y\right) \cdot \mathbf{P}_{\mathbb{H}}\left(q_m \cdot \delta_t \exp Y\right) \cdot \left(\mathbf{P}_{\mathbb{H}}\left(q_m \cdot \delta_t \exp Y\right)\right)^{-1} \cdot f\left(\Delta_{Y,t} q_m\right)\right) \\
&= \mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot q_m \cdot \delta_t \exp Y \cdot \left(\mathbf{P}_{\mathbb{H}}\left(q_m \cdot \delta_t \exp Y\right)\right)^{-1} \cdot f\left(\Delta_{Y,t} q_m\right)\right) \\
&= \left(\mathbf{P}_{\mathbb{H}}(q_m \cdot \delta_t \exp Y\right)^{-1} \cdot f\left(\Delta_{Y,t} q_m\right) \\
&= \left(\mathbf{P}_{\mathbb{H}}(f(m) \cdot \delta_t \exp Y\right)^{-1} \cdot f\left(m \cdot \mathbf{P}_{\mathbb{M}}\left(f(m) \cdot \delta_t \exp Y\right)\right)
\end{aligned}$$

Finally we propose the following definitions

Definition 3.7 Let $\mathbb{M}, \mathbb{H}$ be complementary subgroups in $\mathbb{G}$ and $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$. If $m \in \mathcal{E}$ and Y belongs to the Lie algebra $\mathfrak{m}$ of $\mathbb{M}$, then the *intrinsic difference quotients of f at m along Y*, are

$$\Delta_Y f(m;t) = \delta_{1/t}\left(\left(\mathbf{P}_{\mathbb{H}}(f(m) \cdot \delta_t \exp Y)\right)^{-1} \cdot f\left(m \cdot \mathbf{P}_{\mathbb{M}}(f(m) \cdot \delta_t \exp Y)\right)\right), \tag{19}$$

for all $t > 0$ such that $m \cdot \mathbf{P}_{\mathbb{M}}(f(m) \cdot \delta_t \exp Y) \in \mathcal{E}$.

Remark 3.8 Notice that formally the definition of difference quotient could be given also for $Y \in \mathfrak{h}$. This case is, as it should be, completely not interesting because the difference quotients are 0. Indeed with $Y \in \mathfrak{h}$ it follows $m \cdot \mathbf{P}_{\mathbb{M}}(f(m) \cdot \delta_t \exp Y) = m$

and using the definition in (18)

$$\begin{aligned}\Delta_Y f(m;t) &= \delta_{1/t}\left(\mathbf{P}_{\mathbb{H}}\left(q_m^{-1}\cdot\Delta_{Y,t}q_m\cdot f\left(\Delta_{Y,t}q_m\right)\right)\right)\\ &= \delta_{1/t}\left(\mathbf{P}_{\mathbb{H}}\left(f(m)^{-1}\cdot m^{-1}\cdot m\cdot\mathbf{P}_{\mathbb{M}}\left(f(m)\cdot\delta_t\exp Y\right)\cdot f(m\cdot\mathbf{P}_{\mathbb{M}}\left(f(m)\cdot\delta_t\exp Y\right))\right)\right)\\ &= \delta_{1/t}\left(\mathbf{P}_{\mathbb{H}}\left(f(m)^{-1}\cdot f(m)\right)\right) = 0.\end{aligned}$$

Remark 3.9 Observe that Definition 3.7 gives the same notion of difference quotient as proposed in (17). Indeed, if $f(m) = 0$ then $\mathbf{P}_{\mathbb{H}}(f(m)\cdot\delta_t\exp Y) = 0$ and $m\cdot\mathbf{P}_{\mathbb{M}}(f(m)\cdot\delta_t\exp Y) = m\cdot\delta_t\exp Y$. Hence

$$f(m) = 0 \implies \Delta_Y f(m;t) = \delta_{1/t}f(m\cdot\delta_t\exp Y) \tag{20}$$

and also, if $q = m\cdot f(m)$ then $f_{q^{-1}}(0) = 0$ we get (17)

$$\Delta_Y f(m;t) = \Delta_Y f_{q^{-1}}(0;t) = \delta_{1/t}\left(f_{q^{-1}}(\delta_t\exp Y)\right).$$

Remark 3.10 With the same notations of Definition 3.7 and recalling Remark 2.9, we get

(i) If $\mathbb{M}$ is normal in $\mathbb{G}$ and $Y\in\mathfrak{m}$ then

$$\mathbf{P}_{\mathbb{H}}(f(m)\cdot\delta_t\exp Y) = f(m)$$

and

$$\begin{aligned}m\cdot\mathbf{P}_{\mathbb{M}}(f(m)\delta_t(\exp Y)) &= m\cdot f(m)\cdot\delta_t\exp Y\cdot f(m)^{-1}\\ &= m\cdot\mathrm{Ad}_{f(m)}(\delta_t\exp Y).\end{aligned}$$

Hence if $\mathbb{M}$ is a normal subgroup and $Y\in\mathfrak{m}$

$$\Delta_Y f(m;t) = \delta_{1/t}\left(f(m)^{-1}\cdot f\left(m\cdot\mathrm{Ad}_{f(m)}(\delta_t\exp Y)\right)\right).$$

(ii) If $\mathbb{H}$ is normal in $\mathbb{G}$ then

$$\mathbf{P}_{\mathbb{H}}\left(f(m)\cdot\delta_t\exp Y\right) = (\delta_t\exp Y)^{-1}\cdot f(m)\cdot\delta_t\exp Y$$

and

$$\mathbf{P}_{\mathbb{M}}(f(m)\cdot\delta_t\exp Y) = \delta_t\exp Y.$$

Hence if $\mathbb{H}$ is a normal subgroup and $Y\in\mathfrak{m}$

$$\Delta_Y f(m;t) = \delta_{1/t}\left((\delta_t\exp Y)^{-1}\cdot f(m)^{-1}\cdot\delta_t\exp Y\cdot f(m\cdot\delta_t\exp Y)\right).$$

(iii) If both $\mathbb{M}$ and $\mathbb{H}$ are normal in $\mathbb{G}$ and $\mathbb{G}$ is a direct product of $\mathbb{M}$ and $\mathbb{H}$ then we get the well known expression for the difference quotient:

$$\Delta_Y f(m;t) = \delta_{1/t}\left(f(m)^{-1}\cdot f(m\cdot\delta_t \exp Y)\right).$$

Next Proposition gives a straightforward characterization of intrinsic Lipschitz functions in terms of the boundedness of their difference quotients.

Proposition 3.11 *Let* $\mathbb{M}, \mathbb{H}$ *be complementary subgroups in* $\mathbb{G}$ *and* $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$*. The following statements are equivalent*

(i) f *is intrinsic L-Lipschitz in* $\mathcal{E}$*;*
(ii) there is $L > 0$ *such that, for all* $Y \in \mathfrak{m}$ *and for all* $m \in \mathcal{E}$

$$\|\Delta_Y f(m;t)\| \leq L\,\|\exp Y\|.$$

Proof If $q = mf(m) \in \text{graph}(f)$ then by (17)

$$\|\Delta_Y f(m;t)\| = \left\|\Delta_Y f_{q^{-1}}(0;t)\right\| = \frac{1}{t}\left\|f_{q^{-1}}(\delta_t \exp Y)\right\|,$$

for all $t > 0$ and $Y \in \mathfrak{m}$.

$(i) \implies (ii)$. By (iii) of Proposition 3.5,

$$\|\Delta_Y f(m;t)\| = \frac{1}{t}\left\|f_{q^{-1}}(\delta_t \exp Y)\right\| \leq \frac{L}{t}\,\|\delta_t \exp Y\| = L\,\|\exp Y\|,$$

for $t > 0$ and $Y \in \mathfrak{m}$. Hence (ii) holds.

$(ii) \implies (i)$. Let $\bar{m} \in \mathcal{E}$ and $\bar{q} := \bar{m}f(\bar{m})$. For any $m \in \mathcal{E}_{\bar{q}^{-1}}$ let $Y \in \mathfrak{m}$ be such that $m = \exp Y$. Then

$$\left\|f_{\bar{q}^{-1}}(m)\right\| = \left\|f_{\bar{q}^{-1}}(\exp Y)\right\| = \|\Delta_Y f(\bar{m};1)\| \leq L\,\|\exp Y\| = L\,\|m\|.$$

Hence (iii) of Proposition 3.5 holds and f is intrinsic L-Lipschitz.

□

We conclude this section observing that the limits for $t \to 0^+$ of intrinsic different quotients, when these limits exist and are finite, give origin to a notion of *intrinsic derivative* for functions acting between complementary subgroups. We will show, in Examples 3.16 and 3.17, that these intrinsic derivatives are precisely the operators considered by Serra Cassano and coauthors to characterize intrinsic Lipschitz and intrinsic regular functions inside Heisenberg groups.

Definition 3.12 Let $\mathbb{M}, \mathbb{H}$ be complementary subgroups in $\mathbb{G}$, let $\mathfrak{m}$ be the Lie algebra of $\mathbb{M}$ and $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$. If $m \in \mathcal{E} \subset \mathbb{M}$, the *intrinsic directional* *derivative of f at m along* $Y \in \mathfrak{m}$, is

$$D_Y f(m) := \lim_{t\to 0^+} \Delta_Y f(m;t) = \lim_{t\to 0^+} \Delta_{-Y} f(m;t) \tag{21}$$

provided the two limits on the right exist and are equal.

Remark 3.13 This remark follows directly from (20). Indeed

$$f(m) = 0 \implies \Delta_Y f(m;t) = \delta_{1/t} f(m \cdot \delta_t \exp Y)$$

hence, if the limits in (21) exist,

$$f(m) = 0 \implies D_Y f(m) = Yf(m).$$

3.3 Examples of Difference Quotients and of Intrinsic Derivatives

Example 3.14 (Horizontal Valued Functions Inside Step 2 Groups) Let $\mathbb{G} = (\mathbb{R}^m, \cdot)$ be a step 2 group and denote $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$. Let $\{Z_1, \dots, Z_m\}$ be a base of $\mathfrak{g}$ with

$$\mathfrak{g}^1 = \operatorname{span}\{Z_1, \dots, Z_{m_1}\}, \qquad \mathfrak{g}^2 = \operatorname{span}\{Z_{m_1+1}, \dots, Z_m\}$$

With the notation in (6) we denote

$$q_{i,h} := \mathcal{Q}^2(\exp Z_i, \exp Z_h) \in \mathbb{R}^{m-m_1}, \qquad \text{for } 1 \le i, h \le m.$$

Notice that $q_{i,h} = -q_{h,i}$ and $q_{i,h} = 0$ if $i > m_1$.

We assume (see Example 2.4) that $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ where $\mathbb{H}$ is a k-dimensional horizontal subgroup and $\mathbb{M}$ is a complementary normal subgroup. Moreover we choose the vectors Z_i are chosen such that

$$\mathbb{H} = \exp\left(\operatorname{span}\{Z_1, \dots, Z_k\}\right), \qquad \mathbb{M} = \exp\left(\operatorname{span}\{Z_{k+1}, \dots, Z_m\}\right).$$

Notice that we are assuming that $Z_1, \dots, Z_k$ are commuting vector fields.

Let $f : \mathbb{M} \to \mathbb{H}$ be defined as

$$f(p) := \exp\left(\sum_1^k \varphi_i(p) Z_i\right) = \sum_1^k \varphi_i(p) \exp Z_i \qquad \text{for all } p \in \mathbb{M}.$$

for all $p = (p^1, p^2) \in \mathbb{M}$, where $\varphi_i : \mathbb{M} \to \mathbb{R}$ for $1 \le i \le k$.

Fix an horizontal $Z_j \in \mathfrak{g}^1$, that is with $k+1 \le j \le m_1$. Using (i) of Remark 3.10, we compute

$$\begin{aligned}\Delta_{Z_j} f(p;t) &= \delta_{1/t}\left(f(p)^{-1}\cdot f\left(p\cdot f(p)\cdot\delta_t \exp Z_j\cdot f(p)^{-1}\right)\right)\\ &= \sum_{i=1}^{k}\frac{1}{t}\left(\varphi_i\left(p\cdot f(p)\cdot\delta_t\exp Z_j\cdot f(p)^{-1}\right)-\varphi_i(p)\right)\exp Z_i\end{aligned}$$

Notice that

$$\begin{aligned}&p\cdot f(p)\cdot\delta_t\exp Z_j\cdot f(p)^{-1}\\ &\quad=\left(p^1+t\exp Z_j, p^2+2\mathcal{Q}^2(f(p),t\exp Z_j)+\mathcal{Q}^2(p^1,t\exp Z_j)\right)\\ &\quad=\Big(p^1+t\exp Z_j, p^2+2t\sum_{\ell=1}^{k}\varphi_\ell(p)\mathcal{Q}^2(\exp Z_\ell,\exp Z_j)\\ &\qquad+t\mathcal{Q}^2(\exp(\sum_{\ell=k+1}^{m_1}p_\ell Z_\ell),\exp Z_j)\Big)\\ &\quad=\left(p^1+t\exp Z_j, p^2+2t\sum_{\ell=1}^{k}\varphi_\ell(p)q_{\ell,j}+t\sum_{\ell=k+1}^{m_1}p_\ell q_{\ell,j}\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\Delta_{Z_j} f(p;t)\\ &\quad=\sum_{i=1}^{k}\frac{1}{t}\Bigg(\varphi_i\Big(p^1+t\exp Z_j, p^2+2t\sum_{\ell=1}^{k}\varphi_\ell(p)q_{\ell,j}\\ &\qquad\qquad+t\sum_{\ell=k+1}^{m_1}p_\ell q_{\ell,j}\Big)-\varphi_i(p^1,p^2)\Bigg)\exp Z_i.\end{aligned}$$

Let us specialize the previous example in the case $\mathbb{G}=\mathbb{H}^n$.

Example 3.15 (Horizontal Valued Functions Inside Heisenberg Groups) We recall here the well known definition of Heisenberg groups mainly to fix a few notations.

The n-Heisenberg group $\mathbb{H}^n$ is identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p\in\mathbb{H}^n$ is denoted $p=(p_1,\dots,p_{2n},p_{2n+1})=(p^1,p^2)$, with

$p^1 \in \mathbb{R}^{2n}$ and $p^2 = p_{2n+1} \in \mathbb{R}$. If p and $q \in \mathbb{H}^n$, the group operation is defined as

$$\begin{aligned} p \cdot q &= (p^1 + q^1, p_{2n+1} + q_{2n+1} + \mathcal{Q}^2(p^1, q^1)) \\ &= (p^1 + q^1, p_{2n+1} + q_{2n+1} - \frac{1}{2}\langle Jp^1, q^1\rangle_{\mathbb{R}^{2n}}) \end{aligned}$$

where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is the $(2n \times 2n)$-symplectic matrix.

For any $q \in \mathbb{H}^n$ and for any $r > 0$ left translations $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ and non isotropic dilations $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ are defined as

$$\tau_q(p) := q \cdot p \qquad \text{and as} \qquad \delta_r p := (rp^1, r^2 p_{2n+1}).$$

We denote as $\mathfrak{h}^n$ the Lie algebra of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}^n$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_i - \frac{1}{2}(Jp')_i \partial_{2n+1}, \quad Y_i := \partial_{i+n} + \frac{1}{2}(Jp')_{i+n}\partial_{2n+1}, \quad T := \partial_{2n+1}.$$

The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}^n$ spanned by $X_1, \dots, X_n$ and by $Y_1, \dots, Y_n$. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}^n$ is expressed by

$$\mathfrak{h}^n = \mathfrak{h}_1 \oplus \mathfrak{h}_2. \tag{22}$$

The Lie algebra $\mathfrak{h}^n$ is also endowed with a scalar product $\langle \cdot, \cdot \rangle$ making the vector fields $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ and T orthonormal. Thus (22) turns out to be an orthonormal decomposition of $\mathfrak{h}^n$ as a vector space.

If $p \in \mathbb{H}^n$, we indicate as $\|p\|$ its Koranyi norm, i.e.

$$\|p\| = \sqrt[4]{\|p^1\|^4_{\mathbb{R}^{2n}} + |p_{2n+1}|^2}$$

There are infinite many different couples of complementary subgroups inside $\mathbb{H}^n$. All these couples contain a horizontal subgroup, here denoted as $\mathbb{V}$ of dimension $k \leq n$, isomorphic and isometric to $\mathbb{R}^k$ and a normal subgroup $\mathbb{W}$ of dimension $2n + 1 - k$, containing the centre $\mathbb{T}$.

Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ where $\mathbb{V}$ is a k-dimensional horizontal subgroup and $\mathbb{W}$ a complementary normal subgroup. We assume, for the time being, that $\mathbb{V}$ and $\mathbb{W}$ are in generic position inside $\mathbb{H}^n$, in particular we do not assume that they are orthogonal.

It is always possible to choose a basis $Z := \{Z_1, \dots, Z_{2n}, T = Z_{2n+1}\}$ of the algebra $\mathfrak{h}^n$ such that

$$\mathbb{V} = \exp(\mathrm{span}\{Z_1, \dots, Z_k\}), \qquad \mathbb{W} = \exp(\mathrm{span}\{Z_{k+1}, \dots, Z_{2n}, T\}),$$

where $Z_1, \dots, Z_k$ are commuting horizontal vector fields.

We use coordinates with respect to the Z basis, i.e.

$$\mathbb{H}^n \ni p = \exp\big(\sum_{i=1}^{2n+1} p_i Z_i\big) \simeq (p_1, \dots, p_{2n+1}) \in \mathbb{R}^{2n+1.}$$

With the notation in (6) we denote

$$q_{\ell,h} := \mathcal{Q}^2(\exp Z_\ell, \exp Z_h) \in \mathbb{R}, \qquad \text{for } 1 \le \ell, h \le 2n+1.$$

As in Example 3.14, let

$$f : \mathbb{W} \to \mathbb{V}, \qquad f(p) := \exp\left(\sum_1^k \varphi_i(p) Z_i\right) = \sum_1^k \exp(\varphi_i(p) Z_i).$$

Nothing changes in the computations from the general case of a step 2 group and we get the following expression for the difference quotients for each horizontal Z_j with $j = k+1, \dots, 2n$,

$$\begin{aligned} &\Delta_{Z_j} f(p;t) \\ &= \sum_{i=1}^k \frac{1}{t}\left(\varphi_i\big(p^1 + t\exp Z_j, p^2 + t(2\sum_{\ell=1}^k \varphi_\ell(p) q_{\ell,j}\right. \\ &\qquad\qquad \left. + \sum_{\ell=k+1}^{2n} p_\ell q_{\ell,j}\big) - \varphi_i(p^1,p^2)\right) \exp Z_i. \end{aligned} \tag{23}$$

Moreover

$$\begin{aligned} \Delta_{Z_{2n+1}} f(p;t) &= \delta_{1/t}\left(f(p)^{-1} \cdot f\left(p \cdot f(p) \cdot \delta_t \exp Z_{2n+1} \cdot f(p)^{-1}\right)\right) \\ &= \delta_{1/t}\left(f(p)^{-1} \cdot f\left(p \cdot (0, \dots, 0, t^2)\right)\right) \\ &= \sum_{i=1}^k \frac{1}{t}\left(\varphi_i\big(p_1, \dots, p_{2n}, p_{2n+1} + t^2\big) - \varphi_i(p_1, \dots, p_{2n}, p_{2n+1})\right) \exp Z_i. \end{aligned} \tag{24}$$

Passing to the limit in (23) for $t \to 0^+$, we obtain the following system of $k(2n-k-1)$ non linear (intrinsic) differential operators acting on the k real valued functions $\varphi_1, \dots, \varphi_k$:

$$D_{Z_j}\varphi_i := \partial_j \varphi_i + \big(2\sum_{\ell=1}^{k} \varphi_\ell(p) q_{\ell,j} + \sum_{\ell=k+1}^{2n} p_\ell q_{\ell,j}\big)\partial_{2n+1}\varphi_i \tag{25}$$

here $\partial_j \varphi_i = \frac{\partial \varphi_i}{\partial p_j}$ for $i = 1, \dots, k$ and $j = k+1, \dots, 2n+1$.

Boundedness in (24) gives only a Holder type condition on the last variable of the functions φ_i.

Example 3.16 We further specialize the setting in Example 3.15 assuming that $\mathbb{W}$ and $\mathbb{V}$ are orthogonal in $\mathbb{H}^n$. Precisely, we assume that

$$\{Z_1, \dots, Z_{2n}, Z_{2n+1}\} = \{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}$$

and that, for $1 \le k \le n$,

$$\mathbb{V} = \exp(\mathrm{span}\{X_1, \dots, X_k\}), \qquad \mathbb{W} = \exp(\mathrm{span}\{Z_{k+1}, \dots, Z_{2n}, T\}).$$

The coefficients $q_{\ell,i}$ take the special form

$$q_{\ell,i} := \mathcal{Q}^2(\exp Z_\ell, \exp Z_i) = \begin{cases} q_{h,h+n} = \frac{1}{2} & \text{for } 1 \le h \le n \\ q_{h+n,h} = -\frac{1}{2} & \text{for } 1 \le h \le n \\ q_{\ell,i} = 0 & \text{otherwise.} \end{cases}$$

Hence (25) takes the form

$$\begin{cases} D_{Z_j}\varphi_i := \partial_j\varphi_i + \displaystyle\sum_{\ell=k+1}^{2n} p_\ell q_{\ell,j}\partial_t\varphi_i = \partial_j\varphi_i - \frac{1}{2}p_{j+n}\partial_t\varphi_i \equiv X_j\varphi_i, & k+1 \le j \le n \\ D_{Z_j}\varphi_i := \partial_j\varphi_i + 2\displaystyle\sum_{\ell=1}^{k} \varphi_\ell(p) q_{\ell,j}\partial_t\varphi_i = \partial_j\varphi_i + \varphi_{j-n}\partial_t\varphi_i, & n+1 \le j \le n+k \\ D_{Z_j}\varphi_i := \partial_j\varphi_i + \displaystyle\sum_{\ell=k+1}^{2n} p_\ell q_{\ell,j}\partial_t\varphi_i = \partial_j\varphi_i + \frac{1}{2}p_{j-n}\partial_t\varphi_i \equiv Y_{j-n}\varphi_i, & n+k+1 \le j \le 2n \end{cases} \tag{26}$$

It may be interesting to consider also these special instances of (26).

Example 3.17 With the notations of Example 3.16 let us consider the complementary subsets of $\mathbb{H}^n \equiv (\mathbb{R}^{2n+1}, \cdot)$

$$\mathbb{V} = \exp(\mathrm{span}\{X_1\}), \qquad \mathbb{W} = \exp(\mathrm{span}\{X_2, \dots, X_n, Y_1, \dots, Y_n, T\})$$

and a function $f : \mathbb{W} \to \mathbb{V}$. Then $f(w) := \varphi(w) \exp X_1$ can be identified with the real valued function φ and we speak, with an abuse of language, of a real valued intrinsic function. Here $w := (0, p_2, \dots, p_{2n}, p_{2n+1}) \in \mathbb{W}$.

Then (26) takes the form

$$\begin{cases} D_{Z_j}\varphi \equiv D_{X_j}\varphi := \partial_j\varphi - \frac{1}{2}p_{j+n}\partial_{2n+1}\varphi \equiv X_j\varphi, & 2 \le j \le n \\ D_{Z_{n+1}}\varphi \equiv D_{Y_1}\varphi := \partial_{n+1}\varphi + \varphi\partial_{2n+1}\varphi, & \\ D_{Z_j}\varphi \equiv D_{Y_{j-n}}\varphi := \partial_j\varphi + \frac{1}{2}p_{j-n}\partial_{2n+1}\varphi \equiv Y_{j-n}\varphi, & n+2 \le j \le 2n \end{cases} \tag{27}$$

In $\mathbb{H}^1$ the system (27) reduces to the single non linear Burger type equation

$$D_Y\varphi := \partial_2\varphi + \varphi\partial_3\varphi = \partial_2\varphi + \frac{1}{2}\partial_3\varphi^2. \tag{28}$$

Equation (28) appeared in this context in [18], when studying the regularity of non critical level sets of group-C^1 functions $\mathbb{H}^n \to \mathbb{R}$. There are many works dealing with weak solutions of equations (28) and their relation with intrinsic regular surfaces inside the first Heisenberg group $\mathbb{H}^1$, (see [6–8, 29]).

System (27) is studied in [9] (see also [3]) where the authors characterize intrinsic real valued Lipschitz functions $f : \mathbb{W} \to \mathbb{V}$ as bounded solutions of (27). We notice that our Theorem 3.19 is related with the above mentioned characterization, notwithstanding that the result in [9] is much deeper than the one in here, given that the assumption in [9] is of boundedness of the limits of the intrinsic different quotients and not, as we make in Theorem 3.19, on the difference quotients themselves.

Much less studied are the vector valued analogues of (27) and (28). Consider the complementary subsets of $\mathbb{H}^2 \equiv (\mathbb{R}^5, \cdot)$

$$\mathbb{V} = \exp(\mathrm{span}\{X_1, X_2\}), \qquad \mathbb{W} = \exp(\mathrm{span}\{Y_1, Y_2, T\})$$

and $f : \mathbb{W} \to \mathbb{V}$. Then $f(w) := \varphi_1(w) \exp X_1 + \varphi_2(w) \exp X_2$, where $w := (0, 0, p_3, p_4, p_5) \in \mathbb{W}$.

In this case the equations in the first and last groups of system (26) disappear and we are left only with the non linear part of the system

$$\begin{cases} D_{Z_3}\varphi_1 := \partial_3\varphi_1 + \varphi_1\partial_5\varphi_1, & D_{Z_3}\varphi_2 := \partial_3\varphi_2 + \varphi_1\partial_5\varphi_2, \\ D_{Z_4}\varphi_1 := \partial_4\varphi_1 + \varphi_2\partial_5\varphi_1, & D_{Z_4}\varphi_2 := \partial_4\varphi_2 + \varphi_2\partial_5\varphi_2, \end{cases}$$

that is the vector valued analogous of (28).

If we consider in $\mathbb{H}^3 \equiv (\mathbb{R}^7, \cdot)$ the complementary subgroups

$$\mathbb{V} = \exp(\text{span}\{X_1, X_2\}), \qquad \mathbb{W} = \exp(\text{span}\{X_3, Y_1, Y_2, Y_3, T\}),$$

a function $f(w) := \varphi_1(w) \exp X_1 + \varphi_2(w) \exp X_2 : \mathbb{W} \to \mathbb{V}$, here $w := (0, 0, p_3, \dots, p_6, p_7) \in \mathbb{W}$, then (26) becomes a system of 8 equations acting on the two real valued functions φ_1, φ_2

$$\begin{cases} D_{Z_3}\varphi_1 := X_3\varphi_1, & D_{Z_3}\varphi_2 := X_3\varphi_2 \\ D_{Z_4}\varphi_1 := \partial_4\varphi_1 + \varphi_1(w)\partial_7\varphi_1, & D_{Z_4}\varphi_2 := \partial_4\varphi_2 + \varphi_1(w)\partial_7\varphi_2 \\ D_{Z_5}\varphi_1 := \partial_5\varphi_1 + \varphi_2(w)\partial_7\varphi_1, & D_{Z_5}\varphi_2 := \partial_5\varphi_2 + \varphi_2(w)\partial_7\varphi_2 \\ D_{Z_6}\varphi_1 := Y_3\varphi_1, & D_{Z_6}\varphi_2 := Y_3\varphi_2. \end{cases}$$

Finally we compute the difference quotients and an intrinsic derivative inside a step 3 group.

Example 3.18 (One Dimensional Function Inside Engels Group) The Engels group is $\mathbb{E} = (\mathbb{R}^4, \cdot, \delta_\lambda)$, were the group law is defined as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + (x_1y_2 - x_2y_1)/2 \\ x_4 + y_4 + [(x_1y_3 - x_3y_1) + (x_2y_3 - x_3y_2)]/2 \\ +(x_1 - y_1 + x_2 - y_2)(x_1y_2 - x_2y_1)/12 \end{pmatrix}$$

and the family of dilation is

$$\delta_\lambda(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4).$$

A basis of left invariant vector fields is X_1, X_2, X_3, X_4 defined as

$$\begin{aligned} X_1(p) &:= \partial_1 - (p_2/2)\,\partial_3 + \big(-p_3/2 - (p_1p_2 + p_2^2)/12\big)\,\partial_4 \\ X_2(p) &:= \partial_2 + (p_1/2)\,\partial_3 + \big(-p_3/2 + (p_1^2 + p_1p_2)/12\big)\,\partial_4 \\ X_3(p) &:= \partial_3 - ((p_1 + p_2)/2)\,\partial_4 \\ X_4(p) &:= \partial_4. \end{aligned}$$

The commutation relations are $[X_1, X_2] = X_3, \quad [X_1, X_3] = [X_2, X_3] = X_4$ and all the others commutators are zero. $\mathbb{E}$ is a semidirect product, as $\mathbb{E} = \mathbb{M} \cdot \mathbb{H}$, of the two subgroups $\mathbb{M}$ and $\mathbb{H}$

$$\mathbb{M} := \{m = (0, p_2, p_3, p_4)\} \qquad \mathbb{H} := \{h = (p_1, 0, 0, 0)\} = \{\exp(\mathrm{span} X_1)\}.$$

Let $f : \mathbb{M} \to \mathbb{H}$ where $f(m) := \exp(\varphi(m)X_1)$. Observe that $\mathbb{M}$ is a normal subgroup being $\mathbb{H}$ an horizontal subgroup. Then it follows

$$\begin{aligned}\Delta_{X_2} f(m;t) &= \delta_{1/t}\big(f(m)^{-1} \cdot f(m \cdot \mathrm{Ad}_{\varphi(m)}(\exp tX_2))\big)\\ &= \delta_{1/t}\big(f(m)^{-1} \cdot f(0, p_2+t, p_3+t\varphi(m), p_4+\frac{t}{2}(\varphi(m)^2+p_2\varphi(m) - p_3))\big)\\ &= \exp\big(\frac{1}{t}(\varphi(0, p_2+t, p_3+t\varphi(m), p_4+\frac{t}{2}(\varphi(m)^2+p_2\varphi(m) - p_3)) - \varphi(m))\big)X_1.\end{aligned}$$

Hence, computing $\lim_{t\to 0^+} \Delta_{X_2} f(m;t)$ we obtain the only horizontal intrinsic derivative of the real valued function φ

$$\begin{aligned}D_{X_2}\varphi(m) &:= \partial_2\varphi + \varphi(m)\partial_3\varphi + \frac{1}{2}(\varphi(m)^2 + p_2\varphi(m) - p_3)\partial_4\varphi\\ &= \partial_2\varphi + \frac{1}{2}\partial_3\varphi^2 + \frac{1}{12}\partial_4(2\varphi^3 + 3p_2\varphi^2 - 6p_3\varphi).\end{aligned}$$

3.4 Horizontal Difference Quotients and Lipschitz Functions

In a few noticeable instances the boundedness of difference quotients along the vectors of the horizontal layer of $\mathfrak{m}$ is sufficient to imply intrinsic Lipschitz continuity.

As observed before, this phenomenon is different from the one about functions defined *on* Carnot groups, although it is strictly related to it. It is well known that if $f : \mathbb{G} \to \mathbb{R}$ is such that Yf is bounded for all Y in the horizontal layer of $\mathfrak{g}$, then f is a Lipschitz function, the reason being that the horizontal layer of $\mathfrak{g}$ generates, by commutation, all the algebra. This is not the case for functions acting from $\mathbb{M}$ to $\mathbb{H}$. Indeed $\mathbb{M}$, though a stratified group, is not necessarily a Carnot group because not necessarily the horizontal layer of the algebra generates the entire algebra of $\mathbb{M}$ (see e.g. Example 3.17), on the other side there is a redeeming feature: intrinsic difference quotients and intrinsic derivatives are non linear operators. Finally one does not have to forget that the final result is that the functions are *intrinsic Lipschitz* and not *Lipschitz*.

We present here only two instances of this phenomenon, both of them inside Heisenberg groups. The first one deals with 1-codimensional graphs of functions acting between any two complementary subgroups the second one deals with k-codimensional horizontal graphs of functions acting between *orthogonal* complementary subgroups.

Theorem 3.19 *Let $\mathbb{W}$ and $\mathbb{V}$ be complementary subgroups in $\mathbb{H}^n$ with $\mathbb{V}$ one dimensional and horizontal. Let $L > 0$ and $f : \mathbb{W} \to \mathbb{V}$ be such that*

$$\|\Delta_Z f(w; t)\| \leq L \,\|\exp Z\|$$

for all $w \in \mathbb{W}$ and for all horizontal vector fields $Z \in \mathfrak{w}^1$. Then there is $\tilde{L} = \tilde{L}(L, \mathbb{V}, \mathbb{W}) \geq L$ such that f is $\tilde{L}$-intrinsic Lipschitz in $\mathbb{W}$.

Proof By translation invariance, it is enough to prove

$$\|f(w)\| \leq \tilde{L}\,\|w\| \tag{29}$$

for all $w \in W$ under the additional assumption

$$f(0) = 0.$$

If $f(w) = 0$ there is nothing to prove. Hence let us assume that $v := f(w) \neq 0$. Under this assumption we prove that there are $s, t \in \mathbb{R}$, there are horizontal vectors Z, U in the first layer $\mathfrak{w}^1$ of the algebra $\mathfrak{w}$ of $\mathbb{W}$ and a constant $C = C(\mathbb{V}, \mathbb{W}) > 0$ such that $\|\exp Z\| = \|\exp U\| = 1$,

$$w \cdot Ad_{f(w)}(\delta_s \exp Z) = \delta_t \exp U \tag{30}$$

and

$$|t| \leq \|w\| + |s|; \qquad |s| \leq C\,\|w\|^2/\|f(w)\|. \tag{31}$$

With the notations of (6), $w = (w^1, w^2) = (w^1, w_{2n+1}), f(w) = (f(w)^1, 0)$, $\exp Z = \big((\exp Z)^1, 0\big)$ and $\exp U = \big((\exp U)^1, 0\big)$. Then

$$\begin{aligned} w \cdot \mathrm{Ad}_{f(w)}(\delta_s \exp Z) &= \big(w^1 + (\delta_s \exp Z)^1, w_{2n+1} + 2\mathcal{Q}^2(f(w)^1, (\delta_s \exp Z)^1) \\ &\qquad + \mathcal{Q}^2(w^1, (\delta_s \exp Z)^1)\big) \\ &= \big(w^1 + s(\exp Z)^1, w_{2n+1} + 2\mathcal{Q}^2(f(w)^1, s(\exp Z)^1) \\ &\qquad + \mathcal{Q}^2(w^1, s(\exp Z)^1)\big) \end{aligned}$$

Hence (30) is equivalent to solve in Z, U and t, s the system of $2n + 1$ equations,

$$\begin{cases} w^1 + s(\exp Z)^1 = t(\exp U)^1 \\ w_{2n+1} + 2\mathcal{Q}^2(f(w)^1, s(\exp Z)^1) + \mathcal{Q}^2(w^1, s(\exp Z)^1) = 0. \end{cases} \tag{32}$$

Let $\alpha, \beta \in \mathbb{R}^{2n}$, with $\|\alpha\|_{\mathbb{R}^{2n}} = \|\beta\|_{\mathbb{R}^{2n}} = 1$, be such that

$$w \in \mathbb{W} \iff \sum_{i=1}^{2n} \alpha_i w_i = 0, \qquad v \in \mathbb{V} \iff v = \delta_{\|v\|} \exp \sum_{i=1}^{n} (\beta_i X_i + \beta_{n+i} Y_i).$$

Because $\mathbb{W}$ and $\mathbb{V}$ are complementary subgroups then

$$C = C(\mathbb{W}, \mathbb{V}) := |\sum_{i=1}^{2n} \alpha_i \beta_i| > 0.$$

Let

$$Z := \sum_{i=1}^{n} \alpha_{n+i} X_i - \alpha_i Y_i \in \mathfrak{w}^1,$$

then $\|\exp Z\| = 1$ and for all $v \in \mathbb{V}$ and $w \in \mathbb{W}$

$$\begin{aligned} \left|\mathcal{Q}^2(v^1, (\exp Z)^1)\right| &= \frac{1}{2} \|v\| \, |\sum_{i=1}^{2n} \alpha_i \beta_i| \geq C \|v\|, \\ \mathcal{Q}^2(w^1, (\exp Z)^1) &= -\frac{1}{2} \sum_{i=1}^{n} (w_i \alpha_i + w_{n+i} \alpha_{n+i}) = 0. \end{aligned} \tag{33}$$

With this choice of Z from the last equation of (32), using that $\mathcal{Q}^2(\cdot, \cdot)$ is bilinear, we get

$$|s| \leq C |w_{2n+1}| / \|f(w)\| \leq C \|w\|^2 / \|f(w)\|,$$

where C is a (different) constant depending only on $\mathbb{V}$ and $\mathbb{W}$. The other estimate in (31) follows from the first equations in (32).

Finally let us see that (29) follows from (30) and (31). Indeed, consider the intrinsic difference quotients starting from 0 along U and from $w \cdot f(w)$ along Z

$$\begin{aligned} \nabla_U f(0; t) &= f(\delta_t \exp U), \\ \nabla_Z f(w; s) &= f(w)^{-1} \cdot f(w \cdot Ad_{f(w)}(\delta_s \exp Z)) = f(w)^{-1} \cdot f(\delta_t \exp U) \end{aligned}$$

From the assumption of boundedness of the difference quotients of f

$$\begin{aligned} \|f(\delta_t \exp U)\| &\leq L|t| \\ \left\|f(w)^{-1} \cdot f(w \cdot Ad_{f(w)}(\delta_s \exp Z))\right\| &\leq L|s| \end{aligned}$$

The boundedness of these difference quotients yields an estimate $\|f(w)\|$. Indeed

$$\begin{aligned}\|f(w)\| &\le \left\|f(w)^{-1}f(\delta_t \exp U)\right\| + \|f(\delta_t \exp U)\| \\ &= \left\|f(w)^{-1}f(w \cdot Ad_{f(w)}(\delta_s \exp Z))\right\| + \|f(\delta_t \exp U)\| \\ &\le L(|s| + |t|) \\ &\le L\big(\|w\| + 2C\|w\|^2/\|f(w)\|\big)\end{aligned} \tag{34}$$

that eventually gives

$$\|f(w)\| \le \tilde{L}\,\|w\|$$

with $\tilde{L} = \frac{1}{2}(L + \sqrt{L^2 + 8LC})$. □

Remark 3.20 Observe that in Theorem 3.19 it has been proved that if $\|\Delta_Z f(w;t)\| = 0$ for all $w \in \mathbb{W}$ and for all horizontal vector field $Z \in \mathfrak{w}^1$ then $f : \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz with 0 Lipschitz constant hence it is a constant function. This fact is not anymore true if f is defined on a proper subset of $\mathbb{W}$. The following one is an example: let $\mathbb{W}$ and $\mathbb{V}$ be the complementary subgroups of $\mathbb{H}^1$ defined as

$$\mathbb{W} := \{(0, x_2, x_3)\}, \qquad \mathbb{V} := \{(x_1, 0, 0)\}.$$

Let $\mathcal{A}$ be the neighborhood of the origin in $\mathbb{W}$ defined as $\mathcal{A} := \{(0, x_2, x_3) : x_2 > -1\}$ and let $f : \mathcal{A} \subset \mathbb{W} \to \mathbb{V}$ be defined as

$$f(0, x_2, x_3) := \left(\frac{x_3}{1 + x_2}, 0, 0\right).$$

The horizontal layer of $\mathfrak{w}^1$ is one dimensional and is spanned by the vector $Y := \partial_{x_2}$. Then from Definition 3.7 (see also (i) of Remark 3.10)

$$\Delta_Y f(w;t) = 0, \qquad \text{for all } w \in \mathcal{A} \text{ and } t \ge 0$$

while clearly f is not constant.

Theorem 3.21 *Let $\mathbb{W}$ and $\mathbb{V}$ be the complementary orthogonal subgroups of $\mathbb{H}^n$ considered in Example 3.16. Precisely, for $1 \le k \le n$ let*

$$\mathbb{V} = \exp(\operatorname{span}\{X_1, \dots, X_k\}), \qquad \mathbb{W} = \exp(\operatorname{span}\{Z_{k+1}, \dots, Z_{2n}, T\}).$$

Hence $\mathbb{V}$ is k-dimensional and horizontal. Let $L > 0$ and $f : \mathbb{W} \to \mathbb{V}$ be such that

$$\|\Delta_Z f(w;t)\| \le L\,\|\exp Z\|$$

for all $w \in \mathbb{W}$ *and for all horizontal vector fields* $Z \in \mathfrak{w}^1$. *Then there is* $\tilde{L} = \tilde{L}(L, \mathbb{V}, \mathbb{W}) \geq L$ *such that* f *is* $\tilde{L}$*-intrinsic Lipschitz in* $\mathbb{W}$.

Proof We keep using the notations introduced in Examples 3.15 and 3.16.

Analogously as in the proof of Theorem 3.19, by translation invariance, it is enough to prove

$$\|f(w)\| \leq \tilde{L}\,\|w\| \tag{35}$$

for all $w \in W$ under the additional assumption

$$f(0) = 0.$$

Let be given $w = (w^1, w^2) \in \mathbb{W}$ and $v = (v^1, 0) \in \mathbb{V}$ with $v^1 \neq 0$. Then there is $z = (z^1, 0) \in \mathbb{W}^1$ as

$$w \cdot \mathrm{Ad}_v z \in \mathbb{W}^1. \tag{36}$$

Indeed, let $z = (z_1, \ldots, z_{2n+1})$ be defined such that

$$\begin{aligned} z_i &= 0, \quad \text{for } 1 \leq i \leq n \text{ and for } n+k+1 \leq i \leq 2n+1 \\ z_{n+i} &= \lambda\, \mathrm{sign}(v_i), \quad \text{for } n+1 \leq i \leq n+k. \end{aligned}$$

With this choice of z we have

$$\begin{aligned} \mathcal{Q}^2(w^1, z^1) &= \frac{1}{2}\sum_{i=1}^{n}(w_i z_{n+1} - w_{n+i} z_i) = 0, \\ \mathcal{Q}^2(v^1, z^1) &= \frac{1}{2}\lambda \sum_{i=1}^{k} |v_i|. \end{aligned}$$

Finally choosing $\lambda = -w_{2n+1}/\sum_{i=1}^{k}|v_i|$ we get

$$w \cdot \mathrm{Ad}_v z = \left(w^1 + z^1, w_{2n+1} + 2\mathcal{Q}^2(v^1, z^1) + \mathcal{Q}^2(w^1, z^1)\right) = (w^1 + z^1, 0) \in \mathbb{W}^1.$$

Let us go back to the proof of (35). If $f(w) = 0$ there is nothing to prove. Hence let us assume that $f(w) \neq 0$ and define $v := f(w)$. Now let $Z, U \in \mathfrak{w}^1$ be chosen such that $\|\exp Z\| = \|\exp U\| = 1$ and

$$\delta_s \exp Z = z, \qquad \delta_t \exp U = w \cdot \mathrm{Ad}_{f(w)} z$$

for appropriate $s, t \in \mathbb{R}$. With this choice of s we have

$$|s| \leq C \|w\|^2 / \|f(w)\| .$$

From now on the proof follows the same pattern of the proof of Theorem 3.19. □

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