Applied and Numerical Harmonic Analysis

$$\hat{f}(\gamma) = \int f(x)e^{-2\pi i x \gamma}\,dx$$

Sagun Chanillo, Bruno Franchi
Guozhen Lu, Carlos Perez
Eric T. Sawyer
Editors

Harmonic Analysis, Partial Differential Equations and Applications

In Honor of Richard L. Wheeden

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Editors
Sagun Chanillo
Dept. of Math
Rutgers University
Piscataway
New Jersey, USA

Bruno Franchi
Department of Mathematics
University of Bologna
Bologna, Italy

Guozhen Lu
Dept. of Math.
Univ. of Connecticut
Storrs CT, USA

Carlos Perez
Department of Mathematics
University of Bilbao
Bilbao, Spain

Eric T. Sawyer
Dept of Math and Stat
McMaster University
Hamilton
Ontario, Canada

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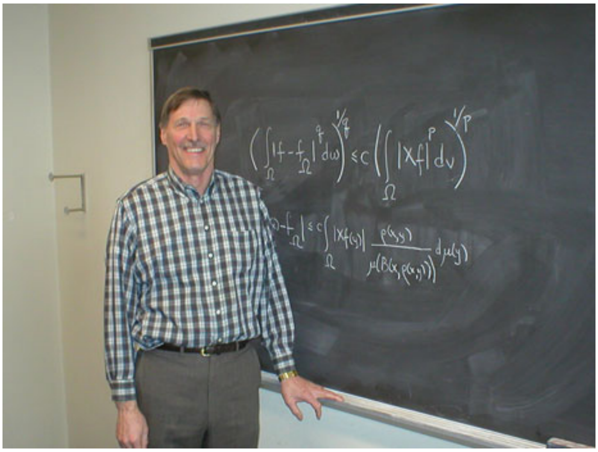

Richard L. Wheeden

Preface

It is a pleasure to bring out this volume of contributed papers on the occasion of the retirement of Richard Wheeden. Dick Wheeden as he is known to his many friends and collaborators spent almost all his professional life at Rutgers University since 1967, other than sabbatical periods at the Institute for Advanced Study, Princeton, Purdue University, and the University of Buenos Aires, Argentina. He has made many original contributions to Potential Theory, Harmonic Analysis, and Partial Differential equations. Many of his papers have profoundly influenced these fields and have had long lasting effects, stimulating research and shedding light. In addition many colleagues and especially young people have benefitted from the generosity of his spirit, where he has shared mathematical insight and provided encouragement. We hope this volume showcases some of the research directions Dick Wheeden was instrumental in pioneering.

1 Potential Theory and Weighted Norm Inequalities for Singular Integrals

Dick Wheeden's work in Analysis can be broken into two periods. The first period consists of his work in Potential Theory, the theory of singular integrals with a deep emphasis on weighted norm inequalities, and a second period from the late 1980s where he and his collaborators successfully applied weighted norm inequalities to the study of degenerate elliptic equations, subelliptic operators, and Monge-Ampère equations.

Wheeden obtained his Ph.D. in 1965 from the University of Chicago under the supervision of Antoni Zygmund. One very productive outcome of this association with Zygmund is the beautiful graduate textbook *Measure and Integral* [36].

Wheeden's thesis dealt with hypersingular integrals. These are singular integrals of the form

$$Tf(x) = \int_{\mathbb{R}^n} (f(x+y) - f(x)) \frac{\Omega(y)}{|y|^{n+\alpha}} dy, \quad 0 < \alpha < 2,$$

where $\Omega(y)$ is homogeneous of degree zero, integrable on $\mathbb{S}^{n-1}$ and satisfies

$$\int_{\mathbb{S}^{n-1}} y_i \Omega(y)\, d\sigma(y) = 0, \quad 1 \le i \le n.$$

Since the singularity of the kernel $\frac{\Omega(y)}{|y|^{n+\alpha}}$ is more than that of a standard Calderón-Zygmund kernel, one needs some smoothness on f to ensure boundedness. A typical result found in [34] is

$$\|Tf\|_{L^p(\mathbb{R}^n)} \le C \|f\|_{W^{\alpha,p}(\mathbb{R}^n)}, \quad 1 < p < \infty,$$

where $W^{\alpha,p}(\mathbb{R}^n)$ is the fractional Sobolev space of order α. These results are developed further in [35].

Another important result that Wheeden obtained at Chicago and in his early time at Rutgers was with Richard Hunt. This work may be viewed as a deep generalization of a classic theorem of Fatou which states that nonnegative harmonic functions in the unit disk in the complex plane have nontangential limits a.e. on the boundary, that is on the unit circle. The theorem of Fatou was generalized to higher dimensions and other domains by Calderón and Carleson. The works [17, 18] extend the Fatou theorem to Lipschitz domains, where now one is dealing with harmonic measure on the boundary. The main result is

Theorem 1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Let $\omega^{P_0}(Q)$, $Q \in \partial\Omega$, denote harmonic measure with respect to a fixed point $P_0 \in \Omega$. Then any nonnegative harmonic function $u(P)$ in Ω has nontangential limits a.e. with respect to harmonic measure ω^{P_0} on $\partial\Omega$.*

The proof relies on constructing clever barriers and in particular on a penetrating analysis using Harnack's inequality on the kernel function $K(P,Q)$, $P \in \Omega$, $Q \in \partial\Omega$, which is the Radon-Nikodym derivative

$$K(P,Q) = \frac{d\omega^P(Q)}{d\omega^{P_0}(Q)}.$$

1.1 Singular Integrals and Weighted Inequalities

In 1967, Wheeden moved to Rutgers University and began a long and fruitful collaboration with his colleague B. Muckenhoupt. Two examples of many seminal results proved by Muckenhoupt and Wheeden are the theorems on weighted norm inequalities for the Hilbert transform and the fractional integral operator. To state these results we recall a definition.

Definition 1 Let $1 < p < \infty$, and let $w \in L^1_{loc}(\mathbb{R}^n)$ be a positive function on $\mathbb{R}^n$. Then $w \in A_p$ if and only if for all cubes Q,

$$\sup_Q \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{-\frac{1}{p-1}}\right)^{p-1} < \infty.$$

The A_p condition had already appeared in Muckenhoupt's pioneering work on the Hardy-Littlewood maximal function [51]. But now Wheeden along with Hunt and Muckenhoupt [19] carried it further. They considered the prototypical one-dimensional singular integral, the Hilbert transform,

$$Hf(x) = p.v. \int_{-\infty}^{\infty} \frac{f(y)}{x-y} dy,$$

and established the following trailblazing theorem.

Theorem 2 *A nonnegative $w \in L^1_{loc}(\mathbb{R})$ satisfies the L^p weighted norm inequality for the Hilbert transform,*

$$\left(\int_{\mathbb{R}} |Hf|^p w\right)^{\frac{1}{p}} \leq C_p \left(\int_{\mathbb{R}} |f|^p w\right)^{\frac{1}{p}},$$

if and only if $w \in A_p$.

Their key difficulty in establishing this result was to prove it when $p = 2$. Then one can adapt the Calderón-Zygmund scheme for singular integrals and finish with an interpolation. The case $p = 2$ had been studied earlier by Helson and Szegö [47] using a completely different function theoretic approach, where they obtained the equivalence of the weighted norm inequality with a subtle decomposition of the weight involving the conjugate function. Theorem 2 finally characterized these two equivalent properties in terms of a remarkably simple and checkable criterion, the A_p condition. Theorem 2 was the forerunner to a deluge of results by Wheeden in the decades since, to multiplier operators by Kurtz and Wheeden [20], to the Lusin square function by Gundy and Wheeden [16] (preceded by Segovia and Wheeden [33]), and the Littlewood-Paley g^*_λ function by Muckenhoupt and Wheeden [24], to name just a few. With Muckenhoupt, Wheeden also initiated a study of the two weight theory for the Hardy-Littlewood maximal operator and Hilbert transform [25] and with Chanillo a study of the two weight theory for the square function [6]. That is one now seeks conditions on nonnegative functions v, w so that one has

$$\left(\int_{\mathbb{R}^n} |Tf|^p v\right)^{\frac{1}{p}} \leq C_p \left(\int_{\mathbb{R}^n} |f|^p w\right)^{\frac{1}{p}},$$

where T could be a singular integral operator, a square function, or the Hardy-Littlewood maximal operator. The papers [25] and [6] stimulated much research in a search for an appropriate two weight theory for singular integrals. In the early 1990s Wheeden returned to this question and undertook a study of two weight problems for the fractional integral. These results are described later in this preface.

The later "one weight" results mentioned above relied on the so-called good-λ inequalities [37], [40], a beautiful stratagem with which Wheeden was wholly won over. We cite two instances of results proved by Wheeden, where good-λ inequalities play a key step in the proofs. The first example is joint work with Chanillo [1] where he investigated a complete theory of differentiation based on the Marcinkiewicz integral

$$Mf(x) = \left(\int_{\mathbb{R}^n} \frac{|f(x+t) + f(x-t) - 2f(x)|^2}{|t|^{n+2}} dt \right)^{\frac{1}{2}}.$$

This work viewed Mf as a rough square function and the aim was to treat it in the spirit of the work of Burkholder and Gundy [37] for the Lusin square function and establish control via a good-λ inequality and maximal functions.

The second work with Muckenhoupt, destined to play a major role in Wheeden's interest in degenerate elliptic PDE in the late 1980s onward, was the paper [23] on fractional integral operators I_α. Define for $0 < \alpha < n$,

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Theorem 3 *Let v be a positive function on $\mathbb{R}^n$. Then for $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{p} + \frac{1}{p'} = 1$, the weighted norm inequality for I_α,*

$$\left(\int_{\mathbb{R}^n} |(I_\alpha f)\, v|^q \right)^{\frac{1}{q}} \leq C_p \left(\int_{\mathbb{R}^n} |f v|^p \right)^{\frac{1}{p}},$$

holds if and only if for all cubes Q,

$$\sup_Q \left(\frac{1}{|Q|} \int_Q v^q \right)^{\frac{1}{q}} \left(\frac{1}{|Q|} \int_Q v^{-p'} \right)^{\frac{1}{p'}} < \infty.$$

The corresponding inequality for the fractional maximal operator

$$M_\alpha f(x) = \sup_{Q:\, x \in Q} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)|\, dy,$$

which is dominated by the fractional integral $I_\alpha |f|(x) = \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy$, can be established by a variety of techniques, and from this, the inequality for the larger (at least when $f \geq 0$) fractional integral I_α can be obtained as a consequence of the *good-λ inequality*:

$$\begin{aligned} &|\{x \in \mathbb{R}^n : M_\alpha f(x) \leq \beta\lambda \text{ and } |I_\alpha f(x)| > 2\lambda\}|_{v^q} \\ &\leq C\beta\, |\{x \in \mathbb{R}^n : |I_\alpha f(x)| > \lambda\}|_{v^q}, \end{aligned}$$

valid for nonnegative functions f, and for all $\lambda > 0$ and $0 < \beta < 1$ and where we have used the notation:

$$|E|_{v^q} = \int_E v^q(x)dx.$$

This striking inequality says, loosely speaking, that the conditional probability of doubling the size of $I_\alpha f$, given a fixed lower threshold, is small unless the maximal function $M_\alpha f$ exceeds a smaller threshold—in other words, $I_\alpha f$ cannot increase by much at a given location unless $M_\alpha f$ is already large there.

The fractional integral operator $I_\alpha f$ plays a major role in the proofs of Sobolev inequalities and localized versions of Sobolev inequalities called Poincaré inequalities. These inequalities, together with the energy inequalities of Cacciopoli, are used to derive via an iteration scheme due to Moser, a fundamental inequality for elliptic PDE, called the Harnack inequality. The Harnack inequality can be then used to obtain regularity in Hölder classes of weak solutions of second order elliptic PDE. Thus Wheeden was now led in a second period to the study of degenerate elliptic PDE and the particular problem of regularity of weak solutions to degenerate elliptic PDE. One of the earliest Poincaré-Sobolev inequalities he obtained was a natural outcome of earlier work for the Peano maximal function [2] and is contained in his paper with Chanillo [3]. To state the main theorem in [3], we need some notation. We consider v, w locally integrable positive functions on $\mathbb{R}^n$. Fix a ball B. We now consider balls $B_r(x_0) \subset B$, centered at x_0 with radius $r > 0$. We assume that v is doubling, i.e.,

$$\int_{B_{2r}(x_0)} v \leq C \int_{B_r(x_0)} v,$$

and we also assume the balance condition (which turns out to be necessary)

$$\frac{r}{h}\left(\frac{\int_{B_r(x_0)} v}{\int_B v}\right)^{\frac{1}{q}} \leq C\left(\frac{\int_{B_r(x_0)} w}{\int_B v}\right)^{\frac{1}{p}}, \tag{1}$$

$$\text{where } h = |B|^{\frac{1}{n}}.$$

For $f \in C^1(B)$, we set $f_{\text{avg}} = \frac{1}{|B|}\int_B f$.

Theorem 4 *Let* $1 < p < q < \infty$. *Assume that* $w \in A_p$, *that* v *is doubling, and that the balance condition (1) holds. Then*

(1) *For* $f \in C_0^1(B)$ *we have the Sobolev inequality,*

$$\left(\frac{1}{\int_B v}\int_B |f|^q\, v\right)^{\frac{1}{q}} \le Ch\left(\frac{1}{\int_B w}\int_B |\nabla f|^p\, w\right)^{\frac{1}{p}}.$$

(2) *For* $f \in C^1(B)$ *we have the Poincaré inequality,*

$$\left(\frac{1}{\int_B v}\int_B \left|f - f_{\mathrm{avg}}\right|^q\, v\right)^{\frac{1}{q}} \le Ch\left(\frac{1}{\int_B w}\int_B |\nabla f|^p\, w\right)^{\frac{1}{p}}.$$

The results in [3] when combined with energy estimates like Cacciopolli's inequality and an appropriate Moser iteration scheme lead to Harnack inequalities [5] and estimates for Green's function for elliptic operators in divergence form [7].

2 Degenerate Elliptic Equations, Subelliptic Operators, and Monge-Ampére Equations

In the early 1990s, Wheeden's interests turned to the study of Sobolev-Poincaré inequalities in the setting of metric spaces, focusing in particular on Carnot-Carathéodory metrics generated by a family of vector fields and on the associated degenerate elliptic equations. Let $X := \{X_1, \ldots, X_m\}$ be a family of Lipschitz continuous vector fields in an open set $\Omega \subset \mathbb{R}^n$, $m \le n$. We can associate with X a metric in Ω—the Carnot-Carathéodory (CC) metric $d_c = d_c(x, y)$ or the control metric—by taking the minimum time we need to go from a point x to a point y along piecewise integral curves of $\pm X_1, \ldots, \pm X_m$ (if such curves exist). The generating vector fields of the Lie algebra of connected and simply connected, stratified nilpotent Lie groups, also called Carnot groups, as well as vector fields of the form $\lambda_1\partial_1, \ldots, \lambda_n\partial_n$, where the λ_j's are Lipschitz continuous nonnegative functions, provide basic examples of vector fields for which the CC distance is always finite. The latter vector fields are said to be of Grushin type in [10].

A (p, q)-Sobolev-Poincaré inequality in this setting is an estimate of the form

$$\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)} |g - g_B|^q\, dx\right)^{\frac{1}{q}} \le C\, r\left(\frac{1}{|B(x_0,r)|}\int_{B(x_0,r)}\left(\sum_j |X_j g|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \tag{2}$$

for all metric balls $B(x_0, r) := \{x\,;\, d_c(x_0, x) < r\}$ and for all smooth functions g with average g_B on $B(x_0, r)$. Moreover, Lebesgue measure on both sides of (2) can be replaced by two different measures which may arise from weight functions. This case gives rise to what is called a weighted (or two-weight) Sobolev-Poincaré inequality. The weight functions are chosen to satisfy conditions akin to [3] and [5].

Wheeden in 1994, in collaboration with Franchi and Gutiérrez [10], proved a two-weight Sobolev-Poincaré inequality for a class of Grushin type vector fields that best illustrates this circle of ideas.

In the Poincaré-Sobolev inequality that follows, the weight function u is assumed doubling. The vector fields are given by, $\nabla_\lambda g(z) = \left(\nabla_x g(z), \lambda(x)\nabla_y g(z)\right)$ for $z = (x, y) \in \mathbb{R}^{n+m}$. $\lambda(x)$ is assumed continuous. The continuity of $\lambda(x)$ allows the notion of a metric $d_c(\cdot,\cdot)$ which is naturally associated with the vector fields $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \lambda(x)\frac{\partial}{\partial y_1}, \ldots, \lambda(x)\frac{\partial}{\partial y_n}$ by means of subunit curves to be defined [42]. To this metric one may now associate balls $B(z_0, r)$ which are balls in $\mathbb{R}^{n+m}$ with center z_0 and radius r defined by $B(z_0, r) := \{z\,;\, d_c(z_0, z) < r\}$. With some further stipulation on the weight v that will be stated later, one has the Poincaré-Sobolev inequality displayed below:

$$\left(\frac{1}{|B(z_0, r)|_u}\int_{B(z_0,r)} |g(z) - g_B|^q\, u(z)\, dz\right)^{\frac{1}{q}}$$

$$\leq Cr\left(\frac{1}{|B(z_0, r)|_v}\int_{B(z_0,r)} |\nabla_\lambda g(z)|^p\, v(z)\, dz\right)^{\frac{1}{p}}$$

Further assumptions on the coefficients of the vector fields are $\lambda(x)$ lies in some Reverse Hölder class, i.e., $\lambda \in RH_\infty$ and $\lambda^n \in \mathit{strong}\ A_\infty$, in the sense of David and Semmes [41] suitably adapted to the Carnot-Caratheodory metric situation. The key assumption on the weight v above is that there exists $w \in \mathit{strong}\ A_\infty$ for which we have

$$v w^{\frac{1}{N}-1} \in A_p\left(w^{1-\frac{1}{N}} dz\right),\ N = n + m.$$

The following "balance condition" which is by now well known to be necessary [3] is also assumed:

$$\frac{r(B)}{r(B_0)}\left(\frac{u(B)}{u(B_0)}\right)^{\frac{1}{q}} \leq C\left(\frac{v(B)}{v(B_0)}\right)^{\frac{1}{p}}, \quad B \subset (1+\varepsilon)B_0.$$

A typical example of a function λ is $\lambda(x) = |x|^\alpha$ for $\alpha > 0$. The results allow weights v that vanish to high order and include new classes of weight functions even in the case $\lambda(x) \equiv 1$. Other important examples arise from weight functions v that are Jacobians of Quasiconformal maps.

The paper [10] contains two remarkable technical results: first of all, it is proved that metric balls for the CC distance satisfy the so-called *Boman condition*, that by now has been proved to be equivalent to several other geometric conditions. This allows one to apply the *Boman chain technique* (as studied by Chua and Bojarski) to suitable metric spaces equipped with doubling measures. This makes it possible to obtain (2) from a "weak" type Sobolev-Poincaré inequality, where in the right-hand side of (2) we replace the ball $B(\bar{x}, r)$ by an "homothetic" ball $B(\bar{x}, \tau r)$, $\tau > 1$. The second important idea in this paper consists in the clever use of a technique inspired by Long and Nie [50], and that will become more or less standard in the future. To illustrate this idea, consider a fractional integral I and let $u \to |Xu|$ be a *local* operator, where $|Xu|$ denotes the norm of the Euclidean gradient or of some generalized gradient $(X_1 u, \dots, X_m u)$. This technique makes it possible to obtain strong type inequalities from weak type inequalities of the form:

$$|\{x \in B := B(\bar{x}, r)\,;\ |u(x) - u_B| > \lambda\}| \leq C \left(\frac{\|I(|Xu|)\|_{L^1(\tau B)}}{\lambda} \right)^{\frac{1}{q}}, \qquad \tau > 1,$$

In particular one obtains $(1, q)$-Sobolev-Poincaré inequalities in situations where one has no recourse to the Marcinkiewicz interpolation theorem. This is achieved by slicing the graph of $u(x) - u_B$ in strips $[2^{-k+1}, 2^{-k}]$. The local character of $|Xu|$ yields that $|Xu|$ vanishes on constants, so that it is possible to reconstruct $|Xu|$ from these slices.

This technique enabled Wheeden in [9] (in collaboration with Franchi and Gallot) and in [11] (in collaboration with Franchi and Lu) to prove Sobolev type inequalities and Sobolev-Poincaré type inequalities on Carnot groups in the *geometric case* $p = 1$, starting from a subrepresentation formula of a compactly supported function (or of a function of zero average on a ball) which expressed the function in terms of a suitable fractional integral of its generalized gradient. In particular, this argument yields forms of Sobolev inequalities which are related to isoperimetric inequalities on Carnot groups.

More generally, on a metric space $(\mathcal{S}, \rho, m)$ endowed with a doubling measure m, we say that a (p, q)-Sobolev-Poincaré inequality holds ($1 \leq p \leq q \leq \infty$) if for any Lipschitz continuous function u there exists $g \in L^p_{\text{loc}}(\mathcal{S})$ such that

$$\left(\frac{1}{|B(\bar{x}, r)|} \int_{B(\bar{x},r)} \left| u - \frac{1}{|B(\bar{x}, r)|} \int_{B(\bar{x},r)} u\, dm \right|^q dm \right)^{\frac{1}{q}} \leq C\, r \left(\frac{1}{|B(\bar{x}, r)|} \int_{B(\bar{x},r)} |g|^p\, dm \right)^{\frac{1}{p}p}, \tag{3}$$

where g depends on u but is independent of $B(\bar{x}, r)$ (notice again we could look for similar inequalities where we replace the measure m by two measures μ, ν). We recall that the metric space $(\mathcal{S}, \rho, m)$ endowed with a measure m is said to be locally doubling, if for the measure m there exists $A > 0$ such that the measure m satisfies the doubling condition $m(B(x, 2r)) \leq Am(x, r)$ for all $x \in \mathcal{S}$ and $r \leq r_0$. That is the doubling condition holds for all balls with small enough radii. The central point in the proof in [12] (see also [8]) consists in establishing the equivalence between (p, q)-Sobolev-Poincaré inequalities in metric measure spaces and subrepresentation formulae. The following result is typical:

Theorem 5 *Let* $(\mathcal{S}, \rho, m)$ *be a complete metric space endowed with a locally doubling measure* m *and satisfying the segment property (i.e., for each pair of points* $x, y \in \mathcal{S}$*, there exists a continuous curve* γ *connecting* x *and* y *such that* $\rho(\gamma(t), \gamma(s)) = |t - s|$*.) Let* μ*,* ν *be locally doubling measures on* $(\mathcal{S}, \rho, m)$*. Let* $B_0 = B(x_0, r_0)$ *be a ball, let* $\tau > 1$ *be a fixed constant, and let* $f, g \in L^1(\tau B_0)$ *be given functions. Assume there exists* $C > 0$ *such that, for all balls* $B \subseteq \tau B_0$*,*

$$\frac{1}{\nu(B)} \int_B |f - f_{B,\nu}| \, d\nu \le C \frac{r(B)}{\mu(B)} \int_B |g| \, d\mu, \tag{4}$$

where $f_{B,\nu} = \frac{1}{\nu(B)} \int_B f d\nu$*. If there is a constant* $\theta(r_0) > 0$ *such that for all balls* B*,* $\tilde{B}$ *with* $\tilde{B} \subseteq B \subseteq \tau B_0$*,*

$$\frac{\mu(B)}{\mu(\tilde{B})} \ge \theta(r_0) \frac{r(B)}{r(\tilde{B})},$$

then for $(d\nu)$*-a.e.* $x \in B_0$*,*

$$|f(x) - f_{B_0,\nu}| \le C \int_{\tau B_0} |g(y)| \frac{\rho(x, y)}{\mu(B(x, \rho(x, y)))} \, d\mu(y). \tag{5}$$

We notice that, by Fubini-Tonelli Theorem, clearly (5) *implies* (4).

The proof of the above result relies on the construction of a suitable chain of balls with controlled overlaps, starting from a ball B and shrinking around a point $x \in B$. Repeated use of the Poincaré inequality (4) yields the subrepresentation formula (5) for any Lebesgue point of u.

Applying results on $L^p - L^q$ continuity for fractional integrals, like for example in [3, 29], from (5) one obtains (two-weight) Sobolev-Poincaré inequalities. In [22] Lu and Wheeden were able to get rid of the constant $\tau > 1$ in the subrepresentation formula (5).

A $(p_0, 1)$-Poincaré-inequality with $p_0 \ge 1$ yielding a (p, q)-Sobolev-Poincaré inequalities (and possibly with weights) is referred to as the *self-improving property* of the Poincaré inequality. This notion had been introduced by Saloff-Coste in [53] in the Riemannian or sub-Riemannian setting. The result in [53] states that the $(p_0, 1)$-Poincaré inequality plus the doubling property of the measure yields (p, q)-Sobolev-Poincaré inequalities. The arguments of [12] can be carried out only in the case $p_0 = 1$, basically since $a(B)$ defined by

$$a(B) = r(B) \left(\frac{1}{|B|} \int_B |g|^{p_0} \, dx \right)^{1/p_0} \qquad (g \text{ and } p_0 \text{ fixed})$$

is not easy to sum, even over a class of disjoint balls B if $p_0 > 1$.

In particular, in [15], this difficulty was overcome by considering a sum operator $T(x)$ which is formed by summing $a(B)$ over an appropriate chain of balls associated with a point x:

$$T(x) = \sum_{B \text{ in a chain for } x} a(B).$$

In case $p_0 = 1$, the sum operator becomes an integral operator. The L^p to L^q mapping properties of the sum operator can be derived in much the same way as those for fractional integral operators, and these norm estimates for T lead to correspondingly more general Poincaré estimates. These results by Wheeden and collaborators for the weighted self-improving property of the Poincaré inequality on general metric spaces may be found in [13–15].

In [21], the authors proved a counterpart of the equivalence between subrepresentation formulae and Sobolev-Poincaré inequality for higher order differential operators. These results are counterparts of earlier results for the gradient derived in [12]. In the higher order case, on the left-hand side of the Poincaré inequality, instead of subtracting a constant given by the average of the function, one subtracts appropriate polynomials, related to the Taylor polynomial on Euclidean spaces, and related Folland-Stein polynomials [43] for the situation on stratified groups.

2.1 Two Weight Norm Inequalities for Fractional Integrals

Beginning in 1992, Wheeden returned to the study of the two weight inequality for fractional integrals,

$$\left(\int_{\mathbb{R}^n} (I_\alpha f)^q \, w\right)^{\frac{1}{q}} \leq C \left(\int_{\mathbb{R}^n} f^p v^p\right)^{\frac{1}{p}}, \quad f \geq 0,$$

and showed with Sawyer [29] that for $1 < p < q < \infty$ and $0 < \alpha < n$, this inequality could be characterized by a simple two weight analogue of Muckenhoupt's condition:

$$A^{\alpha}_{p,q} \equiv \sup_Q |Q|^{1-\frac{\alpha}{n}} \left(\int_Q s_Q^q w\right)^{\frac{1}{q}} \left(\int_Q s_Q^{p'} v^{-p'}\right)^{\frac{1}{p'}} < \infty,$$

where $s_Q(x) \equiv |Q|^{\frac{\alpha}{n}-1} + |x - x_Q|^{\alpha-n}$ is a "tailed" version of the scaled indicator $|Q|^{\frac{\alpha}{n}-1} \mathbf{1}_Q(x)$. This work built on the weak type work of Kokilashvili and Gabidzashvili. Unfortunately, this simple solution fails when $p = q$, but there it was shown that a "bumped-up" version of $A^{\alpha}_{p,q}$ suffices for the fractional integral inequality: there is $r > 1$ such that

$$\sup_{Q} |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}} \left(\frac{1}{|Q|} \int_{Q} w^{r} \right)^{\frac{1}{qr}} \left(\frac{1}{|Q|} \int_{Q} v^{-rp'} \right)^{\frac{1}{p'r}} < \infty.$$

A forerunner to the situation when $p = q$ is the paper by Chanillo and Wheeden [4], where weighted fractional integral inequalities when $p = q$ are derived and then applied to obtain Weyl type eigenvalue estimates for the Schrödinger operator with appropriate potential.

The results on weighted norm inequalities for two weights for fractional integrals and other similar results were then extended in [29] to spaces of homogeneous type. Along the way two discoveries were made which we list:

- The failure of the Besicovitch covering lemma for the Heisenberg group equipped with the usual left invariant metric and where all balls are chosen using this metric (also obtained independently by Koranyi and Reimann).
- A construction of a dyadic grid for spaces of homogeneous type (a variant was also obtained independently and a bit earlier by M. Christ [38], and a precursor of this by G. David even earlier).

2.2 Fefferman-Phong and Hörmander Regularity

The 2006 Memoir of Wheeden with Sawyer [30] is concerned with regularity of solutions to *rough subelliptic* equations. Previously, regularity had been reasonably well understood in two cases:

1. when the equation is subelliptic, and the coefficients are restricted to being smooth,
2. when the equation is elliptic, and the coefficients are quite rough.

In the subelliptic case, there were two main types of result. First, there was the algebraic commutator criterion of Hörmander for sums of squares of smooth vector fields [48]. These operators had a special "sum of squares" form for the second order terms, but no additional restriction on the smooth first order term. Second, there was the geometric "control ball" criterion of Fefferman and Phong that applies to operators with general smooth subelliptic second order terms, but the operators were restricted to be self-adjoint. They obtained the following analogue of the Fefferman-Phong theorem for rough coefficients, namely a quadratic form $\mathcal{Q}(x, \xi) = \xi' Q(x) \xi$ is *subelliptic* (which means that in a quantitative sense we leave unspecified, all weak solutions u to the equation $\nabla' Q(x) \nabla u = \phi$ are Hölder continuous, i.e., $u \in$

C^{α} for $\alpha > 0$), if the control balls $K(x,r)$ relative to $\mathcal{Q}$ satisfy

1. $|K(x,2r)| \leq C|K(x,r)|$ (doubling),
2. $D(x,r) \subset K(x,Cr^{\varepsilon})$ (containment),
3. $\left\{\frac{1}{|K|}\int_K |w|^{2\sigma}\right\}^{\frac{1}{2\sigma}} \leq Cr(K)\left\{\frac{1}{|K|}\int_K \left(\|\nabla w\|_{\mathcal{Q}}^2 + |w|^2\right)\right\}^{\frac{1}{2}}$,
4. $\left\{\frac{1}{|K|}\int_K |w - w_K|^2\right\}^{\frac{1}{2}} \leq Cr(K)\left\{\frac{1}{|K^*|}\int_{K^*} \|\nabla w\|_{\mathcal{Q}}^2\right\}^{\frac{1}{2}}$,

where $w \in W_0^{1,2}(K)$ in the Sobolev inequality and $w \in W^{1,2}(K^*)$ in the Poincaré inequality, where K^* is the double of K.

The containment condition 2. is necessary. The Sobolev inequality 3. is necessary for a related notion of subellipticity for the homogeneous Dirichlet problem for $L = \nabla' Q(x)\nabla$: for all balls B there exists a weak solution u satisfying

$$\begin{cases} Lu = f & in\ B \\ u = 0 & on\ \partial B \end{cases} \text{ and } \sup_B |u| \lesssim \left(\int_B |f|^{\frac{q}{2}}\right)^{\frac{2}{q}}.$$

The Poincare inequality 4. is necessary for a related notion of hypoellipticity for the homogeneous Neumann problem for $\mathbf{n}_Q = \mathbf{n}' Q(x)\nabla$: for all balls B there exists a weak solution u satisfying

$$\begin{cases} Lu = f & in\ B \\ \mathbf{n}_Q u = 0 & on\ \partial B \end{cases} \text{ and } \|u\|_{L^2(B)} \lesssim r(B)^2 \|f\|_{L^2(B)}.$$

The doubling condition 1. is not needed and has been replaced more recently with the theory of nondoubling measures pioneered by Nazarov, Treil, and Volberg.

They also obtained an analogue of the Hörmander theorem for diagonal vector fields with rough coefficients. As a starting point, they showed that if the vector fields $X_j = a_j(x)\frac{\partial}{\partial x_j}$ were analytic, then the X_j satisfied a "flag condition" if and only if they satisfied the Hörmander commutation condition. They then extended the flag condition to rough vector fields and obtained regularity theorems for solutions to the corresponding sums of squares operators.

2.3 *The Monge-Ampére Equation*

Using the regularity theorems in their 2006 Memoir [30] (see also [28], [31] and [32]), Wheeden with Rios and Sawyer [26, 27] obtained the following geometric result: A C^2 convex function u whose graph has smooth Gaussian curvature $k \approx |x|^2$ is itself smooth *if and only* if the sub-Gaussian curvature k_{n-1} of u is positive in Ω.

The question remains open for $C^{1,1}$ convex solutions today—this much regularity is assured for solutions to the Dirichlet problem with smooth data and nonnegative Gaussian curvature k (Guan et al. [46]), but cannot in general be improved to C^2 by the example of Sibony in which the tops and sides of the unit disk are curled up to form a smooth boundary but with second order discontinuities at the start of the curls.

The proof of the regularity theorem for C^2 solutions draws from a broad spectrum of results—an n-dimensional extension of the partial Legendre transform due to the authors [26], Calabi's identity for $\sum u^{ij}\sigma_{ij}$, the Campanato method of Xu and Zuily [54], the Rothschild-Stein lifting theorem for vector fields [52], Citti's idea (see, e.g., [39]) of approximating vector fields by first order Taylor expansions, and earlier work of the authors in [26] generalizing Guan's subelliptic methods in [44, 45]. The proof of the geometric consequence uses the Morse lemma to obtain the sum of squares representation of k. The necessity of $k_{n-1} > 0$ follows an idea of Iaia [49]: the inequality $k \leq (k_{n-1})^{\frac{n}{n-1}}$ shows that for a smooth convex solution u with $k(x) \approx |x|^2$ we must have $k_{n-1} > 0$ at the origin.

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Piscataway, NJ, USA — Sagun Chanillo
Bologna, Italy — Bruno Franchi
Hamilton, ON, Canada — Eric T. Sawyer

Contents

On Some Pointwise Inequalities Involving Nonlocal Operators

Luis A. Caffarelli and Yannick Sire

To Dick Wheeden, with admiration and affection

Abstract The purpose of this paper is threefold: first, we survey on several known pointwise identities involving fractional operators; second, we propose a unified way to deal with those identities; third, we prove some new pointwise identities in different frameworks in particular geometric and infinite-dimensional ones.

1 Introduction

The present paper is devoted to several pointwise inequalities involving several nonlocal operators. We focus on two types of pointwise inequalities: the Córdoba-Córdoba inequality and the Kato inequality. In order to keep the presentation simple, we state the inequalities in question in the case of the fractional laplacian, i.e. $(-\Delta)^s$, in $\mathbb{R}^n$. Actually, in subsequent sections, we will generalize these inequalities to a lot of different contexts. Furthermore, we will present a unified proof for both inequalities based on some extension properties of some nonlocal operators. Our proofs are elementary and simplify the original arguments.

L.A. Caffarelli
Mathematics Department, University of Texas at Austin, 2515 Speedway Stop C1200, Austin, TX 78712–1202, USA
e-mail: caffarel@math.utexas.edu

Y. Sire (✉)
Department of Mathematics, Johns Hopkins University, Krieger Hall, 3400 N. Charles Street, Baltimore, MD 21218, USA
e-mail: sire@math.jhu.edu

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The fractional Laplacian can be defined in various ways, which we review now. It can be defined using Fourier transform by

$$\mathcal{F}((-\Delta)^s v) = |\xi|^{2s}\,\mathcal{F}(v),$$

for $v \in H^s(\mathbb{R}^n)$. It can also be defined through the kernel representation (see the book by Landkof [12])

$$(-\Delta)^s v(x) = C_{n,s}\ \text{P.V.} \int_{\mathbb{R}^n} \frac{v(x) - v(\overline{x})}{|x - \overline{x}|^{n+2s}}\, d\overline{x}, \tag{1}$$

for instance for $v \in \mathcal{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decaying functions. Here we will only consider $s \in (0, 1)$.

The inequalities considered in the present paper are the following

Theorem 1.1 (Córdoba-Córdoba Inequality) *Let φ be a $C^2(\mathbb{R}^n)$ convex function. Assume that u and $\varphi(u)$ are such that $(-\Delta)^s u$ and $(-\Delta)^s\varphi(u)$ exist. Then the following holds*

$$(-\Delta)^s \varphi(u) \le \varphi'(u)(-\Delta)^s\, u. \tag{2}$$

The next theorem is the Kato inequality.

Theorem 1.2 (Kato Inequality) *The following inequality holds in the distributional sense*

$$(-\Delta)^s |u| \le sgn(u)(-\Delta)^s\, u. \tag{3}$$

The previous two theorems are already known: Theorem 1.1 is due to Córdoba and Córdoba (see [8, 9]). Theorem 1.2 is due to Chen and Véron (see [6]). Both original proofs are based on the representation formula given in (1). This formula holds only when the fractional laplacian is defined on $\mathbb{R}^n$. The Córdoba-Córdoba inequality is a very useful result in the study of the quasi-geostrophic equation (see [9]). This inequality has been generalized in several contexts in [10] for instance or [7]. In this line of research we propose a unified way of proving these inequalities based on some extension properties for nonlocal operators.

2 Some New Inequalities

In this section, we derive by a very simple argument several inequalities at the nonlocal level, i.e. without using any extensions, which are not available in these frameworks.

2.1 A Pointwise Inequality for Nonlocal Operators in Non-divergence Form

Nonlocal operators in non-divergence form are defined by

$$\mathcal{I}u(x) = -\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)\,dy$$

for a kernel $K \geq 0$. Denote

$$\delta_y u(x) = -\Big(u(x+y) + u(x-y) - 2u(x)\Big).$$

Then, considering a C^2 convex function φ, one has by the fact that a convex function is above its tangent line

$$\delta_y \varphi(u)(x) = -\Big(\varphi(u(x+y)) + \varphi(u(x-y)) - 2\varphi(u(x))\Big) =$$
$$-\Big(\varphi(u(x+y)) - \varphi(u(x)) + \varphi(u(x-y)) - \varphi(u(x))\Big)$$
$$\leq \varphi'(u(x))\delta_y u(x).$$

Hence for the operator $\mathcal{I}$ one has also an analogue of the original Córdoba-Córdoba estimate.

2.2 The Case of Translation Invariant Kernels

Consider the operator

$$\mathcal{L}u(x) = \int_{\mathbb{R}^n} (u(x) - u(y))K(x-y)\,dy$$

where K is symmetric. Hence one can write

$$\mathcal{L}u(x) = \int_{\mathbb{R}^n} (u(x) - u(x-h))K(h)\,dh$$

or in other words, by a standard change of variables

$$\mathcal{L}u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \delta_h u(x)K(h)\,dh$$

We start with the following lemma, which is a direct consequence of the symmetry of the kernel

Lemma 2.1

$$\int_{\mathbb{R}^n} \mathcal{L}u(x) = 0.$$

The following lemma is consequence of straightforward computations

Lemma 2.2

$$\delta_h uv(x) = u\delta_h v + v\delta_h u +$$
$$(v(x+h) - v(x))(u(x+h) - u(x)) + (v(x-h) - v(x))(u(x-h) - u(x)).$$

Hence by the two previous lemma one has the useful identity

$$0 = \int_{\mathbb{R}^n} \mathcal{L}u^2 = 2\int_{\mathbb{R}^n} u\mathcal{L}u + 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} (u(x) - u(y))^2 K(x-y)\,dxdy.$$

2.3 Some Integral Operators on Geometric Spaces

In this section, we describe new operators involving curvature terms. These operators appear naturally in harmonic analysis, as described below. They are of the form

$$\mathcal{L}u(x) = \int (u(x) - u(y))K(x,y)\,dy$$

where the non-negative kernel K is symmetric and has some geometric meaning. The integral sign runs either over a Lie group or over a Riemannian manifold. By exactly the same argument as in the previous section, one deduces trivially Córdoba-Córdoba estimates for these operators. We now describe these new operators.

2.3.1 The Case of Lie Groups

Let G be a unimodular connected Lie group endowed with the Haar measure dx. By "unimodular", we mean that the Haar measure is left and right-invariant. If we denote by $\mathcal{G}$ the Lie algebra of G, we consider a family

$$\mathbb{X} = \{X_1, \ldots, X_k\}$$

of left-invariant vector fields on G satisfying the Hörmander condition, i.e. $\mathcal{G}$ is the Lie algebra generated by the $X_i's$. A standard metric on G , called the Carnot-Caratheodory metric, is naturally associated with $\mathbb{X}$ and is defined as follows: let $\ell : [0, 1] \to G$ be an absolutely continuous path. We say that ℓ is admissible if there exist measurable functions $a_1, \ldots, a_k : [0, 1] \to \mathbb{C}$ such that, for almost every $t \in [0, 1]$, one has

$$\ell'(t) = \sum_{i=1}^{k} a_i(t) X_i(\ell(t)).$$

If ℓ is admissible, its length is defined by

$$|\ell| = \int_0^1 \left(\sum_{i=1}^{k} |a_i(t)|^2 \, dt \right)^{\frac{1}{2}}.$$

For all $x, y \in G$, define $d(x, y)$ as the infimum of the lengths of all admissible paths joining x to y (such a curve exists by the Hörmander condition). This distance is left-invariant. For short, we denote by $|x|$ the distance between e, the neutral element of the group and x, so that the distance from x to y is equal to $|y^{-1}x|$.

For all $r > 0$, denote by $B(x, r)$ the open ball in G with respect to the Carnot-Caratheodory distance and by $V(r)$ the Haar measure of any ball. There exists $d \in \mathbb{N}^*$ (called the local dimension of $(G, \mathbb{X})$) and $0 < c < C$ such that, for all $r \in (0, 1)$,

$$cr^d \le V(r) \le Cr^d,$$

see [14]. When $r > 1$, two situations may occur (see [11]):

- Either there exist $c, C, D > 0$ such that, for all $r > 1$,

$$cr^D \le V(r) \le Cr^D$$

  where D is called the dimension at infinity of the group (note that, contrary to d, D does not depend on $\mathbb{X}$). The group is said to have polynomial volume growth.
- Or there exist $c_1, c_2, C_1, C_2 > 0$ such that, for all $r > 1$,

$$c_1 e^{c_2 r} \le V(r) \le C_1 e^{C_2 r}$$

  and the group is said to have exponential volume growth.

When G has polynomial volume growth, it is plain to see that there exists $C > 0$ such that, for all $r > 0$,

$$V(2r) \le CV(r), \tag{4}$$

which implies that there exist $C > 0$ and $\kappa > 0$ such that, for all $r > 0$ and all $\theta > 1$,

$$V(\theta r) \leq C\theta^{\kappa} V(r). \tag{5}$$

On a Lie group as previously described, one introduces the Kohn sub-laplacian

$$\Delta_G = \sum_{i=1}^{k} X_i^2.$$

On any Lie group G, it is natural by functional calculus to define the fractional powers $(-\Delta_G)^s$, $s \in (0,1)$ of the Kohn sub-laplacian $-\Delta_G$. It has been proved in [13, 15] (see also [16]) that for Lie groups with polynomial volume

$$\|(-\Delta_G)^{s/2}u\|^2_{L^2(G)} \leq C \int_{G\times G} \frac{|u(x)-u(y)|^2}{V(|y^{-1}x|)|y^{-1}x|^{2s}}\, dx\, dy.$$

It is therefore natural to consider the operator which is the Euler-Lagrange operator of the Dirichlet form in the R.H.S. of the previous equation given by

$$\mathcal{L}u(x) = \int_G \frac{u(x)-u(y)}{V(|y^{-1}x|)|y^{-1}x|^{2s}}\, dy.$$

It defines a new Gagliardo-type norm, suitably designed for Lie groups (of any volume growth). By the structure itself of this norm, one can prove as before a Córdoba-Córdoba inequality.

2.3.2 The Case of Manifolds

Let M be a complete Riemannian manifold of dimension n. Denote $d(x,y)$ the geodesic distance from x to y. Similarly to the previous case it is natural to introduce the new operators, Euler-Lagrange of suitable Gagliardo norms, given by

$$\mathcal{L}u(x) = \int_M \frac{u(x)-u(y)}{d(x,y)^{n+2s}}\, dy$$

These new operators also satisfy Córdoba-Córdoba estimates (see [15] for an account in harmonic analysis where these quantities pop up).

3 A Review of the Extension Property

3.1 The Extension Property in $\mathbb{R}^n$

We first introduce the spaces

$$H^s(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}^n) \ : \ |\xi|^s (\mathcal{F} v)(\xi) \in L^2(\mathbb{R}^n) \right\},$$

where $s \in (0,1)$ and $\mathcal{F}$ denotes Fourier transform. For $\Omega \subset \mathbb{R}^{n+1}_+$ a Lipschitz domain (bounded or unbounded) and $a \in (-1,1)$, we denote

$$H^1(\Omega, y^a) = \left\{ u \in L^2(\Omega, y^a\, dx\, dy) \ : \ |\nabla u| \in L^2(\Omega, y^a\, dx\, dy) \right\}.$$

Let $a = 1 - 2s$. It is well known that the space $H^s(\mathbb{R}^n)$ coincides with the trace on $\partial\mathbb{R}^{n+1}_+$ of $H^1(\mathbb{R}^{n+1}_+, y^a)$. In particular, every $v \in H^s(\mathbb{R}^n)$ is the trace of a function $u \in L^2_{\text{loc}}(\mathbb{R}^{n+1}_+, y^a)$ such that $\nabla u \in L^2(\mathbb{R}^{n+1}_+, y^a)$. In addition, the function u which minimizes

$$\min \left\{ \int_{\mathbb{R}^{n+1}_+} y^a \, |\nabla u|^2 \ dxdy \ : \ u|_{\partial\mathbb{R}^{n+1}_+} = v \right\} \tag{6}$$

solves the Dirichlet problem

$$\begin{cases} L_a u := \operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ u = v & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{7}$$

By standard elliptic regularity, u is smooth in $\mathbb{R}^{n+1}_+$. It turns out that $-y^a u_y(\cdot, y)$ converges in $H^{-s}(\mathbb{R}^n)$ to a distribution $h \in H^{-s}(\mathbb{R}^n)$ as $y \downarrow 0$. That is, u weakly solves

$$\begin{cases} \operatorname{div}(y^a \nabla u) = 0 & \text{in } \mathbb{R}^{n+1}_+ \\ -y^a \partial_y u = h & \text{on } \partial\mathbb{R}^{n+1}_+. \end{cases} \tag{8}$$

Consider the Dirichlet to Neumann operator

$$\begin{aligned} \Gamma_a : H^s(\mathbb{R}^n) &\to H^{-s}(\mathbb{R}^n) \\ v \mapsto \Gamma_a(v) = h &:= -\lim_{y \to 0^+} y^a \partial_y u = \frac{\partial u}{\partial \nu^a}, \end{aligned}$$

where u is the solution of (7). Then, we have:

Theorem 3.1 ([4]) *For every $v \in H^s(\mathbb{R}^n)$,*

$$(-\Delta)^s v = d_s \Gamma_a(v) = -d_s \lim_{y \to 0^+} y^a \partial_y u,$$

where $a = 1 - 2s$, d_s is a positive constant depending only on s, and the equality holds in the distributional sense.

3.2 *The Extension Property in Bounded Domains*

We consider now the case of bounded domains. In this case, two different operators can be defined.

- *The spectral Laplacian:* If one considers the classical Dirichlet Laplacian Δ_Ω on the domain Ω , then the spectral definition of the fractional power of Δ_Ω relies on the following formulas:

$$(-\Delta_\Omega)^s g(x) = \sum_{j=1}^{\infty} \lambda_j^s \, \hat{g}_j \, \phi_j(x) = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\Delta_\Omega} g(x) - g(x) \right) \frac{dt}{t^{1+s}}. \qquad (9)$$

  Here $\lambda_j > 0, j = 1, 2, \ldots$ are the eigenvalues of the Dirichlet Laplacian on Ω with zero boundary conditions, written in increasing order and repeated according to their multiplicity and ϕ_j are the corresponding normalized eigenfunctions, namely

$$\hat{g}_j = \int_\Omega g(x)\phi_j(x)\, dx, \qquad \text{with} \qquad \|\phi_j\|_{L^2(\Omega)} = 1\,.$$

  The first part of the formula is therefore an interpolation definition. The second part gives an equivalent definition in terms of the semigroup associated to the Laplacian. We will denote the operator defined in such a way as $\mathcal{A}_{1,s} = (-\Delta_\Omega)^s$, and call it the *spectral fractional Laplacian*.
- *The restricted fractional laplacian:* On the other hand, one can define a fractional Laplacian operator by using the integral representation in terms of hypersingular kernels already mentioned

$$(-\Delta_{\mathbb{R}^d})^s g(x) = C_{d,s} \text{ P.V.} \int_{\mathbb{R}^n} \frac{g(x) - g(z)}{|x - z|^{n+2s}}\, dz, \qquad (10)$$

  In this case we materialize the zero Dirichlet condition by restricting the operator to act only on functions that are zero outside Ω. We will call the operator defined in such a way the *restricted fractional Laplacian* and use the specific notation

$\mathcal{A}_{2,s} = (-\Delta_{|\Omega})^s$ when needed. As defined, $\mathcal{A}_{2,s}$ is a self-adjoint operator on $L^2(\Omega)$, with a discrete spectrum: we will denote by $\lambda_{s,j} > 0$, $j = 1, 2, \ldots$ its eigenvalues written in increasing order and repeated according to their multiplicity and we will denote by $\{\phi_{s,j}\}_j$ the corresponding set of eigenfunctions, normalized in $L^2(\Omega)$.

- *Common notation.* In the sequel we use $\mathcal{A}$ to refer to any of the two types of operators $\mathcal{A}_{1,s}$ or $\mathcal{A}_{2,s}$, $0 < s < 1$. Each one is defined on a Hilbert space

$$H(\Omega) = \{u = \sum_{k=1}^{\infty} u_k \phi_{s,k} \in L^2(\Omega) \ : \ \|u\|_H^2 = \sum_{k=1}^{\infty} \lambda_{s,k} |u_k|^2 < +\infty\} \subset L^2(\Omega) \tag{11}$$

with values in its dual H^*. The notation in the formula copies the one just used for the second operator. When applied to the first one we put here $\phi_{s,k} = \phi_k$, and $\lambda_{s,k} = \lambda_k^s$. Note that $H(\Omega)$ depends in principle on the type of operator and on the exponent s. Moreover, the operator $\mathcal{A}$ is an isomorphism between H and H^*, given by its action on the eigen-functions. It has been proved in [1] (see also [5]) that

$$H(\Omega) = \begin{cases} H^s(\Omega) & \text{if } s \in (0, 1/2), \\ H_{00}^{1/2}(\Omega) & \text{if } s = 1/2, \\ H_0^s(\Omega) & \text{if } s \in (1/2, 1), \end{cases}$$

We now introduce the Caffarelli-Silvestre extension for these operators. In the case of the restricted fractional laplacian, the extension is precisely the one described in Sect. 3.1. We now concentrate on the case of the spectral fractional laplacian. Let us define

$$\mathcal{C} = \Omega \times (0, +\infty),$$
$$\partial_L \mathcal{C} = \partial\Omega \times [0, +\infty).$$

We write points in the cylinder using the notation $(x, y) \in \mathcal{C} = \Omega \times (0, +\infty)$. Given $s \in (0, 1)$, it has been proved in [5] (see also [3]) that the following holds.

Lemma 3.1 *Consider a weak solution of*

$$\begin{cases} \operatorname{div}(y^{1-2s}\nabla w) = 0 & in \ \ \mathcal{C} = \Omega \times (0, +\infty), \\ w = 0, & on \ \ \partial\Omega \times (0, +\infty) \end{cases} \tag{12}$$

Then $-\lim_{y\to 0} y^{1-2s}\partial_y w = \mathcal{A}w(\cdot, 0)$. *where $\mathcal{A}$ is the spectral fractional laplacian.*

3.3 The Extension Property in General Frameworks

To generalize the inequalities under consideration, one has to invoke a rather general version of the Caffarelli-Silvestre extension proved by Stinga and Torrea [17]. Their approach, based on semi-group theory, allows to prove the previous results in quite general ambient spaces, like Riemannian manifolds or Lie groups.

In the following theorem, we will consider three cases later for the object $\mathcal{M}$:

(1) The case of complete Riemannian manifolds and the Laplace-Beltrami operator
(2) The case of Lie groups and the Kohn laplacian
(3) The case of the Wiener space and the Ornstein-Uhlenbeck operator

Let $\mathcal{L}$ be a positive and self-adjoint operator in $L^2(\mathcal{M})$. One can define its fractional powers by means of the standard formula in spectral theory

$$\mathcal{L}^s = \frac{1}{\Gamma(-s)} \int_0^\infty \left(e^{t\mathcal{L}} - \mathrm{Id}\right) \frac{dt}{t^{1+s}},$$

where $s \in (0, 1)$ and $e^{t\mathcal{L}}$ denotes the heat semi-group on $\mathcal{M}$. Then one has

Theorem 3.2 *Let $u \in dom(\mathcal{L}^s)$. A solution of the extension problem*

$$\begin{cases} \mathcal{L}v + \dfrac{1-2s}{y}\partial_y v + \partial_y^2 v = 0 & \text{on } \mathcal{M} \times \mathbb{R}^+ \\ \\ v(x,0) = u & \text{on } \mathcal{M}, \end{cases}$$

is given by

$$v(x,y) = \frac{1}{\Gamma(s)} \int_0^\infty e^{t\mathcal{L}}(\mathcal{L}^s u)(x) e^{-y^2/4t} \frac{dt}{t^{1-s}}$$

and furthermore, one has at least in the distributional sense

$$-\lim_{y \to 0+} y^{1-2s}\partial_y v(x,y) = \frac{2s\Gamma(-s)}{4^s\Gamma(s)} \mathcal{L}^s u(x). \tag{13}$$

4 Proofs of Theorems 1.1 and 1.2

4.1 Proof of Theorem 1.1

We now come to the proof of Theorem 1.1. We introduce the function

$$\tilde{w} = \varphi(w) - v$$

where w is the Caffarelli-Silvestre extension of u and v the Caffarelli-Silvestre extension of $\varphi(u)$. Then $\tilde{w}$ satisfies

$$\begin{cases} L_a\tilde{w} = y^a\varphi''(w)|\nabla w|^2 \geq 0, & \text{in } \mathbb{R}^{n+1}_+ \\ \tilde{w} = 0 & \text{on } \partial\mathbb{R}^{n+1}_+ \end{cases}$$

since φ is convex. Hence by the Hopf lemma in [2] (see also the Appendix) (notice $\tilde{w} \geq 0$ by the weak maximum principle) , one has $\frac{\partial \tilde{w}}{\partial \nu^a} > 0$, hence the result.

4.2 Proof of Theorem 1.2

We now turn to the proof of the Kato inequality in Theorem 1.2. This is a consequence of the Cordoba-Cordoba inequality. Indeed consider the convex function

$$\varphi_\epsilon(x) = \sqrt{x^2 + \epsilon^2}.$$

Then the result follows by Theorem 1.1 and a standard approximation argument.

4.3 The Results in Bounded Domains

In the case of the spectral laplacian, the Córdoba-Córdoba estimate has been proved by Constantin and Ignatova [7] by a rather involved use of semi-group theory. Our proof has the same flavour as the one of Theorem 1.1. Furthermore, in our framework, one can also prove the Córdoba-Córdoba estimate in the case of the restricted laplacian, which is not covered by [7].

Theorem 4.1 *Let φ be a $C^2(\mathbb{R}^n)$ convex function. Assume that u and $\varphi(u)$ are such that $\mathcal{A}u$ and $\mathcal{A}\varphi(u)$ exist where $\mathcal{A}$ is either the restricted or spectral fractional laplacian. Then the following holds*

$$\mathcal{A}\varphi(u) \leq \varphi'(u)\mathcal{A}\, u \tag{14}$$

Proof The case of the restricted laplacian is fully covered by the proof of Theorem 1.1 verbatim. In the case of the spectral fractional laplacian, one considers as before

$$\tilde{w} = \varphi(w) - v$$

where w is the Caffarelli-Silvestre extension of u and v the Caffarelli-Silvestre extension of $\varphi(u)$ where the Caffarelli-Silvestre extension is the one described in

Sect. 3.1. Then $\tilde{w}$ satisfies

$$\begin{cases} L_a \tilde{w} = y^a \varphi''(w)|\nabla w|^2 \geq 0, & \text{in } \mathcal{C} \\ \tilde{w} = 0 \quad \text{on } \partial_L \mathcal{C} \\ \tilde{w} = 0 \quad \text{on } \{y = 0\} \end{cases}$$

By the weak maximum principle, one has $\tilde{w} \geq 0$ in $\mathcal{C}$ and one concludes with the Hopf lemma in the appendix. □

Remark 4.2 Our proof of the estimate is the same as the one in Córdoba and Martínez in [10] for the Dirichlet-to-Neumann operator. However, their proof covers only the case $1/2$ and for power-like convex functions. The argument can be actually generalized as we mentioned. Furthermore, it unifies all the possible proofs of the Córdoba-Córdoba estimates.

5 Geometric Ambient Spaces

5.1 The Case of Manifolds

The case of compact manifolds, through a parabolic argument, has been proved by Cordoba and Martínez [10]. Our proof once again completely unifies the several approaches. Consider a complete Riemannian manifold $\mathcal{M}$ and its Laplace-Beltrami operator

$$\mathcal{L} = -\Delta_g$$

Invoking now the extension of Stinga and Torrea described in Sect. 3.3, one proves

Theorem 5.1 *Let φ be a $C^2(\mathbb{R}^n)$ convex function. Assume that u and $\varphi(u)$ are such that $\mathcal{L}u$ and $\mathcal{L}\varphi(u)$ exist. Then the following holds*

$$\mathcal{L}\varphi(u) \leq \varphi'(u)\mathcal{L}\, u \tag{15}$$

We then recover the case of compact manifolds in [10] and even generalize it to complete non-compact manifolds. The proof of the previous theorem is identical, once the extension is well defined as described above (see [17]), to the proof of Theorem 1.1.

5.2 The Case of Lie Groups

Consider a Lie group G with its Kohn Laplacian

$$\mathcal{L} = -\Delta_G$$

Invoking now the extension of Stinga and Torrea described in Sect. 3.3, one proves

Theorem 5.2 *Let φ be a $C^2(\mathbb{R}^n)$ convex function. Assume that u and $\varphi(u)$ are such that $\mathcal{L}u$ and $\mathcal{L}\varphi(u)$ exist. Then the following holds*

$$\mathcal{L}\varphi(u) \leq \varphi'(u)\mathcal{L}\, u \tag{16}$$

5.3 The Case of the Wiener Space

We start by recalling the basic notions about the Wiener space and its associated operators. An abstract Wiener space is defined as a triple (X, γ, H) where X is a separable Banach space, endowed with the norm $\|\cdot\|_X$, γ is a nondegenerate centred Gaussian measure, and H is the Cameron–Martin space associated with the measure γ, that is, H is a separable Hilbert space densely embedded in X, endowed with the inner product $[\cdot,\cdot]_H$ and with the norm $|\cdot|_H$. The requirement that γ is a centred Gaussian measure means that for any $x^* \in X^*$, the measure $x^*_{\#}\gamma$ is a centred Gaussian measure on the real line $\mathbb{R}$, that is, the Fourier transform of γ is given by

$$\hat{\gamma}(x^*) = \int_X e^{-i\langle x,x^*\rangle}\, d\gamma(x) = \exp\left(-\frac{\langle Qx^*, x^*\rangle}{2}\right), \qquad \forall x^* \in X^*;$$

here the operator $Q \in \mathcal{L}(X^*, X)$ is the covariance operator and it is uniquely determined by the formula

$$\langle Qx^*, y^*\rangle = \int_X \langle x, x^*\rangle\langle x, y^*\rangle d\gamma(x), \qquad \forall x^*, y^* \in X^*.$$

The nondegeneracy of γ implies that Q is positive definite: the boundedness of Q follows by Fernique's Theorem, asserting that there exists a positive number $\beta > 0$ such that

$$\int_X e^{\beta\|x\|^2} d\gamma(x) < +\infty.$$

This implies also that the maps $x \mapsto \langle x, x^*\rangle$ belong to $L^p_\gamma(X)$ for any $x^* \in X^*$ and $p \in [1, +\infty)$, where $L^p_\gamma(X)$ denotes the space of all γ-measurable functions $f : X \to \mathbb{R}$ such that

$$\int_X |f(x)|^p d\gamma(x) < +\infty.$$

In particular, any element $x^* \in X^*$ can be seen as a map $x^* \in L^2_\gamma(X)$, and we denote by $R^* : X^* \to \mathcal{H}$ the identification map $R^*x^*(x) := \langle x, x^*\rangle$. The space $\mathcal{H}$ given by the closure of R^*X^* in $L^2_\gamma(X)$ is usually called reproducing kernel. By considering the map $R : \mathcal{H} \to X$ defined as

$$R\hat{h} := \int_X \hat{h}(x)x\,d\gamma(x),$$

we obtain that R is an injective γ–Radonifying operator, which is Hilbert–Schmidt when X is Hilbert. We also have $Q = RR^* : X^* \to X$. The space $H := R\mathcal{H}$, equipped with the inner product $[\cdot,\cdot]_H$ and norm $|\cdot|_H$ induced by $\mathcal{H}$ via R, is the Cameron-Martin space and is a dense subspace of X. The continuity of R implies that the embedding of H in X is continuous, that is, there exists $c > 0$ such that

$$\|h\|_X \le c|h|_H, \qquad \forall h \in H.$$

We have also that the measure γ is absolutely continuous with respect to translation along Cameron–Martin directions; in fact, for $h \in H$, $h = Qx^*$, the measure $\gamma_h(B) = \gamma(B - h)$ is absolutely continuous with respect to γ with density given by

$$d\gamma_h(x) = \exp\left(\langle x, x^*\rangle - \frac{1}{2}|h|_H^2\right) d\gamma(x).$$

For $j \in \mathbb{N}$ we choose $x_j^* \in X^*$ in such a way that $\hat{h}_j := R^*x_j^*$, or equivalently $h_j := R\hat{h}_j = Qx_j^*$, form an orthonormal basis of H. We order the vectors x_j^* in such a way that the numbers $\lambda_j := \|x_j^*\|_{X^*}^{-2}$ form a non-increasing sequence. Given $m \in \mathbb{N}$, we also let $H_m := \langle h_1, \dots, h_m\rangle \subseteq H$, and $\Pi_m : X \to H_m$ be the closure of the orthogonal projection from H to H_m

$$\Pi_m(x) := \sum_{j=1}^m \left\langle x, x_j^*\right\rangle h_j \qquad x \in X.$$

The map Π_m induces the decomposition $X \simeq H_m \oplus X_m^\perp$, with $X_m^\perp := \ker(\Pi_m)$, and $\gamma = \gamma_m \otimes \gamma_m^\perp$, with γ_m and $\gamma_m^\perp$ Gaussian measures on H_m and $X_m^\perp$ respectively, having H_m and $H_m^\perp$ as Cameron–Martin spaces. When no confusion is possible we identify H_m with $\mathbb{R}^m$; with this identification the measure $\gamma_m = \Pi_{m\#}\gamma$ is the standard Gaussian measure on $\mathbb{R}^m$. Given $x \in X$, we denote by $\underline{x}_m \in H_m$ the projection $\Pi_m(x)$, and by $\overline{x}_m \in X_m^\perp$ the infinite dimensional component of x, so that $x = \underline{x}_m + \overline{x}_m$. When we identify H_m with $\mathbb{R}^m$ we rather write $x = (\underline{x}_m, \overline{x}_m) \in \mathbb{R}^m \times X_m^\perp$.

We say that $u : X \to \mathbb{R}$ is a *cylindrical function* if $u(x) = v(\Pi_m(x))$ for some $m \in \mathbb{N}$ and $v : \mathbb{R}^m \to \mathbb{R}$. We denote by $\mathcal{F}C_b^k(X)$, $k \in \mathbb{N}$, the space of all C_b^k cylindrical functions, that is, functions of the form $v(\Pi_m(x))$ with $v \in C^k(\mathbb{R}^n)$, with continuous and bounded derivatives up to the order k. We denote by $\mathcal{F}C_b^k(X, H)$ the space generated by all functions of the form uh, with $u \in \mathcal{F}C_b^k(X)$ and $h \in H$.

Given $u \in L^2_\gamma(X)$, we consider the canonical cylindrical approximation $\mathbb{E}_m$ given by

$$\mathbb{E}_m u(x) = \int_{X_m^\perp} u(\Pi_m(x), y)\, d\gamma_m^\perp(y). \tag{17}$$

Notice that $\mathbb{E}_m u$ depends only on the first m variables and $\mathbb{E}_m u$ converges to u in $L^p_\gamma(X)$ for all $1 \le p < \infty$.

We let

$$\nabla_\gamma u := \sum_{j\in\mathbb{N}} \partial_j u\, h_j \qquad \text{for } u \in \mathcal{F}C^1_b(X)$$

$$\mathrm{div}_\gamma \varphi := \sum_{j\ge 1} \partial_j^* [\varphi, h_j]_H \text{ for } \varphi \in \mathcal{F}C^1_b(X, H)$$

$$\Delta_\gamma u := \mathrm{div}_\gamma \nabla_\gamma u \qquad \text{for } u \in \mathcal{F}C^2_b(X)$$

where $\partial_j := \partial_{h_j}$ and $\partial_j^* := \partial_j - \hat{h}_j$ is the adjoint operator of ∂_j. With this notation, the following integration by parts formula holds:

$$\int_X u\, \mathrm{div}_\gamma \varphi\, d\gamma = -\int_X [\nabla_\gamma u, \varphi]_H\, d\gamma \qquad \forall \varphi \in \mathcal{F}C^1_b(X, H). \tag{18}$$

In particular, thanks to (18), the operator ∇_γ is closable in $L^p_\gamma(X)$, and we denote by $W^{1,p}_\gamma(X)$ the domain of its closure. The Sobolev spaces $W^{k,p}_\gamma(X)$, with $k \in \mathbb{N}$ and $p \in [1, +\infty]$, can be defined analogously, and $\mathcal{F}C^k_b(X)$ is dense in $W^{j,p}_\gamma(X)$, for all $p < +\infty$ and $k, j \in \mathbb{N}$ with $k \ge j$.

Given a vector field $\varphi \in L^p_\gamma(X; H)$, $p \in (1, \infty]$, using (18) we can define $\mathrm{div}_\gamma \varphi$ in the distributional sense, taking test functions u in $W^{1,q}_\gamma(X)$ with $\frac{1}{p} + \frac{1}{q} = 1$. We say that $\mathrm{div}_\gamma \varphi \in L^p_\gamma(X)$ if this linear functional can be extended to all test functions $u \in L^q_\gamma(X)$. This is true in particular if $\varphi \in W^{1,p}_\gamma(X; H)$.

Let $u \in W^{2,2}_\gamma(X)$, $\psi \in \mathcal{F}C^1_b(X)$ and $i, j \in \mathbb{N}$. From (18), with $u = \partial_j u$ and $\varphi = \psi h_i$, we get

$$\int_X \partial_j u\, \partial_i \psi\, d\gamma = \int_X -\partial_i(\partial_j u)\, \psi + \partial_j u\, \psi \langle x, x_i^* \rangle d\gamma \tag{19}$$

Let now $\varphi \in \mathcal{F}C^1_b(X, H)$. If we apply (19) with $\psi = [\varphi, h_j]_H =: \varphi^j$, we obtain

$$\int_X \partial_j u\, \partial_i \varphi^j\, d\gamma = \int_X -\partial_j(\partial_i u)\, \varphi^j + \partial_j u\, \varphi^j \langle x, x_i^* \rangle d\gamma$$

which, summing up in j, gives

$$\int_X [\nabla_\gamma u, \partial_i \varphi]_H \, d\gamma = \int_X -[\nabla_\gamma(\partial_i u), \varphi]_H + [\nabla_\gamma u, \varphi]_H \langle x, x_i^* \rangle d\gamma$$

for all $\varphi \in \mathcal{F}C_b^1(X, H)$.

The operator $\Delta_\gamma : W_\gamma^{2,p}(X) \to L_\gamma^p(X)$ is usually called the Ornstein-Uhlenbeck operator on X. Notice that, if u is a cylindrical function, that is $u(x) = v(y)$ with $y = \Pi_m(x) \in \mathbb{R}^m$ and $m \in \mathbb{N}$, then

$$\Delta_\gamma u = \sum_{j=1}^m \partial_{jj} u - \langle x, x_j^* \rangle \partial_j u = \Delta v - \langle \nabla v, y \rangle_{\mathbb{R}^m} .$$

We write $u \in C(X)$ if $u : X \to \mathbb{R}$ is continuous and $u \in C^1(X)$ if both $u : X \to \mathbb{R}$ and $\nabla_\gamma u : X \to H$ are continuous.

For simplicity of notation, from now on we omit the explicit dependence on γ of operators and spaces. We also indicate by $[\cdot, \cdot]$ and $|\cdot|$ respectively the inner product and the norm in H.

By means of Sect. 3.3, one can prove an extension property for the operator $(-\Delta_\gamma)^s$ and one proves in this case also a Córdoba-Córdoba estimate.

Appendix

In this appendix, we provide the Hopf lemma, which is crucial in the proof of the estimates. We state the theorem in the case of $\mathbb{R}^n$ as stated in [4]. However, an inspection of the proof shows that it is extendable to cylinders $\mathcal{M} \times (0, +\infty)$ where $\mathcal{M}$ is one of the cases covered in the present note and the associated operators. Indeed, the geometry is always the same and the Hopf lemma just depends on the structure of the equation.

We start with some notations. We introduce

$$B_R^+ = \{(x, y) \in \mathbb{R}^{n+1} : y > 0, |(x, y)| < R\},$$
$$\Gamma_R^0 = \{(x, 0) \in \partial\mathbb{R}_+^{n+1} : |x| < R\},$$
$$\Gamma_R^+ = \{(x, y) \in \mathbb{R}^{n+1} : y \geq 0, |(x, y)| = R\}.$$

Lemma 1 *Consider the cylinder $\mathcal{C}_{R,1} = \Gamma_R^0 \times (0, 1) \subset \mathbb{R}_+^{n+1}$ where Γ_R^0 is the ball of center 0 and radius R in $\mathbb{R}^n$. Let $u \in C(\overline{\mathcal{C}_{R,1}}) \cap H^1(\mathcal{C}_{R,1}, y^a)$ satisfy*

$$\begin{cases} L_a u \leq 0 & \text{in } \mathcal{C}_{R,1} \\ u > 0 & \text{in } \mathcal{C}_{R,1} \\ u(0, 0) = 0. \end{cases}$$

Then,

$$\limsup_{y\to 0+} -y^a \frac{u(0,y)}{y} < 0.$$

In addition, if $y^a u_y \in C(\overline{\mathcal{C}_{R,1}})$*, then*

$$\partial_{\nu^a} u(0,0) < 0.$$

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The Incompressible Navier Stokes Flow in Two Dimensions with Prescribed Vorticity

Sagun Chanillo, Jean Van Schaftingen, and Po-Lam Yung

To Dick Wheeden in friendship and appreciation

Abstract We study the incompressible two dimensional Navier–Stokes equation with initial vorticity in the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^2)$. This complements our earlier work for the case when the initial vorticity is in the inhomogeneous Sobolev space $W^{1,1}(\mathbb{R}^2)$.

The two-dimensional incompressible Navier–Stokes equation

$$\begin{cases} \mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} = \nu \Delta \mathbf{v} - \nabla p, \\ \qquad\qquad \nabla \cdot \mathbf{v} = 0, \end{cases} \tag{1}$$

models an incompressible flow of a fluid whose velocity and mechanical pressure at position $x \in \mathbb{R}^2$ and time $t \in \mathbb{R}$ are represented by the vector $\mathbf{v}(x,t) \in \mathbb{R}^2$ and the scalar $p(x,t) \in \mathbb{R}$; here ν is the kinematic viscosity coefficient. Note we have divided the Navier–Stokes equation by the constant density of the fluid ρ and thus ν in our equation is the dynamic viscosity coefficient divided by the density, assumed constant. Throughout this paper, ∇ will refer only to the spatial derivatives (i.e. derivative in the x variables). We also sometimes use the notation ∂_x to denote a derivative in the x variables when we have no need to be specific which space variable we are differentiating in.

S. Chanillo
Department of Mathematics, Rutgers, The State University of New Jersey, Piscataway, NJ 08854, USA
e-mail: chanillo@math.rutgers.edu

J. Van Schaftingen
Institut de Recherche en Mathématique et en Physique, Université catholique de Louvain, Chemin du Cyclotron 2 bte L7.01.01, 1348 Louvain-la-Neuve, Belgium
e-mail: Jean.VanSchaftingen@uclouvain.be

P.-L. Yung (✉)
Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong
e-mail: plyung@math.cuhk.edu.hk

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The vorticity of the Navier–Stokes flow is a scalar in the two-dimensional case, defined by

$$\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$$

where we wrote $\mathbf{v} = (v_1, v_2)$. It propagates according to the convection-diffusion equation

$$\omega_t - \nu \Delta \omega = -\nabla \cdot (\mathbf{v}\omega),$$

which one obtains from (1) by taking the curl of both sides. Formally the velocity $\mathbf{v}$ in the Navier–Stokes equation can be expressed in terms of the vorticity through the Biot–Savart relation

$$\mathbf{v} = (-\Delta)^{-1}(\partial_{x_2}\omega, -\partial_{x_1}\omega). \tag{2}$$

This follows formally by differentiating $\omega = \partial_{x_1} v_2 - \partial_{x_2} v_1$, and using that $\nabla \cdot \mathbf{v} = 0$.

Our theorems concern the solution of the vorticity equation when the initial vorticity ω_0 is in the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^2)$. Here $\dot{W}^{1,1}(\mathbb{R}^2)$ is the completion of $C_c^\infty(\mathbb{R}^2)$ under the norm $\|u\|_{\dot{W}^{1,1}(\mathbb{R}^2)} := \|\nabla u\|_{L^1(\mathbb{R}^2)}$. The theorems are as follows:

Theorem 1 *Consider the two-dimensional vorticity equation*

$$\omega_t - \nu \Delta \omega = -\nabla \cdot (\mathbf{v}\omega), \tag{3}$$

where $\mathbf{v}$ *is defined through the Biot–Savart relation (2). Suppose we are given an initial vorticity* $\omega_0 \in \dot{W}^{1,1}(\mathbb{R}^2)$ *at time* $t = 0$. *If*

$$\|\nabla \omega_0(x)\|_{L^1(\mathbb{R}^2)} \leq A_0,$$

then there exists a unique solution to the integral formulation of this vorticity equation up to time $t_0 = C\nu/A_0^2$, *such that*

$$\sup_{t \leq t_0} \|\nabla \omega(x,t)\|_{L^1(\mathbb{R}^2)} \leq 2A_0. \tag{4}$$

Moreover, the solution ω *depends continuously on the initial data* ω_0, *in the sense that if* $\omega_0^{(i)}$ *converges to* ω_0 *in* $\dot{W}^{1,1}(\mathbb{R}^2)$ *as* $i \to \infty$, *then the sequence of solutions* $\omega^{(i)}(x,t)$ *to (3) with initial data* $\omega_0^{(i)}$ *converges to* $\omega(x,t)$ *in* $L^\infty([0,t_0), \dot{W}^{1,1}(\mathbb{R}^2))$ *as* $i \to \infty$.

Theorem 2 *Let* $\omega_0 \in \dot{W}^{1,1}(\mathbb{R}^2)$, *and* ω *be the solution to the integral formulation of the vorticity equation (3) given by Theorem 1, with initial vorticity* ω_0. *Define a velocity vector* $\mathbf{v}$ *by the Biot–Savart relation (2). Then* $\mathbf{v}$ *is a distributional* *solution to the two-dimensional incompressible Navier–Stokes (1) up to time* $t_0 := C\nu\|\nabla\omega_0\|_{L^1(\mathbb{R}^2)}^{-2}$, *in the sense that*

$$\begin{cases} \int_0^{t_0}\int_{\mathbb{R}^2}[\mathbf{v}\cdot\partial_t\Phi+\nu\mathbf{v}\cdot\Delta\Phi+\mathbf{v}\cdot(\mathbf{v}\cdot\nabla)\Phi]\,dxdt=-\int_{\mathbb{R}^2}\mathbf{v}(x,0)\cdot\Phi(x,0)dx \\ \int_0^{t_0}\int_{\mathbb{R}^2}\mathbf{v}\cdot\nabla\psi\,dxdt=0 \end{cases}$$

holds for any $\psi \in C_c^\infty(\mathbb{R}^2\times[0,t_0),\mathbb{R})$, *and any* $\Phi \in C_c^\infty(\mathbb{R}^2\times[0,t_0),\mathbb{R}^2)$ *that satisfies* $\nabla\cdot\Phi = 0$ *for all* $t\in[0,t_0)$. *We also have*

$$\sup_{t\le t_0}\|\mathbf{v}(x,t)\|_{L^\infty(\mathbb{R}^2)}+\sup_{t\le t_0}\|\nabla\mathbf{v}(x,t)\|_{L^2(\mathbb{R}^2)}\le c\|\nabla\omega_0\|_{L^1(\mathbb{R}^2)}, \tag{5}$$

and the pressure $p(x,t) := (-\Delta)^{-1}\nabla\cdot((\mathbf{v}\cdot\nabla)\mathbf{v})$ *satisfies*

$$\sup_{t\le t_0}\|\nabla p(x,t)\|_{L^2(\mathbb{R}^2)}\le c\|\nabla\omega_0\|_{L^1(\mathbb{R}^2)}^2. \tag{6}$$

Note that in these theorems, we are only assuming that the initial vorticity ω_0 is in the homogeneous Sobolev space $\dot{W}^{1,1}(\mathbb{R}^2)$, contrary to [6] where we assumed the stronger assumption that the initial vorticity is in the inhomogeneous Sobolev space $W^{1,1}(\mathbb{R}^2)$. Giga et al. [8] and Kato [9] showed that the vorticity equation is globally well-posed under the hypothesis that the initial vorticity is a measure; see also an alternative approach in Ben-Artzi [1], and a stronger uniqueness result in Brezis [4]. We point out though that the scaling of their results is different from ours: we are assuming that the *gradient* of the initial vorticity is in L^1. This explains why the solution constructed by Kato satisfies the estimate

$$\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}+\|\nabla\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)}\le Ct^{-\frac{1}{2}},\ t\to 0$$

(see Eq. (0.5) of [9]), whereas we can obtain bounds on $\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}$ and $\|\nabla\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)}$ that are uniform in t. Indeed it is known that they could not have done better, without further assumptions on the vorticity: the famous example of the *Lamb–Oseen vortex* for $\nu = 1$ [10] consists of an initial vorticity $\omega_0 = \alpha_0\delta_0$, a Dirac mass at the origin of $\mathbb{R}^2$ where α_0 is a constant (called the total circulation of the vortex). The corresponding solution ω to the vorticity equation (3) with this initial vorticity, and its corresponding velocity $\mathbf{v}$, are given by

$$\omega(x,t)=\frac{\alpha_0}{4\pi t}e^{-\frac{|x|^2}{4t}},\qquad \mathbf{v}(x,t)=\frac{\alpha_0}{2\pi}\frac{(-x_2,x_1)}{|x|^2}\Big(1-e^{-\frac{|x|^2}{4t}}\Big).$$

We then have

$$\|\omega(\cdot,t)\|_{W^{1,1}(\mathbb{R}^2)}\sim\|\mathbf{v}(\cdot,t)\|_{L^\infty(\mathbb{R}^2)}\sim\|\nabla\mathbf{v}(\cdot,t)\|_{L^2(\mathbb{R}^2)}\sim ct^{-\frac{1}{2}},\ t\to 0.$$

Hence the assumption that the initial vorticity is a measure cannot yield an estimate like in Theorems 1 or 2.

We mention in passing a result in [7] where an estimate was established for systems of wave equations with divergence-free inhomogeneties.

In order to prove Theorem 1, we rely on a basic proposition that follows from the work of Bourgain and Brezis [2, 3].

Proposition 3 *If* $\omega(\cdot, t) \in \dot{W}^{1,1}(\mathbb{R}^2)$ *at a time t, then one can define a vector-valued function* $\mathbf{v}(\cdot, t)$ *via the Biot–Savart relation (2) at this time t, in which case we will have*

$$\|\mathbf{v}(\cdot, t)\|_{L^\infty(\mathbb{R}^2)} + \|\nabla \mathbf{v}(\cdot, t)\|_{L^2(\mathbb{R}^2)} \leq C\|\nabla\omega(\cdot, t)\|_{L^1(\mathbb{R}^2)}$$

at this time t. Here C is a constant independent of t and ω*.*

Proof of Proposition 3 For simplicity, let's fix the time t, and drop the dependence of ω and $\mathbf{v}$ on t in the notation. Note that $(-\partial_{x_2}\omega, \partial_{x_1}\omega)$ is a vector field in $\mathbb{R}^2$ with vanishing divergence. The desired conclusion then follows from (2) and the two-dimensional result of Bourgain and Brezis [3] (see also [11], [5] and [7]).

Since the proof of the two-dimensional result of Bourgain and Brezis [3] is actually quite simple, we adapt it here in our particular setting, for the convenience of the reader.

The main point here is that if $\omega \in C_c^\infty(\mathbb{R}^2)$, then $\mathbf{v} = (v_1, v_2) := (-\Delta)^{-1}(\partial_{x_2}\omega, -\partial_{x_1}\omega)$ satisfies

$$v_1(x) := \frac{1}{2\pi}\int_{\mathbb{R}^2} \partial_2\omega(x-y)\log\frac{1}{|y|}dy = \frac{1}{2\pi}\int_{\mathbb{R}^2}\omega(x-y)\frac{-y_2}{|y|^2}dy$$

so

$$|v_1(x)| \leq \frac{1}{2\pi}\int_{\mathbb{R}^2}|\omega(x-y)|\frac{1}{|y|}dy \leq c\|\nabla\omega(x-\cdot)\|_{L^1(\mathbb{R}^2)}$$

the last inequality following from an application of Hardy's inequality (alternatively, one can see that the last inequality holds, by writing $\frac{1}{|y|}$ as $\nabla \cdot \frac{y}{|y|}$, and by integrating by parts). This shows

$$\|v_1\|_{L^\infty(\mathbb{R}^2)} \leq c\|\nabla\omega\|_{L^1(\mathbb{R}^2)}.$$

Similarly one shows $\|v_2\|_{L^\infty(\mathbb{R}^2)} \leq c\|\nabla\omega\|_{L^1(\mathbb{R}^2)}$, so

$$\|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)} \leq c\|\nabla\omega\|_{L^1(\mathbb{R}^2)}.$$

Finally,

$$\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)} \leq \|\nabla^2(-\Delta)^{-1}\omega\|_{L^2(\mathbb{R}^2)} \leq \|\omega\|_{L^2(\mathbb{R}^2)} \leq c\|\nabla\omega\|_{L^1(\mathbb{R}^2)},$$

the last inequality following from the Gagliardo–Nirenberg inequality. The above proves the desired conclusion of the proposition under the extra assumption that $\omega \in C_c^\infty(\mathbb{R}^2)$. Since such functions are dense in $\dot{W}^{1,1}(\mathbb{R}^2)$, a standard approximation argument shows that these estimates extend to the general case when $\omega \in \dot{W}^{1,1}(\mathbb{R}^2)$. Hence the full proposition follows. □

Proof of Theorem 1 In the sequel by a scaling we may assume without loss of generality that the viscosity coefficient $\nu = 1$.

Let K_t be the heat kernel on $\mathbb{R}^2$, i.e.

$$K_t(x) = \frac{1}{4\pi t} e^{-\frac{|x|^2}{4t}}.$$

Rewriting (3) as an integral equation for ω using Duhamel's theorem, where ω_0 is the initial vorticity, we have,

$$\omega(x,t) = K_t \star \omega_0(x) + \int_0^t \partial_x K_{t-s} \star [\mathbf{v}\omega(x,s)] ds \tag{7}$$

where $\mathbf{v}$ is given by (2).

We apply a Banach fixed point argument to the operator T given by

$$T\omega(x,t) = K_t \star \omega_0(x) + \int_0^t \partial_x K_{t-s} \star [\mathbf{v}\omega(x,s)] ds, \tag{8}$$

where again $\mathbf{v}$ is given by (2). Let us set

$$E = \Big\{ g : \mathbb{R}^2 \times [0,t_0] \to \mathbb{R} \mid \sup_{0<t<t_0} \|\nabla g(x,t)\|_{L^1(\mathbb{R}^2)} \le A \Big\}.$$

We will first show that T maps E into itself, for t_0 chosen as in the theorem.

Differentiating (8) in the space variable once, we get

$$(T\omega(x,t))_x = K_t \star (\omega_0)_x + \int_0^t \partial_x K_{t-s} \star \big(\mathbf{v}_x\omega\big) ds + \int_0^t \partial_x K_{t-s} \star \big(\mathbf{v}\omega_x\big) ds.$$

By Young's convolution inequality, we have

$$\|(T\omega(\cdot,t))_x\|_{L^1(\mathbb{R}^2)} \le \|(\omega_0)_x\|_{L^1(\mathbb{R}^2)} + C \int_0^t (t-s)^{-1/2} (\|\mathbf{v}_x\omega\|_{L^1(\mathbb{R}^2)} + \|\mathbf{v}\omega_x\|_{L^1(\mathbb{R}^2)}) ds.$$

Now we apply Proposition 3 to each of the terms in the integral on the right hand side. For the first term we have, by Cauchy-Schwarz followed by Gagliardo–Nirenberg, that

$$\|\mathbf{v}_x\omega\|_{L^1(\mathbb{R}^2)} \le C\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)}\|\omega\|_{L^2(\mathbb{R}^2)} \le C\|\nabla\mathbf{v}\|_{L^2(\mathbb{R}^2)}\|\omega_x\|_{L^1(\mathbb{R}^2)}.$$

We control $\|\nabla \mathbf{v}\|_{L^2(\mathbb{R}^2)}$ with Proposition 3: this gives

$$\|\mathbf{v}_x \omega\|_{L^1(\mathbb{R}^2)} \leq C\|\omega_x\|^2_{L^1(\mathbb{R}^2)}.$$

Similarly, by Proposition 3, for the second term, we have

$$\|\mathbf{v}\omega_x\|_{L^1(\mathbb{R}^2)} \leq \|\mathbf{v}\|_{L^\infty(\mathbb{R}^2)}\|\omega_x\|_{L^1(\mathbb{R}^2)} \leq C\|\omega_x\|^2_{L^1(\mathbb{R}^2)}.$$

Altogether, we have,

$$\|(T\omega)_x\|_{L^1(\mathbb{R}^2)} \leq \|\nabla \omega_0\|_{L^1(\mathbb{R}^2)} + C\int_0^t (t-s)^{-1/2}\|\nabla \omega\|^2_{L^1(\mathbb{R}^2)} ds.$$

Thus if $\|\nabla \omega_0\|_{L^1(\mathbb{R}^2)} \leq A_0$, then since $\omega \in E$, we have

$$\sup_{0\leq t\leq t_0} \|\nabla(T\omega)(x,t)\|_{L^1(\mathbb{R}^2)} \leq A_0 + Ct_0^{1/2}A^2.$$

By choosing A so that $A_0 = A/2$ and $t_0 = 1/(2CA)^2$, we see that if $\omega \in E$, then

$$\sup_{0\leq t\leq t_0} \|\nabla_x(T\omega)(x,t)\|_{L^1(\mathbb{R}^2)} \leq A,$$

i.e. $T\omega \in E$. It remains to show that T is a contraction on E.

For this let $\omega_1(x,t), \omega_2(x,t) \in E$. We just need to observe that from Proposition 3, we get

$$\|\mathbf{v}_1 - \mathbf{v}_2\|_\infty + \|\nabla \mathbf{v}_1 - \nabla \mathbf{v}_2\|_2 \leq C\|\nabla(\omega_1 - \omega_2)\|_{L^1(\mathbb{R}^2)}.$$

Thus repeating our earlier computations, we see that

$$\sup_{0\leq t\leq t_0} \|\nabla(T\omega_1 - T\omega_2)\|_{L^1(\mathbb{R}^2)} \leq Ct_0^{1/2}A \sup_{0\leq t\leq t_0} \|\nabla(\omega_1 - \omega_2)\|_{L^1(\mathbb{R}^2)}.$$

By the choice of t_0, it is seen that T is a contraction. Thus using the Banach fixed point theorem on E, we obtain our operator T has a fixed point and so the integral equation (7) has a unique solution in E. The continuous dependence on initial data can be proved in an identical way, and we will not repeat the details here. □

Proof of Theorem 2 Let $\omega_0 \in \dot{W}^{1,1}(\mathbb{R}^2)$, and $\omega(x,t)$ be the unique solution to (7) given above. Let $\mathbf{v}(x,t)$ be defined by the Biot–Savart relation (2) as in Proposition 3. If $\omega_0^{(i)}$ is a sequence of functions in $C_c^\infty(\mathbb{R}^2)$ converging to ω_0 in $\dot{W}^{1,1}(\mathbb{R}^2)$, then the corresponding solution $\omega^{(i)}(x,t)$ to the vorticity equation (3) converges to $\omega(x,t)$ in $L^\infty([0,t_0), \dot{W}^{1,1}(\mathbb{R}^2))$. Thus the velocities $\mathbf{v}^{(i)} := (-\Delta)^{-1}(-\partial_{x_2}\omega^{(i)}, \partial_{x_1}\omega^{(i)})$ converges in $L^\infty([0,t_0); L^\infty(\mathbb{R}^2))$ to $\mathbf{v}$. But since $\omega_0^{(i)} \in C_c^\infty(\mathbb{R}^2)$, which are in particular in the inhomogeneous Sobolev space $W^{1,1}(\mathbb{R}^2)$, so we may apply

Theorem II of Kato [9] as in [6], and conclude that the $\mathbf{v}^{(i)}$ defined above solves the Navier–Stokes equation (1), at least in the distributional sense. We can now pass to limit as $i \to \infty$, using the convergence of $\mathbf{v}^{(i)}$ to $\mathbf{v}$ in $L^\infty([0,t_0), L^\infty(\mathbb{R}^2))$ we obtained above, and appealing to the dominated convergence theorem: this shows that $\mathbf{v}(x,t)$ is also a distributional solution to (1) up to time t_0, in the sense that

$$\begin{cases} \int_0^{t_0} \int_{\mathbb{R}^2} [\mathbf{v} \cdot \partial_t \Phi + \mathbf{v} \cdot \Delta \Phi + \mathbf{v} \cdot (\mathbf{v} \cdot \nabla)\Phi] \, dxdt = -\int_{\mathbb{R}^2} \mathbf{v}(x,0) \cdot \Phi(x,0) dx \\ \int_0^{t_0} \int_{\mathbb{R}^2} \mathbf{v} \cdot \nabla \psi \, dxdt = 0 \end{cases}$$

holds for any $\psi \in C_c^\infty(\mathbb{R}^2 \times [0,t_0), \mathbb{R})$, and any $\Phi \in C_c^\infty(\mathbb{R}^2 \times [0,t_0), \mathbb{R}^2)$ that satisfies $\nabla \cdot \Phi = 0$ for all $t \in [0,t_0)$. The estimate (5) then follows from Propositions 3 and (4). Lastly we observe that the estimate (6) follows, from the fact that the pressure $p(x,t)$ satisfies the equation

$$-\Delta p = \nabla \cdot ((\mathbf{v} \cdot \nabla)\mathbf{v}),$$

which is a consequence of taking the divergence of the Navier–Stokes equation. □

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Weighted Inequalities of Poincaré Type on Chain Domains

Seng-Kee Chua

Abstract We provide an abstract version of a chain argument used in deriving Poincaré type inequalities on Boman domains. No doubling conditions need to be assumed for this abstract version. It unifies various results on chain domains that include ϕ-John domains. Besides Poincaré type inequalities, it works also for fractional Poincaré inequalities on quasimetric spaces.

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1 Introduction

Poincaré inequalities are essential tools in many applications. For examples, they imply various Sobolev embedding, compact embedding theorems. Of course, they are also basic tools in the studies of Elliptic and parabolic differential equations. In this paper, we will establish Poincaré type inequalities on bounded irregular domains via an abstract argument arisen from a chain argument used more than 30 years ago in the study of Poincaré inequalities on Boman domains.

Let Ω be a bounded domain in $\mathbb{R}^n$, $f \in \mathrm{Lip}(\Omega)$, μ and w be measures/weights. The following is usually known as weighted Poincaré inequalities:

$$\|f - C(f,\Omega)\|_{L^q_\mu(\Omega)} \leq C\|\nabla f\|_{L^p_w(\Omega)} \tag{*}$$

where $C(f,\Omega)$ is a constant depending of f and Ω, For example, sharp conditions have been obtained for convex domains in [10, 11, 13] when $\mu = \mathrm{dist}(x,\Omega)^a$, $w = \mathrm{dist}(x,\Omega)^b$, $a \geq 0, b \in \mathbb{R}$ or $\mu = w$ is a power of a concave weight. Moreover, it has also been discussed in [6, 12, 14, 16, 20, 23, 24, 26] for less

S.-K. Chua (✉)
Department of Mathematics, National University of Singapore, 10, Lower Kent Ridge Road, Singapore 119076, Singapore
e-mail: matcsk@nus.edu.sg

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regular domains. While [20, 26] consider the case where both μ and w are power type weights as above and Ω an s-John domain, [16] considers the case of reverse doubling weights on 1-John domains (which are just Boman domains). Most of the earlier studies assume $1 \le p \le q$ until more recently where some studies involving $q < p$ for Poincaré and fractional Poincaré inequalities were done [12, 14, 22, 23]. Chua and Wheeden [12, 14] considered general measures on ϕ-John domains (see Definition 1.11) on homogeneous spaces and more general power type weights and also the case $1 \le q < p$. The studies in [12, 14] rely mostly on an abstract version of self improving properties of Poincaré type inequality [12, Theorems 1.2 and 1.10] that originated from [18] and [19].

The Poincaré inequality (*) is usually deduced via the assumption that it holds on balls in Ω,

$$\|f - C(f, B)\|_{L^q_\mu(B)} \le C\|\nabla f\|_{L^p_w(B)}, \tag{1}$$

and a doubling condition for μ. It was first appeared in [25] for Boman domains Ω in $\mathbb{R}^n$ with $q = p$ and $\mu = w$ being an Muckenhoupt weight with a general function g instead of $|\nabla f|$, that is,

$$\|f - C(f, \Omega)\|_{L^p_\mu(\Omega)} \le C\|g\|_{L^p_\mu(\Omega)}; \tag{2}$$

see also [1]. It was then extended in [5] for $q \ge p$, with μ being a doubling weight instead of a Muckenhoupt weight (note that Muckenhoupt weights are doubling) and (1) holds for all δ-balls (balls that are 'deep' inside Ω, see Definition 1.2) with B being replaced by a fixed enlargement of B on the right (i.e., $\|\nabla f\|_{L^p_w(B)}$ being replaced by $\|\nabla f\|_{L^p_w(\tau B)}$, $\tau > 1$).

Moreover, it is observed that the above assumption could be relaxed to just all balls in a Boman cover of Ω instead of all δ-balls; see [7, Theorem 2.11]. It is also noted in [12, Theorem 2.9] that (2) holds with $g = |\nabla f|$ for doubling measure (see Definition 1.3) μ if (1) holds for all balls in Boman covering for John domains in quasimetric spaces.

Indeed, [14] considered more than just 1-John domains, it considered s-John domains ($s \ge 1$) (see Definition 2.1) in quasimetric spaces. Moreover, the assumption on μ was further relaxed to just reverse doubling on the domain (see also [16]). However, it required a more complicated assumption besides (1) [14, Eqs. (1–13)]. Note that in $\mathbb{R}^n$ or metric spaces with "geodesic path" property, Boman domains are 1-John domains [3].

On the other hand, Poincaré type inequalities on Boman domains in non-Euclidean metric spaces probably first appeared in [27], where Poincaré inequalities were established on Boman domains defined by the metric associated with the Hörmander vector fields. Indeed, weighted Poincaré inequalities on Boman domains were further proved for vector fields by Franchi, Lu and Wheeden [17]. Furthermore, tt has been known that a fractional type representation formula holds on any Boman domains as shown by Lu and Wheeden [28]. Essentially any weighted Poincaré type inequality holds on Boman domains as long as the fractional integral operators are bounded with respect to the weight such that the balance condition (introduced by Chanillo and Wheeden [4]) holds. In fact, even high order representation formulas are established on Boman domains by Lu and Wheeden [29, 30]. These representation formulas immediately lead to the Poincaré inequality. However, their methods are quite different from our approach.

In this note, we will instead establish an abstract version of the above. While assumption is simple, our abstract version is surprisingly powerful. It unifies consideration of various Poincaré type inequalities (including fractional Poincaré inequalities) on various chain domains. Using our abstract version, we are able to obtain results obtained in various papers (for example, [14, 22, 23, 26]) that used a number of different techniques. We do not even need to assume the measures involved are doubling, reverse doubling or majorized by a ball set function that satisfies "ratio condition (R)" [14, (1–5)] (but of course, they are still needed in certain specific results). Furthermore, our theorem has simpler assumptions compared to that of [12, 14].

Our abstract version has setting on an abstract measure space $\langle X, \Sigma, \sigma\rangle$. Let $\mathcal{W} \subset \Sigma$ and

(C) $\mathcal{W}$ has a "central set" Q' such that for all $Q \in \mathcal{W}$, there exists a chain of sets $\{Q_0 = Q, Q_1, \cdots, Q_N = Q'\} \subset \mathcal{W}$ connecting Q to Q' (where N may depend on Q) such that

$$0 < \sigma(Q_i \cup Q_{i+1}) \le c_\sigma \sigma(Q_i \cap Q_{i+1}) \;\text{ for all } i = 0, \cdots, N-1. \tag{3}$$

We will say $\mathcal{W}$ satisfies (C) w.r.t. σ and center Q'. In metric spaces, $\mathcal{W}$ usually consists of metric balls. Such condition has appeared (in quasimetric spaces) in [12, 15]. An obvious example of $\mathcal{W}$ is the collection of a (fixed) slight enlargement of dyadic cubes of the Whitney decomposition of any domain Ω in $\mathbb{R}^n$ and σ is doubling on Ω (with c_σ depending on the doubling constant of σ), then $\mathcal{W}$ satisfies (C) w.r.t. σ and any fixed Q' in $\mathcal{W}$. For more examples of $\mathcal{W}$ that satisfies condition (C), see Definition 1.4.

Remark 1.1

(1) In general, $\mathcal{W}$ could consist of Boman domains in a metric space X. For example, in $\mathbb{R}^n$, such idea (sequence of Boman domains) has been employed in [23]. Note that in general, metric balls need not be Boman domains.
(2) Suppose Q is a closed unit cube in $\mathbb{R}^n$, one can find a finite cover W consists of closed cubes (inside Q) of the same edge length that is less than $\delta/\sqrt{n}$ and such that $|Q_i \cap Q_j| \ge c|Q_i \cup Q_j|$ $(c > 0)$ whenever $Q_i, Q_j \in W$ have nonempty intersection. Then it is easy to see that W satisfies condition (C) w.r.t. the Lebesgue measure with any choice of $Q' \in W$ as "center". Note that $Q' \subset B(x, \delta)$ for all $x \in Q'$.

We now define quasimetric spaces.

Definition 1.2 A pair $\langle H, d\rangle$ is a quasimetric space with quasimetric constant κ if for all $x, y, z \in H$,

(1) $d(x,y) > 0$ if $x \neq y$, $d(x,x) = 0$ and
(2) $d(x,y) \leq \kappa[d(x,z) + d(y,z)]$.

Moreover, if $d(x,y) = d(y,x)$ for all $x, y \in H$, then we say d is symmetric. Quasimetric spaces with symmetric quasimetric are quite well studied; see [12, 33]. Note that for any quasimetric space, one can find an equivalent symmetric quasimetric on the same space [33, p. 34]. However, sometimes we may not like the extra constant that arises in this process especially when doubling is not assumed; for example, calculations are done using a nonsymmetric metric in [13].

For a quasimetric space $\langle H, d\rangle$ (may not be symmetric), any $x \in H$ and $r > 0$, we write

$$B_r(x) = B(x,r) = \{y \in H : d(y,x) < r\}$$

and call $B(x,r)$ the ball with center x and radius r. If $B = B(x,r)$ is a ball and c is a positive constant, we use cB to denote $B(x, cr)$. If B is a ball, we use $r(B)$ and x_B to denote the radius and center of B.

For simplicity, all quasimetric balls in this note will be called balls. If $0 < \delta' < \delta \leq 1/(2\kappa^3)$, a ball $B_r(x)$ is called a δ-ball of a given set Ω if $0 < r/\delta \leq d(\Omega, x) = \inf\{d(y,x) : y \notin \Omega\}$. It is called a (δ', δ)-Whitney ball (of Ω) if

$$\delta' d(\Omega, x) \leq r \leq \delta d(\Omega, x).$$

Alternatively, one could work with "closed" balls $\overline{B}_r(x) = \{y : d(y,x) \leq r\}$. However, for simplicity, we will only work with balls (and not "closed" balls) defined earlier.

Some useful properties of δ-balls can be found in Proposition 2.2.

We next define various notions of doubling and reverse doubling.

Definition 1.3 Let $\langle H, d\rangle$ be a quasimetric space. A nonnegative finite functional σ defined on balls in H, i.e., $\sigma : \{B : B \text{ is a ball in } H\} \to [0, \infty)$, will be called a ball set function (a measure will always be a ball set function in this paper). For any given family $\mathcal{F}$ of balls, we say σ is doubling on $\mathcal{F}$ if $\sigma(5\kappa^2 B) \leq D_\sigma \sigma(B)$ with $D_\sigma > 1$ for all $B \in \mathcal{F}$. If $\mathcal{F}$ is the collection of all δ-balls in $\Omega \subset H$, we say σ is weak δ-doubling on Ω (cf. [15, (1.5)]). The notion of weak δ-doubling seems to be weaker than the usual notion of δ-doubling used in [12, 14]. If $\mathcal{F}$ is the collection of all balls with center in Ω, we say σ is doubling on Ω. Finally, if $\mathcal{F}$ consists of all balls in H, we say simply σ is doubling.

We say that a collection of balls has *bounded intercepts* if there exists a constant N such that each ball in the collection intersects at most N balls in the collection. Such a collection also has bounded overlaps in the pointwise sense since no point belongs to more than N balls in the collection.

We now give examples of domains that have a cover satisfying chain condition (C).

Definition 1.4 We say a bounded set Ω is a chain domain in a quasimetric space $\langle H, d\rangle$ if it has a bounded overlapping cover $\mathcal{W}$ of (δ', δ)-Whitney balls ($0 < \delta' < \delta \leq 1/(2\kappa^3)$, δ is usually a fixed multiple of δ' depending on κ); with a fixed $B' \in \mathcal{W}$ (usually called central ball) such that for any other $Q \in \mathcal{W}$, there exists a chain of balls, $\{Q_0 = Q, Q_1, \cdots, Q_L = B'\} \subset \mathcal{W}$ (where L may depend on Q) such that $Q_i \cap Q_{i+1}$ contains a (δ', δ)-Whitney ball R_i with $Q_i \cup Q_{i+1} \subset M_1 R_i$ for each i ($M_1 > 1$). We will write $\Omega \in \mathcal{F}_d(\delta', \delta, M_1)$ and $\mathcal{W}$ will be called a chain cover of Ω. In particular, $\mathcal{W}$ will satisfy condition (C) w.r.t. σ and center B' if σ is doubling on (δ', δ)-Whitney balls.

Next, let $\tilde{\mathcal{W}} = \{\tilde{Q}_\alpha : Q_\alpha \in \mathcal{W}\}$ be such that for all α, $\tilde{Q}_\alpha$ is a ball with the same center as Q_α and the following property holds:

for any $Q \in \mathcal{W}$, if $\{Q = Q_0, Q_1, \cdots, Q_L = Q'\}$ is a chain connecting Q to Q' in condition (C), then $Q = Q_0 \subset \tilde{Q}_i \ \forall i$.

If there exists $M_2 > 1$ such that $\tilde{Q}_\alpha \subset M_2 Q_\alpha$ for all $Q_\alpha \in \mathcal{W}$, then Ω is known as Boman domain with Boman cover $\mathcal{W}$ and we will write $\Omega \in \mathcal{F}_d(\delta', \delta, M_1, M_2)$.

Throughout the paper, positive constants will be denoted by C or c and their dependence on important parameters will be indicated. Note that all measures are defined on a fixed σ-algebra that contains all balls.

First, for easy comparison, let us recall one of the more recent extension of [25, Theorem 3] mentioned above.

Theorem 1.5 ([12, Theorem 2.9] cf. [14, Remark 2.5]) *Let Ω be a Boman domain in a symmetric quasimetric space $\langle H, d\rangle$ with quasimetric constant κ. Let $0 < \delta < 1/(2\kappa^2)$. Suppose $\mathcal{W}$ is a Boman cover of Ω. Let f be a (measurable) function on Ω and $C(f, B)$ be an associated constant for every $B \in W$. If μ is a doubling measure on Ω and $1 \leq q < \infty$, then*

$$\|f - C(f, B')\|^q_{L^q_\mu(\Omega)} \leq C \sum_{B \in W} \|f - C(f, B)\|^q_{L^q_\mu(B)} \tag{4}$$

where B' is the "central ball" in Ω.

However, there are gaps in the proof that has been pointed out in [14]. Unfortunately, there is still one problem in [14] that it failed to fix, namely, it did not justify why maximal functions used there are measurable. Nevertheless, the above theorem will be a consequence of our result here. Indeed, we will extend the theorem here so that it does not require any doubling condition.

Let us now state our main theorem.

Theorem 1.6 *Let σ and μ be measures on a σ-algebra Σ of subsets of X. Let $M \geq 1$ and $\mathcal{W} = \{Q_\alpha \in \Sigma : \alpha \in I\}$ with $\sum_{\alpha \in I} \chi_{Q_\alpha} \leq M$. Let $1 \leq q < \infty$ and assume $\mathcal{W}$ satisfies condition (C) w.r.t σ and center Q'. Moreover, suppose*

$\tilde{\mathcal{W}} = \{\tilde{Q}_\alpha \in \Sigma : \alpha \in I\}$ is such that $Q = Q_0 \subset \tilde{Q}_i$ for any Q_i in the chain (C) of sets (that satisfies (3)). Assume further that

(C2) *There exist $c_\mu, c_\alpha \geq 1$ ($\alpha \in I$) such that for all $\{a_\alpha\}_{\alpha\in I} \subset \mathbb{R}_+$,*

$$\Big\| \sum_{\alpha\in I} a_\alpha \chi_{\tilde{Q}_\alpha} \Big\|_{L^q_\mu} \leq c_\mu \Big\| \sum_{\alpha\in I} c_\alpha a_\alpha \chi_{Q_\alpha} \Big\|_{L^q_\mu}. \tag{5}$$

If f is a measurable function and $C(f, Q_\alpha)$ is a constant corresponds to Q_α for each $\alpha \in I$, then

$$\|f - C(f, Q')\|^q_{L^q_\mu(\bigcup_{\alpha\in I} Q_\alpha)} \leq 2^{q-1} \Bigg(\sum_{\alpha\in I} \|f - C(f, Q_\alpha)\|^q_{L^q_\mu(Q_\alpha)} + (c_\mu c_\sigma^{1/p_0} M \max\{2^{1/p_0}, 2\})^q \sum_{\alpha\in I} c_\alpha^q \frac{\mu(Q_\alpha)}{\sigma(Q_\alpha)^{q/p_0}} \|f - C(f, Q_\alpha)\|^q_{L^{p_0}_\sigma(Q_\alpha)} \Bigg) \tag{6}$$

for any $0 < p_0 < \infty$.

Remark 1.7

(1) When $X = H$ is a quasimetric space and $\mathcal{W} = \{Q_\alpha\}_{\alpha\in I}, \tilde{\mathcal{W}} = \{\tilde{Q}_\alpha\}_{\alpha\in I}$ are collections of balls, c_α in (C2) can be chosen as $\mu(\tilde{Q}_\alpha)/\mu(Q_\alpha)$ in many cases. Indeed, it always hold when $q = 1$ with $c_\mu = 1$. For $q > 1$, by results from [9] (see Theorems 2.3 and 2.4), it holds when both Q_α and $\tilde{Q}_\alpha$ are concentric balls ($\forall\alpha \in I$) in Euclidean space such that either $r(\tilde{Q}_\alpha) > mr(Q_\alpha)$, $m > 1$ ($\forall\alpha$) or μ is known to be doubling on $\tilde{\mathcal{W}}$.
(2) To obtain Theorem 1.5, we take $q = p_0$, $\sigma = \mu$, then $\mu(\tilde{Q}_\alpha) = \mu(M_2 Q_\alpha) \leq C\mu(Q_\alpha)$ for all $\alpha \in I$. (4) then follows from (6).

In particular, when $X = \mathbb{R}^n$, we have an easy consequence on domain Ω with a chain cover.

Theorem 1.8 *Let $0 < p_0 < \infty, 1 \leq q < \infty$ and let $\mathcal{W} = \{Q_\alpha : \alpha \in I\}$ be a family of bounded overlapping balls in $\mathbb{R}^n$. Let μ, σ be measures on Ω and let $\mathcal{W}$ satisfy condition (C) w.r.t. σ and center Q'. Suppose $\tilde{\mathcal{W}} = \{\tilde{Q}_\alpha : \alpha \in I\}$ such that $\tilde{Q}_i \supset Q_0$ for all Q_i in the chain (C). Suppose either (i) $\tilde{Q}_\alpha$ and Q_α are concentric balls with $r(\tilde{Q}_\alpha) > mr(Q_\alpha)$ for all $\alpha \in I$ or (ii) μ is doubling on $\tilde{\mathcal{W}}$, i.e., $\mu(5\tilde{Q}_\alpha) \leq c_1\mu(\tilde{Q}_\alpha)$ for all $\alpha \in I$. Then there exists a constant C such that*

$$\|f - C(f, Q')\|^q_{L^q_\mu(\cup_{\alpha\in I} Q_\alpha)} \leq C \sum_{\alpha\in I} \|f - C(f, Q_\alpha)\|^q_{L^q_\mu(Q_\alpha)} + C \sum_{\alpha\in I} \left(\frac{\mu(\tilde{Q}_\alpha)}{\mu(Q_\alpha)}\right)^q \frac{\mu(Q_\alpha)}{\sigma(Q_\alpha)^{q/p_0}} \|f - C(f, Q_\alpha)\|^q_{L^{p_0}_\sigma(Q_\alpha)} \tag{7}$$

Remark 1.9

(1) When Ω is a ϕ-John domain (Definition 2.1) in a quasimetric space with symmetric quasimetric and there exists a measure that is doubling on all δ-balls in Ω, then [12, Proposition 2.6] assures us that $\Omega \in \mathcal{F}_d(\delta', \delta, M_1)$ $(0 < \delta' < \delta < 1/(2\kappa^2)$, with $\delta' = C(\kappa)\delta$, $M > 1)$ with a chain cover $\mathcal{W}$ consisting of (δ', δ)-Whitney balls. Moreover, the family of balls $\{\tau B : B \in \mathcal{W}\}$ has bounded overlaps when $\tau < 1/(2\delta\kappa^2)$. Furthermore, if Ω is an 1-John domain, it also assures us that $\Omega \in \mathcal{F}_d(\delta', \delta, M_1, M_2)$ $(M_2 > 1)$ and hence Ω is a Boman domain. Indeed, the proof there can be modified for nonsymmetric quasimetric when $\delta \le 1/(2\tau\kappa^3)$, $\tau \ge 1$. In particular, the proof in [12] works if $\delta\tau \le 1/(2\kappa^2)$ for symmetric quasimetric spaces.

(2) Balls in the above could be of course either open or closed and they can be quasimetric balls defined by a (nondegenerate) convex Minkowski functional; see [13] or [9] for its definition.

(3) In case $\mu = \sigma$ is doubling on a Boman domain Ω (with a Boman cover $\mathcal{W}$), since one could take $\tilde{Q}_\alpha = M_2 Q_\alpha \cap \Omega$ for $Q_\alpha \in \mathcal{W}$ where $M_2 > 1$ is a fixed constant, we see that Theorem 1.5 is a consequence of Theorem 1.8 (when $\Omega \subset \mathbb{R}^n$).

(4) Let us also state an easy application of Theorem 1.8 on fractional Poincaré inequalities. The result is not new. But it is now an easy consequence of Theorem 1.8.

Corollary 1.10 *Let* $1 \le q < \infty$, $1 < p < \infty$, $0 < \tilde{\delta} \le \sqrt{n}/2$, *$Q$ be a cube in* $\mathbb{R}^n$ *and* $0 < s < 1$ *such that* $\frac{1}{q} \ge \frac{1}{p} - \frac{s}{n}$. *Then for any* $u \in L^1(Q)$, $u_Q = \int_Q u dx/|Q|$, *we have*

$$\|u - u_Q\|_{L^q(Q))} \le C(\tilde{\delta})|Q|^{\frac{1}{q}-\frac{1}{p}+\frac{s}{n}} \left(\int_Q \int_{Q \cap B(x, \tilde{\delta} l(Q))} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dy dx \right)^{1/p}. \tag{8}$$

The corollary follows from the fact that

$$\|u - u_Q\|_{L^q(Q)} \le C|Q|^{\frac{1}{q}-\frac{1}{p}+\frac{s}{n}} \|u\|_{W^{s,p}(Q)} \tag{9}$$

[2], where

$$\|u\|_{W^{s,p}(Q)} = \left(\int_Q \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dy dx \right)^{1/p};$$

see also [31, 32]. Indeed, for $\frac{1}{q} = \frac{1}{p} - \frac{s}{n}$, it has been pointed out in [2] that (9) follows from a classical observation on fractional Poincaré together with the note after the proof of [2, Remark 1]. Indeed, it is well known that if $\frac{1}{q} \geq \frac{1}{p} - \frac{s}{n}$, then

$$\|u\|_{L^q(Q))} \leq C(\|u\|_{L^p(Q)} + \|u\|_{W^{s,p}(Q)}).$$

Combining this with the fact that (9) holds with $q = p$, one could deduce (9) for $1/q \geq 1/p - s/n$.

To prove Corollary 1.10, we will choose $\mathcal{W}$ to be the one in Remark 1.1 (2). Similar argument can be found in [23] using a partition instead of our bounded overlapping cover. However, we obtain it by direct application of our abstract version.

(5) We can apply the conclusion of Corollary 1.10 to see that (for $\frac{1}{q} \geq \frac{1}{p} - \frac{s}{n}$)

$$\|u - u_{Q'}\|_{L^q(\Omega)} \leq C \left(\int_\Omega \int_{B(x,\delta d(x,\Omega))} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dydx \right)^{1/p} \tag{10}$$

for any John domain $\Omega \subset \mathbb{R}^n$ (and $\delta \leq 1/2$) as any John domain has a Boman cover W (for example, a slight enlargement of the Whitney decomposition) satisfying condition (C) w.r.t. Lebesgue measure and center cube Q', and for each Q there exists $\tilde{Q} = M_2 Q$ such that $\tilde{Q}_i \supset Q_0$ for all Q_i in the chain (C). Note that we could choose $\tilde{\delta} > 0$ small enough such that $B(x, \delta d(x, \Omega)) \supset B(x, \tilde{\delta} l(Q))$ for all $x \in Q$, $Q \in W$.

Let us now state some more consequences. For simplicity, for any $1 \leq q < \infty$, let us write $\|a(Q_\alpha)\|_{l^q(\alpha \in I)}$ to denote $\left(\sum_{\alpha \in I} |a(Q_a)|^q \right)^{1/q}$.

Theorem 1.11 *Let Ω be a domain in a quasimetric space $\langle H, d\rangle$ with quasimetric constant κ. Let $0 < \delta' < \delta \leq 1/(2\kappa^3)$, $1 \leq \tau \leq 1/(2\delta\kappa^3)$, $0 < p_0 < \infty$, $1 \leq p, q, M_1, M < \infty$. Let $\Omega \in \mathcal{F}_d(\delta', \delta, M_1)$ in H with chain cover $\mathcal{W}$, central ball B' and $\sum_{B \in \mathcal{W}} \chi_{\tau B} \leq M$.*

Let σ be a measure doubling on (δ', δ)-Whitney balls with doubling constant D_σ (hence $\mathcal{W}$ satisfies assumption (C) w.r.t. σ and center B') and w, μ be measures on Ω. Suppose f and g are fixed measurable functions on Ω such that

$$\left(\frac{1}{\mu(Q_\alpha)} \int_B |f - C(f, Q_\alpha)|^q d\mu \right)^{\frac{1}{q}} \leq a_*(Q_\alpha) \left(\int_{\tau Q_\alpha} |g|^p dw \right)^{1/p} \tag{11}$$

and

$$\left(\frac{1}{\sigma(Q_\alpha)} \int_{Q_\alpha} |f - C(f, Q_\alpha)|^{p_0} d\sigma \right)^{\frac{1}{p_0}} \leq a_*(Q_\alpha) \left(\int_{\tau Q_\alpha} |g|^p dw \right)^{1/p} \tag{12}$$

for all $Q_\alpha \in \mathcal{W}$ where $C(f, Q_\alpha)$ is a constant depending on Q_α. Let $\tilde{\mathcal{W}} = \{\tilde{Q}_\alpha : Q_\alpha \in \mathcal{W}\}$ such that $Q_0 \subset \tilde{Q}_i$ for all Q_i in the chain (C) such that (C2) of Theorem 1.6 *holds. Suppose also*

$$c_\alpha \mu(Q_\alpha)^{1/q} a_*(Q_\alpha) \le c_0 \text{ for all } Q_\alpha \in \mathcal{W} \text{ when } q \ge p \text{ and}, \tag{13}$$

$$\|c_\alpha \mu(Q_\alpha)^{1/q} a_*(Q_\alpha)\|_{l^{pq/(p-q)}(Q_\alpha \in \mathcal{W})} \le c_0 \text{ when } 1 \le q < p. \tag{14}$$

Then

$$\|f - C(f, B')\|_{L^q_\mu(\Omega)} \le C(D_\sigma, M_1, q) c_0 c_\mu M^{1+1/p} \|g\|_{L^p_w(\Omega)}. \tag{15}$$

Remark 1.12

(1) Note that one usually only need to check (14) for any disjoint family of balls in $\mathcal{W}$ as balls in $\mathcal{W}$ can usually be written as bounded families of pairwise disjoint balls.
(2) It is observed in [12, Proposition 2.6] that a ϕ-John domain (that include s-John domains; see Definition 2.1) is in $\mathcal{F}_d(\delta', \delta, M_1)$ if there exists a measure that is doubling on δ-balls (d is assumed to be symmetric there, however by similar technique, one could get the same conclusion with nonsymmetric quasimetric). Hence, Theorem 1.11 is a generalization of [14, Theorem 1.8] where it is assumed $\mu = \sigma$ which is doubling on all δ-balls and $q \ge p$ there.
(3) If $q > p$, $g = |\nabla f|$ and μ is reverse doubling, (11) will follow from (12) such that $\mu(B)^{1/q} a_*(B) \le C$ for all balls (provided $a_*(B)$ has special monotonicity property [14, Remark 1.7(4)]); see also [14, Theorem 1.6 and Remark 1.11]. However, the quasimetric has been assumed to be symmetric there.
(4) In quasimetric spaces, the constant τ in (11) and (12) is only known to be > 1 when balls are not known to be Boman domains.
(5) Similar to Corollary 1.10, we could take ($d(x) = \inf\{d(y, x) : y \in \Omega^c\}$)

$$g(x) = \left(\int_{B(x,\delta d(x))} \frac{|f(x) - f(z)|^p}{\mu(B(x, \delta d(z, x))) d(z, x)^{sp}} d\mu(z) \right)^{1/p}, \tag{16}$$

we will then obtain fractional Sobolev inequalities. Indeed, one could take

$$\|f\|_{W^{s,p}_w(\Omega)} = \left(\int_\Omega \int_{B(x,\delta d(x)} \frac{|f(x) - f(z)|^p}{\mu(B(x, \delta d(z, x))) d(z, x)^{sp}} d\mu(z) dw(x) \right)^{1/p}$$

and then RHS of (15) can be replaced by $\|f\|_{W^{s,p}_w(\Omega)}$ if corresponding inequalities (11) and (12) hold with g defined in (16).

The following corollary generalizes [14, Theorem 1.8(i)] and with simpler condition.

Corollary 1.13 *Under the notation of Theorem 1.11, let $M_2 \ge 1$ and suppose $\Omega \in \mathcal{F}_d(\delta', \delta, M_1, M_2)$ is a Boman domain with a Boman cover $\mathcal{W}$. Suppose conditions (11) and (12) hold as in Theorem 1.11. Let $\rho(x) = \inf\{d(z, x) : z \in \Omega_0\}$,*

$\Omega_0 \subset \Omega^c$ *and* $w_b(E) = \int_E \rho(x)^b dw$ *for* $b \in \mathbb{R}$. *Let* $\bar{\rho}(E) = \sup\{\rho(x) : x \in E\}$ *for any* E *and*

$$(\mu(M_2 B_\alpha \cap \Omega)/\mu(B_\alpha))\mu(B_\alpha)^{1/q} a_*(B_\alpha) \le c_0 r(B_\alpha)^{\beta'} \ , \beta' \ge 0 \ \text{for all } B_\alpha \in \mathcal{W}. \tag{17}$$

Suppose condition (C2) holds with $c_\alpha = \mu(M_2 B_\alpha \cap \Omega)/\mu(B_\alpha)$.

(1) *If* $1 \le p \le q$ *and* $\beta' - b/p \ge 0$, *then*

$$\|f - C(f, B')\|_{L^q_\mu(\Omega)} \le C(D_\sigma, M_1, M_2, \beta', q, p, b) c_\mu c_0 M^{1+1/p} \bar{\rho}(\Omega)^{\beta' - b/p} \|g\|_{L^p_{w_b}(\Omega)}. \tag{18}$$

(2) *Suppose* $1 \le q < p$ *and there exist positive constants* M_3, η *such that the number of pairwise disjoint balls in* $\mathcal{W}$ *with radius more than* 2^k *is less than* $M_3 2^{-\eta k}$ *for all* $k \in \mathbb{Z}$. *If*

$$(p - q)\eta/(pq) < \min\{\beta', \beta' - b/p\}, \tag{19}$$

then

$$\|f - C(f, B')\|_{L^q_\mu(\Omega)} \le C(D_\sigma, M_1, M_2, M_3, , \beta', \eta, q, p, b) c_\mu c_0 \times$$

$$r(\Omega)^{[\min\{\beta' - b/p, \beta'\}\frac{pq}{p-q} - \eta]\frac{p-q}{pq}} \bar{\rho}(\Omega)^{\max\{-b/p, 0\}} \|g\|_{L^p_{w_b}(\Omega)} \tag{20}$$

where $r(\Omega) = \sup\{r(B) : B \in \mathcal{W}\}$. *Moreover, if* $\Omega_0 = \Omega^c$, *then* $\bar{\rho}(\Omega) \sim r(\Omega)$ *and hence the RHS of condition (19) can be replaced by just* $\beta' - b/p$.

Remark 1.14

(a) The condition in (2) holds with any η that is less than the upper Minkowski content of $\partial\Omega$ w.r.t. $\Omega \subset \mathbb{R}^n$, that is,

$$\limsup_{r \to 0} | \cup_{x \in \partial\Omega} B_r(x) \cap \Omega | / r^{n-\eta} < \infty.$$

Indeed, if Ω is a Lipschitz domain, then one can take $\eta = n - 1$. Note that in general, if $n + a - \eta > 0$ ($a < 0$), then

$$\rho^a(\Omega) \le \sum_{k=-\infty}^{L} \sum_{B \in \mathcal{W}: 2^k \le r(B) < 2^{k+1}} \rho^a(B) \ (\text{with } 2^L \sim r(\Omega))$$

$$\le C \sum_{k=-\infty}^{L} M_3 2^{k(n+a)} 2^{-k\eta} \le C M_3 r(\Omega)^{n+a-\eta} < \infty.$$

(b) The above can easily be applied to the measure μ_a (instead of μ) when $a > 0$ since $\mu_a(M_2B \cap \Omega) \le M_2^a \bar{\rho}(B)^a \mu(M_2B \cap \Omega)$ for δ-balls B. Hence if $\mu|_\Omega$ is doubling on the family $\{M_2B : B \in \mathcal{W}\}$, then $\mu_a|_\Omega$ is also doubling on the same family for any $a > 0$.

(c) The following observation is often useful in changing the constant $C(f, B')$ in the Poincaré type inequality (18) and (20). Suppose $\mathcal{D}$ is any measurable set such that $\sigma(B' \cap \mathcal{D}) > 0$ and $0 < p_0 < \infty$. Then

$$\|f - C(f,\mathcal{D})\|_{L^q_\mu(\Omega)} \le \|f - C(f,B')\|_{L^q_\mu(\Omega)} + \frac{\max\{2^{1/p-1},1\}\mu(\Omega)^{1/q}}{\sigma(B'\cap\mathcal{D})^{1/p}}\left[\|f - C(f,\mathcal{D})\|_{L^p_\sigma(\mathcal{D})} + \|f - C(f,B')\|_{L^p_\sigma(B')}\right].$$

The above follows immediately from the triangle inequality and the following computation:

$$|C(f,B') - C(f,\mathcal{D})| = \frac{1}{\sigma(B'\cap\mathcal{D})^{1/p}}\|C(f,B') - C(f,\mathcal{D})\|_{L^p_\sigma(B'\cap\mathcal{D})}$$
$$\le \frac{\max\{2^{1/p-1},1\}}{\sigma(B'\cap\mathcal{D})^{1/p}}\left[\|f - C(f,\mathcal{D})\|_{L^p_\sigma(B'\cap\mathcal{D})} + \|C(f,B') - f\|_{L^p_\sigma(B'\cap\mathcal{D})}\right]$$
$$\le \frac{\max\{2^{1/p-1},1\}}{\sigma(B'\cap\mathcal{D})^{1/p}}\left[\|f - C(f,\mathcal{D})\|_{L^p_\sigma(\mathcal{D})} + \|f - C(f,B')\|_{L^p_\sigma(B')}\right].$$

As a consequence, we provide a simple balance condition for Poincaré inequality to hold on Boman domains without assuming that the measure satisfies any doubling or reverse doubling condition.

Theorem 1.15 *Let $\Omega \subset \mathbb{R}^n$, $\Omega \in \mathcal{F}_d(\delta', \delta, M_1, M_2)$, $0 < \delta' < \delta \le 1/2$, $M_1, M_2 \ge 1$, (hence a Boman domain). Let $f \in \mathrm{Lip}_{loc}(\mathbb{R}^n)$ and μ be a Borel measure on $\mathbb{R}^n$. Suppose w is a weight such that (recall $r(\Omega) = \sup\{r(B) : B \in W\}$)*

$$\mu(B\cap\Omega)r(B)^{1-n}\|w^{-1}\|_{L^\infty(B\cap\Omega)} \le c_1\mu(\Omega)r(\Omega)^{1-n}\|w^{-1}\|_{L^\infty(\Omega)} \tag{21}$$

for all balls B in $\mathbb{R}^n$ with center in Ω. Suppose further that

$$\|f - f_B\|_{L^1_\mu(B)} \le C\mu(B)r(B)^{1-n}\|w^{-1}\|_{L^\infty(B)}\|\nabla f\|_{L^1_w(B)} \tag{22}$$

where $f_B = \int_B f dx/|B|$ for all balls B in a Boman cover of Ω. Then

$$\|f - f_\Omega\|_{L^1_\mu(\Omega)} \le C\mu(\Omega)r(\Omega)^{1-n}\|w^{-1}\|_{L^\infty(\Omega)}\|\nabla f\|_{L^1_w(\Omega)} \tag{23}$$

where $f_\Omega = \int_\Omega f dx/|\Omega|$.

Remark 1.16

(1) Let $f \in \mathrm{Lip}_{loc}(\mathbb{R}^n)$. Then for all balls B in $\mathbb{R}^n$,

$$\frac{1}{|B|}\|f - f_B\|_{L^1(B)} \leq Cr(B)^{1-n}\|\nabla f\|_{L^1(B)},\ f_B = \int_B f dx/|B|, \tag{24}$$

by the unweighted Poincaré inequality. It then follows from Hölder's inequality that (24) will also hold with RHS being replaced by

$$Cr(B)^{1-n}\|w^{-1}\|_{L^\infty(B)}\|\nabla f\|_{L^1_w(B)}.$$

Thus (12) holds with $\sigma = 1$, $p = p_0 = 1$, $g = |\nabla f|$ and

$$a_*(B) = Cr(B)^{1-n}\|w^{-1}\|_{L^\infty(B)}.$$

(2) When μ and w are both reverse doubling weights, it has been established in [16] that (23) holds for any 1-John domains Ω under assumption (21).

(3) Let Φ be a fixed (nondegenerate) Minkowski functional (a positive homogeneous convex function which is zero only at the origin, see [13, Sect. 2]). Then the collection of all Φ-balls is just the collection of all translations of some dilations of a fixed open convex set in $\mathbb{R}^n$. By similar argument, if (21) holds for all Φ-balls, then (22) holds for all Φ-balls if and only if (23) holds for all John domains.

Our result can of course be applied to s-John domains (see Definition 2.1) for $s = \mathfrak{s} > 1$. However, for a simple illustration, we will just consider a special case. We will consider the following typical s-cusp domain, which is an s-John domain:

$$\mathcal{D} = \{(z, z') \in \mathbb{R} \times \mathbb{R}^{n-1} : 0 < z < 4, |z'| < z^{\mathfrak{s}}\}.$$

Instead of considering positive power distant weights as in [26, Example 2.4] and [14, Theorem 1.14], we will only consider a case (negative weight) that has not been studied.

Theorem 1.17 *Let $\mathcal{D}$ be the s-cusp domain above and $\rho(x) = d(x, \partial\mathcal{D})$. Suppose $a > -1$, $b \in \mathbb{R}$, $1 < p < \infty$, and*

$$\lambda = a + \frac{1}{\mathfrak{s}} + s - \frac{b+1}{p} > 0. \tag{25}$$

Then for all $f \in L^1_{loc}(\mathcal{D})$,

$$\|f - C(f, \mathcal{D})\|_{L^1_{\rho^a dx}(\mathcal{D})} \leq C\|f\|_{\hat{W}^{s,p}_{\rho^b dx}(\mathcal{D})}, \tag{26}$$

where

$$\|f\|_{\hat{W}^{s,p}_{\rho^b dx}(\mathcal{D})} = \left(\int_\Omega \int_{B(x,\rho(x)/2)} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}} dy \rho(x)^b dx\right)^{1/p}.$$

Remark 1.18 (1) Of course, our technique will also work for the usual weighted Poincaré inequality (on s-cusp) of the following form:

$$\|f - C(f,\mathcal{D})\|_{L^q_{\rho^a dx}(\mathcal{D})} \leq C\|\nabla f\|_{L^p_{\rho^b dx}(\mathcal{D})}. \tag{27}$$

(2) Similar to [14], we could also study the case $\rho(x) = d(\mathcal{D}_0, x) = \inf\{|y - x_0| : y \in \mathcal{D}_0\}$ with $\partial\mathcal{D} \cap B(0,\varepsilon) \subset D_0 \subset \partial\mathcal{D}$.

2 Preliminaries

In this section, we will list definitions, terminology and establish basic properties on quasimetric spaces. We will also state some results concerning condition (C2).

First, let us state a definition similar to [14, Definition 1.2] and [8, 15].

Definition 2.1 Let $\langle H, d\rangle$ be a quasimetric space. Fix $\Omega \subset H$ and $x \in H$, set

$$d(x) = \inf_{y\in\Omega^c} d(y,x).$$

Let ϕ be a strictly increasing function on $[0,\infty)$ such that $\phi(0) = 0$ and $\phi(t) < t$ for all $t > 0$. We say that Ω is a (weak) ϕ-John domain with central point (or 'center') $x' \in \Omega$ if for all $x \in \Omega$ with $x \neq x'$, there is a curve $\gamma : [0,l] \to \Omega$ such that $\gamma(0) = x$, $\gamma(l) = x'$,

$$d(\gamma(b),\gamma(a)) \leq b - a \ \text{ for all } [a,b] \subset [0,l], \text{ and} \tag{28}$$

$$d(\gamma(t)) > \phi(d(\gamma(t),x)) \ \text{ for all } t \in [0,l]. \tag{29}$$

If Ω is a ϕ-John domain for the function $\phi = \phi_s$ defined by $\phi_s(t) = c_s t^s$ for $t \leq 1$ and $\phi_s(t) = c_s t$ for $t > 1$, with $s \geq 1$, we say Ω is an (weak) s-John domain. We may assume that $0 < c_s < 1$. When $s = 1$, it is usually known as a (weak) John domain. This definition is essentially the same as those in [34] and [20] with (29) slightly different; indeed, this has been adopted in [15, Definition 1.3] (where the quasimetric is assumed to be symmetric). The corresponding definition in [12, 14] replaces (29) by $d(\gamma(t)) > \phi(t)$, which is nominally a stronger assumption since ϕ is increasing and $d(\gamma(t),x) = d(\gamma(t),\gamma(0)) \leq t$ by (28). The weak version (29) was first given by Väisälä [35] in $\mathbb{R}^n$ when $\Phi(t) = ct$ and shown to be equivalent to the strong version in $\mathbb{R}^n$. It was extended to metric spaces in [21] and [8]. We do not know an example when the weak version is true and the strong version is false. In general, this weak version is easier to apply.

Note that (29) implies that $d(x) > 0$ for all $x \in \Omega$.

Proposition 2.2 *Let $\langle H, d\rangle$ be a quasimetric space with quasimetric constant κ. Let $\Omega \subset H$ and $0 < \delta \le 1/(2\kappa^3)$. Let $\Omega_0 \subset \Omega^c$ and define $\rho(x) = \inf\{d(z,x) : z \in \Omega_0\}$.*

(1) *If $z \in B(x,r)$, then*

$$B(z,r) \subset (\kappa + \kappa^2)B(x,r).$$

(2) *Let $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$ be balls with $B_1 \cap B_2 \neq \emptyset$. Then*

(a) *$B_2 \subset B(x_1, \kappa^2(r_1 + 2r_2))$. In particular, when $r_2 \le 2r_1$, then $B_2 \subset 5\kappa^2 B_1$.*
(b) *If in addition both B_1 and B_2 are δ-balls, $\delta \le 1/2\kappa^3$, then*

$$\frac{1}{2\kappa + 1} \le \frac{d(x_2)}{d(x_1)} \le 2\kappa + 1.$$

Thus if B_1 and B_2 are intersecting δ-Whitney balls, then

$$\frac{1}{2\kappa + 1} \le \frac{r(B_2)}{r(B_1)} \le 2\kappa + 1.$$

(c) *If z is in a δ-ball $B(x_0, r)$, then*

$$\frac{1}{\kappa + \kappa\delta} \le \frac{\rho(x_0)}{\rho(z)} \le \frac{\kappa}{1 - \kappa^2\delta}.$$

Proof Let $z \in B(x,r)$. Then for any $y \in B(z,r)$,

$$d(y,x) \le \kappa[d(y,z) + d(x,z)] < (\kappa + \kappa^2)r.$$

This proves (1).

Next, for (2), let $z \in B_1 \cap B_2$. If $u \in B_2$,

$$\begin{aligned} d(u,x_1) &\le \kappa(d(u,z) + d(x_1,z)) \\ &\le \kappa[\kappa(d(u,x_2) + d(z,x_2)) + \kappa d(z,x_1)] \\ &< 2\kappa^2 r_2 + \kappa^2 r_1 \end{aligned}$$

and 2(a) is now clear. If in addition B_1 and B_2 are both δ-balls and $z \in B_1 \cap B_2$, then

$$d(x_1) \le \kappa(d(x_2)+d(x_1,x_2)) \le \kappa(d(x_2)+\kappa[d(x_1,z)+d(x_2,z)]) < \kappa(d(x_2)+\kappa^2(r_1+r_2)).$$

Since $r_1 + r_2 \le \delta(d(x_1) + d(x_2))$, a simple computation based on our assumption that $\kappa^3\delta \le 1/2$ then gives

$$d(x_1) < \frac{\kappa + \kappa^3\delta}{1 - \kappa^3\delta} d(x_2) \le (2\kappa + 1)d(x_2).$$

By interchanging the roles of B_1 and B_2, we also have $d(x_2) < (2\kappa+1)d(x_1)$, which proves the first part of 2(b). The remaining part of 2(b) follows by the fact that $r_i = \delta d(x_i)$.

Finally, part (c) can be proved by using the quasi-triangle property and the fact that $\delta \le 1/(2\kappa^3)$.

We now state two results concerning the validity of (C2) from [9].

Theorem 2.3 ([9, Theorem 1.3]) *Let $\langle H, d\rangle$ be a quasimetric space. Suppose $\mathcal{W} = \{B_\alpha\}_{\alpha\in I}$, $\tilde{\mathcal{W}} = \{\tilde{B}_\alpha\}_{\alpha\in I}$ are families of countable collection of (quasimetric) balls in H such that $B_\alpha \subset \tilde{B}_\alpha$ for all $\alpha \in I$. Let balls in $\tilde{\mathcal{W}}$ be of bounded radius. Let μ be a measure.*

Suppose there exists $c_1 > 1$ such that either

(Ai) μ is doubling on $\tilde{\mathcal{W}}$, that is, $\mu(5\kappa^2\tilde{B}) \le c_1\mu(\tilde{B})$ for all $\tilde{B} \in \tilde{\mathcal{W}}$, or
(Aii) for any family of balls $\mathcal{F} \subset \mathcal{W}$, there exists $\mathcal{F}_0 \subset \mathcal{F}$ with $\cup_{B\in\mathcal{F}}B \subset \cup_{B\in\mathcal{F}_0}\tilde{B}$ such that $\sum_{\tilde{B}_\alpha\in F_0} \chi_{\tilde{B}_\alpha} \le c_1$.

Then for any $1 < q < \infty$, there exists $c_\mu \ge 1$ such that

$$\| \sum_{\alpha\in I} a_\alpha \chi_{\tilde{B}_\alpha} \|_{L^q_\mu} \le c_\mu \| \sum_{\alpha\in I} \frac{\mu(\tilde{B}_\alpha)}{\mu(B_\alpha)} a_\alpha \chi_{\tilde{B}_\alpha} \|_{L^q_\mu}$$

for all $\{a_\alpha\}_{\alpha\in I} \subset \mathbb{R}^+ \cup \{0\}$.

It is being proved (in [9]) with the help of the following maximal function.

$$h^*(x) = \sup\{\frac{1}{\mu(\tilde{Q}_\alpha)} \int_{\tilde{Q}_\alpha} |h(x)| d\mu : x \in Q_\alpha\}.$$

Under assumption (Ai), we will just use a Vitali-type covering lemma to obtain a week type estimate (for L^1).

Assumption (Aii) is just an extension of Besicovitvh covering lemma. It is known to hold in $\mathbb{R}^n$. The following is a special case of [9, Theorem 1.4].

Theorem 2.4 *Suppose $\mathcal{W} = \{B_\alpha\}_{\alpha\in I}, \tilde{\mathcal{W}} = \{\tilde{B}_\alpha\}_{\alpha\in I}$ are balls in $\mathbb{R}^n$ such that there exists $m > 1$ such that $r(\tilde{B}_\alpha) \ge mr(B_\alpha)$ for all $\alpha \in I$. Then $\mathcal{W}, \tilde{\mathcal{W}}$ satisfy condition (Aii), that is, given any family $\mathcal{F}$ of balls in $\mathcal{W}$, there exists $\mathcal{F}_0 \subset \mathcal{F}$ such that $\bigcup_{B_\alpha\in\mathcal{F}} B_\alpha \subset \bigcup_{\tilde{B}_\alpha\in\mathcal{F}_0} \tilde{B}_\alpha$ with $\sum_{\tilde{B}_\alpha\in\mathcal{F}_0} \chi_{\tilde{B}_\alpha} \le c_1$ where c_1 depends only on m, η and n.*

3 Proof of Main Theorems

Proof of Theorem 1.6 For any fixed $Q \in \mathcal{W}$, let $\{Q_0 = Q, Q_1, \cdots, Q_N = Q'\}$ be the chain given in (C). Then by standard argument, we have (see for example the proof of [12, Lemma 3.1])

$$|C(f,Q) - C(f,Q')| \le \sum_{i=0}^{N-1} |C(f,Q_i) - C(f,Q_{i+1})|$$

$$\le \sum_{i=0}^{N} \frac{C_\sigma^{1/p_0} \max\{2^{1/p_0}, 2\}}{\sigma(Q_i)^{1/p_0}} \|f - C(f,Q_i)\|_{L_\sigma^{p_0}(Q_i)}.$$

It follows that

$$|C(f,Q) - C(f,Q')|\chi_Q(x) \le c_\sigma^{1/p_0} \max\{2^{1/p_0}, 2\} \sum_{\alpha\in I} \frac{\chi_{\tilde{Q}_\alpha}(x)}{\sigma(Q_\alpha)^{1/p_0}} \|f - C(f,Q_\alpha)\|_{L_\sigma^{p_0}(Q_\alpha)}.$$

Thus, by condition (C2) and the fact that $\sum_{\alpha\in I} \chi_{Q_\alpha} \le M$,

$$\sum_{\alpha\in I} \|C(f,Q_\alpha) - C(f,Q')\|^q_{L^q_\mu(Q_\alpha)} \le \sum_{\alpha\in I} \int |C(f,Q_\alpha) - C(f,Q')|^q \chi_{Q_\alpha} d\mu$$

$$\le M(c_\sigma^{1/p_0} \max\{2^{1/p_0}, 2\})^q \int |\sum_{\alpha\in I} \frac{\chi_{\tilde{Q}_\alpha}}{\sigma(Q_\alpha)^{1/p_0}} \|f - C(f,Q_\alpha)\|_{L_\sigma^{p_0}(Q_\alpha)}|^q d\mu$$

$$\le M(c_\mu c_\sigma^{1/p_0} \max\{2^{1/p_0}, 2\})^q \int |\sum_{\alpha\in I} \frac{c_\alpha \chi_{Q_\alpha}}{\sigma(Q_\alpha)^{1/p_0}} \|f - C(f,Q_\alpha)\|_{L_\sigma^{p_0}(Q_\alpha)}|^q d\mu$$

$$\le M(c_\mu c_\sigma^{1/p_0} \max\{2^{1/p_0}, 2\})^q \int \left(\sum_{\alpha\in I} (\frac{c_\alpha \chi_{Q_\alpha}}{\sigma(Q_\alpha)^{1/p_0}} \|f - C(f,Q_\alpha)\|_{L_\sigma^{p_0}(Q_\alpha)})^q\right)$$

$$\times \left(\sum_{\alpha\in I} (\chi_{Q_\alpha})^{q'}\right)^{q/q'} d\mu$$

by Hölder's inequality

$$\le (Mc_\mu c_\sigma^{1/p_0} \max\{2^{1/p_0}, 2\})^q \sum_{\alpha\in I} \frac{c_\alpha^q \mu(Q_\alpha)}{\sigma(Q_\alpha)^{q/p_0}} \|f - C(f,Q_\alpha)\|^q_{L_\sigma^{p_0}(Q_\alpha)}.$$

Finally, note that

$$\|f - C(f,Q')\|^q_{L^q_\mu(\bigcup Q_\alpha)} \le \sum_{\alpha\in I} \|f - C(f,Q')\|^q_{L^q_\mu(Q_\alpha)}$$
$$\le 2^{q-1}\Big(\sum_{\alpha\in I} \|f - C(f,Q_\alpha)\|^q_{L^q_\mu(Q_\alpha)} + \sum_{\alpha\in I} \|C(f,Q_\alpha) - C(f,Q')\|^q_{L^q_\mu(Q_\alpha)}\Big).$$

Theorem 1.6 is now clear.

Proof of Theorem 1.8 It is enough to observe that (C2) holds with $c_\alpha = \mu(\tilde{Q}_\alpha)/\mu(Q_\alpha)$. This is clear when $q = 1$. For $q > 1$, see Theorems 2.3 and 2.4.

Proof of Theorem 1.11 First note that (C) holds with $c_\sigma = C(D_\sigma)$ where D_σ is the doubling constant of σ.

When $q \ge p$, by (6), (11), (12), we have

$$\|f - C(f,B')\|^q_{L^q_\mu(\Omega)} \le C(M_1, q, D_\sigma, p_0) M^q c^q_\mu \sum_{B\in W} \big(c_\alpha \mu(B)^{1/q} a_*(B) \|g\|_{L^p_w(\tau B)}\big)^q.$$

We now use the fact that

$$\sum_{B\in W} \chi_{\tau B} \le M \quad \text{and} \quad c_\alpha \mu(B)^{1/q} a_*(B) \le c_0$$

to conclude (15). However, for the case $q < p$, we will also need to apply Hölder's inequality. Indeed,

$$\|f - C(f,B')\|^q_{L^q_\mu(\Omega)} \le C(M_1, q, D_\sigma, p_0) M^q c^q_\mu \sum_{B\in W} \big(c_\alpha \mu(B)^{1/q} a_*(B) \|g\|_{L^p_w(\tau B)}\big)^q$$
$$\le C M^q c^q_\mu \|c_\alpha \mu(B)^{1/q} a_*(B)\|^q_{l^{pq/(p-q)}(B\in W)} \Big(\sum_{B\in W} \|g\|^p_{L^p_w(\tau B)}\Big)^{q/p}$$

by Hölder's inequality

$$\le C(M_1, q, D_\sigma, p_0) (c_0 M^{1+1/p} c_\mu)^q \|g\|^q_{L^p_w(\Omega)}$$

by condition (2) of the theorem and this completes the proof of the theorem.

Proof of Corollary 1.13 First, recall that $\rho(x) \sim \bar{\rho}(B)$ on any δ-ball B (by Proposition 2.2) with constants depending only on δ and κ, we have by (11),

$$\frac{1}{\mu(B)^{1/q}} \|f - C(f,B)\|_{L^q_\mu(B)} \le C(\kappa, \delta, D_\mu, p, b) a_*(B) \bar{\rho}(B)^{-b/p} \|g\|_{L^p_{w_b}(\tau B)}.$$

Next, similarly by (12), we have

$$\frac{1}{\sigma(B)^{1/p_0}}\|f-C(f,B)\|_{L^{p_0}_\sigma(B)} \le Ca_*(B)\bar{\rho}(B)^{-b/p}\|g\|_{L^p_{w_b}(\tau B)}.$$

Now observe that by (17),

$$\frac{\mu(M_2B\cap\Omega)}{\mu(B)}\mu(B)^{1/q}a_*(B)\bar{\rho}(B)^{-b/p} \le C(\delta,\kappa,b,p)c_0\bar{\rho}(B)^{-b/p}r(B)^{\beta'} \text{ when } B\in W.$$

Hence,

$$\sum_{B\in\mathcal{W}}\left|\frac{\mu(M_2B\cap\Omega)}{\mu(B)}\mu(B)^{1/q}a_*(B)\bar{\rho}(B)^{-b/p}\right|^{\frac{pq}{p-q}}$$

$$\le C(\delta,\kappa,b,p)\sum_{k=-\infty}^{L}\sum_{B\in\mathcal{W}:2^k\le r(B)<2^{k+1}}|c_0\bar{\rho}(B)^{-b/p}r(B)^{\beta'}|^{\frac{pq}{p-q}} \text{ (where } 2^L\sim r(\Omega))$$

$$\le C(\delta,\kappa,b,p,M_3)c_0^{\frac{pq}{p-q}}\bar{\rho}(\Omega)^{\max\{-b/p,0\}\frac{pq}{p-q}}\sum_{k=-\infty}^{L}2^{k\min\{\beta',\beta'-b/p\}\frac{pq}{p-q}-k\eta}$$

$$\le Cc_0^{\frac{pq}{p-q}}\bar{\rho}(\Omega)^{\max\{-b/p,0\}\frac{pq}{p-q}}r(\Omega)^{\min\{\beta',\beta'-b/p\}\frac{pq}{p-q}-\eta}$$

by (19). Next, when $\Omega_0=\Omega^c$, as $\mathcal{W}$ consists of only (δ',δ)-Whitney balls, $\bar{\rho}(B)\sim r(B)$ for all $B\in W$ and hence

$$\sum_{B\in\mathcal{W}}\left|\frac{\mu(M_2B\cap\Omega)}{\mu(B)}\mu(B)^{1/q}a_*(B)\bar{\rho}(B)^{-b/p}\right|^{\frac{pq}{p-q}}$$

$$\le C(\delta,\kappa,b,p)\sum_{k=-\infty}^{L}\sum_{B\in\mathcal{W}:2^k\le r(B)<2^{k+1}}|c_0\bar{\rho}(B)^{-b/p}r(B)^{\beta'}|^{\frac{pq}{p-q}}$$

$$\le C(\delta,\kappa,b,p,M_3)c_0^{\frac{pq}{p-q}}\sum_{k=-\infty}^{L}2^{k(\beta'-b/p)\frac{pq}{p-q}-k\eta}$$

$$\le Cc_0^{\frac{pq}{p-q}}r(\Omega)^{(\beta'-b/p)\frac{pq}{p-q}-\eta}.$$

This completes the proof of the corollary by Theorem 1.11.

Proof of Theorem 1.15 First, it follows from 1–1 unweighted Poincaré inequality and Hölder's inequality that

$$\frac{1}{|B|}\|f-f_B\|_{L^1(B)} \le Cr(B)^{1-n}\|w^{-1}\|_{L^\infty(B)}\|\nabla f\|_{L^1_w(B)},\ f_B=\int_B f dx/|B| \tag{30}$$

for all balls. Hence (12) holds for all balls with $\sigma = 1, p = p_0 = \tau = 1, a_*(B) = Cr(B)^{1-n}\|w^{-1}\|_{L^\infty(B)}, g = |\nabla f|$ and $C(f, B) = f_B$.

Let Ω be any Boman domain. Note that for all $B \in \mathcal{W}$ ($\mathcal{W}$ is a Boman cover of Ω),

$$\mu(M_2 B \cap \Omega) r(B)^{1-n}\|w^{-1}\|_{L^\infty(B)} \le M_2^{n-1}\mu(M_2 B \cap \Omega) r(M_2 B)^{1-n}\|w^{-1}\|_{L^\infty(M_2 B\cap\Omega)}.$$

Hence, (17) will hold with $q = 1$, $\beta' = 0$ and

$$c_0 = c_1 C(M_2)\mu(\Omega) r(\Omega)^{1-n}\|w^{-1}\|_{L^\infty(\Omega)}$$

by (21). By Corollary 1.13, we have

$$\|f - f_{B'}\|_{L^1_\mu(\Omega)} \le C\mu(\Omega) r(\Omega)^{1-n}\|w^{-1}\|_{L^\infty(\Omega)}\|\nabla f\|_{L^1_w(\Omega)}. \tag{31}$$

Finally, observe that by the triangle inequality,

$$\begin{aligned} &\|f - f_\Omega\|_{L^1_\mu(\Omega)} \\ &\le \|f - f_{B'}\|_{L^1_\mu(\Omega)} + \frac{\mu(\Omega)}{|\Omega|}\|f - f_{B'}\|_{L^1(\Omega)} \\ &\le \|f - f_{B'}\|_{L^1_\mu(\Omega)} + C\mu(\Omega) r(\Omega)^{1-n}\|\nabla f\|_{L^1(\Omega)} \\ &\qquad \text{by applying (31) to } w = 1 \\ &\le \|f - f_{B'}\|_{L^1_\mu(\Omega)} + C\mu(\Omega) r(\Omega)^{1-n}\|w^{-1}\|_{L^\infty(\Omega)}\|\nabla f\|_{L^1_w(\Omega)}. \end{aligned}$$

This conclude the proof of the theorem.

Proof of Theorem 1.17 For convenience, let us denote

$$g(x) = \left(\int_{B(x,\rho(x)/2)} \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}} dy\right)^{1/p}.$$

Next, let W_0 be the Whitney decomposition of Ω and $W = \{\frac{9}{8}Q : Q \in W_0\}$. It is easy to see that condition (C) holds w.r.t. the Lebesgue measure and center Q' (a cube in W that contains the point $(2, 0)$). Recall that for each $Q \in W$, we have from (8) (see Remark 1.9 (4)) with $q = 1$ and $\tilde{\delta}$ sufficiently small such that $\delta = 1/2 \ge \frac{7}{9}\tilde{\delta}/\sqrt{n}$,

$$\frac{1}{|Q|}\|f - f_Q\|_{L^1(Q)} \le C|Q|^{\frac{s}{n}-\frac{1}{p}}\|g\|_{L^p(Q)}$$

since $\frac{7}{9}\sqrt{n}l(Q) \le \rho(x) \le 5\sqrt{n}l(Q)$ for $x \in Q$, $Q \in W$. It is easy to see that if $2^{k-1} < l(Q) \le 2^k$, $Q \in W$, $k \in \mathbb{Z}$, then $\rho^a(Q) \sim 2^{(n+a)k}$.

Let $d\mu = \rho^a dx$ and $dw = \rho^b dx$. Since $\rho(x) \sim \bar{\rho}(Q) = \sup\{\rho(y) : y \in Q\}$, for $x \in Q$, $Q \in W$, we have $\rho^a(Q) \sim l(Q)^{(n+a)}$ and

$$\frac{1}{\mu(Q)}\|f - f_Q\|_{L^1_\mu(Q)} \le C|Q|^{s/n}w(Q)^{-1/p}\|g\|_{L^p_w(Q)}. \tag{32}$$

Observe that if $Q \in W$ is such that its length $l(Q) = \frac{9}{8}2^k$ with $k < -2$, $k \in \mathbb{Z}$, such that with $(z, z') \in Q$, $z < 1/4$, then $\tilde{Q}$ can be assumed to be $8 \times 2^{k(\frac{1}{\mathfrak{s}}-1)}Q$, i.e., $l(\tilde{Q}) = 9 \times 2^{k/\mathfrak{s}}$. Let us consider the "tip" of the cusp $D_r = \{(z, z') \in \Omega : z < r\}$, $r < 1$. Observe that

$$\mu(D_r) = C\int_0^r \int_0^{z^{\mathfrak{s}}} \int_{\sigma\in S^{n-2}} \rho(z, y\sigma)^a y^{n-2} d\sigma dy dz.$$

However, it is clearly less than

$$C\int_0^r \int_0^{z^{\mathfrak{s}}} (z^{\mathfrak{s}} - y)^a y^{n-2} dy dz \le C\int_0^r \int_0^{z^{\mathfrak{s}}} (z^{\mathfrak{s}} - y)^a z^{\mathfrak{s}(n-2)} dy dz = Cr^{1+\mathfrak{s}(n+a-1)}.$$

Now, suppose $Q \in W$ with $Q \subset D_1$ and $l(Q) = r/8$. We see that

$$\mu(\tilde{Q} \cap \mathcal{D}) \le \mu(D_{r^{1/\mathfrak{s}}}) \le Cr^{(n+a-1)+1/\mathfrak{s}}.$$

Next, for any other cube $Q \in \mathcal{W}$ such that $l(Q) \le 2^{-4}$, it is easy to see that one can take $\tilde{Q} = 8Q$ and $\tilde{Q} \cap \mathcal{D}_{1/2} = \emptyset$. Hence,

$$\mu(\tilde{Q} \cap \mathcal{D}) \le C|\tilde{Q}|^{1+a/n} = Cr^{n+a} \text{ where } l(Q) = r.$$

Moreover, for each $k \in \mathbb{Z}$, it is easy to see that the number of cubes in W with edgelength more than 2^k is less than $C2^{(1-n)k}$. Hence

$$\begin{aligned}
&\sum_{Q\in W}[\mu(\tilde{Q} \cap \mathcal{D})|Q|^{s/n}w(Q)^{-1/p}]^{p'} \\
\le &\sum_{Q\in \mathcal{W}, l(Q)>1/8}[\mu(\tilde{Q} \cap \mathcal{D})|Q|^{s/n}w(Q)^{-1/p}]^{p'} \\
&+ \sum_{k=-\infty}^{-4}\ \sum_{Q\in W: 2^k<l(Q)\le 2^{k+1}}[\mu(\tilde{Q} \cap \mathcal{D})|Q|^{s/n}w(Q)^{-1/p}]^{p'}
\end{aligned}$$

$$\leq C + C \sum_{k=-\infty}^{-4} \sum_{Q \in W, 2^k < l(Q) \leq 2^{k+1}} [2^{k[(n+a-1)+1/s]} 2^{ks} 2^{-k(n+b)/p}]^{p'}$$

$$\leq C + C \sum_{k=-\infty}^{-4} 2^{k[(1-n)+\{n+a-1+\frac{1}{\mathfrak{s}}+s-\frac{n+b}{p}\}p']} = C + C2^{\lambda p'}$$

as $\lambda = a + \frac{1}{\mathfrak{s}} + s - \frac{b+1}{p} > 0$. Theorem 1.17 now follows from Theorem 1.11 with $c_\alpha = \mu(\tilde{Q}_\alpha \cap \mathcal{D})/\mu(Q_\alpha)$ and

$$C(f, \mathcal{D}) = f_{Q'} = \int_{Q'} f dx/|Q'|.$$

However, by standard argument (such as the one employed at the end of the proof of Theorem 1.15) we can replace $f_{Q'}$ by $\int_\Omega f d\mu/\mu(\Omega)$ in the above.

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Smoluchowski Equation with Variable Coefficients in Perforated Domains: Homogenization and Applications to Mathematical Models in Medicine

Bruno Franchi and Silvia Lorenzani

To Dick, in friendship and admiration of his mathematics

Abstract In this paper, we study the homogenization of a Smoluchowski system of periodic discrete diffusion-coagulation equations, when the diffusion coefficients depend on all variables, in particular on the microscopic variable. This system modelizes the aggregation and diffusion of the β-amyloid peptide $A\beta_{42}$ in the cerebral tissue, a process associated with the development of Alzheimer's disease. Our homogenization result, based on Allaire-Nguetseng two-scale convergence, is meant to pass from a microscopic model to a macroscopic one.

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1 Introduction

This paper is devoted to the homogenization of a set of Smoluchowski's discrete diffusion-coagulation equations [17] over periodically perforated domains. This type of equations, describing the evolving densities of diffusing particles that are prone to coagulate in pairs, models various physical phenomena: the evolution of a system of solid or liquid particles suspended in a gas, polymerization, aggregation of colloidal particles, formation of stars and planets as well as biological populations, behavior of fuel mixtures in engines, etc. (see, e.g. [8, 12]). Quite often, starting

B. Franchi (✉)
Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy
e-mail: bruno.franchi@unibo.it

S. Lorenzani
Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy
e-mail: silvia.lorenzani@polimi.it

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from a microscopic description of a problem, we seek a macroscopic, or averaged, description. As a matter of fact, while being closer to the actual physical nature, a mathematical model for a physical system that resolves smaller scales is usually more complicated and sometimes even virtually impossible to solve. Moreover, experimental data are often available for macroscale quantities only, but not for the microscale. Therefore, for quite a long time, the key issue has been how to formulate laws on a scale that is larger than the microscale and to justify these laws on the basis of a microscopic approach. To do that, in the seventies, mathematicians have developed a new method called homogenization [6]. This method allows to perform certain limits of the solutions of partial differential equations describing media with microstructures and to determine equations which the limits are solution of. Roughly speaking, what one does is to consider media with microstructures, to average out the physical and chemical processes arising at the microscale and to calculate effective properties of the media on the macroscale. This is precisely what has been done in the present work, where the homogenization method has been applied to the model presented below.

Let Ω be a bounded open set in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$. Let Y be the unit periodicity cell $[0,1[^N$ having the paving property. We perforate Ω by removing from it a set T_ϵ of periodically distributed holes defined as follows. Let us denote by T an open subset of Y with a smooth boundary Γ, such that $\overline{T} \subset \operatorname{Int} Y$. Set $Y^* = Y \setminus T$ which is called in the literature the solid or material part. We define $\tau(\epsilon\overline{T})$ to be the set of all translated images of $\epsilon\overline{T}$ of the form $\epsilon(k+\overline{T})$, $k \in \mathbb{Z}^N$. Then,

$$T_\epsilon := \Omega \cap \tau(\epsilon\overline{T}).$$

Introduce now the periodically perforated domain Ω_ϵ defined by

$$\Omega_\epsilon = \Omega \setminus \overline{T}_\epsilon.$$

For the sake of simplicity, we make the following standard assumption on the holes [7]:

there exists a 'security' zone around $\partial\Omega$ without holes, i.e.

$$\exists\, \delta > 0 \text{ such that } dist\,(\partial\Omega, T_\epsilon) \geq \delta. \tag{1}$$

Therefore, Ω_ϵ is a connected set [7]. The boundary $\partial\Omega_\epsilon$ of Ω_ϵ is then composed of two parts. The first one is the union of the boundaries of the holes strictly contained in Ω. It is denoted by Γ_ϵ and is defined by

$$\Gamma_\epsilon := \cup\left\{\partial(\epsilon(k+\overline{T})) \mid \epsilon(k+\overline{T}) \subset \Omega\right\}.$$

The second part of $\partial\Omega_\epsilon$ is its fixed exterior boundary denoted by $\partial\Omega$. It is easily seen that (see [3, Eq. (3)])

$$\lim_{\epsilon\to 0} \epsilon \mid \Gamma_\epsilon|_{N-1} =| \Gamma|_{N-1} \frac{| \Omega |_N}{| Y |_N} \tag{2}$$

where $| \cdot |_{N-1}$ and $| \cdot |_N$ are the $(N-1)$-dimensional and the N-dimensional Hausdorff measure, respectively.

Throughout this paper, ϵ will denote the general term of a sequence of positive reals which converges to zero. From now on, let $M \in \mathbb{N}$ be fixed. We consider in the following a system of anisotropic diffusion-coagulation equations in Ω_ϵ (the so-called Smoluchowski system with diffusion) which describes the dynamics of cluster growth. In particular, we introduce the vector-valued function $u^\epsilon : [0,T]\times\Omega_\epsilon \to \mathbb{R}^M$, $u^\epsilon = (u_1^\epsilon, \dots, u_M^\epsilon)$ where the variable $u_m^\epsilon \geq 0$ $(1 \leq m < M)$ represents the concentration of m-clusters, that is, clusters consisting of m identical elementary particles (monomers), while $u_M^\epsilon \geq 0$ takes into account aggregations of more than $M-1$ monomers. We assume that the only reaction allowing clusters to coalesce to form larger clusters is a binary coagulation mechanism, while the movement of clusters leading to aggregation results only from a diffusion process described by a matrix $D_m(t,x,\frac{x}{\epsilon})$ $(1 \leq m \leq M)$ with non-constant coefficients. Similar results for constant diffusion matrices have been obtained in [9] (see also the comments in Sect. 4).

Under these assumptions, our system reads:

$$\begin{cases} \dfrac{\partial u_1^\epsilon}{\partial t} - div(D_1(t,x,\frac{x}{\epsilon})\,\nabla_x u_1^\epsilon) + u_1^\epsilon \sum_{j=1}^M a_{1,j}u_j^\epsilon = 0 & \text{in } [0,T]\times\Omega_\epsilon \\ \left[D_1(t,x,\frac{x}{\epsilon})\nabla_x u_1^\epsilon\right]\cdot n = 0 & \text{on } [0,T]\times\partial\Omega \\ \left[D_1(t,x,\frac{x}{\epsilon})\nabla_x u_1^\epsilon\right]\cdot n = \epsilon\,\psi(t,x,\frac{x}{\epsilon}) & \text{on } [0,T]\times\Gamma_\epsilon \\ u_1^\epsilon(0,x) \equiv U_1 > 0 & \text{in } \Omega_\epsilon; \end{cases} \tag{3}$$

if $1 < m < M$,

$$\begin{cases} \dfrac{\partial u_m^\epsilon}{\partial t} - div(D_m(t,x,\frac{x}{\epsilon})\,\nabla_x u_m^\epsilon) + u_m^\epsilon \sum_{j=1}^M a_{m,j}u_j^\epsilon = f_m^\epsilon & \text{in } [0,T]\times\Omega_\epsilon \\ \left[D_m(t,x,\frac{x}{\epsilon})\nabla_x u_m^\epsilon\right]\cdot n = 0 & \text{on } [0,T]\times\partial\Omega \\ \left[D_m(t,x,\frac{x}{\epsilon})\nabla_x u_m^\epsilon\right]\cdot n = 0 & \text{on } [0,T]\times\Gamma_\epsilon \\ u_m^\epsilon(0,x) = 0 & \text{in } \Omega_\epsilon \end{cases} \tag{4}$$

and eventually

$$\begin{cases} \dfrac{\partial u_M^\epsilon}{\partial t} - div(D_M(t,x,\frac{x}{\epsilon})\,\nabla_x u_M^\epsilon) = g^\epsilon & \text{in } [0,T]\times\Omega_\epsilon \\ \left[D_M(t,x,\frac{x}{\epsilon})\nabla_x u_M^\epsilon\right]\cdot n = 0 & \text{on } [0,T]\times\partial\Omega \\ \left[D_M(t,x,\frac{x}{\epsilon})\nabla_x u_M^\epsilon\right]\cdot n = 0 & \text{on } [0,T]\times\Gamma_\epsilon \\ u_M^\epsilon(0,x) = 0 & \text{in } \Omega_\epsilon, \end{cases} \tag{5}$$

where the gain terms f_m^ϵ and g^ϵ in (4) and (5) are given by

$$f_m^\epsilon = \frac{1}{2}\sum_{j=1}^{m-1} a_{j,m-j}\, u_j^\epsilon\, u_{m-j}^\epsilon \qquad \text{and} \qquad g^\epsilon = \frac{1}{2}\sum_{\substack{j+k\geq M\\ k<M\\ j<M}} a_{j,k}\, u_j^\epsilon\, u_k^\epsilon. \tag{6}$$

The kinetic coefficients $a_{i,j}$ represent a reaction in which an $(i+j)$-cluster is formed from an i-cluster and a j-cluster. Therefore, they can be interpreted as "coagulation rates" and are symmetric $a_{i,j} = a_{j,i} > 0$, $i, j = 1, \ldots, M$, but $a_{M,M} = 0$. Let us remark that the meaning of u_M^ϵ differs from that of u_m^ϵ $(m < M)$, since it describes the sum of the densities of all the 'large' assemblies. It is assumed that large assemblies exhibit all the same coagulation properties and do not coagulate with each other.

Here

$$(t, x, y) \in [0, T] \times \Omega \times Y \to D_m(t, x, y)$$

is a matrix-valued map with entries $d_{i,j}^m$, $i, j = 1, \ldots, N$ and $m = 1, \ldots, M$.

We assume that:

(H.1) the diffusion coefficients $d_{i,j}^m$ are continuously differentiable in $[0, T] \times \overline{\Omega} \times Y$ for $i, j = 1, \ldots, N$, $m = 1, \ldots, M$, and are y-periodic on Y. We put $\Lambda^\star := \max_{i,j,m} \|d_{i,j}^m\|_{C^1([0,T]\times\overline{\Omega}\times Y)}$.

In particular, (see [2, Definition 1.4 and Remark 1.5]) the map $(t, x) \to D_m(t, x, \frac{x}{\epsilon})$ is measurable on Ω_ϵ, and

$$\lim_{\epsilon\to 0}\int_0^T\int_{\Omega_\epsilon} |d_{i,j}^m(t, x, \frac{x}{\epsilon})|^2\, dt\, dx = \int_0^T\int_\Omega\int_{Y^*} |d_{i,j}^m(t, x, y)|^2\, dt\, dx\, dy \tag{7}$$

(H.2) $d_{i,j}^m = d_{j,i}^m$, for $i, j = 1, \ldots, N$, $m = 1, \ldots, M$;

(H.3) there exists $0 < \lambda \leq \Lambda$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^N d_{i,j}^m \xi_i \xi_j \leq \Lambda|\xi|^2$$

for all $\xi \in \mathbb{R}^N$, $m = 1, \ldots, M$.

Moreover, ψ, appearing in (3), is a given bounded function satisfying the following conditions:

(H.4) $\psi(t, x, \frac{x}{\epsilon}) \in C^1(0, T; B)$ with $B = C^1[\overline{\Omega}; C_\#^1(Y)]$, where $C_\#^1(Y)$ is the subset of $C^1(\mathbb{R}^N)$ of Y-periodic functions;

(H.5) $\psi(t = 0, x, \frac{x}{\epsilon}) = 0$

and U_1 is a positive constant such that

$$U_1 \leq \|\psi\|_{L^\infty(0,T;B)}. \tag{8}$$

In the Sect. 2 we show preliminarily that the system (3)–(5) has a unique classic solution $u^\epsilon \in C^{1+\alpha/2,2+\alpha}([0,T] \times \Omega_\epsilon, \mathbb{R}^M)$ for any $\epsilon > 0$. The core of this note is the study of the asymptotic behavior of u^ϵ as $\epsilon \to 0$ in the framework of the so-called *two-scale convergence*. This method, introduced by Gabriel Nguetseng [15] and Gregoire Allaire [2], relies on the following compactness theorem:

Theorem 1.1 *Let $(v^\epsilon)_{\epsilon>0}$ be a bounded sequence in $L^2([0,T] \times \Omega)$. There exists a subsequence, still denoted by $(v^\epsilon)_{\epsilon>0}$, and a function $v_0(t,x,y)$ in $L^2([0,T] \times \Omega \times Y)$ such that*

$$\lim_{\epsilon \to 0} \int_0^T \int_\Omega v^\epsilon(t,x)\, \phi\left(t, x, \frac{x}{\epsilon}\right) dt\, dx = \int_0^T \int_\Omega \int_Y v_0(t,x,y)\, \phi(t,x,y)\, dt\, dx\, dy \tag{9}$$

for all $\phi \in C^1([0,T] \times \overline{\Omega}; C^1_\#(Y))$.

A sequence $(v^\epsilon)_{\epsilon>0}$ satisfying (9) is said to two-scale converge to $v_0(t,x,y)$.

Within the general setting of two-scale convergence, we can state our main homogenization result:

Theorem 1.2 *Let $u^\epsilon_m(t,x)$ $(1 \le m \le M)$ be a family of classical solutions to problems (3)–(5). Denote by a tilde the extension by zero outside Ω_ϵ of a function defined in Ω_ϵ and let $\chi(y)$ represent the characteristic function of Y^*.*

Then, the sequences $(\widetilde{u^\epsilon_m})_{\epsilon>0}$ and $(\widetilde{\nabla_x u^\epsilon_m})_{\epsilon>0}$ $(1 \le m \le M)$ two-scale converge to: $[\chi(y)\, u_m(t,x)]$ and $[\chi(y)(\nabla_x u_m(t,x) + \nabla_y u^1_m(t,x,y))]$ $(1 \le m \le M)$, respectively. The limiting functions $(u_m(t,x), u^1_m(t,x,y))$ $(1 \le m \le M)$ are the unique solutions in $L^2(0,T; H^1(\Omega)) \times L^2([0,T] \times \Omega; H^1_\#(Y)/\mathbb{R})$ of the following two-scale homogenized systems:

If $m = 1$ we have:

$$\begin{cases} \theta\, \dfrac{\partial u_1}{\partial t}(t,x) - div_x\left[D^\star_1(t,x)\, \nabla_x u_1(t,x)\right] & \\ +\theta\, u_1(t,x) \sum_{j=1}^M a_{1,j}\, u_j(t,x) = \displaystyle\int_\Gamma \psi(t,x,y)\, d\sigma(y) & \text{in } [0,T] \times \Omega \\ [D^\star_1(t,x)\, \nabla_x u_1(t,x)] \cdot n = 0 & \text{on } [0,T] \times \partial\Omega \\ u_1(0,x) = U_1 & \text{in } \Omega \end{cases} \tag{10}$$

if $1 < m < M$ we have

$$\begin{cases} \theta\, \dfrac{\partial u_m}{\partial t}(t,x) - div_x\left[D^\star_m(t,x)\, \nabla_x u_m(t,x)\right] & \\ +\theta\, u_m(t,x) \sum_{j=1}^M a_{m,j}\, u_j(t,x) & \\ = \dfrac{\theta}{2} \sum_{j=1}^{m-1} a_{j,m-j}\, u_j(t,x)\, u_{m-j}(t,x) & \text{in } [0,T] \times \Omega \\ [D^\star_m(t,x)\, \nabla_x u_m(t,x)] \cdot n = 0 & \text{on } [0,T] \times \partial\Omega \\ u_m(0,x) = 0 & \text{in } \Omega \end{cases} \tag{11}$$

if $m = M$ we have:

$$\begin{cases} \theta \dfrac{\partial u_M}{\partial t}(t,x) - div_x\Big[D^\star_M(t,x)\,\nabla_x u_M(t,x)\Big] \\ = \dfrac{\theta}{2}\sum_{\substack{j+k\geq M\\ k<M\\ j<M}} a_{j,k}\, u_j(t,x)\, u_k(t,x) & in\ [0,T]\times\Omega \\ [D^\star_M(t,x)\,\nabla_x u_M(t,x)]\cdot n = 0 & on\ [0,T]\times\partial\Omega \\ u_M(0,x) = 0 & in\ \Omega \end{cases} \tag{12}$$

where

$$u^1_m(t,x,y) = \sum_{i=1}^{N} w_i(t,x,y)\,\frac{\partial u_m}{\partial x_i}(t,x) \quad (1\leq m\leq M),$$

$$\theta = \int_Y \chi(y)dy = |Y^*|$$

is the volume fraction of material, and $D^\star_m(t,x)$ is a matrix defined by

$$(D^\star_m)_{ij}(t,x) = \int_{Y^*} D_m(t,x,y)(\nabla_y w_i(t,x,y)+\hat{e}_i)\cdot(\nabla_y w_j(t,x,y)+\hat{e}_j)\,dy$$

with $\hat{e}_i$ being the i-th unit vector in $\mathbb{R}^N$, and $(w_i)_{1\leq i\leq N}$ the family of solutions of the cell problem

$$\begin{cases} -div_y(D_m(t,x,y)[\nabla_y w_i(t,x,y)+\hat{e}_i]) = 0 & in\ Y^* \\ D_m(t,x,y)[\nabla_y w_i(t,x,y)+\hat{e}_i]\cdot n = 0 & on\ \Gamma \\ y\to w_i(t,x,y) \quad Y-periodic \end{cases} \tag{13}$$

2 The Problem at ϵ-Scale: Existence and Regularity

The system (3)–(5) admits a local positive classical solution. Indeed, by Amann [4] and the usual parabolic comparison principle, we have:

Theorem 2.1 *Suppose (H.1)–(H.5) hold. If $\epsilon > 0$, then the system* (3)–(5) *admits a unique maximal classical solution $u^\epsilon = (u^\epsilon_1,\dots,u^\epsilon_M)$, that is defined in a relatively open interval $J\subset[0,T]$ such that $0\in J$. More precisely,*

$$u^\epsilon \in C^0(J\times\bar{\Omega}_\epsilon)\cap C^1((J\setminus\{0\})\times\bar{\Omega}_\epsilon)\cap C^2((J\setminus\{0\})\times\Omega_\epsilon).$$

Moreover

$$u_j^\epsilon(t,x) > 0 \quad \textit{for } (t,x) \in (J \setminus \{0\}) \times \Omega_\epsilon, j = 1,\ldots,M.$$

We are now faced with several questions that will turn out to be deeply interconnected. In particular, we want to show that

- for fixed $\epsilon > 0$, the local solution u^ϵ is in fact a *global* solution on $[0, T]$;
- u^ϵ satisfies sharp regularity estimates, i.e. $u_j^\epsilon \in C^{1+\alpha/2,2+\alpha}([0,T] \times \Omega)$ for $j = 1,\ldots,M$.

Moreover, in order to establish our homogenization results, we have to prove

- a priori estimates for the sequences $(u_j^\epsilon)_{\epsilon>0}$, $(\nabla_x u_j^\epsilon)_{\epsilon>0}$, $(\partial_t u_j^\epsilon)_{\epsilon>0}$ in $L^2([0,T] \times \Omega_\epsilon)$, that are independent of ϵ.

The first and crucial step will consist of proving that the u_j^ϵ are equibounded in $L^\infty([0,T] \times \Omega_\epsilon)$ for $j = 1,\ldots,M$. The uniform boundedness of $u_1^\epsilon(t,x)$ in $L^\infty([0,T] \times \Omega_\epsilon)$ is provided by the following statement:

Theorem 2.2 *Take* $0 < T_{\max} < \sup J$ *and let* u_1^ϵ *be a classical solution of (3). Then,*

$$\|u_1^\epsilon\|_{L^\infty(0,T_{\max};L^\infty(\Omega_\epsilon))} \le |U_1| + c\,\|\psi\|_{L^\infty(0,T_{\max};B)}, \tag{14}$$

where c is independent of ϵ.

Proof Since

$$div(D_1(t,x,\frac{x}{\epsilon})\,\nabla_x u_1^\epsilon) - \frac{\partial u_1^\epsilon}{\partial t} \ge 0,$$

by the classical maximum principle the following estimate holds:

$$\|u_1^\epsilon\|_{L^\infty(0,T_{\max};L^\infty(\Omega_\epsilon))} \le |U_1| + \|u_1^\epsilon\|_{L^\infty(0,T_{\max};L^\infty(\Gamma_\epsilon))}. \tag{15}$$

Thus, (14) will follow once we prove that

$$\|u_1^\epsilon\|_{L^\infty(0,T_{\max};L^\infty(\Gamma_\epsilon))} \le c\,\|\psi\|_{L^\infty(0,T_{\max};B)} \tag{16}$$

Let now $k \ge 0$ be fixed. Define: $u_\epsilon^{(k)}(t) := (u_1^\epsilon(t) - k)_+$ for $t \ge 0$, with derivatives:

$$\frac{\partial u_\epsilon^{(k)}}{\partial t} = \frac{\partial u_1^\epsilon}{\partial t}\,\mathbb{1}_{\{u_1^\epsilon > k\}} \tag{17}$$

$$\nabla_x u_\epsilon^{(k)} = \nabla_x u_1^\epsilon\,\mathbb{1}_{\{u_1^\epsilon > k\}}. \tag{18}$$

Moreover,

$$u_\epsilon^{(k)} \mid_{\partial\Omega} = (u_1^\epsilon \mid_{\partial\Omega} - k)_+ \tag{19}$$

$$u_\epsilon^{(k)} \mid_{\Gamma_\epsilon} = (u_1^\epsilon \mid_{\Gamma_\epsilon} - k)_+ \tag{20}$$

Let us assume $k \geq \hat{k}$, where $\hat{k} := \|\psi\|_{L^\infty(0,T_{\max};B)}$. Then, by (8),

$$u_1^\epsilon(0,x) = U_1 \leq \hat{k} \leq k. \tag{21}$$

For $t \in [0, T_1]$ with $T_1 \leq T_{\max}$, we get

$$\begin{aligned}\frac{1}{2}\int_{\Omega_\epsilon} |u_\epsilon^{(k)}(t)|^2\,dx &= \int_0^t \frac{d}{ds}\left[\frac{1}{2}\int_{\Omega_\epsilon} |u_\epsilon^{(k)}(s)|^2\,dx\right] ds \\ &= \int_0^t ds \int_{\Omega_\epsilon} \frac{\partial u_\epsilon^{(k)}(s)}{\partial s}\, u_\epsilon^{(k)}(s)\,dx.\end{aligned} \tag{22}$$

Taking into account (3), (17) and (18), we obtain that for all $s \in [0, T_1]$

$$\begin{aligned}&\int_{\Omega_\epsilon} \frac{\partial u_\epsilon^{(k)}(s)}{\partial s}\, u_\epsilon^{(k)}(s)\,dx = \int_{\Omega_\epsilon} \frac{\partial u_1^\epsilon(s)}{\partial s}\, u_\epsilon^{(k)}(s)\,dx \\ &= \int_{\Omega_\epsilon}\left[div(D_1(s,x,\frac{x}{\epsilon})\,\nabla_x u_1^\epsilon) - u_1^\epsilon \sum_{j=1}^M a_{1,j}u_j^\epsilon\right] u_\epsilon^{(k)}(s)\,dx \\ &= -\int_{\Omega_\epsilon} u_1^\epsilon(s) \sum_{j=1}^M a_{1,j}u_j^\epsilon(s)\,u_\epsilon^{(k)}(s)\,dx + \epsilon\int_{\Gamma_\epsilon} \psi\left(s,x,\frac{x}{\epsilon}\right) u_\epsilon^{(k)}(s)\,d\sigma_\epsilon(x) \\ &-\int_{\Omega_\epsilon}\left\langle D_1(s,x,\frac{x}{\epsilon})\nabla_x u_\epsilon^{(k)}(s), \nabla_x u_\epsilon^{(k)}(s)\right\rangle dx\end{aligned} \tag{23}$$

By the assumption (H.3) and Lemma 7.1 in [9], one has

$$\begin{aligned}&\int_{\Omega_\epsilon} \frac{\partial u_\epsilon^{(k)}(s)}{\partial s}\, u_\epsilon^{(k)}(s)\,dx \leq \epsilon\int_{\Gamma_\epsilon} \psi\left(s,x,\frac{x}{\epsilon}\right) u_\epsilon^{(k)}(s)\,d\sigma_\epsilon(x) \\ &-\lambda\int_{\Omega_\epsilon} |\nabla_x u_\epsilon^{(k)}(s)|^2\,dx \leq \frac{\epsilon}{2}\int_{B_k^\epsilon(s)} \left|\psi\left(s,x,\frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x) \\ &+\frac{C_1}{2}\int_{A_k^\epsilon(s)} |u_\epsilon^{(k)}(s)|^2\,dx - \left(\lambda - \frac{C_1\epsilon^2}{2}\right)\int_{\Omega_\epsilon} |\nabla_x u_\epsilon^{(k)}(s)|^2\,dx\end{aligned} \tag{24}$$

where we denote by $A_k^\epsilon(t)$ and $B_k^\epsilon(t)$ the set of points in Ω_ϵ and on Γ_ϵ, respectively, at which $u_1^\epsilon(t, x) > k$. It holds:

$$|A_k^\epsilon(t)| \leq |\Omega_\epsilon|$$
$$|B_k^\epsilon(t)| \leq |\Gamma_\epsilon|$$

with $|\cdot|$ being the Hausdorff measure.

Plugging (24) into (22) and varying over t, we arrive at the estimate:

$$\sup_{0\leq t\leq T_1}\left[\frac{1}{2}\int_{\Omega_\epsilon}|u_\epsilon^{(k)}(t)|^2\,dx\right]+\left(\lambda-\frac{C_1\,\epsilon^2}{2}\right)\int_0^{T_1}dt\int_{\Omega_\epsilon}|\nabla u_\epsilon^{(k)}(t)|^2\,dx$$
$$\leq\frac{C_1}{2}\int_0^{T_1}dt\int_{A_k^\epsilon(t)}|u_\epsilon^{(k)}(t)|^2\,dx+\frac{\epsilon}{2}\int_0^{T_1}dt\int_{B_k^\epsilon(t)}\left|\psi\left(t,x,\frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x) \tag{25}$$

Introducing the following norm

$$\|u\|^2_{Q_\epsilon(T_{\max})} := \sup_{0\leq t\leq T_{\max}}\int_{\Omega_\epsilon}|u(t)|^2\,dx+\int_0^{T_{\max}}dt\int_{\Omega_\epsilon}|\nabla u(t)|^2\,dx \tag{26}$$

the inequality (25) can be rewritten as follows

$$\min\left\{\frac{1}{2},\left(\lambda-\frac{C_1\,\epsilon^2}{2}\right)\right\}\|u_\epsilon^{(k)}\|^2_{Q_\epsilon(T_1)}\leq\frac{C_1}{2}\int_0^{T_1}dt\int_{A_k^\epsilon(t)}|u_\epsilon^{(k)}(t)|^2\,dx$$
$$+\frac{\epsilon}{2}\int_0^{T_1}dt\int_{B_k^\epsilon(t)}\left|\psi\left(t,x,\frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x) \tag{27}$$

We estimate the right-hand side of (27). From Hölder's inequality we obtain

$$\int_0^{T_1}dt\int_{A_k^\epsilon(t)}|u_\epsilon^{(k)}(t)|^2\,dx\leq\|u_\epsilon^{(k)}\|^2_{L^{\overline{r}_1}(0,T_1;L^{\overline{q}_1}(\Omega_\epsilon))}\,\|\mathbb{1}_{A_k^\epsilon}\|_{L^{r_1'}(0,T_1;L^{q_1'}(\Omega_\epsilon))} \tag{28}$$

with $r_1' = \dfrac{r_1}{r_1-1}$, $q_1' = \dfrac{q_1}{q_1-1}$, $\overline{r}_1 = 2\,r_1$, $\overline{q}_1 = 2\,q_1$, where, for $N > 2$, $\overline{r}_1 \in (2,\infty)$ and $\overline{q}_1 \in (2, \frac{2N}{(N-2)})$ have been chosen such that

$$\frac{1}{\overline{r}_1}+\frac{N}{2\,\overline{q}_1}=\frac{N}{4}$$

In particular, $r_1', q_1' < \infty$, so that (28) yields

$$\int_0^{T_1}dt\int_{A_k^\epsilon(t)}|u_\epsilon^{(k)}(t)|^2\,dx\leq\|u_\epsilon^{(k)}\|^2_{L^{\overline{r}_1}(0,T_1;L^{\overline{q}_1}(\Omega_\epsilon))}\,|\Omega|^{1/q_1'}\,T_1^{1/r_1'}. \tag{29}$$

If we choose

$$T_1^{1/r_1'} < \frac{\min\{1,\lambda\}}{2C_1} |\Omega|^{-1/q_1'} \le \frac{\min\left\{\frac{1}{2}, \left(\lambda - \frac{C_1\,\epsilon^2}{2}\right)\right\}}{C_1} |\Omega|^{-1/q_1'},$$

then from Eq. (117) in [9], it follows that

$$\frac{C_1}{2}\int_0^{T_1} dt \int_{A_k^\epsilon(t)} |u_\epsilon^{(k)}(t)|^2\, dx \le \frac{1}{2}\min\left\{\frac{1}{2}, \left(\lambda - \frac{C_1\,\epsilon^2}{2}\right)\right\} \|u_\epsilon^{(k)}\|^2_{Q_\epsilon(T_1)}. \tag{30}$$

Analogously, from Hölder's inequality we have, for $k \ge \hat{k}$

$$\begin{aligned}\frac{\epsilon}{2}\int_0^{T_1} dt \int_{B_k^\epsilon(t)} \left|\psi\left(t,x,\frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x) &\le \frac{\epsilon\, k^2}{2}\left(\frac{\hat{k}^2}{k^2}\right) \|\mathbb{1}_{B_k^\epsilon}\|_{L^1(0,T_1;L^1(\Gamma_\epsilon))} \\ &\le \frac{\epsilon\, k^2}{2}\int_0^{T_1} dt\, |B_k^\epsilon(t)|.\end{aligned} \tag{31}$$

Thus (27) yields

$$\|u_\epsilon^{(k)}\|^2_{Q_\epsilon(T_1)} \le \epsilon\,\gamma\, k^2 \int_0^{T_1} dt\, |B_k^\epsilon(t)|. \tag{32}$$

Now, as in [9, Theorem 5.2], relying on arguments that go back to [11, 16], it follows from (32) that

$$\|u_1^\epsilon\|_{L^\infty(0,T_1;L^\infty(\Gamma_\epsilon))} \le 2\,\mu\,\hat{k}$$

where the positive constant μ is independent of ϵ. Analogous arguments are valid for the cylinder $[T_s, T_{s+1}] \times \Omega_\epsilon$, $s = 1, 2, \ldots, p-1$ with

$$\left[T_{s+1} - T_s\right]^{1/r_1'} < \frac{\min\{1,\lambda\}}{2C_1} |\Omega|^{-1/q_1'}$$

and $T_p \equiv T_{\max}$. Thus, after a finite number of steps, we get the estimate (16), completing the proof of Theorem 2.2. □

Following the inductive argument presented in [18, Lemma 2.2], we obtain eventually the global L^∞ estimate for local classical solutions of (4) and (5).

Theorem 2.3 *Let $u_j^\epsilon(t,x)$ ($1 \le j \le M$) be a classical solution of (3)–(5). Then there exists $K > 0$ such that*

$$\|u_j^\epsilon\|_{L^\infty(0,T_{\max};L^\infty(\Omega_\epsilon))} \le K \tag{33}$$

uniformly with respect to ϵ.

A first consequence of the estimates (14) and (33) is that, for any fixed $\epsilon > 0$, $J = [0, T)$ and u^ϵ satisfies sharp Hölder estimates (see also [10]).

Theorem 2.4 *Let $\epsilon > 0$ be fixed. Then*

i) $T_{\max} = T$*, i.e.* $J = [0, T)$*;*
ii) *there exists* $\alpha \in (0, 1)$*,* α *depending only on* $N, \lambda, \Lambda^\star$*, and* ϵ*, such that* $u^\epsilon \in C^{1+\alpha/2,2+\alpha}([0, T] \times \Omega_\epsilon, \mathbb{R}^M)$ *and*

$$\|u^\epsilon\|_{C^{1+\alpha/2,2+\alpha}([0,T]\times\Omega_\epsilon,\mathbb{R}^M)} \leq C_0 = C_0(U_1, \|\psi\|_{L^\infty(0,T;B)}, K, \epsilon, \alpha). \tag{34}$$

Proof First of all, we notice that, if we prove (34) in J, then, in particular, if $N < p < \infty$, we have

$$\|u^\epsilon\|_{L^\infty(J,W^{2,p}(\Omega_\epsilon)^M)} < \infty.$$

Thus i) follows by Amann [4, p. 154].

Let us prove ii). To avoid cumbersome notations, let us set

$$F_m(t, x, u^\epsilon) = \begin{cases} -u_1^\epsilon \displaystyle\sum_{j=1}^{M} a_{1,j} u_j^\epsilon & \text{if } \ m = 1 \\ -u_m^\epsilon \displaystyle\sum_{j=1}^{M} a_{m,j} u_j^\epsilon + f_m^\epsilon & \text{if } \ 2 \leq m < M \\ \quad g^\epsilon & \text{if } \ m = M \end{cases} \tag{35}$$

and $F := (F_1, \ldots, F_M)$.

We can use a modified version for the parabolic Neumann-Cauchy problem of the classical Hölder estimates for the corresponding Dirichlet-Cauchy problem, as one can find, for instance, in [13, Theorem 6.44]. If $D[r]$ is an arbitrary parabolic cylinder

$$D[r] = \{(x, t) \ ; \ |x - x_0| < r, |t - t_0| < r^2\} \cap \big(\Omega \times [0, T]\big),$$

we obtain

$$\mathrm{osc}_{D[r]} u_m^\epsilon \leq C\, r^\alpha \left[\sup_{[0,T]\times\Omega} |u_m^\epsilon| + \sup_{[0,T]\times\Omega} |F_m(t, x, u^\epsilon)| \right] \tag{36}$$

for any $m = 1, \ldots, M$ and $0 < r < 1$, where C depends on Ω, T, λ and Λ. Thus, by Theorem 2.3,

$$\begin{aligned} \|u^\epsilon\|_{C^{\alpha/2,\alpha}([0,T]\times\Omega_\epsilon,\mathbb{R}^M)} &\leq \sup_{(t,x),(\tau,\xi)\in[0,T]\times\Omega} \frac{|u_m^\epsilon(t, x) - u_m^\epsilon(\tau, \xi)|}{|x - \xi|^\alpha + |t - \tau|^{\alpha/2}} + K \\ &\leq C \left[\sup_{[0,T]\times\Omega} |u^\epsilon| + \sup_{[0,T]\times\Omega} |F(t, x, u^\epsilon)| \right] + K \leq C\,(1 + K^2). \end{aligned} \tag{37}$$

We write now Eqs. (3)–(5) in non-divergence form, and then we apply classic Hölder estimates as in [11, Theorem 5.2] and [13, Theorem 5.18]. Eventually, keeping in mind (37), (34) follows. This achieves the proof of the theorem. □

We stress that all the constants involved in these Hölder estimates depend also on the space derivatives of the diffusion coefficients. Since in (3)–(5) the diffusion coefficients have the form $d^m_{i,j}(t, x, x/\epsilon)$, then our Hölder estimates turn out to depend on ϵ.

3 Homogenization

In order to prove that the solutions u^ϵ of our Neumann-Cauchy problem at the scale ϵ converge to a solution of the homogenized problem described in Theorem 1.2, we need a priori L^2-estimates of the derivatives of u^ϵ, *that are independent of* $\epsilon > 0$. Unfortunately, the bounds in (34) are not uniform in ϵ, and therefore the compactness Theorem 1.1 does not apply

To overcome this difficulty, in the sequel we shall prove weaker estimates, that nevertheless are uniform in ϵ.

Theorem 3.1 *The sequence* $(\nabla_x u^\epsilon_m)_{\epsilon>0}$ *(*$1 \le m \le M$*) is bounded in* $L^2([0, T] \times \Omega_\epsilon)$*, uniformly in* ϵ*.*

Proof Case $m = 1$: let us multiply the first equation in (3) by the function $u^\epsilon_1(t, x)$. Integrating, by divergence theorem and assumption (H.3), one has

$$\begin{aligned}\frac{1}{2}\int_{\Omega_\epsilon}\frac{\partial}{\partial t}|u^\epsilon_1|^2\,dx + \lambda\int_{\Omega_\epsilon}|\nabla_x u^\epsilon_1|^2\,dx \\ \le \epsilon\int_{\Gamma_\epsilon}\psi\left(t, x, \frac{x}{\epsilon}\right)u^\epsilon_1(t, x)\,d\sigma_\epsilon(x)\end{aligned} \tag{38}$$

Let us now estimate the term on the right-hand side of (38). It follows from Lemma 7.4 in [9] that

$$\epsilon\int_{\Gamma_\epsilon}|\psi(t, x, \frac{x}{\epsilon})|^2\,d\sigma_\epsilon(x) \le C_2\,\|\psi(t)\|^2_B \tag{39}$$

where C_2 is a positive constant independent of ϵ and $B = C^1[\overline{\Omega}; C^1_\#(Y)]$. Hence, by Hölder's and Young's inequalities and Lemma 7.1 in [9], we deduce

$$\begin{aligned}&\int_{\Omega_\epsilon}\frac{\partial}{\partial t}|u^\epsilon_1|^2\,dx + (2\,\lambda - \epsilon^2\,C_1)\int_{\Omega_\epsilon}|\nabla_x u^\epsilon_1|^2\,dx \\ &\le C_2\,\|\psi(t)\|^2_B + C_1\int_{\Omega_\epsilon}|u^\epsilon_1|^2\,dx\end{aligned} \tag{40}$$

Integrating over $[0, t]$ with $t \in [0, T]$, and taking into account that the sequence $(u_1^\epsilon)_{\epsilon>0}$ is bounded in $L^\infty(0, T; L^\infty(\Omega_\epsilon))$, we get

$$(2\lambda - \epsilon^2 C_1) \|\nabla_x u_1^\epsilon\|^2_{L^2(0,T;L^2(\Omega_\epsilon))} \le C_3, \tag{41}$$

where C_1 and C_3 are positive constants independent of ϵ.

Thus the boundedness of $\nabla_x u_1^\epsilon(t, x)$ follows, provided that ϵ is close to zero.

The proof for the case $1 < m \le M$ is achieved by applying exactly the same arguments considered when $m = 1$. □

Theorem 3.2 *The sequence* $(\partial_t u_m^\epsilon)_{\epsilon>0}$ *($1 \le m \le M$) is bounded in* $L^2([0, T] \times \Omega_\epsilon)$*, uniformly in* ϵ.

Proof Case $m = 1$: let us multiply the first equation in (3) by the function $\partial_t u_1^\epsilon(t, x)$. Integrating, the divergence theorem yields

$$\begin{aligned}
&\int_{\Omega_\epsilon} \left|\frac{\partial u_1^\epsilon(t,x)}{\partial t}\right|^2 dx + \frac{1}{2}\int_{\Omega_\epsilon} \frac{\partial}{\partial t}\Big\langle D_1(t,x,\frac{x}{\epsilon})\nabla_x u_1^\epsilon, \nabla_x u_1^\epsilon\Big\rangle dx \\
&- \frac{1}{2}\int_{\Omega_\epsilon} \Big\langle \partial_t D_1\, \nabla_x u_1^\epsilon, \nabla_x u_1^\epsilon\Big\rangle dx + \int_{\Omega_\epsilon} \left(\sum_{j=1}^{M} a_{1,j}\, u_1^\epsilon\, u_j^\epsilon\right) \frac{\partial u_1^\epsilon}{\partial t}\, dx \\
&= \epsilon \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) \frac{\partial u_1^\epsilon}{\partial t}\, d\sigma_\epsilon(x)
\end{aligned} \tag{42}$$

From Hölder's and Young's inequalities, exploiting the boundedness of $u_l^\epsilon(t, x)$ ($1 \le l \le M$) in $L^\infty(0, T; L^\infty(\Omega_\epsilon))$, one has

$$\begin{aligned}
&\int_{\Omega_\epsilon} \left|\frac{\partial u_1^\epsilon(t,x)}{\partial t}\right|^2 dx + \frac{\partial}{\partial t}\int_{\Omega_\epsilon} \Big\langle D_1(t,x,\frac{x}{\epsilon})\nabla_x u_1^\epsilon, \nabla_x u_1^\epsilon\Big\rangle dx \\
&- \int_{\Omega_\epsilon} \Big\langle \partial_t D_1\, \nabla_x u_1^\epsilon, \nabla_x u_1^\epsilon\Big\rangle dx \le C_1 + 2\epsilon \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) \frac{\partial u_1^\epsilon}{\partial t}\, d\sigma_\epsilon(x)
\end{aligned} \tag{43}$$

where C_1 is a positive constant independent of ϵ. Integrating over $[0, t]$ with $t \in [0, T]$ and keeping in mind assumption (H.3), we obtain

$$\begin{aligned}
&\int_0^t ds \int_{\Omega_\epsilon} \left|\frac{\partial u_1^\epsilon}{\partial s}\right|^2 dx + \lambda \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon(t,x)|^2\, dx \le C_1\, T \\
&+ \Lambda^\star \int_0^t ds \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon|^2\, dx + 2\epsilon \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) u_1^\epsilon(t,x)\, d\sigma_\epsilon(x) \\
&- 2\epsilon \int_0^t ds \int_{\Gamma_\epsilon} \frac{\partial}{\partial s}\psi\left(s, x, \frac{x}{\epsilon}\right) u_1^\epsilon(s,x)\, d\sigma_\epsilon(x)
\end{aligned} \tag{44}$$

since $\psi\left(t = 0, x, \frac{x}{\epsilon}\right) \equiv 0$. Now we estimate the last two terms on the right-hand side of (44).

From Hölder's and Young's inequalities, taking into account (39) and Lemma 7.1 in [9], one has

$$2\,\epsilon \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) u_1^\epsilon(t,x)\, d\sigma_\epsilon(x) \leq C_2 + \epsilon^2\, C_3 \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon|^2\, dx \tag{45}$$

where C_2 is a positive constant independent of ϵ since $\psi \in L^\infty(0,T;B)$ and u_1^ϵ is bounded in $L^\infty(0,T;L^\infty(\Omega_\epsilon))$. Analogously, we get the following inequality

$$\begin{aligned} &2\,\epsilon \int_0^t ds \int_{\Gamma_\epsilon} \frac{\partial}{\partial s}\psi\left(s, x, \frac{x}{\epsilon}\right) u_1^\epsilon(s,x)\, d\sigma_\epsilon(x) \\ &\leq C_4\, T + C_5 \int_0^t ds \|u_1^\epsilon(s)\|^2_{L^2(\Omega_\epsilon)} + \epsilon^2\, C_5 \int_0^t ds \|\nabla_x u_1^\epsilon(s)\|^2_{L^2(\Omega_\epsilon)} \\ &\leq C_6 \end{aligned} \tag{46}$$

where $C_6 \geq 0$ is a constant independent of ϵ, since $(u_1^\epsilon)_{\epsilon>0}$ is bounded in $L^\infty(0,T;L^\infty(\Omega_\epsilon))$, $(\nabla_x u_1^\epsilon)_{\epsilon>0}$ is bounded in $L^2(0,T;L^2(\Omega_\epsilon))$ and

$$\epsilon \int_{\Gamma_\epsilon} \left|\partial_t \psi\left(t, x, \frac{x}{\epsilon}\right)\right|^2 d\sigma_\epsilon(x) \leq \tilde{C}\, \|\partial_t \psi(t)\|_B^2 \leq C_4$$

with $\tilde{C}$ and C_4 independent of ϵ. Combining the estimates (45) and (46) with (44) we obtain

$$\int_0^t ds \int_{\Omega_\epsilon} \left|\frac{\partial u_1^\epsilon}{\partial s}\right|^2 dx + (\lambda - \epsilon^2 C_3) \int_{\Omega_\epsilon} |\nabla_x u_1^\epsilon|^2\, dx \leq C_7 \tag{47}$$

For a sequence ϵ of positive numbers going to zero: $(\lambda - \epsilon^2 C_3) \geq 0$. Then, the second term on the left-hand side of (47) is nonnegative, and one has

$$\|\partial_t u_1^\epsilon\|^2_{L^2(0,T;L^2(\Omega_\epsilon))} \leq C \tag{48}$$

where $C \geq 0$ is a constant independent of ϵ.

The proof for the case $1 < m \leq M$ is achieved by applying exactly the same arguments considered when $m = 1$.

□

Proof of Theorem 1.2 In view of Theorems 2.2, 2.3 and 3.1, the sequences $(\widetilde{u_m^\epsilon})_{\epsilon>0}$ and $(\widetilde{\nabla_x u_m^\epsilon})_{\epsilon>0}$ $(1 \leq m \leq M)$ are bounded in $L^2([0,T] \times \Omega)$, and by application of Theorems 7.1 and 7.3 in [9], they two-scale converge, up to a subsequence, to: $[\chi(y)\, u_m(t,x)]$ and $[\chi(y)(\nabla_x u_m(t,x)+\nabla_y u_m^1(t,x,y))]$ $(1 \le m \le M)$. Similarly, in view of Theorem 3.2, it is possible to prove that the sequence $\left(\widetilde{\frac{\partial u_m^\epsilon}{\partial t}}\right)_{\epsilon>0}$ $(1 \le m \le M)$ two-scale converges to: $\left[\chi(y)\, \frac{\partial u_m}{\partial t}(t,x)\right]$ $(1 \le m \le M)$.

We can now find the homogenized equations satisfied by $u_m(t,x)$ and $u_m^1(t,x,y)$ $(1 \le m \le M)$.

Case $m = 1$: let us multiply the first equation of (3) by the test function

$$\phi_\epsilon \equiv \phi(t,x) + \epsilon\, \phi_1\left(t, x, \frac{x}{\epsilon}\right)$$

where $\phi \in C^1([0,T] \times \overline{\Omega})$ and $\phi_1 \in C^1([0,T] \times \overline{\Omega}; C^1_\#(Y))$. Integrating, the divergence theorem yields

$$\begin{aligned} &\int_0^T \int_{\Omega_\epsilon} \frac{\partial u_1^\epsilon}{\partial t}\, \phi_\epsilon(t,x,\frac{x}{\epsilon})\, dt\, dx + \int_0^T \int_{\Omega_\epsilon} \left\langle D_1(t,x,\frac{x}{\epsilon}) \nabla_x u_1^\epsilon, \nabla \phi_\epsilon \right\rangle dt\, dx \\ &+ \int_0^T \int_{\Omega_\epsilon} u_1^\epsilon \sum_{j=1}^{M} a_{1,j}\, u_j^\epsilon\, \phi_\epsilon\, dt\, dx = \epsilon \int_0^T \int_{\Gamma_\epsilon} \psi\left(t, x, \frac{x}{\epsilon}\right) \phi_\epsilon\, dt\, d\sigma_\epsilon(x) \end{aligned} \tag{49}$$

Passing to the two-scale limit we get

$$\begin{aligned} &\int_0^T \int_\Omega \int_{Y^*} \frac{\partial u_1}{\partial t}(t,x)\, \phi(t,x)\, dt\, dx\, dy \\ &+ \int_0^T \int_\Omega \int_{Y^*} D_1(t,x,y)[\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y)] \\ &\qquad \cdot [\nabla_x \phi(t,x) + \nabla_y \phi_1(t,x,y)]\, dt\, dx\, dy \\ &+ \int_0^T \int_\Omega \int_{Y^*} u_1(t,x) \sum_{j=1}^{M} a_{1,j}\, u_j(t,x)\, \phi(t,x)\, dt\, dx\, dy \\ &= \int_0^T \int_\Omega \int_\Gamma \psi(t,x,y)\, \phi(t,x)\, dt\, dx\, d\sigma(y) \end{aligned} \tag{50}$$

where assumption (H.1) has been taken into account. The last term on the left-hand side of (50) has been obtained by using Theorem 1.8 in [2], while the term on the right-hand side has been attained by application of Theorem 2.1 in [3]. An integration by parts shows that (50) is a variational formulation associated to the following homogenized system:

$$-div_y[D_1(t,x,y)(\nabla_x u_1(t,x)+\nabla_y u_1^1(t,x,y))] = 0 \qquad \text{in } [0,T]\times\Omega\times Y^* \tag{51}$$

$$[D_1(t,x,y)(\nabla_x u_1(t,x)+\nabla_y u_1^1(t,x,y))]\cdot n = 0 \qquad \text{on } [0,T]\times\Omega\times\Gamma \tag{52}$$

$$\begin{aligned} &\theta\,\frac{\partial u_1}{\partial t}(t,x) - div_x\Big[\int_{Y^*} D_1(t,x,y)(\nabla_x u_1(t,x) + \nabla_y u_1^1(t,x,y))dy\Big] \\ &+ \theta\, u_1(t,x)\sum_{j=1}^{M} a_{1,j}\, u_j(t,x) - \int_\Gamma \psi(t,x,y)\, d\sigma(y) = 0 \qquad \text{in } [0,T]\times\Omega \end{aligned} \tag{53}$$

$$\Big[\int_{Y^*} D_1(t,x,y)(\nabla_x u_1(t,x)+\nabla_y u_1^1(t,x,y))\, dy\Big]\cdot n = 0 \qquad \text{on } [0,T]\times\partial\Omega \tag{54}$$

where

$$\theta = \int_Y \chi(y)dy = |Y^*|$$

is the volume fraction of material. To conclude, by continuity, we have that

$$u_1(0,x) = U_1 \qquad \text{in } \Omega\,.$$

The function $u_1^1(t,x,y)$, satisfying (51) and (52), can be expressed as follows

$$u_1^1(t,x,y) = \sum_{i=1}^{N} w_i(t,x,y)\,\frac{\partial u_1}{\partial x_i}(t,x) \tag{55}$$

where $(w_i)_{1\le i\le N}$ is the family of solutions of the cell problem

$$\begin{cases} -div_y(D_1(t,x,y)[\nabla_y w_i(t,x,y) + \hat{e}_i]) = 0 & \text{in } Y^* \\ D_1(t,x,y)[\nabla_y w_i(t,x,y) + \hat{e}_i]\cdot n = 0 & \text{on } \Gamma \\ y \to w_i(t,x,y) \qquad Y-\text{periodic} \end{cases} \tag{56}$$

By using the relation (55) in Eqs. (53) and (54), we get

$$\begin{aligned} &\theta\,\frac{\partial u_1}{\partial t}(t,x) - div_x\Big[D_1^\star\,\nabla_x u_1(t,x)\Big] + \theta\, u_1(t,x)\sum_{j=1}^{M} a_{1,j}\, u_j(t,x) \\ &- \int_\Gamma \psi(t,x,y)\, d\sigma(y) = 0 \qquad \text{in } [0,T]\times\Omega \end{aligned} \tag{57}$$

$$[D_1^\star\nabla_x u_1(t,x)]\cdot n = 0 \qquad \text{on } [0,T]\times\partial\Omega \tag{58}$$

where the entries of the matrix $D_1^\star$ are given by

$$(D_1^\star)_{ij}(t,x) = \int_{Y^*} D_1(t,x,y)[\nabla_y w_i(t,x,y) + \hat{e}_i] \cdot [\nabla_y w_j(t,x,y) + \hat{e}_j]\, dy.$$

The proof for the case $1 < m \leq M$ is achieved by applying exactly the same arguments considered when $m = 1$. □

4 A Mathematical Model in Medicine

Recently, the Smoluchowski equation with diffusion has been introduced for the study of a mathematical model in medicine [1, 5, 9, 14]: the diffusion and the aggregation of the β-amyloid in the cerebral tissue of patients affected by Alzheimer's Disease (AD). The β amyloid (shortly, Aβ) is a peptide with different isoforms that is naturally produced by neurons. Nowadays, the microscopic description of the AD relies on the so-called *amyloid cascade hypothesis*, that is largely accepted: roughly speaking, the Aβ-peptide is produced normally by the intramembranous proteolysis of APP (amyloid precursor protein) throughout life, but a change in the metabolism (due to unknown reasons, partially genetic) may increase the total production of the monomeric isoform Aβ_{42}, that is highly toxic for neurons. Thus, high concentrations of Aβ_{42} lead to neuronal death, synaptic degeneration and eventually to dementia. Successively, Aβ_{42} oligomers are subject to agglomeration (leading ultimately to the formation of long, insoluble amyloid fibrils, which accumulate in microscopic deposits known as senile plaques) and to diffusion through the microscopic tortuosities of the brain tissue.

Mathematically, this process can be modeled at a microscopic level through the system (3)–(5). More precisely, we define the periodically perforated domain Ω_ϵ, obtained by removing from the fixed domain Ω (the cerebral tissue) infinitely many small holes of size ϵ (the neurons), which support a non-homogeneous Neumann boundary condition describing the production of Aβ_{42} by the neuron membranes. Then, we prove that, when $\epsilon \to 0$, the solution of this micro-model two-scale converges to the solution of a macro-model asymptotically consistent with the original one. Indeed, the information given on the microscale by the non-homogeneous Neumann boundary condition is transferred into a source term appearing in the limiting (homogenized) equations. Furthermore, on the macroscale, the geometric structure of the perforated domain induces a correction in the diffusion matrix of the limit problem.

A similar approach to the transition from the microscopic model to the macroscopic one has been carried out starting from constant diffusion coefficients in [9]. Here, we have considered the case of diffusion matrices depending on time, on the macroscopic variable $x \in \Omega$ and, most of all, on the microscopic variable $y \in Y$. Indeed, aging (as well as the AD itself) yields an atrophy of the cerebral tissue, that induces changes in the diffusion rate of the amyloid fibrils. Analogously, this rate may vary for different regions of the brain. Finally, the dependence on the microscopic variable makes possible to include in the model the specific features of the diffusion. Indeed, the Aβ_{42}-polymers do not diffuse freely in an uniform fluid: the cerebral tissue consists of large non-neuronal support cells (the macroglia) and the Aβ polymers move within the cerebrospinal fluid along the interstices between these cells.

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Form-Invariance of Maxwell Equations in Integral Form

Cristian E. Gutiérrez

To Richard Wheeden on the occasion of his retirement

Abstract We find transformation formulas for weak solutions to Maxwell's equations in integral form by general changes of coordinates obtaining that the equations are also "form invariant" as in the standard case. Solutions are defined using test functions.

1 Introduction

In this note we consider weak solutions to Maxwell equations and show their invariance under changes of coordinates. These changes are assumed to be locally Lipschitz functions and therefore they might be not differentiable in a set of Lebesgue measure zero. In this formulation, the fields **E** and **H** and the permittivity $\epsilon(x)$ and the permeability $\mu(x)$ might be discontinuous and only need to satisfy Lebesgue integrability conditions. From the invariance, we recover the remarkable fact, that standard Maxwell's equations preserve their form under smooth coordinate transformations, see [9] and [7, Chap. 5].[1]

[1]I would like to thank Professor Ulf Leonhardt for pointing out a connection between his paper [6, Sect. 3.2] with our results. I also like to thank the referee for pointing out a connection between our results and Maxwell's vacuum equations in general relativity, and for providing reference [1, Chap. 10, Sect. 11].

C.E. Gutiérrez (✉)
Department of Mathematics, Temple University, Philadelphia, PA 19122, USA
e-mail: gutierre@temple.edu; c.e.a.gutierrez@gmail.com

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2 Maxwell Equations

These are the equations (see for example, [5, Chap I, p. 35])

$$\nabla \cdot (\epsilon(x)\mathbf{E}(x,t)) = \rho(x,t), \tag{1}$$

$$\nabla \cdot (\mu(x)\mathbf{H}(x,t)) = 0, \tag{2}$$

$$\nabla \times \mathbf{H}(x,t) = \epsilon(x)\frac{\partial \mathbf{E}}{\partial t}(x,t) + \mathbf{J}(x,t), \tag{3}$$

$$\nabla \times \mathbf{E}(x,t) = -\mu(x)\frac{\partial \mathbf{H}}{\partial t}(x,t), \tag{4}$$

valid for (x,t) in a bounded domain $\Omega \subset \mathbb{R}^4$, $x = (x_1, x_2, x_3)$, where we initially assume the tensors $\epsilon(x)$ and $\mu(x)$, and the fields $\mathbf{E}$ and $\mathbf{H}$ are continuously differentiable in Ω; the divergence operator $\nabla = (\partial_{x_1}, \partial_{x_2}, \partial_{x_3})$.

Following [5, Sect. 2, Chap. VI], we will first rewrite these equations in a form that the fields $\mathbf{E}$ and $\mathbf{H}$, and the permittivity coefficient $\epsilon(x)$ and the permeability coefficient $\mu(x)$, only need to satisfy Lebesgue integrability conditions and therefore might not be differentiable and might be discontinuous.

2.1 Maxwell Equations in Integral Form

Multiplying (1) by $\phi \in C_0^1(\Omega)$ [2] and integrating we get

$$\int_\Omega \phi(x,t)\, \nabla \cdot (\epsilon(x)\mathbf{E}(x,t))\; dxdt = \int_\Omega \rho(x,t)\, \phi(x,t)\, dxdt.$$

We set $\epsilon(x)\mathbf{E}(x,t) = (F_1(x,t), F_2(x,t), F_3(x,t))$ and we write

$$\begin{aligned}\int_\Omega \phi(x,t)\, \nabla \cdot (\epsilon(x)\mathbf{E}(x,t))\; dxdt &= \int_t \int_{\Omega_t} \phi(x,t)\, \nabla \cdot (F_1(x,t), F_2(x,t), F_3(x,t))\, dxdt \\ &= \int_t \int_{\Omega_t} \phi(x,t) \sum_{i=1}^{3} \frac{\partial F_i}{\partial x_i}(x,t)\, dxdt \\ &= \int_t \int_{\Omega_t} \left(\sum_{i=1}^{3} \frac{\partial(\phi F_i)}{\partial x_i}(x,t) - \sum_{i=1}^{3} \frac{\partial \phi}{\partial x_i}(x,t) F_i(x,t) \right) dxdt\end{aligned}$$

[2] $C_0^1(\Omega)$ denotes as usual the class of functions having continuous derivatives of first order in Ω with compact support contained in Ω.

$$= \int_t \left(\int_{\partial \Omega_t} \phi F \cdot \nu \, d\sigma(x) \right) dt - \int_\Omega \nabla \phi(x,t) \cdot F(x,t) \, dxdt$$
$$= -\int_\Omega \nabla \phi(x,t) \cdot F(x,t) \, dxdt,$$

from the divergence theorem; $\Omega_t = \{x : (x,t) \in \Omega\}$. Therefore, we obtain the equation

$$\int_\Omega \nabla \phi(x,t) \cdot (\epsilon(x)\mathbf{E}(x,t)) \; dxdt + \int_\Omega \rho(x,t)\, \phi(x,t) \, dxdt = 0, \tag{5}$$

valid for all $\phi \in C_0^1(\Omega)$. Notice that to write (5) we only need that $\epsilon(x)\mathbf{E}(x,t)$ and $\rho(x,t)$ are locally integrable functions in Ω in the Lebesgue sense; in particular, they might be discontinuous. In other words, *the field* $\mathbf{E}$ *is a generalized solution to the Eq.* (1) *in* Ω *if* $\epsilon(x)\mathbf{E}(x,t)$ *and* $\rho(x,t)$ *are locally integrable functions in* Ω *in the Lebesgue sense, and* (5) *holds for all* $\phi \in C_0^1(\Omega)$.

Treating (2) in the same way yields

$$\int_\Omega \nabla \phi(x,t) \cdot (\mu(x)\mathbf{H}(x,t)) \; dxdt = 0, \tag{6}$$

valid for all $\phi \in C_0^1(\Omega)$.

Next, multiplying (3) by $\phi \in C_0^1(\Omega)$ and integrating yields

$$\int_\Omega \left(\phi(x,t)\, \nabla \times \mathbf{H}(x,t) - \phi(x,t)\, \epsilon(x) \frac{\partial \mathbf{E}}{\partial t}(x,t) - \phi(x,t)\, \mathbf{J}(x,t) \right) dxdt = 0.$$

Applying the divergence theorem we obtain that

$$\begin{aligned}
\int_\Omega \phi(x,t)\, \nabla \times \mathbf{H}(x,t) \, dxdt = \mathbf{i} &\int_\Omega \phi(x,t) \left(\frac{\partial \mathbf{H}_3}{\partial x_2} - \frac{\partial \mathbf{H}_2}{\partial x_3} \right) dxdt \\
&- \mathbf{j} \int_\Omega \phi(x,t) \left(\frac{\partial \mathbf{H}_3}{\partial x_1} - \frac{\partial \mathbf{H}_1}{\partial x_3} \right) dxdt \\
&+ \mathbf{k} \int_\Omega \phi(x,t) \left(\frac{\partial \mathbf{H}_2}{\partial x_1} - \frac{\partial \mathbf{H}_1}{\partial x_2} \right) dxdt \\
= -\mathbf{i} &\int_\Omega \left(\frac{\partial \phi}{\partial x_2}(x,t)\mathbf{H}_3(x,t) - \frac{\partial \phi}{\partial x_3}(x,t)\mathbf{H}_2(x,t) \right) dxdt \\
&+ \mathbf{j} \int_\Omega \left(\frac{\partial \phi}{\partial x_1}(x,t)\mathbf{H}_3(x,t) - \frac{\partial \phi}{\partial x_3}(x,t)\mathbf{H}_1(x,t) \right) dxdt
\end{aligned}$$

$$-\mathbf{k}\int_{\Omega}\left(\frac{\partial\phi}{\partial x_1}(x,t)\mathbf{H}_2(x,t)-\frac{\partial\phi}{\partial x_2}(x,t)\mathbf{H}_1(x,t)\right)dxdt$$
$$=-\int_{\Omega}\nabla\phi(x,t)\times\mathbf{H}(x,t)\,dxdt.$$

Also

$$\int_{\Omega}\phi(x,t)\,\epsilon(x)\frac{\partial\mathbf{E}}{\partial t}(x,t)\,dxdt=\int_{\Omega}\frac{\partial\left(\phi(x,t)\,\epsilon(x)\mathbf{E}\right)}{\partial t}(x,t)\,dxdt$$
$$-\int_{\Omega}\epsilon(x)\,\mathbf{E}(x,t)\,\frac{\partial\phi}{\partial t}(x,t)\,dxdt$$
$$=-\int_{\Omega}\epsilon(x)\,\mathbf{E}(x,t)\,\frac{\partial\phi}{\partial t}(x,t)\,dxdt.$$

Therefore we can write (3) in integral form as

$$\int_{\Omega}\left(\nabla\phi(x,t)\times\mathbf{H}(x,t)-\epsilon(x)\,\mathbf{E}(x,t)\,\frac{\partial\phi}{\partial t}(x,t)+\phi(x,t)\,\mathbf{J}(x,t)\right)dxdt=0, \tag{7}$$

for all $\phi\in C_0^1(\Omega)$. Once again, to write this equation we only need the fields $\mathbf{H}$, $\epsilon(x)\mathbf{E}$ and $\mathbf{J}$ be locally integrable over Ω in the Lebesgue sense.

Finally, to write (4) in integral form multiplying by ϕ and integrating yields

$$\int_{\Omega}\left(\phi(x,t)\,\nabla\times\mathbf{E}(x,t)+\phi(x,t)\,\mu(x)\frac{\partial\mathbf{H}}{\partial t}(x,t)\right)dxdt=0,$$

for each $\phi\in C_0^1(\Omega)$, and proceeding as before we obtain

$$\int_{\Omega}\left(\nabla\phi(x,t)\times\mathbf{E}(x,t)+\mu(x)\,\mathbf{H}(x,t)\,\frac{\partial\phi}{\partial t}(x,t)\right)dxdt=0, \tag{8}$$

for all $\phi\in C_0^1(\Omega)$.

We say that *the fields* $\mathbf{E}$ *and* $\mathbf{H}$ *satisfy Maxwell's equations in integral form if the set of Eqs.* (5)–(8) *are satisfied for all test functions* $\phi\in C_0^1(\Omega)$. [3]

We remark that in case $\mathbf{E}$, $\mathbf{H}$, ϵ and μ are all differentiable, and $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell equations in integral form hold, then the integration procedure described above can be clearly reversed and therefore $\mathbf{E}$ and $\mathbf{H}$ satisfy Maxwell equations (1)–(4) in the standard sense.

[3] The term integral form of Maxwell equations is used sometimes to write the equations in terms of surface integrals, see for example [8, Chap. 1, §4], [5, Chap. I, Sect. 3] and [3, Sect. 5.2]. This requires to restrict the fields and coefficients to surfaces which does not make sense in general when they are only Lebesgue integrable, in particular, when they are discontinuous.

3 Changes of Coordinates

We consider transformations

$$T : \Omega^* \to \Omega$$

given by $T(x,t) = (q(x),t) = (q_1(x), q_2(x), q_3(x), t)$; $x = (x_1, x_2, x_3)$; Ω, Ω^* are bounded domains in $\mathbb{R}^4$. We assume that

(a) T is bijective;
(b) T is locally Lipschitz in Ω^*;
(c) $T^{-1} : \Omega \to \Omega^*$, the inverse of T is locally Lipschitz in Ω.

Therefore, the components $q_i(x)$, $i = 1, 2, 3$ are locally Lipschitz continuous and by a theorem of Rademacher, [2, Theorem 2, p. 81], q_i are differentiable except possibly on a set of Lebesgue measure zero. Clearly under these circumstances the functions q_i are continuous but they are allowed to have kinks.

If we let $\psi(x,t) = \phi(q(x),t)$ with $\phi \in C_0^1(\Omega)$, then ψ is differentiable almost everywhere in Ω^* and

$$\frac{\partial \psi}{\partial z_i}(z,t) = \sum_{k=1}^{3} \frac{\partial \phi}{\partial x_k}(q(z),t)\, \frac{\partial q_k}{\partial z_i}(z)$$

for $i = 1, 2, 3$ and for almost every $z \in \Omega^*$. Therefore the matrix

$$J(z) = \begin{pmatrix} \frac{\partial q_1}{\partial z_1} & \frac{\partial q_2}{\partial z_1} & \frac{\partial q_3}{\partial z_1} \\ \frac{\partial q_1}{\partial z_2} & \frac{\partial q_2}{\partial z_2} & \frac{\partial q_3}{\partial z_2} \\ \frac{\partial q_1}{\partial z_3} & \frac{\partial q_2}{\partial z_3} & \frac{\partial q_3}{\partial z_3} \end{pmatrix}$$

is well defined for almost every $z \in \Omega^*$, and we have

$$\nabla \psi(z,t) = J(z)\, \nabla \phi(q(z),t),$$

for a.e. $z \in \Omega^*$.

We recall the following two results (Ω, Ω^* are bounded domains in $\mathbb{R}^4$):

(i) if $f : \Omega^* \to \Omega$ and $g : \Omega^* \to \Omega$ are locally Lipschitz maps with $f(g(x)) = x$ and $g(f(x)) = x$, then

$$Dg(f(x))Df(x) = Id \text{ (3} \times \text{3 identity matrix)} \tag{9}$$

for almost every $x \in \Omega^*$, where D denotes the Jacobian matrix

$$Df(x) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{pmatrix},$$

see [2, Corollary 1, p. 84].

(ii) if $f : \Omega^* \to \Omega$ is a locally Lipschitz map and u is a Lebesgue integrable function in Ω, then

$$\int_{\Omega^*} u(f(x)) \, |\det Df(x)| \, dx = \int_{f(\Omega^*)} u(y) \, N_f(y, \Omega^*) \, dy$$

where $N_f(y, \Omega^*) = \#\{f^{-1}(y) \cap \Omega^*\}$,[4] see [2, Theorem 2, p. 99] or [4, Appendix].

4 Invariance Properties of (5)–(8)

4.1 Invariance of (5) and (6) by Changes of Coordinates

With the results (i) and (ii) from Sect. 3 in hand we change variables in the Eq. (5), and let $x = q(z)$. Since T is bijective and locally Lipschitz, we have $N_T(y, \Omega^*) = 1$ and then

$$\int_{\Omega^*} \nabla\phi(q(z), t) \cdot (\epsilon(q(z))\mathbf{E}(q(z), t)) \; |\det Dq(z)| \, dzdt \\ + \int_{\Omega^*} \rho(q(z), t) \, \phi(q(z), t) \, |\det Dq(z)| \, dzdt = 0. \tag{10}$$

Notice that $J(z) = [(Dq)(z)]^t$ and so $\nabla\psi(z, t) = [(Dq)(z)]^t \, \nabla\phi(q(z), t)$.[5] On the other hand, from (i) above, $(Dq^{-1})(q(z))Dq(z) = Id$ for a.e. z and so $[Dq(z)]^{-1} = (Dq^{-1})(q(z))$, and $\det Dq(z) = \dfrac{1}{\det(Dq^{-1})(q(z))}$. Therefore

$$\nabla\phi(q(z), t) = [Dq(z)]^{-t} \, \nabla\psi(z, t) = \left[(Dq^{-1})(q(z))\right]^t \, \nabla\psi(z, t), \tag{11}$$

[4] $\#E$ denotes the number of elements in the set E if it is finite; and $\#E = +\infty$ when E has an infinite number of elements.

[5] A^t denotes the transpose of the matrix A.

for a.e. $z \in \Omega^*$. Hence the first integral in (10) equals

$$\int_{\Omega^*} \left[(Dq^{-1})(q(z))\right]^t \nabla\psi(z,t) \cdot (\epsilon(q(z))\mathbf{E}(q(z),t)) \; |\det Dq(z)|\, dzdt$$

$$= \int_{\Omega^*} \nabla\psi(z,t) \cdot \left[(Dq^{-1})(q(z))\right] (\epsilon(q(z))\mathbf{E}(q(z),t)) \; |\det Dq(z)|\, dzdt$$

$$= \int_{\Omega^*} \nabla\psi(z,t) \cdot \left[(Dq^{-1})(q(z))\right]$$

$$\left(\epsilon(q(z)) \left[(Dq^{-1})(q(z))\right]^t \left[(Dq^{-1})(q(z))\right]^{-t} \mathbf{E}(q(z),t)\right) \; |\det Dq(z)|\, dzdt.$$

If we set

$$\epsilon'(x) = \frac{\left[(Dq^{-1})(x)\right] \epsilon(x) \left[(Dq^{-1})(x)\right]^t}{|\det(Dq^{-1})(x)|}, \qquad \mathbf{E}'(x,t) = \left[(Dq^{-1})(x)\right]^{-t} \mathbf{E}(x,t),$$

and

$$\rho'(x,t) = \frac{\rho(x,t)}{|\det(Dq^{-1})(x)|},$$

then (10) can be written as

$$\int_{\Omega^*} \nabla\psi(z,t) \cdot \left(\epsilon'(q(z))\, \mathbf{E}'(q(z),t)\right) \, dzdt + \int_{\Omega^*} \rho'(q(z),t)\, \psi(z,t)\, dzdt = 0, \qquad (12)$$

showing the invariance of the notion of generalized solution given by Eq. (5).

Similarly, we obtain that

$$\int_{\Omega^*} \nabla\psi(z,t) \cdot \left(\mu'(q(z))\, \mathbf{H}'(q(z),t)\right) \, dzdt = 0, \qquad (13)$$

with

$$\mu'(x) = \frac{\left[(Dq^{-1})(x)\right] \mu(x) \left[(Dq^{-1})(x)\right]^t}{|\det(Dq^{-1})(x)|}, \quad \text{and} \quad \mathbf{H}'(x,t) = \left[(Dq^{-1})(x)\right]^{-t} \mathbf{H}(x,t).$$

4.2 *Invariance of* (7) *and* (8) *by Changes of Coordinates*

Assuming the change of variables $q(x)$ satisfies the conditions of Sect. 3, with similar calculations we see that the new term to analyze is $\nabla\phi(q(z),t) \times \mathbf{H}(q(z),t)$.

By (11) we have

$$
\begin{aligned}
&\nabla\phi(q(z),t)\times\mathbf{H}(q(z),t)\\
&=\left[(Dq^{-1})(q(z))\right]^t\,\nabla\psi(z,t)\times\mathbf{H}(q(z),t)\\
&=\left[(Dq^{-1})(q(z))\right]^t\,\nabla\psi(z,t)\times\left[(Dq^{-1})(q(z))\right]^t\left[(Dq^{-1})(q(z))\right]^{-t}\mathbf{H}(q(z),t)\\
&=\det\left[(Dq^{-1})(q(z))\right]\,\left[(Dq^{-1})(q(z))\right]^{-1}\,\left(\nabla\psi(x,t)\times\left[(Dq^{-1})(q(z))\right]^{-t}\mathbf{H}(q(z),t)\right),
\end{aligned}
$$

for a.e. $z\in\Omega^*$ and where we have used the formula

$$(Mu)\times(Mv)=(\det M)\;M^{-t}\,(u\times v),$$

valid for any invertible matrix M and any vectors u,v, see [5, p. 120] for a proof. Hence

$$
\begin{aligned}
&\int_\Omega\nabla\phi(x,t)\times\mathbf{H}(x,t)\,dxdt\\
&=\int_{\Omega^*}\nabla\phi(q(z),t)\times\mathbf{H}(q(z),t)\,|\det Dq(z)|\,dzdt\\
&=\int_{\Omega^*}\left[(Dq^{-1})(q(z))\right]^{-1}\,\left(\nabla\psi(x,t)\times\left[(Dq^{-1})(q(z))\right]^{-t}\mathbf{H}(q(z),t)\right)\,\det\\
&\quad\left[(Dq^{-1})(q(z))\right]\,|\det Dq(z)|\,dzdt\\
&=\int_{\Omega^*}\left[(Dq^{-1})(q(z))\right]^{-1}\,\left(\nabla\psi(x,t)\times\left[(Dq^{-1})(q(z))\right]^{-t}\mathbf{H}(q(z),t)\right)\\
&\quad\frac{|\det Dq(z)|}{\det Dq(z)}\,dzdt.
\end{aligned}
$$

Also

$$
\begin{aligned}
&\int_\Omega\epsilon(x)\,\mathbf{E}(x,t)\,\frac{\partial\phi}{\partial t}(x,t)\,dxdt\\
&=\int_{\Omega^*}\epsilon(q(z))\,\mathbf{E}(q(z),t)\,\frac{\partial\phi}{\partial t}(q(z),t)\,|\det Dq(z)|\,dzdt\\
&=\int_{\Omega^*}\epsilon(q(z))\,\mathbf{E}(q(z),t)\,\frac{\partial\psi}{\partial t}(z,t)\,|\det Dq(z)|\,dzdt
\end{aligned}
$$

$$= \int_{\Omega^*} \left[(Dq^{-1})(q(z))\right]^{-1} \left[(Dq^{-1})(q(z))\right] \epsilon(q(z)) \left[(Dq^{-1})(q(z))\right]^t$$

$$\left[(Dq^{-1})(q(z))\right]^{-t} \mathbf{E}(q(z), t) \frac{\partial \psi}{\partial t}(z, t) \, |\det Dq(z)| \, dzdt$$

$$= \int_{\Omega^*} \left[(Dq^{-1})(q(z))\right]^{-1} \epsilon'(q(z)) \, \mathbf{E}'(q(z), t) \frac{\partial \psi}{\partial t}(z, t) \, dzdt.$$

Putting all together we get

$$\int_{\Omega^*} \left[(Dq^{-1})(q(z))\right]^{-1} \left(\nabla\psi(x, t) \times \left[(Dq^{-1})(q(z))\right]^{-t} \mathbf{H}(q(z), t)\right) \frac{|\det Dq(z)|}{\det Dq(z)} \, dzdt$$

$$- \int_{\Omega^*} \left[(Dq^{-1})(q(z))\right]^{-1} \epsilon'(q(z)) \, \mathbf{E}'(q(z), t) \frac{\partial \psi}{\partial t}(z, t) \, dzdt$$

$$+ \int_{\Omega^*} \phi(q(z), t) \, \mathbf{J}(q(z), t) \, |\det Dq(z)| dzdt = 0,$$

which we can write as[6]

$$\int_{\Omega^*} \left[(Dq^{-1})(q(z))\right]^{-1}$$

$$\left\{\nabla\psi(x, t) \times \mathbf{H}'(q(z), t) \frac{|\det Dq(z)|}{\det Dq(z)} - \epsilon'(q(z)) \, \mathbf{E}'(q(z), t) \frac{\partial \psi}{\partial t}(z, t) + \mathbf{J}'(q(z), t)\psi(z, t)\right\}$$

$$dzdt = 0, \tag{14}$$

where

$$\mathbf{H}'(x, t) = \left[(Dq^{-1})(x)\right]^{-t} \mathbf{H}(x, t), \qquad \mathbf{J}'(x, t) = \frac{\mathbf{J}(x, t)}{|\det(Dq^{-1})(x)|}.$$

Finally, to show the invariance of (8) we proceed in the same manner to obtain

$$\int_{\Omega^*} \left[(Dq^{-1})(q(z))\right]^{-1} \tag{15}$$

$$\left\{\nabla\psi(x, t) \times \mathbf{E}'(q(z), t) \frac{|\det Dq(z)|}{\det Dq(z)} + \mu'(q(z)) \, \mathbf{H}'(q(z), t) \frac{\partial \psi}{\partial t}(z, t)\right\} \, dzdt = 0.$$

[6]Notice that if the change of variables preserves orientation, then the coefficient $\frac{|\det Dq(z)|}{\det Dq(z)} = 1$.

4.3 Conclusion

We show form-invariance conditions for Maxwell equations in integral form; they are conditions (12)–(15). The changes of variables are allowed to be non differentiable except on a set of Lebesgue measure zero. This is important in the applications for media having discontinuous refractive indices and so discontinuous fields. As a consequence of these invariance conditions, the result from [9] follows. Indeed, if $\mathbf{E}$, $\mathbf{H}$, ϵ and μ are differentiable, and the change of variables $x = q(x)$ is smooth and preserves orientation, then it follows from the remark made at the end of Sect. 2.1, that the form of Maxwell equations is preserved now with the fields $\mathbf{E}'$, $\mathbf{H}'$, the coefficients ϵ', μ', and ρ' and $\mathbf{J}'$ all defined above.

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Chern-Moser-Weyl Tensor and Embeddings into Hyperquadrics

Xiaojun Huang and Ming Xiao

Dedicated to our friend Dick Wheeden

1 Introduction

A central problem in Mathematics is the classification problem. Given a set of objects and an equivalence relation, loosely speaking, the problem asks how to find an accessible way to tell whether two objects are in the same equivalence class. A general approach to this problem is to find a complete set of (geometric, analytic or algebraic) invariants. In the subject of Several Complex Variables and Complex Geometry, a fundamental problem is to classify complex manifolds or more generally, normal complex spaces under the action of biholomorphic transformations. When the normal complex spaces are open and have strongly pseudo-convex boundary, by the Fefferman-Bochner theorem, one needs only to classify the corresponding boundary strongly pseudoconvex CR manifolds under the application of CR diffeomorphisms. The celebrated Chern-Moser theory is a theory which gives two different constructions of a complete set of invariants for such a classification problem. Among various aspects of the Chern-Moser theory (especially the geometric aspect of the theory), the Chern-Moser-Weyl tensor plays a key role. However, this trace-free tensor is defined in a very complicated manner. This makes it hard to apply in the applications. The majority of first several sections in this article surveys some work done in papers of Chern-Moser [3], Huang-Zhang [14], Huang-Zaitsev [13]. Here, we give a simple and more accessible account on the Chern-Moser-Weyl tensor. We also make an immediate application of the monotonicity property for this tensor to the study of CR embedding problem for the positive signature case.

X. Huang (✉)
Department of Mathematics, Rutgers University, New Brunswick, NJ 08903, USA
e-mail: huangx@math.rutgers.edu

M. Xiao
Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL 61801, USA
e-mail: mingxiao@illinois.edu

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In the last section of this paper, we present new materials. We will show that the family of compact strongly pseudo-convex algebraic hypersurfaces constructed in [15] cannot be locally holomorphically embedded into a sphere of any dimension. The argument is based on the rationality result established in [15] and the Segre geometry associated with such a family. This gives a negative answer to a long standing folklore conjecture concerning the embeddability of compact strongly pseudo-convex algebraic hypersurfaces into a sphere of sufficiently high dimension. For an extensive discussion on the history on the CR embeddability into spheres, we refer the reader to the introduction section of a recent joint paper of the first author with Zaistev [13].

2 Chern-Moser-Weyl Tensor for a Levi Non-degenerate Hypersurface

In this article, we assume that the CR manifolds under consideration are already embedded as hypersurfaces in the complex Euclidean spaces. We first consider the case where the manifolds are even Levi non-degenerate.

We use $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ for the coordinates of $\mathbb{C}^{n+1}$. We always assume that $n \geq 2$, for otherwise the Chern-Moser-Weyl tensor is identically zero. In that setting, one has to consider the Cartan curvature functions instead, which we will not touch in this article.

Let M be a smooth real hypersurface. We say that M is Levi non-degenerate at $p \in M$ with signature $\ell \leq n/2$ if there is a local holomorphic change of coordinates, that maps p to the origin, such that in the new coordinates, M is defined near 0 by an equation of the form:

$$r = v - |z|_\ell^2 + o(|z|^2 + |zu|) = 0 \tag{1}$$

Here, we write $u = \Re w, v = \Im w$ and $< a, \bar{b} >_\ell = -\sum_{j \leq \ell} a_j \bar{b}_j + \sum_{j=\ell+1}^n a_j \bar{b}_j, |z|_\ell^2 =< z, \bar{z} >_\ell$. When $\ell = 0$, we regard $\sum_{j \leq l} a_j \bar{b}_j = 0$.

Assume that M is Levi non-degenerate with the same signature ℓ at any point in M. For a point $p \in M$, a real non-vanishing 1-form θ_p at $p \in M$ is said to be appropriate contact form at p if θ_p annihilates $T_p^{(1,0)} + T_p^{(0,1)}M$ and the Levi form L_{θ_p} associated with θ_p at $p \in M$ has ℓ negative eigenvalues and $n - \ell$ positive eigenvalues. Here we recall the definition of the Levi-form L_{θ_p} at p as follows: We first extend θ_p to a smooth 1-form θ near p such that $\theta|_q$ annihilates $T_q^{(1,0)} + T_q^{(0,1)}M$ at any point $q \approx p$. For any $X_\alpha, X_\beta \in T_p^{(1,0)}$, we define

$$L_{\theta_p}(X_\alpha, X_\beta) := -i < d\theta|_p, X_\alpha \wedge \overline{X}_\beta > . \tag{2}$$

One can easily verify that L_{θ_p} is a well-defined Hermitian form in the tangent space of type $(1, 0)$ of M at p, which is independent of the choice of the extension of the

1-form θ. In the literature, any smooth non-vanishing 1-form θ along M is called a smooth contact form, if $\theta|_q$ annihilates $T_q^{(1,0)}M$ for any $q \in M$. If $\theta|_q$ is appropriate at $q \in M$, we call θ an appropriate smooth contact 1-form along M. Write E_p for the set of appropriate contact 1-forms at p defined above, and E for the disjoint union of E_p. Then two elements in E_p are proportional by a positive constant for the case of $\ell < n/2$; and are proportional by a non zero constant when $\ell = n/2$. There is a natural smooth structure over E which makes E into a R^+ fiber bundle over M when $\ell < n/2$, or a R^*-bundle over M when $\ell = n/2$. When M is defined near 0 by an equation of the form as in (1), then $i\partial r$ is an appropriate contact form of M near 0. In particular, for any appropriate contact 1-form θ_0 at $0 \in M$, there is a constant $c \neq 0$ such that $\theta_0 = ic\partial r|_0$. And $c > 0$ when $\ell < n/2$. Applying further a holomorphic change of coordinates $(z, w) \to (\sqrt{|c|}z, cw)$ and the permutation transformation $(z_1, \cdots, z_n, w) \to (z_n, \cdots, z_1, w)$ if necessary, we can simply have $\theta_0 = i\partial r|_0$. Assign the weight of $z, \overline{z}$ to be 1 and that of u, v, w to be 2. We say $h(z, \overline{z}, u) = o_{wt}(k)$ if $\frac{h(tz, \overline{tz}, t^2u)}{t^k} \to 0$ uniformly on compact sets in (z, u) near the origin. We write $h^{(k)}(z, w)$ for a weighted homogeneous holomorphic polynomial of weighted degree k and $h^{(k)}(z, \overline{z}, u)$ for a weighted homogeneous polynomial of weighted degree k. We first have the following special but crucial case of the Chern-Moser normalization theorem:

Proposition 2.1 *Let $M \subset \mathbb{C}^n \times \mathbb{C}$ be a smooth Levi non-degenerate hypersurface. Let $\theta_p \in E_p$ be an appropriate real 1-form at $p \in M$. Then there is a biholomorphic map F from a neighborhood of p to a neighborhood of 0 such that $F(p) = 0$ and $F(M)$ near 0 is defined by an equation of the following normal form (up to fourth order):*

$$r = v - |z|_\ell^2 + \frac{1}{4}s(z, \overline{z}) + R(z, \overline{z}, u) = v - |z|_\ell^2 + \frac{1}{4}\sum s^0_{\alpha\bar{\beta}\gamma\bar{\delta}} z_\alpha \overline{z}_\beta z_\gamma \overline{z}_\delta + R(z, \overline{z}, u) = 0. \tag{3}$$

Here $s(z, \overline{z}) = \sum s^0_{\alpha\bar{\beta}\gamma\bar{\delta}} z_\alpha \overline{z}_\beta z_\gamma \overline{z}_\delta$, $s^0_{\alpha\bar{\beta}\gamma\bar{\delta}} = s^0_{\gamma\bar{\beta}\alpha\bar{\delta}} = s^0_{\gamma\bar{\delta}\alpha\bar{\beta}}$, $\overline{s^0_{\alpha\bar{\beta}\gamma\bar{\delta}}} = s^0_{\beta\bar{\alpha}\delta\bar{\gamma}}$ *and*

$$\sum_{\alpha,\beta=1}^{n} s^0_{\alpha\bar{\beta}\gamma\bar{\delta}} g_0^{\bar{\beta}\alpha} = 0 \tag{4}$$

where $g_0^{\bar{\beta}\alpha} = 0$ *for* $\beta \neq \alpha$, $g_0^{\bar{\beta}\beta} = 1$ *for* $\beta > \ell$, $g_0^{\bar{\beta}\beta} = -1$ *for* $\beta \leq \ell$. *Also* $R(z, \overline{z}, u) = o_{wt}(|(z, u)|^4) \cap o(|(z, u)|^4)$. *Moreover, we have* $i\partial r|_0 = (F^{-1})^*\theta_p$.

Proof of Proposition 2.1 By what we discussed above, we can assume that $p = 0$ and M near $p = 0$ is defined by an equation of the form as in (1). We first show that we can get rid of all weighted third order degree terms. For this purpose, we choose a transformation of the form $f = z + f^{(2)}(z, w)$ and $g = w + g^{(3)}(z, w)$. Suppose that $F = (f_1, \cdots, f_n, g) = (f, g)$ maps $(M, p = 0)$ to a hypersurface near 0 defined by an equation of the form as in (1) but without weighted degree 3 terms in the right hand side. Substituting F into the new equation and comparing terms of weighted degree three, we get

$$\Im\left(g^{(3)} - 2i < \overline{z}, f^{(2)} >_\ell\right)|_{w=u+i|z|_\ell} = G^{(3)}(z, \overline{z}, u)$$

where $G^{(3)}$ is a certain given real-valued polynomial of weighted degree 3 in $(z, \overline{z}, u)$. Write $G^{(3)}(z, \overline{z}, u) = \Im\{a^{(1)}(z)w + \sum_{j=1}^{n} b_j^{(2)}(z)\overline{z_j}\}$. Choosing $g^{(3)} = a^{(1)}(z)w$ and $f_j^{(2)} = \frac{i}{2} b_j^{(2)}(z)$, it then does our job.

Next, we choose a holomorphic transformation of the form $f = z + f^{(3)}(z, w)$ and $g = w + g^{(4)}(z, w)$ to simplify the weighted degree 4 terms in the defining equation of $(M, p = 0)$. Suppose that M is originally defined by

$$r = v - |z|_\ell^2 + A^{(4)}(z, \overline{z}, u) + o_{wt}(4) = 0$$

and is transformed to an equation of the form:

$$r = v - |z|_\ell^2 + N^{(4)}(z, \overline{z}, u) + o_{wt}(4) = 0.$$

substituting the map F and collecting terms of weighted degree 4, we get the equation:

$$\Im\left(g^{(4)} - 2i < \overline{z}, f^{(3)} >_\ell\right)|_{w=u+i|z|_\ell} = N^{(4)}(z, \overline{z}, u) - A^{(4)}(z, \overline{z}, u).$$

Now, we like to make $N^{(4)}$ as simple as possible by choosing F. Write

$$-A^{(4)} = \Im\{b^{(4)}(z) + b^{(2)}(z)u + b^{(0)}u^2 + \sum_{j=1}^{n} c_j^{(3)}(z)\overline{z_j} + \sum_{|\alpha|=|\beta|=2} \widetilde{c_{\alpha\overline{\beta}}} z^\alpha \overline{z^\beta}\}.$$

Let

$$\begin{aligned} X^{(4)}(z, w) &= b^{(4)}(z) + b^{(2)}(z)w + b^{(0)}w^2, \ -2i\delta_{j\ell} Y_j^{(3)}(z, w) \\ &= c_j^{(3)}(z) - ib^{(2)}(z)z_j - 2ib^{(0)}z_j w, \\ Y^{(3)} &= (Y_1^{(3)}, \cdots, Y_n^{(3)}), \end{aligned}$$

where $\delta_{j\ell}$ is 1 for $j > \ell$ and is -1 otherwise. Then $\Im\left(Y^{(4)} - 2i < \overline{z}, X^{(3)} >_\ell\right) + A^{(4)}(z, \overline{z}, u) = -\Im(b^{(0)})|z|_\ell^4 + \sum_{|\alpha|=|\beta|=2} d_{\alpha\overline{\beta}} z^\alpha \overline{z^\beta}$. By the Fischer decomposition theorem [19], write in the unique way

$$-\Im(b^{(0)})|z|_\ell^4 + \sum_{|\alpha|=|\beta|=2} d_{\alpha\overline{\beta}} z^\alpha \overline{z^\beta} = h^{(2)}(z, \overline{z})|z|_\ell + h^{(4)}(z, \overline{z}).$$

Here $h^{(2)}(z,\overline{z})$ and $h^{(4)}(z,\overline{z})$ are real-valued, bi-homogeneous in $(z,\overline{z})$ and $\Delta_\ell h^{(4)}(z,\overline{z}) = 0$. Here, we write $\triangle_\ell = -\sum_{j\le\ell}\frac{\partial^2}{\partial z_j\partial\overline{z}_j} + \sum_{j=\ell+1}^n \frac{\partial^2}{\partial z_j\partial\overline{z}_j}$. Notice that $h^{(2)}$ has no harmonic terms, we can find $Z^{(1)}(z)$ such that $\Re(<\overline{z}, Z^{(1)}(z)>_\ell) = 0$ and $\Im(2<\overline{z}, Z^{(1)}>) = h^{(2)}(z,\overline{z})$. Finally, if we define $f = z + X^{(4)}(z,w) + Z^{(1)}(z)w$ and $g^{(4)} = w + Y^{(4)}$, then (f,g) maps $(M,0)$ to a hypersurface with $R(z,\overline{z},u) = o_{wt}(4) \cap O(|(z,u)|^3)$. Now suppose that the terms with non-weighted degree of 3 or 4 in R are uniquely written as $ub^{(3)}(z,\overline{z}) + u^2\Im(b^{(1)}(z)) + b^{(0)}u^3 + c^{(0)}u^4$ with $b^{(3)}(z,\overline{z}) = \Im(c^{(3)}(z) + \sum_{|\alpha|=2,|\beta|=1} d_{\alpha\overline{\beta}} z^\alpha \overline{z}^\beta)$. Then we need to make further change of variables as follows to make $R = o_{wt}(4) \cap o(|(z,u)|^4)$ without changing $N^{(4)}(z,\overline{z})$:

$$w' = w + wc^{(3)}(z) + w^2 b^{(1)}(z) + ib^{(0)}w^3 + ic^{(0)}w^4,$$

$$z'_j = z_j + \delta_{j,\ell} w b^{(1)}(z) z_j + \frac{i}{2}\sum_{|\alpha|=2} w d_{\alpha,\overline{j}} z^\alpha + \delta_{j,\ell}\frac{3i}{2} w^2 z_j b^{(0)}.$$

Now, the trace-free condition in (4) is equivalent to the following condition :

$$\triangle_\ell s(z,\overline{z}) \equiv 0.$$

Indeed, this follows from the following fact: Let $\Delta_H = \sum_{l,k=1}^n h^{l\overline{k}}\partial_l\overline{\partial}_k$ with $\overline{h^{l\overline{k}}} = h^{k\overline{l}}$ for any l,k. Then

$$\Delta_H s^0(z,\overline{z}) = 4\sum_{\gamma,\delta=1}^n \sum_{\alpha,\beta=1}^n h^{\alpha\overline{\beta}} s^0_{\alpha\overline{\beta}\gamma\overline{\delta}} z_\gamma \overline{z_\delta}. \tag{5}$$

This proves the proposition. ■

We assume the notation and conclusion in Proposition 2.1. The Chern-Moser-Weyl tensor at p associated with the appropriate 1-form θ_p is defined as the 4th order tensor S_{θ_p} acting over $T_p^{(1,0)}M \otimes T_p^{(0,1)}M \otimes T_p^{(1,0)}M \otimes T_p^{(0,1)}M$. More precisely, for each $X_p, Y_p.Z_p, W_p \in T_p^{(1,0)}M$, we have the following definition:

Let F be the biholomorphic map sending M near p to the normal form as in Proposition 2.1 with $F(p) = 0$, and write $F_*(X_p) = \sum_{j=1}^n a^j \frac{\partial}{\partial z_j}|_0 := X_p^0$, $F_*(Y_p) = \sum_{j=1}^n b^j \frac{\partial}{\partial z_j}|_0 := Y_p^0$, $F_*(Z_p) = \sum_{j=1}^n c^j \frac{\partial}{\partial z_j}|_0 := Z_p^0$, and $F_*(W_p) = \sum_{j=1}^n d^j \frac{\partial}{\partial z_j}|_0 := W_p^0$. Then

$$S_{\theta_p}(X_p, \overline{Y_p}, Z_p, \overline{W_p}) := \sum_{\alpha,\beta,\gamma,\delta=1}^n s^0_{\alpha\overline{\beta}\gamma\overline{\delta}} a^\alpha \overline{b^\beta} c^\gamma \overline{d^\delta}, \text{ which is denoted by } S_{\theta_0}(X_p^0, \overline{Y_p^0}, Z_p^0, \overline{W_p^0}). \tag{6}$$

Since the normalization map F is not unique, we have to verify that the tensor S_{θ_p} is well-defined. Namely, we need to show that it is independent of the choice of the normal coordinates. We do this in the next section. For the rest of this section, we assume this fact and derive some basic properties for the tensor.

For a basis $\{X_\alpha\}_{\alpha=1}^n$ of $T_p^{(1,0)}M$ with $p \in M$, write $(S_{\theta_p})_{\alpha\bar\beta\gamma\bar\delta} = S_{\theta_p}(X_\alpha, \overline{X}_\beta, X_\gamma, \overline{X}_\delta)$. From the definition, we then have the following symmetric properties:

$$(S_{\theta_p})_{\alpha\bar\beta\gamma\bar\delta} = (S_{\theta_p})_{\gamma\bar\beta\alpha\bar\delta} = (S_{\theta_p})_{\gamma\bar\delta\alpha\bar\beta}$$
$$\overline{(S_{\theta_p})_{\alpha\bar\beta\gamma\bar\delta}} = (S_{\theta_p})_{\beta\bar\alpha\delta\bar\gamma},$$

and the following trace-free condition:

$$\sum_{\beta,\alpha=1}^{n} g^{\bar\beta\alpha}(S_{\theta_p})_{\alpha\bar\beta\gamma\bar\delta} = 0. \tag{7}$$

Here

$$g_{\alpha\bar\beta} = L_{\theta|_p}(X_\alpha, X_\beta) := -i < (d\theta)|_p, X_\alpha \wedge \overline{X}_\beta > \tag{8}$$

is the Levi form of M associated with θ_p and θ is a smooth extension of θ_p as a proper contact form of M near p. Also, $(g^{\bar\beta\alpha})$ is the inverse matrix of $(g_{\alpha\bar\beta})$. In the following, we write $\widetilde{\theta} = (F^{-1})^*(\theta)$.

To see the trace-free property in (7), we write that $F_*(X_\alpha) = \sum_{k=1}^n a_\alpha^k \frac{\partial}{\partial z_k}|_0$. Then $g_{\alpha\bar\beta} = L_{\theta_p}(X_\alpha, X_\beta) = -i < (d\theta)|_p, X_\alpha \wedge \overline{X}_\beta >= -i < (dF^*(\widetilde{\theta}))|_p, X_\alpha \wedge \overline{X}_\beta >= -i < (i\bar\partial\partial r|_0, F_*(X_\alpha) \wedge \overline{F_*(X_\beta)} >= (g_0)_{k\bar l} a_\alpha^k \overline{a_\beta^l}$. Here $(g_0)_{k\bar l}$ is defined as before. Write $G = (g_{\alpha\beta}), G^0 = (g_0)_{\alpha\beta}, A = (a_k^l), B = A^{-1} := (b_k^l)$. Then we have the matrix relation: $G = AG^0\overline{A}^t$. Thus $G^{-1} = (\overline{A^t})^{-1}(G^0)^{-1}A^{-1}$, from which we have $g^{\gamma\overline{\beta}} = \overline{b_l^\beta}(g_0)^{j\bar l}b_j^\gamma$. Thus,

$$g^{\alpha\overline{\beta}}S_{\alpha\overline{\beta}\gamma\bar\delta} = \overline{b_l^\beta}(g_0)^{j\bar l}b_j^\alpha s^0_{\tilde k\bar{\tilde j}\tilde l\bar{\tilde m}}\widetilde{a_\alpha^k}\overline{\widetilde{a_\beta^j}}\widetilde{a_\gamma^l}\overline{\widetilde{a_\delta^m}} = (g_0)^{j\bar l}s^0_{\widetilde{j\bar l l \bar m}}\widetilde{a_\gamma^l}\overline{\widetilde{a_\delta^m}} = 0.$$

We should mention the above argument can also be easily adapted to show the biholomorphic invariance of the appropriateness. Namely, if F is a CR diffeomorphism between two Levi non-degenerate hypersurfaces M and $\widetilde{M}$ of signature ℓ. For $\widetilde{\theta_q}$ is an appropriate contact 1-form at $q \in \widetilde{M}$, then $F^*(\widetilde{\theta_q})$ is also an appropriate contact 1-form at $F^{-1}(q) \in M$.

For a smooth vector field X, Y, Z, W of type $(1,0)$ and an appropriate smooth contact form along M, $\mathcal{S}_\theta(X, \overline{Y}, Z, \overline{W})$ is also a smooth function along M. One easy way to see this is to use the Webster-Chern-Moser-Weyl formula obtained in [21] through the curvature tensor of the Webster pseudo-Hermitian metric, whose constructions are done by only applying the algebraic and differentiation operations on the defining function of M. Another more direct way is to trace the dependence of the tensor on the base points under the above normalization procedure.

Assume that $\ell > 0$ and define

$$\mathcal{C}_\ell = \{z \in \mathbb{C}^n : |z|_\ell^2 = 0\}.$$

Then $\mathcal{C}_\ell$ is a real algebraic variety of real codimension 1 in $\mathbb{C}^n$ with the only singularity at 0. For each $p \in M$, write $\mathcal{C}_\ell T_p^{(1,0)}M = \{v_p \in T_p^{(1,0)}M : < (d\theta)|_p, v_p \wedge \bar{v}_p >= 0\}$. Apparently, $\mathcal{C}_l T_p^{(1,0)}M$ is independent of the choice of θ_p. Let F be a CR diffeomorphism from M to M'. We also have $F_*(\mathcal{C}_\ell T_p^{(1,0)}M) = \mathcal{C}_\ell T_{F(p)}^{(1,0)}M'$. Write $\mathcal{C}_\ell T^{(1,0)}M = \coprod_{p\in M} \mathcal{C}_\ell T_p^{(1,0)}M$ with the natural projection π to M. We say that X is a smooth section of $\mathcal{C}_\ell T^{(1,0)}M$ if X is a smooth vector field of type $(1,0)$ along M such that $X|_p \in \mathcal{C}_\ell T_p^{(1,0)}M$ for each $p \in M$. $\mathcal{C}_\ell T^{(1,0)}M$ is a kind of smooth bundle with each fiber isomorphic to $\mathcal{C}_\ell$.

$\mathcal{C}_\ell$ is obviously a uniqueness set for holomorphic functions. The following lemma shows that it is also a uniqueness set for the Chern-Moser-Weyl curvature tensor. (For the proof, see Lemma 2.1 of [14].)

Proposition 2.2 (Huang-Zhang [14]) *(I). Suppose that $H(z,\bar{z})$ is a real real-analytic function in $(z,\bar{z})$ near 0. Assume that $\triangle_\ell H(z,\bar{z}) \equiv 0$ and $H(z,\bar{z})|_{\mathcal{C}_\ell} = 0$. Then $H(z,\bar{z}) \equiv 0$ near 0. (II). Assume the above notation and $\ell > 0$. If $S_{\theta_p}(X,\overline{X},X,\overline{X}) = 0$ for any $X \in \mathcal{C}_\ell T_p^{(1,0)}M$, then $S_{\theta|_p} \equiv 0$.*

3 Transformation Law for the Chern-Moser-Weyl Tensor

We next show that the Chern-Moser-Weyl tensor defined in the previous section is well-defined by proving a transformation law. We follow the approach and expositions developed in Huang-Zhang [14].

Let $\widetilde{M} \subset \mathbb{C}^{N+1} = \{(z,w) \in \mathbb{C}^n \times \mathbb{C}\}$ be also a Levi non-degenerate real hypersurface near 0 of signature $\ell \geq 0$ defined by an equation of the form:

$$\widetilde{r} = \Im\widetilde{w} - |\widetilde{z}|_\ell^2 + o(|\widetilde{z}|^2 + |\widetilde{zu}|) = 0. \tag{9}$$

Let $F := (f_1, \ldots, f_n, \phi, g) : M \to \widetilde{M}$ be a smooth CR diffeomorphism. Then, as in [12] and [1], we can write

$$\begin{aligned} \tilde{z} &= \tilde{f}(z,w) = (f_1(z,w), \ldots, f_n(z,w)) = \lambda z U + \vec{a}w + O(|(z,w)|^2) \\ \tilde{w} &= g(z,w) = \sigma\lambda^2 w + O(|(z,w)|^2). \end{aligned} \tag{10}$$

Here $U \in SU(n,\ell)$. (Namely $< XU, Y\overline{U} >_\ell = < X, Y >_\ell$ for any $X, Y \in \mathbb{C}^n$). Moreover, $\vec{a} \in \mathbb{C}^n$, $\lambda > 0$ and $\sigma = \pm 1$ with $\sigma = 1$ for $\ell < \frac{n}{2}$. When $\sigma = -1$, by considering $F \circ \tau_{n/2}$ instead of F, where $\tau_{\frac{n}{2}}(z_1, \ldots, z_{\frac{n}{2}}, z_{\frac{n}{2}+1}, \ldots, z_n, w) =$

$(z_{\frac{n}{2}+1}, \ldots, z_n, z_1, \ldots, z_{\frac{n}{2}}, -w)$, we can make $\sigma = 1$. Hence, we will assume in what follows that $\sigma = 1$.

Write $r_0 = \frac{1}{2}\Re\{g''_{ww}(0)\}$, $q(\tilde{z}, \tilde{w}) = 1 + 2i < \tilde{z}, \lambda^{-2}\overline{\vec{a}} >_\ell + \lambda^{-4}(r_0 - i|\vec{a}|^2_\ell)\tilde{w}$,

$$T(\tilde{z}, \tilde{w}) = \frac{(\lambda^{-1}(\tilde{z} - \lambda^{-2}\vec{a}\tilde{w})U^{-1}, \lambda^{-2}\tilde{w})}{q(\tilde{z}, \tilde{w})}. \tag{11}$$

Then

$$F^\sharp(z, w) = (\tilde{f}^\sharp, g^\sharp)(z, w) := T \circ F(z, w) = (z, w) + O(|(z, w)|^2) \tag{12}$$

with $\Re\{g^{\sharp''}_{ww}(0)\} = 0$.

Assume that $\widetilde{M}$ is also defined in the Chern-Moser normal form up to the 4th order:

$$\tilde{r} = \Im\tilde{w} - |\tilde{z}|^2_\ell + \frac{1}{4}\tilde{s}(\tilde{z}, \bar{\tilde{z}}) + o_{wt}(|(\tilde{z}, \widetilde{u})|^4) = 0. \tag{13}$$

Then $M^\sharp = T(\widetilde{M})$ is defined by

$$r^\sharp = \Im w^\sharp - |z^\sharp|^2_\ell + \frac{1}{4}s^\sharp(z^\sharp, \bar{z^\sharp}) + o_{wt}(|(z^\sharp, w^\sharp)|^4) = 0 \tag{14}$$

with $s^\sharp(z^\sharp, \bar{z^\sharp}) = \lambda^{-2}\tilde{s}(\lambda z^\sharp U, \lambda\overline{z^\sharp U})$.

One can verify that

$$(-\sum_{j=1}^{\ell} \frac{\partial^2}{\partial z^\sharp_j \partial \bar{z}^\sharp_j} + \sum_{j=\ell+1}^{N} \frac{\partial^2}{\partial z^\sharp_j \partial \bar{z}^\sharp_j}) s^\sharp(z^\sharp, \overline{z^\sharp}) = 0. \tag{15}$$

Therefore (14) is also in the Chern-Moser normal form up to the 4th order. Write $F^\sharp(z, w) = \sum_{k=1}^{\infty} F^{\sharp(k)}(z, w)$. Since $F^\sharp$ maps M into $M^\sharp = T(\widetilde{M})$, we get the following

$$\begin{aligned} &\Im\{\sum_{k\geq 2} g^{\sharp(k+1)}(z, w) - 2i \sum_{k\geq 2} < f^{\sharp(k)}(z, w), \bar{z} >_\ell\} \\ &\quad = \sum_{k_1,\, k_2\geq 2} < f^{\sharp(k_1)}(z, w), \overline{f^{\sharp(k_2)}(z, w)} >_\ell + \frac{1}{4}(s(z, \bar{z}) - s^\sharp(z, \bar{z})) + o_{wt}(4) \end{aligned} \tag{16}$$

over $\Im w = |z|^2_\ell$. Here, we write $F^\sharp(z, w) = (f^\sharp(z, w), g^\sharp(z, w))$.

Collecting terms of weighted degree 3 in (16), we get

$$\Im\{g^{\sharp(3)}(z, w) - 2i < f^{\sharp(2)}(z, w), \bar{z} >_\ell\} = 0 \text{ on } \Im w = |z|^2_\ell.$$

By Huang [12], we get $g^{\sharp(3)} \equiv 0, f^{\sharp(2)} \equiv 0$.

Collecting terms of weighted degree 4 in (16), we get

$$\Im\{g^{\sharp(4)}(z,w) - 2i < f^{\sharp(3)}(z,w), \bar{z} >_\ell\} = \frac{1}{4}(s(z,\bar{z}) - s^\sharp(z,\bar{z})).$$

Similar to the argument in [12] and making use of the fact that $\Re\{\frac{\partial^2 g^{\sharp(4)}}{\partial w^2}(0)\} = 0$, we get the following:

$$\begin{aligned} g^{\sharp(4)} \equiv 0,\ f^{\sharp(3)}(z,w) &= \frac{i}{2}a^{(1)}(z)w, \\ < a^{(1)}(z), \bar{z} >_\ell |z|_\ell^2 = \frac{1}{4}(s(z,\bar{z}) - s^\sharp(z,\bar{z})) &= \frac{1}{4}(s(z,\bar{z}) - \lambda^{-2}\widetilde{s}(\lambda z U, \overline{\lambda z U})). \end{aligned} \tag{17}$$

Since the right hand side of the above equation is annihilated by Δ_ℓ and the left hand side of the above equation is divisible by $|z|_\ell^2$. We conclude that $f^{\sharp(3)}(z,w) = 0$ and

$$s(z,\bar{z}) = \lambda^{-2}\widetilde{s}(\lambda z U, \overline{\lambda z U}). \tag{18}$$

Write $\theta_0 = i\partial r|_0$ and $\widetilde{\theta_0} = i\partial\widetilde{r}|_0$. Then $F^*(\widetilde{\theta_0}) = \lambda^2\theta_0$. For any $X = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j}|_0, F_*(X) = \lambda(z_1\frac{\partial}{\partial \widetilde{z_1}}|_0, \cdots, z_n\frac{\partial}{\partial \widetilde{z_n}}|_0)U$. Under this notation, (19) can be written as

$$S^0_{F^*(\widetilde{\theta_0})}(X,\overline{X},X,\overline{X}) = S^0_{\widetilde{\theta_0}}(F_*(X), \overline{F_*(X)}, F_*(X), \overline{F_*(X)}).$$

This immediately gives the following transformation law and thus the following theorem, too.

$$S^0_{F^*(\widetilde{\theta_0})}(X,\overline{Y},Z,\overline{W}) = S^0_{\widetilde{\theta_0}}(F_*(X), \overline{F_*(Y)}, F_*(Z), \overline{F_*(W)}),\ \text{for } X,Y,Z,W \in T_0^{(1,0)}M. \tag{19}$$

Theorem 3.1 *(1). The Chern-Moser-Weyl tensor defined in the previous section is independent of the choice of the normal coordinates and thus is a well-defined fourth order tensor. (2). Let F be a CR diffeomorphism between two Levi non-degenerate hypersurfaces* $M, M' \subset \mathbb{C}^{n+1}$. *Suppose* $F(p) = q$. *Then, for any appropriate contact 1-form* $\widetilde{\theta_q}$ *of* $\widetilde{M}$ *at q and a vector* $v \in T_p^{(1,0)}M$, *we have the following transformation formula for the corresponding Chern-Moser-Weyl tensor:*

$$\tilde{S}_{\tilde{\theta}_p}(F_*(v_1), \overline{F_*(v_2)}, F_*(v_3), \overline{F_*(v_4)}) = S_{F^*(\tilde{\theta}_q)}(v_1, \overline{v_2}, v_3, \overline{v_4}). \tag{20}$$

Proof Let θ_p be an appropriate contact form of M at p, and let F_1, F_2 be two normalization (up to fourth order) of M at p. Suppose that $F_1(M)$ and $F_2(M)$ are defined near 0 by equations $r_1 = 0$ and $r_2 = 0$ as in (1), respectively. Write

$\Phi = F_2 \circ F_1^{-1}$ and $\theta_0^1 = i\partial r_1$, $\theta_0^2 = i\partial r_2$. We also assume that $F_1^*(\theta_0^1) = \theta_p$ and $F_2^*(\theta_0^2) = \theta_p$. Then for any $X_p, Y_p, Z_p, W_p \in T_p^{(1,0)}M$, we have

$$S^1_{\theta_p}(X_p, \overline{Y_p}, Z_p, \overline{W_p}) = S^1_{\theta_0^1}((F_1)_*(X_p), \overline{(F_1)_*(Y_p)}, (F_1)_*(Z_p), \overline{(F_1)_*(W_p)})$$

if we define the tensor at p by applying F_2. We also have

$$S^2_{\theta_p}(X_p, \overline{Y_p}, Z_p, \overline{W_p}) = S^2_{\theta_0^2}((F_2)_*(X_p), \overline{(F_2)_*(Y_p)}, (F_2)_*(Z_p), \overline{(F_2)_*(W_p)}),$$

if we define the tensor at p by applying F_2. Since $\theta_0^2 = \Phi^*(\theta_0^1)$, and $\Phi_*((F_1)_*(X_p)) = (F_2)_*(X_p)$, by the transformation law obtained in (19), we see the proof in Part I of the theorem. The proof in Part II of the theorem also follows easily from the formula in (19).

4 A Monotonicity Theorem for the Chern-Moser-Weyl Tensor

We now let $M_\ell \subset \mathbb{C}^{n+1}$ be a Levi non-degenerate hypersurface with signature $\ell > 0$ defined in the normal form as in (3). Let $F = (f_1, \cdots, f_N, g)$ be a CR-transversal CR embedding from M_ℓ into $\mathbb{H}_\ell^{N+1}$ with $N \geq n$. Then again as in Sect. 3, a simple linear algebra argument [14] shows that after a holomorphic change of variables, we can make F into the following preliminary normal form:

$$\begin{aligned} \tilde{z} &= \tilde{f}(z,w) = (f_1(z,w), \ldots, f_N(z,w)) = \lambda z U + \vec{a} w + O(|(z,w)|^2) \\ \tilde{w} &= g(z,w) = \sigma\lambda^2 w + O(|(z,w)|^2). \end{aligned} \tag{21}$$

Here U can be extended to an $N \times N$ matrix $\widetilde{U} \in SU(N,\ell)$. Moreover, $\vec{a} \in \mathbb{C}^N$, $\lambda > 0$ and $\sigma = \pm 1$ with $\sigma = 1$ for $\ell < \frac{n}{2}$. When $\sigma = -1$, qs discussed before, by considering $F \circ \tau_{n/2}$ instead of F, where $\tau_{\frac{n}{2}}(z_1, \ldots, z_{\frac{n}{2}}, z_{\frac{n}{2}+1}, \ldots, z_n, w) = (z_{\frac{n}{2}+1}, \ldots, z_n, z_1, \ldots, z_{\frac{n}{2}}, -w)$, we can make $\sigma = 1$. Hence, we will assume that $\sigma = 1$.

Write $r_0 = \frac{1}{2}\Re\{g''_{ww}(0)\}$, $q(\tilde{z}, \tilde{w}) = 1 + 2i < \tilde{z}, \lambda^{-2}\overline{\vec{a}} >_\ell + \lambda^{-4}(r_0 - i|\vec{a}|_\ell^2)\tilde{w}$,

$$T(\tilde{z}, \tilde{w}) = \frac{(\lambda^{-1}(\tilde{z} - \lambda^{-2}\vec{a}\tilde{w})\widetilde{U}^{-1}, \lambda^{-2}\tilde{w})}{q(\tilde{z}, \tilde{w})}. \tag{22}$$

Then

$$F^\sharp(z,w) = (\tilde{f}^\sharp, g^\sharp)(z,w) := T \circ F(z,w) = (z, 0, w) + O(|(z,w)|^2) \tag{23}$$

with $\Re\{g^{\sharp''}_{ww}(0)\} = 0$. Now, $T(\mathbb{H}^{N+1}_\ell) = \mathbb{H}^{N+1}_\ell$. With the same argument as in the previous section, we also arrive at the following:

$$g^{\sharp(3)} = g^{\sharp(4)} \equiv 0,\ f^{\sharp(3)}(z,w) = \frac{i}{2}a^{(1)}(z)w,$$
$$< a^{(1)}(z), \bar{z} >_\ell |z|^2_\ell = |\phi^{\sharp(2)}(z)|^2 + \frac{1}{4}s(z,\bar{z}). \tag{24}$$

In the above equation, if we let z be such that $|z|_\ell = 0$, we see that $s(z,\bar{z}) \le 0$. Now, if F is not CR transversal but not totally non-degenerate in the sense that F does not map an open subset of $\mathbb{C}^n$ into $\mathbb{H}^N_\ell$ (see [14]), then one can apply this result on a dense open subset of M [2] where F is CR transversal and then take a limit as did in [14]. Then we have the following special case of the monotonicity theorem for the Chern-Moser-Weyl tensor obtained in Huang-Zhang [14]:

Theorem 4.1 ([14]) *Let $M_\ell \subset \mathbb{C}^{n+1}$ be a Levi non-degenerate real hypersurface of signature ℓ. Suppose that F is a holomorphic mapping defined in a (connected) open neighborhood U of M in $\mathbf{C}^{n+1}$ that sends M_ℓ into $\mathbf{H}^{N+1}_\ell \subset \mathbb{C}^{N+1}$. Assume that $F(U) \not\subset \mathbf{H}^{N+1}_\ell$. Then when $\ell < \frac{n}{2}$, the Chern-Moser-Weyl curvature tensor with respect to any appropriate contact form θ is pseudo semi-negative in the sense that for any $p \in M$, the following holds:*

$$\mathcal{S}_{\theta|_p}(v_p, \overline{v_p}, v_p, \overline{v_p}) \le 0,\ \ for\ v_p \in \mathcal{C}_\ell T^{(1,0)}_p M. \tag{25}$$

When $\ell = \frac{n}{2}$, along a certain contact form θ, $\mathcal{S}_\theta$ is pseudo negative.

5 Counter-Examples to the Embeddability Problem for Compact Algebraic Levi Non-degenerate Hypersurfaces with Positive Signature into Hyperquadrics

In this section, we apply Theorem 4.1 to construct a compact Levi-nondegenerate hypersurface in a projective space, for which any piece of it can not be holomorphically embedded into a hyperquadric of any dimension with the same signature. This section is based on the work in the last section of Huang-Zaitsev [13].

Let n, ℓ be two integers with $1 < \ell \le n/2$. For any ϵ, define

$$M_\epsilon := \left\{ [z_0, \cdots, z_{n+1}] \in \mathbb{P}^{n+1} : |z|^2 \left(-\sum_{j=0}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2 \right) + \epsilon \left(|z_1|^4 - |z_{n+1}|^4 \right) = 0 \right\}.$$

Here $|z|^2 = \sum_{j=0}^{n+1} |z_j|^2$ as usual. For $\epsilon = 0$, M_ϵ reduces to the generalized sphere with signature ℓ, which is the boundary of the generalized ball

$$\mathbb{B}_\ell^{n+1} := \left\{ \{[z_0, \cdots, z_{n+1}] \in \mathbb{P}^{n+1} : -\sum_{j=0}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2 < 0 \right\}.$$

The boundary $\partial\mathbb{B}_\ell^{n+1}$ is locally holomorphically equivalent to the hyperquadric $\mathbb{H}_\ell^{n+1} \subset \mathbb{C}^{n+1}$ of signature ℓ defined by $\Im z_{n+1} = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n+1} |z_j|^2$, where $(z_1, \cdots, z_{n+1})$ is the coordinates of $\mathbb{C}^{n+1}$.

For $0 < \epsilon << 1$, M_ϵ is a compact smooth real-algebraic hypersurface with Levi form non-degenerate of the same signature ℓ.

Theorem 5.1 ([13]) *There is an $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$, the following holds: (i) M_ϵ is a smooth real-algebraic hypersurface in $\mathbb{P}^{n+1}$ with non-degenerate Levi form of signature ℓ at every point. (ii) There does not exist any holomorphic embedding from any open piece of M_ℓ into $\mathbb{H}_\ell^{N+1}$.*

When $0 < \epsilon << 1$, since M_ϵ is a small algebraic deformation of the generalized sphere, we see that M_ϵ must also be a compact real-algebraic Levi non-degenerate hypersurface in $\mathbb{P}^{n+1}$ with signature ℓ diffeomorphic to the generalized sphere which is the boundary of the generalized ball $\mathbb{B}_\ell^{n+1} \subset \mathbb{P}^{n+1}$.

Proof of Theorem 5.1 The proof uses the following algebraicity of the first author:

Theorem 5.2 ([11], Corollary in Sect. 2.3.5) *Let $M_1 \subset \mathbb{C}^n$ and $M_2 \subset \mathbb{C}^N$ with $N \geq n \geq 2$ be two Levi non-degenerate real-algebraic hypersurfaces. Let $p \in M_1$ and U_p be a small connected open neighborhood of p in $\mathbf{C}^n$ and F be a holomorphic map from U_p into $\mathbf{C}^N$ such that $F(U_p \cap M_1) \subset M_2$ and $F(U_p) \not\subset M_2$. Suppose that M_1 and M_2 have the same signature ℓ at p and $F(p)$, respectively. Then F is algebraic in the sense that each component of F satisfies a nontrivial holomorphic polynomial equation.*

Next, we compute the Chern-Moser-Weyl tensor of M_ϵ at the point

$$P_0 := [\xi_0^0, \cdots, \xi_{n+1}^0], \quad \xi_j^0 = 0 \text{ for } j \neq 0, \ell+1, \quad \xi_0^0 = 1, \quad \xi_{\ell+1}^0 = 1,$$

and consider the coordinates

$$\xi_0 = 1, \quad \xi_j = \frac{\eta_j}{1+\sigma}, \quad j = 1, \cdots, \ell, \quad \xi_{\ell+1} = \frac{1-\sigma}{1+\sigma}, \quad \xi_{j+1} = \frac{\eta_j}{1+\sigma},$$
$$j = \ell+1, \cdots, n.$$

Then in the (η, σ)-coordinates, P_0 becomes the origin and M_ϵ is defined near the origin by an equation in the form:

$$\rho = -4\Re\sigma - \sum_{j=1}^{\ell} |\eta_j|^2 + \sum_{j=\ell+1}^{n} |\eta_j|^2 + a(|\eta_1|^4 - |\eta_n|^4) + o(|\eta|^4) = 0, \tag{26}$$

for some $a > 0$. Now, let $Q(\eta, \overline{\eta}) = -a(|\eta_1|^4 - |\eta_n|^4)$ and make a standard ℓ-harmonic decomposition [19]:

$$Q(\eta, \overline{\eta}) = N^{(2,2)}(\eta, \overline{\eta}) + A^{(1,1)}(\eta, \overline{\eta})|\eta|_\ell^2. \tag{27}$$

Here $N^{(2,2)}(\eta, \eta)$ is a $(2, 2)$-homogeneous polynomial in $(\eta, \overline{\eta})$ such that $\Delta_\ell N^{(2,2)}(\eta, \overline{\eta}) = 0$ with Δ_ℓ as before. Now $N^{(2,2)}$ is the Chern-Moser-Weyl tensor of M_ϵ at 0 (with respect to an obvious contact form) with $N^{(2,2)}(\eta, \overline{\eta}) = Q(\eta, \overline{\eta})$ for any $\eta \in \mathcal{C}T_0^{(1,0)} M_e$. Now the value of the Chern-Moser-Weyl tension has negative and positive value at $X_1 = \frac{\partial}{\partial \eta_1} + \frac{\partial}{\partial \eta_{\ell+1}}|_0$ and $X_2 = \frac{\partial}{\partial \eta_2} + \frac{\partial}{\partial \eta_n}|_0$, respectively. If $\ell > 1$, then both X_1 and X_2 are in $\mathcal{C}T_0^{(1,0)} M_e$. We see that the Chern-Moser-Weyl tensor can not be pseudo semi-definite near the origin in such a coordinate system.

Next, suppose an open piece U of M_ϵ can be holomorphically and transversally embedded into the $\mathbf{H}_\ell^{N+1}$ for $N > n$ by F. Then by the algebraicity result in Theorem 5.2, F is algebraic. Since the branching points of F and the points where F is not defined (poles or points of indeterminacy of F) are contained in a complex-algebraic variety of codimension at most one, F extends holomorphically along a smooth curve γ starting from some point in U and ending up at some point $p^*(\approx 0) \in M_\ell$ in the (η, σ)-space where the Chern-Moser-Weyl tensor of M_ϵ is not pseudo-semi-definite. By the uniqueness of real-analytic functions, the extension of F must also map an open piece of p^* into $\mathbf{H}_\ell^{N+1}$. The extension is not totally degenerate. By Theorem 4.1, we get a contradiction. ■

6 Non-embeddability of Compact Strongly Pseudo-Convex Real Algebraic Hypersurfaces into Spheres

As discussed in the previous sections, spheres serve as the model of strongly pseudoconvex real hypersurfaces where the Chern-Moser-Weyl tensor vanishes. An immediate application of the invariant property for the Chern-Moser-Weyl tensor is that very rare strongly pseudoconvex real hypersurfaces can be biholomorphically mapped to a unit sphere. Motivated by various embedding theorems in geometries (Nash embedding, Remmert embedding theorems, etc), a natural question to pursue in Several Complex Variables is to determine when a real hypersurface in $\mathbb{C}^n$ can be holomorphically embedded into the unit sphere $\mathbb{S}^{2N-1} = \{Z \in \mathbb{C}^N : ||Z||^2 = 1\}$.

By a holomorphic embedding of $M \subset \mathbb{C}^n$ into $M' \subset \mathbb{C}^N$, we mean a holomorphic embedding of an open neighborhood U of M into a neighborhood U' of M', sending M into M'. We also say M is locally holomorphically embeddable into M' at $p \in M$, if there is a neighborhood V of p and a holomorphic embedding $F : V \to \mathbb{C}^N$ sending $M \cap V$ into M'.

A real hypersurface holomorphically embeddable into a sphere is necessarily strongly pseudoconvex and real-analytic. However, due to results by Forstnerić [9] (See a recent work [10] for further result) and Faran [7], not every strongly pseudoconvex real-analytic hypersurface can be embedded into a sphere. Explicit examples of non-embeddable strongly pseudoconvex real-analytic hypersurfaces constructed much later in [23]. Despite a vast of literature devoted to the embeddability problem, the following question remains an open question of long standing. Here recall a smooth real hypersurface in an open subset U of $\mathbb{C}^n$ is called real-algebraic, if it has a real-valued polynomial defining function.

Question 6.1 *Is every compact real-algebraic strongly pseudoconvex real hypersurface in $\mathbb{C}^n$ holomorphically embeddable into a sphere of sufficiently large dimension?*

Part of the motivation to study this embeddability problem is a well-known result due to Webster [22] which states that every real-algebraic Levi-nondegenerate hypersurface admits a transversal holomorphic embedding into a non-degenerate hyperquadric in sufficiently large complex space. (See also [17] for further study along this line.) Notice that in [13], the authors showed that there are many compact real-algebraic pseudoconvex real hypersurfaces with just one weakly pseudoconvex point satisfying the following property: Any open piece of them cannot be holomorphically embedded into any compact real-algebraic strongly pseudoconvex hypersurfaces which, in particular, includes spheres. Many other related results can be found in the work of Ebenfelt-Son [6], Fornaess [8], etc.

In [15], the authors constructed the following family of compact real-algebraic strongly pseudoconvex real hypersurfaces:

$$M_\epsilon = \{(z, w) \in \mathbb{C}^2 : \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \epsilon|z|^2 - 1 = 0\}, \quad 0 < \epsilon < 1. \tag{28}$$

Here, $2 < c < \frac{16}{7}$, $\varepsilon_0 > 0$ is a sufficiently small number such that M_ε is smooth for all $0 \le \epsilon < 1$. An easy computation shows that for any $0 < \epsilon < 1, M_\epsilon$ is strongly pseudoconvex. M_ϵ is indeed a small algebraic deformation of the boundary of the famous Kohn-Nirenberg domain [16]. It is shown in [15] that for any integer N, there exists a small number $0 < \epsilon(N) < 1$, such that for any $0 < \epsilon < \epsilon(N)$, M_ϵ cannot be locally holomorphically embedded into the unit sphere $\mathbb{S}^{2N-1}$ in $\mathbb{C}^N$. More precisely, any holomorphic map sending an open piece of M_ϵ to $\mathbb{S}^{2N-1}$ must be a constant map. We will write

$$\rho_\epsilon = \rho_\epsilon(z, w, \overline{z}, \overline{w}) := \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |w|^2 + |z|^{10} + \epsilon|z|^2 - 1.$$

We first fix some notations. Let $M \subset \mathbb{C}^n$ be a real-algebraic subset defined by a family of real-valued polynomials $\{\rho_\alpha(Z, \overline{Z}) = 0\}$, where Z is the coordinates of $\mathbb{C}^n$. Then the complexification $\mathcal{M}$ of M is the complex-algebraic subset in $\mathbb{C}^n \times \mathbb{C}^n$ defined by $\rho_\alpha(Z, W) = 0$ for each α, $(Z, W) \in \mathbb{C}^n \times \mathbb{C}^n$. Then for $p \in \mathbb{C}^n$, the Segre variety of M associated with the point p is defined by $Q_p := \{Z \in \mathbb{C}^n : (Z, \overline{p}) \in \mathcal{M}\}$. The geometry of Segre varieties of a real-analytic hypersurface has been used in many literatures since the work of Segre [18] and Webster [20].

In this note, fundamentally based on our previous joint work with Li [15], we show that M_ϵ cannot be locally holomorphically embedded into any unit sphere. The other important observation we need is the fact that for some $p \in M_\epsilon$, the associated Segre variety Q_p cuts M_ϵ along a one dimensional real analytic subvariety inside M_ϵ. The geometry related to intersection of the Segre variety with the boundary plays an important role in the study of many problems in Several Complex Variables. (We mention, in particular, the work of D'Angelo-Putinar [5], Huang-Zaitsev [13]).

This then provides a counter-example to a long standing open question—Question 6.1. (See [13] for more discussions on this matter).

Theorem 6.2 *There exist compact real-algebraic strongly pseudoconvex real hypersurfaces in* $\mathbb{C}^2$*, diffeomorphic to the sphere, that are not locally holomorphically embeddable into any sphere. In particular, for sufficiently small positive* $\varepsilon_0, \epsilon, M_\epsilon$ *cannot be locally holomorphically embedded into any sphere. More precisely, a local holomorphic map sending an open piece of* M_ϵ *to a unit sphere must be a constant map.*

Write $D_\epsilon = \{\rho_\epsilon < 0\}$ as the interior domain enclosed by M_ϵ. Since M_ϵ is a small smooth deformation of $\{|z|^{10} + |w|^2 = 1\}$ for small ε_0 and ϵ. This implies M_ϵ is diffeomorphic to the unit sphere $\mathbb{S}^3$ for sufficiently small ε_0 and ϵ. Consequently, M_ϵ separates $\mathbb{C}^2$ into two connected components D_ϵ and $\mathbb{C}^2 \setminus \overline{D_\epsilon}$.

Proposition 6.3 *Let* $p_0 = (0, 1) \in M_\epsilon$*. Let* Q_{p_0} *be the Segre variety of* M_ϵ *associated to* p_0*. There exists* $\widetilde{\epsilon} > 0$ *such that for each* $0 < \epsilon < \widetilde{\epsilon}$*,* $Q_{p_0} \cap M_\epsilon$ *is a real analytic subvariety of dimension one.*

Proof of Proposition 6.3 It suffices to show that there exists $q \in Q_{p_0}$ such that $q \in D_\epsilon$. Note that $Q_{p_0} = \{(z, w) : w = 1\}$. Set

$$\psi(z, \epsilon) = \varepsilon_0(|z|^8 + c\mathrm{Re}|z|^2 z^6) + |z|^{10} + \epsilon|z|^2,\ 0 \leq \epsilon < 1.$$

Note $q = (\mu_0, 1) \in D_\epsilon$ if and only if $\psi(\mu_0, \epsilon) < 0$. Now, set $\phi(\lambda, \epsilon) = \varepsilon_0\lambda^8(1 - c) + \lambda^{10} + \epsilon\lambda^2, 0 \leq \epsilon < 1$. First we note there exists small $\lambda' > 0$, such that $\phi(\lambda', 0) < 0$. Consequently, we can find $\widetilde{\epsilon} > 0$ such that for each $0 < \epsilon \leq \widetilde{\epsilon}, \phi(\lambda', \epsilon) < 0$. Write $\mu_0 = \lambda' e^{i\frac{\pi}{6}}$. It is easily to see that $\psi(\mu_0, \epsilon) < 0$ if $0 < \epsilon \leq \widetilde{\epsilon}$. This establishes Proposition 6.3. ■

Proposition 6.4 *Let* $M := \{Z \in \mathbb{C}^n : \rho(Z, \overline{Z}) = 0\}, n \geq 2$*, be a compact, connected, strongly pseudo-convex real-algebraic hypersurface. Assume that there exists a point* $p \in M$ *such that the associated Segre variety* Q_p *of* M *is irreducible* *and Q_p intersects M at infinitely many points. Let F be a holomorphic rational map sending an open piece of M to the unit sphere $\mathbb{S}^{2N-1}$ in some $\mathbb{C}^N$. Then F is a constant map.*

Proof of Proposition 6.4 Let D be the interior domain enclosed by M. From the assumption and a theorem of Chiappari [4], we know F is holomorphic in a neighborhood U of $\overline{D}$ and sends M to $\mathbb{S}^{2N-1}$. Consequently, if we write $\mathcal{S}$ as the singular set of F, then it does not intersect U. Write Q'_q for the Segre variety of $\mathbb{S}^{2N-1}$ associated to $q \in \mathbb{C}^N$. We first conclude by complexification that for a small neighborhood V of p,

$$F(Q_p \cap V) \subset Q'_{F(p)}. \tag{29}$$

Note that $\mathcal{S} \cap Q_p$ is a Zariski close proper subset of Q_p. Notice that Q_p is connected as it is irreducible. We conclude by unique continuation that if $\widetilde{p} \in Q_p$ and F is holomorphic at $\widetilde{p}$, then $F(\widetilde{p}) \in Q'_{F(p)}$. In particular, if $\widetilde{p} \in Q_p \cap M$, then $F(\widetilde{p}) \in Q'_{F(p)} \cap \mathbb{S}^{2N-1} = \{F(p)\}$. That is, $F(\widetilde{p}) = F(p)$.

Notice by assumption that $Q_p \cap M$ is a compact set and contains infinitely many points. Let $\hat{p}$ be an accumulation point of $Q_p \cap M$. Clearly, by what we argued above, F is not one-to-one in any neighborhood of $\hat{p}$. This shows that F is constant. Indeed, suppose F is not a constant map. We then conclude that F is a holomorphic embedding near $\hat{p}$ by a standard Hopf lemma type argument (see [11], for instance) for both M_ϵ and $\mathbb{S}^{2N-1}$ are strongly pseudo-convex. This completes the proof of Proposition 6.4. ■

Proof of Theorem 6.2 Pick $p_0 = (0, 1) \in M_\epsilon$. Notice that the associated Segre variety $Q_{p_0} = \{(z, 1) : z \in \mathbb{C}\}$ is an irreducible complex variety in $\mathbb{C}^2$. Let ϵ, ε_0 be sufficiently small such that Proposition 6.3 holds.

Now, let F be a holomorphic map defined in a small neighborhood U of some point $q \in M_\epsilon$ that sends an open piece of M_ϵ into $\mathbb{S}^{2N-1}, N \in \mathbb{N}$. It is shown in [15] that F is a rational map. Then it follows from Proposition 6.4 that F is a constant map. We have thus established Theorem 6.2. ■

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The Focusing Energy-Critical Wave Equation

Carlos Kenig

To Dick Wheeden with friendship and appreciation.

Abstract We survey recent results related to soliton resolution.

Classification: 35L15

Since the 1970s there has been a widely held belief that "coherent structures" describe the long-time asymptotic behavior of general solutions to nonlinear hyperbolic/dispersive equations.

This belief has come to be known as the soliton resolution conjecture. This is one of the grand challenges in partial differential equations. Loosely speaking, this conjecture says that the long-time evolution of a general solution of most hyperbolic/dispersive equations, asymptotically in time decouples into a sum of modulated solitons (traveling wave solutions) and a free radiation term (linear solution) which disperses to 0.

This is a beautiful, remarkable conjecture which postulates a "simplification" of the very complicated dynamics into a superposition of simple "nonlinear objects," namely traveling wave solutions, and radiation, a linear object.

Until recently, the only cases in which these asymptotics had been proved was for integrable equations (which reduce the nonlinear problem to a collection of linear ones) and in perturbative regimes.

In 2012–2013, Duyckaerts–Kenig–Merle [14, 15] broke the impasse by establishing the desired asymptotic decomposition for radial solutions of the energy critical wave equation in three space dimensions, first for a well-chosen sequence of times, and then for general times.

C. Kenig (✉)
Department of Mathematics, University of Chicago, Chicago, IL 60637, USA
e-mail: cek@math.uchicago.edu

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This is the equation

$$\begin{cases} \partial_t^2 u - \Delta u - |u|^{\frac{4}{N-2}} u = 0, & (x,t) \in \mathbb{R}^N \times I \\ u|_{t=0} = u_0 \in \dot{H}^1, & \partial_t u|_{t=0} = u_1 \in L^2, \end{cases} \tag{NLW}$$

$N = 3, 4, 5, 6 \ldots$ Here, I is an interval, $0 \in I$.

In this problem, small data yield global solutions which "scatter," while for large data, we have solutions $u \in C(I; \dot{H}^1 \times L^2)$, with a maximal interval of existence $(T_-(u), T_+(u))$ and $u \in L^{\frac{2(N+1)}{N-2}}(\mathbb{R}^N \times I')$ for each $I' \Subset I$.

The energy norm is "critical" since for all $\lambda > 0$, $u_\lambda(x,t) := \lambda^{-\frac{N-2}{2}} u(x/\lambda, t/\lambda)$ is also a solution and $\|(u_{0,\lambda}, u_{1,\lambda})\|_{\dot{H}^1 \times L^2} = \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}$. The equation is focusing, the conserved energy is

$$E(u_0, u_1) = \frac{1}{2} \int |\nabla u_0|^2 + |u_1|^2 dx - \frac{N-2}{2N} \int |u_0|^{\frac{2N}{N-2}} dx.$$

It is easy to construct solutions which blow-up in finite time say at $T = 1$, by considering the ODE. For instance, when $N = 3$, $u(x,t) = \left(\frac{3}{4}\right)^{1/4} (1-t)^{-1/2}$ is a solution, and using finite speed of propagation it is then easy to construct solutions with $T_+ = 1$, such that $\lim_{t \uparrow T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} = \infty$. This is called type I blow-up. There exist also type II blow-up solutions, i.e. solutions for which $T_+ < \infty$, and $\sup_{0<t<T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty$. Here the break-down occurs by "concentration." The existence of such solutions is a typical feature of energy critical problems.

The first example of such solutions (radial) were constructed for $N = 3$ by Krieger–Schlag–Tataru [27], then for $N = 4$ by Hillairet–Raphael [19], and recently by Jendrej [20] for $N = 5$.

For this equation one expects soliton resolution for type II solutions, i.e. solutions such that $\sup_{0<t<T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} < \infty$, where T_+ may be finite or infinite.

Some examples of type II solutions when $T_+ = \infty$ are scattering solutions that is:

Definition 1 A scattering solution is a solution such that $T_+ = \infty$, and $\exists (u_0^+, u_1^+) \in \dot{H}^1 \times L^2$, such that

$$\lim_{t \to \infty} \left\| (u(t), \partial_t u(t)) - \left(S(t)(u_0^+, u_1^+), \partial_t S(t)(u_0^+, u_1^+) \right) \right\|_{\dot{H}^1 \times L^2} = 0,$$

where $S(t)(u_0^+, u_1^+)$ is the solution to the associated linear equation with data (u_0^+, u_1^+).

For example, for (u_0, u_1) small in $\dot{H}^1 \times L^2$, we have a scattering solution.

Other examples of type II solutions of (NLW) with $T_+ = \infty$ are the stationary solutions, that is the solutions $Q \neq 0$ of the elliptic equation $\Delta Q + |Q|^{4/(N-2)} Q = 0$, $Q \in \dot{H}^1$ (we say $Q \in \Sigma$).

For example,

$$W(x) = \left(1 + \frac{|x|^2}{N(N-2)}\right)^{-\frac{N-2}{2}}$$

is such a solution. These stationary solutions do not scatter (if u scatters then $\int_{|x|<1} |\nabla_{x,t} u(x,t)|^2 dx \to 0$ as $t \to \infty$). W has several important characterizations: up to sign and scaling it is the only radial, non-zero solution. Up to translation and scaling it is also the only positive solution.

However, there is a continuum of variable sign, non-radial $Q \in \Sigma$ [8–10]. W also has a variational characterization as the extremizer for the Sobolev embedding $\|f\|_{L^{\frac{2N}{N-2}}} \leq C_N \|\nabla f\|_{L^2}$. It is referred to as the "ground state."

In 2008, Kenig–Merle [23] established the following "ground state conjecture" for (NLW). For u a solution of (NLW) with $E(u_0, u_1) < E(W, 0)$, the following dichotomy holds: if $\|\nabla u_0\| < \|\nabla W\|$ then $T_+ = \infty$, $T_- = -\infty$, and u scatters in both time directions, while if $\|\nabla u_0\| > \|\nabla W\|$, then $T_+ < \infty$ and $T_- > -\infty$. The case $\|\nabla u_0\| = \|\nabla W\|$ is vacuous because of variational considerations. The threshold case $E(u_0, u_1) = E(W, 0)$ was completely described by Duyckaerts–Merle [11] in an important work.

The proof of the "ground state conjecture" was obtained through the "concentration-compactness/rigidity theorem" method, introduced by Kenig–Merle for this purpose, which has since become the standard tool to understand the global in time behavior of solutions, below the ground-state threshold, for critical dispersive problems.

Other non-scattering solutions, with $T_+ = \infty$, are the traveling wave solutions. They are obtained as Lorentz transforms of $Q \in \Sigma$. Let $\vec{\ell} \in \mathbb{R}^N$, $|\vec{\ell}| < 1$. Then,

$$\begin{aligned} Q_{\vec{\ell}}(x,t) &= Q_{\vec{\ell}}(x - t\vec{\ell}, 0) \\ &= Q\left(\left[\frac{-t}{\sqrt{1-|\vec{\ell}|^2}} + \frac{1}{|\vec{\ell}|^2}\left(\frac{1}{\sqrt{1-|\vec{\ell}|^2}} - 1\right)\vec{\ell}\cdot x\right]\vec{\ell} + x\right) \end{aligned}$$

is a traveling wave solution of (NLW).

When Kenig–Merle introduced the "concentration-compactness/rigidity theorem" method to study critical dispersive problems, the ultimate goal was to establish the soliton resolution conjecture.

As I said earlier, for (NLW) one expects to have soliton resolution for type II solutions. Thus, if u is a type II solution, one would want to show that:

$\exists J \in \mathbb{N} \cup \{0\}$, Q_j, $j = 1, \ldots, J$, $Q_j \in \Sigma$, $\vec{\ell}_j \in \mathbb{R}^N$, $|\vec{\ell}_j| < 1$, $1 \leq j \leq N$, such that, if $t_n \uparrow T_+$ (which may be finite or infinite), there exist $\lambda_{j,n} > 0$, $x_{j,n} \in \mathbb{R}^N$, $j = 1, \ldots, J$, with $\frac{\lambda_{j,n}}{\lambda_{j',n}} + \frac{\lambda_{j',n}}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{j',n}|}{\lambda_{j,n}} \to_n \infty$ for $j \neq j'$ (orthogonality of the parameters) and a linear solution $v_L(x,t)$ (the radiation term) such that

$$\begin{aligned}
&(u(t_n), \partial_t u(t_n)) \\
&\quad = \sum_{j=1}^{J} \left(\frac{1}{\lambda_{j,n}^{(N-2)/2}} Q^j_{\vec{\ell}_j} \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, 0 \right), \frac{1}{\lambda_{j,n}^{N/2}} \partial_t Q^j_{\vec{\ell}_j} \left(\frac{x - x_{j,n}}{\lambda_{j,n}}, 0 \right) \right) \\
&\quad + (v_L(x, t_n), \partial_t v_L(x, t_n)) + o_n(1)
\end{aligned}$$

as $n \to \infty$.

This has been proven in the radial case, $N = 3$ (DKM 12', 13') [14, 15], and in the general case, $N = 3, 5$ when $T_+ < \infty$ and u is "close" to W, (DKM 12') [13]. The "orthogonality of parameters" in such decompositions shows that the profiles are "decoupled." Such orthogonality conditions originate, in the elliptic setting in the work of Brézis–Coron [2], see also Gérard [18]. For dispersive settings they originate in work of Bahouri–Gérard [1] and Merle–Vega [28].

Let me discuss now the radial results. In DKM 12' [14], the decomposition was proved for a well-chosen sequence of times $\{t_n\}_n$, while in DKM 13' [15] it was proven for any sequence of times $\{t_n\}_n$.

Let me first quickly describe the proof of the 13' result. The key new idea was the use of the "channel of energy" method introduced by DKM, which was used to quantify the ejection of energy as we approach the final time of existence T_+ [12].

The main new fact shown was that if u is a radial, type II, non-scattering solution, not a rescaled W, $\exists r_0 > 0$, $\eta > 0$, and a small (in $\dot{H}^1 \times L^2$ norm) radial global solution $\tilde{u}$, with $u(r,t) = \tilde{u}(r,t)$, for $r \geq r_0 + |t|$, $t \in I_{max}(u)$, such that $\forall t \geq 0$ or $\forall t \leq 0$,

$$\int_{|x| \geq |t| + r_0} |\nabla_{x,t} \tilde{u}(x,t)|^2 dx \geq \eta.$$

The key tool for proving this is what I like to call "outer energy lower bounds," which are valid for solutions of the linear wave equation. Let $N = 3$, for $r_0 > 0$, $P_{r_0} = \left\{ (ar^{-1}, 0) : a \in \mathbb{R}, r \geq r_0 \right\}$ be a subspace of $\dot{H}^1 \times L^2(r \geq r_0)$. Let $\pi^\perp_{r_0}$ be the orthogonal projection onto the orthogonal complement of P_{r_0}.

Then: for v a radial solution of the linear wave equation, $\forall t \geq 0$ or $\forall t \leq 0$, we have [12]

$$\int_{|x|\geq|t|+r_0} |\nabla_{x,t} v|^2 \geq c \left\| \pi_{r_0}^{\perp}(v_0, v_1) \right\|^2_{\dot{H}^1 \times L^2(r \geq r_0)}. \tag{1}$$

We remark that $\|\pi_{r_0}^{\perp}(v_0, v_1)\|^2_{\dot{H}^1\times L^2(r\geq r_0)} = \int_{r_0}^{\infty} (\partial_r(rv_0))^2 + r^2 v_1^2 \, dr$ (see [24]).

In the non-radial case, we have for $N = 3, 5, 7, \ldots$ for v a solution of the linear wave equation, $\forall t \geq$ or $\forall t \leq 0$ [13]

$$\int_{|x|\geq|t|} |\nabla v_{x,t}|^2 dx \geq c \int |\nabla v_0|^2 + |v_1|^2 dx. \tag{2}$$

When $r_0 = 0$, the two inequalities coincide.

The inequality (1) has an interesting application in connection with the strong Huygens principle. We recall that this principle states (in odd dimensions) that if (v_0, v_1) is supported in the ball of radius ρ, then for $t \geq 0$ $(v(x,t), \partial_t v(x,t))$ is supported in $\{t - \rho \leq |x| \leq t + \rho\}$. Let

$$\rho(v_0, v_1) = \inf\{r > 0 : |\{s > r : (v_0(s), v_1(s)) \neq (0,0)\}| = 0\},$$

i.e. the radius of the essential support of (v_0, v_1). Inequality (1) gives that if (v_0, v_1) is compactly supported, either for $t \geq 0$, or $t \leq 0$,

$$\rho(v(t), \partial_t v(t)) = |t| + \rho(v_0, v_1).$$

The fact that $\rho(v(t), \partial_t v(t)) \leq |t| + \rho(v_0, v_1)$ is an immediate consequence of the strong Huygens principle (or even just finite speed of propagation). The strong Huygens principle gives $\rho(v(t), \partial_t v(t)) \geq |t| - \rho(v_0, v_1)$. Inequality (1), as we will see, implies that, for $t \geq 0$, or for $t \leq 0$,

$$\rho(v(t), \partial_t v(t)) \geq |t| + \rho(v_0, v_1).$$

To see this, we can assume that $0 < \rho_0 = \rho(v_0, v_1) < \infty$, i.e. $(v_0, v_1) \neq (0,0)$ and is compactly supported. Note that, for $A > 0$,

$$\int_A^{\rho_0} (\partial_r v_0)^2 r^2 dr = \int_A^{\rho_0} (\partial_r(rv_0))^2 \, dr + A v_0^2(A).$$

Consider $\rho_0 > A > \rho_0/2$. Then $A v_0^2(A) = A \left(\int_A^{\rho_0} \partial_r v_0 dr\right)^2 \leq A \left(\int_A^{\rho_0} (\partial_r v_0)^2 r^2 dr\right) \frac{1}{A^2}(\rho_0 - A) \leq \frac{2}{\rho_0}(\rho_0 - A) \int_A^{\rho_0} (\partial_r v_0)^2 r^2 dr$. Hence we can choose $A_0 \geq \rho_0/2$ such that for $\rho_0 > A \geq A_0$, we have $\int_A^{\rho_0} (\partial_r(rv_0))^2 \, dr + \int_A^{\rho_0} v_1^2 r^2 dr \geq$

$\frac{1}{2}\int_A^{\rho_0}\left[(\partial_r v_0)^2 + v_1^2\right] r^2 dr > 0$. Thus, if (say) (1) holds for $t > 0$, we have

$$\int_{r \geq t+A} |\nabla_{x,t} v(x,t)|^2 dx > 0,$$

which gives $\rho(v(t), \partial_t v(t)) \geq t + A$. Letting $A \to \rho_0$, we obtain $\rho(v(t), \partial_t v(t)) \geq t + \rho_0$ and so

$$\rho(v(t), \partial_t v(t)) = t + \rho(v_0, v_1) \text{ for } t > 0,$$

as claimed. This means that the support of $\vec{v}(t)$ expands at exactly speed 1 for either $t \geq 0$ or $t \leq 0$ [according to (1)].

For the proof in DKM 13' [15], say when $T_+ = 1$ (the case $T_+ = \infty$ is similar) we first consider $(v_0, v_1) =$ weak limit of $(u(t), \partial_t u(t))$ as $t \uparrow 1$, in $\dot{H}^1 \times L^2$, which can be shown to exist. Then, v_L is the linear solution with data (v_0, v_1) at time 1, the "radiation" term. We let v be the nonlinear solution with data (v_0, v_1) at time 1, so that, with $\vec{v}(t) = (v(t), \partial_t v(t))$, $\vec{v}_L(t) = (v_L(t), \partial_t v_L(t))$, $\|\vec{v}(t) - \vec{v}_L(t)\|_{\dot{H}^1 \times L^2} \to 0$ as $t \to 1$. It is easy to see, from finite speed of propagation, that for t near 1, $\text{supp}(\vec{u}(x,t) - \vec{v}(x,t)) \subseteq \{|x| \leq 1 - t\}$. We then break up $\vec{u}(t_n) - \vec{v}(t_n)$ into a sum of "blocks" (technically, nonlinear profiles U^j associated to a Bahouri–Gérard profile decomposition) plus a "dispersive" error $\vec{w}_n$ which is small in a weaker "dispersive" norm [1].

If one of the "blocks" U^j is not $\pm W$, by (1), it will send energy outside the light cone at $t = 1$ (case $t \geq 0$) , a contradiction to the support property of $\vec{u} - \vec{v}$, or arbitrarily close to the boundary of the inverted light cone, at $t = 0$ (case $t \leq 0$), also a contradiction. Finally, one uses (1) again to show that the dispersive error has to be small in energy, by a similar argument.

The argument in DKM 12' [14], for a well-chosen sequence of times, was different. The first step, say again in the case $T_+ = 1$, was to show that "no self-similar blow-up" is possible. This means to show, for each $0 < \lambda < 1$, that

$$\lim_{t \uparrow 1} \int_{\lambda(1-t) < |x| < 1-t} |\nabla_{x,t} u(x,t)|^2 dx = 0.$$

The proof of this used (1).

One then combines this with virial identities: if $2^* = 2N/(N-2)$, and φ is a suitable cut-off, we have:

$$\partial_t \int \varphi u \partial_t u dx = \int |\partial_t u|^2 dx - \int [|\nabla u|^2 - |u|^{2^*}] dx + \text{error}, \tag{3}$$

$$\partial_t \int \varphi x \cdot \nabla u \partial_t u dx = -\frac{N}{2} \int |\partial_t u|^2 dx + \frac{N-2}{2} \int [|\nabla u|^2 dx - |u|^{2^*}] dx + \text{error}. \tag{4}$$

When $N = 3$, adding $\frac{1}{2}$(3) + (4), we obtain (using no self-similar blow-up)

$$\partial_t \left(\int \varphi u \partial_t u dx + \int \varphi x \cdot \nabla u \partial_t u dx \right) = - \int |\partial_t u|^2 dx + \text{error}$$

which gives us

$$\lim_{t \uparrow 1} \frac{1}{1-t} \int_t^1 \int_{|x| \leq 1-s} |\partial_t u|^2 dx ds = 0.$$

Using this fact, one can show that each nonlinear block U^j is time independent, and hence $\pm W$, and that the dispersive error $\vec{w}_n$ has time derivative going to 0 in L^2, for a well chosen sequence of times. One can then use (2) to show $w_n \to 0$ in $\dot{H}^1$.

We next turn our attention to higher dimensions and the non-radial case. Before doing so, let me mention that the techniques just explained have found important applications to the study of equivariant wave maps and to the defocusing energy critical wave equation with a trapping potential, in works of Côte, Lawrie, Schlag, Liu, Jia, Kenig, etc.

Now we should mention an important fact, proved by Côte–Kenig–Schlag 13' [4]: (1) and (2) fail for all even N, radial solutions. However, (2) holds for $N = 4, 8, 12, \ldots$ for $(v_0, v_1) = (v_0, 0)$ and for $N = 6, 10, 14, \ldots$ for $(v_0, v_1) = (0, v_1)$, but not necessarily otherwise.

Moreover, Kenig–Lawrie–Liu–Schlag [25] have shown that an analogue of (1) holds for all odd N, u radial, and applied this to a stable soliton resolution for exterior wave maps on $\mathbb{R}^3$ [26].

In 14', Casey Rodriguez [29] used this analogue of (1) for all odd N to prove the radial case of soliton resolution along a well-chosen sequence of times for (NLW) in all odd dimensions, following the argument in DKM 12'.

What to do for N even, radial case, non-radial case? We start by discussing the radial case for $N = 4$, which is very close in spirit to co-rotational wave maps from $\mathbb{R}^2$ into the sphere $\mathbb{S}^2$. The first obstacle is that, due to the failure of (1), we did not know that self-similar blow-up is ruled out, which is the first thing to do in order to implement the strategy of DKM 12' for a well-chosen sequence of times.

This was not a difficulty in the work of Côte–Kenig–Lawrie–Schlag 13' [5, 7] on co-rotational wave maps, due to classical results of Christodoulou, Shatah, and Tahvildar–Zadeh from the 90s, who showed it by integration by parts, exploiting the finiteness of the flux, a consequence of the fact that the energy density is non-negative [3, 30].

This obviously does not hold for (NLW) and is a major difficulty. This difficulty was overcome by Côte–Kenig–Lawrie–Schlag 14' [7], by reversing the usual analogy with co-rotational wave maps.

We observed that if u is a radial solution to the energy critical wave equation on $\mathbb{R}^4$, then $\psi(r,t) = ru(r,t)$ solves

$$\partial_t^2 \psi - \partial_r^2 \psi - \frac{1}{r}\partial_r \psi + \frac{\psi - \psi^3}{r^2} = 0.$$

We let $f(\psi) = \psi - \psi^3$, $F(\psi) = \int_0^{\psi} f(\alpha)d\alpha = \frac{\psi^2}{2}\left[1 - \frac{\psi^2}{2}\right]$, and the "energy" is

$$\frac{1}{2}\int_0^\infty \left[(\partial_t \psi)^2 + (\partial_r \psi)^2 + \frac{2F(\psi)}{r^2}\right] rdr,$$

which is conserved. Note that if $|\psi| \leq \sqrt{2}$, $F(\psi) \geq 0$.

Now recall that we have the "radiation" v such that $\mathrm{supp}(\vec{u}(t) - \vec{v}(t)) \subset \{0 < r \leq 1-t\}$ and since v is a "regular" solution at $t = 1$, $v \to 0$, for $r = 1-t$, $t \to 1$. Thus, the same holds for u, which shows that, for λ_0 close to 1, $\lambda_0(1-t) < r < (1-t)$ we have the non-negativity of $F(\psi)$, and the classical argument applies, also yielding that now $\psi \to 0$ on $r = \lambda_0(1-t)$.

An iterative argument in λ_0 now gives the lack of self-similar blow-up. Hence we could start the process in DKM 12', and use the fact that on $\mathbb{R}^4$ (2) holds for data of the form $(v_0, 0)$, which is the type that we have for the dispersive error, and everything works.

What do we do when $N = 6$, when the good data in (2) are of the form $(0, v_1)$? This was the same difficulty one encountered for 2-equivariant wave maps into the sphere, and for radial Yang–Mills in $\mathbb{R}^4$.

All of this was overcome in recent work of Hao Jia–Kenig [22], who proved the analog of DKM 12' for a well-chosen sequence of times in all dimensions, and also dealt with all equivariant classes for wave maps and radial Yang–Mills in $\mathbb{R}^4$. This was done by not using the "channels of energy."

The first step is to prove lack of self-similar blow-up. The argument I sketched in $\mathbb{R}^4$ in fact applies to all dimensions, yielding a decomposition with blocks that are $\pm W$ and a dispersive error, for a well-chosen sequence of times $\{t_n^1\}_n$.

We then use again the second virial (4) which now gives

$$\lim_{t \uparrow 1} \frac{1}{1-t}\int_t^1 \int_{|x| \leq 1-s} [|\nabla u|^2 - |u|^{2^*}]dxds = 0.$$

On static solutions, $\int [|\nabla Q|^2 - |Q|^{2^*}]dx = 0$, and thus we obtain, by real variable arguments that, for a possibly different, well-chosen sequence of times $\{t_n^2\}_n$ we have $\overline{\lim} \int [|\nabla w_n|^2 - |w_n|^{2^*}]dx \leq 0$. But, for the dispersive error $\int |w_n|^{2^*} dx \to 0$, which concludes the argument.

I would like to conclude with some recent results in the non-radial setting. In the summer of 2015, Hao Jia [21] was able to extend the analogy with wave maps to the non-radial setting and in particular control the flux, when $T_+ < \infty$, i.e. the type II blow-up case.

This allowed him to obtain a Morawetz type identity (adapted from the wave maps one), to find a well-chosen sequence of times $t_n \to T_+ < \infty$, so that the desired decomposition holds in the non-radial case when $T_+ < \infty$, with an error tending to 0 in the dispersive sense. Here he also used the idea of combining virial identities I just explained.

In the case $T_+ = \infty$, one new difficulty is the extraction of the linear radiation term. This has been done recently by DKM 16' [16]. Moreover, very recently, in the joint work of D–Jia–K–M 16' [17] we have obtained the soliton resolution for a well-chosen sequence of times, for general type II solutions, both in the case $T_+ < \infty$ and $T_+ = \infty$. The result is:

Theorem 2 *Let $u \in C([0, T_+), \dot{H}^1 \times L^2(\mathbb{R}^N))$, $3 \le N \le 6$, be such that*

$$\sup_{0 \le t < T_+} \|(u(t), \partial_t u(t))\|_{\dot{H}^1 \times L^2} \le M.$$

Case 1: $T_+ < \infty$. Consider the singular set S, which is a finite set of points, and $x^* \in S$. Then $\exists J^* \in \mathbb{N}, J \ge 1, r^* > 0, v \in \dot{H}^1 \times L^2$ a regular solution near T_+, $t_n \uparrow T_+$, scales $\lambda_n^j, 0 < \lambda_n^j \ll (T_+ - t_n)$, positions $c_n^j \in \mathbb{R}^N$ such that $c_n^j \in B_{\beta(T_+ - t_n)}(x^*)$, $\beta \in (0, 1)$ with $\vec{\ell}_j = \lim_n (c_n^j - x^*)/(T_+ - t_n)$ well defined and traveling waves $Q^j_{\vec{\ell}_j}$ for $1 \le j \le J^*$ such that in the ball $B_{r^*}(x^*)$ we have

$$\begin{aligned}\vec{u}(t_n) &= \vec{v}(t_n)\\ &+ \sum_{j=1}^{J^*} \Big((\lambda_n^j)^{-\frac{N-2}{2}} Q^j_{\vec{\ell}_j}\Big(\frac{x - c_n^j}{\lambda_n^j}, 0\Big), (\lambda_n^j)^{-\frac{N}{2}} \partial_t Q^j_{\vec{\ell}_j}\Big(\frac{x - c_n^j}{\lambda_n^j}, 0\Big)\Big)\\ &+ o_{\dot{H}^1 \times L^2}(1) \quad \text{as } n \to \infty,\end{aligned}$$

and $\lambda_n^j/\lambda_n^{j'} + \lambda_n^{j'}/\lambda_n^j + |c_n^j - c_n^{j'}|/\lambda_n^j \to_n \infty$, $1 \le j \ne j' \le J^*$.

Case 2: $T_+ = \infty$. $\exists$ a linear solution u^L such that

$$\lim_{t \to \infty} \int_{|x| \ge t - A} |\nabla(u - u^L)|(x, t)^2 + |\partial_t (u - u^L)|(x, t)^2 dx = 0,$$

for all $A > 0$. Moreover, $\exists J^* \in \mathbb{N}, t_n \uparrow \infty, \lambda_n^j, 0 < \lambda_n^j \ll t_n, c_n^j \in \mathbb{R}^N$ such that $c_n^j \in B_{\beta t_n}(0)$, $\beta \in (0, 1)$ with $\vec{\ell}_j = \lim_n c_n^j / t_n$ well defined and traveling waves $Q^j_{\vec{\ell}_j}$ for $1 \le j \le J^*$ such that

$$\begin{aligned}\vec{u}(t_n) &= \vec{u}^L(t_n)\\ &+ \sum_{j=1}^{J^*} \Big((\lambda_n^j)^{-\frac{N-2}{2}} Q^j_{\vec{\ell}_j}\Big(\frac{x - c_n^j}{\lambda_n^j}, 0\Big), (\lambda_n^j)^{-\frac{N}{2}} \partial_t Q^j_{\vec{\ell}_j}\Big(\frac{x - c_n^j}{\lambda_n^j}, 0\Big)\Big)\\ &+ o_{\dot{H}^1 \times L^2}(1) \quad \text{as } n \to \infty,\end{aligned}$$

and $\lambda_n^j/\lambda_n^{j'} + \lambda_n^{j'}/\lambda_n^j + |c_n^j - c_n^{j'}|/\lambda_n^j \to_n \infty$, $1 \le j \ne j' \le J^*$.

The passage to arbitrary time sequences seems to require substantially different arguments.

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Densities with the Mean Value Property for Sub-Laplacians: An Inverse Problem

Giovanni Cupini and Ermanno Lanconelli

Dedicated to Richard L. Wheeden

Abstract Inverse problem results, related to densities with the mean value property for the harmonic functions, were recently proved by the authors. In the present paper we improve and extend them to the sub-Laplacians on stratified Lie groups.

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1 Introduction

Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 3$, $\mathbb{R}^n \setminus \overline{\Omega} \neq \emptyset$ and let $w : \Omega \to [0, \infty]$ be a lower semicontinuous function such that $\text{int}\{w = 0\} = \emptyset$. We say that w is a *density with the mean value property* for nonnegative harmonic functions in Ω if

(i) $w(\Omega) := \int_\Omega w(y)\, dy < \infty$,
(ii) there exists a point $x_0 \in \Omega$ such that

$$u(x_0) = \frac{1}{w(\Omega)} \int_\Omega u(y)w(y)\, dy$$

for every harmonic function u in Ω, $u \geq 0$.

G. Cupini • E. Lanconelli (✉)
Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy
e-mail: giovanni.cupini@unibo.it; ermanno.lanconelli@unibo.it

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For the sake of simplicity, if w is a density with the mean value property for the harmonic nonnegative functions in Ω, we say that

$$(\Omega, w, x_0) \text{ is a } \Delta\text{-triple;}$$

as usual Δ denotes the classical Laplace operator in $\mathbb{R}^n$.

A basic example of Δ-triple is $(B_r(x_0), 1, x_0)$, where $B_r(x_0)$ is the Euclidean ball with center x_0 and radius r. Indeed, by the Gauss Theorem,

$$u(x_0) = \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} u(y)\, dy, \qquad \forall u \in \mathcal{H}(B_r(x_0)), u \geq 0,$$

where $|B_r(x_0)|$ stands for the Lebesgue measure of $B_r(x_0)$ and $\mathcal{H}(B_r(x_0))$ denotes the space of the harmonic functions in $B_r(x_0)$.

More general Δ-triples can be obtained using the densities with the mean value property for harmonic functions constructed by Hansen-Netuka [10] and Aikawa [2, 3], see also [6]. In particular, for every bounded $C^{1,\epsilon}$-open set Ω, and for every $x_0 \in \Omega$, there exists a (non-unique) density w such that (Ω, w, x_0) is a Δ-triple.

The problem of the *best harmonic L^1-approximation* of subharmonic functions, see [9], suggests the following inverse problem:

(IP) if (Ω, w, x_0) and (D, w', x_0) are Δ-triples, such that

$$\frac{w}{w(\Omega)} = \frac{w'}{w'(D)} \qquad \text{in } \Omega \cap D,$$

is it true that $\Omega = D$?

Positive answers to (IP), in the case Ω is a Euclidean ball, were given by Epstein [7], Epstein-Schiffer [8], Kuran [12]. In our language, their results can be stated as follows:

let $D \subset \mathbb{R}^n$ be an open connected set with finite Lebesgue measure. If $(D, 1, x_0)$ is a Δ-triple and $r > 0$ is such that $\frac{1}{|D|} = \frac{1}{|B_r(x_0)|}$, then $D = B_r(x_0)$.

Notice that the Euclidean balls play a privileged role here; indeed $(B_r(x_0), 1, x_0)$ is not only a Δ-triple, as previously observed, but it has the following extra-property: if we denote by Γ the fundamental solution with pole at 0 of the Laplace operator in $\mathbb{R}^n$, $n \geq 3$, then

$$\Gamma(x_0 - x) > \frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} \Gamma(y - x)\, dy \qquad \text{for all } x \in B_r(x_0) \setminus \{x_0\}. \tag{1}$$

Moreover, the Euclidean balls also have some trivial, but important for our aims, topological properties: $B_r(x_0) = \operatorname{int} \overline{B_r(x_0)}$ and $\mathbb{R}^n \setminus \overline{B_r(x_0)}$ is connected.

These properties of the Euclidean balls lead us to give the following definition.

Definition 1.1 Let Ω be an open set in $\mathbb{R}^n$, $n \geq 3$. We say that (Ω, w, x_0) is a *strong* Δ-triple if

(a) (Ω, w, x_0) is a Δ-triple,
(b) $\Gamma(x_0 - x) > \dfrac{1}{w(\Omega)} \displaystyle\int_{\Omega} \Gamma(y - x) w(y)\, dy$, for every $x \in \Omega \setminus \{x_0\}$.

Moreover, we say that Ω is solid, if $\Omega = \operatorname{int} \overline{\Omega}$ and $\mathbb{R}^n \setminus \overline{\Omega}$ is a connected, not empty set.

In the very recent paper [6], we proved a result (Theorem 1.1) implying, as a corollary, the following theorem.

Theorem 1.2 *Let Ω, D be bounded, open sets in $\mathbb{R}^n$, $n \geq 3$. Assume that*

(i) (Ω, w, x_0) *is a* strong Δ*-triple and* $x \mapsto \displaystyle\int_{\mathbb{R}^n} \Gamma(y - x) w(y)\, dy$ *is continuous,*
(ii) (D, w', x_0) *is a* Δ*-triple,*
(iii) $\dfrac{w}{w(\Omega)} = \dfrac{w'}{w'(D)}$ *in* $\Omega \cap D$,
(iv) Ω *is a solid set.*

Then $D = \Omega$ *and* $w' = \dfrac{w'(D)}{w(\Omega)} w$.

In the present paper we will prove a more general version of [6, Theorem 1.1], see Theorem 3.4, so also obtaining a more general result than Theorem 1.2, see Theorem 3.1. Precisely, we will improve the results in [6] in two directions: the involved operators will be not only the classical Laplacian, but any sub-Laplacian on a stratified group; moreover, the boundedness assumptions on Ω and D, and the continuity assumption in (i) will be removed.

The plan of the paper is the following. In the next section, we will introduce the sub-Laplacian operators $\mathcal{L}$ and we will recall some of their fundamental properties. Moreover, we will give the definitions of $\mathcal{L}$-triples, strong $\mathcal{L}$-triples and, correspondingly, Γ-triples and strong Γ-triples, with Γ the fundamental solution of $\mathcal{L}$. We will also exhibit examples of strong $\mathcal{L}$-triples, see Theorem 2.4. In Sect. 3 we will state our results on the inverse problem (Theorem 3.1, Corollary 3.2, Theorem 3.4 and Corollary 3.5) and in Sect. 4 we will prove them. In Appendix, for reader's convenience, we will recall the definition and list some properties of the $\mathcal{L}$-superharmonic functions as presented in [5, Chap. 8].

2 Sub-Laplacians and Related Triples

A sum of squares operator

$$\mathcal{L} = \sum_{i=1}^{m} X_j^2, \tag{2}$$

is a sub-Laplacian in $\mathbb{R}^n$ if the following conditions hold.

(H1) The X_j's are smooth vector fields in $\mathbb{R}^n$ and generate a Lie algebra $\mathfrak{a}$ satisfying rank $\mathfrak{a}(x) = \dim \mathfrak{a} = n$ at any point $x \in \mathbb{R}^n$.

(H2) There exists a group of dilations $(\delta_\lambda)_{\lambda>0}$ in $\mathbb{R}^n$ such that every vector field X_j is δ_λ-homogeneous of degree one.

A group of dilations in $\mathbb{R}^n$ is a family of diagonal linear functions $(\delta_\lambda)_{\lambda>0}$ of the kind

$$\delta_\lambda(x_1, \ldots, x_n) = (\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_n} x_n),$$

where the σ_j's are natural numbers.

Due to the rank condition in (H1), the operator $\mathcal{L}$ is hypoelliptic, see [11], so that the $\mathcal{L}$-harmonic functions, i.e., the solutions to $\mathcal{L}u = 0$, are smooth.

Conditions (H1) and (H2) imply the existence of a group law $\circ$ making $\mathbb{G} = (\mathbb{R}^n, \circ, \delta_\lambda)$ a stratified Lie group on which every vector field X_j is left translation invariant, see [4]. The natural number $Q := \sigma_1 + \ldots + \sigma_N$ is called the homogeneous dimension of $\mathbb{G}$. If $Q = 2$, $\mathbb{G}$ is the Euclidean group and $\mathcal{L}$, up to a linear transformation, is the usual Laplace operator. From now on, we assume, without further comment, that $Q \geq 3$.

One of the main features of a sub-Laplacian $\mathcal{L}$ is the existence of a gauge function playing for it the same role played by the Euclidean norm for the classical Laplace operator. A $\mathcal{L}$-gauge is a continuous function $d : \mathbb{G} \to [0, \infty[$, $\mathbb{G}$-symmetric, i.e., $d(x^{-1}) = d(x)$ for every $x \in \mathbb{G}$, strictly positive and smooth outside the origin, which is δ_λ-homogeneous of degree one, and such that

$$\gamma(x) := \frac{1}{d(x)^{Q-2}},$$

is $\mathcal{L}$-harmonic in $\mathbb{G} \setminus \{0\}$.

The d-balls $B_r^d(x) := \left\{ y \in \mathbb{G} : d(x^{-1} \circ y) < r \right\}$ support averaging operators which characterize the $\mathcal{L}$-harmonic functions the same way as the usual mean value operators on Euclidean balls characterize classical harmonic functions. To be precise, define in $\mathbb{G} \setminus \{0\}$

$$\psi := |\nabla_{\mathcal{L}} d|^2, \qquad \nabla_{\mathcal{L}} := (X_1, \ldots, X_m), \tag{3}$$

and

$$M_r(u)(x) := \frac{m_d}{r^Q} \int_{B_r^d(x)} u(y)\psi(x^{-1} \circ y)\, dy, \tag{4}$$

where

$$m_d := Q(Q-2)\beta_d$$

and

$$(\beta_d)^{-1} := Q(Q-2) \int_{B_1^d(0)} \psi(y)\, dy. \tag{5}$$

Then a continuous function $u : O \to \mathbb{R}$, $O \subseteq \mathbb{G}$ open, is smooth and satisfies $\mathcal{L}u = 0$ in O if and only if

$$u(x) = M_r(u)(x) \quad \forall \overline{B_r^d(x_0)} \subseteq O.$$

This is Gauss-Koebe's Theorem for $\mathcal{L}$, see [5, Theorem 5.6.3].

The fundamental solution of $\mathcal{L}$ with pole at the origin is

$$\Gamma := \beta_d \gamma,$$

see [5, Theorem 5.5.6].

We now give some definitions: $\mathcal{L}$-triples, strong $\mathcal{L}$-triples and, correspondingly, Γ-triples and strong Γ-triples.

Definition 2.1 Let Ω be an open subset of $\mathbb{G}$, such that $\mathbb{G} \setminus \overline{\Omega} \neq \emptyset$ and let $w : \Omega \to [0, \infty]$ be a lower semicontinuous function with $\operatorname{int}\{w = 0\} = \emptyset$.

We say that (Ω, w, x_0) is a $\mathcal{L}$-triple if

(i) $w(\Omega) := \int_\Omega w(y)\, dy < \infty$,
(ii) there exists $x_0 \in \Omega$ such that

$$u(x_0) = \frac{1}{w(\Omega)} \int_\Omega u(y) w(y)\, dy$$

for every $\mathcal{L}$-harmonic function u in Ω, $u \geq 0$.

If, moreover,

(iii) $\Gamma(x^{-1} \circ x_0) > \dfrac{1}{w(\Omega)} \displaystyle\int_\Omega \Gamma(x^{-1} \circ y) w(y)\, dy$, for every $x \in \Omega \setminus \{x_0\}$,

then we say that (Ω, w, x_0) is a *strong* $\mathcal{L}$-triple.

A variant of the notion of $\mathcal{L}$-triple is the following definition of Γ-triple. Before stating it, we recall that if Ω is an open subset of $\mathbb{G}$, μ is a nonnegative Radon measure in $\mathbb{G}$, $\mu(\Omega^c) = 0$, then the Γ*-potential* of μ is defined as follows:

$$\Gamma_\mu(x) := \int_\Omega \Gamma(x^{-1} \circ y)\, d\mu(y) \qquad x \in \mathbb{G}.$$

Definition 2.2 Let Ω be an open subset of $\mathbb{G}$, x_0 a point of Ω and let μ be a nonnegative Radon measure in $\mathbb{G}$, $\mu(\Omega) = 1$ and $\mu(\Omega^c) = 0$.

We say that (Ω, μ, x_0) is a Γ*-triple* if

$$\Gamma_\mu(x) = \Gamma(x^{-1} \circ x_0) \qquad \forall\, x \in \Omega^c. \tag{6}$$

If, moreover,

$$\Gamma_\mu(x) < \Gamma(x^{-1} \circ x_0) \qquad \forall\, x \in \Omega \setminus \{x_0\}, \tag{7}$$

then we say that (Ω, μ, x_0) is a *strong* Γ*-triple*.

Remark 2.3 Let (Ω, w, x_0) be a $\mathcal{L}$-triple. Extend w to $\mathbb{G}$ by letting w be 0 in Ω^c and define μ the measure

$$d\mu(y) = \frac{w(y)}{w(\Omega)} dy.$$

Then (Ω, μ, x_0) is a Γ-triple. Indeed, $\mu(\Omega) = 1$ and, fixed $x \in \Omega^c$, (ii) in Definition 2.1, applied with $u(y) := \Gamma(x^{-1} \circ y)$, implies (6) in Definition 2.2.

We stress that the present definition in the case of $\mathcal{L} = \Delta$, the classical Laplacian, is more general that the one given in [6]: indeed, we don't require anymore the boundedness of Ω and the continuity of Γ_μ.

The Gauss-type Theorem for sub-Laplacians recalled above implies that $(B^d_r(x_0), \psi(x_0^{-1} \circ \cdot), x_0)$ is a $\mathcal{L}$-triple. Actually, it is a strong $\mathcal{L}$-triple; as a matter of fact, more general strong $\mathcal{L}$-triples can be defined on every d-ball, as the following theorem shows.

Theorem 2.4 *Let $f :]0, \infty[\to]0, \infty[$ be a continuous function, such that*

$$F(r) := \int_0^r f(\rho)\, d\rho < \int_0^\infty f(\rho)\, d\rho = \infty \qquad \forall r \in]0, \infty[.$$

Define

$$w_f(y) := \frac{f(d(y))}{d(y)^{Q-1}} \psi(y), \qquad y \in \mathbb{G} \setminus \{0\}, \tag{8}$$

where ψ is the function in (3).

Then $\mathrm{int}\{w_f = 0\} = \emptyset$ *and, for every* $x_0 \in \mathbb{G}$ *and* $r > 0$*,*

$$(B_r^d(x_0), w_f(x_0^{-1} \circ \cdot), x_0)$$

is a strong $\mathcal{L}$*-triple.*

We agree to say that the function

$$y \mapsto w_f(x_0^{-1} \circ \cdot) \qquad y \in \mathbb{G} \setminus \{x_0\},$$

with w_f defined in (8), is $\mathcal{L}$*-radially symmetric* with respect to x_0.

Proof of Theorem 2.4 We first observe that $\{w_f = 0\} = \{\psi = 0\}$ and this last set has empty interior as proved in [5, p. 262].

Let us now prove that $(B_r^d(x_0), w_f(x_0^{-1} \circ \cdot), x_0)$ is a strong $\mathcal{L}$-triple. The proof relies on [5, Theorem 9.5.2] and the coarea formula.

Let us first prove that $(B_r^d(x_0), w_f(x_0^{-1} \circ \cdot), x_0)$ is a $\mathcal{L}$-triple.

By formula [5, (9.22)], for every $\mathcal{L}$-harmonic nonnegative function u in $B_r^d(x_0)$ and for every $\rho < r$,

$$u(x_0) = \mathcal{M}_\rho(u)(x_0), \tag{9}$$

where $\mathcal{M}_\rho$ is the surface average operator defined in [5, (5.46)]; i.e.,

$$\mathcal{M}_\rho(u)(x_0) := \frac{(Q-2)\beta_d}{\rho^{Q-1}} \int_{\partial B_\rho^d(x_0)} u(y) \frac{\psi(x_0^{-1} \circ y)}{|\nabla d(x_0^{-1} \circ y)|} \, d\sigma(y), \tag{10}$$

with β_d as in (5).

Let us multiply (9) by $\frac{f(\rho)}{F(r)}$ and integrate w.r.t. ρ on $]0, r[$. By the coarea formula we get

$$\begin{aligned} u(x_0) &= \frac{1}{F(r)} \int_0^r f(\rho) \mathcal{M}_\rho(u)(x_0) \, d\rho \\ &= \frac{(Q-2)\beta_d}{F(r)} \int_0^r \left(\int_{\partial B_\rho^d(x_0)} u(y) \frac{f(d(x_0^{-1} \circ y))}{d(x_0^{-1} \circ y)^{Q-1}} \psi(x_0^{-1} \circ y) \frac{d\sigma(y)}{|\nabla d(x_0^{-1} \circ y)|} \right) d\rho \\ &= \frac{(Q-2)\beta_d}{F(r)} \int_{B_r^d(x_0)} u(y) w_f(x_0^{-1} \circ y) \, dy. \end{aligned} \tag{11}$$

If we take $u = 1$ in the previous identities, we obtain

$$1 = \frac{(Q-2)\beta_d}{F(r)} \int_{B_r^d(x_0)} w_f(x_0^{-1} \circ y) \, dy;$$

i.e.,

$$w_f(x_0^{-1} \circ B_r^d(x_0)) := \frac{F(r)}{(Q-2)\beta_d} < \infty. \tag{12}$$

Therefore, $(B_r^d(x_0), w_f(x_0^{-1} \circ \cdot), x_0)$ is a $\mathcal{L}$-triple.

To show that this triple is *strong*, we only need to prove that w_f satisfies (iii) in Definition 2.1.

For every $x \in \mathbb{G}$ let us define

$$u_x(y) := \Gamma(x^{-1} \circ y) \qquad y \in \mathbb{G}.$$

We remark that u_x is a $\mathcal{L}$-superharmonic function and

$$\mathcal{L}u_x = -\delta_x \qquad \text{in the sense of distributions,}$$

where δ_x is the Dirac measure at $\{x\}$. By Poisson-Jensen's formula [5, Theorem 9.5.2], for every $\rho > 0$,

$$u_x(x_0) = \mathcal{M}_\rho(u_x)(x_0) + \int_{B_\rho^d(x_0)} \big(\Gamma(x_0^{-1} \circ y) - \Gamma(\rho)\big)\, d\delta_x(y), \tag{13}$$

where $\mathcal{M}_\rho$ is the surface average operator in (10). We have

$$\int_{B_\rho^d(x_0)} \big(\Gamma(x_0^{-1} \circ y) - \Gamma(\rho)\big)\, d\delta_x(y) = \begin{cases} 0 & \text{if } d(x_0^{-1} \circ x) \geq \rho \\ \Gamma(x_0^{-1} \circ x) - \Gamma(\rho) & \text{if } 0 < d(x_0^{-1} \circ x) < \rho. \end{cases}$$

Therefore, (13) and the equality $\Gamma(x^{-1} \circ x_0) = \Gamma(x_0^{-1} \circ x)$, give

$$\mathcal{M}_\rho(u_x)(x_0) = \begin{cases} u_x(x_0) = \Gamma(x^{-1} \circ x_0) & \text{if } d(x_0^{-1} \circ x) \geq \rho \\ \Gamma(\rho) < \Gamma(x_0^{-1} \circ x) & \text{if } 0 < d(x_0^{-1} \circ x) < \rho. \end{cases} \tag{14}$$

Let us now consider $x \in B_r^d(x_0) \setminus \{x_0\}$.

By (11), (14) and (12),

$$\begin{aligned} &\int_{B_r^d(x_0)} \Gamma(x^{-1} \circ y) w_f(x_0^{-1} \circ y)\, dy = \frac{1}{(Q-2)\beta_d} \int_0^r f(\rho) \mathcal{M}_\rho(u_x)(x_0)\, d\rho \\ &= \int_0^{d(x_0^{-1} \circ x)} \frac{f(\rho)}{(Q-2)\beta_d} \mathcal{M}_\rho(u_x)(x_0)\, d\rho + \int_{d(x_0^{-1} \circ x)}^r \frac{f(\rho)}{(Q-2)\beta_d} \mathcal{M}_\rho(u_x)(x_0)\, d\rho \\ &< \Gamma(x^{-1} \circ x_0) \frac{1}{(Q-2)\beta_d} \int_0^r f(\rho)\, d\rho = \Gamma(x^{-1} \circ x_0) w_f(x_0^{-1} \circ B_r^d(x_0)). \end{aligned}$$

This concludes the proof. □

3 Results on the Inverse Problem for $\mathcal{L}$

In this section we state and prove our main results regarding the analogue of the inverse problem (IP) for the sub-Laplacian, see Theorems 3.1 and 3.4. As an application, we show that the d-balls are the only open sets supporting $\mathcal{L}$-radially symmetric densities with the mean value property for $\mathcal{L}$, see Corollary 3.2 and the related Corollary 3.5.

Theorem 3.1 *Let Ω, D be open sets in $\mathbb{G}$, such that $\left(\overline{\Omega} \cup \overline{D}\right)^c \neq \emptyset$.*
Assume that

(i) (Ω, w, x_0) *is a* strong $\mathcal{L}$*-triple,*
(ii) (D, w', x_0) *is a $\mathcal{L}$-triple,*
(iii) $\dfrac{w}{w(\Omega)} = \dfrac{w'}{w'(D)}$ *in $\Omega \cap D$,*
(iv) Ω *is a solid set.*

Then $D = \Omega$ and $w' = \dfrac{w'(D)}{w(\Omega)} w$.

If we apply this theorem to the strong $\mathcal{L}$-triples given by Theorem 2.4, we obtain the following *$\mathcal{L}$-harmonic characterization* of the d-balls.

Corollary 3.2 *Let D be an open set in $\mathbb{G}$ such that $\overline{D}^c$ is unbounded. Let w_f be the $\mathcal{L}$-radially symmetric function in (8) and assume that, for some $x_0 \in D$,*

(a) $w_f(x_0^{-1} \circ D) < \infty$,
(b) $u(x_0) = \dfrac{1}{w_f(x_0^{-1} \circ D)} \displaystyle\int_D u(y) w_f(x_0^{-1} \circ y)\, dy$ *for every nonnegative $\mathcal{L}$-harmonic function in D.*

Then

$$D = B_r^d(x_0),$$

where r is the only real positive number such that

$$w_f(x_0^{-1} \circ D) = w_f(x_0^{-1} \circ B_r^d(x_0)). \tag{15}$$

Remark 3.3 The unique number r such that the equality (15) holds is, by (12), the only real positive number such that

$$\int_0^r f(\rho)\, d\rho = (Q-2)\beta_d w_f(x_0^{-1} \circ D). \tag{16}$$

In the particular case of $f(\rho) = \rho^{Q-1}$ and D a bounded set, Corollary 3.2 is Theorem 1.1 in [13]. Moreover, if $\mathbb{G}$ is the Euclidean group $\mathbb{R}^n$, d is the Euclidean norm, $\mathcal{L}$ is the classical Laplace operator and D is a connected set with finite Lebesgue measure, then Corollary 3.2 was proved in [12].

Theorem 3.1 above is a consequence of a result on Γ-triples, that, even in the case of the Laplacian, is more general than the analogue [6, Theorem 1.1].

Before stating it, we recall that the support of a measure μ can be defined as follows:

$$\operatorname{supp}\mu := \{x \in \mathbb{G} \,:\, (A \text{ open set}, x \in A) \;\Rightarrow\; \mu(A) > 0\}.$$

Theorem 3.4 *Let Ω and D be open sets in $\mathbb{G}$ containing x_0, $\left(\overline{\Omega} \cup \overline{D}\right)^c \neq \emptyset$.*
Assume that

(i) (Ω, μ, x_0) *is a strong* Γ*-triple,*
(ii) (D, ν, x_0) *is a* Γ*-triple,*
(iii) $\mu\llcorner(\Omega \cap D) = \nu\llcorner(\Omega \cap D)$*,*
(iv) $\partial D \subseteq \operatorname{supp}\nu$*,*
(v) Ω *is a solid set.*

Then $D = \Omega$ and $\nu = \mu$.

Examples given in [6] for the Laplace operator show that the assumptions are essentially sharp: the request that (Ω, μ, x_0) is a strong Γ-triple cannot be weakened by assuming that (Ω, μ, x_0) is simply a Γ-triple and neither (iii) nor (iv) can be removed.

If we apply this theorem to the strong Γ-triples given by Theorem 2.4 and Remark 2.3 we obtain the following characterization of the d-balls.

Corollary 3.5 *Let D be an open set in $\mathbb{G}$ such that $\overline{D}^c$ is unbounded. Let w_f be the $\mathcal{L}$-radially symmetric function in (8) and assume that, for some $x_0 \in D$,*

(a) $w_f(x_0^{-1} \circ D) < \infty$*,*
(b) $\Gamma(x^{-1} \circ x_0) = \dfrac{1}{w_f(x_0^{-1} \circ D)} \displaystyle\int_D \Gamma(x^{-1} \circ y) w_f(x_0^{-1} \circ y)\, dy$ *for every $x \notin D$.*

Then

$$D = B_r^d(x_0),$$

where r is the only real positive number such that

$$w_f(x_0^{-1} \circ D) = w_f(x_0^{-1} \circ B_r^d(x_0)). \tag{17}$$

In the particular case of $f(\rho) = \rho^{Q-1}$ and D is a bounded set, Corollary 3.5 is Theorem 1.2 in [13]. Moreover, if $\mathbb{G}$ is the Euclidean group $\mathbb{R}^n$, d is the Euclidean norm, $\mathcal{L}$ is the classical Laplace operator and D is bounded, then Corollary 3.5 is a result by Aharonov-Schiffer-Zalcman in [1].

4 Proofs of the Results on the Inverse Problem

In this section we prove the results stated in Sect. 3.

Proof of Theorem 3.4 We split the proof in four steps.

STEP 1. Let us prove that $\Gamma_\mu \le \Gamma_\nu$ in $\mathbb{G} \setminus \{x_0\}$.

Assumption (i) and (ii) imply

$$\Gamma_\mu(x) \le \Gamma(x^{-1} \circ x_0) < \infty \quad \forall x \in \mathbb{G} \setminus \{x_0\}, \quad \Gamma_\nu(x) = \Gamma(x^{-1} \circ x_0) \quad \forall x \in D^c. \tag{18}$$

Then, since $x_0 \notin D^c$,

$$\Gamma_\mu(x) \le \Gamma_\nu(x) \qquad \forall\, x \in D^c.$$

It remains to prove that $\Gamma_\mu \le \Gamma_\nu$ in $D \setminus \{x_0\}$.

We first remark that, by the first chain of inequalities in (18),

$$\Gamma_\mu - \Gamma_\nu \ \text{ is well defined and } \ < \infty \text{ in } D \setminus \{x_0\}.$$

Moreover, by using (iii), one easily recognizes that

$$\Gamma_\mu(x) - \Gamma_\nu(x) = \int_{\Omega\setminus D} \Gamma(x^{-1} \circ y)\, d\mu(y) - \int_{D\setminus\Omega} \Gamma(x^{-1} \circ y)\, d\nu(y) \qquad \forall x \in D \setminus \{x_0\}. \tag{19}$$

Hereafter we agree to let an integral be equal to zero, if the integration domain is empty.

The functions

$$h(x) := \int_{\Omega\setminus D} \Gamma(x^{-1} \circ y)\, d\mu(y), \qquad x \in D$$

and

$$v(x) := \int_{D\setminus\Omega} \Gamma(x^{-1} \circ y)\, d\nu(y), \qquad x \in D$$

are, respectively, $\mathcal{L}$-harmonic and $\mathcal{L}$-superharmonic in D, see Appendix. As a consequence,

$$\tilde{u} := h - v \ \text{ is } \mathcal{L}\text{-subharmonic in } D;$$

moreover, keeping in mind (19),

$$\tilde{u} = \Gamma_\mu - \Gamma_\nu \quad \text{in } D \setminus \{x_0\}.$$

On the other hand, by the first item in (18) and the lower semicontinuity of Γ_ν, for every $x \in \partial D$,

$$\begin{aligned}\limsup_{D \ni y \to x} \tilde{u}(y) &= \limsup_{D \ni y \to x} (\Gamma_\mu - \Gamma_\nu)(y) \leq \limsup_{D \ni y \to x} (\Gamma(y^{-1} \circ x_0) - \Gamma_\nu(y)) \\ &\leq \Gamma(x^{-1} \circ x_0) - \Gamma_\nu(x) = 0,\end{aligned}$$

since (D, ν, x_0) is a Γ-triple and $x \notin D$. Moreover,

$$\limsup_{D \ni y \to \infty} \tilde{u}(y) \leq \limsup_{D \ni y \to \infty} (\Gamma(y^{-1} \circ x_0) - \Gamma_\nu(y)) \leq \limsup_{y \to \infty} \Gamma(y^{-1} \circ x_0) = 0.$$

By the maximum principle for subharmonic functions (see [5, Theorem 8.2.19 (ii)]) we get $\tilde{u} \leq 0$ in D; hence $\Gamma_\mu \leq \Gamma_\nu$ in $D \setminus \{x_0\}$.

STEP 2. Let us prove that $\partial D \subseteq \overline{\Omega}$.

By contradiction, assume there exists a point $x \in \partial D$ such that $x \notin \overline{\Omega}$. Then $x \in \operatorname{supp} \nu$ (by assumption (iv)) and $\mathbb{G} \setminus \overline{\Omega}$ is an open set containing x. As a consequence

$$\nu(\mathbb{G} \setminus \overline{\Omega}) > 0. \tag{20}$$

Since μ has its support contained in $\overline{\Omega}$, Γ_μ is $\mathcal{L}$-harmonic in $\mathbb{G} \setminus \overline{\Omega}$, see Appendix, so that

$$\Gamma_\mu - \Gamma_\nu \ \text{ is } \mathcal{L}\text{-subharmonic in } \mathbb{G} \setminus \overline{\Omega}.$$

On the other hand, by what we proved in Step 1, $\Gamma_\mu - \Gamma_\nu \leq 0$ in $\mathbb{G} \setminus \overline{\Omega}$. Moreover, since (i) and (ii) imply

$$\Gamma_\mu = \Gamma_\nu \qquad \text{in } \Omega^c \cap D^c,$$

then $(\Gamma_\mu - \Gamma_\nu)(x) = 0$.

By (v) $\mathbb{G} \setminus \overline{\Omega}$ is a connected set, so, the strong maximum principle for subharmonic functions (see in [5, Theorem 8.2.19 (i)]) imply

$$\Gamma_\mu - \Gamma_\nu = 0 \qquad \text{in } \mathbb{G} \setminus \overline{\Omega},$$

so that

$$\mathcal{L}(\Gamma_\mu - \Gamma_\nu) = 0 \qquad \text{in } \mathbb{G} \setminus \overline{\Omega}.$$

On the other hand, in $\mathbb{G} \setminus \overline{\Omega}$, $\mathcal{L}(\Gamma_\mu - \Gamma_\nu) = \nu$. Therefore, $\nu(\mathbb{G} \setminus \overline{\Omega}) = 0$, in contradiction with (20).

STEP 3. Let us prove that $D \subseteq \Omega$.

We have

$$\mathbb{G} \setminus \overline{\Omega} = (D \cup D^c) \setminus \overline{\Omega} = (D \setminus \overline{\Omega}) \cup (\partial D \setminus \overline{\Omega}) \cup (\overline{D}^c \cap \overline{\Omega}^c) = (D \setminus \overline{\Omega}) \cup (\overline{D} \cup \overline{\Omega})^c.$$

By assumption, $(\overline{D} \cup \overline{\Omega})^c$ is not empty. Moreover, $D \setminus \overline{\Omega}$ and $(\overline{D} \cup \overline{\Omega})^c$ are open, disjoint sets. The set $\mathbb{G} \setminus \overline{\Omega}$ is connected by (v), then $D \setminus \overline{\Omega}$ must be empty. Therefore $D \subseteq \overline{\Omega}$. By (v) we have that $\operatorname{int} \overline{\Omega} = \Omega$, thus we obtain $D \subseteq \Omega$.

STEP 4. Let us prove that $\Omega \subseteq D$. We argue by contradiction; i.e., we assume that there exists $x \in \Omega \setminus D$. In particular, $x \neq x_0$. By Step 3, $D \subseteq \Omega$. Therefore, by (i), (iii) and (ii), we have

$$\begin{aligned}\Gamma(x^{-1} \circ x_0) > \Gamma_\mu(x) &= \int_D \Gamma(x^{-1} \circ y)\, d\mu(y) + \int_{\Omega \setminus D} \Gamma(x^{-1} \circ y)\, d\mu(y) \\ &\geq \int_D \Gamma(x^{-1} \circ y)\, d\mu(y) = \int_D \Gamma(x^{-1} \circ y)\, d\nu(y) = \Gamma_\nu(x) = \Gamma(x^{-1} \circ x_0).\end{aligned}$$

This is an absurd.

We have so proved that $D = \Omega$ and, consequently, that $\mu = \nu$. □

As a corollary of Theorem 3.4, we get Theorem 3.1

Proof of Theorem 3.1 Let us extend w and w' with 0 to all $\mathbb{G}$ and define the measures μ, ν as follows:

$$d\mu(y) = \frac{w(y)}{w(\Omega)} dy, \qquad d\nu(y) = \frac{w'(y)}{w'(D)} dy.$$

By (i), (ii) and Remark 2.3, (D, ν, x_0) is a Γ-triple and (Ω, μ, x_0) is a strong Γ-triple. By (iii), $\mu \llcorner (\Omega \cap D) = \nu \llcorner (\Omega \cap D)$. Moreover, by definition of $\mathcal{L}$-triple, since $\operatorname{int}\{y \in D \ : \ w'(y) = 0\} = \emptyset$, then $\partial D \subseteq \operatorname{supp} \nu$. The conclusion follows by Theorem 3.4. □

We now are ready to prove Corollary 3.2.

Proof of Corollary 3.2 In order to apply Theorem 3.1 it is convenient to introduce the following notation:

$$\begin{aligned}&\Omega := B_r^d(x_0) \quad \text{with } r > 0 \text{ given by (16)}, \\ &w(y) := w_f(x_0^{-1} \circ y), \quad y \in \Omega, \\ &w'(y) := w_f(x_0^{-1} \circ y) \quad y \in D.\end{aligned}$$

Since $\overline{D}^c$ is unbounded, then $\left(\overline{\Omega} \cup \overline{D}\right)^c \neq \emptyset$. Moreover,

(i) (Ω, w, x_0) is a *strong* $\mathcal{L}$-triple (by Theorem 2.4)
(ii) (D, w', x_0) is a $\mathcal{L}$-triple (by hypotheses (a) and (b)),
(iii) $\dfrac{w}{w(\Omega)} = \dfrac{w'}{w'(D)}$ in $\Omega \cap D$ (since $w = w'$ in $\Omega \cap D$ and, by (15), $w(\Omega) = w'(D)$),
(iv) Ω is a solid set.

As far as (iv) is concerned, it is quite obvious that $B_r^d(x_0) = \operatorname{int} \overline{B_r^d(x_0)}$; the second condition, $\mathbb{G} \setminus \overline{B_r^d(x_0)}$ is connected, can be proved as follows. Without loss of generality we can assume $x_0 = O$.

Let B be an Euclidean ball containing $\overline{B_r^d(x_0)}$. Then for every $x, y \in \mathbb{G} \setminus \overline{B_r^d(x_0)}$,

$$(\mathbb{G} \setminus B) \cup \{\delta_\lambda(x) \,:\, \lambda \geq 1\} \cup \{\delta_\lambda(y) \,:\, \lambda \geq 1\}$$

is a connected subset in $\mathbb{G} \setminus \overline{B_r^d(x_0)}$ and it contains x and y.

Then, all the assumptions of Theorem 3.1 are satisfied; hence $D = \Omega$; i.e.,

$$D = B_r^d(x_0).$$

□

We now turn to the proof of Corollary 3.5.

Proof of Corollary 3.5 Define

$$d\nu(y) := \frac{w_f(x_0^{-1} \circ y)}{w_f(x_0^{-1} \circ D)} \chi_D(y)\, dy.$$

By (a) and (b) (D, ν, x_0) is a Γ-triple. Since by Theorem 2.4 $\operatorname{int}\{y \in D \,:\, w_f(x_0^{-1} \circ y) = 0\} = \emptyset$, then $\partial D \subseteq \overline{D} = \operatorname{supp}(\nu)$.

Let us choose $r > 0$ such that (17) holds and define

$$d\mu(y) := \frac{w_f(x_0^{-1} \circ y)}{w_f(x_0^{-1} \circ B_r^d(x_0))} \chi_{B_r^d(x_0)}(y)\, dy.$$

In particular, (iii) in Theorem 3.4 holds.

By Theorem 2.4 $(B_r^d(x_0), w_f(x_0^{-1} \circ \cdot), x_0)$ is a strong $\mathcal{L}$-triple, therefore, by Remark 2.3, $(B_r^d(x_0), \mu, x_0)$ is a strong Γ-triple. Taking also into account that $B_r^d(x_0)$ is a solid set (see the proof of Corollary 3.2) we have that all the assumptions of Theorem 3.4 are satisfied with $\Omega = B_r^d(x_0)$. The conclusion follows. □

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Appendix: $\mathcal{L}$-Superharmonic Functions

In this section we recall the definition and list some properties of the $\mathcal{L}$-superharmonic functions, as presented in [5, Chap. 8].

Let $\Omega \subseteq \mathbb{G}$ be open and let $u : \Omega \to]-\infty, \infty]$ be lower semicontinuous. We say that u is $\mathcal{L}$-superharmonic in Ω if

(a) $u \in L^1_{\text{loc}}(\Omega)$ and $\mathcal{L}(u) \leq 0$ in Ω in the weak sense of distributions,
(b) u is M_r-continuous; i.e.,

$$u(x) = \lim_{r\to 0+} M_r(u)(x) \qquad \forall x \in \Omega\,.$$

Here M_r denotes the average operator in (4).

A function $v : \Omega \to [-\infty, \infty[$ is $\mathcal{L}$-subharmonic if $-v$ is $\mathcal{L}$-superharmonic. We say that v is $\mathcal{L}$-harmonic if v is smooth and $\mathcal{L}v = 0$.

Let Γ be the fundamental solution of $\mathcal{L}$ and let μ be a nonnegative Radon measure in $\mathbb{G}$. The Γ-potential of μ is defined as follows

$$\Gamma_\mu(x) := \int_{\mathbb{G}} \Gamma(x^{-1} \circ y)\, d\mu(y), \qquad x \in \mathbb{G}.$$

Obviously, if Ω is an open set such that $\mu(\Omega^c) = 0$,

$$\Gamma_\mu(x) = \int_{\Omega} \Gamma(x^{-1} \circ y)\, d\mu(y), \qquad x \in \Omega\,.$$

The function Γ_μ is nonnegative and lower semicontinuous; it is $\mathcal{L}$-superharmonic in $\mathbb{G}$ if and only if there exists $z \in \mathbb{G}$ such that $\Gamma_\mu(z) < \infty$, see [5, Theorem 9.3.2].

In this case, see [5, Theorem 9.3.5],

$$\mathcal{L}\Gamma_\mu = -\mu \quad \text{in the sense of distributions}$$

and

$$\Gamma_\mu \text{ is } \mathcal{L}\text{-harmonic in } \mathbb{G} \setminus \operatorname{supp} \mu.$$

For our purposes, the following remark is crucial.

Remark Let (Ω, μ, x_0) be a Γ-triple (see Definition 2.2) and let $A \subseteq \Omega$ be a Borel set. Then the function

$$\mathbb{G} \ni x \mapsto \Gamma_{\mu_A}(x) := \int_A \Gamma(x^{-1} \circ y)\, d\mu(y)$$

is the Γ-potential of $\mu_A := \mu \llcorner A$ and satisfies

$$\Gamma_{\mu_A}(x) \le \Gamma_\mu(x) = \Gamma(x^{-1} \circ x_0) < \infty \quad \forall x \in \Omega^c.$$

Moreover, Γ_{μ_A} is $\mathcal{L}$-superharmonic in $\mathbb{G}$ and

$$\Gamma_{\mu_A} \text{ is } \mathcal{L}\text{-harmonic in } O$$

for every open set $O \subseteq A^c$. Indeed $O \subseteq A^c$ implies $O \subseteq \overline{A}^c \subseteq (\operatorname{supp} \mu_A)^c$.

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A Good-λ Lemma, Two Weight $T1$ Theorems Without Weak Boundedness, and a Two Weight Accretive Global Tb Theorem

Eric T. Sawyer, Chun-Yen Shen, and Ignacio Uriarte-Tuero

This paper is dedicated to Dick Wheeden on the occasion of his retirement from Rutgers University, and for all of his fundamental contributions to the theory of weighted inequalities, in particular for the beautiful paper of Hunt, Muckenhoupt and Wheeden that started it all back in 1973.

Abstract Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$, let T^α be a standard α-fractional Calderón-Zygmund operator on $\mathbb{R}^n$ with $0 \leq \alpha < n$, and assume as side conditions the $\mathcal{A}_2^\alpha$ conditions, punctured A_2^α conditions, and certain α*-energy conditions*. Then the weak boundedness property associated with the operator T^α and the weight pair (σ, ω), is 'good-λ' controlled by the testing conditions and the Muckenhoupt and energy conditions. As a consequence, assuming the side conditions, we can eliminate the weak boundedness property from Theorem 1 of Sawyer et al. (A two weight fractional singular integral theorem with side conditions, energy and k-energy dispersed. arXiv:1603.04332v2) to obtain that T^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$ if and only if the testing conditions hold for T^α and its dual. As a corollary we give a simple derivation of a two weight accretive global Tb theorem from a related $T1$ theorem. The role of two different parameterizations of the family of dyadic grids, by scale and by translation, is highlighted in simultaneously exploiting both goodness and NTV surgery with families of grids that are *common* to both measures.

E.T. Sawyer (✉)
Department of Mathematics & Statistics, McMaster University, 1280 Main Street West, Hamilton, ON, Canada L8S 4K1
e-mail: sawyer@mcmaster.ca

C.-Y. Shen
Department of Mathematics, National Central University, Chungli 32054, Taiwan
e-mail: cyshen@math.ncu.edu.tw

I. Uriarte-Tuero
Department of Mathematics, Michigan State University, East Lansing, MI, USA
e-mail: ignacio@math.msu.edu

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1 Introduction

The theory of weighted norm inequalities burst into the general mathematical consciousness with the celebrated theorem of Hunt et al. [5] that extended boundedness of the Hilbert transform to measures more general than Lebesgue's, namely showing that H was bounded on the weighted space $L^2(\mathbb{R}^n; w)$ if and only if the A_2 condition of Muckenhoupt,

$$\left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q \frac{1}{w(x)}dx\right) \lesssim 1\ ,$$

holds when taken uniformly over all cubes Q in $\mathbb{R}^n$. The ensuing thread of investigation culminated in the theorem of Coifman and Fefferman [3] that characterized those nonnegative weights w on $\mathbb{R}^n$ for which all of the 'nicest' of the $L^2(\mathbb{R}^n)$ bounded singular integrals T above are bounded on weighted spaces $L^2(\mathbb{R}^n; w)$, and does so in terms of the above A_2 condition of Muckenhoupt.

Attention then turned to the corresponding two weight inequalities for singular integrals, which turned out to be considerably more complicated. For example, Cotlar and Sadosky gave a beautiful function theoretic characterization of the weight pairs (σ, ω) for which H is bounded from $L^2(\mathbb{R}; \sigma)$ to $L^2(\mathbb{R}; \omega)$, namely a two-weight extension of the Helson-Szegö theorem, which illuminated a deep connection between two quite different function theoretic conditions, but failed to shed much light on when either of them held.[1] On the other hand, the two weight inequality for positive fractional integrals, Poisson integrals and maximal functions were characterized using testing conditions by one of us in [24] (see also [6] for the Poisson inequality with 'holes') and [23], but relying in a very strong way on the positivity of the kernel, something the Hilbert kernel lacks. In a groundbreaking series of papers including [16, 18] and [19], Nazarov, Treil and Volberg used weighted Haar decompositions with random grids, introduced their 'pivotal' condition, and proved that the Hilbert transform is bounded from $L^2(\mathbb{R}; \sigma)$ to $L^2(\mathbb{R}; \omega)$ if and only if a variant of the A_2 condition 'on steroids' held, and the norm inequality and its dual held when tested locally over indicators of cubes—but **only** under the side assumption that their pivotal conditions held.

The last dozen years have seen a resurgence in the investigation of two weight inequalities for singular integrals, beginning with the aforementioned work of NTV, and due in part to applications of the two weight $T1$ theorem in operator theory, such as in [14], where embedding measures are characterized for model spaces K_θ, where θ is an inner function on the disk, and where norms of composition operators are characterized that map K_θ into Hardy and Bergman spaces. A $T1$ theorem could also have implications for a number of problems that are higher dimensional analogues of those connected to the Hilbert transform (rank

[1]However, the testing conditions in Theorem 1 are subject to the same criticism due to the highly unstable nature of singular integrals acting on measures.

one perturbations [20, 32]; products of two densely defined Toeplitz operators; subspaces of the Hardy space invariant under the inverse shift operator [15, 32]; orthogonal polynomials [21, 22, 33]; and quasiconformal theory [1, 2, 8, 11]), and we refer the reader to [28] for more detail on these applications.

Following the groundbreaking work of Nazarov, Treil and Volberg, two of us, Sawyer and Uriarte-Tuero, together with Lacey in [12], showed that the pivotal conditions were not necessary in general, and introduced instead a necessary 'energy' condition as a substitute, along with a hybrid merging of these two conditions that was shown to be sufficient for use as a side condition. The resurgence was then capped along the way with a resolution—involving the work of Nazarov, Treil and Volberg in [19], the authors and M. Lacey in the two part paper [9, 13] and T. Hytönen in [6]—of the two weight Hilbert transform conjecture of Nazarov, Treil and Volberg [32]:

Theorem 1 *The Hilbert transform is bounded from $L^2(\mathbb{R};\sigma)$ to $L^2(\mathbb{R};\omega)$, i.e.*

$$\|H(f\sigma)\|_{L^2(\mathbb{R};\omega)} \lesssim \|f\|_{L^2(\mathbb{R};\sigma)}, \quad f\in L^2(\mathbb{R};\sigma), \tag{1}$$

if and only if the two weight A_2 condition with holes holds,

$$\frac{|Q|_\sigma}{|Q|}\left(\frac{1}{|Q|}\int_{\mathbb{R}\setminus Q}\mathbf{s}_Q^2 d\omega(x)\right)+\left(\frac{1}{|Q|}\int_{\mathbb{R}\setminus Q}\mathbf{s}_Q^2 d\sigma(x)\right)\frac{|Q|_\omega}{|Q|}\lesssim 1,$$

uniformly over all cubes Q, and the two testing conditions hold,

$$\left\|\mathbf{1}_Q H(\mathbf{1}_Q\sigma)\right\|_{L^2(\mathbb{R};\omega)} \lesssim \|\mathbf{1}_Q\|_{L^2(\mathbb{R};\sigma)}=\sqrt{|Q|_\sigma},$$
$$\left\|\mathbf{1}_Q H^*(\mathbf{1}_Q\omega)\right\|_{L^2(\mathbb{R};\sigma)} \lesssim \|\mathbf{1}_Q\|_{L^2(\mathbb{R};\omega)}=\sqrt{|Q|_\omega},$$

uniformly over all cubes Q.

Here $Hf(x)=\int_{\mathbb{R}}\frac{f(y)}{y-x}dy$ is the Hilbert transform on the real line $\mathbb{R}$, and σ and ω are locally finite positive Borel measures on $\mathbb{R}$. The two weight A_2 condition with holes is also a testing condition in disguise, in particular it follows from

$$\left\|H(\mathbf{s}_Q\sigma)\right\|_{L^2(\mathbb{R};\omega)} \lesssim \|\mathbf{s}_Q\|_{L^2(\mathbb{R};\sigma)},$$

tested over all 'indicators with tails' $\mathbf{s}_Q(x)=\frac{\ell(Q)}{\ell(Q)+|x-c_Q|}$ of intervals Q in $\mathbb{R}$. Below we discuss the precise interpretation of the above inequalities involving the singular integral H.

At this juncture, attention naturally turned to the analogous two weight inequalities for *higher dimensional* singular integrals, as well as α-fractional singular integrals such as the Cauchy transform in the plane. A variety of results were obtained, e.g. [10, 14, 26] and [27], in which a $T1$ theorem was proved under certain side conditions that implied the energy conditions. However, in [28], the authors have recently shown that the energy conditions are *not* in general necessary for elliptic singular integrals.

The aforementioned higher dimensional results require refinements of the various one-dimensional conditions associated with the norm inequalities, namely the A_2 conditions, the testing conditions, the weak boundedness property and energy conditions. The purpose of this paper is to prove in higher dimensions that the weak boundedness constant $\mathcal{WBP}_{T^\alpha}(\sigma,\omega)$ that is associated with an α-fractional singular integral T^α and a weight pair (σ,ω) in $\mathbb{R}^n$, is 'good-λ' controlled by the usual testing conditions $\mathfrak{T}_{T^\alpha}(\sigma,\omega)$, $\mathfrak{T}^*_{T^\alpha}(\sigma,\omega)$ and two side conditions on weight pairs, namely the Muckenhoupt conditions $\mathfrak{A}_2^\alpha(\sigma,\omega)$ and the energy conditions $\mathcal{E}_\alpha^{\text{strong}}(\sigma,\omega)$, $\mathcal{E}_\alpha^{\text{strong},*}(\sigma,\omega)$: more precisely, for every $0<\lambda<\frac{1}{2}$, we have the Good-λ Lemma:

$$\mathcal{WBP}_{T^\alpha}(\sigma,\omega)\leq C_\alpha\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}^*_{T^\alpha}+\mathcal{E}_\alpha^{\text{strong}}+\mathcal{E}_\alpha^{\text{strong},*}+\sqrt[4]{\lambda}\,\mathfrak{N}_{T^\alpha}\right).$$

The first instance of this type of conclusion appears in Lacey and Wick in [10])—see Remark 1 in Sect. 2.1 below.

Applications of the Good-λ Lemma are then given to obtain both $T1$ and Tb theorems for two weights. We now turn to a description of the higher dimensional conditions appearing in the above display. As the Good-λ Lemma, along with its corollaries, hold in the more general setting of quasicubes, we describe them first. But the reader interested only in cubes can safely ignore this largely cosmetic generalization (but crucial for our 'measure on a curve' $T1$ theorem in [26]) by simply deleting the prefix 'quasi' wherever it appears.

1.1 Quasicubes

We begin by recalling the notion of quasicube used in [27]—a special case of the classical notion used in quasiconformal theory.

Definition 1 We say that a homeomorphism $\Omega:\mathbb{R}^n\to\mathbb{R}^n$ is a globally biLipschitz map if

$$\|\Omega\|_{Lip}\equiv\sup_{x,y\in\mathbb{R}^n}\frac{\|\Omega(x)-\Omega(y)\|}{\|x-y\|}<\infty, \tag{2}$$

and $\left\|\Omega^{-1}\right\|_{Lip}<\infty$.

Notation 1 *We define $\mathcal{P}^n$ to be the collection of half open, half closed cubes in $\mathbb{R}^n$ with sides parallel to the coordinate axes. A half open, half closed cube Q in $\mathbb{R}^n$ has the form $Q=Q(c,\ell)\equiv\prod_{k=1}^n\left[c_k-\frac{\ell}{2},c_k+\frac{\ell}{2}\right)$ for some $\ell>0$ and $c=(c_1,\dots,c_n)\in\mathbb{R}^n$. The cube $Q(c,\ell)$ is described as having center c and sidelength ℓ.*

Definition 2 Suppose that $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ is a globally biLipschitz map.

(1) If E is a measurable subset of $\mathbb{R}^n$, we define $\Omega E \equiv \{\Omega (x) : x \in E\}$ to be the image of E under the homeomorphism Ω.

  (a) In the special case that $E = Q$ is a cube in $\mathbb{R}^n$, we will refer to ΩQ as a quasicube (or Ω-quasicube if Ω is not clear from the context).
  (b) We define the center $c_{\Omega Q} = c(\Omega Q)$ of the quasicube ΩQ to be the point Ωc_Q where $c_Q = c(Q)$ is the center of Q.
  (c) We define the side length $\ell(\Omega Q)$ of the quasicube ΩQ to be the sidelength $\ell(Q)$ of the cube Q.
  (d) For $r > 0$ we define the 'dilation' $r\Omega Q$ of a quasicube ΩQ to be $\Omega r Q$ where rQ is the usual 'dilation' of a cube in $\mathbb{R}^n$ that is concentric with Q and having side length $r\ell(Q)$.

(2) If $\mathcal{K}$ is a collection of cubes in $\mathbb{R}^n$, we define $\Omega\mathcal{K} \equiv \{\Omega Q : Q \in \mathcal{K}\}$ to be the collection of quasicubes ΩQ as Q ranges over $\mathcal{K}$.
(3) If $\mathcal{F}$ is a grid of cubes in $\mathbb{R}^n$, we define the inherited quasigrid structure on $\Omega\mathcal{F}$ by declaring that ΩQ is a child of $\Omega Q'$ in $\Omega\mathcal{F}$ if Q is a child of Q' in the grid $\mathcal{F}$.

Note that if ΩQ is a quasicube, then $|\Omega Q|^{\frac{1}{n}} \approx |Q|^{\frac{1}{n}} = \ell(Q) = \ell(\Omega Q)$. For a quasicube $J = \Omega Q$, we will generally use the expression $|J|^{\frac{1}{n}}$ in the various estimates arising in the proofs below, but will often use $\ell(J)$ when defining collections of quasicubes. Moreover, there are constants R_{big} and R_{small} such that we have the comparability containments

$$Q + \Omega x_Q \subset R_{big}\Omega Q \text{ and } R_{small}\Omega Q \subset Q + \Omega x_Q .$$

Example 1 Quasicubes can be wildly shaped, as illustrated by the standard example of a logarithmic spiral in the plane $f_\varepsilon(z) = z|z|^{2\varepsilon i} = ze^{i\varepsilon \ln(z\bar{z})}$. Indeed, $f_\varepsilon : \mathbb{C} \to \mathbb{C}$ is a globally biLipschitz map with Lipschitz constant $1 + C\varepsilon$ since $f_\varepsilon^{-1}(w) = w|w|^{-2\varepsilon i}$ and

$$\nabla f_\varepsilon = \left(\frac{\partial f_\varepsilon}{\partial z}, \frac{\partial f_\varepsilon}{\partial \bar{z}}\right) = \left(|z|^{2\varepsilon i} + i\varepsilon |z|^{2\varepsilon i}, i\varepsilon \frac{z}{\bar{z}} |z|^{2\varepsilon i}\right).$$

On the other hand, f_ε behaves wildly at the origin since the image of the closed unit interval on the real line under f_ε is an infinite logarithmic spiral.

1.2 Standard Fractional Singular Integrals and the Norm Inequality

Let $0 \leq \alpha < n$. We define a standard α-fractional CZ kernel $K^\alpha(x, y)$ to be a real-valued function defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the following fractional size and smoothness conditions of order $1+\delta$ for some $\delta>0$: For $x\neq y$,

$$|K^{\alpha}(x,y)|\leq C_{CZ}|x-y|^{\alpha-n} \text{ and } |\nabla K^{\alpha}(x,y)|\leq C_{CZ}|x-y|^{\alpha-n-1},$$

$$\left|\nabla K^{\alpha}(x,y)-\nabla K^{\alpha}\left(x',y\right)\right|\leq C_{CZ}\left(\frac{|x-x'|}{|x-y|}\right)^{\delta}|x-y|^{\alpha-n-1}, \quad \frac{|x-x'|}{|x-y|}\leq\frac{1}{2}, \tag{3}$$

and the last inequality also holds for the adjoint kernel in which x and y are interchanged. We note that a more general definition of kernel has only order of smoothness $\delta>0$, rather than $1+\delta$, but the use of the Monotonicity and Energy Lemmas in arguments below, which involve first order Taylor approximations to the kernel functions $K^{\alpha}(\cdot,y)$, requires order of smoothness more than 1 to handle remainder terms.

1.2.1 Defining the Norm Inequality

We now turn to a precise definition of the weighted norm inequality

$$\|T^{\alpha}_{\sigma}f\|_{L^2(\omega)}\leq\mathfrak{N}_{T^{\alpha}_{\sigma}}\|f\|_{L^2(\sigma)}, \quad f\in L^2(\sigma). \tag{4}$$

For this we introduce a family $\left\{\eta^{\alpha}_{\delta,R}\right\}_{0<\delta<R<\infty}$ of nonnegative functions on $[0,\infty)$ so that the truncated kernels $K^{\alpha}_{\delta,R}(x,y)=\eta^{\alpha}_{\delta,R}(|x-y|)K^{\alpha}(x,y)$ are bounded with compact support for fixed x or y. Then the truncated operators

$$T^{\alpha}_{\sigma,\delta,R}f(x)\equiv\int_{\mathbb{R}^n}K^{\alpha}_{\delta,R}(x,y)f(y)\,d\sigma(y), \quad x\in\mathbb{R}^n,$$

are pointwise well-defined, and we will refer to the pair $\left(K^{\alpha},\left\{\eta^{\alpha}_{\delta,R}\right\}_{0<\delta<R<\infty}\right)$ as an α-fractional singular integral operator, which we typically denote by T^{α}, suppressing the dependence on the truncations.

Definition 3 We say that an α-fractional singular integral operator $T^{\alpha}=\left(K^{\alpha},\left\{\eta^{\alpha}_{\delta,R}\right\}_{0<\delta<R<\infty}\right)$ satisfies the norm inequality (4) provided

$$\left\|T^{\alpha}_{\sigma,\delta,R}f\right\|_{L^2(\omega)}\leq\mathfrak{N}_{T^{\alpha}_{\sigma}}\|f\|_{L^2(\sigma)}, \quad f\in L^2(\sigma), 0<\delta<R<\infty.$$

It turns out that, in the presence of Muckenhoupt conditions, the norm inequality (4) is essentially independent of the choice of truncations used, and we now explain this in some detail. A *smooth truncation* of T^{α} has kernel $\eta_{\delta,R}(|x-y|)K^{\alpha}(x,y)$ for a smooth function $\eta_{\delta,R}$ compactly supported in (δ,R), $0<\delta<R<\infty$, and satisfying standard CZ estimates. A typical example of an

α-fractional transform is the α-fractional *Riesz* vector of operators

$$\mathbf{R}^{\alpha,n} = \left\{ R_\ell^{\alpha,n} : 1 \le \ell \le n \right\}, \quad 0 \le \alpha < n.$$

The Riesz transforms $R_\ell^{n,\alpha}$ are convolution fractional singular integrals $R_\ell^{n,\alpha} f \equiv K_\ell^{n,\alpha} * f$ with odd kernel defined by

$$K_\ell^{\alpha,n}(w) \equiv \frac{w^\ell}{|w|^{n+1-\alpha}} \equiv \frac{\Omega_\ell(w)}{|w|^{n-\alpha}}, \quad w = \left(w^1, \ldots, w^n\right).$$

However, in dealing with energy considerations, and in particular in the Monotonicity Lemma below where first order Taylor approximations are made on the truncated kernels, it is necessary to use the *tangent line truncation* of the Riesz transform $R_\ell^{\alpha,n}$ whose kernel is defined to be $\Omega_\ell(w)\,\psi_{\delta,R}^\alpha(|w|)$ where $\psi_{\delta,R}^\alpha$ is continuously differentiable on an interval $(0,S)$ with $0 < \delta < R < S$, and where $\psi_{\delta,R}^\alpha(r) = r^{\alpha-n}$ if $\delta \le r \le R$, and has constant derivative on both $(0,\delta)$ and (R,S) where $\psi_{\delta,R}^\alpha(S) = 0$. Here S is uniquely determined by R and α. Finally we set $\psi_{\delta,R}^\alpha(S) = 0$ as well, so that the kernel vanishes on the diagonal and common point masses do not 'see' each other. Note also that the tangent line extension of a $C^{1,\delta}$ function on the line is again $C^{1,\delta}$ with no increase in the $C^{1,\delta}$ norm.

It was shown in the one dimensional case with no common point masses in [13], that boundedness of the Hilbert transform H with one set of appropriate truncations together with the A_2^α condition without holes, is equivalent to boundedness of H with any other set of appropriate truncations, and this was extended to $\mathbf{R}^{\alpha,n}$ and more general operators in higher dimensions, permitting common point masses as well. Thus we are free to use the tangent line truncations throughout the proofs of our results.

1.3 Quasicube Testing Conditions

The following 'dual' quasicube testing conditions are necessary for the boundedness of T^α from $L^2(\sigma)$ to $L^2(\omega)$,

$$\mathfrak{T}_{T^\alpha}^2 \equiv \sup_{Q \in \Omega\mathcal{P}^n} \frac{1}{|Q|_\sigma} \int_Q \left| T^\alpha \left(\mathbf{1}_Q \sigma\right)\right|^2 \omega < \infty,$$

$$\left(\mathfrak{T}_{T^\alpha}^*\right)^2 \equiv \sup_{Q \in \Omega\mathcal{P}^n} \frac{1}{|Q|_\omega} \int_Q \left| (T^\alpha)^* \left(\mathbf{1}_Q \omega\right)\right|^2 \sigma < \infty,$$

and where we interpret the right sides as holding uniformly over all tangent line truncations of T^α. Equally necessary are the following 'full' testing conditions where the integrations are taken over the entire space $\mathbb{R}^n$:

$$\mathfrak{F}\mathfrak{T}_{T^\alpha}^2 \equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{1}{|Q|_\sigma} \int_{\mathbb{R}^n} \left|T^\alpha\left(\mathbf{1}_Q\sigma\right)\right|^2 \omega < \infty,$$

$$\left(\mathfrak{F}\mathfrak{T}_{T^\alpha}^*\right)^2 \equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{1}{|Q|_\omega} \int_{\mathbb{R}^n} \left|(T^\alpha)^*\left(\mathbf{1}_Q\omega\right)\right|^2 \sigma < \infty,$$

1.4 Quasiweak Boundedness and Indicator/Touching Property

The quasiweak boundedness property for T^α with constant C is given by

$$\left|\int_Q T^\alpha\left(1_{Q'}\sigma\right) d\omega\right| \leq \mathcal{WBP}_{T^\alpha}\sqrt{|Q|_\omega\,|Q'|_\sigma}, \tag{5}$$

$$\text{for all quasicubes } Q, Q' \text{ with } \frac{1}{C} \leq \frac{\ell(Q)}{\ell(Q')} \leq C,$$

$$\text{and either } Q \subset 3Q' \setminus Q' \text{ or } Q' \subset 3Q \setminus Q,$$

and where we interpret the left side above as holding uniformly over all tangent line trucations of T^α. This condition is used in our $T1$ theorem with an energy side condition in [27], but will be removed in our $T1$ theorem with an energy side condition obtained here as a corollary of the Good-λ Lemma.

We say that two quasicubes Q and Q' in $\Omega\mathcal{P}^n$ are *touching quasicubes* if the intersection of their closures is nonempty and contained in the boundary of the larger quasicube. Finally, let $\mathfrak{I}_{T^\alpha} = \mathfrak{I}_{T^\alpha}(\sigma,\omega)$ be the best constant in the *indicator/touching* inequality for the bilinear form corresponding to T

$$\left|\mathcal{T}^\alpha\left(\mathbf{1}_Q, \mathbf{1}_{Q'}\right)\right| \leq \mathfrak{I}_{T^\alpha}(\sigma,\omega)\left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}\left\|\mathbf{1}_{Q'}\right\|_{L^2(\omega)}\ , \tag{6}$$

$$\text{for all touching quasicubes } Q, Q' \in \mathcal{P}^n,$$

$$\text{with } \frac{1}{C} \leq \frac{\ell(Q)}{\ell(Q')} \leq C,$$

$$\text{and either } Q \subset 3Q' \setminus Q' \text{ or } Q' \subset 3Q \setminus Q.$$

1.5 Poisson Integrals and $\mathcal{A}_2^\alpha$

Let μ be a locally finite positive Borel measure on $\mathbb{R}^n$, and suppose Q is an Ω-quasicube in $\mathbb{R}^n$. Recall that $|Q|^{\frac{1}{n}} \approx \ell(Q)$ for a quasicube Q. The two α-fractional

Poisson integrals of μ on a quasicube Q are given by:

$$\mathrm{P}^{\alpha}(Q,\mu) \equiv \int_{\mathbb{R}^n} \frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}}+|x-x_Q|\right)^{n+1-\alpha}} d\mu(x),$$

$$\mathcal{P}^{\alpha}(Q,\mu) \equiv \int_{\mathbb{R}^n} \left(\frac{|Q|^{\frac{1}{n}}}{\left(|Q|^{\frac{1}{n}}+|x-x_Q|\right)^{2}}\right)^{n-\alpha} d\mu(x),$$

where we emphasize that $|x-x_Q|$ denotes Euclidean distance between x and x_Q and $|Q|$ denotes the Lebesgue measure of the quasicube Q. We refer to P^{α} as the *standard* Poisson integral and to $\mathcal{P}^{\alpha}$ as the *reproducing* Poisson integral.

We say that the pair K, K' in $\mathcal{P}^n$ are *neighbours* if K and K' live in a common dyadic grid and both $K \subset 3K' \setminus K'$ and $K' \subset 3K \setminus K$, and we denote by $\mathcal{N}^n$ the set of pairs (K, K') in $\mathcal{P}^n \times \mathcal{P}^n$ that are neighbours. Let

$$\Omega\mathcal{N}^n = \left\{\left(\Omega K, \Omega K'\right) : \left(K, K'\right) \in \mathcal{N}^n\right\}$$

be the corresponding collection of neighbour pairs of quasicubes. Let σ and ω be locally finite positive Borel measures on $\mathbb{R}^n$, and suppose $0 \le \alpha < n$. Then we define the classical *offset* A_2^{α} *constants* by

$$A_2^{\alpha}(\sigma,\omega) \equiv \sup_{(Q,Q')\in\Omega\mathcal{N}^n} \frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q'|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}}. \tag{7}$$

Since the cubes in $\mathcal{P}^n$ are products of half open, half closed intervals $[a, b)$, the neighbouring quasicubes $(Q, Q') \in \Omega\mathcal{N}^n$ are disjoint, and any common point masses of σ and ω do not simultaneously appear in each factor.

We now define the *one-tailed* $\mathcal{A}_2^{\alpha}$ constant using $\mathcal{P}^{\alpha}$. The energy constants $\mathcal{E}_{\alpha}^{\text{strong}}$ introduced below will use the standard Poisson integral P^{α}.

Definition 4 The one-tailed constants $\mathcal{A}_2^{\alpha}$ and $\mathcal{A}_2^{\alpha,*}$ for the weight pair (σ, ω) are given by

$$\mathcal{A}_2^{\alpha} \equiv \sup_{Q\in\Omega\mathcal{P}^n} \mathcal{P}^{\alpha}\left(Q, \mathbf{1}_{Q^c}\sigma\right) \frac{|Q|_{\omega}}{|Q|^{1-\frac{\alpha}{n}}} < \infty,$$

$$\mathcal{A}_2^{\alpha,*} \equiv \sup_{Q\in\Omega\mathcal{P}^n} \mathcal{P}^{\alpha}\left(Q, \mathbf{1}_{Q^c}\omega\right) \frac{|Q|_{\sigma}}{|Q|^{1-\frac{\alpha}{n}}} < \infty.$$

Note that these definitions are the analogues of the corresponding conditions with ‘holes’ introduced by Hytönen [6] in dimension $n = 1$—the supports of the measures $\mathbf{1}_{Q^c}\sigma$ and $\mathbf{1}_{Q}\omega$ in the definition of $\mathcal{A}_2^{\alpha}$ are disjoint, and so the common point masses of σ and ω do not appear simultaneously in each factor. Note also that, unlike in [29], where common point masses were not permitted, we can no longer assert the equivalence of $\mathcal{A}_2^\alpha$ with holes taken over *quasicubes* with $\mathcal{A}_2^\alpha$ with holes taken over *cubes*.

1.5.1 Punctured A_2^α Conditions

The *classical* A_2^α characteristic $\sup_{Q\in\Omega\mathcal{Q}^n} \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}$ fails to be finite when the measures σ and ω have a common point mass—simply let Q in the sup above shrink to a common mass point. But there is a substitute that is quite similar in character that is motivated by the fact that for large quasicubes Q, the sup above is problematic only if just *one* of the measures is *mostly* a point mass when restricted to Q.

Given an at most countable set $\mathfrak{P} = \{p_k\}_{k=1}^\infty$ in $\mathbb{R}^n$, a quasicube $Q \in \Omega\mathcal{P}^n$, and a locally finite positive Borel measure μ, define as in [27],

$$\mu(Q, \mathfrak{P}) \equiv |Q|_\mu - \sup\{\mu(p_k) : p_k \in Q \cap \mathfrak{P}\},$$

where the supremum is actually achieved since $\sum_{p_k\in Q\cap\mathfrak{P}} \mu(p_k) < \infty$ as μ is locally finite. The quantity $\mu(Q, \mathfrak{P})$ is simply the $\widetilde{\mu}$ measure of Q where $\widetilde{\mu}$ is the measure μ with its largest point mass from $\mathfrak{P}$ in Q removed. Given a locally finite measure pair (σ, ω), let $\mathfrak{P}_{(\sigma,\omega)} = \{p_k\}_{k=1}^\infty$ be the at most countable set of common point masses of σ and ω. Then the weighted norm inequality (4) typically implies finiteness of the following *punctured* Muckenhoupt conditions (see [27]):

$$A_2^{\alpha,\text{punct}}(\sigma, \omega) \equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{\omega\left(Q, \mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}} \frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}},$$

$$A_2^{\alpha,*,\text{punct}}(\sigma, \omega) \equiv \sup_{Q\in\Omega\mathcal{P}^n} \frac{|Q|_\omega}{|Q|^{1-\frac{\alpha}{n}}} \frac{\sigma\left(Q, \mathfrak{P}_{(\sigma,\omega)}\right)}{|Q|^{1-\frac{\alpha}{n}}}.$$

Now we turn to the definition of a quasiHaar basis of $L^2(\mu)$.

1.6 A Weighted QuasiHaar Basis

We will use a construction of a quasiHaar basis in $\mathbb{R}^n$ that is adapted to a measure μ (c.f. [18] for the nonquasi case). Given a dyadic quasicube $Q \in \Omega\mathcal{D}$, where $\mathcal{D}$ is a dyadic grid of cubes from $\mathcal{P}^n$, let $\triangle_Q^\mu$ denote orthogonal projection onto the finite dimensional subspace $L_Q^2(\mu)$ of $L^2(\mu)$ that consists of linear combinations of the indicators of the children $\mathfrak{C}(Q)$ of Q that have μ-mean zero over Q:

$$L_Q^2(\mu) \equiv \left\{ f = \sum_{Q'\in\mathfrak{C}(Q)} a_{Q'} \mathbf{1}_{Q'} : a_{Q'} \in \mathbb{R}, \int_Q f d\mu = 0 \right\}.$$

Then we have the important telescoping property for dyadic quasicubes $Q_1 \subset Q_2$ that arises from the martingale differences associated with the projections $\triangle_Q^\mu$:

$$\mathbf{1}_{Q_0}(x)\left(\sum_{Q\in[Q_1,Q_2]}\triangle_Q^\mu f(x)\right)=\mathbf{1}_{Q_0}(x)\left(\mathbb{E}_{Q_0}^\mu f-\mathbb{E}_{Q_2}^\mu f\right),\quad Q_0\in\mathfrak{C}(Q_1),\ f\in L^2(\mu). \tag{8}$$

We will at times find it convenient to use a fixed orthonormal basis $\{h_Q^{\mu,a}\}_{a\in\Gamma_n}$ of $L_Q^2(\mu)$ where $\Gamma_n \equiv \{0,1\}^n \setminus \{\mathbf{1}\}$ is a convenient index set with $\mathbf{1}=(1,1,\ldots,1)$. Then $\{h_Q^{\mu,a}\}_{a\in\Gamma_n \text{ and } Q\in\Omega\mathcal{D}}$ is an orthonormal basis for $L^2(\mu)$, with the understanding that we add the constant function $\mathbf{1}$ if μ is a finite measure. In particular we have for an infinite measure

$$\|f\|_{L^2(\mu)}^2=\sum_{Q\in\Omega\mathcal{D}}\left\|\triangle_Q^\mu f\right\|_{L^2(\mu)}^2=\sum_{Q\in\Omega\mathcal{D}}\sum_{a\in\Gamma_n}\left|\widehat{f}(Q)\right|^2,\quad \left|\widehat{f}(Q)\right|^2\equiv\sum_{a\in\Gamma_n}\left|\left\langle f,h_Q^{\mu,a}\right\rangle_\mu\right|^2,$$

where the measure is suppressed in the notation $\widehat{f}$. Indeed, this follows from (8) and Lebesgue's differentiation theorem for quasicubes. We also record the following useful estimate. If I' is any of the 2^n $\Omega\mathcal{D}$-children of I, and $a\in\Gamma_n$, then

$$\left|\mathbb{E}_{I'}^\mu h_I^{\mu,a}\right|\le\sqrt{\mathbb{E}_{I'}^\mu\left(h_I^{\mu,a}\right)^2}\le\frac{1}{\sqrt{|I'|_\mu}}. \tag{9}$$

1.7 The Strong Quasienergy Conditions

Given a dyadic quasicube $K\in\Omega\mathcal{D}$ and a positive measure μ we define the quasiHaar projection $\mathsf{P}_K^\mu\equiv\sum_{J\in\Omega\mathcal{D}:\ J\subset K}\triangle_J^\mu$ on K by

$$\mathsf{P}_K^\mu f=\sum_{J\in\Omega\mathcal{D}:\ J\subset K}\sum_{a\in\Gamma_n}\left\langle f,h_J^{\mu,a}\right\rangle_\mu h_J^{\mu,a}\text{ so that }\left\|\mathsf{P}_K^\mu f\right\|_{L^2(\mu)}^2=\sum_{J\in\Omega\mathcal{D}:\ J\subset K}\sum_{a\in\Gamma_n}\left|\left\langle f,h_J^{\mu,a}\right\rangle_\mu\right|^2,$$

and where a quasiHaar basis $\{h_J^{\mu,a}\}_{a\in\Gamma_n \text{ and } J\in\mathcal{D}\Omega}$ adapted to the measure μ was defined in the subsection on a weighted quasiHaar basis above.

Now we define various notions for quasicubes which are inherited from the same notions for cubes. The main objective here is to use the familiar notation that one uses for cubes, but now extended to Ω-quasicubes. We have already introduced the notions of quasigrids $\Omega\mathcal{D}$, and center, sidelength and dyadic associated to quasicubes $Q\in\Omega\mathcal{D}$, as well as quasiHaar functions, and we will continue to extend to quasicubes the additional familiar notions related to cubes as we come across them. We begin with the notion of *deeply embedded*. Fix a quasigrid $\Omega\mathcal{D}$. We say that a dyadic quasicube J is $(\mathbf{r}, \varepsilon)$-*deeply embedded* in a (not necessarily dyadic) quasicube K, which we write as $J \Subset_{\mathbf{r},\varepsilon} K$, when $J \subset K$ and both

$$\ell(J) \leq 2^{-\mathbf{r}}\ell(K), \qquad (10)$$
$$\operatorname{qdist}(J, \partial K) \geq \frac{1}{2}\ell(J)^{\varepsilon}\,\ell(K)^{1-\varepsilon},$$

where we define the quasidistance $\operatorname{qdist}(E, F)$ between two sets E and F to be the Euclidean distance $\operatorname{dist}\left(\Omega^{-1}E, \Omega^{-1}F\right)$ between the preimages $\Omega^{-1}E$ and $\Omega^{-1}F$ of E and F under the map Ω, and where we recall that $\ell(J) \approx |J|^{\frac{1}{n}}$. For the most part we will consider $J \Subset_{\mathbf{r},\varepsilon} K$ when J and K belong to a common quasigrid $\Omega\mathcal{D}$, but an exception is made when defining the strong energy constants below.

Recall that in dimension $n = 1$, and for $\alpha = 0$, the energy condition constant was defined by

$$\mathcal{E}^2 \equiv \sup_{I=\dot{\cup} I_r} \frac{1}{|I|_\sigma}\sum_{r=1}^{\infty}\left(\frac{\mathrm{P}^{\alpha}(I_r, \mathbf{1}_I\sigma)}{|I_r|}\right)^2 \left\|\mathsf{P}^{\omega}_{I_r}\mathbf{x}\right\|^2_{L^2(\omega)},$$

where I, I_r and J are intervals in the real line. The extension to higher dimensions we use here is that of 'strong quasienergy condition' defined in [27] and recalled below.

We define a quasicube K (not necessarily in $\Omega\mathcal{D}$) to be an *alternate* $\Omega\mathcal{D}$-quasicube if it is a union of 2^n $\Omega\mathcal{D}$-quasicubes K' with side length $\ell(K') = \frac{1}{2}\ell(K)$ (such quasicubes were called shifted in [29], but that terminology conflicts with the more familiar notion of shifted quasigrid). Thus for any $\Omega\mathcal{D}$-quasicube L there are exactly 2^n alternate $\Omega\mathcal{D}$-quasicubes of twice the side length that contain L, and one of them is of course the $\Omega\mathcal{D}$-parent of L. We denote the collection of alternate $\Omega\mathcal{D}$-quasicubes by $\mathcal{A}\Omega\mathcal{D}$.

The extension of the energy conditions to higher dimensions in [29] used the collection

$$\mathcal{M}_{\mathbf{r},\varepsilon-\mathrm{deep}}(K) \equiv \{\text{maximal dyadic } J \Subset_{\mathbf{r},\varepsilon} K\}$$

of *maximal* $(\mathbf{r}, \varepsilon)$-deeply embedded dyadic subquasicubes of a quasicube K (a subquasicube J of K is a *dyadic* subquasicube of K if $J \in \Omega\mathcal{D}$ when $\Omega\mathcal{D}$ is a dyadic quasigrid containing K). This collection of dyadic subquasicubes of K is of course a pairwise disjoint decomposition of K. We also defined there a refinement and extension of the collection $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$ for certain K and each $\ell \geq 1$. For an alternate quasicube $K \in \mathcal{A}\Omega\mathcal{D}$, define $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K)$ to consist of the *maximal* $\mathbf{r}$-deeply embedded $\Omega\mathcal{D}$-dyadic subquasicubes J of K. (In the special case that K itself belongs to $\Omega\mathcal{D}$, then $\mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K) = \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)$.) Then in [29] for $\ell \geq 1$ we defined the refinement

$$\mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}(K) \equiv \Big\{J \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep},\Omega\mathcal{D}}\left(\pi^{\ell}K'\right) \text{ for some } K' \in \mathfrak{C}_{\Omega\mathcal{D}}(K) :$$
$$J \subset L \text{ for some } L \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\mathrm{deep}}(K)\Big\},$$

where $\mathfrak{C}_{\Omega\mathcal{D}}(K)$ is the obvious extension to alternate quasicubes of the set of $\Omega\mathcal{D}$-dyadic children. Thus $\mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\text{deep},\Omega\mathcal{D}}(K)$ is the union, over all quasichildren K' of K, of those quasicubes in $\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}\left(\pi^{\ell}K'\right)$ that happen to be contained in some $L \in \mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep},\Omega\mathcal{D}}(K)$. We then define the *strong* quasienergy condition as follows.

Definition 5 Let $0 \leq \alpha < n$ and fix 'goodness' parameters $(\mathbf{r},\varepsilon)$. Suppose σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$. Then the *strong* quasienergy constant $\mathcal{E}^{\text{strong}}_{\alpha}$ is defined by

$$\left(\mathcal{E}^{\text{strong}}_{\alpha}\right)^2 \equiv \sup_{I=\dot{\cup} I_r} \frac{1}{|I|_{\sigma}} \sum_{r=1}^{\infty} \sum_{J\in\mathcal{M}_{\mathbf{r},\varepsilon-\text{deep}}(I_r)} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_I\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\omega}_J\mathbf{x}\right\|^2_{L^2(\omega)}$$

$$+ \sup_{\Omega\mathcal{D}} \sup_{I\in\mathcal{A}\Omega\mathcal{D}} \sup_{\ell\geq 0} \frac{1}{|I|_{\sigma}} \sum_{J\in\mathcal{M}^{\ell}_{(\mathbf{r},\varepsilon)-\text{deep},\Omega\mathcal{D}}(I)} \left(\frac{\mathrm{P}^{\alpha}\left(J,\mathbf{1}_I\sigma\right)}{|J|^{\frac{1}{n}}}\right)^2 \left\|\mathsf{P}^{\omega}_J\mathbf{x}\right\|^2_{L^2(\omega)} .$$

Similarly we have a dual version of $\mathcal{E}^{\text{strong}}_{\alpha}$ denoted $\mathcal{E}^{\text{strong},*}_{\alpha}$, and both depend on $\mathbf{r}$ and ε as well as on n and α. An important point in this definition is that the quasicube I in the second line is permitted to lie *outside* the quasigrid $\Omega\mathcal{D}$, but only as an alternate dyadic quasicube $I \in \mathcal{A}\Omega\mathcal{D}$. In the setting of quasicubes we continue to use the linear function $\mathbf{x}$ in the final factor $\left\|\mathsf{P}^{\omega}_J\mathbf{x}\right\|^2_{L^2(\omega)}$ of each line, and not the pushforward of $\mathbf{x}$ by Ω. The reason of course is that this condition is used to capture the first order information in the Taylor expansion of a singular kernel.

2 The Good-λ Lemma

The basic new result of this paper is the following 'Good-λ Lemma' whose utility will become evident when we pursue its corollaries below. Set *fraktur* A^{α}_2 to be the sum of the four A^{α}_2 conditions:

$$\mathfrak{A}^{\alpha}_2 = \mathcal{A}^{\alpha}_2 + \mathcal{A}^{\alpha,*}_2 + A^{\alpha,\text{punct}}_2 + A^{\alpha,*,\text{punct}}_2 .$$

Lemma 1 (The Good-λ Lemma) *Suppose that T^{α} is a standard α-fractional singular integral in $\mathbb{R}^n$, and that σ and ω are locally finite positive Borel measures on $\mathbb{R}^n$. For every $\lambda \in \left(0,\frac{1}{2}\right)$, we have*

$$\mathcal{WBP}_{T^{\alpha}}(\sigma,\omega) \tag{11}$$
$$\leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_2(\sigma,\omega)} + \left(\mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^{*}_{T^{\alpha}}\right)(\sigma,\omega) + \left(\mathcal{E}^{\text{strong}}_{\alpha} + \mathcal{E}^{\text{strong},*}_{\alpha}\right)(\sigma,\omega) + \sqrt[4]{\lambda}\,\mathfrak{N}_{T^{\alpha}}(\sigma,\omega)\right).$$

Thus the effect of the Good-λ Lemma is to 'good-λ replace' the quasiweak boundedness property with just the usual testing conditions in the presence of the side conditions of Muckenhoupt and energy on the weight pair. However, in dimension $n = 1$ a much stronger inequality can be proved (see e.g. [19] and [12]):

$$\mathcal{WBP}_{T^\alpha} \leq C_\alpha \left(\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* \right).$$

2.1 Corollaries

Now we come to the corollaries of the Good-λ Lemma. We first remove the hypothesis of the quasiweak boundedness property from the conclusion of part (1) of Theorem 1 in [27].

Remark 1 In [10], Lacey and Wick have removed the weak boundedness property from their $T1$ theorem by using NTV surgery with two independent grids, one for each function f and g in $\langle T_\sigma^\alpha f, g \rangle$, in the course of their argument. The use of independent grids for each of f and g greatly simplifies the NTV surgery, but does not accommodate our control of functional energy by Muckenhoupt and energy conditions.

Theorem 2 *Suppose $0 \leq \alpha < n$, that T^α is a standard α-fractional singular integral operator on $\mathbb{R}^n$, and that ω and σ are locally finite positive Borel measures on $\mathbb{R}^n$. Set $T_\sigma^\alpha f = T^\alpha(f\sigma)$ for any smooth truncation of T_σ^α. Let $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ be a globally biLipschitz map. Then the operator T_σ^α is bounded from $L^2(\sigma)$ to $L^2(\omega)$, i.e.*

$$\|T_\sigma^\alpha f\|_{L^2(\omega)} \leq \mathfrak{N}_{T_\sigma^\alpha} \|f\|_{L^2(\sigma)},$$

uniformly in smooth truncations of T^α, and moreover

$$\mathfrak{N}_{T_\sigma^\alpha} \leq C_\alpha \left(\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} \right),$$

provided that the two dual $\mathcal{A}_2^\alpha$ conditions and the two dual punctured Muckenhoupt conditions all hold, and the two dual quasitesting conditions for T^α hold, and provided that the two dual strong quasienergy conditions hold uniformly over all dyadic quasigrids $\Omega\mathcal{D} \subset \Omega\mathcal{P}^n$, i.e. $\mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} < \infty$, and where the goodness parameters $\mathbf{r}$ and ε implicit in the definition of the collections $\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep}}(K)$ and $\mathcal{M}_{(\mathbf{r},\varepsilon)-\text{deep},\Omega\mathcal{D}}^\ell(K)$ appearing in the strong energy conditions, are fixed sufficiently large and small respectively depending only on n and α.

Proof Let $T_{\delta,R}^\alpha$ be a tangent line approximation to T^α as introduced above. Then $\mathfrak{N}_{T_{\delta,R}^\alpha} < \infty$, indeed $\mathfrak{N}_{T_{\delta,R}^\alpha} \leq C_{n,\alpha,\delta,R}\sqrt{\mathfrak{A}_2^\alpha}$ by an easy argument, and by part (1) of

Theorem 1 in [27] applied to the α-fractional singular integral $T^{\alpha}_{\delta,R}$ we have

$$\mathfrak{N}_{T^{\alpha}_{\delta,R}} \leq C_{\alpha}\left(\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}_{\delta,R}} + \mathfrak{T}^{*}_{T^{\alpha}_{\delta,R}} + \mathcal{E}^{\text{strong}}_{\alpha} + \mathcal{E}^{\text{strong},*}_{\alpha} + \mathcal{WBP}_{T^{\alpha}_{\delta,R}}\right),$$

with C_{α} independent of δ and R. We obtain from the Good-λ Lemma applied to $T^{\alpha}_{\sigma,\delta,R}$ in place of T^{α},

$$\mathcal{WBP}_{T^{\alpha}_{\delta,R}} \leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}_{\delta,R}} + \mathfrak{T}^{*}_{T^{\alpha}_{\delta,R}} + \mathcal{E}^{\text{strong}}_{\alpha} + \mathcal{E}^{\text{strong},*}_{\alpha} + \sqrt[4]{\lambda}\,\mathfrak{N}_{T^{\alpha}_{\delta,R}}\right),$$

and then combining inequalities gives

$$\mathfrak{N}_{T^{\alpha}_{\delta,R}} \leq C'_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}_{\delta,R}} + \mathfrak{T}^{*}_{T^{\alpha}_{\delta,R}} + \mathcal{E}^{\text{strong}}_{\alpha} + \mathcal{E}^{\text{strong},*}_{\alpha} + \sqrt[4]{\lambda}\,\mathfrak{N}_{T^{\alpha}_{\delta,R}}\right),$$

with C'_{α} independent of δ and R. Since $\mathfrak{N}_{T^{\alpha}_{\delta,R}} < \infty$, we can absorb the term $C'_{\alpha}\sqrt[4]{\lambda}\,\mathfrak{N}_{T^{\alpha}_{\delta,R}}$ on the right hand side above into the left hand side for $\lambda > 0$ sufficiently small. Since $T^{\alpha}_{\delta,R}$ is an arbitrary tangent line approximation to T^{α}, the proof of Theorem 2 is complete. □

The first case of the following $T1$ theorem was proved in [26], and the second case is a corollary of Theorem 2 above and Theorem 2 in [27].

Theorem 3 *Suppose $0 \leq \alpha < n$, that T^{α} is a standard α-fractional singular integral operator on $\mathbb{R}^n$, and that ω and σ are locally finite positive Borel measures on $\mathbb{R}^n$. Set $T^{\alpha}_{\sigma}f = T^{\alpha}(f\sigma)$ for any smooth truncation of T^{α}_{σ}. Let $\Omega : \mathbb{R}^n \to \mathbb{R}^n$ be a globally biLipschitz map. Then*

$$\mathfrak{N}_{T^{\alpha}_{\sigma}} \approx \sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^{*}_{T^{\alpha}},$$

in the following two cases:

(1) when T^{α} is a strongly elliptic standard α-fractional singular integral operator on $\mathbb{R}^n$, and one of the weights σ or ω is supported on a compact $C^{1,\delta}$ curve in $\mathbb{R}^n$,
(2) when $T^{\alpha} = \mathbf{R}^{\alpha}$ is the vector of α-fractional Riesz transforms, and both weights σ and ω are k-energy dispersed where $0 \leq k \leq n-1$ satisfies

$$\begin{cases} n-k < \alpha < n,\ \alpha \neq n-1 & \text{if } 1 \leq k \leq n-2 \\ 0 \leq \alpha < n,\ \alpha \neq 1, n-1 & \text{if } \quad k = n-1 \end{cases}.$$

There is a further corollary that can be easily obtained, namely a **two weight** accretive global Tb theorem whenever a two weight $T1$ theorem holds for strictly comparable weight pairs. We say that two weight pairs (σ, ω) and $(\widetilde{\sigma}, \widetilde{\omega})$ are *strictly comparable* if $\widetilde{\sigma} = h_1\sigma$ and $\widetilde{\omega} = h_2\omega$ where each h_i is a function bounded between two positive constants. The simple proof of the following accretive global

Tb theorem uses only the *statement* of a related $T1$ theorem. We say that a complex-valued function b is *accretive* on $\mathbb{R}^n$ if

$$0 < c_b \leq \operatorname{Re} b(x) \leq |b(x)| \leq C_b < \infty, \qquad x \in \mathbb{R}^n .$$

Theorem 4 *Suppose $0 \leq \alpha < n$, that T^α is a standard α-fractional singular integral operator on $\mathbb{R}^n$, and that ω and σ are locally finite positive Borel measures on $\mathbb{R}^n$ for which we have the '$T1$ theorem' for strictly comparable weight pairs, i.e.*

$$\mathfrak{N}_{T_\sigma^\alpha}(\widetilde{\sigma},\widetilde{\omega}) \approx \sqrt{\mathfrak{A}_2^\alpha(\widetilde{\sigma},\widetilde{\omega})} + \mathfrak{T}_{T^\alpha}(\widetilde{\sigma},\widetilde{\omega}) + \mathfrak{T}_{T^\alpha}^*(\widetilde{\sigma},\widetilde{\omega}), \tag{12}$$

whenever (σ,ω) and $(\widetilde{\sigma},\widetilde{\omega})$ are strictly comparable. Finally, let b and b^* be two accretive functions on $\mathbb{R}^n$. Then the best constant $\mathfrak{N}_{T_\sigma^\alpha} = \mathfrak{N}_{T_\sigma^\alpha}(\sigma,\omega)$ in the two weight norm inequality

$$\|T_\sigma^\alpha f\|_{L^2(\omega)} \leq \mathfrak{N}_{T_\sigma^\alpha} \|f\|_{L^2(\sigma)} ,$$

taken uniformly over tangent line truncations of T^α, satisfies

$$\mathfrak{N}_{T_\sigma^\alpha} \approx \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha}^b + \mathfrak{T}_{T^\alpha}^{b^*,*}, \tag{13}$$

where the two dual b-testing conditions for T^α are given by

$$\int_Q \left|T_\sigma^\alpha(\mathbf{1}_Q b)\right|^2 d\omega \leq \mathfrak{T}_{T^\alpha}^b |Q|_\sigma , \quad \text{for all cubes } Q,$$

$$\int_Q \left|T_\omega^{\alpha,*}(\mathbf{1}_Q b^*)\right|^2 d\sigma \leq \mathfrak{T}_{T^\alpha}^{b^*,*} |Q|_\omega , \quad \text{for all cubes } Q,$$

and where we interpret the left sides above as holding uniformly over all tangent line truncations of T^α.

Note that Theorem 4 applies in particular to both cases (1) and (2) of Theorem 3.

Proof We first note that since the kernel K^α is real-valued,

$$\int_Q \left|T_\sigma^\alpha(\mathbf{1}_Q \operatorname{Re} b)\right|^2 d\omega = \int_Q \left|\operatorname{Re} T_\sigma^\alpha(\mathbf{1}_Q b)\right|^2 d\omega \leq \int_Q \left|T_\sigma^\alpha(\mathbf{1}_Q b)\right|^2 d\omega \leq \mathfrak{T}_{T^\alpha}^b |Q|_\sigma ,$$

$$\int_Q \left|T_\omega^{\alpha,*}(\mathbf{1}_Q \operatorname{Re} b^*)\right|^2 d\sigma = \int_Q \left|\operatorname{Re} T_\omega^{\alpha,*}(\mathbf{1}_Q b^*)\right|^2 d\sigma \leq \int_Q \left|T_\omega^{\alpha,*}(\mathbf{1}_Q b^*)\right|^2 d\sigma \leq \mathfrak{T}_{T^\alpha}^{b^*,*} |Q|_\omega ,$$

and if we now define measures

$$\widetilde{\omega} \equiv \left(\operatorname{Re} b^*\right)\omega \text{ and } \widetilde{\sigma} \equiv (\operatorname{Re} b)\,\sigma ,$$

we see that the operator T^{α} and the weight pair $(\widetilde{\sigma}, \widetilde{\omega})$ satisfy (12). But it follows that $\mathfrak{T}_{T^{\alpha}}(\widetilde{\sigma}, \widetilde{\omega}) \leq \mathfrak{T}_{T^{\alpha}}^{b}(\sigma, \omega)$ and $\mathfrak{T}_{T^{\alpha}}^{*}(\widetilde{\sigma}, \widetilde{\omega}) \leq \mathfrak{T}_{T^{\alpha}}^{b^{*},*}(\sigma, \omega)$, and since the Muckenhoupt A_2 conditions are clearly comparable for strictly comparable weight pairs, we have the equivalence

$$\mathfrak{N}_{T_{\sigma}^{\alpha}}(\widetilde{\sigma}, \widetilde{\omega}) \approx \sqrt{\mathfrak{A}_{2}^{\alpha}(\sigma, \omega)} + \mathfrak{T}_{T^{\alpha}}^{b}(\sigma, \omega) + \mathfrak{T}_{T^{\alpha}}^{b^{*},*}(\sigma, \omega).$$

Finally, since $0 < c \leq \operatorname{Re} b, \operatorname{Re} b^{*} \leq C$, we see that $\mathfrak{N}_{T_{\sigma}^{\alpha}}(\widetilde{\sigma}, \widetilde{\omega}) \approx \mathfrak{N}_{T_{\sigma}^{\alpha}}(\sigma, \omega)$, and this completes the proof of (13). □

Note that the presence of a (b, b^{*})-variant of the weak boundedness property here would complicate matters, since in general,

$$\operatorname{Re} \int_{Q} T^{\alpha}\left(1_{Q'} b\sigma\right) b^{*} d\omega \neq \int_{Q} T^{\alpha}\left(1_{Q'} \operatorname{Re} b\sigma\right) \operatorname{Re} b^{*} d\omega.$$

To remind the reader of the versatility of even a *global Tb* theorem, we reproduce a proof of the boundedness of the Cauchy integral on $C^{1,\delta}$ curves.

2.1.1 Boundedness of the Cauchy Integral on $C^{1,\delta}$ Curves

Here we point out how the above *Tb* theorem can apply to obtain the boundedness of the Cauchy integral on $C^{1,\delta}$ curves in the plane (which can be obtained in many other easy ways as well, see e.g. [31, Sect. 4 of Chap. VII]). Recall that the problem reduces to boundedness on $L^{2}(\mathbb{R})$ of the singular integral operator C_A with kernel

$$K_{A}(x, y) \equiv \frac{1}{x-y+i(A(x)-A(y))},$$

where the curve has graph $\{x+iA(x) : x \in \mathbb{R}\}$. Now $b(x) \equiv 1+iA'(x)$ is accretive and we have the b-testing condition

$$\int_{I}\left|C_{A}\left(\mathbf{1}_{I} b\right)(x)\right|^{2} dx \leq \mathfrak{T}_{H}^{b}|I|,$$

and its dual. Indeed, if $I=[\alpha, \beta]$, then

$$\begin{aligned} C_{A}\left(\mathbf{1}_{I} b\right)(x) &= \int_{\alpha}^{\beta} \frac{1+iA'(y)}{x-y+i(A(x)-A(y))} dy \\ &= -\log(x-y+i(A(x)-A(y)))\,\Big|_{\alpha}^{\beta} \\ &= \log\left(\frac{x-\alpha+i(A(x)-A(\alpha))}{x-\beta+i(A(x)-A(\beta))}\right), \end{aligned}$$

gives

$$|C_A(\mathbf{1}_I b)(x)|^2 \approx \ln \frac{x-\alpha}{\beta - x}, \qquad x \in I = [\alpha, \beta],$$

and it follows that

$$\begin{aligned}\int_I |C_A(\mathbf{1}_I b)(x)|^2 \, dx &\approx \int_I \left| \ln \frac{x-\alpha}{\beta-x} \right|^2 dx \approx \int_0^{\beta-\alpha} \left| \ln \frac{x}{\beta-\alpha} \right|^2 dx \\ &= (\beta-\alpha) \int_0^1 |\ln w|^2 \, dw = C\,|I|.\end{aligned}$$

Since the kernel K_A is $C^{1,\delta}$, the *Tb* theorem above applies with $T = C_A$ and $\sigma = \omega = dx$ Lebesgue measure, to show that C_A is bounded on $L^2(\mathbb{R})$. Of course this proof just misses the case of Lipschitz curves since our two weight *Tb* theorem does not apply to kernels that fail to be $C^{1,\delta}$.

3 Proof of the Good-λ Lemma

We will prove the Good-λ Lemma by first replacing the quasiweak boundedness constant on the left hand side of (11) with the indicator/touching constant introduced in (6) above. To control the indicator/touching constant, we will need to tweak the usual good/bad technology of NTV a bit in the following subsection.

3.1 Good/Bad Technology

First we recall the good/bad cube technology of Nazarov, Treil and Volberg [32] as in [25], but with a small simplification introduced in the real line by Hytönen in [6]. This simplification does not impact the validity of the arguments in [30], but will facilitate the use of NTV surgery in later subsections.

Following [6], we momentarily fix a large positive integer $M \in \mathbb{N}$, and consider the tiling of $\mathbb{R}^n$ by the family of cubes $\mathbb{D}_M \equiv \{I_\alpha^M\}_{\alpha \in Z^n}$ having side length 2^{-M} and given by $I_\alpha^M \equiv I_0^M + 2^{-M}\alpha$ where $I_0^M = \left[0, 2^{-M}\right)^n$. A *dyadic grid* $\mathcal{D}$ built on $\mathbb{D}_M$ is defined to be a family of cubes $\mathcal{D}$ satisfying:

(1) Each $I \in \mathcal{D}$ has side length $2^{-\ell}$ for some $\ell \in \mathbb{Z}$ with $\ell \leq M$, and I is a union of $2^{n(M-\ell)}$ cubes from the tiling $\mathbb{D}_M$,
(2) For $\ell \leq M$, the collection $\mathcal{D}_\ell$ of cubes in $\mathcal{D}$ having side length $2^{-\ell}$ forms a pairwise disjoint decomposition of the space $\mathbb{R}^n$,
(3) Given $I \in \mathcal{D}_i$ and $J \in \mathcal{D}_j$ with $j \leq i \leq M$, it is the case that either $I \cap J = \emptyset$ or $I \subset J$.

We now momentarily fix a *negative* integer $N \in -\mathbb{N}$, and restrict the above grids to cubes of side length at most 2^{-N}:

$$\mathcal{D}^N \equiv \left\{ I \in \mathcal{D} : \text{side length of } I \text{ is at most } 2^{-N} \right\}.$$

We refer to such grids $\mathcal{D}^N$ as a (truncated) dyadic grid $\mathcal{D}$ built on $\mathbb{D}_M$ of size 2^{-N}. There are now two traditional means of constructing probability measures on collections of such dyadic grids.

Construction #1: Consider first the special case of dimension $n = 1$. Then for any

$$\beta = \{\beta_i\}_{i \in_M^N} \in \omega_M^N \equiv \{0, 1\}^{\mathbb{Z}_M^N},$$

where $\mathbb{Z}_M^N \equiv \{\ell \in \mathbb{Z} : N \le \ell \le M\}$, define the dyadic grid $\mathcal{D}_\beta$ built on $\mathbb{D}_M$ of size 2^{-N} by

$$\mathcal{D}_\beta = \left\{ 2^{-\ell} \left([0, 1) + k + \sum_{i:\ \ell < i \le M} 2^{-i+\ell} \beta_i \right) \right\}_{N \le \ell \le M,\, k \in \mathbb{Z}}.$$

Place the uniform probability measure ρ_M^N on the finite index space $\omega_M^N = \{0, 1\}^{\mathbb{Z}_M^N}$, namely that which charges each $\beta \in \omega_M^N$ equally. This construction is then extended to Euclidean space $\mathbb{R}^n$ by taking products in the usual way and using the product index space $\Omega_M^N \equiv \left(\omega_M^N\right)^n$ and the uniform product probability measure $\mu_M^N = \rho_M^N \times \ldots \times \rho_M^N$.

Construction #2: Momentarily fix a (truncated) dyadic grid $\mathcal{D}$ built on $\mathbb{D}_M$ of size 2^{-N}. For any

$$\gamma = (\gamma_1, \ldots, \gamma_n) \in \Gamma_M^N \equiv \left\{ 2^{-M} \mathbb{Z}_+^n : |\gamma_i| < 2^{-N} \right\},$$

where $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, define the dyadic grid $\mathcal{D}^\gamma$ built on $\mathbb{D}_M$ of size 2^{-N} by

$$\mathcal{D}^\gamma \equiv \mathcal{D} + \gamma.$$

Place the uniform probability measure ν_M^N on the finite index set Γ_M^N, namely that which charges each multiindex γ in Γ_M^N equally.

The two probability spaces $\left(\left\{ \mathcal{D}_\beta \right\}_{\beta \in \Omega_M^N}, \mu_M^N \right)$ and $\left(\{\mathcal{D}^\gamma\}_{\gamma \in \Gamma_M^N}, \nu_M^N \right)$ are isomorphic since both collections $\left\{ \mathcal{D}_\beta \right\}_{\beta \in \Omega_M^N}$ and $\{\mathcal{D}^\gamma\}_{\gamma \in \Gamma_M^N}$ describe the set $\boldsymbol{A}_M^N$ of **all** (truncated) dyadic grids $\mathcal{D}^\gamma$ built on $\mathbb{D}_M$ of size 2^{-N}, and since both measures μ_M^N and ν_M^N are the uniform measure on this space. Indeed, it suffices to verify this in the case $n = 1$. The first construction may be thought of as being *parameterized by scales*—each component β_i in $\beta = \{\beta_i\}_{i \in_M^N} \in \omega_M^N$ amounting to a choice of the two possible tilings at level i that respect the choice of tiling at the level below—and since any grid in $\boldsymbol{A}_M^N$ is determined by a choice of scales , we see that $\{\mathcal{D}_\beta\}_{\beta\in\Omega_M^N} = \boldsymbol{A}_M^N$. The second construction may be thought of as being *parameterized by translation*—each $\gamma \in \Gamma_M^N$ amounting to a choice of translation of the grid $\mathcal{D}$ fixed in construction #2–and since any grid in $\boldsymbol{A}_M^N$ is determined by any of the cubes at the top level, i.e. with side length 2^{-N}, we see that $\{\mathcal{D}^\gamma\}_{\gamma\in\Gamma_M^N} = \boldsymbol{A}_M^N$ as well, since every cube at the top level in $\boldsymbol{A}_M^N$ has the form $Q+\gamma$ for some $\gamma \in \Gamma_M^N$ and $Q \in \mathcal{D}$ at the top level in $\boldsymbol{A}_M^N$ (i.e. every cube at the top level in $\boldsymbol{A}_M^N$ is a union of small cubes in $\mathbb{D}_M$, and so must be a translate of some $Q \in \mathcal{D}$ by an amount 2^{-M} times an element of $\mathbb{Z}_+^n$). Note also that in all dimensions, $\#\Omega_M^N = \#\Gamma_M^N = 2^{n(M-N)}$. We will use $\mathbb{E}_{\Omega_M^N}$ to denote expectation with respect to this common probability measure on $\boldsymbol{A}_M^N$.

The usual NTV probabilistic reduction to 'good' cubes will be implemented below for each positive integer M and each negative integer N assuming that the functions f and g are supported in a large cube L with $\int_L f d\sigma = 0 = \int_L g d\omega$, and moreover assuming that $-N$ is sufficiently large compared to $\ell(L)$ that the small probability estimates claimed below hold ($-N > \ell(L) + \mathbf{r}$ will work where $\mathbf{r}$ is the goodness constant), and finally assuming that f and g are constant on each cube Q in the tiling $\mathbb{D}_M$. Recall that we can always reduce to the case $\int_L f d\sigma = 0 = \int_L g d\omega$ by simply subtracting off averages and controlling the resulting error terms by the testing conditions (see e.g. [32]).

Notation 2 *For purposes of notation and clarity, we often suppress all reference to M and N in our families of grids, and in the notations Ω and Γ for the parameter sets, and we will use $\mathbb{P}_\Omega$ and $\mathbb{E}_\Omega$ to denote probability and expectation, and instead proceed as if all grids considered are unrestricted. The careful reader can supply the modifications necessary to handle the assumptions made above on the grids $\mathcal{D}$ and the functions f and g regarding M and N. In fact, we will exploit the integers M and N explicitly in the subsubsections on NTV surgery below.*

In the case of one independent family of grids, as is the case here, the main result is the following *conditional* probability estimate: for every $I \in \mathcal{P}^n$,

$$\mathbb{P}_\Omega\{\mathcal{D} : I \text{ is a } \textit{bad} \text{ cube in } \mathcal{D} \mid I \in \mathcal{D}\} \le C2^{-\varepsilon\mathbf{r}}. \tag{14}$$

Provided we obtain estimates independent of M and N, this will be sufficient for our proof—this follows the procedure with *two* independent grids initiated by Hytönen for the Hilbert transform inequality in [6]. The key point of introducing the two different parameterizations above of the same probability space, is that construction #1 is well-adapted to the reduction to good cubes in a *single* independent family of grids, as used in the proof of the main theorem in [30], which is in turn needed below, while construction #2 facilitates the use of NTV surgery below when combined with the construction of Q-good *grids,* to which we next turn.

3.1.1 Q-Good Quasicubes and Q-Good Quasigrids

We first introduce these notions for usual cubes, and later pass to quasicubes. Let $Q \in \mathcal{P}^n$ be an arbitrary cube in $\mathbb{R}^n$ with sides parallel to the coordinate axes. For technical reasons associated to our application below, we also want to consider the 'siblings' of Q, i.e. the 'triadic children' of $3Q$.

Definition 6 We say that a cube $I \in \mathcal{P}^n$ is *Q-good* if either $\ell(I) > 2^{-\rho}\ell(Q)$, or for every sibling Q' of Q, we have

$$\operatorname{dist}\left(I, \partial Q'\right) \geq \frac{1}{2}\ell(I)^{\varepsilon}\,\ell\left(Q'\right)^{1-\varepsilon}$$

when $\ell(I) \leq 2^{-\rho}\ell(Q)$. We say $I \in \mathcal{P}^n$ is *Q-bad* if I is **not** Q-good.

Note that for a fixed cube $Q \in \mathcal{P}^n$, we do **not** have a conditional probability estimate $\mathbb{P}_\Omega\{\mathcal{D} : I \in \mathcal{D} \text{ and } I \text{ is } Q\text{-bad}\} \leq C2^{-\varepsilon\mathbf{r}}$ since the property of a cube I being Q-bad is independent of which grids $\mathcal{D}$ it belongs to. To rectify this complication we will introduce below a *second independent* family of grids—but this second family will also be used to simultaneously Haar-decompose both $f \in L^2(\sigma)$ and $g \in L^2(\omega)$.[2]

We next wish to capture the idea of a grid $\mathcal{D}$ being '*Q-good*' with respect to this fixed cube Q, and the idea will be to require that Q is I-good for all sufficiently larger cubes I in the grid $\mathcal{D}$. Here we *will* obtain a 'goodness' estimate in Lemma 2 below.

Definition 7 Let $\mathbf{r}$ and ε be goodness constants as in [25]. For $Q \in \mathcal{P}^n$ we declare a grid $\mathcal{D}$ to be *Q-good* if for every sibling Q' of Q and for every $I \in \mathcal{D}$ with $\ell(I) \geq 2^{\mathbf{r}}\ell(Q)$, the following holds: the distance from the cube Q' to the boundary of the cube I satisfies the 'deeply embedded' inequality,

$$\operatorname{dist}\left(Q', \partial I\right) \geq \frac{1}{2}\ell\left(Q'\right)^{\varepsilon}\ell(I)^{1-\varepsilon}.$$

We say the grid $\mathcal{D}$ is *Q-bad* if it is not Q-good.

Note that Q is fixed in this definition and it is easy to see, using the translation parameterization in construction #2 above, that the collection of grids $\mathcal{D}$ that are Q-bad occur with small probability. Indeed, if $I \supset Q$ has side length at least $2^{\mathbf{r}}$ times that of Q, then the translates of I satisfy $Q \Subset_{\mathbf{r}} I$ with probability near 1.

Lemma 2 *Fix a cube $Q \in \mathcal{P}^n$. Then $\mathbb{P}_\Omega\{\mathcal{D} : \mathcal{D} \text{ is } Q\text{-bad}\} \leq C2^{-\varepsilon\mathbf{r}}$.*

[2]Traditionally, two independent grids are applied to f and g separately, something we *avoid* since the treatment of functional energy in the arguments of [27, 30] (which we use here) relies on using a *common* grid for f and g.

The following is our tweaking of the good/bad technology of NTV [32]. Fix a cube $Q \in \mathcal{P}^n$ and let $\mathcal{D}$ be randomly selected. Define linear operators (depending on the grid $\mathcal{D}$),

$$\mathsf{P}^{\sigma}_{Q;\mathrm{good}} f \equiv \begin{cases} \sum_{I\in\mathcal{D}:\ I \text{ is } \mathbf{r}\text{-good in } \mathcal{D}} \triangle^{\sigma}_{I} f & \text{if } \mathcal{D} \text{ is } Q\text{-good} \\ 0 & \text{if } \mathcal{D} \text{ is } Q\text{-bad} \end{cases},$$

$$\mathsf{P}^{\sigma}_{Q;\mathrm{bad}} f \equiv f - \mathsf{P}^{\sigma}_{Q;\mathrm{good}} f ,$$

and likewise for $\mathsf{P}^{\omega}_{Q;\mathrm{good}} g$ and $\mathsf{P}^{\omega}_{Q;\mathrm{bad}} g$.

Proposition 1 *Fix a cube $Q \in \mathcal{P}^n$. Then we have the estimates*

$$\mathbb{E}_{\Omega} \left\| \mathsf{P}^{\sigma}_{Q;\mathrm{bad}} f \right\|_{L^2(\sigma)} \leq C 2^{-\frac{\varepsilon \mathbf{r}}{2}} \left\| f \right\|_{L^2(\sigma)} ,$$

$$\mathbb{E}_{\Omega} \left\| \mathsf{P}^{\omega}_{Q;\mathrm{bad}} g \right\|_{L^2(\omega)} \leq C 2^{-\frac{\varepsilon \mathbf{r}}{2}} \left\| g \right\|_{L^2(\omega)} .$$

Proof We have from (14) and Lemma 2 that

$$\begin{aligned} \mathbb{E}_{\Omega} \left\| \mathsf{P}^{\sigma}_{\mathrm{bad}} f \right\|^2_{L^2(\sigma)} &= \mathbb{E}_{\Omega} \left(\mathbf{1}_{\{\mathcal{D} \text{ is } Q\text{-good}\}} \sum_{I\in\mathcal{D} \text{ is bad}} \left\| \triangle^{\sigma}_{I} f \right\|^2_{L^2(\sigma)} \right) + \mathbb{E}_{\Omega} \left(\mathbf{1}_{\{\mathcal{D} \text{ is } Q\text{-bad}\}} \sum_{I\in\mathcal{D}} \left\| \triangle^{\sigma}_{I} f \right\|^2_{L^2(\sigma)} \right) \\ &\leq C 2^{-\varepsilon \mathbf{r}} \sum_{I\in\mathcal{D}} \left\| \triangle^{\sigma}_{I} f \right\|^2_{L^2(\sigma)} + \mathbb{E}_{\Omega} \left(\mathbf{1}_{\{\mathcal{D} \text{ is } Q\text{-bad}\}} \right) \sum_{I\in\mathcal{D}} \left\| \triangle^{\sigma}_{I} f \right\|^2_{L^2(\sigma)} \lesssim C 2^{-\varepsilon \mathbf{r}} \left\| f \right\|^2_{L^2(\sigma)} . \end{aligned}$$

□

From this we conclude that there is an absolute choice of $\mathbf{r}$ depending on $0 < \varepsilon < 1$ so that the following holds. Let $T : L^2(\sigma) \to L^2(\omega)$ be a bounded linear operator, and let $Q \in \mathcal{P}^n$ be a fixed cube. We then have

$$\left\| T \right\|_{L^2(\sigma)\to L^2(\omega)} \leq 2 \sup_{\|f\|_{L^2(\sigma)}=1} \sup_{\|g\|_{L^2(\omega)}=1} \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{good}} f, \mathsf{P}^{\omega}_{Q;\mathrm{good}} g \right\rangle_{\omega} \right| . \tag{15}$$

Indeed, we can choose $f \in L^2(\sigma)$ of norm one, and $g \in L^2(\omega)$ of norm one so that

$$\begin{aligned} \left\| T \right\|_{L^2(\sigma)\to L^2(\omega)} &= \langle Tf, g \rangle_{\omega} \\ &\leq \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{good}} f, \mathsf{P}^{\omega}_{Q;\mathrm{good}} g \right\rangle_{\omega} \right| + \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{bad}} f, \mathsf{P}^{\omega}_{Q;\mathrm{good}} g \right\rangle_{\omega} \right| \\ &\quad + \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{good}} f, \mathsf{P}^{\omega}_{Q;\mathrm{bad}} g \right\rangle_{\omega} \right| + \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{bad}} f, \mathsf{P}^{\omega}_{Q;\mathrm{bad}} g \right\rangle_{\omega} \right| \\ &\leq \mathbb{E}_{\Omega} \left| \left\langle T \mathsf{P}^{\sigma}_{Q;\mathrm{good}} f, \mathsf{P}^{\omega}_{Q;\mathrm{good}} g \right\rangle_{\omega} \right| + 3C \cdot 2^{-\frac{\mathbf{r}\varepsilon}{16}} \left\| T \right\|_{L^2(\sigma)\to L^2(\omega)} , \end{aligned}$$

And this proves (15) for $\mathbf{r}$ sufficiently large depending on $\varepsilon > 0$.

Clearly, all of this extends automatically to the quasiworld.

Implication: *Given a quasicube $Q \in \Omega\mathcal{P}^n$, it suffices to consider only Q-good quasigrids and Q-good quasicubes in these quasigrids, and to prove an estimate for $\|T_\sigma\|_{L^2(\sigma)\to L^2(\omega)}$ that is independent of these assumptions.*

3.2 Control of the Indicator/Touching Property

Recall the indicator/touching constant $\mathfrak{I}_{T^\alpha}$ defined in (6) above. Here we will prove that

$$\mathfrak{I}_{T^\alpha} \le C_\alpha \left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} + \sqrt[4]{\lambda}\,\mathfrak{N}_{T^\alpha} \right), \tag{16}$$

from which it easily follows that we have the same inequality for the weak boundedness property constant $\mathcal{WBP}_{T^\alpha}$ defined in (5) above,

$$\mathcal{WBP}_{T^\alpha} \le C_\alpha \left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} + \sqrt[4]{\lambda}\,\mathfrak{N}_{T^\alpha} \right). \tag{17}$$

Indeed an elementary argument shows that $\mathcal{WBP}_{T^\alpha} \lesssim \mathfrak{I}_{T^\alpha} + \sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha}$. For the proof of (16) we assume the reader is already familiar with the proof of the main theorem in [30] or [27], and we now review the parts of this proof that are pertinent here.

We first recall the basic setup in [30]. Let $\Omega\mathcal{D}^\sigma = \Omega\mathcal{D}^\omega$ be a quasigrid on $\mathbb{R}^n$, and let $\left\{h_I^{\sigma,a}\right\}_{I\in\Omega\mathcal{D}^\sigma,\ a\in\Gamma_n}$ and $\left\{h_J^{\omega,b}\right\}_{J\in\Omega\mathcal{D}^\omega,\ b\in\Gamma_n}$ be corresponding quasiHaar bases, so that $f \in L^2(\sigma)$ and $g \in L^2(\omega)$ can be written $f = f_{\text{good}} + f_{\text{bad}}$ and $g = g_{\text{good}} + g_{\text{bad}}$ where

$$f = \sum_{I\in\Omega\mathcal{D}^\sigma} \triangle_I^\sigma f \text{ and } g = \sum_{J\in\Omega\mathcal{D}^\omega} \triangle_J^\omega g\,,$$

$$f_{\text{good}} = \sum_{I\in\Omega\mathcal{D}^\sigma_{\text{good}}} \triangle_I^\sigma f \text{ and } g_{\text{good}} = \sum_{J\in\Omega\mathcal{D}^\omega_{\text{good}}} \triangle_J^\omega g\,,$$

and where $\Omega\mathcal{D}^\sigma_{\text{good}} = \Omega\mathcal{D}^\omega_{\text{good}}$ is the $(\mathbf{r},\varepsilon)$-good subgrid, and where the quasiHaar projections $\triangle_I^\sigma f_{\text{good}}$ and $\triangle_J^\omega g_{\text{good}}$ *vanish* if the quasicubes I and J are not good in $\Omega\mathcal{D}^\sigma = \Omega\mathcal{D}^\omega$. Note that we use a *single* independent family of grids $\Omega\mathcal{D}^\sigma = \Omega\mathcal{D}^\omega$ and only include the different superscripts σ and ω to emphasize which measure the grid is being used with in a given situation.

Remark 2 In [27] and [30], the quasiHaar projections $\triangle_I^\sigma f_{\text{good}}$ and $\triangle_J^\omega g_{\text{good}}$ are required to vanish if the quasicubes I and J are not $\boldsymbol{\tau}$-good in $\Omega\mathcal{D}^\sigma = \Omega\mathcal{D}^\omega$, where a quasicube I is $\boldsymbol{\tau}$-good in a quasigrid $\Omega\mathcal{D}$ if I together with its children and its ancestors up to order $\boldsymbol{\tau}$ are all good. This more restrictive condition doesn't affect what is done here.

For future reference note that the argument in [30] applies just as well to the smaller projections $\mathsf{P}^{\sigma}_{Q;\text{good}}f$ and $\mathsf{P}^{\omega}_{Q;\text{good}}g$ in place of f_{good} and g_{good} respectively. We fix $f = f_{\text{good}}$ and $g = g_{\text{good}}$. For now we continue to work with general functions f and g and the projections f_{good} and g_{good}, but keeping in mind that in order to prove (16), we will later specialize to the cases of indicator functions $f = \mathbf{1}_Q$ and $g = \mathbf{1}_R$, and we will then also include the restriction to Q-good grids $\Omega\mathcal{D}_{Q;\text{good}}$ and projections $\mathsf{P}^{\sigma}_{Q;\text{good}}f$ and $\mathsf{P}^{\omega}_{Q;\text{good}}g$ for a fixed quasicube Q - the quasicube Q in the projection $\mathsf{P}^{\sigma}_{Q;\text{good}}f$ is chosen to coincide with the quasicube Q in the indicator $\mathbf{1}_Q$ in order to achieve the three critical reductions in Sect. 3.2.1 below. Continuing with [27, 30], we then proved there the bilinear inequality

$$
\begin{aligned}
\left|\mathcal{T}^{\alpha}(f,g)\right| &= \left|\sum_{I\in\Omega\mathcal{D}^{\sigma}_{\text{good}} \text{ and } J\in\Omega\mathcal{D}^{\omega}_{\text{good}}} \mathcal{T}^{\alpha}\left(\Delta^{\sigma}_{I}f,\Delta^{\omega}_{J}g\right)\right| \qquad (18)\\
&\leq C_{\alpha}\left(\sqrt{\mathfrak{A}^{\alpha}_2}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}^{*}_{T^{\alpha}}+\mathcal{E}^{\text{strong}}_{\alpha}+\mathcal{E}^{\text{strong},*}_{\alpha}+\mathcal{WBP}_{T^{\alpha}}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)},
\end{aligned}
$$

uniformly over grids $\mathcal{D}$, and we now discuss the salient features of this proof for us.

As in [27, 30] let

$$
\begin{aligned}
\mathcal{NTV}_{\alpha} &\equiv \sqrt{\mathfrak{A}^{\alpha}_2}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}^{*}_{T^{\alpha}}+\mathcal{WBP}_{T^{\alpha}}\ ,\\
\mathfrak{A}^{\alpha}_2 &\equiv \mathcal{A}^{\alpha}_2+\mathcal{A}^{\alpha,*}_2+A^{\alpha,\text{punct}}_2+A^{\alpha,*,\text{punct}}_2\ ,
\end{aligned}
$$

and recall the following brief schematic diagram of the decompositions involved in the proof given in [30], with bounds in $\square$:

$$
\boxed{
\begin{array}{ccccccc}
\left\langle T^{\alpha}_{\sigma}f,g\right\rangle_{\omega} & & & & & & \\
\downarrow & & & & & & \\
\mathsf{B}_{\Subset_{\rho}}(f,g) & + & \mathsf{B}_{_{\rho}\Supset}(f,g) & + & \mathsf{B}_{\cap}(f,g) & + & \mathsf{B}_{\diagup}(f,g) \\
\downarrow & & \boxed{\text{dual}} & & \boxed{\mathcal{NTV}_{\alpha}} & & \boxed{\mathcal{NTV}_{\alpha}} \\
\downarrow & & & & & & \\
\mathsf{T}_{\text{diagonal}}(f,g) & + & \mathsf{T}_{\text{far below}}(f,g) & + & \mathsf{T}_{\text{far above}}(f,g) & + & \mathsf{T}_{\text{disjoint}}(f,g) \\
\downarrow & & \downarrow & & \boxed{\emptyset} & & \boxed{\emptyset} \\
\downarrow & & \downarrow & & & & \\
\mathsf{B}^{A}_{\Subset_{\rho}}(f,g) & & \mathsf{T}^{1}_{\text{far below}}(f,g) & + & \mathsf{T}^{2}_{\text{far below}}(f,g) & & \\
\downarrow & & \boxed{\mathcal{NTV}_{\alpha}+\mathcal{E}^{\text{strong}}_{\alpha}} & & \boxed{\mathcal{NTV}_{\alpha}} & & \\
\downarrow & & & & & & \\
\mathsf{B}^{A}_{stop}(f,g) & + & \mathsf{B}^{A}_{paraproduct}(f,g) & + & \mathsf{B}^{A}_{neighbour}(f,g) & & \\
\boxed{\mathcal{E}^{\text{strong}}_{\alpha}+\sqrt{A^{\alpha}_2}} & & \boxed{\mathfrak{T}_{T^{\alpha}}} & & \boxed{\sqrt{A^{\alpha}_2}} & &
\end{array}
}
$$

With reference to this diagram, we now make a sweeping and crucial claim.

The **only** two places in our proof of the main theorem in [30] where the **weak boundedness property** $\mathcal{WBP}_{T^\alpha}$ is used, is

(1) in proving the estimates for terms A_1 and A_2 involving $\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right), \triangle_J^\omega g\right\rangle_\omega$ that arise in estimating the form $\mathsf{B}_{\diagup}(f,g)$ at the top right of the schematic diagram, and
(2) and in the estimates for the inner products $\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right), \triangle_J^\omega g\right\rangle_\omega$ in the form $\mathsf{T}^2_{\text{far below}}(f,g)$ for which I are J are close in both scale and position,
(3) and *even then* in these two cases, only for certain child quasicubes I_θ and $J_{\theta'}$ when they *touch*, i.e. their interiors are disjoint but their closures intersect (even in just a point). In all other instances where $\mathcal{NTV}_\alpha$ appears in the schematic diagram, the weak boundedness property is *not* used.

In order to make the application of the quasiweak boundedness property in these arguments clear, we reproduce the relevant portions of the arguments from [30] that deal with the forms $\mathsf{B}_{\diagup}(f,g)$ and $\mathsf{T}^2_{\text{far below}}(f,g)$. Recall also that the parameters $\boldsymbol{\rho}, \boldsymbol{\tau}, \mathbf{r}$ in [30, Definition 12 on p. 40] were fixed to satisfy

$$\boldsymbol{\tau} > \mathbf{r} \text{ and } \boldsymbol{\rho} > \boldsymbol{\tau} + \mathbf{r} .$$

1: Here is the beginning of the proof of (6.1) on page 28 dealing with $\mathsf{B}_{\diagup}(f,g)$ in the statement of Lemma 9 in [30].

Extract from pages 28 and 29 of [30]:

Note that in (6.1) we have used the parameter $\boldsymbol{\rho}$ in the exponent rather than $\mathbf{r}$*, and this is possible because the arguments we use here only require that there are finitely many levels of scale separating I and J. To handle this term we first decompose it into*

$$\left\{\sum_{\substack{(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega:\ J\subset 3I\\ 2^{-\boldsymbol{\rho}}\ell(I)\le\ell(J)\le 2^{\boldsymbol{\rho}}\ell(I)}} + \sum_{\substack{(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega:\ I\subset 3J\\ 2^{-\boldsymbol{\rho}}\ell(I)\le\ell(J)\le 2^{\boldsymbol{\rho}}\ell(I)}} + \sum_{\substack{(I,J)\in\Omega\mathcal{D}^\sigma\times\Omega\mathcal{D}^\omega\\ 2^{-\boldsymbol{\rho}}\ell(I)\le\ell(J)\le 2^{\boldsymbol{\rho}}\ell(I)\\ J\not\subset 3I \text{ and } I\not\subset 3J}}\right\}\left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right), \triangle_J^\omega g\right\rangle_\omega\right|$$
$$\equiv A_1 + A_2 + A_3.$$

The proof of the bound for term A_3 is similar to that of the bound for the left side of (6.2), and so we will defer the bound for A_3 until after (6.2) has been proved.

We now consider term A_1 as term A_2 is symmetric. To handle this term we will write the quasiHaar functions h_I^σ and h_J^ω as linear combinations of the indicators of the children of their supporting quasicubes, denoted I_θ and $J_{\theta'}$ respectively. Then we use the quasitesting condition on I_θ and $J_{\theta'}$ when they overlap, i.e. their interiors intersect; we use the quasiweak boundedness property on I_θ and $J_{\theta'}$ when they touch, i.e. their interiors are disjoint but their closures intersect (even in just a point); and finally we use the A_2^α condition when I_θ and $J_{\theta'}$ are separated, i.e. their closures are disjoint. We will suppose initially that the side length of J is at most *the side length I, i.e.* $\ell(J) \leq \ell(I)$, *the proof for* $J = \pi I$ *being similar but for one point mentioned below. So suppose that* I_θ *is a child of I and that* $J_{\theta'}$ *is a child of J. If* $J_{\theta'} \subset I_\theta$ *we have from (9) that,*

$$\begin{aligned}
\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta} \triangle_I^\sigma f\right), \mathbf{1}_{J_{\theta'}} \triangle_J^\omega g\right\rangle_\omega\right| &\lesssim \sup_{a,a' \in \Gamma_n} \frac{\left|\left\langle f, h_I^{\sigma,a}\right\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right), \mathbf{1}_{J_{\theta'}}\right\rangle_\omega\right| \frac{\left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right|}{\sqrt{|J_{\theta'}|_\omega}} \\
&\lesssim \sup_{a,a' \in \Gamma_n} \frac{\left|\left\langle f, h_I^{\sigma,a}\right\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \left(\int_{J_{\theta'}} \left|T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right)\right|^2 d\omega\right)^{\frac{1}{2}} \left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right| \\
&\lesssim \sup_{a,a' \in \Gamma_n} \frac{\left|\left\langle f, h_I^{\sigma,a}\right\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \mathfrak{T}_{T_\alpha} \left|I_\theta\right|_\sigma^{\frac{1}{2}} \left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right| \\
&\lesssim \sup_{a,a' \in \Gamma_n} \mathfrak{T}_{T_\alpha} \left|\left\langle f, h_I^{\sigma,a}\right\rangle_\sigma\right| \left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right|.
\end{aligned}$$

The point referred to above is that when $J = \pi I$ *we write* $\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right), \mathbf{1}_{J_{\theta'}}\right\rangle_\omega = \left\langle \mathbf{1}_{I_\theta}, T_\omega^{\alpha,*}\left(\mathbf{1}_{J_{\theta'}}\right)\right\rangle_\sigma$ *and get the dual quasitesting constant* $T_{T_\alpha}^*$. *If* $J_{\theta'}$ *and* I_θ *touch, then* $\ell(J_{\theta'}) \leq \ell(I_\theta)$ *and we have* $J_{\theta'} \subset 3I_\theta \setminus I_\theta$, *and so*

$$\begin{aligned}
\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta} \triangle_I^\sigma f\right), \mathbf{1}_{J_{\theta'}} \triangle_J^\omega g\right\rangle_\omega\right| &\lesssim \sup_{a,a' \in \Gamma_n} \frac{\left|\left\langle f, h_I^{\sigma,a}\right\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right), \mathbf{1}_{J_{\theta'}}\right\rangle_\omega\right| \frac{\left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right|}{\sqrt{|J_{\theta'}|_\omega}} \qquad (19) \\
&\lesssim \sup_{a,a' \in \Gamma_n} \frac{\left|\left\langle f, h_I^{\sigma,a}\right\rangle_\sigma\right|}{\sqrt{|I_\theta|_\sigma}} \mathcal{WBP}_{T^\alpha} \sqrt{|I_\theta|_\sigma |J_{\theta'}|_\omega} \frac{\left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right|}{\sqrt{|J_{\theta'}|_\omega}} \\
&= \sup_{a,a' \in \Gamma_n} \mathcal{WBP}_{T^\alpha} \left|\left\langle f, h_I^{\sigma,a}\right\rangle_\sigma\right| \left|\left\langle g, h_J^{\omega,a'}\right\rangle_\omega\right|.
\end{aligned}$$

The only place where the quasiweak boundedness property $\mathcal{WBP}_{T^\alpha}$ was used above was in the second line of the display (19) when we invoked

$$\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I_\theta}\right), \mathbf{1}_{J_{\theta'}}\right\rangle_\omega\right| \leq \mathcal{WBP}_{T^\alpha} \sqrt{|I_\theta|_\sigma |J_{\theta'}|_\omega}$$

for quasicubes $I_\theta \in \mathfrak{C}(I)$ and $J_{\theta'} \in \mathfrak{C}(J)$ that touch.

2: Here is the beginning of the proof on page 41 that controls the form $\mathsf{T}_{\text{far below}}(f,g)$ in [30].

Extract from page 41 of [30]:

The far below term $T_{\text{far below}}(f,g)$ *is bounded using the Intertwining Proposition and the control of functional energy condition by the energy condition given in the* *next two sections. Indeed, assuming these two results, we have from* $\tau < \rho$ *that*

$$\begin{aligned}
\mathsf{T}_{\text{far below}}(f,g) &= \sum_{\substack{A,B\in\mathcal{A}\\ B\subsetneqq A}} \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\tau-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega \\
&= \sum_{B\in\mathcal{A}} \sum_{A\in\mathcal{A}:\ B\subsetneqq A} \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\mathbf{fi}-\text{shift}}\\ J\Subset_{\rho,\varepsilon} I}} \left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega \\
&= \sum_{B\in\mathcal{A}} \sum_{A\in\mathcal{A}:\ B\subsetneqq A} \sum_{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\tau-\text{shift}}} \left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega \\
&\quad - \sum_{B\in\mathcal{A}} \sum_{A\in\mathcal{A}:\ B\subsetneqq A} \sum_{\substack{I\in\mathcal{C}_A \text{ and } J\in\mathcal{C}_B^{\mathbf{fi}-\text{shift}}\\ J\not\Subset_{\rho,\varepsilon} I}} \left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega \\
&= \mathsf{T}_{\text{far below}}^1(f,g) - \mathsf{T}_{\text{far below}}^2(f,g).
\end{aligned}$$

Now $\mathsf{T}_{\text{far below}}^2(f,g)$ *is bounded by* NTV_α *by Lemma 9 since J is good if* $\triangle_J^\omega g \neq 0$.

The only place where the quasiweak boundedness property $\mathcal{WBP}_{T^\alpha}$ was used above[3] was in bounding the inner products $\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma f\right),\left(\triangle_J^\omega g\right)\right\rangle_\omega$ by Lemma 9 of [30] when in addition I and J were close in both scale and position, and this reduces to the previous extract from pages 28 and 29 of [30] treated above.

Thus we may split the sum in (18) as follows:

$$\begin{aligned}
\mathcal{T}^\alpha(f,g) &= \sum_{I\in\Omega\mathcal{D}_{\text{good}}^\sigma \text{ and } J\in\Omega\mathcal{D}_{\text{good}}^\omega} \mathcal{T}^\alpha\left(\triangle_I^\sigma f,\triangle_J^\omega g\right) \\
&= \left\{ \sum_{\substack{(I,J)\in\Omega\mathcal{D}_{\text{good}}^\sigma\times\Omega\mathcal{D}_{\text{good}}^\omega:\ J\subset 3I\\ 2^{-\rho}\ell(I)\le\ell(J)\le 2^\rho\ell(I)}} + \sum_{\substack{(I,J)\in\Omega\mathcal{D}_{\text{good}}^\sigma\times\Omega\mathcal{D}_{\text{good}}^\omega:\ J\subset 3I\\ 2^{-\rho}\ell(I)\le\ell(J)\le 2^\rho\ell(I)}} \right\} \mathcal{T}^\alpha\left(\triangle_I^\sigma f,\triangle_J^\omega g\right) \\
&\quad + \mathcal{R}^\alpha(f,g) \\
&\equiv \left\{A_1(f,g)+A_2(f,g)\right\} + \mathcal{R}^\alpha(f,g),
\end{aligned}$$

where we are including in the terms $A_1(f,g)+A_2(f,g)$ the corresponding inner products from the form $\mathsf{T}_{\text{far below}}^2(f,g)$ to which Lemma 9 of [30] was applied. Then

[3] On page 41 of [30], there was a typo in that $J \Subset_{\mathbf{j},\varepsilon} I$ appeared in the fourth line of the display instead of $J \not\Subset_{\rho,\varepsilon} I$ as corrected here.

the remainder form $\mathcal{R}^{\alpha}(f,g)$ satisfies the estimate

$$\left|\mathcal{R}^{\alpha}(f,g)\right| \leq C_{\alpha}\left(\sqrt{\mathfrak{A}_2^{\alpha}}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}_{T^{\alpha}}^{*}+\mathcal{E}_{\alpha}^{\text{strong}}+\mathcal{E}_{\alpha}^{\text{strong},*}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}. \tag{20}$$

The key point here is that the quasiweak boundedness constant $\mathcal{WBP}_{T^{\alpha}}$ does **not** appear on the right hand side of this estimate, and this is because the arguments in [30] that are used to bound $\mathcal{R}^{\alpha}(f,g)$ do not use the quasiweak boundedness property at all, as a patient reader can verify. This constitutes the deepest part of our argument to prove (16).

We now turn to the 'good-λ' argument that will substitute for the use of the quasiweak boundedness property in (18) in order to prove (16). First we observe that the constant C in (6) can be taken to be 2^{ρ}, and then an application of the inequality

$$\left|\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{I_{\theta}}\right),\mathbf{1}_{J_{\theta'}}\right\rangle_{\omega}\right| \leq \mathfrak{I}_{T^{\alpha}}\sqrt{\left|I_{\theta}\right|_{\sigma}\left|J_{\theta'}\right|_{\omega}},$$

to the display in (19) above, shows that

$$\begin{aligned}\left|\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{I_{\theta}}\triangle_I^{\sigma}f\right),\mathbf{1}_{J_{\theta'}}\triangle_J^{\omega}g\right\rangle_{\omega}\right| &\lesssim \sup_{a,a'\in\Gamma_n}\frac{\left|\left\langle f,h_I^{\sigma,a}\right\rangle_{\sigma}\right|}{\sqrt{\left|I_{\theta}\right|_{\sigma}}}\left|\left\langle T_{\sigma}^{\alpha}\left(\mathbf{1}_{I_{\theta}}\right),\mathbf{1}_{J_{\theta'}}\right\rangle_{\omega}\right|\frac{\left|\left\langle g,h_J^{\omega,a'}\right\rangle_{\omega}\right|}{\sqrt{\left|J_{\theta'}\right|_{\omega}}}\\ &\lesssim \sup_{a,a'\in\Gamma_n}\frac{\left|\left\langle f,h_I^{\sigma,a}\right\rangle_{\sigma}\right|}{\sqrt{\left|I_{\theta}\right|_{\sigma}}}\mathfrak{I}_{T^{\alpha}}\sqrt{\left|I_{\theta}\right|_{\sigma}\left|J_{\theta'}\right|_{\omega}}\frac{\left|\left\langle g,h_J^{\omega,a'}\right\rangle_{\omega}\right|}{\sqrt{\left|J_{\theta'}\right|_{\omega}}}\\ &= \sup_{a,a'\in\Gamma_n}\mathfrak{I}_{T^{\alpha}}\left|\left\langle f,h_I^{\sigma,a}\right\rangle_{\sigma}\right|\left|\left\langle g,h_J^{\omega,a'}\right\rangle_{\omega}\right|.\end{aligned}$$

From this we obtain the following *crude* estimate valid for any $f \in L^2(\sigma)$ and $g \in L^2(\omega)$:

$$\left|A_1(f,g)+A_2(f,g)\right| \leq C_{\alpha}\left(\sqrt{\mathfrak{A}_2^{\alpha}}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}_{T^{\alpha}}^{*}+\mathfrak{I}_{T^{\alpha}}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}. \tag{21}$$

Definition 8 We say that two quasicubes K and L have η*-comparable* side lengths, or simply that $\ell(K)$ and $\ell(L)$ are η*-comparable*, if

$$2^{-\eta}\ell(K) \leq \ell(L) \leq 2^{\eta}\ell(K).$$

Furthermore, we say that K and L are η*-close* if they have η*-comparable* side lengths, and if they belong to a common quasigrid $\Omega\mathcal{D}$ and are touching quasicubes that satisfy either $K \subset 3L$ or $L \subset 3K$.

Now consider the special indicator case $f = \mathbf{1}_Q$ and $g = \mathbf{1}_R$ where Q and R are ρ*-close* in some $\Omega\mathcal{D}$. For this case we will be able to do much better than (21). In fact, for each $0 < \lambda < \frac{1}{2}$ we claim that the following 'good-λ' inequality holds:

$$\left|A_1\left(\mathbf{1}_Q, \mathbf{1}_R\right)\right| + \left|A_2\left(\mathbf{1}_Q, \mathbf{1}_R\right)\right| \leq C_\alpha \left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \sqrt[4]{\lambda}\mathfrak{N}_{T^\alpha}\right) \left\|\mathbf{1}_Q\right\|_{L^2(\sigma)} \left\|\mathbf{1}_R\right\|_{L^2(\omega)}. \tag{22}$$

With (22) proved, we can use it and (20) to complete the proof of the estimate for the indicator/touching property (16) by taking expectations $\mathbb{E}_\Omega$ as usual:

$$\begin{aligned}
&\mathbb{E}_\Omega \left|\sum_{I \in \Omega\mathcal{D}^\sigma \text{ and } J \in \Omega\mathcal{D}^\omega} \mathcal{T}_\sigma^\alpha \left(\triangle_I^\sigma \mathbf{1}_Q, \triangle_J^\omega \mathbf{1}_R\right)\right| \\
&\leq \mathbb{E}_\Omega \left(|A_1| + |A_2|\right) + \mathbb{E}_\Omega \left|\mathcal{R}^\alpha\left(\mathbf{1}_Q, \mathbf{1}_R\right)\right| \\
&\leq C_\alpha \left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \sqrt{\lambda}\mathfrak{N}_{T^\alpha}\right) \left\|\mathbf{1}_Q\right\|_{L^2(\sigma)} \left\|\mathbf{1}_R\right\|_{L^2(\omega)} \\
&\quad + C_\alpha \left(\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*}\right) \left\|\mathbf{1}_Q\right\|_{L^2(\sigma)} \left\|\mathbf{1}_R\right\|_{L^2(\omega)} \\
&\leq C_\alpha \left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} + \sqrt[4]{\lambda}\mathfrak{N}_{T^\alpha}\right) \left\|\mathbf{1}_Q\right\|_{L^2(\sigma)} \left\|\mathbf{1}_R\right\|_{L^2(\omega)},
\end{aligned}$$

which gives (16) upon taking the supremum over such Q and R to get

$$\mathfrak{I}_{T^\alpha} \leq C'_\alpha \left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} + \sqrt[4]{\lambda}\mathfrak{N}_{T^\alpha}\right).$$

Notation 3 *The remainder of this paper is devoted to proving (22) for touching and $\boldsymbol{\rho}$-close quasicubes Q and R. To simplify notation and geometric constructions, we consider only the case of* ordinary *cubes in $\mathcal{P}^n$, and note that the extension to the quasiworld is then routine.*

To prove the claim (22) we use the *parameterization by translation* introduced above. Essentially this approach was used in the averaging technique employed in [23], which in turn was borrowed from Fefferman and Stein [4], later refined in [6], and further refined here in this paper. It suffices to prove that

$$\left|\mathcal{T}^\alpha\left(\left(\mathbf{1}_Q\right)_{\text{good}}, \left(\mathbf{1}_R\right)_{\text{good}}\right)\right| \leq C_\alpha \left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha} + \mathfrak{T}_{T^\alpha} + \mathfrak{T}_{T^\alpha}^* + \mathcal{E}_\alpha^{\text{strong}} + \mathcal{E}_\alpha^{\text{strong},*} + \sqrt{\lambda}\mathfrak{N}_{T^\alpha}\right) \times \left\|\mathbf{1}_Q\right\|_{L^2(\sigma)} \left\|\mathbf{1}_R\right\|_{L^2(\omega)},$$

for all $Q, R \in \mathcal{P}^n$ that are $\boldsymbol{\rho}$-close, uniformly over Q-good grids, and where

$$\mathcal{T}^\alpha\left(\left(\mathbf{1}_Q\right)_{\text{good}}, \left(\mathbf{1}_R\right)_{\text{good}}\right) = \sum_{I \in \mathcal{D}_{Q;\text{good}}^\sigma \text{ and } J \in \mathcal{D}_{Q;\text{good}}^\omega} \mathcal{T}^\alpha\left(\triangle_I^\sigma \mathbf{1}_Q, \triangle_J^\omega \mathbf{1}_R\right).$$

The grids $\mathcal{D}^{\sigma}_{Q;\text{good}} = \mathcal{D}^{\omega}_{Q;\text{good}}$ are those arising in the projections $\mathsf{P}^{\sigma}_{Q;\text{good}}f$ and $\mathsf{P}^{\omega}_{Q;\text{good}}g$ above. Moreover, due to the key observation above regarding where the weak boundedness property arises in the proof of the main theorem in [30], it suffices to prove

$$\mathbb{E}_{\Omega}\left\{\sum_{\substack{(I,J)\in\mathcal{D}^{\sigma}_{Q;\text{good}}\times\mathcal{D}^{\omega}_{Q;\text{good}}:\ J\subset 3I\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)}}+\sum_{\substack{(I,J)\in\mathcal{D}^{\sigma}_{Q;\text{good}}\times\mathcal{D}^{\omega}_{Q;\text{good}}:\ I\subset 3J\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)}}\right\}\left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right),\triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}\right|$$
$$\leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}^{*}_{T^{\alpha}}+\sqrt[4]{\lambda}\mathfrak{N}_{T^{\alpha}}\right)\left\|\mathbf{1}_{Q}\right\|_{L^{2}(\sigma)}\left\|\mathbf{1}_{R}\right\|_{L^{2}(\omega)},$$

under the assumption that we sum over only Q-good cubes I and J that belong to Q-good grids in the above sums, and where we recall that we may realize the underlying probability space as translations of any fixed grid, say the standard dyadic grid. Note that R is contained in $3Q$, and this accounts for our inclusion of siblings in Definition 7 above.

By symmetry it suffices to prove for all $0<\lambda<\frac{1}{2}$ that

$$\mathbb{E}_{\Omega}\sum_{\substack{(I,J)\in\mathcal{D}^{\sigma}_{Q;\text{good}}\times\mathcal{D}^{\omega}_{Q;\text{good}}:\ J\subset 3I\\ 2^{-\rho}\ell(I)\leq\ell(J)\leq 2^{\rho}\ell(I)\\ I\text{ and }J\text{ touch}}}\left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right),\triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}\right| \tag{23}$$
$$\leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}^{*}_{T^{\alpha}}+\sqrt[4]{\lambda}\mathfrak{J}_{T^{\alpha}}\right)\left\|\mathbf{1}_{Q}\right\|_{L^{2}(\sigma)}\left\|\mathbf{1}_{R}\right\|_{L^{2}(\omega)},$$

for all cubes $Q,R\in\mathcal{P}^{n}$ that are $\boldsymbol{\rho}$-close (we are including the testing conditions here because we are including children I_{θ} and $J_{\theta'}$ in the display (19) that coincide as well).

3.2.1 Three Critical Reductions

Now we make three critical reductions that permit the application of NTV surgery, and lie at the core of the much better estimate (22).

(1) We must have that I 'cuts across the boundary' of Q, i.e. $|I\cap Q|>0$ and $|I\cap Q^{c}|>0$ (or else $\triangle^{\sigma}_{I}\mathbf{1}_{Q}=0$),
(2) We must have that J 'cuts across the boundary' of R, i.e. $|J\cap R|>0$ and $|J\cap R^{c}|>0$ (or else $\triangle^{\omega}_{J}\mathbf{1}_{R}=0$),
(3) By the assumed 'Q-goodness' in Definition 7, together with reductions (1) and (2) above, we *cannot* have either $\ell(I)\geq 2^{\mathbf{r}}\ell(Q)$ or $\ell(J)\geq 2^{\mathbf{r}}\ell(R)$.

From these reductions, we are left to prove

$$\mathbb{E}_{\Omega} \sum_{\substack{(I,J)\in\mathcal{D}^{\sigma}_{Q;\text{good}}\times\mathcal{D}^{\omega}_{Q;\text{good}}:\ J\subset 3I \\ I \text{ and } J \text{ are } \boldsymbol{\rho}\text{-close} \\ \ell(I)<2^{\mathbf{r}}\ell(Q) \text{ and } \ell(J)<2^{\mathbf{r}}\ell(R)}} \left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right), \triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}\right| \tag{24}$$

$$\leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^{*}_{T^{\alpha}} + \sqrt[4]{\lambda}\mathfrak{N}_{T^{\alpha}}\right)\left\|\mathbf{1}_{Q}\right\|_{L^{of 2}(\sigma)}\left\|\mathbf{1}_{R}\right\|_{L^{2}(\omega)} ,$$

for all $\boldsymbol{\rho}$-close $Q, R \in \mathcal{P}^{n}$.

The *small* pairs of cubes (I, J), i.e. those with both $\ell(I) < 2^{-\mathbf{r}}\ell(Q)$ and $\ell(J) < 2^{\mathbf{r}}\ell(R)$, pose a difficulty and our next task is to further reduce matters to proving the more restricted estimate:

$$\mathbb{E}_{\Omega} \sum_{\substack{(I,J)\in\mathcal{D}^{\sigma}_{Q;\text{good}}\times\mathcal{D}^{\omega}_{Q;\text{good}}:\ J\subset 3I \\ I \text{ and } J \text{ are } \boldsymbol{\rho}\text{-close} \\ \ell(I) \text{ and } \ell(Q) \text{ are } r\text{-comparable} \\ \ell(J) \text{ and } \ell(R) \text{ are } r\text{-comparable}}} \left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right), \triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}\right| \tag{25}$$

$$\leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^{*}_{T^{\alpha}} + \sqrt[4]{\lambda}\mathfrak{N}_{T^{\alpha}}\right)\left\|\mathbf{1}_{Q}\right\|_{L^{2}(\sigma)}\left\|\mathbf{1}_{R}\right\|_{L^{2}(\omega)} ,$$

for all $Q, R \in \mathcal{P}^{n}$ that are $\boldsymbol{\rho}$-close. The difference between (25) and (24) is that in (25), we do *not* permit small pairs of (I, J), i.e. those with $\ell(I) < 2^{-\mathbf{r}}\ell(Q)$ or $\ell(J) < 2^{-\mathbf{r}}\ell(RQ)$.

3.2.2 Elimination of Small Pairs

To eliminate the small pairs from (24), we apply for a second time our proof from [30] as outlined above, but this time to each inner product $\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right), \triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}$ appearing in the sum in (24) inside the expectation $\mathbb{E}_{\Omega}$. In other words, for fixed I, J, Q and R, we take $f = \triangle^{\sigma}_{I}\mathbf{1}_{Q}$ and $g = \triangle^{\omega}_{J}\mathbf{1}_{R}$, and we obtain that

$$\mathbb{E}_{\Omega}\mathbb{E}_{\Omega'}\left|\left\langle T^{\alpha}_{\sigma}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right), \triangle^{\omega}_{J}\mathbf{1}_{R}\right\rangle_{\omega}\right|$$

$$\leq C_{\alpha}\left(\sqrt{\mathfrak{A}^{\alpha}_{2}} + \mathfrak{T}_{T^{\alpha}} + \mathfrak{T}^{*}_{T^{\alpha}} + +\mathcal{E}^{\text{strong}}_{\alpha} + \mathcal{E}^{\text{strong},*}_{\alpha} + 2^{-\varepsilon\mathbf{r}}\mathfrak{N}_{T^{\alpha}}\right)\left\|\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right\|_{L^{2}(\sigma)}\left\|\triangle^{\sigma}_{J}\mathbf{1}_{R}\right\|_{L^{2}(\omega)}$$

$$+\mathbb{E}_{\Omega}\mathbb{E}_{\Omega'} \sum_{\substack{(K,L)\in\mathcal{D}'_{Q;\text{good}}\times\mathcal{D}'_{Q;\text{good}}:\ L\subset 3K \\ K \text{ and } L \text{ are } \mathbf{j}\text{-close} \\ \ell(K)<2^{\mathbf{r}}\ell(I) \text{ and } \ell(L)<2^{\mathbf{r}}\ell(JR)}} \left|\left\langle T^{\alpha}_{\sigma}\triangle^{\sigma}_{K}\left(\triangle^{\sigma}_{I}\mathbf{1}_{Q}\right), \triangle^{\omega}_{L}\left(\triangle^{\omega}_{J}\mathbf{1}_{R}\right)\right\rangle_{\omega}\right| ,$$

where here the expectation $\mathbb{E}_{\Omega'}$ is taken to be independent of $\mathbb{E}_{\Omega}$.

But now we may further assume that the pair of grids $(\mathcal{D},\mathcal{D}')$, for which $(I,J)\in\mathcal{D}\times\mathcal{D}$ and $(K,L)\in\mathcal{D}'\times\mathcal{D}'$, are *mutually good*.[4] Thus we cannot have $\ell(K)<2^{-\boldsymbol{\rho}}\ell(I)$ because K is I-good, and this eliminates the inclusion of small pairs (K,L), i.e. those with $\ell(K)<2^{-\boldsymbol{\rho}}\ell(I)$. Note that the term $2^{-\varepsilon\mathbf{r}}\mathfrak{N}_{T^\alpha}$ arises from the bad Haar projections $\triangle_K^\sigma$ and $\triangle_L^\omega$ of $\triangle_I^\sigma\mathbf{1}_Q$ and $\triangle_J^\omega\mathbf{1}_R$ respectively. Finally, we note that $f=\triangle_I^\sigma\mathbf{1}_Q$ is constant on the children of I and that $\left\|\triangle_I^\sigma\mathbf{1}_Q\right\|_{L^2(\sigma)}^2=\sum_{I'\in\mathfrak{C}(I)}\int_{I'}\left|\mathbb{E}_{I'}^\sigma\mathbf{1}_Q-\mathbb{E}_I^\sigma\mathbf{1}_Q\right|^2d\sigma$. Thus it suffices to prove the following estimate,

$$\mathbb{E}_{\Omega'}\sum_{\substack{(K,L)\in\mathcal{D}'_{Q;\mathrm{good}}\times\mathcal{D}'_{Q;\mathrm{good}}:\ L\subset 3K\\ K\text{ and }L\text{ are }\boldsymbol{\rho}\text{-close}\\ \ell(K)\text{ and }\ell(I)\text{ are }\mathbf{r}\text{-comparable}\\ \ell(L)\text{ and }\ell(J)\text{ are }\mathbf{r}\text{-comparable}}}\ \sum_{\substack{I'\in\mathfrak{C}(I)\\ J'\in\mathfrak{C}(J)}}\left|\left\langle T_\sigma^\alpha\,\triangle_K^\sigma\left(\left[\mathbb{E}_{I'}^\sigma\,\triangle_I^\sigma\,\mathbf{1}_Q\right]\mathbf{1}_{I'}\right),\triangle_L^\omega\left(\left[\mathbb{E}_{J'}^\omega\,\triangle_J^\omega\,\mathbf{1}_R\right]\mathbf{1}_{J'}\right)\right\rangle_\omega\right|$$

$$\leq C_\alpha\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}_{T^\alpha}^*+\sqrt[4]{\lambda}\mathfrak{N}_{T^\alpha}\right)\sum_{\substack{I'\in\mathfrak{C}(I)\\ J'\in\mathfrak{C}(J)}}\left|\mathbb{E}_{I'}^\sigma\,\triangle_I^\sigma\,\mathbf{1}_Q\right|\,\left|\mathbb{E}_{J'}^\omega\,\triangle_J^\omega\,\mathbf{1}_R\right|\,\|\mathbf{1}_{I'}\|_{L^2(\sigma)}\|\mathbf{1}_{J'}\|_{L^2(\omega)},$$

which we can write simply as

$$\mathbb{E}_{\Omega'}\sum_{\substack{(K,L)\in\mathcal{D}'_{Q;\mathrm{good}}\times\mathcal{D}'_{Q;\mathrm{good}}:\ L\subset 3K\\ K\text{ and }L\text{ are }\boldsymbol{\rho}\text{-close}\\ \ell(K)\text{ and }\ell(I')\text{ are }\mathbf{r}\text{-comparable}\\ \ell(L)\text{ and }\ell(J')\text{ are }\mathbf{r}\text{-comparable}}}\left|\left\langle T_\sigma^\alpha\,\triangle_K^\sigma\left(\mathbf{1}_{I'}\right),\triangle_L^\omega\left(\mathbf{1}_{J'}\right)\right\rangle_\omega\right|$$

$$\leq C_\alpha\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}+\mathfrak{T}_{T^\alpha}+\mathfrak{T}_{T^\alpha}^*+\sqrt[4]{\lambda}\mathfrak{N}_{T^\alpha}\right)\|\mathbf{1}_{I'}\|_{L^2(\sigma)}\|\mathbf{1}_{J'}\|_{L^2(\omega)}$$

for each $I'\in\mathfrak{C}(I)$ and $J'\in\mathfrak{C}(J)$. Now relabel I' and J' as Q and R respectively (and then also K and L as I and J respectively) to obtain (25).

3.2.3 NTV Surgery

Now in order to prove (25), we invoke the technique of NTV surgery as used in [7, 17] and [10]. Given $0<\lambda<\frac{1}{2}$, define

$$J_\lambda\equiv\{x\in J:\operatorname{dist}(x,\partial J)>\lambda\ell(J)\}.$$

Then we write

$$\begin{aligned}\left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma\mathbf{1}_Q\right),\triangle_J^\omega\mathbf{1}_R\right\rangle_\omega\right|&\leq\left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma\mathbf{1}_Q\right),\mathbf{1}_{J_\lambda}\,\triangle_J^\omega\,\mathbf{1}_R\right\rangle_\omega\right|+\left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma\mathbf{1}_Q\right),\mathbf{1}_{J\setminus J_\lambda}\,\triangle_J^\omega\,\mathbf{1}_R\right\rangle_\omega\right|\\&\equiv A_1+A_2.\end{aligned}$$

[4]Both I and J belong to the common grid $\mathcal{D}$, while K and L belong to the independent common grid $\mathcal{D}'$—in contrast to the traditional use of two independent grids where $I\in\mathcal{D}$ and $J\in\mathcal{D}'$.

Now we use first the fact that I and J_λ are separated by a distance at least $\lambda\ell(J) > 0$ in order to bound the first term A_1 by

$$\begin{aligned} A_1 &= \left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I'}\,\triangle_I^\sigma\,\mathbf{1}_Q\right), \mathbf{1}_{J_\lambda}\,\triangle_J^\omega\,\mathbf{1}_R\right\rangle_\omega\right| \\ &\lesssim \frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}\,\left\|\triangle_I^\sigma\mathbf{1}_Q\right\|_{L^2(\sigma)}\left\|\triangle_J^\omega\mathbf{1}_R\right\|_{L^2(\omega)} \leq \frac{1}{\lambda}\sqrt{\mathfrak{A}_2^\alpha}\,\left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}\left\|\mathbf{1}_R\right\|_{L^2(\omega)}\ . \end{aligned} \tag{26}$$

We further dominate the square of the second term A_2 by

$$\begin{aligned} A_2^2 &= \left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma\mathbf{1}_Q\right), \mathbf{1}_{J\setminus J_\lambda}\,\triangle_J^\omega\,\mathbf{1}_R\right\rangle_\omega\right|^2 \\ &= \left|\left\langle T_\sigma^\alpha\left(\sum_{I'\in\mathfrak{C}(I)}\mathbf{1}_{I'}\,\triangle_I^\sigma\,\mathbf{1}_Q\right), \mathbf{1}_{J\setminus J_\delta}\sum_{J'\in\mathfrak{C}(J)}\mathbf{1}_{J'}\,\triangle_J^\omega\,\mathbf{1}_R\right\rangle_\omega\right|^2 \\ &\lesssim \sum_{I'\in\mathfrak{C}(I)}\sum_{J'\in\mathfrak{C}(J)}\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I'}\,\triangle_I^\sigma\,\mathbf{1}_Q\right), \mathbf{1}_{J'\setminus J_\lambda}\,\triangle_J^\omega\,\mathbf{1}_R\right\rangle_\omega\right|^2 \\ &\lesssim \sum_{I'\in\mathfrak{C}(I)}\sum_{J'\in\mathfrak{C}(J)}\mathfrak{N}_{T^\alpha}^2\left\|\mathbf{1}_{I'}\,\triangle_I^\sigma\,\mathbf{1}_Q\right\|_{L^2(\sigma)}^2\left\|\mathbf{1}_{J'\setminus J_\lambda}\,\triangle_J^\omega\,\mathbf{1}_R\right\|_{L^2(\omega)}^2 \\ &\lesssim \mathfrak{N}_{T^\alpha}^2\left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}^2\sum_{J'\in\mathfrak{C}(J)}\left\|\mathbf{1}_{J'\setminus J_\lambda}\,\triangle_J^\omega\,\mathbf{1}_R\right\|_{L^2(\omega)}^2 = \mathfrak{N}_{T^\alpha}^2\left\|\mathbf{1}_Q\right\|_{L^2(\sigma)}^2\int_{J'\setminus J_\lambda}\left|\triangle_J^\omega\mathbf{1}_R\right|^2 d\omega\ . \end{aligned} \tag{27}$$

Then we note the fact that, using the *translation parameterization* of Ω indexed by $\gamma \in \Gamma$, we have

$$\mathbb{E}_\Omega\left|R\cap\left[(J+\gamma)'\setminus(J+\gamma)_\lambda\right]\right|_\omega \leq C_\alpha\lambda\left|R\right|_\omega\,, \tag{28}$$

which follows upon taking the average over certain translates $\mathcal{D}_0 + \gamma$ where $\mathcal{D}_0$ is a fixed grid containing J. This is of course equivalent to taking instead the average over the same translates $\omega + \gamma$ of the measure ω, and it is in this latter form that (28) is evident.

Now we will apply (28), together with an argument to resolve the difficulty associated with the appearance of J in *both* $J'\setminus J_\lambda$ and $\triangle_J^\omega\mathbf{1}_R$, to obtain the following key estimate for every $0 < \lambda < \frac{1}{2}$:

$$\mathbb{E}_\Omega\int_{J'\setminus J_\lambda}\left|\triangle_J^\omega\mathbf{1}_R\right|^2 d\omega \leq C_\alpha\sqrt{\lambda}\left|R\right|_\omega\,, \tag{29}$$

for the expected value of the final integral on the right hand side of (27). With (29) and (26) in hand, we will obtain that

$$\begin{aligned} &\mathbb{E}_\Omega\left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma\mathbf{1}_Q\right), \triangle_J^\omega\mathbf{1}_R\right\rangle_\omega\right|^2 \\ &\lesssim \mathbb{E}_\Omega\left|\left\langle T_\sigma^\alpha\left(\triangle_I^\sigma\mathbf{1}_Q\right), \mathbf{1}_{J_\lambda}\,\triangle_J^\omega\,\mathbf{1}_R\right\rangle_\omega\right|^2 + \mathbb{E}_\Omega\sum_{I'\in\mathfrak{C}(I)}\sum_{J'\in\mathfrak{C}(J)}\left|\left\langle T_\sigma^\alpha\left(\mathbf{1}_{I'}\,\triangle_I^\sigma\,\mathbf{1}_Q\right), \mathbf{1}_{J'\setminus J_\lambda}\,\triangle_J^\omega\,\mathbf{1}_R\right\rangle_\omega\right|^2 \end{aligned}$$

$$\leq C_\alpha^2 \frac{1}{\lambda^2} \mathfrak{A}_2^\alpha \left\| \mathbf{1}_Q \right\|_{L^2(\sigma)}^2 \left\| \mathbf{1}_R \right\|_{L^2(\omega)}^2 + \mathbb{E}_\Omega \sum_{I' \in \mathfrak{C}(I)} \sum_{J' \in \mathfrak{C}(J)} \mathfrak{N}_{T^\alpha}^2 \left\| \mathbf{1}_{I'} \bigtriangleup_I^\sigma \mathbf{1}_Q \right\|_{L^2(\sigma)}^2 \left\| \mathbf{1}_{J' \setminus J_\lambda} \bigtriangleup_J^\omega \mathbf{1}_R \right\|_{L^2(\omega)}^2$$

$$\leq C_\alpha^2 \frac{1}{\lambda^2} \mathfrak{A}_2^\alpha \left\| \mathbf{1}_Q \right\|_{L^2(\sigma)}^2 \left\| \mathbf{1}_R \right\|_{L^2(\omega)}^2 + \sqrt{\lambda} \mathfrak{N}_{T^\alpha}^2 \left\| \mathbf{1}_Q \right\|_{L^2(\sigma)}^2 \left\| \mathbf{1}_R \right\|_{L^2(\omega)}^2 ,$$

as required. Thus the proof of (16), and hence also that of the Good-λ Lemma, will be complete once we have proved the estimate (29), to which we now turn.

Remark 3 In the third line above we have used the norm inequality $\left|\langle T_\sigma^\alpha f, g\rangle_\omega\right| \leq \mathfrak{N}_{T^\alpha} \|f\|_{L^2(\sigma)} \|g\|_{L^2(\omega)}$ with $f = \mathbf{1}_{I'} \bigtriangleup_I^\sigma \mathbf{1}_Q$ and $g = \mathbf{1}_{J' \setminus J_\lambda} \bigtriangleup_J^\omega \mathbf{1}_R$, and where g is a constant multiple of an indicator of a 'rectangle' $J' \setminus J_\lambda$. This prevents us from using the smaller bound $\lambda \mathfrak{I}_{T^\alpha}^2$ in place of $\lambda \mathfrak{N}_{T^\alpha}^2$.

In order to illuminate the main ideas in the proof of (29), we first prove the simplest case of dimension $n = 1$. So let

$$J \setminus J_\lambda = J_\lambda^{\text{left}} \cup J_\lambda^{\text{right}},$$

where $J_\lambda^{\text{left}} = J_- \setminus J_\lambda$ and $J_\lambda^{\text{right}} = J_+ \setminus J_\lambda$, and write

$$\mathbb{E}_\Omega \int_{J' \setminus J_\lambda} \left| \bigtriangleup_J^\omega \mathbf{1}_R \right|^2 d\omega = \mathbb{E}_\Omega \int_{J_\lambda^{\text{left}}} \left| \bigtriangleup_J^\omega \mathbf{1}_R \right|^2 d\omega + \mathbb{E}_\Omega \int_{J_\lambda^{\text{right}}} \left| \bigtriangleup_J^\omega \mathbf{1}_R \right|^2 d\omega = \mathit{Left} + \mathit{Right}. \tag{30}$$

Now we recall the parameterization of the expectation by translations $\gamma \in \Gamma_M^N$ of step size 2^{-M}, and let $\eta = \lambda 2^M$ where λ is the side length of the interval $J' \setminus J_\lambda$. Then, by using the 'average of an average' principle, we can rewrite the expectation in terms of the larger step size $\eta 2^{-M}$. We continue to use γ to denote the new step size $\eta 2^{-M}$. Then we further decompose the expectation *Left* in (30) as

$$\begin{aligned}
\mathit{Left} &= \mathbb{E}_\Omega \int_{J_\lambda^{\text{left}}} \left| \bigtriangleup_J^\omega \mathbf{1}_R \right|^2 d\omega = \mathbb{E}_\Omega \int_{(J+\gamma)_\lambda^{\text{left}}} \left| \bigtriangleup_{J+\gamma}^\omega \mathbf{1}_R \right|^2 d\omega \\
&= \mathbb{E}_\Omega \mathbf{1}_{\left\{\gamma : (J+\gamma)_\lambda^{\text{left}} \subset R\right\}} \int_{(J+\gamma)_\lambda^{\text{left}}} \left| \bigtriangleup_{J+\gamma}^\omega \mathbf{1}_R \right|^2 d\omega \\
&\quad + \mathbb{E}_\Omega \mathbf{1}_{\left\{\gamma : (J+\gamma)_\lambda^{\text{left}} \text{ lies to the left of } R\right\}} \int_{(J+\gamma)_\lambda^{\text{left}}} \left| \bigtriangleup_{J+\gamma}^\omega \mathbf{1}_R \right|^2 d\omega \\
&\equiv A_3 + A_4 ,
\end{aligned}$$

where because of our change of step size, we have that $\left\{(J+\gamma)_\lambda^{\text{left}}\right\}_\gamma$ is a pairwise disjoint covering of the top interval containing J that has side length 2^{-N} (see the beginning of Sect. 3.1 above).

For term A_3 we use the elementary estimate

$$\left|\triangle^{\omega}_{J+\gamma}\mathbf{1}_R\right| = \left|\mathbb{E}_{(J+\gamma)_-}\mathbf{1}_R - \mathbb{E}_{(J+\gamma)}\mathbf{1}_R\right| \leq 1$$

together with the estimate in (28), to obtain

$$\begin{aligned} A_3 &= \mathbb{E}_{\Omega}\mathbf{1}_{\{\gamma:(J+\gamma)^{\mathrm{left}}_{\lambda}\subset R\}} \int_{(J+\gamma)^{\mathrm{left}}_{\lambda}} \left|\triangle^{\omega}_{J+\gamma}\mathbf{1}_R\right|^2 d\omega \\ &\leq \mathbb{E}_{\Omega}\left|R\cap(J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega} \leq C_{\alpha}\lambda\,|R|_{\omega} \ . \end{aligned}$$

For term A_4 we proceed as follows. We suppose that $(J+\gamma)^{\mathrm{left}}_{\lambda}$ lies to the left of R, since the case when $(J+\gamma)^{\mathrm{right}}_{\lambda}$ lies to the right of R is similar. We have

$$\begin{aligned} \int_{(J+\gamma)^{\mathrm{left}}_{\lambda}} \left|\triangle^{\omega}_{J+\gamma}\mathbf{1}_R\right|^2 d\omega &= \int_{(J+\gamma)^{\mathrm{left}}_{\lambda}} \left|\mathbb{E}_{(J+\gamma)_-}\mathbf{1}_R - \mathbb{E}_{(J+\gamma)}\mathbf{1}_R\right|^2 d\omega \\ &= \int_{(J+\gamma)^{\mathrm{left}}_{\lambda}} \left|\frac{|R\cap(J+\gamma)_-|_{\omega}}{|(J+\gamma)_-|_{\omega}} - \frac{|R\cap(J+\gamma)|_{\omega}}{|J+\gamma|_{\omega}}\right|^2 d\omega \\ &\leq 2\left|(J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega}\left(\frac{|R\cap(J+\gamma)_-|_{\omega}}{|(J+\gamma)_-|_{\omega}}\right)^2 \\ &\quad +2\left|(J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega}\left(\frac{|R\cap(J+\gamma)|_{\omega}}{|J+\gamma|_{\omega}}\right)^2 \ . \end{aligned}$$

We now estimate the sum of the first terms above since the sum of the second terms can be estimated with the same argument.

For the sum of the first terms we write

$$\begin{aligned} &\sum_{\gamma:\ (J+\gamma)^{\mathrm{left}}_{\lambda}\text{ is left of }R} \left|(J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega}\left(\frac{|R\cap(J+\gamma)_-|_{\omega}}{|(J+\gamma)_-|_{\omega}}\right)^2 \\ &\leq \left(\sum_{\gamma:\ (J+\gamma)^{\mathrm{left}}_{\lambda}\text{ is left of }R} \frac{\left|(J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega}}{|(J+\gamma)_-|_{\omega}}\frac{|R\cap(J+\gamma)_-|_{\omega}}{|(J+\gamma)_-|_{\omega}}\right)|R|_{\omega} \ , \end{aligned}$$

and let $J+\gamma_1$ be the leftmost translate of J such that

$$\frac{\left|(J+\gamma)^{\mathrm{left}}_{\lambda}\right|_{\omega}}{|(J+\gamma)_-|_{\omega}}\frac{|R\cap(J+\gamma)_-|_{\omega}}{|(J+\gamma)_-|_{\omega}} > \delta, \tag{31}$$

where $\delta > 0$ will be chosen later to be $\sqrt{\lambda}$. We suppose the translations γ are ordered to be increasing. Note that we have both

$$1 \geq \frac{|R \cap (J+\gamma_1)_-|_\omega}{|(J+\gamma_1)_-|_\omega} > \delta$$

and

$$(J+\gamma)_\lambda^{\text{left}} \subset (J+\gamma_1)_- \ ,$$
$$\text{if both } \gamma > \gamma_1 \text{ and } (J+\gamma)_\lambda^{\text{left}} \text{ is left of } R.$$

Thus we compute that

$$\mathbb{E}_\Omega \int_{(J+\gamma)_\lambda^{\text{left}}} \left| \triangle_{J+\gamma}^\omega \mathbf{1}_R \right|^2 d\omega = \frac{1}{\Lambda} \left\{ \sum_{\gamma<\gamma_1} + \sum_{\gamma>\gamma_1:\ (J+\gamma)_\lambda^{\text{left}} \text{ is left of } R} \right\} \tag{32}$$

$$\int_{(J+\gamma)_\lambda^{\text{left}}} \left| \triangle_{J+\gamma}^\omega \mathbf{1}_R \right|^2 d\omega \tag{33}$$

$$\leq \frac{1}{\Lambda} \sum_{\gamma<\gamma_1} \frac{\left|(J+\gamma)_\lambda^{\text{left}}\right|_\omega |R \cap (J+\gamma)_-|_\omega}{|(J+\gamma)_-|_\omega^2} |R|_\omega + \frac{1}{\Lambda} \sum_{\gamma>\gamma_1:\ (J+\gamma)_\lambda^{\text{left}} \text{ is left of } R} \left|(J+\gamma)_\lambda^{\text{left}}\right|_\omega$$

$$\leq \frac{1}{\Lambda} \delta \, \#\{\gamma < \gamma_1\} \, |R|_\omega + \frac{1}{\Lambda} |(J+\gamma_1)_-|_\omega \leq \delta \, |R|_\omega + \frac{1}{\Lambda} \frac{1}{\delta} |R \cap (J+\gamma_1)_-|_\omega$$

$$\leq \left(\delta + \frac{\lambda}{\delta} \right) |R|_\omega = 2\sqrt{\lambda} \, |R|_\omega \, ,$$

if we choose $\delta = \sqrt{\lambda}$. This completes the proof of (29) in dimension $n = 1$.

3.2.4 Higher Dimensions

In the case of $n > 1$ dimensions we decompose the 'corner-like' pieces $J' \setminus J_\lambda$ for each child $J' \in \mathfrak{C}(J)$ into faces $S + \gamma$ of width λ (when $n = 1$ there are only two such faces $S + \gamma$, namely the intervals $(J+\gamma)_\lambda^{\text{left}}$ and $(J+\gamma)_\lambda^{\text{right}}$). Then we apply the above argument for $(J+\gamma)_\lambda^{\text{left}}$ to $S + \gamma$ for each face S of width λ in $J' \setminus J_\lambda$, but using only translations perpendicular to the face S, and finally apply the 'average of an average' principle, to obtain (29). We illustrate the proof in the case $n = 2$ since the general case $n \geq 2$ is no different.

For a square K in the plane, let K_- denote the lower left child of K. Now fix squares J and R in the plane with $\boldsymbol{\rho}$-comparable side lengths and such that $J \subset 3R$. For $\gamma \in \mathcal{H}_\lambda$, where $\mathcal{H}_\lambda$ is the set of *horizontal* translations γ of step size λ with

$|\gamma| \leq C\ell(R)$, denote by $(J+\gamma)_{\lambda}^{\text{lower left}}$ the L-shaped 'corner'

$$(J+\gamma)_{\lambda}^{\text{lower left}} \equiv (J+\gamma)_{-} \setminus J_{\lambda} ,$$

and by $(J+\gamma)_{\lambda}^{\text{left}}$ the *vertical* portion of the L-shaped set $(J+\gamma)_{\lambda}^{\text{lower left}}$ (this is one of the faces $S+\gamma$ introduced above). We will show that

$$\frac{1}{\#\mathcal{H}_{\lambda}} \sum_{\gamma \in \mathcal{H}_{\lambda}} \int_{(J+\gamma)_{\lambda}^{\text{left}}} \left| \triangle_{J+\gamma}^{\omega} \mathbf{1}_R \right|^2 d\omega \lesssim \sqrt{\lambda}, \tag{34}$$

where $\#\mathcal{H}_{\lambda} \approx \frac{C\ell(R)}{\lambda}$, and then by the 'average of an average' principle we obtain (29). To prove (34) we will apply the one-dimensional argument from the previous subsubsection, but with modifications to accommodate the fact that $(J+\gamma)_{\lambda}^{\text{left}}$ can now spill out over the top of R as well as to the left of R (recall that in the one-dimensional setting, $(J+\gamma)_{\lambda}^{\text{left}}$ occurred to the left of the interval R if it was not contained in R). As in dimension $n = 1$, let $J+\gamma_1$ be the leftmost horizontal translate of J such that

$$\frac{\left|(J+\gamma)_{\lambda}^{\text{left}}\right|_{\omega}}{|(J+\gamma)_{-}|_{\omega}} \frac{|R \cap (J+\gamma)_{-}|_{\omega}}{|(J+\gamma)_{-}|_{\omega}} > \delta, \tag{35}$$

so that we have

$$1 \geq \frac{|R \cap (J+\gamma_1)_{-}|_{\omega}}{|(J+\gamma_1)_{-}|_{\omega}} > \delta.$$

Then with notation analogous to the case $n = 1$ we have a similar calculation to that in (33):

$$\begin{aligned}
&\frac{1}{\Lambda} \left\{ \sum_{\gamma<\gamma_1} + \sum_{\gamma>\gamma_1:\ (J+\gamma)_{\lambda}^{\text{left}} \subset (J+\gamma_1)_{-}} \right\} \int_{(J+\gamma)_{\lambda}^{\text{left}}} \left| \triangle_{J+\gamma}^{\omega} \mathbf{1}_R \right|^2 d\omega \\
&\leq \frac{1}{\Lambda} \sum_{\gamma<\gamma_1} \frac{\left|(J+\gamma)_{\lambda}^{\text{left}}\right|_{\omega} |R \cap (J+\gamma)_{-}|_{\omega}}{|(J+\gamma)_{-}|_{\omega}^2} |R|_{\omega} + \frac{1}{\Lambda} \sum_{\gamma>\gamma_1:\ (J+\gamma)_{\lambda}^{\text{left}} \subset (J+\gamma_1)_{-}} \left|(J+\gamma)_{\lambda}^{\text{left}}\right|_{\omega} \\
&\leq \frac{1}{\Lambda} \delta \, \#\{\gamma < \gamma_1\} |R|_{\omega} + \frac{1}{\Lambda} |(J+\gamma_1)_{-}|_{\omega} \leq \delta |R|_{\omega} + \frac{1}{\Lambda} \frac{1}{\delta} |R \cap (J+\gamma_1)_{-}|_{\omega} \\
&\leq \left(\delta + \frac{\lambda}{\delta}\right) |R|_{\omega} = 2\sqrt{\lambda} |R|_{\omega} ,
\end{aligned}$$

if we choose $\delta = \sqrt{\lambda}$. Thus we have so far successfully estimated the sum over translations γ that satisfy either $\gamma < \gamma_1$ or $(J+\gamma)_{\lambda}^{\text{left}} \subset (J+\gamma_1)_{-}$.

Now we simply repeat the last step considering only the remaining horizontal translations. Since the side lengths of J and R are comparable, there are at most a fixed number of such steps left, and adding up the results, and using the 'average of an average' principle, then gives

$$\mathbb{E}_{\Omega}\int_{(J+\gamma)_{\lambda}^{\text{left}}}\left|\triangle_{J+\gamma}^{\omega}\mathbf{1}_{R}\right|^{2}d\omega\leq C_{\alpha}\sqrt{\lambda}.$$

This completes the proof of (29) in the case of dimension $n=2$, and as mentioned earlier, the above two-dimensional argument easily adapts to the case $n\geq 3$.

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Appendix

We assume notation as above. Define the bilinear form

$$\mathcal{B}(f,g)\equiv\langle T_{\sigma}^{\alpha}f,g\rangle_{\omega}\ ,\quad f\in L^{2}(\sigma),g\in L^{2}(\omega),$$

restricted to functions f and g of compact support and mean zero. For each dyadic grid $\mathcal{D}$ we then have

$$\mathcal{B}(f,g)=\sum_{I,J\in\mathcal{D}}\left\langle T_{\sigma}^{\alpha}\ \triangle_{I}^{\sigma}f,\triangle_{J}^{\omega}g\right\rangle_{\omega}\ .$$

Now define the bilinear forms

$$\mathcal{C}_{\mathcal{D}}(f,g)=\sum_{I,J\in\mathcal{D}:\ I\text{ and }J\text{ are }\mathbf{r}\text{-close}}\left\langle T_{\sigma}^{\alpha}\ \triangle_{I}^{\sigma}f,\triangle_{J}^{\omega}g\right\rangle_{\omega}\ ,\quad f\in L^{2}(\sigma),g\in L^{2}(\omega).$$

Thus the form $\mathcal{C}_{\mathcal{D}}(f,g)$ sums over those pairs of cubes in the grid $\mathcal{D}$ that are close in both scale and position, these being the only pairs where the need for a weak boundedness property traditionally arises. We also consider the subbilinear form

$$\mathcal{S}_{\mathcal{D}}(f,g)=\sum_{I,J\in\mathcal{D}:\ I\text{ and }J\text{ are }\mathbf{r}\text{-close}}\left|\left\langle T_{\sigma}^{\alpha}\ \triangle_{I}^{\sigma}f,\triangle_{J}^{\omega}g\right\rangle_{\omega}\right|\ ,\quad f\in L^{2}(\sigma),g\in L^{2}(\omega),$$

which dominates $\mathcal{C}_{\mathcal{D}}(f,g)$, i.e. $|\mathcal{C}_{\mathcal{D}}(f,g)|\leq\mathcal{S}_{\mathcal{D}}(f,g)$ for all $f\in L^{2}(\sigma),g\in L^{2}(\omega)$. The main results above can be organized into the following two part theorem.

Theorem 5 *With notation as above, we have:*

(1) *For f and g of compact support and mean zero,*

$$\begin{aligned}&\mathbb{E}_{\Omega}\left|\mathcal{B}(f,g)-\mathcal{C}_{\mathcal{D}}(f,g)\right| \\ &\leq C_{\alpha}\left(\sqrt{\mathfrak{A}_2^{\alpha}}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}_{T^{\alpha}}^{*}+\mathcal{E}_{\alpha}^{\mathrm{strong}}+\mathcal{E}_{\alpha}^{\mathrm{strong},*}+2^{-\varepsilon\mathbf{r}}\mathfrak{N}_{T^{\alpha}}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)} \\ &\qquad +C_{\alpha}\mathbb{E}_{\Omega}\mathcal{S}_{\mathcal{D}}(f,g)\,.\end{aligned}$$

(2) *For f and g of compact support and mean zero, and for* $0<\lambda<\frac{1}{2}$,

$$\mathbb{E}_{\Omega}\mathcal{S}_{\mathcal{D}}(f,g)\leq C_{\alpha}\left(\frac{1}{\lambda}\sqrt{\mathfrak{A}_2^{\alpha}}+\mathfrak{T}_{T^{\alpha}}+\mathfrak{T}_{T^{\alpha}}^{*}+\sqrt[4]{\lambda}\mathfrak{N}_{T^{\alpha}}\right)\|f\|_{L^2(\sigma)}\|g\|_{L^2(\omega)}\,.$$

The reason for emphasizing the two estimates in this way, is that a different proof strategy might produce a *different* bound for $\mathbb{E}_{\Omega}\left|\mathcal{B}(f,g)-\mathcal{C}_{\mathcal{D}}(f,g)\right|$, which can then be combined with the bound for $\mathbb{E}_{\Omega}\mathcal{S}_{\mathcal{D}}(f,g)$ to control $\left|\mathcal{B}(f,g)\right|$. Note also that the term $C_{\alpha}\mathbb{E}_{\Omega}\mathcal{S}_{\mathcal{D}}(f,g)$ is included in part (1) of the theorem, to allow for some of the inner products in the definition of $\mathcal{C}_{\mathcal{D}}(f,g)$ to be added back into the form $\mathcal{B}(f,g)-\mathcal{C}_{\mathcal{D}}(f,g)$ during the course of the proof of estimate (1). Indeed, this was done when controlling the form $\mathsf{T}^2_{\text{far below}}(f,g)$ above.

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Intrinsic Difference Quotients

Raul Paolo Serapioni

Abstract An alternative characterizations of intrinsic Lipschitz functions within Carnot groups through the boundedness of appropriately defined difference quotients is provided. It is also shown how intrinsic difference quotients along horizontal directions are naturally related with the intrinsic derivatives, introduced e.g. in Franchi et al. (Comm Anal Geom 11(5):909–944, 2003) and Ambrosio et al. (J Geom Anal 16:187–232, 2006) and used to characterize intrinsic real valued C^1 functions inside Heisenberg groups. Finally the question of the equivalence of the two conditions: (1) boundedness of horizontal intrinsic difference quotients and (2) intrinsic Lipschitz continuity is addressed in a few cases.

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1 Introduction

The aim of this paper is to contribute to the theory of *intrinsic Lipschitz functions* within Carnot groups.

We provide here an alternative characterizations of intrinsic Lipschitz functions through the boundedness of appropriately defined difference quotients. We show also how intrinsic difference quotients are strictly related with the intrinsic derivatives, introduced in [3, 15] and used by Serra Cassano et al. to characterize intrinsic real valued C^1 functions inside Heisenberg groups. Finally in the last section we attach the related question when the boundedness of only horizontal intrinsic difference quotients yields intrinsic Lipschitz continuity.

For a first description of Carnot groups we refer to the beginning of next section and to the literature there indicated. We anticipate here that we identify a Carnot group $\mathbb{G}$ with $\mathbb{R}^n$ endowed with a non commutative polynomial group

R.P. Serapioni (✉)
Dipartimento di Matematica, Università di Trento, Via Sommarive 14, Povo, 38123 Trento, Italy
e-mail: serapion@science.unitn.it

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operation denoted as $\cdot$ (see (1) and (2)). Moreover (non commutative) Carnot groups, endowed with their natural Carnot-Carathéodory distance (see Definition 2.1) are not Riemannian manifolds being also non Riemannian at any scale [26].

In the last years, there has been a general attempt aimed to carry on geometric analysis in non-Euclidean structures, and, in particular, to develop a good notion of rectifiable sets in sub-Riemannian metric structures and, specifically, in Carnot groups. For different notions of rectifiability in these general settings see [1, 2, 4, 14, 18, 23–25] and the references therein.

We recall that in Euclidean spaces, rectifiable sets are obtained, up to a negligible subset, by 'gluing up' countable families of graphs of C^1 or of Lipschitz functions. Hence, understanding the objects that, within Carnot groups, naturally take the role of C^1 or of Lipschitz functions seems to be preliminary in order to develop a satisfactory theory of *intrinsic* rectifiable sets. It has been clear for a long time that considering Euclidean notions, even in the simplest Carnot groups i.e. the Heisenberg groups, may be both too general and too restrictive (see [22] for a striking example). More intrinsic definitions are necessary.

Observe that, the adjective "intrinsic" is meant to emphasize the role played by the algebra of the group, in particular by its horizontal layer, and by group translations and dilations. In other words, "intrinsic" notions or properties in $\mathbb{G}$ are those depending only on the structure of its Lie algebra $\mathfrak{g}$. In particular, an intrinsic geometric property, such as e.g. being an intrinsic graph, or an intrinsic regular graph, or an intrinsic Lipschitz graph, must be invariant under group translations and group dilations. By this we mean that, after a translation or a dilation, they keep being graphs or regular graphs or Lipschitz graphs.

The notion of graph *within* Carnot groups is somehow more delicate than in Euclidean spaces, since Carnot groups in general are not cartesian products of subgroups (unlike Euclidean spaces). A notion of *intrinsic graph* fitting the structure of the group $\mathbb{G}$ is needed.

An intrinsic graph inside $\mathbb{G}$ is associated with a decomposition of the ambient group $\mathbb{G}$ as a product $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ of two homogeneous *complementary subgroups* $\mathbb{M}$, $\mathbb{H}$ (Definition 2.2) and the idea is the following one: let $\mathbb{M}$, $\mathbb{H}$ be complementary homogeneous subgroups of a group $\mathbb{G}$, then the intrinsic (left) graph of $f : \mathcal{A} \subset \mathbb{M} \to \mathbb{H}$ is the set

$$\text{graph}\,(f) = \{g \cdot f(g) : g \in \mathcal{A}\}.$$

Intrinsic graphs appeared naturally in [5, 17, 19] in relation with the study of non critical level sets of differentiable functions from $\mathbb{G}$ to $\mathbb{R}^k$. Indeed, implicit function theorems for groups [14, 15, 18] can be rephrased stating precisely that non critical level sets are always, locally, intrinsic graphs.

What are then appropriate *intrinsic* notions of Lipschitz functions or of differentiable functions when dealing with functions acting between complementary subgroups?

Both these notions were originally given in a somewhat indirect way as intrinsic geometric properties of the graphs of the functions in question. Precisely, a function acting between complementary subgroups is an intrinsic Lipschitz function when its graph does not intersect appropriately defined cones (see Definitions 3.2 and 3.3). Analogously, a function is an intrinsically differentiable function when its graph admits an appropriately defined tangent homogeneous subgroup at each point (see [5, 20, 27]).

Both these notions, are invariant under group translations and dilations, hence they are intrinsic and seem to be the right ones to be considered inside groups (see e.g [24]).

On the other hand, in the Euclidean setting, the most common and elementary definition of Lipschitz function is through the boundedness of the difference quotients of the function itself and the natural definition of a differentiable function is through existence and continuity of its partial derivatives.

As anticipated before, we introduce here an analogous definition of *intrinsic difference quotients* (see Definition 3.7). These intrinsic difference quotients, though their form may be algebraically complicated, can be explicitly computed given the group $\mathbb{G}$ and the couple of complementary subgroups $\mathbb{M}$ and $\mathbb{H}$. Moreover it is easy to characterize intrinsic Lipschitz functions as intrinsic functions with bounded intrinsic difference quotients (see Proposition 3.11).

The problem of characterizing intrinsic differentiable or intrinsic C^1 graphs in terms of intrinsic differentiability properties of their underlying functions, is definitely much more complicated. The available results are up to now limited to the case of hypersurfaces inside Heisenberg groups that is to the case of 'real valued' functions inside Heisenberg groups. By this we mean precisely that $\mathbb{G}$ is an Heisenberg group and that the target space $\mathbb{H}$, in the decomposition $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$, is 1-dimensional and horizontal.

Moreover the actual form of the intrinsic derivatives (in many significant cases they are first order non linear differential operators) was obtained in the above mentioned cases, in a rather indirect way through the use of Dini theorem. We observe here as, in perfect analogy with Euclidean calculus, intrinsic derivatives of functions acting between complementary subgroups of $\mathbb{G}$ can be obtained as *limits of intrinsic difference quotients along horizontal directions* (when these limits exist). So we provide an explicit way of computing the form of intrinsic derivatives, given the group $\mathbb{G}$ and the couple of complementary subgroups $\mathbb{M}$ and $\mathbb{H}$.

Finally we observe that it is not clear when informations on boundedness or continuity of intrinsic derivatives of $f : \mathbb{M} \to \mathbb{H}$ are sufficient to yield that the graph of f is intrinsic Lipschitz or intrinsic differentiable in $\mathbb{G}$. Related to this is the fact that in many significant instances the homogeneous subgroup $\mathbb{M}$, though a stratified group, is not a Carnot group. The validity of an intrinsic Lipschitz continuity result, such as in Theorem 3.21, that does not have up to now a corresponding result in term of continuity or boundedness of intrinsic derivatives, might suggest that also in this case such a result might hold true.

Finally it is a pleasure to thank for their interest in this work and for many pleasant and useful conversations Bruno Franchi, Francesco Serra Cassano and Sebastiano Nicolussi Golo.

2 Notations and Definitions

2.1 Carnot Groups

We recall here only the notions about Carnot groups that will be used in the following of this paper. For general accounts on Carnot groups, see e.g. [10–12, 21].

A *graded group* of step κ is a connected, simply connected Lie group $\mathbb{G}$ whose finite dimensional Lie algebra $\mathfrak{g}$ is the direct sum of k subspaces $\mathfrak{g}_i$, $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_\kappa$, such that

$$\left[\mathfrak{g}_i, \mathfrak{g}_j\right] \subset \mathfrak{g}_{i+j}, \quad \text{for } 1 \le i, j \le \kappa,$$

where $\mathfrak{g}_i = 0$ for $i > \kappa$. We denote as n the dimension of $\mathfrak{g}$ and as n_j the dimension of $\mathfrak{g}_j$, for $1 \le j \le \kappa$.

A *Carnot group* $\mathbb{G}$ of step κ is a graded group of step κ, where $\mathfrak{g}_1$ generates all of $\mathfrak{g}$. That is $[\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}$, for $i = 1, \dots, \kappa$.

Let $X_1, \dots, X_n$ be a base for $\mathfrak{g}$ such that $X_1, \dots, X_{m_1}$ is a base for $\mathfrak{g}_1$ and, for $1 < j \le \kappa$, $X_{m_{j-1}+1}, \dots, X_{m_j}$ is a base for $\mathfrak{g}_j$. Here we have $m_0 = 0$ and $m_j - m_{j-1} = n_j$, for $1 \le j \le \kappa$.

Because the exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a one to one diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$, any $p \in \mathbb{G}$ can be written, in a unique way, as $p = \exp(p_1X_1 + \cdots + p_nX_n)$ and we identify p with the n-tuple $(p_1, \dots, p_n) \in \mathbb{R}^n$ and $\mathbb{G}$ with $(\mathbb{R}^n, \cdot)$, i.e. $\mathbb{R}^n$ endowed with the product $\cdot$. The identity of $\mathbb{G}$ is denoted as $0 = (0, \dots, 0)$.

If $\mathbb{G}$ is a graded group, for all $\lambda > 0$, the *(non isotropic) dilations* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ are automorphisms of $\mathbb{G}$ defined as

$$\delta_\lambda(p_1, \dots, p_n) = (\lambda^{\alpha_1} p_1, \lambda^{\alpha_2} p_2, \dots, \lambda^{\alpha_n} p_n),$$

where $\alpha_i = j$, if $m_{j-1} < i \le m_j$. We denote the product of p and $q \in \mathbb{G}$ as $p \cdot q$ (or sometimes as pq). The explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula. It has the form

$$p \cdot q = p + q + \mathcal{Q}(p, q), \qquad \text{for all } p, q \in \mathbb{R}^n, \tag{1}$$

where $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_n) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. Each $\mathcal{Q}_i$ is a homogeneous polynomial of degree α_i with respect to the intrinsic dilations of $\mathbb{G}$. That is

$$\mathcal{Q}_i(\delta_\lambda p, \delta_\lambda q) = \lambda^{\alpha_i} \mathcal{Q}_i(p, q), \quad \text{for all } p, q \in \mathbb{G} \text{ and } \lambda > 0. \tag{2}$$

We collect now further properties of $\mathcal{Q}$ following from Campbell-Hausdorff formula. First of all $\mathcal{Q}$ is antisymmetric, that is

$$\mathcal{Q}_i(p, q) = -\mathcal{Q}_i(-q, -p), \qquad \text{for all } p, q \in \mathbb{G}.$$

Each $\mathcal{Q}_i(p,q)$ depends only on a section of the components of p and q. Precisely

$$\begin{aligned} &\mathcal{Q}_1(p,q) = \ldots = \mathcal{Q}_{m_1}(p,q) = 0 \\ &\mathcal{Q}_j(p,q) = \mathcal{Q}_j(p_1,\ldots,p_{m_{i-1}},q_1,\ldots,q_{m_{i-1}}), \end{aligned} \tag{3}$$

if $m_{i-1} < j \le m_i$ and $2 \le i$. By Proposition 2.2.22 (4) in [10], for $m_1 < i \le n$ we can write

$$\mathcal{Q}_i(p,q) = \sum_{k,h} \mathcal{R}^i_{k,h}(p,q)(p_k q_h - p_h q_k), \tag{4}$$

where the functions $\mathcal{R}^i_{k,h}$ are polynomials, homogenous of degree $\alpha_i - \alpha_k - \alpha_h$ with respect to group dilations, and the sum is extended to all h,k such that $\alpha_h + \alpha_k \le \alpha_i$. From (4) it follows in particular that

$$\mathcal{Q}_i(p,0) = \mathcal{Q}_i(0,q) = 0 \quad \text{and} \quad \mathcal{Q}_i(p,p) = \mathcal{Q}_i(p,-p) = 0. \tag{5}$$

Finally, it is useful to think $\mathbb{G} = \mathbb{G}^1 \oplus \mathbb{G}^2 \oplus \cdots \oplus \mathbb{G}^\kappa$, where $\mathbb{G}^i = \exp(\mathfrak{g}_i) = \mathbb{R}^{n_i}$ is the ith layer of $\mathbb{G}$ and to write $p \in \mathbb{G}$ as $(p^1,\ldots,p^\kappa)$, with $p^i \in \mathbb{G}^i$. $\mathbb{G}^1$ is denoted as the *horizontal layer* of $\mathbb{G}$.

Accordingly we also denote $\mathcal{Q} = (\mathcal{Q}^1,\ldots,\mathcal{Q}^\kappa)$ where $\mathcal{Q}^1 \equiv 0$ and for $2 \le i \le \kappa$ each $\mathcal{Q}^i$ is a vector valued polynomial homogeneous of degree i with respect to the intrinsic dilations of $\mathbb{G}$. With this notation (1) becomes

$$p \cdot q = \left(p^1 + q^1, p^2 + q^2 + \mathcal{Q}^2(p,q), \ldots, p^\kappa + q^\kappa + \mathcal{Q}^\kappa(p,q)\right), \qquad \text{for all } p,q \in \mathbb{G}. \tag{6}$$

An homogeneous norm in $\mathbb{G}$ is a function $\|\cdot\| : \mathbb{G} \to \mathbb{R}^+$ such that for all $p,q \in \mathbb{G}$ and for all $\lambda \ge 0$

$$\|p \cdot q\| \le \|p\| + \|q\|, \qquad \|\delta_\lambda p\| = \lambda \|p\|.$$

Homogeneous norms exist. A convenient one (see [16, Theorem 5.1]) is

$$\|p\| := \max_{j=1,\ldots,\kappa} \{\varepsilon_j \left\|p^j\right\|^{1/j}_{\mathbb{R}^{n_j}}\}, \qquad \text{for all } p = (p^1,\ldots,p^\kappa) \in \mathbb{G}, \tag{7}$$

where $\varepsilon_1 = 1$, and $\varepsilon_2, \ldots \varepsilon_\kappa \in (0,1]$ are suitable positive constants depending on $\mathbb{G}$.

Definition 2.1 An absolutely continuous curve $\gamma : [0,T] \to \mathbb{G}$ is a *sub-unit curve* if there exist measurable real functions $c_1(s),\ldots,c_{m_1}(s)$, $s \in [0,T]$ such that $\sum_j c_j^2 \le 1$ and

$$\dot\gamma(s) = \sum_{j=1}^{m_1} c_j(s) X_j(\gamma(s)), \qquad \text{for a.e. } s \in [0,T].$$

If $p, q \in \mathbb{G}$, we define their *Carnot-Carathéodory distance* as

$$d_c(p,q) := \inf\{T > 0 : \text{there exists a sub-unit curve } \gamma \text{ with } \gamma(0) = p,\ \gamma(T) = q\}\,.$$

By Chow's Theorem, the set of sub-unit curves joining p and q is not empty, furthermore d_c is a distance on $\mathbb{G}$ that induces the Euclidean topology (see Chap. 19 in [10]).

More generally, given any homogeneous norm $\|\cdot\|$, it is possible to define a distance in $\mathbb{G}$ as

$$d(p,q) = d(q^{-1} \cdot p, 0) = \left\| q^{-1} \cdot p \right\|, \qquad \text{for all } p, q \in \mathbb{G}. \tag{8}$$

The distance d in (8) is comparable with the Carnot-Carathéodory distance of $\mathbb{G}$ and

$$d(g \cdot p, g \cdot q) = d(p,q) \quad , \quad d(\delta_\lambda(p), \delta_\lambda(q)) = \lambda d(p,q) \tag{9}$$

for all $p, q, g \in \mathbb{G}$ and all $\lambda > 0$.

2.2 Complementary Subgroups and Graphs

From now on $\mathbb{G}$ will always be a Carnot group, identified with $\mathbb{R}^n$ through exponential coordinates.

Definition 2.2 A *homogeneous subgroup* of $\mathbb{G}$ (see [28, 5.2.4]) is a Lie subgroup $\mathbb{H}$ such that $\delta_\lambda g \in \mathbb{H}$, for all $g \in \mathbb{H}$ and for all $\lambda \geq 0$. Homogeneous subgroups are linear subspaces of $\mathbb{G} \equiv \mathbb{R}^n$.

Two homogeneous subgroups $\mathbb{M}, \mathbb{H}$ of $\mathbb{G}$ are *complementary subgroups* in $\mathbb{G}$, if $\mathbb{M} \cap \mathbb{H} = \{0\}$ and if for all $g \in \mathbb{G}$, there are $m \in \mathbb{M}$ and $h \in \mathbb{H}$ such that $g = m \cdot h$. If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$ we say that $\mathbb{G}$ is the *product of* $\mathbb{M}$ *and* $\mathbb{H}$ and we denote this as

$$\mathbb{G} = \mathbb{M} \cdot \mathbb{H}.$$

If $\mathbb{M}, \mathbb{H}$ are complementary subgroups of $\mathbb{G} = (\mathbb{R}^n, \cdot)$ then they are also complementary linear subspaces of $\mathbb{R}^n$ and we denote this as $\mathbb{G} = \mathbb{M} \oplus \mathbb{H}$. If one of them is a normal subgroup then $\mathbb{G}$ is said to be the *semi-direct product* of $\mathbb{M}$ and $\mathbb{H}$. If both $\mathbb{M}$ and $\mathbb{H}$ are normal subgroups then $\mathbb{G}$ is said to be the *direct product* of $\mathbb{M}$ and $\mathbb{H}$.

Remark 2.3 If $\mathbb{M}$ is an homogeneous subgroup of $\mathbb{G}$ then also $\mathbb{M}$ is a stratified group, but it is not necessarily a Carnot group. If $\mathbb{M}, \mathbb{H}$ are complementary subgroups of $\mathbb{G}$ then $\mathbb{G}^i = \mathbb{M}^i \oplus \mathbb{H}^i$, for $i = 1, \dots, \kappa$.

Example 2.4 Complementary subgroups always exist in any Carnot group $\mathbb{G}$. Indeed, choose any horizontal homogeneous subgroup $\mathbb{H} = \mathbb{H}^1 \subset \mathbb{G}^1$ and a subgroup $\mathbb{M} = \mathbb{M}^1 \oplus \cdots \oplus \mathbb{M}^\kappa$ such that: $\mathbb{H} \oplus \mathbb{M}^1 = \mathbb{G}^1$, and $\mathbb{G}^j = \mathbb{M}^j$ for all $2 \le j \le \kappa$. Then it is easy to check that $\mathbb{M}$ and $\mathbb{H}$ are complementary subgroups in $\mathbb{G}$ and that the product $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ is semidirect because $\mathbb{M}$ is a normal subgroup.

Given two complementary subgroups $\mathbb{M}, \mathbb{H}$ of $\mathbb{G}$, then for any $g \in \mathbb{G}$ the elements $m \in \mathbb{M}$ and $h \in \mathbb{H}$ such that $g = mh$ are unique because $\mathbb{M} \cap \mathbb{H} = \{0\}$. These elements are denoted as *components* of g along $\mathbb{M}$ and $\mathbb{H}$ or as *projections* of g on $\mathbb{M}$ and $\mathbb{H}$.

Proposition 2.5 *If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$ there is $c_0 = c_0(\mathbb{M}, \mathbb{H}) > 0$ such that for all $g = mh$*

$$c_0 \left(\|m\| + \|h\|\right) \le \|g\| \le \|m\| + \|h\| \,. \tag{10}$$

From now on, we will keep the following convention: when $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, $\mathbb{M}$ will always be the first 'factor' and $\mathbb{H}$ the second one, hence $g_\mathbb{M} \in \mathbb{M}$ and $g_\mathbb{H} \in \mathbb{H}$ are the unique elements such that

$$g = g_\mathbb{M} g_\mathbb{H}. \tag{11}$$

We stress that this notation is ambiguous because $g_\mathbb{M}$ and $g_\mathbb{H}$ depend on both the complementary subgroups $\mathbb{M}$ and $\mathbb{H}$ and also on the order under which they are taken.

The projection maps $\mathbf{P}_\mathbb{M} : \mathbb{G} \to \mathbb{M}$ and $\mathbf{P}_\mathbb{H} : \mathbb{G} \to \mathbb{H}$ are defined as

$$\mathbf{P}_\mathbb{M}(g) := g_\mathbb{M}, \qquad \mathbf{P}_\mathbb{H}(g) := g_\mathbb{H} \tag{12}$$

Proposition 2.6 *Let $\mathbb{M}$, $\mathbb{H}$ be complementary subgroups of $\mathbb{G}$, then the projection maps $\mathbf{P}_\mathbb{M} : \mathbb{G} \to \mathbb{M}$ and $\mathbf{P}_\mathbb{H} : \mathbb{G} \to \mathbb{H}$ defined in* (12) *are polynomial maps. More precisely, if κ is the step of $\mathbb{G}$, there are 2κ matrices $A^1, \dots, A^\kappa, B^1, \dots, B^\kappa$, depending on $\mathbb{M}$ and $\mathbb{H}$, such that*

(i) *A^j and B^j are (n_j, n_j)-matrices, for all $1 \le j \le \kappa$,*

and, with the notations of (1)*,*

(ii) $\mathbf{P}_\mathbb{M} g = \big(A^1 g^1, A^2(g^2 - \mathcal{Q}^2(A^1 g^1, B^1 g^1)), \dots, A^\kappa(g^\kappa - \mathcal{Q}^\kappa(A^1 g^1, \dots, B^{\kappa-1} g^{\kappa-1}))\big)$;
(iii) $\mathbf{P}_\mathbb{H} g = \big(B^1 g^1, B^2(g^2 - \mathcal{Q}^2(A^1 g^1, B^1 g^1)), \dots, B^\kappa(g^\kappa - \mathcal{Q}^\kappa(A^1 g^1, \dots, B^{\kappa-1} g^{\kappa-1}))\big)$;
(iv) *A^j is the identity on $\mathbb{M}^j$, and B^j is the identity on $\mathbb{H}^j$, for $1 \le j \le \kappa$.*

Recall that n_j is the dimension of the layer $\mathfrak{g}_j$.

Definition 2.7 Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$. We say that a set $S \subset \mathbb{G}$ is a *(left)* $\mathbb{H}$-*graph* (or a *left graph in direction* $\mathbb{H}$) if S intersects each left coset of $\mathbb{H}$ in one point, at most.

If $\mathcal{A} \subset \mathbb{G}$ parametrizes the left cosets of $\mathbb{H}$—in particular if $\mathcal{A}$ itself intersect each left coset of $\mathbb{H}$ at most one time—and if S is an $\mathbb{H}$-graph, then there is a unique function $f : \mathcal{E} \subset \mathcal{A} \to \mathbb{H}$ such that S *is the graph of* f, that is

$$S = \text{graph}\,(f) := \{\xi \cdot f(\xi) : \xi \in \mathcal{E}\}.$$

Conversely, for any $\psi : \mathcal{D} \subset \mathcal{A} \to \mathbb{H}$ the set graph (ψ) is an $\mathbb{H}$-graph.

One has an important special case when $\mathbb{H}$ admits a complementary subgroup $\mathbb{M}$. Indeed, in this case, $\mathbb{M}$ naturally parametrizes the left cosets of $\mathbb{H}$ and we have that

$$S \text{ is a } \mathbb{H}\text{-graph if and only if } S = \text{graph}\,(f)$$

for $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$. By uniqueness of the components along $\mathbb{M}$ and $\mathbb{H}$, if $S =$ graph (f) then f is uniquely determined among all functions from $\mathbb{M}$ to $\mathbb{H}$.

If a set $S \subset \mathbb{G}$ is an intrinsic graph then it keeps being an intrinsic graph after left translations or group dilations.

Proposition 2.8 *Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$. If S is a $\mathbb{H}$-graph then, for all $\lambda > 0$ and for all $q \in \mathbb{G}$, $\delta_\lambda S$ and $q \cdot S$ are $\mathbb{H}$-graphs.*

If, in particular, $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, if $S =$ graph (f) with $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$, then

$$\begin{aligned} &\textit{For all } \lambda > 0, \delta_\lambda S = \text{graph}\,(f_\lambda), \textit{with} \\ &f_\lambda : \delta_\lambda \mathcal{E} \subset \mathbb{M} \to \mathbb{H} \textit{ and} \\ &f_\lambda(m) = \delta_\lambda f(\delta_{1/\lambda} m), \textit{ for } m \in \delta_\lambda \mathcal{E}. \end{aligned} \tag{13}$$

$$\begin{aligned} &\textit{For any } q \in \mathbb{G}, \quad q \cdot S = \text{graph}\,(f_q), \textit{ where} \\ &f_q : \mathcal{E}_q \subset \mathbb{M} \to \mathbb{H}, \quad \mathcal{E}_q = \{m : \mathbf{P}_\mathbb{M}(q^{-1} \cdot m) \in \mathcal{E}\} \textit{ and} \\ &f_q(m) = \big(\mathbf{P}_\mathbb{H}(q^{-1} \cdot m)\big)^{-1} \cdot f\big(\mathbf{P}_\mathbb{M}(q^{-1} \cdot m)\big), \textit{ for all } m \in \mathcal{E}_q. \end{aligned} \tag{14}$$

Remark 2.9 The algebraic expression of f_q in (14) is more explicit when $\mathbb{G}$ is a semi-direct product of $\mathbb{M}, \mathbb{H}$. Precisely

(i) If $\mathbb{M}$ is normal in $\mathbb{G}$ then $f_q(m) = q_\mathbb{H} f\big((q^{-1}m)_\mathbb{M}\big)$, for $m \in \mathcal{E}_q = q\mathcal{E}(q_\mathbb{H})^{-1}$.
(ii) If $\mathbb{H}$ is normal in $\mathbb{G}$ then $f_q(m) = (q^{-1}m)_\mathbb{H}^{-1} f(q_\mathbb{M}^{-1} m)$, for $m \in \mathcal{E}_q = q_\mathbb{M}\mathcal{E}$.

If both $\mathbb{M}$ and $\mathbb{H}$ are normal in $\mathbb{G}$—that is if $\mathbb{G}$ is a direct product of $\mathbb{M}$ and $\mathbb{H}$—then we get the well known Euclidean formula

(iii) $f_q(m) = q_{\mathbb{H}} f(q_{\mathbb{M}}^{-1} m), \quad$ for $m \in \mathcal{E}_q = q_{\mathbb{M}}\mathcal{E}$.

See also [5, Proposition 3.6].

3 Intrinsic Lipschitz Functions

3.1 General Definitions

As anticipated in the introduction, *intrinsic Lipschitz functions* in $\mathbb{G}$ are functions, acting between complementary subgroups of $\mathbb{G}$, with graphs non intersecting naturally defined cones. Hence, the notion of *intrinsic Lipschitz graph* respects strictly the geometry of the ambient group $\mathbb{G}$. Intrinsic Lipschitz functions appeared for the first time in [14] and were studied, more diffusely, in [13, 18, 19, 30].

We begin with two definitions of intrinsic (closed) cones.

Definition 3.1 Let $\mathbb{H}$ be a homogeneous subgroup of $\mathbb{G}$, $q \in \mathbb{G}$. Then, the *cones* $X(q, \mathbb{H}, \alpha)$ with *axis* $\mathbb{H}$*, vertex* q*, opening* α, $0 \le \alpha \le 1$ are defined as

$$X(q, \mathbb{H}, \alpha) = q \cdot X(0, \mathbb{H}, \alpha), \text{ where } X(0, \mathbb{H}, \alpha) = \left\{ p : \operatorname{dist}(p, \mathbb{H}) \le \alpha \, \|p\| \right\}.$$

Notice that Definition 3.1 does not require that $\mathbb{H}$ is a complemented subgroup.

Frequently, while working with functions acting between complementary subgroups, it will be convenient to consider also the following family of cones.

Definition 3.2 If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, $q \in \mathbb{G}$ and $\beta \ge 0$, the cones $C_{\mathbb{M},\mathbb{H}}(q, \beta)$, with base $\mathbb{M}$, axis $\mathbb{H}$, vertex q, opening β are defined as

$$C_{\mathbb{M},\mathbb{H}}(q, \beta) = q \cdot C_{\mathbb{M},\mathbb{H}}(0, \beta), \text{ where } C_{\mathbb{M},\mathbb{H}}(0, \beta) = \{ p : \|p_{\mathbb{M}}\| \le \beta \, \|p_{\mathbb{H}}\| \} .$$

Observe that

$$\mathbb{H} = X(0, \mathbb{H}, 0) = C_{\mathbb{M},\mathbb{H}}(0, 0), \qquad \mathbb{G} = X(0, \mathbb{H}, 1) = \overline{\cup_{\beta > 0} C_{\mathbb{M},\mathbb{H}}(0, \beta)}.$$

Moreover, the cones $C_{\mathbb{M},\mathbb{H}}(q, \beta)$ are equivalent with the cones $X(q, \mathbb{H}, \alpha)$ that is: for any $\alpha \in (0, 1)$ there is $\beta \ge 1$, depending on α, $\mathbb{M}$ and $\mathbb{H}$, such that

$$C_{\mathbb{M},\mathbb{H}}(q, 1/\beta) \subset X(q, \mathbb{H}, \alpha) \subset C_{\mathbb{M},\mathbb{H}}(q, \beta), \tag{15}$$

Now we introduce the basic definition of this paragraph.

Definition 3.3

(i) Let $\mathbb{H}$ be an homogeneous subgroup, not necessarily complemented in $\mathbb{G}$. We say that an $\mathbb{H}$-graph S is an *intrinsic Lipschitz* $\mathbb{H}$*-graph* if there is $\alpha \in (0,1)$ such that,

$$S \cap X(p, \mathbb{H}, \alpha) = \{p\}, \qquad \text{for all } p \in S.$$

(ii) If $\mathbb{M}, \mathbb{H}$ are complementary subgroups in $\mathbb{G}$, we say that $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is *intrinsic Lipschitz* in $\mathcal{E}$ when graph (f) is an *intrinsic Lipschitz* $\mathbb{H}$*-graph*.

(iii) We say that $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is *intrinsic L-Lipschitz* in $\mathcal{E}$ if there is $L > 0$ such that

$$C_{\mathbb{M},\mathbb{H}}(p, 1/L) \cap \operatorname{graph}(f) = \{p\}, \quad \text{for all } p \in \operatorname{graph}(f). \tag{16}$$

The Lipschitz constant of f in $\mathcal{E}$ is the infimum of the $L > 0$ such that (16) holds.

It follows immediately from (15) that f is intrinsic Lipschitz in $\mathcal{E}$ if and only if it is intrinsic L-Lipschitz for an appropriate constant L, depending on α, f and $\mathbb{M}$.

Because of Proposition 2.8 and Definition 3.2 left translations of intrinsic Lipschitz $\mathbb{H}$-graphs, or of intrinsic L-Lipschitz functions, are intrinsic Lipschitz $\mathbb{H}$-graphs, or intrinsic L-Lipschitz functions. We state these facts in the following theorem.

Theorem 3.4 *If $\mathbb{G}$ is a Carnot group, then for all $q \in \mathbb{G}$,*

(i) *$S \subset \mathbb{G}$ is an intrinsic Lipschitz $\mathbb{H}$-graph $\Longrightarrow$ $q{\cdot}S$ is an intrinsic Lipschitz $\mathbb{H}$-graph;*
(ii) *$f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ is intrinsic L-Lipschitz, $\Longrightarrow$ $f_q : \mathcal{E}_q \subset \mathbb{M} \to \mathbb{H}$ is intrinsic L-Lipschitz.*

The geometric definition of intrinsic Lipschitz graphs has equivalent algebraic forms (see also [5, 17, 19]).

Proposition 3.5 *Let $\mathbb{M}, \mathbb{H}$ be complementary subgroups in $\mathbb{G}$, $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ and $L > 0$. Then* (i) *to* (iii) *are equivalent.*

(i) *f is intrinsic L-Lipschitz in $\mathcal{E}$.*
(ii) $\left\|\mathbf{P}_{\mathbb{H}}(\bar{q}^{-1}q)\right\| \le L \left\|\mathbf{P}_{\mathbb{M}}(\bar{q}^{-1}q)\right\|, \qquad$ *for all* $q, \bar{q} \in \operatorname{graph}(f)$.
(iii) $\left\|f_{\bar{q}^{-1}}(m)\right\| \le L \|m\|, \qquad$ *for all* $\bar{q} \in \operatorname{graph}(f)$ *and* $m \in \mathcal{E}_{\bar{q}^{-1}}$.

Remark 3.6 If $\mathbb{G}$ is the semi-direct product of $\mathbb{M}$ and $\mathbb{H}$, (ii) of Proposition 3.5 takes a more explicit form. Indeed, from Remark 2.9, we get

(i) If $\mathbb{M}$ is normal in $\mathbb{G}$ then f is intrinsic L-Lipschitz if and only if

$$\left\| f(\bar{m})^{-1} f(m) \right\| \leq L \left\| f(\bar{m})^{-1} \bar{m}^{-1} m f(\bar{m}) \right\|, \qquad \text{for all } m, \bar{m} \in \mathcal{E}.$$

(ii) If $\mathbb{H}$ is normal in $\mathbb{G}$ then f is intrinsic L-Lipschitz if and only if

$$\left\| m^{-1} \bar{m} f(\bar{m})^{-1} \bar{m}^{-1} m f\big(m\big) \right\| \leq L \left\| \bar{m}^{-1} m \right\|, \qquad \text{for all } m, \bar{m} \in \mathcal{E}.$$

(iii) If $\mathbb{G}$ is a direct product of $\mathbb{M}$ and $\mathbb{H}$ we get the well known expression for Lipschitz functions

$$\left\| f(\bar{m})^{-1} f(m) \right\| \leq L \left\| \bar{m}^{-1} m \right\|, \qquad \text{for all } m, \bar{m} \in \mathcal{E}.$$

Hence in this case intrinsic Lipschitz functions are the same as the usual metric Lipschitz functions from $(\mathbb{M}, d_\infty)$ to $(\mathbb{H}, d_\infty)$.

3.2 Intrinsic Difference Quotients

A different new characterization of intrinsic Lipschitz functions can be given in terms of boundedness of appropriately defined *intrinsic difference quotients*. Let us begin with this notion. In the spirit of the previous paragraphs, first we propose the definition in the particular case of a function vanishing in the origin of the group and then we get the general definition extending the particular case in a translation invariant way.

Let $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$ and $Y \in \mathfrak{m}$. Assume $0 \in \mathcal{E}$ and $f(0) = 0$. In this case the *difference quotients* $\Delta_Y f(0; t)$ of f (from $0 \in \mathcal{E}$ in direction Y) are defined as

$$\Delta_Y f(0; t) := \delta_{1/t} f(\delta_t \exp Y)$$

for all $t > 0$ such that $\delta_t \exp Y \in \mathcal{E}$. Then we extend this definition to any $m \in \mathcal{E}$. Let $q := m \cdot f(m) \in \text{graph}\,(f)$, then $f_{q^{-1}}$ vanishes in $0 \in \mathcal{E}_{q^{-1}}$ and we define

$$\Delta_Y f(m; t) := \Delta_Y f_{q^{-1}}(0; t) = \delta_{1/t} f_{q^{-1}}(\delta_t \exp Y) \tag{17}$$

once more for all $t > 0$ such that $\delta_t \exp Y \in \mathcal{E}_{q^{-1}}$.

To make the previous definition less implicit, i.e. given directly on the function f and not on its translated $f_{q^{-1}}$, we consider the following steps making also more transparent the underlying geometry of the construction.

- Let $f : \mathbb{M} \to \mathbb{H}$. Fix $m \in \mathbb{M}$ and $Y \in \mathfrak{m}$. Then consider the line from $q_m := m \cdot f(m)$

$$s \mapsto q_m \cdot \delta_s \exp Y \qquad \text{for } 0 \leq s$$

and its projection on $\mathbb{M}$

$$s \mapsto \mathbf{P}_{\mathbb{M}}\left(q_m \cdot \delta_s \exp Y\right) \qquad \text{for } 0 \le s.$$

Let

$$\Delta_{Y,t} q_m := \mathbf{P}_{\mathbb{M}}\left(q_m \cdot \delta_t \exp Y\right) = m \cdot \mathbf{P}_{\mathbb{M}}\left(f(m) \cdot \delta_t \exp Y\right).$$

- Consider the projection on $\mathbb{H}$ of the difference between the two points on graph (f): $\Delta_{Y,t} q_m \cdot f\left(\Delta_{Y,t} q_m\right)$ and $q_m := m \cdot f(m)$,

$$\mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \Delta_{Y,t} q_m \cdot f\left(\Delta_{Y,t} q_m\right)\right) = \mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \Delta_{Y,t} q_m\right) \cdot f\left(\Delta_{Y,t} q_m\right)$$

- Finally the *intrinsic difference quotient* of f from m in direction Y is

$$\Delta_Y f(m;t) := \delta_{1/t}\left(\mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \Delta_{Y,t} q_m \cdot f\left(\Delta_{Y,t} q_m\right)\right)\right). \tag{18}$$

The previous definition of $\Delta_Y f(m;t)$ can be given a different expression.

$$\begin{aligned}
&\mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \Delta_{Y,t} q_m \cdot f\left(\Delta_{Y,t} q_m\right)\right)\\
&= \mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \mathbf{P}_{\mathbb{M}}\left(q_m \cdot \delta_t \exp Y\right) \cdot f\left(\Delta_{Y,t} q_m\right)\right)\\
&= \mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot \mathbf{P}_{\mathbb{M}}\left(q_m \cdot \delta_t \exp Y\right) \cdot \mathbf{P}_{\mathbb{H}}\left(q_m \cdot \delta_t \exp Y\right) \cdot \left(\mathbf{P}_{\mathbb{H}}\left(q_m \cdot \delta_t \exp Y\right)\right)^{-1} \cdot f\left(\Delta_{Y,t} q_m\right)\right)\\
&= \mathbf{P}_{\mathbb{H}}\left(q_m^{-1} \cdot q_m \cdot \delta_t \exp Y \cdot \left(\mathbf{P}_{\mathbb{H}}\left(q_m \cdot \delta_t \exp Y\right)\right)^{-1} \cdot f\left(\Delta_{Y,t} q_m\right)\right)\\
&= \left(\mathbf{P}_{\mathbb{H}}(q_m \cdot \delta_t \exp Y\right)^{-1} \cdot f\left(\Delta_{Y,t} q_m\right)\\
&= \left(\mathbf{P}_{\mathbb{H}}(f(m) \cdot \delta_t \exp Y\right)^{-1} \cdot f\left(m \cdot \mathbf{P}_{\mathbb{M}}\left(f(m) \cdot \delta_t \exp Y\right)\right)
\end{aligned}$$

Finally we propose the following definitions

Definition 3.7 Let $\mathbb{M}, \mathbb{H}$ be complementary subgroups in $\mathbb{G}$ and $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$. If $m \in \mathcal{E}$ and Y belongs to the Lie algebra $\mathfrak{m}$ of $\mathbb{M}$, then the *intrinsic difference quotients of f at m along Y*, are

$$\Delta_Y f(m;t) = \delta_{1/t}\Big(\left(\mathbf{P}_{\mathbb{H}}(f(m) \cdot \delta_t \exp Y)\right)^{-1} \cdot f\big(m \cdot \mathbf{P}_{\mathbb{M}}(f(m) \cdot \delta_t \exp Y)\big)\Big), \tag{19}$$

for all $t > 0$ such that $m \cdot \mathbf{P}_{\mathbb{M}}(f(m) \cdot \delta_t \exp Y) \in \mathcal{E}$.

Remark 3.8 Notice that formally the definition of difference quotient could be given also for $Y \in \mathfrak{h}$. This case is, as it should be, completely not interesting because the difference quotients are 0. Indeed with $Y \in \mathfrak{h}$ it follows $m \cdot \mathbf{P}_{\mathbb{M}}(f(m) \cdot \delta_t \exp Y) = m$

and using the definition in (18)

$$\begin{aligned}\Delta_Y f(m;t) &= \delta_{1/t}\left(\mathbf{P}_{\mathbb{H}}\left(q_m^{-1}\cdot\Delta_{Y,t}q_m\cdot f\left(\Delta_{Y,t}q_m\right)\right)\right)\\ &= \delta_{1/t}\left(\mathbf{P}_{\mathbb{H}}\left(f(m)^{-1}\cdot m^{-1}\cdot m\cdot\mathbf{P}_{\mathbb{M}}\left(f(m)\cdot\delta_t\exp Y\right)\cdot f(m\cdot\mathbf{P}_{\mathbb{M}}\left(f(m)\cdot\delta_t\exp Y\right))\right)\right)\\ &= \delta_{1/t}\left(\mathbf{P}_{\mathbb{H}}\left(f(m)^{-1}\cdot f(m)\right)\right)=0.\end{aligned}$$

Remark 3.9 Observe that Definition 3.7 gives the same notion of difference quotient as proposed in (17). Indeed, if $f(m) = 0$ then $\mathbf{P}_{\mathbb{H}}(f(m)\cdot\delta_t\exp Y) = 0$ and $m\cdot\mathbf{P}_{\mathbb{M}}(f(m)\cdot\delta_t\exp Y) = m\cdot\delta_t\exp Y$. Hence

$$f(m)=0 \implies \Delta_Y f(m;t)=\delta_{1/t}f(m\cdot\delta_t\exp Y) \tag{20}$$

and also, if $q = m\cdot f(m)$ then $f_{q^{-1}}(0) = 0$ we get (17)

$$\Delta_Y f(m;t) = \Delta_Y f_{q^{-1}}(0;t) = \delta_{1/t}\left(f_{q^{-1}}(\delta_t\exp Y)\right).$$

Remark 3.10 With the same notations of Definition 3.7 and recalling Remark 2.9, we get

(i) If $\mathbb{M}$ is normal in $\mathbb{G}$ and $Y\in\mathfrak{m}$ then

$$\mathbf{P}_{\mathbb{H}}(f(m)\cdot\delta_t\exp Y)=f(m)$$

and

$$\begin{aligned}m\cdot\mathbf{P}_{\mathbb{M}}(f(m)\delta_t(\exp Y)) &= m\cdot f(m)\cdot\delta_t\exp Y\cdot f(m)^{-1}\\ &= m\cdot\mathrm{Ad}_{f(m)}(\delta_t\exp Y).\end{aligned}$$

Hence if $\mathbb{M}$ is a normal subgroup and $Y\in\mathfrak{m}$

$$\Delta_Y f(m;t)=\delta_{1/t}\left(f(m)^{-1}\cdot f\left(m\cdot\mathrm{Ad}_{f(m)}(\delta_t\exp Y)\right)\right).$$

(ii) If $\mathbb{H}$ is normal in $\mathbb{G}$ then

$$\mathbf{P}_{\mathbb{H}}\left(f(m)\cdot\delta_t\exp Y\right)=(\delta_t\exp Y)^{-1}\cdot f(m)\cdot\delta_t\exp Y$$

and

$$\mathbf{P}_{\mathbb{M}}(f(m)\cdot\delta_t\exp Y)=\delta_t\exp Y.$$

Hence if $\mathbb{H}$ is a normal subgroup and $Y\in\mathfrak{m}$

$$\Delta_Y f(m;t)=\delta_{1/t}\left((\delta_t\exp Y)^{-1}\cdot f(m)^{-1}\cdot\delta_t\exp Y\cdot f(m\cdot\delta_t\exp Y)\right).$$

(iii) If both $\mathbb{M}$ and $\mathbb{H}$ are normal in $\mathbb{G}$ and $\mathbb{G}$ is a direct product of $\mathbb{M}$ and $\mathbb{H}$ then we get the well known expression for the difference quotient:

$$\Delta_Y f(m; t) = \delta_{1/t}\left(f(m)^{-1} \cdot f(m \cdot \delta_t \exp Y)\right).$$

Next Proposition gives a straightforward characterization of intrinsic Lipschitz functions in terms of the boundedness of their difference quotients.

Proposition 3.11 *Let* $\mathbb{M}, \mathbb{H}$ *be complementary subgroups in* $\mathbb{G}$ *and* $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$. *The following statements are equivalent*

(i) f *is intrinsic L-Lipschitz in* $\mathcal{E}$;
(ii) *there is* $L > 0$ *such that, for all* $Y \in \mathfrak{m}$ *and for all* $m \in \mathcal{E}$

$$\|\Delta_Y f(m; t)\| \le L \|\exp Y\|.$$

Proof If $q = mf(m) \in \operatorname{graph}(f)$ then by (17)

$$\|\Delta_Y f(m; t)\| = \left\|\Delta_Y f_{q^{-1}}(0; t)\right\| = \frac{1}{t}\left\|f_{q^{-1}}(\delta_t \exp Y)\right\|,$$

for all $t > 0$ and $Y \in \mathfrak{m}$.

$(i) \implies (ii)$. By (iii) of Proposition 3.5,

$$\|\Delta_Y f(m; t)\| = \frac{1}{t}\left\|f_{q^{-1}}(\delta_t \exp Y)\right\| \le \frac{L}{t}\|\delta_t \exp Y\| = L\|\exp Y\|,$$

for $t > 0$ and $Y \in \mathfrak{m}$. Hence (ii) holds.

$(ii) \implies (i)$. Let $\bar{m} \in \mathcal{E}$ and $\bar{q} := \bar{m}f(\bar{m})$. For any $m \in \mathcal{E}_{\bar{q}^{-1}}$ let $Y \in \mathfrak{m}$ be such that $m = \exp Y$. Then

$$\left\|f_{\bar{q}^{-1}}(m)\right\| = \left\|f_{\bar{q}^{-1}}(\exp Y)\right\| = \|\Delta_Y f(\bar{m}; 1)\| \le L\|\exp Y\| = L\|m\|.$$

Hence (iii) of Proposition 3.5 holds and f is intrinsic L-Lipschitz.

□

We conclude this section observing that the limits for $t \to 0^+$ of intrinsic different quotients, when these limits exist and are finite, give origin to a notion of *intrinsic derivative* for functions acting between complementary subgroups. We will show, in Examples 3.16 and 3.17, that these intrinsic derivatives are precisely the operators considered by Serra Cassano and coauthors to characterize intrinsic Lipschitz and intrinsic regular functions inside Heisenberg groups.

Definition 3.12 Let $\mathbb{M}, \mathbb{H}$ be complementary subgroups in $\mathbb{G}$, let $\mathfrak{m}$ be the Lie algebra of $\mathbb{M}$ and $f : \mathcal{E} \subset \mathbb{M} \to \mathbb{H}$. If $m \in \mathcal{E} \subset \mathbb{M}$, the *intrinsic directional* *derivative of f at m along $Y \in \mathfrak{m}$*, is

$$D_Y f(m) := \lim_{t \to 0^+} \Delta_Y f(m;t) = \lim_{t \to 0^+} \Delta_{-Y} f(m;t) \tag{21}$$

provided the two limits on the right exist and are equal.

Remark 3.13 This remark follows directly from (20). Indeed

$$f(m) = 0 \implies \Delta_Y f(m;t) = \delta_{1/t} f(m \cdot \delta_t \exp Y)$$

hence, if the limits in (21) exist,

$$f(m) = 0 \implies D_Y f(m) = Yf(m).$$

3.3 Examples of Difference Quotients and of Intrinsic Derivatives

Example 3.14 (Horizontal Valued Functions Inside Step 2 Groups) Let $\mathbb{G} = (\mathbb{R}^m, \cdot)$ be a step 2 group and denote $\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2$. Let $\{Z_1, \dots, Z_m\}$ be a base of $\mathfrak{g}$ with

$$\mathfrak{g}^1 = \text{span}\{Z_1, \dots, Z_{m_1}\}, \qquad \mathfrak{g}^2 = \text{span}\{Z_{m_1+1}, \dots, Z_m\}$$

With the notation in (6) we denote

$$q_{i,h} := \mathcal{Q}^2(\exp Z_i, \exp Z_h) \in \mathbb{R}^{m-m_1}, \qquad \text{for } 1 \le i, h \le m.$$

Notice that $q_{i,h} = -q_{h,i}$ and $q_{i,h} = 0$ if $i > m_1$.

We assume (see Example 2.4) that $\mathbb{G} = \mathbb{M} \cdot \mathbb{H}$ where $\mathbb{H}$ is a k-dimensional horizontal subgroup and $\mathbb{M}$ is a complementary normal subgroup. Moreover we choose the vectors Z_i are chosen such that

$$\mathbb{H} = \exp\left(\text{span}\{Z_1, \dots, Z_k\}\right), \qquad \mathbb{M} = \exp\left(\text{span}\{Z_{k+1}, \dots, Z_m\}\right).$$

Notice that we are assuming that $Z_1, \dots, Z_k$ are commuting vector fields.

Let $f : \mathbb{M} \to \mathbb{H}$ be defined as

$$f(p) := \exp\left(\sum_1^k \varphi_i(p) Z_i\right) = \sum_1^k \varphi_i(p) \exp Z_i \qquad \text{for all } p \in \mathbb{M}.$$

for all $p = (p^1, p^2) \in \mathbb{M}$, where $\varphi_i : \mathbb{M} \to \mathbb{R}$ for $1 \le i \le k$.

Fix an horizontal $Z_j \in \mathfrak{g}^1$, that is with $k+1 \le j \le m_1$. Using (i) of Remark 3.10, we compute

$$\Delta_{Z_j} f(p;t) = \delta_{1/t}\left(f(p)^{-1} \cdot f\left(p \cdot f(p) \cdot \delta_t \exp Z_j \cdot f(p)^{-1}\right)\right)$$
$$= \sum_{i=1}^{k} \frac{1}{t}\left(\varphi_i\left(p \cdot f(p) \cdot \delta_t \exp Z_j \cdot f(p)^{-1}\right) - \varphi_i(p)\right) \exp Z_i$$

Notice that

$$\begin{aligned} p \cdot f(p) \cdot \delta_t \exp Z_j \cdot f(p)^{-1} \\ &= \left(p^1 + t \exp Z_j, p^2 + 2\mathcal{Q}^2(f(p), t \exp Z_j) + \mathcal{Q}^2(p^1, t \exp Z_j)\right) \\ &= \left(p^1 + t \exp Z_j, p^2 + 2t \sum_{\ell=1}^{k} \varphi_\ell(p) \mathcal{Q}^2(\exp Z_\ell, \exp Z_j)\right. \\ &\quad \left. + t\mathcal{Q}^2(\exp(\sum_{\ell=k+1}^{m_1} p_\ell Z_\ell), \exp Z_j)\right) \\ &= \left(p^1 + t \exp Z_j, p^2 + 2t \sum_{\ell=1}^{k} \varphi_\ell(p) q_{\ell,j} + t \sum_{\ell=k+1}^{m_1} p_\ell q_{\ell,j}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \Delta_{Z_j} f(p;t) \\ &= \sum_{i=1}^{k} \frac{1}{t} \left(\varphi_i\big(p^1 + t \exp Z_j, p^2 + 2t \sum_{\ell=1}^{k} \varphi_\ell(p) q_{\ell,j} \right. \\ &\qquad \left. + t \sum_{\ell=k+1}^{m_1} p_\ell q_{\ell,j}\big) - \varphi_i(p^1, p^2) \right) \exp Z_i. \end{aligned}$$

Let us specialize the previous example in the case $\mathbb{G} = \mathbb{H}^n$.

Example 3.15 (Horizontal Valued Functions Inside Heisenberg Groups) We recall here the well known definition of Heisenberg groups mainly to fix a few notations.

The n-Heisenberg group $\mathbb{H}^n$ is identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted $p = (p_1, \dots, p_{2n}, p_{2n+1}) = (p^1, p^2)$, with

$p^1 \in \mathbb{R}^{2n}$ and $p^2 = p_{2n+1} \in \mathbb{R}$. If p and $q \in \mathbb{H}^n$, the group operation is defined as

$$\begin{aligned} p \cdot q &= (p^1 + q^1, p_{2n+1} + q_{2n+1} + \mathcal{Q}^2(p^1, q^1)) \\ &= (p^1 + q^1, p_{2n+1} + q_{2n+1} - \frac{1}{2}\langle Jp^1, q^1\rangle_{\mathbb{R}^{2n}}) \end{aligned}$$

where $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ is the $(2n \times 2n)$-symplectic matrix.

For any $q \in \mathbb{H}^n$ and for any $r > 0$ left translations $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ and non isotropic dilations $\delta_r : \mathbb{H}^n \to \mathbb{H}^n$ are defined as

$$\tau_q(p) := q \cdot p \qquad \text{and as} \qquad \delta_r p := (rp^1, r^2 p_{2n+1}).$$

We denote as $\mathfrak{h}^n$ the Lie algebra of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}^n$ is given, for $i = 1, \dots, n$, by

$$X_i := \partial_i - \frac{1}{2}(Jp')_i \partial_{2n+1}, \quad Y_i := \partial_{i+n} + \frac{1}{2}(Jp')_{i+n}\partial_{2n+1}, \quad T := \partial_{2n+1}.$$

The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}^n$ spanned by $X_1, \dots, X_n$ and by $Y_1, \dots, Y_n$. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}^n$ is expressed by

$$\mathfrak{h}^n = \mathfrak{h}_1 \oplus \mathfrak{h}_2. \tag{22}$$

The Lie algebra $\mathfrak{h}^n$ is also endowed with a scalar product $\langle \cdot, \cdot \rangle$ making the vector fields $X_1, \dots, X_n$ and $Y_1, \dots, Y_n$ and T orthonormal. Thus (22) turns out to be an orthonormal decomposition of $\mathfrak{h}^n$ as a vector space.

If $p \in \mathbb{H}^n$, we indicate as $\|p\|$ its Koranyi norm, i.e.

$$\|p\| = \sqrt[4]{\|p^1\|^4_{\mathbb{R}^{2n}} + |p_{2n+1}|^2}$$

There are infinite many different couples of complementary subgroups inside $\mathbb{H}^n$. All these couples contain a horizontal subgroup, here denoted as $\mathbb{V}$ of dimension $k \le n$, isomorphic and isometric to $\mathbb{R}^k$ and a normal subgroup $\mathbb{W}$ of dimension $2n + 1 - k$, containing the centre $\mathbb{T}$.

Let $\mathbb{H}^n = \mathbb{W} \cdot \mathbb{V}$ where $\mathbb{V}$ is a k-dimensional horizontal subgroup and $\mathbb{W}$ a complementary normal subgroup. We assume, for the time being, that $\mathbb{V}$ and $\mathbb{W}$ are in generic position inside $\mathbb{H}^n$, in particular we do not assume that they are orthogonal.

It is always possible to choose a basis $Z := \{Z_1, \dots, Z_{2n}, T = Z_{2n+1}\}$ of the algebra $\mathfrak{h}^n$ such that

$$\mathbb{V} = \exp(\text{span}\{Z_1, \dots, Z_k\}), \qquad \mathbb{W} = \exp(\text{span}\{Z_{k+1}, \dots, Z_{2n}, T\}),$$

where $Z_1, \dots, Z_k$ are commuting horizontal vector fields.

We use coordinates with respect to the Z basis, i.e.

$$\mathbb{H}^n \ni p = \exp\big(\sum_{i=1}^{2n+1} p_i Z_i\big) \simeq (p_1, \dots, p_{2n+1}) \in \mathbb{R}^{2n+1.}$$

With the notation in (6) we denote

$$q_{\ell,h} := \mathcal{Q}^2(\exp Z_\ell, \exp Z_h) \in \mathbb{R}, \qquad \text{for } 1 \le \ell, h \le 2n+1.$$

As in Example 3.14, let

$$f : \mathbb{W} \to \mathbb{V}, \qquad f(p) := \exp\left(\sum_1^k \varphi_i(p) Z_i\right) = \sum_1^k \exp(\varphi_i(p) Z_i).$$

Nothing changes in the computations from the general case of a step 2 group and we get the following expression for the difference quotients for each horizontal Z_j with $j = k+1, \dots, 2n$,

$$\begin{aligned} &\Delta_{Z_j} f(p;t) \\ &= \sum_{i=1}^k \frac{1}{t}\Bigg(\varphi_i\big(p^1 + t\exp Z_j, p^2 + t(2\sum_{\ell=1}^k \varphi_\ell(p) q_{\ell,j} \\ &\qquad + \sum_{\ell=k+1}^{2n} p_\ell q_{\ell,j}) - \varphi_i(p^1, p^2)\Bigg) \exp Z_i. \end{aligned} \tag{23}$$

Moreover

$$\begin{aligned} \Delta_{Z_{2n+1}} f(p;t) &= \delta_{1/t}\left(f(p)^{-1} \cdot f\left(p \cdot f(p) \cdot \delta_t \exp Z_{2n+1} \cdot f(p)^{-1}\right)\right) \\ &= \delta_{1/t}\left(f(p)^{-1} \cdot f\left(p \cdot (0, \dots, 0, t^2)\right)\right) \\ &= \sum_{i=1}^k \frac{1}{t}\left(\varphi_i\big(p_1, \dots, p_{2n}, p_{2n+1} + t^2\big) - \varphi_i(p_1, \dots, p_{2n}, p_{2n+1})\right) \exp Z_i. \end{aligned} \tag{24}$$

Passing to the limit in (23) for $t \to 0^+$, we obtain the following system of $k(2n-k-1)$ non linear (intrinsic) differential operators acting on the k real valued functions $\varphi_1, \dots, \varphi_k$:

$$D_{Z_j}\varphi_i := \partial_j\varphi_i + \big(2\sum_{\ell=1}^{k}\varphi_\ell(p)q_{\ell,j} + \sum_{\ell=k+1}^{2n} p_\ell q_{\ell,j}\big)\partial_{2n+1}\varphi_i \tag{25}$$

here $\partial_j\varphi_i = \frac{\partial\varphi_i}{\partial p_j}$ for $i = 1, \dots, k$ and $j = k+1, \dots, 2n+1$.

Boundedness in (24) gives only a Holder type condition on the last variable of the functions φ_i.

Example 3.16 We further specialize the setting in Example 3.15 assuming that $\mathbb{W}$ and $\mathbb{V}$ are orthogonal in $\mathbb{H}^n$. Precisely, we assume that

$$\{Z_1, \dots, Z_{2n}, Z_{2n+1}\} = \{X_1, \dots, X_n, Y_1, \dots, Y_n, T\}$$

and that, for $1 \le k \le n$,

$$\mathbb{V} = \exp(\mathrm{span}\{X_1, \dots, X_k\}), \qquad \mathbb{W} = \exp(\mathrm{span}\{Z_{k+1}, \dots, Z_{2n}, T\}).$$

The coefficients $q_{\ell,i}$ take the special form

$$q_{\ell,i} := \mathcal{Q}^2(\exp Z_\ell, \exp Z_i) = \begin{cases} q_{h,h+n} = \frac{1}{2} & \text{for } 1 \le h \le n \\ q_{h+n,h} = -\frac{1}{2} & \text{for } 1 \le h \le n \\ q_{\ell,i} = 0 & \text{otherwise.} \end{cases}$$

Hence (25) takes the form

$$\begin{cases} D_{Z_j}\varphi_i := \partial_j\varphi_i + \displaystyle\sum_{\ell=k+1}^{2n} p_\ell q_{\ell,j}\partial_t\varphi_i = \partial_j\varphi_i - \frac{1}{2}p_{j+n}\partial_t\varphi_i \equiv X_j\varphi_i, & k+1 \le j \le n \\ D_{Z_j}\varphi_i := \partial_j\varphi_i + 2\displaystyle\sum_{\ell=1}^{k}\varphi_\ell(p)q_{\ell,j}\partial_t\varphi_i = \partial_j\varphi_i + \varphi_{j-n}\partial_t\varphi_i, & n+1 \le j \le n+k \\ D_{Z_j}\varphi_i := \partial_j\varphi_i + \displaystyle\sum_{\ell=k+1}^{2n} p_\ell q_{\ell,j}\partial_t\varphi_i = \partial_j\varphi_i + \frac{1}{2}p_{j-n}\partial_t\varphi_i \equiv Y_{j-n}\varphi_i, & n+k+1 \le j \le 2n \end{cases} \tag{26}$$

It may be interesting to consider also these special instances of (26).

Example 3.17 With the notations of Example 3.16 let us consider the complementary subsets of $\mathbb{H}^n \equiv (\mathbb{R}^{2n+1}, \cdot)$

$$\mathbb{V} = \exp(\mathrm{span}\{X_1\}), \qquad \mathbb{W} = \exp(\mathrm{span}\{X_2, \dots, X_n, Y_1, \dots, Y_n, T\})$$

and a function $f : \mathbb{W} \to \mathbb{V}$. Then $f(w) := \varphi(w) \exp X_1$ can be identified with the real valued function φ and we speak, with an abuse of language, of a real valued intrinsic function. Here $w := (0, p_2, \dots, p_{2n}, p_{2n+1}) \in \mathbb{W}$.

Then (26) takes the form

$$\begin{cases} D_{Z_j}\varphi \equiv D_{X_j}\varphi := \partial_j \varphi - \frac{1}{2} p_{j+n} \partial_{2n+1}\varphi \equiv X_j \varphi, & 2 \le j \le n \\ D_{Z_{n+1}}\varphi \equiv D_{Y_1}\varphi := \partial_{n+1}\varphi + \varphi \partial_{2n+1}\varphi, & \\ D_{Z_j}\varphi \equiv D_{Y_{j-n}}\varphi := \partial_j \varphi + \frac{1}{2} p_{j-n} \partial_{2n+1}\varphi \equiv Y_{j-n}\varphi, & n+2 \le j \le 2n \end{cases} \tag{27}$$

In $\mathbb{H}^1$ the system (27) reduces to the single non linear Burger type equation

$$D_Y \varphi := \partial_2 \varphi + \varphi \partial_3 \varphi = \partial_2 \varphi + \frac{1}{2} \partial_3 \varphi^2. \tag{28}$$

Equation (28) appeared in this context in [18], when studying the regularity of non critical level sets of group-C^1 functions $\mathbb{H}^n \to \mathbb{R}$. There are many works dealing with weak solutions of equations (28) and their relation with intrinsic regular surfaces inside the first Heisenberg group $\mathbb{H}^1$, (see [6–8, 29]).

System (27) is studied in [9] (see also [3]) where the authors characterize intrinsic real valued Lipschitz functions $f : \mathbb{W} \to \mathbb{V}$ as bounded solutions of (27). We notice that our Theorem 3.19 is related with the above mentioned characterization, notwithstanding that the result in [9] is much deeper than the one in here, given that the assumption in [9] is of boundedness of the limits of the intrinsic different quotients and not, as we make in Theorem 3.19, on the difference quotients themselves.

Much less studied are the vector valued analogues of (27) and (28). Consider the complementary subsets of $\mathbb{H}^2 \equiv (\mathbb{R}^5, \cdot)$

$$\mathbb{V} = \exp(\mathrm{span}\{X_1, X_2\}), \qquad \mathbb{W} = \exp(\mathrm{span}\{Y_1, Y_2, T\})$$

and $f : \mathbb{W} \to \mathbb{V}$. Then $f(w) := \varphi_1(w) \exp X_1 + \varphi_2(w) \exp X_2$, where $w := (0, 0, p_3, p_4, p_5) \in \mathbb{W}$.

In this case the equations in the first and last groups of system (26) disappear and we are left only with the non linear part of the system

$$\begin{cases} D_{Z_3}\varphi_1 := \partial_3 \varphi_1 + \varphi_1 \partial_5 \varphi_1, & D_{Z_3}\varphi_2 := \partial_3 \varphi_2 + \varphi_1 \partial_5 \varphi_2, \\ D_{Z_4}\varphi_1 := \partial_4 \varphi_1 + \varphi_2 \partial_5 \varphi_1, & D_{Z_4}\varphi_2 := \partial_4 \varphi_2 + \varphi_2 \partial_5 \varphi_2, \end{cases}$$

that is the vector valued analogous of (28).

If we consider in $\mathbb{H}^3 \equiv (\mathbb{R}^7, \cdot)$ the complementary subgroups

$$\mathbb{V} = \exp(\mathrm{span}\{X_1, X_2\}), \qquad \mathbb{W} = \exp(\mathrm{span}\{X_3, Y_1, Y_2, Y_3, T\}),$$

a function $f(w) := \varphi_1(w)\exp X_1 + \varphi_2(w)\exp X_2 : \mathbb{W} \to \mathbb{V}$, here $w := (0, 0, p_3, \dots, p_6, p_7) \in \mathbb{W}$, then (26) becomes a system of 8 equations acting on the two real valued functions φ_1, φ_2

$$\begin{cases} D_{Z_3}\varphi_1 := X_3\varphi_1, & D_{Z_3}\varphi_2 := X_3\varphi_2 \\ D_{Z_4}\varphi_1 := \partial_4\varphi_1 + \varphi_1(w)\partial_7\varphi_1, & D_{Z_4}\varphi_2 := \partial_4\varphi_2 + \varphi_1(w)\partial_7\varphi_2 \\ D_{Z_5}\varphi_1 := \partial_5\varphi_1 + \varphi_2(w)\partial_7\varphi_1, & D_{Z_5}\varphi_2 := \partial_5\varphi_2 + \varphi_2(w)\partial_7\varphi_2 \\ D_{Z_6}\varphi_1 := Y_3\varphi_1, & D_{Z_6}\varphi_2 := Y_3\varphi_2. \end{cases}$$

Finally we compute the difference quotients and an intrinsic derivative inside a step 3 group.

Example 3.18 (One Dimensional Function Inside Engels Group) The Engels group is $\mathbb{E} = (\mathbb{R}^4, \cdot, \delta_\lambda)$, were the group law is defined as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 + (x_1y_2 - x_2y_1)/2 \\ x_4 + y_4 + [(x_1y_3 - x_3y_1) + (x_2y_3 - x_3y_2)]/2 \\ +(x_1 - y_1 + x_2 - y_2)(x_1y_2 - x_2y_1)/12 \end{pmatrix}$$

and the family of dilation is

$$\delta_\lambda(x_1, x_2, x_3, x_4) = (\lambda x_1, \lambda x_2, \lambda^2 x_3, \lambda^3 x_4).$$

A basis of left invariant vector fields is X_1, X_2, X_3, X_4 defined as

$$\begin{aligned} X_1(p) &:= \partial_1 - (p_2/2)\,\partial_3 + \big(-p_3/2 - (p_1p_2 + p_2^2)/12\big)\,\partial_4 \\ X_2(p) &:= \partial_2 + (p_1/2)\,\partial_3 + \big(-p_3/2 + (p_1^2 + p_1p_2)/12\big)\,\partial_4 \\ X_3(p) &:= \partial_3 - ((p_1 + p_2)/2)\,\partial_4 \\ X_4(p) &:= \partial_4. \end{aligned}$$

The commutation relations are $[X_1, X_2] = X_3$, $\quad [X_1, X_3] = [X_2, X_3] = X_4$ and all the others commutators are zero. $\mathbb{E}$ is a semidirect product, as $\mathbb{E} = \mathbb{M} \cdot \mathbb{H}$, of the two subgroups $\mathbb{M}$ and $\mathbb{H}$

$$\mathbb{M} := \{m = (0, p_2, p_3, p_4)\} \qquad \mathbb{H} := \{h = (p_1, 0, 0, 0)\} = \{\exp(\operatorname{span} X_1)\}.$$

Let $f : \mathbb{M} \to \mathbb{H}$ where $f(m) := \exp(\varphi(m)X_1)$. Observe that $\mathbb{M}$ is a normal subgroup being $\mathbb{H}$ an horizontal subgroup. Then it follows

$$\begin{aligned}\Delta_{X_2} f(m;t) &= \delta_{1/t}\big(f(m)^{-1} \cdot f(m \cdot \mathrm{Ad}_{\varphi(m)}(\exp tX_2))\big) \\ &= \delta_{1/t}\big(f(m)^{-1} \cdot f(0, p_2+t, p_3+t\varphi(m), p_4+\frac{t}{2}(\varphi(m)^2+p_2\varphi(m)-p_3))\big) \\ &= \exp\big(\frac{1}{t}(\varphi(0, p_2+t, p_3+t\varphi(m), p_4+\frac{t}{2}(\varphi(m)^2+p_2\varphi(m)-p_3)) - \varphi(m))\big)X_1.\end{aligned}$$

Hence, computing $\lim_{t\to 0^+} \Delta_{X_2} f(m;t)$ we obtain the only horizontal intrinsic derivative of the real valued function φ

$$\begin{aligned}D_{X_2}\varphi(m) &:= \partial_2\varphi + \varphi(m)\partial_3\varphi + \frac{1}{2}(\varphi(m)^2 + p_2\varphi(m) - p_3)\partial_4\varphi \\ &= \partial_2\varphi + \frac{1}{2}\partial_3\varphi^2 + \frac{1}{12}\partial_4(2\varphi^3 + 3p_2\varphi^2 - 6p_3\varphi).\end{aligned}$$

3.4 Horizontal Difference Quotients and Lipschitz Functions

In a few noticeable instances the boundedness of difference quotients along the vectors of the horizontal layer of $\mathfrak{m}$ is sufficient to imply intrinsic Lipschitz continuity.

As observed before, this phenomenon is different from the one about functions defined *on* Carnot groups, although it is strictly related to it. It is well known that if $f : \mathbb{G} \to \mathbb{R}$ is such that Yf is bounded for all Y in the horizontal layer of $\mathfrak{g}$, then f is a Lipschitz function, the reason being that the horizontal layer of $\mathfrak{g}$ generates, by commutation, all the algebra. This is not the case for functions acting from $\mathbb{M}$ to $\mathbb{H}$. Indeed $\mathbb{M}$, though a stratified group, is not necessarily a Carnot group because not necessarily the horizontal layer of the algebra generates the entire algebra of $\mathbb{M}$ (see e.g. Example 3.17), on the other side there is a redeeming feature: intrinsic difference quotients and intrinsic derivatives are non linear operators. Finally one does not have to forget that the final result is that the functions are *intrinsic Lipschitz* and not *Lipschitz*.

We present here only two instances of this phenomenon, both of them inside Heisenberg groups. The first one deals with 1-codimensional graphs of functions acting between any two complementary subgroups the second one deals with k-codimensional horizontal graphs of functions acting between *orthogonal* complementary subgroups.

Theorem 3.19 *Let $\mathbb{W}$ and $\mathbb{V}$ be complementary subgroups in $\mathbb{H}^n$ with $\mathbb{V}$ one dimensional and horizontal. Let $L > 0$ and $f : \mathbb{W} \to \mathbb{V}$ be such that*

$$\|\Delta_Z f(w;t)\| \leq L \|\exp Z\|$$

for all $w \in \mathbb{W}$ and for all horizontal vector fields $Z \in \mathfrak{w}^1$. Then there is $\tilde{L} = \tilde{L}(L, \mathbb{V}, \mathbb{W}) \geq L$ such that f is $\tilde{L}$-intrinsic Lipschitz in $\mathbb{W}$.

Proof By translation invariance, it is enough to prove

$$\|f(w)\| \leq \tilde{L} \|w\| \tag{29}$$

for all $w \in W$ under the additional assumption

$$f(0) = 0.$$

If $f(w) = 0$ there is nothing to prove. Hence let us assume that $v := f(w) \neq 0$. Under this assumption we prove that there are $s, t \in \mathbb{R}$, there are horizontal vectors Z, U in the first layer $\mathfrak{w}^1$ of the algebra $\mathfrak{w}$ of $\mathbb{W}$ and a constant $C = C(\mathbb{V}, \mathbb{W}) > 0$ such that $\|\exp Z\| = \|\exp U\| = 1$,

$$w \cdot Ad_{f(w)}(\delta_s \exp Z) = \delta_t \exp U \tag{30}$$

and

$$|t| \leq \|w\| + |s|; \qquad |s| \leq C \|w\|^2 / \|f(w)\|. \tag{31}$$

With the notations of (6), $w = (w^1, w^2) = (w^1, w_{2n+1}), f(w) = (f(w)^1, 0)$, $\exp Z = \big((\exp Z)^1, 0\big)$ and $\exp U = \big((\exp U)^1, 0\big)$. Then

$$\begin{aligned} w \cdot \mathrm{Ad}_{f(w)}(\delta_s \exp Z) &= \big(w^1 + (\delta_s \exp Z)^1, w_{2n+1} + 2\mathcal{Q}^2(f(w)^1, (\delta_s \exp Z)^1) \\ &\quad + \mathcal{Q}^2(w^1, (\delta_s \exp Z)^1)\big) \\ &= \big(w^1 + s(\exp Z)^1, w_{2n+1} + 2\mathcal{Q}^2(f(w)^1, s(\exp Z)^1) \\ &\quad + \mathcal{Q}^2(w^1, s(\exp Z)^1)\big) \end{aligned}$$

Hence (30) is equivalent to solve in Z, U and t, s the system of $2n + 1$ equations,

$$\begin{cases} w^1 + s(\exp Z)^1 = t(\exp U)^1 \\ w_{2n+1} + 2\mathcal{Q}^2(f(w)^1, s(\exp Z)^1) + \mathcal{Q}^2(w^1, s(\exp Z)^1) = 0. \end{cases} \tag{32}$$

Let $\alpha, \beta \in \mathbb{R}^{2n}$, with $\|\alpha\|_{\mathbb{R}^{2n}} = \|\beta\|_{\mathbb{R}^{2n}} = 1$, be such that

$$w \in \mathbb{W} \iff \sum_{i=1}^{2n} \alpha_i w_i = 0, \qquad v \in \mathbb{V} \iff v = \delta_{\|v\|} \exp \sum_{i=1}^{n} (\beta_i X_i + \beta_{n+i} Y_i).$$

Because $\mathbb{W}$ and $\mathbb{V}$ are complementary subgroups then

$$C = C(\mathbb{W}, \mathbb{V}) := |\sum_{i=1}^{2n} \alpha_i \beta_i| > 0.$$

Let

$$Z := \sum_{i=1}^{n} \alpha_{n+i} X_i - \alpha_i Y_i \in \mathfrak{w}^1,$$

then $\|\exp Z\| = 1$ and for all $v \in \mathbb{V}$ and $w \in \mathbb{W}$

$$\begin{aligned} \left|\mathcal{Q}^2(v^1, (\exp Z)^1)\right| &= \frac{1}{2} \|v\| \, |\sum_{i=1}^{2n} \alpha_i \beta_i| \geq C \|v\|, \\ \mathcal{Q}^2(w^1, (\exp Z)^1) &= -\frac{1}{2} \sum_{i=1}^{n} (w_i \alpha_i + w_{n+i} \alpha_{n+i}) = 0. \end{aligned} \tag{33}$$

With this choice of Z from the last equation of (32), using that $\mathcal{Q}^2(\cdot, \cdot)$ is bilinear, we get

$$|s| \leq C |w_{2n+1}| / \|f(w)\| \leq C \|w\|^2 / \|f(w)\|,$$

where C is a (different) constant depending only on $\mathbb{V}$ and $\mathbb{W}$. The other estimate in (31) follows from the first equations in (32).

Finally let us see that (29) follows from (30) and (31). Indeed, consider the intrinsic difference quotients starting from 0 along U and from $w \cdot f(w)$ along Z

$$\begin{aligned} \nabla_U f(0; t) &= f(\delta_t \exp U), \\ \nabla_Z f(w; s) &= f(w)^{-1} \cdot f(w \cdot Ad_{f(w)}(\delta_s \exp Z)) = f(w)^{-1} \cdot f(\delta_t \exp U) \end{aligned}$$

From the assumption of boundedness of the difference quotients of f

$$\begin{aligned} &\|f(\delta_t \exp U)\| \leq L|t| \\ &\left\| f(w)^{-1} \cdot f(w \cdot Ad_{f(w)}(\delta_s \exp Z)) \right\| \leq L|s| \end{aligned}$$

The boundedness of these difference quotients yields an estimate $\|f(w)\|$. Indeed

$$
\begin{aligned}
\|f(w)\| &\leq \left\|f(w)^{-1}f(\delta_t \exp U)\right\| + \|f(\delta_t \exp U)\| \\
&= \left\|f(w)^{-1}f(w \cdot Ad_{f(w)}(\delta_s \exp Z))\right\| + \|f(\delta_t \exp U)\| \\
&\leq L(|s| + |t|) \\
&\leq L\big(\|w\| + 2C\|w\|^2/\|f(w)\|\big)
\end{aligned} \tag{34}
$$

that eventually gives

$$\|f(w)\| \leq \tilde{L}\, \|w\|$$

with $\tilde{L} = \frac{1}{2}(L + \sqrt{L^2 + 8LC})$. □

Remark 3.20 Observe that in Theorem 3.19 it has been proved that if $\|\Delta_Z f(w;t)\| = 0$ for all $w \in \mathbb{W}$ and for all horizontal vector field $Z \in \mathfrak{w}^1$ then $f : \mathbb{W} \to \mathbb{V}$ is intrinsic Lipschitz with 0 Lipschitz constant hence it is a constant function. This fact is not anymore true if f is defined on a proper subset of $\mathbb{W}$. The following one is an example: let $\mathbb{W}$ and $\mathbb{V}$ be the complementary subgroups of $\mathbb{H}^1$ defined as

$$\mathbb{W} := \{(0, x_2, x_3)\}, \qquad \mathbb{V} := \{(x_1, 0, 0)\}.$$

Let $\mathcal{A}$ be the neighborhood of the origin in $\mathbb{W}$ defined as $\mathcal{A} := \{(0, x_2, x_3) : x_2 > -1\}$ and let $f : \mathcal{A} \subset \mathbb{W} \to \mathbb{V}$ be defined as

$$f(0, x_2, x_3) := \left(\frac{x_3}{1 + x_2}, 0, 0\right).$$

The horizontal layer of $\mathfrak{w}^1$ is one dimensional and is spanned by the vector $Y := \partial_{x_2}$. Then from Definition 3.7 (see also (i) of Remark 3.10)

$$\Delta_Y f(w;t) = 0, \qquad \text{for all } w \in \mathcal{A} \text{ and } t \geq 0$$

while clearly f is not constant.

Theorem 3.21 *Let $\mathbb{W}$ and $\mathbb{V}$ be the complementary orthogonal subgroups of $\mathbb{H}^n$ considered in Example 3.16. Precisely, for $1 \leq k \leq n$ let*

$$\mathbb{V} = \exp(\text{span}\{X_1, \dots, X_k\}), \qquad \mathbb{W} = \exp(\text{span}\{Z_{k+1}, \dots, Z_{2n}, T\}).$$

Hence $\mathbb{V}$ is k-dimensional and horizontal. Let $L > 0$ and $f : \mathbb{W} \to \mathbb{V}$ be such that

$$\|\Delta_Z f(w;t)\| \leq L\, \|\exp Z\|$$

for all $w \in \mathbb{W}$ and for all horizontal vector fields $Z \in \mathfrak{w}^1$. Then there is $\tilde{L} = \tilde{L}(L, \mathbb{V}, \mathbb{W}) \geq L$ such that f is $\tilde{L}$-intrinsic Lipschitz in $\mathbb{W}$.

Proof We keep using the notations introduced in Examples 3.15 and 3.16.

Analogously as in the proof of Theorem 3.19, by translation invariance, it is enough to prove

$$\|f(w)\| \leq \tilde{L}\,\|w\| \tag{35}$$

for all $w \in W$ under the additional assumption

$$f(0) = 0.$$

Let be given $w = (w^1, w^2) \in \mathbb{W}$ and $v = (v^1, 0) \in \mathbb{V}$ with $v^1 \neq 0$. Then there is $z = (z^1, 0) \in \mathbb{W}^1$ as

$$w \cdot \mathrm{Ad}_v z \in \mathbb{W}^1. \tag{36}$$

Indeed, let $z = (z_1, \dots, z_{2n+1})$ be defined such that

$$\begin{aligned} z_i &= 0, \qquad \text{for } 1 \leq i \leq n \text{ and for } n+k+1 \leq i \leq 2n+1 \\ z_{n+i} &= \lambda \operatorname{sign}(v_i), \quad \text{for } n+1 \leq i \leq n+k. \end{aligned}$$

With this choice of z we have

$$\begin{aligned} \mathcal{Q}^2(w^1, z^1) &= \frac{1}{2}\sum_{i=1}^{n}(w_i z_{n+1} - w_{n+i} z_i) = 0, \\ \mathcal{Q}^2(v^1, z^1) &= \frac{1}{2}\lambda \sum_{i=1}^{k} |v_i|. \end{aligned}$$

Finally choosing $\lambda = -w_{2n+1}/\sum_{i=1}^{k}|v_i|$ we get

$$w \cdot \mathrm{Ad}_v z = \big(w^1 + z^1, w_{2n+1} + 2\mathcal{Q}^2(v^1, z^1) + \mathcal{Q}^2(w^1, z^1)\big) = (w^1 + z^1, 0) \in \mathbb{W}^1.$$

Let us go back to the proof of (35). If $f(w) = 0$ there is nothing to prove. Hence let us assume that $f(w) \neq 0$ and define $v := f(w)$. Now let $Z, U \in \mathfrak{w}^1$ be chosen such that $\|\exp Z\| = \|\exp U\| = 1$ and

$$\delta_s \exp Z = z, \qquad \delta_t \exp U = w \cdot \mathrm{Ad}_{f(w)} z$$

for appropriate $s, t \in \mathbb{R}$. With this choice of s we have

$$|s| \leq C \|w\|^2 / \|f(w)\| .$$

From now on the proof follows the same pattern of the proof of Theorem 3.19. □

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Multilinear Weighted Norm Inequalities Under Integral Type Regularity Conditions

Lucas Chaffee, Rodolfo H. Torres, and Xinfeng Wu

Abstract Weighted norm inequalities for operators corresponding to non-smooth versions of Calderón-Zygmund and fractional integral multilinear operators are revisited and improved in a unified way. Graded classes of weights matching the amount of regularity assumptions on the operators are also studied.

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1 Introduction

It is great pleasure for us to contribute to this volume in honor of Prof. Richard L. Wheeden with some results about multilinear weighted estimates, which can be traced back to his important collaboration with D. Kurtz on weighted norm inequalities for multipliers [16].

L. Chaffee
Department of Mathematics, University of Kansas, Lawrence, KS 66045-7523, USA

Department of Mathematics, Western Washington University, 516 High Street, Bellingham, WA 98225, USA
e-mail: lucas.chaffee@wwu.edu

R.H. Torres (✉)
Department of Mathematics, University of Kansas, Lawrence, KS 66045-7523, USA
e-mail: torres@ku.edu, torres@math.ku.edu

X. Wu
Department of Mathematics, China University of Mining & Technology, Beijing 100083, China

Department of Mathematics, University of Kansas, Lawrence, KS 66045-7523, USA
e-mail: wuxf@ku.edu

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At the core of both linear and multilinear versions of Calderón-Zygmund theory is the use of appropriate regularity assumptions in the kernel of the operators to extend their boundedness from some initial Lebesgue spaces to all those in a full range of exponents, including weak-type end-points. When possible, one tries to extend the boundedness properties to weighted spaces as well. The regularity assumptions have been modified in the literature in several ways and results for quite rough operators of very particular forms exist. Nonetheless, when the focus is on weighted norm inequalities for general operators, as it is in this article, the regularity assumptions inexorably circle around the so-called Hörmander type integral conditions. This is because the conditions are easy to verify but also because they tend to parametrize very well the classes of weights used. The natural classes of weights are of course related to the Muckenhoupt classes A_p, as they are characterized by the boundedness on weighted Lebesgue spaces of smooth (non-degenerate) Calderón-Zygmund operators and the maximal functions in the theory. Experience tells us, however, that once the amount of regularity in the kernels of the operators studied starts to diminish away from some standard conditions, one cannot hope for $L^p(w)$ estimates to hold for all $w \in A_p$, but at most for a smaller class of weights.

In this work we shall revisit Hörmander's integral conditions and associated classes of weights in the multilinear setting. Ultimately, the goal is to explore the effect of such regularity assumptions and to adapt the general template of the linear theory to the multivariable one. Our motivation comes from several sources. One is the work [25] of Pérez and the second named author in this article, where end-point estimates for bilinear operators were studied under variations of Hörmander's condition. Another motivation is provided by the works of Fujita and Tomita [7] and Bui and Duong [1] on weighted norm inequalities, as well as further recent results on multilinear multipliers such as the ones by Li and Sun [18]. Although the techniques we will use are very close to the one in those works, we are able to consider both Calderón-Zygmund and potential type multilinear operators. Our results improve the ones in [1] for the former operators, are new for the latter, and will be presented in a unified way. We will also precisely identify the graded classes of multiple weights needed to be used for all the different types of multilinear operators we consider. In this regard, and like numerous other results on the subject, we are influenced by the original article of Kurtz and Wheeden for Fourier multiplier [16], which was one of the first works to relate a reduced amount of regularity of multipliers with the classes of weights allowed.

We would like to take this opportunity to congratulate and thank Prof. Wheeden (Dick) for his numerous pioneering contributions in several areas of harmonic analysis and partial differential equations, and in particular for those in the study of weighted norm inequalities, many of which have helped shape the subject and continue to be of relevance today.

2 Preliminaries and Statement of the Main Result

As in [25], and for the reader's convenience, we repeat here some basic facts of the linear theory to put our results in context. If T is linear operator of the form

$$Tf(x) = \int K(x,y)f(y)\,dy$$

(properly interpreted) which is bounded in $L^{p_0}(\mathbb{R}^n)$ for some $p_0 > 1$, then the typical Hörmander or integral regularity condition

$$\sup_Q \sup_{x\in Q} \int_{\mathbb{R}^n\setminus Q^*} |K(x,y) - K(x_Q,y)|\,dy < \infty, \tag{1}$$

gives also the boundedness of T on $L^p(\mathbb{R}^n)$ for all $p_0 < p < \infty$. Here Q is a cube in $\mathbb{R}^n$ with sides parallel to the axes and center x_Q, and Q^* is an appropriate dilation of Q. The condition (1) is easily implied by a pointwise regularity assumption such as

$$|\nabla_x K(x,y)| \lesssim \frac{1}{|x-y|^{n+1}} \quad \text{for} \quad x \neq y, \tag{2}$$

or other Hölder regularity variations of it. In the translation invariant case, that is when $K(x,y) = k(x-y)$, (1) can be obtained from well-known regularity estimates for the *multiplier* of T, which we denote by $m = \hat{k}$. Namely, when $\widehat{Tf} = m\hat{f}$, then the pointwise condition

$$|\partial^\alpha m(\xi)| \lesssim_\alpha |\xi|^{-|\alpha|} \tag{3}$$

considered by Marcinkiewicz [20] and Mihlin [22] or its average (Sobolev type) variation

$$\sup_{R>0} \left(R^{2|\alpha|-n} \int_{R<|x|<2R} |\partial^\alpha m(\xi)|^2\,d\xi \right)^{1/2} < C \tag{4}$$

considered by Hörmander [13], yield (1) if enough derivatives of m (roughly $n/2$ derivatives) can be estimated in the above fashion. Such operators are then bounded on L^p for all $1 < p < \infty$.

It has been known for a long time (going back to the work of Kurtz-Wheeden [16]), however, that if the smoothness on the multiplier is only enough to obtain integral regularity conditions on the kernel side but not pointwise ones, then the operator, though bounded in the full range of unweighted L^p spaces, it may only satisfy $L^p(w)$ estimates for weights w in more restricted classes than A_p. This is even the case if (1) is replaced by stronger L^r-Hörmander's conditions of the form

$$\sum_{j=1}^{\infty}(2^j|Q|)^{1/r'}\left(\int_{2^jQ^*\setminus 2^{j-1}Q^*}|K(x,y)-K(x_Q,y)|^r\,dy\right)^{1/r}<C \tag{5}$$

for all $x \in Q$, and where $r > 1$ and r' is its dual exponent. Note that $r = 1$ gives back (1). In addition to [16], we direct the reader to the works by Rubio de Francia-Ruiz-Torrea [26], and Martell-Pérez-Trujillo [21] for detailed results as well as further references to earlier work in the subject. Moreover, we remark that it was shown in [21] that (1) is not enough to obtain $L^p(w)$ estimates even if $w \in A_1$. See also [21] for negative results for other values of r in (5). We also want to point out the work by Kurtz [15], where the case $L^p \to L^q$, $p \neq q$, was considered. We will develop analogous results to [15] for multilinear fractional integral operators.

Much of the above has been studied in the multilinear setting already, generating some expected analogs of the linear theory as well as some unexpected interesting new phenomena. Grafakos and Torres developed a fairly complete multilinear Calderón-Zygmund theory in [11] based on multilinear analogs of the pointwise gradient estimates (2). The operators are now of the form

$$T(\mathbf{f})(x)=\int_{(\mathbb{R}^n)^N}K(x,y_1,\cdots,y_N)\mathbf{f}(y)\,d\mathbf{y},$$

at least for $x \notin \cap_{i=1}^N \mathrm{supp} f_i$, where $\mathbf{f} = (f_1,\ldots,f_N)$ and $d\mathbf{y} = dy_1 \ldots dy_N$. Relevant references to earlier work by other authors are given in [11] too. In particular, when applied to kernels of the form $K(x-y_1,\ldots,x-y_N)$, studied also by Kenig and Stein [14], this theory recovers and extends the founding multilinear results of Coifman and Meyer [4, 5] for the Fourier multipliers that nowadays bear their names. That is

$$T_m(\mathbf{f})(x)=\int_{\mathbb{R}^{nN}}e^{ix\cdot(\xi_1+\cdots+\xi_N)}m(\xi_1,\ldots,\xi_N)\widehat{f}_1(\xi_1)\cdots\widehat{f}_N(\xi_N)d\xi_1\cdots\xi_N,$$

where m satisfies

$$|\partial_{\xi_1}^{\alpha_1}\cdots\partial_{\xi_N}^{\alpha_N}m(\xi_1,\cdots,\xi_N)|\lesssim_{\alpha_1,\cdots,\alpha_N}(|\xi_1|+\cdots+|\xi_N|)^{-(|\alpha_1|+\cdots+|\alpha_N|)}. \tag{6}$$

Clearly, (6) is the analog of the Marcinkiewicz-Mihlin condition (3). If (6) holds for enough derivatives, then pointwise gradient estimates on the corresponding kernel $K(x-y_1,\ldots,x-y_N)$ follow. The operator T_m is then bounded from $L^{p_1}(\mathbb{R}^n)\times\cdots\times L^{p_N}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p_1,\cdots,p_N < \infty$ satisfying $1/p_1+\cdots+1/p_N = 1/p$. It was natural then to determine the minimum amount of regularity that would yield such boundedness result. This has been done by Tomita [27] and Grafakos and Si [10]. Moreover, they also studied analogous of (4), which were normalized on various Sobolev spaces.

The weighted theory in the multilinear setting has several interesting twists. Under pointwise regularity estimates on the kernel of the operators, the theory was extended in [12] and [24] to weighted Lebesgue spaces with weights in the product of A_p classes. Though these seemed at first the natural weights to use, it was later discovered by Lerner et al. [17] that they were far from optimal. In fact, a much larger class of multiple weights $\mathbf{A_P}$ can be used. Moreover, such classes characterize the boundedness of appropriate maximal functions and smooth (non-degenerate) multilinear Calderón-Zygmund operators.

Going back to multipliers, in [7], Fujita and Tomita studied the weighted boundedness of T_m with product of scalar A_p weights, while Li and Sun [18] did it with multiple weights, both under minimal regularity assumptions on the multiplier. Interestingly, different forms of Sobolev regularity appear to determine whether product of scalar weights or multiple weights $\mathbf{A_P}$ could be used for multipliers. A detailed summary of positive results and some negative ones is given in the recent work of Fujita and Tomita [8]. Li and Sun [19] also considered an extension to pseudodifferential operators. For operators which are not defined in terms of a symbol, the most general results available so far in this context are the ones in the already mentioned contribution of Bui-Duong [1]. They introduced the following version of (5) in the multilinear setting[1]:

$$\begin{aligned}\left(\int_{S_{j_m}(Q^*)}\cdots\int_{S_{j_1}(Q^*))}|K(x,\mathbf{y})-K(z,\mathbf{y})|^{r'}d\mathbf{y}\right)^{1/r'}\\ \lesssim \frac{|x-z|^{N(\delta-n/p_0)}}{|Q|^{N\delta/n}}2^{-N\delta\max\{j_1,\dots,j_m\}}\end{aligned}\tag{7}$$

for all $x,z\in Q$ and $(j_1,\dots,j_N)\neq(0,\dots,0)$, where $S_j(Q^*)=2^jQ^*\setminus 2^{j-1}Q^*$ if $j\geq 1$ and $S_0(Q^*)=Q^*$. They proved that this condition together with an a priori weak-type estimate imply multiple weighted boundedness for reduced classes of multiple weights. As an application, they subsumed previous results based on pointwise regularity estimates for multipliers too. We will present a different form of (7) and we will also allow for off diagonal operators as Kurtz [15] did in the linear situation.

For $0\leq\alpha<nN$, assume that T_α is a multilinear operator initially defined on the m-fold product of Schwartz spaces and taking values into the space of tempered distributions

$$T_\alpha:\mathcal{S}(\mathbb{R}^n)\times\cdots\times\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n).$$

[1] In [1] the condition is written in terms of balls but the change to cubes is of no significance.

Assume also that for a certain kernel function K,

$$T_\alpha(\mathbf{f})(x) = \int_{(\mathbb{R}^n)^N} K(x, y_1, \cdots, y_N) f_1(y_1) \cdots f_N(y_N) d\mathbf{y},$$

as an absolutely convergent integral whenever $x \notin \cap_{i=1}^N \mathrm{supp} f_i$.

Definition 2.1 The operator T_α satisfies the *multilinear $L^{r',\alpha}$-Hörmander condition* if

$$\sup_Q \sup_{x,z\in\frac{1}{2}Q} \sum_{k=0}^{\infty} |2^k Q|^{\frac{N}{r}-\frac{\alpha}{n}} \left(\int_{(2^{k+1}Q)^N \setminus (2^k Q)^N} |K(x,\mathbf{y}) - K(z,\mathbf{y})|^{r'} d\mathbf{y} \right)^{1/r'}$$

is finite. When $r = 1$, the above condition is understood as

$$\sup_Q \sup_{x,z\in\frac{1}{2}Q} \sum_{k=0}^{\infty} |2^k Q|^{N-\frac{\alpha}{n}} \sup_{\mathbf{y}\in(2^{k+1}Q)^N \setminus (2^k Q)^N} |K(x,\mathbf{y}) - K(z,\mathbf{y})| < \infty.$$

Note that when $N = 1$ (the linear case) and $\alpha = 0$, the above condition reduces to the classical $L^{r'}$-Hörmander's condition (5).

We now introduce the relevant multiple kind of weights for our work.

Definition 2.2 For $1 \leq r < p_1, \ldots, p_N < \infty$, $1/p = 1/p_1 + \cdots + 1/p_N$, and $r/N < p \leq q < \infty$, we say that a vector of weights $\mathbf{w} = (w_1, \ldots, w_N)$ is in the class $\mathbf{A}(\mathbf{P}, q, r)$, or that it satisfies the $\mathbf{A}(\mathbf{P}, q, r)$ condition, if

$$[\mathbf{w}]_{\mathbf{A}(\mathbf{P},q,r)} := \sup_Q \left(\fint_Q w(x)^q dx \right)^{\frac{1}{q}} \prod_{i=1}^{N} \left(\fint_Q w_i(x)^{-p_i r/(p_i - r)} dx \right)^{\frac{p_i - r}{p_i r}} < \infty,$$

where $w(x) = \prod_{i=1}^N w_i(x)$.

Our main result is the following

Theorem 2.3 *Let $1 < p_1, \ldots, p_N < \infty$, $1/p = 1/p_1 + \cdots + 1/p_N$, $0 \leq \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $1 \leq r < \min(p_1, \ldots, p_N, nN/\alpha, Np)$ and $1/r^* = N/r - \alpha/n$. If*

$$T_\alpha : L^r \times \cdots \times L^r \to L^{r^*,\infty} \tag{8}$$

and satisfies the multilinear $L^{r',\alpha}$-Hörmander condition, then

$$T_\alpha : L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_N}(w_N^{p_N}) \to L^q(w^q)$$

for all $\mathbf{w} \in \mathbf{A}(\mathbf{P}, q, r)$.

The case $\alpha = 0$ should be seen as the Calderón-Zygmund situation and represents an improvement over [1]. The case $\alpha > 0$ correspond to operators of multilinear fractional integral type and is a new result. Clearly the result applies to smooth multilinear Calderón-Zygmund operators as in [11] and also to fractional integral ones as in [14]. The interest, however, is that the regularity conditions are weaker, and they apply also to multiplier operators with minimal Sobolev regularity.

In the next section we introduce additional notation and prove several results for the classes $\mathbf{A}(\mathbf{P}, q, r)$, relating them to other classes of weights already studied in the literature. In Sect. 3 we present the proof of our main theorem while in Sect. 4 we apply the results to multiplier operators.

3 The Classes $\mathbf{A}(\mathbf{P}, q, r)$

We start by recalling other classes of multiple weights introduced in the literature.

Let $1 < p_1, \ldots, p_N < \infty$, $1/p = 1/p_1 + \cdots + 1/p_N$, $\mathbf{P} = (p_1, \ldots, p_n)$, and $1/N < p < \infty$. As defined in [17], a vector of weights $\mathbf{w} = (w_1, \ldots, w_N)$ is in the class $\mathbf{A_P}$ if

$$\sup_Q \left(\fint_Q \left(\prod_{i=1}^N w_i^{1/p_i} \right)^p \right)^{\frac{1}{p}} \prod_{i=1}^N \left(\fint_Q w_i^{1-p_i'} \right)^{1/p_i'} < \infty. \tag{9}$$

On the other hand Moen [23] (see also Chen-Xue [3]) considered for $1 < p_1, \ldots, p_N < \infty$, $1/p = 1/p_1 + \cdots + 1/p_N$, and $1/N < p \le q < \infty$, the classes $\mathbf{A}(\mathbf{P}, q)$ of vector weights $\mathbf{w} = (w_1, \ldots, w_N)$ such that

$$\sup_Q \left(\fint_Q \left(\prod_{i=1}^N w_i \right)^q \right)^{\frac{1}{q}} \prod_{i=1}^N \left(\fint_Q w_i^{-p_i'} \right)^{1/p_i'} < \infty. \tag{10}$$

Note that the normalization in (9) and (10) is different, so for $q = p$, $\mathbf{w} = (w_1, \ldots, w_N) \in \mathbf{A}(\mathbf{P}, p)$ if and only if $(w_1^{p_1}, \ldots, w_N^{p_N}) \in \mathbf{A_P}$.

Next, we observe that it is immediate to verify the following properties.

Lemma 3.1 *Let $1 < r < s < p_1, \ldots, p_N, Np < \infty$ and $1 \le q < \infty$. The following properties hold.*

(i) $\mathbf{A}(\mathbf{P}, q, 1) = \mathbf{A}(\mathbf{P}, q)$ *and* $[w]_{\mathbf{A}(\mathbf{P},p,1)} = [w^p]_{A_p}^{1/p}$ *when* $N = 1$.[2]

(ii) $\mathbf{w} \in \mathbf{A}(\mathbf{P}, q, r)$ *if and only if* $\mathbf{w}^r := (w_1^r, \ldots, w_N^r) \in \mathbf{A}(\frac{\mathbf{P}}{r}, \frac{q}{r})$.

(iii) $\mathbf{A}(\mathbf{P}, q, s) \subset \mathbf{A}(\mathbf{P}, q, r) \subset \mathbf{A}(\mathbf{P}, q)$.

[2]Recall that in the scalar case $[w]_{A_p} = \sup_Q \left(\fint_Q w \right) \left(\fint_Q w^{-p'} \right)^{p-1}$.

We collect now a few other properties of the classes $\mathbf{A}(\mathbf{P}, q, r)$ which can essentially be found in the literature. We provide only a few missing details. The following characterization of $\mathbf{A}(\mathbf{P}, q, r)$ holds.

Lemma 3.2 *The weight* $\mathbf{w} \in \mathbf{A}(\mathbf{P}, q, r)$ *if and only if*

$$w^q \in A_{Nq/r} \quad \textit{and} \quad w_i^{-\frac{p_i r}{p_i - r}} \in A_{N(\frac{p_i}{r})'} \quad i = 1, \dots, N. \tag{11}$$

Proof Using Lemma 3.1 (ii), we can see that the necessity part is [23, Theorem 3.4]. So our contribution here is to prove the converse implication. We use the ideas in [17]. Assume that (11) holds. We claim

$$\left(\fint_Q w^{-\frac{qr}{Nq-r}} \right)^{\frac{N}{r} - \frac{1}{q}} \prod_{i=1}^{N} \left(\fint_Q w_i^{\frac{p_i r}{(N-1)p_i + r}} \right)^{\frac{(N-1)p_i + r}{p_i r}} \geq 1. \tag{12}$$

Let

$$\epsilon = \frac{rpq}{N^2 pq + rq - rp} \quad \text{and} \quad \tau = \frac{qr}{Nq - r}.$$

Clearly $\tau > \epsilon > 0$. Then by Hölder's inequality,

$$\begin{aligned} 1 &= \left(\fint_Q w^{-\epsilon} w^{\epsilon} \right)^{\frac{1}{\epsilon}} \\ &\leq \left(\fint_Q w^{-\epsilon(\frac{\tau}{\epsilon})} \right)^{\frac{1}{\epsilon(\frac{\tau}{\epsilon})}} \left(\fint_Q w^{\epsilon(\frac{\tau}{\epsilon})'} \right)^{\frac{1}{\epsilon(\frac{\tau}{\epsilon})'}} \\ &= \left(\fint_Q w^{-\tau} \right)^{\frac{1}{\tau}} \left(\fint_Q w^{\frac{\epsilon\tau}{\tau-\epsilon}} \right)^{\frac{1}{\epsilon} - \frac{1}{\tau}}. \end{aligned}$$

Let

$$\beta_i = \frac{(pN(N-1) + r)p_i}{(p_i(N-1) + r)p} = \frac{N(N-1) + r/p}{(N-1) + r/p_i}.$$

Then $\min(\beta_i) \geq 1$ and $\sum_{i=1}^{N} 1/\beta_i = 1$. Note also that

$$\frac{\epsilon\tau}{\tau - \epsilon} = \frac{r}{N(N-1) + r/p},$$

so by Hölder's inequality,

$$\left(\fint_Q w^{\frac{\epsilon\tau}{\tau-\epsilon}}\right)^{\frac{1}{\epsilon}-\frac{1}{\tau}} \le \prod_{i=1}^N \left(\fint_Q w_i^{\frac{\beta_i\epsilon\tau}{\tau-\epsilon}}\right)^{\frac{1}{\beta_i}(\frac{1}{\epsilon}-\frac{1}{\tau})}$$
$$= \prod_{i=1}^N \left(\fint_Q w_i^{\frac{p_i r}{(N-1)p_i+r}}\right)^{\frac{(N-1)p_i+r}{p_i r}},$$

and (12) follows. Combining (12) and (11), we finally obtain

$$\begin{aligned}
[w]_{\mathbf{A}(\mathbf{P},q,r)} &= \sup_Q \left(\fint_Q w^q\right)^{\frac{1}{q}} \prod_{i=1}^N \left(\fint_Q w^{-\frac{p_i r}{p_i-r}}\right)^{\frac{p_i-r}{p_i r}} \\
&\le \sup_Q \left(\fint_Q w^q\right)^{\frac{1}{q}} \left(\fint_Q w^{-\frac{qr}{Nq-r}}\right)^{\frac{N}{r}-\frac{1}{q}} \\
&\quad \times \prod_{i=1}^N \sup_Q \left(\fint_Q w_i^{\frac{p_i r}{(N-1)p_i+r}}\right)^{\frac{(N-1)p_i+r}{p_i r}} \left(\fint_Q w_i^{-\frac{p_i r}{p_i-r}}\right)^{\frac{p_i-r}{p_i r}} \\
&\le \sup_Q \left(\fint_Q w^q\right)^{\frac{1}{q}} \left(\fint_Q (w^q)^{-\frac{1}{Nq/r-1}}\right)^{(\frac{Nq}{r}-1)\frac{1}{q}} \\
&\quad \times \prod_{i=1}^N \sup_Q \left(\fint_Q \left(w_i^{-\frac{p_i r}{p_i-r}}\right)^{\frac{r-p_i}{(N-1)p_i+r}}\right)^{\frac{(N-1)p_i+r}{p_i r}} \left(\fint_Q w_i^{-\frac{p_i r}{p_i-r}}\right)^{\frac{p_i-r}{p_i r}} \\
&= [w^q]_{A_{Nq/r}}^{1/q} \prod_{i=1}^N [w_i^{-\frac{p_i r}{p_i-r}}]_{A_{N(\frac{p_i}{r})'}}^{\frac{p_i-r}{p_i r}},
\end{aligned}$$

since

$$\frac{r-p_i}{(N-1)p_i+r} = \frac{1-p_i/r}{(N-1)p_i/r+1} = -\frac{1}{N\frac{p_i/r}{p_i/r-1}-1} = -\frac{1}{N(\frac{p_i}{r})'-1}.$$

□

A reverse Hölder's property for $\mathbf{A_P}$ was obtained in [17] while one for $\mathbf{A}(\mathbf{P},q)$ was proved by Chen and Wu [2]. For convenience we provide the details for a similar property in terms of the class $\mathbf{A}(\mathbf{P},q,r)$.

Lemma 3.3 (Reverse Hölder's Property for $\mathbf{A}(\mathbf{P},q,r)$) *Assume that* $\mathbf{w} \in \mathbf{A}(\mathbf{P},q,r)$. *Then there exists* $t > r$ *such that* $\mathbf{w} \in \mathbf{A}(\mathbf{P},q,t)$.

Proof By Lemma 3.2, we have that $w_i^{-\frac{p_i r}{p_i-r}} \in A_\infty$ and by the reverse Hölder property for such weights,

$$\left(\fint_Q w_i^{-\frac{\gamma_i p_i r}{p_i-r}}\right)^{1/\gamma_i} \le c_i \fint_Q w_i^{-\frac{p_i r}{p_i-r}}$$

for some $c_i, \gamma_i > 1$. Moreover, it is possible to select γ_i close to 1 and numbers t_i, $r < t_i < p_i$ such that

$$\frac{1}{1/t_i - 1/p_i} = \frac{\gamma_i}{1/r - 1/p_i}$$

or equivalently,

$$\frac{p_i t_i}{p_i - t_i} = \frac{\gamma_i p_i r}{p_i - r}.$$

Let $t = \min\{t_1, \ldots, t_N\}$. We have

$$\begin{aligned}\left(\fint_Q w^q\right)^{1/q} \prod_{i=1}^N \left(\fint_Q w_i^{-\frac{p_i t}{p_i-t}}\right)^{\frac{p_i-t}{p_i t}} &= \left(\fint_Q w^q\right)^{1/q} \prod_{i=1}^N \left(\fint_Q w_i^{-\frac{\gamma_i p_i r}{p_i-r}}\right)^{\frac{1}{\gamma_i}\frac{p_i-r}{p_i r}} \\ &\le \left(\fint_Q w^q\right)^{1/q} \prod_{i=1}^N c_i^{\frac{p_i-r}{p_i r}} \left(\fint_Q w_i^{-\frac{p_i r}{p_i-r}}\right)^{\frac{p_i-r}{p_i r}} \\ &\lesssim [\mathbf{w}]_{\mathbf{A}(\mathbf{P},q,r)}.\end{aligned}$$

□

We will use several maximal functions. For $0 < \alpha < n$ and $f \in L^1_{loc}(\mathbb{R}^n)$, the fractional maximal function $M_\alpha(f)$ is defined by

$$M_\alpha(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)|dy.$$

For $1 \le r < \frac{n}{\alpha}$, let $M_{\alpha,r}(f)(x) = (M_{r\alpha}(|f|^r)(x))^{1/r}$. We also recall the sharp maximal function defined by

$$M^\sharp(f)(x) = \sup_{Q \ni x} \inf_{c \in \mathbb{R}} \frac{1}{|Q|} \int_Q |f(y) - c|dy.$$

and for $\delta > 0$, define $M^\sharp_\delta(f) = M^\sharp(|f|^\delta)^{1/\delta}$.

For $0 \leq \alpha < nN$ and $f = (f_1, \ldots, f_N) \in (L^1_{loc}(\mathbb{R}^n))^N$, the multi(sub)linear fractional maximal function $\mathcal{M}_\alpha(\mathbf{f})$ is defined by

$$\mathcal{M}_\alpha(\mathbf{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{N} \frac{1}{|Q|^{1-\alpha/(nN)}} \int_Q |f_i(y_i)| dy_i.$$

Finally, for $1 \leq r < \frac{nN}{\alpha}$, we also define

$$\mathcal{M}_{\alpha,r}(\mathbf{f})(x) = \sup_{Q \ni x} \prod_{i=1}^{N} \left(\frac{1}{|Q|^{1-r\alpha/(nN)}} \int_Q |f_i(y_i)|^r dy_i \right)^{1/r}.$$

We have the following weighted estimate for $\mathcal{M}_{\alpha,r}(\mathbf{f})$.

Lemma 3.4 *Let* $1 \leq r < p_1, \ldots, p_N, Np < \infty$, $1/p = 1/p_1 + \cdots + 1/p_N$, $0 \leq \alpha < \frac{n}{p}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then* $\mathcal{M}_{\alpha,r}$ *is bounded from* $L^{p_1}(w_1^{p_1}) \times \cdots \times L^{p_N}(w_N^{p_N})$ *to* $L^q(w^q)$ *if and only if* $\mathbf{w} \in \mathbf{A}(\mathbf{P}, q, r)$.

Proof Note that

$$\|\mathcal{M}_{\alpha,r}(\mathbf{f})\|_{L^q(w^q)} = \|\mathcal{M}_{r\alpha}(\mathbf{f}^r)\|^{\frac{1}{r}}_{L^{q/r}(w^q)},$$

where for $\mathbf{f} = (f_1, \cdots, f_N)$, we define $\mathbf{f}^r = (|f_1|^r, \cdots, |f_N|^r)$. For $\alpha = 0$ the result follows then from [17, Theorem 3.7], cf. also [1, Proposition 2.3]. For $\alpha > 0$, the result is a consequence of [23, Theorem 3.6]. □

4 Proof of Theorem 2.3

Once the point-wise estimate of the next lemma is established, the proof of Theorem 2.3 will follow a familiar pattern in the theory using the Fefferman-Stein estimate

$$\int_{\mathbb{R}^n} (M_\delta f)(x)^p w(x) dx \lesssim \int_{\mathbb{R}^n} (M_\delta^\sharp f)(x)^p w(x) dx, \tag{13}$$

which holds for any $0 < p, \delta < \infty$ and any $w \in A_\infty$; see [6].

Lemma 4.1 *Let* T_α *satisfy the hypothesis of Theorem 2.3. If* $0 < \delta < \min(1, \frac{rn}{nN - r\alpha})$, *then for all* $\mathbf{f} \in \prod_{i=1}^N L^{l_i}$ *with* $r < l_1, \ldots, l_N < \infty$,

$$M_\delta^\sharp(T_\alpha(\mathbf{f})) \lesssim \mathcal{M}_{\alpha,r}(\mathbf{f}).$$

Proof Fix $Q \subset \mathbb{R}^n$ and $x \in Q$. It it is enough to show that

$$\left(\frac{1}{|Q|}\int_Q |T_\alpha(\mathbf{f})(z) - c_Q|^\delta dz\right)^{\frac{1}{\delta}} \lesssim \mathcal{M}_{\alpha,r}(\mathbf{f})(x).$$

for some c_Q. The arguments follow a familiar pattern. For $i = 1, \ldots, N$, let $f_i^0 = f_i\chi_{Q^*}$ and $f_i^\infty = f_i(1 - \chi_{Q^*})$. Then $f_i = f_i^0 + f_i^\infty$ and

$$\begin{aligned}\prod_{i=1}^N f_i(y_i) &= \prod_{i=1}^N (f_i^0(y_i) + f_i^\infty(y_i)) \\ &= \prod_{i=1}^N f_i^0(y_i) + \sum_{(\beta_1,\cdots,\beta_N)\in I} f_1^{\beta_1}(y_1)\cdots f_N^{\beta_N}(y_N),\end{aligned}$$

where I is the collection of all N-tuples $\beta = (\beta_1, \cdots, \beta_N)$ with each $\beta_i = 0$ or ∞ and at least on $\beta_j = \infty$. We can write then

$$T_\alpha(\mathbf{f})(z) = T_\alpha(f_1^0, \cdots, f_N^0)(z) + \sum_{\beta\in I} T_\alpha(f_1^{\beta_1}, \cdots, f_N^{\beta_N})(z).$$

Using the weak-type estimate on T_α and applying Kolmogorov's inequality we have for $\mathbf{f}^0 = (f_1^0, \cdots, f_N^0)$,

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q |T_\alpha(\mathbf{f}^0)(z)|^\delta dz\right)^{\frac{1}{\delta}} &\lesssim \|T_\alpha(\mathbf{f}^0)\|_{L^{r^*,\infty}(Q,dz/|Q|)} \\ &\lesssim \frac{1}{|Q^*|^{-\frac{\alpha}{n}}}\left(\prod_{i=1}^N \frac{1}{|Q^*|}\int_{Q^*} |f_i(y_i)|^r dy_i\right)^{1/r} \\ &\lesssim \mathcal{M}_{\alpha,r}(\mathbf{f})(x).\end{aligned}$$

In order to estimate the other terms, we select the constant c_Q to be

$$c_Q = \sum_{\beta\in I} T_\alpha(f_1^{\beta_1}, \ldots, f_N^{\beta_N})(x)$$

and we will show that

$$\sum_{\beta\in I} |T_\alpha(\mathbf{f}^\beta)(z) - T_\alpha(\mathbf{f}^\beta)(x)| \lesssim \mathcal{M}_{\alpha,r}(\mathbf{f})(x), \tag{14}$$

for all $z \in Q$, where $\mathbf{f}^\beta = (f_1^{\beta_1}, \ldots, f_N^{\beta_N})$.

We observe that for any $J = 1, \dots, N-1$,

$$(Q^*)^{N-J} \times (\mathbb{R}^n \setminus Q^*)^J \subset \mathbb{R}^{nN} \setminus (Q^*)^N = \sum_{k=0}^{\infty} (2^{k+1}Q^*)^N \backslash (2^k Q^*)^N,$$

and similarly $(\mathbb{R}^n \setminus Q^*)^N \subset \sum_{k=0}^{\infty} (2^{k+1}Q^*)^N \backslash (2^k Q^*)^N$. Using this observation, Hölder's inequality, and the $L^{r',\alpha}$-Hörmander condition, we have

$$\begin{aligned}
&|T_\alpha(\mathbf{f}^\beta)(z) - T_\alpha(\mathbf{f}^\beta)(x)| \\
&\leq \int_{\mathbb{R}^{nN} \backslash (Q^*)^N} |K(z, \mathbf{y}) - K(x, \mathbf{y})| \prod_{i=1}^{N} |f_i(y_i)| dy_i \\
&\leq \sum_{k=0}^{\infty} \int_{(2^{k+1}Q^*)^N \backslash (2^k Q^*)^N} |K(z, \mathbf{y}) - K(x, \mathbf{y})| \prod_{i=1}^{N} |f_i(y_i)| dy_i \\
&\leq \sum_{k=0}^{\infty} \Big(\int_{(2^{k+1}Q^*)^N \backslash (2^k Q^*)^N} |K(z, \mathbf{y}) - K(x, \mathbf{y})|^{r'} d\mathbf{y} \Big)^{1/r'} \\
&\qquad \times \prod_{i=1}^{N} \Big(\int_{2^{k+1}Q^*} |f_i(y_i)|^r dy_i \Big)^{1/r} \\
&\lesssim \sum_{k=0}^{\infty} |2^k Q^*|^{\frac{N}{r} - \frac{\alpha}{n}} \Big(\int_{(2^{k+1}Q^*)^N \backslash (2^k Q^*)^N} |K(z, \mathbf{y}) - K(x, \mathbf{y})|^{r'} d\mathbf{y} \Big)^{1/r'} \\
&\qquad \times \prod_{i=1}^{N} \Big(\frac{1}{|2^{k+1}Q^*|^{1-\frac{\alpha r}{nN}}} \int_{2^{k+1}Q^*} |f_i(y_i)|^r dy_i \Big)^{1/r} \\
&\lesssim \mathcal{M}_{\alpha,r}(\mathbf{f})(x),
\end{aligned} \tag{15}$$

which gives (14). ☐

Proof of Theorem 2.3. Note that we always have $w^q \in A^\infty$, so applying (13), Lemma 4.1, and then Lemma 3.4, we have

$$\begin{aligned}
\|T_\alpha(\mathbf{f})\|_{L^q(w^q)} \leq& \|M_\delta(T_\alpha(\mathbf{f}))\|_{L^q(w^q)} \leq C \|M_\delta^\sharp(T_\alpha(\mathbf{f}))\|_{L^q(w^q)} \\
\leq& C \|\mathcal{M}_{\alpha,r}(\mathbf{f})\|_{L^q(w^q)} \leq C \prod_{i=1}^{N} \|f_i\|_{L^{p_i}(w_i^{p_i})},
\end{aligned}$$

as we wanted to show. ☐

5 Multipliers

When $\alpha = 0$, it is easy to see that our L^r-Hörmander's condition is weaker than that in [1]. Moreover, our results also extend to the case $\alpha > 0$. In particular, we obtain some new applications to multilinear Fourier multipliers in this last case, extending the results in [19].

We will consider Fourier multiplier T_m given by

$$T_m(\mathbf{f})(x) = \int_{\mathbb{R}^{nN}} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m(\xi_1,\dots,\xi_N)\widehat{f_1}(\xi_1)\dots\widehat{f_N}(\xi_N)d\xi_1\dots d\xi_N$$

for $f_1,\dots,f_N \in \mathcal{S}(\mathbb{R}^n)$, and where the function m satisfies some regularity property defined in terms of Sobolev space estimates.

Recall that for $s \in \mathbb{R}$, the Sobolev space $W^s(\mathbb{R}^{nN})$ consists of all $F \in \mathcal{S}'(\mathbb{R}^{nN})$ such that

$$\|F\|_{W^s(\mathbb{R}^{nN})} = \left(\int_{\mathbb{R}^{nN}} (1+|\xi|^2)^s|\widehat{F}(\xi)|^2\, d\xi_1\dots d\xi_N\right)^{1/2} < \infty,$$

where $\xi = (\xi_1,\dots,\xi_N) \in \mathbb{R}^n \times\cdots\times \mathbb{R}^n$.

Let $\Psi \in \mathcal{S}(\mathbb{R}^{nN})$ be such that supp $\Psi \subset \{\xi \in \mathbb{R}^{nN} : 1/2 \le |\xi| \le 2\}$ and $\sum_{k\in\mathbb{Z}} \Psi(\xi/2^k) = 1$ for all $\xi \in \mathbb{R}^{nN}\backslash\{\mathbf{0}\}$. For a function m, $\alpha > 0$ and $k \in \mathbb{Z}$ we set

$$m_k^\alpha(\xi) = 2^{k\alpha} m(2^k\xi)\Psi(\xi).$$

We will use Theorem 2.3 to prove the next result. The version for $\alpha = 0$ is in [19]. Our proof relies on some computations used in that article and those in [7].

Theorem 5.1 *Let* $\mathbf{P} = (p_1,\dots,p_N)$ *with* $1 < p_1,\dots,p_N < \infty$ *and* $1/p = \sum_{i=1}^N 1/p_i$, $0 < \alpha < n/p$, $1/q = 1/p - \alpha/n$, $s > nN/2$, *and* $r = \max\left\{1, \frac{nN}{s+\alpha}\right\} < p_1,\dots,p_N, pN$. *Suppose that* m *satisfies*

$$\sup_{k\in\mathbb{Z}} \|m_k^\alpha\|_{W^s(\mathbb{R}^{nN})} < \infty. \tag{16}$$

Then for all $\mathbf{w} = (w_1,\dots,w_N) \in \mathbf{A}(\mathbf{P}, q, r)$ *and* $w = \prod_{i=1}^N w_i$, T_m *is bounded from* $L^{p_1}(w_1^{p_1}) \times\cdots\times L^{p_N}(w_N^{p_N})$ *to* $L^q(w^q)$ *with* $\|T_m\| \lesssim \sup_{k\in\mathbb{Z}} \|m_k^\alpha\|_{W^s(\mathbb{R}^n)}$.

Proof of Theorem 5.1. Since $\mathbf{w} \in \mathbf{A}(\mathbf{P}, q, r)$, by Lemma 3.3, there exists $r < \bar{r} < p_1,\cdots,p_N, pN$ such that $\mathbf{w} \in \mathbf{A}(\mathbf{P}, q, \bar{r})$. In order to prove Theorem 5.1, if suffices to prove that the hypothesis in Theorem 2.3 are satisfied with r replaced by $\bar{r}$.

We will establish the unweighted $L^{p_1} \times\cdots\times L^{p_N} \to L^q$ boundedness of the multiplier for any $(N+1)$-tuples $(p_1,\dots,p_N, q)$ with $1/q = 1/p - \alpha/n$ and $r = \max\left\{1, \frac{nN}{s+\alpha}\right\} < p_1,\dots,p_N, pN$. This gives a fortiori the weak-type estimate in the hypotheses of Theorem 2.3 for $\bar{r}$.

We need first to borrow and/or adapt to our situation some lemmata from [7], which we include here for the reader's convenience.

Lemma 5.2 *Let* $1 < p, q, r < \infty$ *with* $1/q = 1/p - \alpha/n$. *Then*

$$\Big\| \Big\{ \sum_{k\in\mathbb{Z}} (M_\alpha f_k)^r \Big\}^{1/r} \Big\|_{L^q} \lesssim \Big\| \Big\{ \sum_{k\in\mathbb{Z}} |f_k|^r \Big\}^{1/r} \Big\|_{L^p}$$

for all sequences $\{f_k\}_{k\in\mathbb{Z}}$ *of locally integrable functions on* $\mathbb{R}^n$.

Proof This is a standard result that will be used in place of [7, Lemma 2.3]. Indeed, let $\mathcal{I}_\alpha$ be the fractional integral operator defined by

$$\mathcal{I}_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

It is well know that $\|\mathcal{I}_\alpha(f)\|_{L^q} \lesssim \|f\|_{L^p}$. Since $\mathcal{I}_\alpha$ is a positive operator, the above inequality has a vector-valued extension (see for example [9, Proposition 4.5.10]):

$$\Big\| \Big\{ \sum_{k\in\mathbb{Z}} (\mathcal{I}_\alpha |f_k|)^r \Big\}^{1/r} \Big\|_{L^q} \lesssim \Big\| \Big\{ \sum_{k\in\mathbb{Z}} |f_k|^r \Big\}^{1/r} \Big\|_{L^p}.$$

The result to be proved follows now from pointwise estimate $M_\alpha f_k(x) \lesssim \mathcal{I}_\alpha(|f_k|)(x)$. □

Lemma 5.3 ([27, Lemma 3.3] and [7, Lemma A.1]) *Let* $R > 0$, $2 \le l < \infty$, $s \ge 0$ *and* $w_s(u) = (1+|u|^2)^{s/2}$ *for* $u \in \mathbb{R}^{nN}$. *Then there exists a constant* $C > 0$ *such that*

$$\|\widehat{F}\|_{L^l(w_s)} \le C\|F\|_{W^{s/l}} \quad \textit{for all } F \in W^{s/l} \textit{ with supp } F \subset \{|u| \le R\}.$$

Lemma 5.4 (cf. [7, Lemma 3.2]) *Let* $R > 0$, $s > nN/2$, $0 < \alpha < nN/2$, *and* $\max\{1, \frac{nN}{s+\alpha}\} < l < 2$ *for* $1 \le i \le N$. *Then*

$$\Big| \int_{\mathbb{R}^{Nn}} 2^{j(Nn-\alpha)} \widehat{m}(2^j(x-y_1), \dots, 2^j(x-y_N)) f_1(y_1)\dots f_N(y_N) dy_1 \dots dy_N \Big|$$

$$\lesssim \|m\|_{W^s} \prod_{i=1}^N M_{\alpha l/N}(|f_i|^l)(x)^{1/l}$$

for all $j \in \mathbb{Z}$, *all* $m \in W^s(\mathbb{R}^{Nn})$ *with* $\operatorname{supp} m \subset \{\sqrt{|\xi_1|^2 + \dots + |\xi_N|^2} \le R\}$, *and all* $f_1, \dots, f_N \in \mathcal{S}(\mathbb{R}^n)$.

Proof For $x \in \mathbb{R}^n$, let $\bar{\mathbf{x}} = (x, \dots, x) \in \mathbb{R}^{Nn}$. By Hölder's inequality

$$\left| \int_{\mathbb{R}^{Nn}} 2^{j(Nn-\alpha)} \widehat{m}(2^j(x-y_1), \dots, 2^j(x-y_N)) f_1(y_1) \dots f_N(y_N) dy_1 \dots dy_N \right|$$
$$= 2^{j(Nn-\alpha)} \left| \int_{\mathbb{R}^{Nn}} \hat{m}(2^j(\bar{\mathbf{x}}-\mathbf{y})) \frac{(1+2^j|\bar{\mathbf{x}}-\mathbf{y}|)^s}{(1+2^j|\bar{\mathbf{x}}-\mathbf{y}|)^s} f_1(y_1) \dots f_N(y_N) dy_1 \dots dy_N \right|$$
$$\leq 2^{j(\frac{Nn}{l}-\alpha)} \left(\int_{\mathbb{R}^{Nn}} |\hat{m}(\mathbf{y})|^{l'} (1+|\mathbf{y}|)^{sl'} d\mathbf{y} \right)^{\frac{1}{l'}} \left(\int_{\mathbb{R}^{Nn}} \frac{|f_1(y_1) \dots f_N(y_N)|^l}{(1+2^j|\bar{\mathbf{x}}-\mathbf{y}|)^{sl}} d\mathbf{y} \right)^{\frac{1}{l}}$$
$$\lesssim \|\widehat{m}\|_{L^{l'}(w_{sl'})} \times \prod_{i=1}^{N} \left(\int_{\mathbb{R}^n} \frac{2^{j(n-l\alpha/N)} |f_i(y_i)|^l}{(1+2^j|x-y_i|)^{sl/N}} dy_i \right)^{\frac{1}{l}}$$
$$\lesssim \|\widehat{m}\|_{L^{l'}(w_{sl'})} \times \prod_{i=1}^{N} M_{l\alpha/N}(|f_i|^l)(x)^{1/l},$$

where in the last inequality we have used the basic estimate

$$\sup_{R>0} \left(R^{n-\beta} \int_{\mathbb{R}^n} \frac{|f(y)|}{(1+R|x-y|)^{n-\beta+\epsilon}} dy \right) \lesssim M_\beta f(x)$$

with $R = 2^j$, $0 < \beta = \alpha l/N < n$ and $\epsilon = sl/N - (n - l\alpha/N) > 0$ because of the choice of l. Since also $2 < l_i' < \infty$, it follows from Lemma 5.3 that $\|\widehat{m}\|_{L^{l'}(w_{sl'})} \leq C\|m\|_{W^s}$, which concludes the proof. □

We are now ready to prove the unweighted $L^{p_1} \times \cdots \times L^{p_N} \to L^q$ boundedness mentioned before. We will decompose T_m following the same idea as in the proof of [7, Theorem 6.2], from where we borrow the following auxiliary functions.

Let ϕ_1 be a $\mathcal{C}^\infty$-function on $[0, \infty)$ satisfying

$$\phi_1(t) = 1 \quad \text{on} \quad [0, 1/(4N)], \qquad \operatorname{supp} \phi_1 \subset [0, 1/(2N)].$$

Let $\phi_2(t) = 1 - \phi_1(t)$ and for $(i_1, \dots, i_N) \in \{1, 2\}^N$, define $\Phi_{i_1,\dots,i_N}$ on $\mathbb{R}^{Nn} \backslash \{0\}$ by

$$\Phi_{(i_1,\dots,i_N)}(\xi) = \phi_{i_1}(|\xi_1|/|\xi|)\phi_{i_2}(|\xi_2|/|\xi|) \dots \phi_{i_N}(|\xi_N|/|\xi|), \tag{17}$$

where $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^n \times \cdots \times \mathbb{R}^n$. As noted in [7], $\Phi_{(1,1,\dots,1)} = 0$.

Lemma 5.5 ([7, Lemma 3.1]) *Let $\Phi_{(i_1,\ldots,i_N)}$ be the same as in (17). Then the following are true:*

(1) *For $\xi \in \mathbb{R}^{Nn}\backslash\{\mathbf{0}\}$,*

$$\sum_{\substack{(i_1,i_2,\ldots,i_N)\in\{1,2\}^N\\(i_1,i_2,\ldots,i_N)\neq(1,\ldots,1)}} \Phi_{(i_1,i_2,\ldots,i_N)}(\xi) = 1.$$

(2) *For $(i_1,\ldots,i_N) \in \{1,2\}^N$ and $(\alpha_1,\ldots,\alpha_N) \in \mathbb{Z}_+^n \times \cdots \times \mathbb{Z}_+^n$ there exists a constant $C_{i_1,i_2,\ldots,i_N}^{\alpha_1,\ldots,\alpha_N} > 0$ such that*

$$|\partial_{\xi_1}^{\alpha_1}\ldots\partial_{\xi_N}^{\alpha_N}\Phi_{(i_1,\ldots,i_N)}(\xi)| \le C_{i_1,i_2,\ldots,i_N}^{\alpha_1,\ldots,\alpha_N}(|\xi_1|+\cdots+|\xi_N|)^{-(|\alpha_1|+\cdots+|\alpha_N|)}$$

for all $\xi \in \mathbb{R}^{Nn}\backslash\{\mathbf{0}\}$

(3) *If $i_j = 2$ for some $1 \le j \le N$ and $i_k = 1$ for all $1 \le k \le N$ with $k \neq j$, then supp $\Phi_{(i_1,\ldots,i_N)} \subset \{(\xi_1,\ldots,\xi_N) : |\xi_k| \le |\xi_j|/N$ for $k \neq j\}$. If $i_j = i_{j'} = 2$ for some $1 \le j,j' \le N$ with $j \neq j'$, then supp $\Phi_{(i_1,\ldots,i_N)} \subset \{(\xi_1,\ldots,\xi_N) : |\xi_j|/(4N) \le |\xi_{j'}| \le 4N|\xi_j|, |\xi_k| \le 4N|\xi_j|$ for $k \neq j,j'\}$.*

We also choose $\varphi, \psi, \widetilde{\psi}, \zeta, \widetilde{\zeta} \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\begin{aligned}
&\text{supp } \varphi \subset \{|\eta| \le 16N\}, && \varphi = 1 \text{ on } \{|\eta| \le 8N\},\\
&\text{supp } \psi \subset \{1/2 \le |\eta| \le 2\}, && \sum_{j\in\mathbb{Z}} \psi(\eta/2^j) = 1 \quad \text{for all } \eta \neq 0,\\
&\text{supp } \widetilde{\psi} \subset \{1/4 \le |\eta| \le 4\}, && \widetilde{\psi} = 1 \text{ on } \{1/2 \le |\eta| \le 2\},\\
&\text{supp } \zeta \subset \{1/(16N) \le |\eta| \le 16N\}, && \zeta = 1 \text{ on } \{1/(8N) \le |\eta| \le 8N\},\\
&\text{supp } \widetilde{\zeta} \subset \{1/(32N) \le |\eta| \le 32N\}, && \widetilde{\zeta} = 1 \text{ on } \{1/(16N) \le |\eta| \le 16N\}.
\end{aligned}$$

If ρ is any of the above functions, then $(\rho(D/2^j)f)(x)$ will denote as usual the convolution operator defined by

$$(\rho(D/2^j)f)\widehat{\ }(\xi) = \rho(\xi/2^j)\hat{f}(\xi)$$

We decompose m as follows.

$$m(\xi) = \sum_{\substack{(i_1,\cdots,i_N)\in\{1,2\}^N\\(i_1,\cdots,i_N)\neq(1,\cdots,1)}} \Phi_{(i_1,\cdots,i_N)}(\xi)m(\xi) =: \sum_{\substack{(i_1,\cdots,i_N)\in\{1,2\}^N\\(i_1,\cdots,i_N)\neq(1,\cdots,1)}} m_{(i_1,\cdots,i_N)}(\xi),$$

where $\Phi_{(i_1,\cdots,i_N)}$ are the same as in (17).

The computations are similar as in [7, Sect. 4] so we consider two separate cases where $(i_1, \ldots, i_N)$ satisfies $\sharp\{j : i_j = 2\} = 1$ and $\sharp\{j : i_j = 2\} \geq 2$ which we briefly describe. The reader should consult [7] for more details.

For the first one, we may assume that $i_1 = 2$ and $i_k = 1$ for $2 \leq k \leq N$. Lemma 5.5 (3) implies supp $m_{(i_1,\ldots,i_N)} \subset \{|\xi_k| \leq |\xi_1|/N \text{ for } 2 \leq k \leq N\}$ and we can write for some k_0 such that $2^{k_0} > N$,

$$\begin{aligned}
&\psi(D/2^j)T_{m_{(i_1,\ldots,i_N)}}(f_1, \ldots, f_N)(x) \\
&= \int_{\mathbb{R}^{Nn}} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m_{(i_1,\ldots,i_N)}(\xi)\psi((\xi_1 + \cdots + \xi_N)/2^j)\widehat{f_1}(\xi_1)\ldots\widehat{f_N}(\xi_N)d\xi \\
&= \sum_{k=-2}^{k_0+1} \int_{\mathbb{R}^{Nn}} e^{ix\cdot(\xi_1+\cdots+\xi_N)} m_{(i_1,\ldots,i_N)}(\xi)\psi((\xi_1 + \cdots + \xi_N)/2^j) \\
&\times \psi(\xi_1/2^{j+k})\varphi(\xi_2/2^{j+k})\ldots\varphi(\xi_N/2^{j+k})\widetilde{\psi}(\xi_1/2^{j+k})\widehat{f_1}(\xi_1)\ldots\widehat{f_N}(\xi_N)d\xi \\
&= \sum_{k=-2}^{k_0+1} \int_{\mathbb{R}^{Nn}} 2^{(j+k)(Nn-\alpha)}\widehat{m_{(i_1,\ldots,i_N)}^{j,k,\alpha}}(2^{j+k}(x-y_1), \ldots, 2^{j+k}(x-y_N)) \\
&\times (\widetilde{\psi}(D/2^{j+k})f_1)(y_1)\ldots f_N(y_N)dy_1\ldots dy_N,
\end{aligned}$$

where

$$\begin{aligned}
&m_{(i_1,\ldots,i_N)}^{j,k,\alpha}(\xi) = \\
&2^{(j+k)\alpha}m_{(i_1,\ldots,i_N)}(2^{j+k}\xi)\psi(2^k(\xi_1 + \cdots + \xi_N))\psi(\xi_1)\varphi(\xi_2)\ldots\varphi(\xi_N).
\end{aligned}$$

As in [7, (4.7)],

$$\sup_{\substack{j\in\mathbb{Z},\\ -2\leq k|\leq k_0+1}} \|m_{(i_1,\ldots,i_N)}^{j,k,\alpha}\|_{W^s} \lesssim \sup_{k\in\mathbb{Z}} \|m_k^\alpha\|_{W^s}.$$

We choose l such that $r = \max\left\{1, \frac{nN}{s+\alpha}\right\} < l < \min\{2, p_1, \ldots, p_N\}$. Since supp $m_{(i_1,\ldots,i_N)}^{j,k,\alpha} \subset \{|\xi| \leq 16N^{\frac{3}{2}}\}$, it follows from Lemma 5.4 that

$$\begin{aligned}
&|\psi(D/2^j)T_{m_{(i_1,\ldots,i_M)}}(f_1, \ldots, f_N)(x)| \\
&\lesssim \sum_{k=-2}^{k_0+1} \|m_{(i_1,\ldots,i_M)}^{j,k,\alpha}\|_{W^s} \left(M_{l\alpha/N}(|\widetilde{\psi}(D/2^{j+k})f_1|^l)(x)\right)^{\frac{1}{l}} \\
&\times \left(M_{l\alpha/N}(|f_2|^l)(x)\right)^{\frac{1}{l}} \ldots \left(M_{l\alpha/N}(|f_N|^l)(x)\right)^{\frac{1}{l}}
\end{aligned}$$

$$\lesssim \left(\sup_{k\in\mathbb{Z}} \|m_k^\alpha\|_{W^s}\right) \sum_{k=-2}^{k_0+1} \left(M_{l\alpha/N}(|\widetilde{\psi}(D/2^{j+k})f_1|^l)(x)\right)^{\frac{1}{l}}$$
$$\times \left(M_{l\alpha/N}(|f_2|^l)(x)\right)^{\frac{1}{l}} \dots \left(M_{l\alpha/N}(|f_N|^l)(x)\right)^{\frac{1}{l}} .$$

Let q_i satisfy $1/q_i = 1/p_i - \alpha/Nn$ for $1 \le i \le N$. Then $1/q = \sum_{i=1}^N 1/q_i$. By Fujita and Tomita [7, Remark 2.6], Hölder's inequality, Lemmas 5.4 and 5.2,

$$\|T_{m_{(i_1,\dots,i_N)}}(f_1,\dots,f_N)\|_{L^q}$$
$$\lesssim \left\| \left(\sum_{j\in\mathbb{Z}} |\psi(D/2^j)T_{m_{(i_1,\dots,i_N)}}(f_1,\dots,f_N)|^2 \right)^{1/2} \right\|_{L^q}$$
$$\lesssim \sum_{k=-2}^{k_0+1} \left(\sup_{k\in\mathbb{Z}} \|m_k^\alpha\|_{W^s}\right) \left\| \left(\sum_{j\in\mathbb{Z}} M_{l\alpha/N}(|\widetilde{\psi}(D/2^{j+k})f_1|^l)^{2/l} \right)^{1/2} \right\|_{L^{q_1}}$$
$$\times \|M_{l\alpha/N}(|f_2|^l)^{\frac{1}{l}}\|_{L^{q_2}} \dots \|M_{l\alpha/N}(|f_N|^l)^{\frac{1}{l}}\|_{L^{q_N}}$$
$$\lesssim \sum_{k=-2}^{k_0+1} \left(\sup_{k\in\mathbb{Z}} \|m_k^\alpha\|_{W^s}\right) \left\| \left(\sum_{j\in\mathbb{Z}} |\widetilde{\psi}(D/2^{j+k})f_1|^2 \right)^{1/2} \right\|_{L^{p_1}} \|f_2\|_{L^{p_2}} \dots \|f_N\|_{L^{p_N}}$$
$$\lesssim \left(\sup_{k\in\mathbb{Z}} \|m_k^\alpha\|_{W^s}\right) \|f_1\|_{L^{p_1}} \dots \|f_N\|_{L^{p_N}},$$

Consider now the case where $\sharp\{j : i_j = 2\} \ge 2$. Assume that $i_1 = i_2 = 2$. By Lemma 5.5, $\operatorname{supp} m_{(i_1,\dots,i_N)} \subset \{|\xi_1|/(4N) \le |\xi_2| \le 4N|\xi_1|,\ |\xi_i| \le 4N|\xi_1| \text{ for } 3 \le i \le N\}$. This implies that if $\xi_1 \in \operatorname{supp} \psi(\cdot/2^k)$ and $(\xi_1,\dots,\xi_N) \in \operatorname{supp} m_{(i_1,\dots,i_N)}$, then $2^{k-3}/N \le |\xi_2| \le 2^{k+3}N$ and $|\xi_i| \le 2^{k+3}N$, and consequently $\zeta(\xi_2/2^k) = 1$ and $\varphi(\xi_i/2^k) = 1$, for $3 \le i \le N$. Hence,

$$T_{m_{(i_1,\dots,i_N)}}(f_1,\dots,f_N)(x)$$
$$= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^{Nn}} e^{ix\cdot(\xi_1+\dots+\xi_N)} m_{(i_1,\dots,i_N)}(\xi)\psi(\xi_1/2^k)\zeta(\xi_2/2^k)\varphi(\xi_3/2^k)\dots\varphi(\xi_N/2^k)$$
$$\times (\widetilde{\psi}(\xi_1/2^k)\widehat{f_1}(\xi_1))(\widetilde{\zeta}(\xi_1/2^k)\widehat{f_1}(\xi_1))\widehat{f_3}(\xi_3)\dots\widehat{f_N}(\xi_N)d\xi_1\dots d\xi_N$$
$$= \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^{Nn}} 2^{k(Nn-\alpha)} \widehat{m_{(i_1,\dots,i_N)}^{k,\alpha}}(2^k(x-y_1),\dots,2^k(x-y_N))$$
$$\times \widetilde{\psi}(D/2^k)f_1(y_1)\widetilde{\zeta}(D/2^k)f_2(y_2)f_3(y_3)\dots f_N(y_N)dy_1\dots dy_N,$$

where

$$m^{k,\alpha}_{(i_1,\dots,i_N)}(\xi) = 2^{k\alpha} m_{(i_1,\dots,i_N)}(2^k\xi)\psi(\xi_1)\zeta(\xi_2)\varphi(\xi_3/2^k)\dots\varphi(\xi_N/2^k).$$

Utilizing Lemmas 5.3 and 5.4, we have with l as before

$$\begin{aligned}
&|T_{m_{(i_1,\dots,i_M)}}(f_1,\dots,f_N)(x)| \\
&\lesssim \sum_{k\in\mathbb{Z}} \|m^{k,\alpha}_{(i_1,\dots,i_M)}\|_{W^s} \left(M_{l\alpha/N}(|f_3|^l)(x)\right)^{\frac{1}{l}} \dots \left(M_{l\alpha/N}(|f_N|^l)(x)\right)^{\frac{1}{l}} \\
&\quad \times \left(M_{l\alpha/N}(|\widetilde{\psi}(D/2^k)f_1|^l)(x)\right)^{\frac{1}{l}} \left(M_{l\alpha/N}(|\widetilde{\zeta}(D/2^k)f_2|^l)(x)\right)^{\frac{1}{l}} \\
&\lesssim \left(\sup_{k\in\mathbb{Z}} \|m^{k,\alpha}_{(i_1,\dots,i_M)}\|_{W^s}\right) \left(M_{l\alpha/N}(|f_3|^l)(x)\right)^{\frac{1}{l}} \dots \left(M_{l\alpha/N}(|f_N|^l)(x)\right)^{\frac{1}{l}} \\
&\quad \times \left(\sum_{k\in\mathbb{Z}} M_{l\alpha/N}(|\widetilde{\psi}(D/2^k)f_1|^l)(x)^{\frac{2}{l}}\right)^{\frac{1}{2}} \left(\sum_{k\in\mathbb{Z}} M_{l\alpha/N}(|\widetilde{\zeta}(D/2^k)f_2|^l)(x)^{\frac{2}{l}}\right)^{\frac{1}{2}}.
\end{aligned}$$

We can now proceed as in the previous case. The rest of the cases are similar too. We finally obtain

$$\|T_{m_{(i_1,\dots,i_N)}}\|_{L^{p_1}\times\dots\times L^{p_N}\to L^q} \lesssim \sup_{k\in\mathbb{Z}} \|m^{k,\alpha}_{(i_1,\dots,i_M)}\|_{W^s} \lesssim \sup_{k\in\mathbb{Z}} \|m^{\alpha}_k\|_{W^s}.$$

It remains to show that T_m verifies the multilinear $L^{\bar{r}',\alpha}$-Hörmander condition. We need to find estimate involving the kernel of T_M using the information on m. The computations are in general familiar in the subject, and in particular similar to those in [18]. Let $m_j = m(\cdot)\psi(\cdot/2^j)$, where $\psi \in \mathcal{S}(\mathbb{R}^{nN})$ with supp $\{\xi \in \mathbb{R}^{nN} : 1/2 \le |\xi| \le 2\}$ and $\sum_{j\in\mathbb{Z}} \psi(2^{-j}\xi) = 1$ for $\xi \ne 0$. Let $K_j(x,\mathbf{y}) = \check{m}_j(x-y_1,\dots,x-y_N)$. We write

$$\begin{aligned}
&\sum_{k=0}^{\infty} |2^k Q|^{\frac{N}{\bar{r}}-\frac{\alpha}{n}} \left(\int_{(2^{k+1}Q)^N\setminus(2^kQ)^N} |K(x,\mathbf{y})-K(z,\mathbf{y})|^{\bar{r}'} d\mathbf{y}\right)^{1/\bar{r}'} \\
&\quad \le \sum_{k=0}^{\infty} |2^k Q|^{\frac{N}{\bar{r}}-\frac{\alpha}{n}} \sum_{j\in\mathbb{Z}} \left(\int_{(2^{k+1}Q)^N\setminus(2^kQ)^N} |K_j(z,\mathbf{y})-K_j(x,\mathbf{y})|^{\bar{r}'} d\mathbf{y}\right)^{\frac{1}{\bar{r}'}} \\
&\quad := \sum_{k=0}^{\infty} |2^k Q|^{\frac{N}{\bar{r}}-\frac{\alpha}{n}} \sum_{j\in\mathbb{Z}} J_{j,k}.
\end{aligned}$$

Let $h = z - x$ and $\widetilde{Q} = x - Q$. As done before, for any $u \in \mathbb{R}^n$, we write $\bar{\mathbf{u}} = (u, \ldots, u) \in \mathbb{R}^{nN}$. We have

$$
\begin{aligned}
J_{j,k} &= \left(\int_{(2^{k+1}Q)^N \setminus (2^k Q)^N} |\check{m}_j(\bar{\mathbf{z}} - \mathbf{y}) - \check{m}_j(\bar{\mathbf{x}} - \mathbf{y})|^{\bar{r}'} d\mathbf{y} \right)^{\frac{1}{\bar{r}'}} \\
&= \left(\int_{(2^{k+1}\widetilde{Q})^N \setminus (2^k \widetilde{Q})^N} |\check{m}_j(\bar{\mathbf{h}} + \mathbf{y}) - \check{m}_j(\mathbf{y})|^{\bar{r}'} d\mathbf{y} \right)^{\frac{1}{\bar{r}'}} \\
&\lesssim \left(\int_{|\mathbf{y}| \sim 2^k \ell(Q)} |\check{m}_j(\mathbf{y})|^{\bar{r}'} d\mathbf{y} \right)^{\frac{1}{\bar{r}'}} \\
&\lesssim (2^k \ell(Q))^{-s} \left(\int_{\mathbb{R}^{nN}} |2^{-j}\mathbf{y}|^{s\bar{r}'} |\check{m}_j(2^{-j}\mathbf{y})|^{\bar{r}'} 2^{-jnN} d\mathbf{y} \right)^{\frac{1}{\bar{r}'}} \\
&\lesssim (2^k \ell(Q))^{-s} \left(\int_{\mathbb{R}^{nN}} (1 + |\mathbf{y}|^2)^{\frac{s\bar{r}'}{2}} |2^{j\alpha} 2^{-jnN} \check{m}_j(2^{-j}y)|^{\bar{r}'} d\mathbf{y} \right)^{\frac{1}{\bar{r}'}} \\
&\quad \times 2^{j(\frac{nN}{\bar{r}} - s - \alpha)} \\
&\lesssim (2^k \ell(Q))^{-s} 2^{j(\frac{nN}{\bar{r}} - s - \alpha)} \|m_j^\alpha\|_{W^s}.
\end{aligned}
$$

Let $\ell(Q) \sim 2^{-l}$. Then

$$
\sum_{j \geq l} J_{j,k} \lesssim \sup_{j \in \mathbb{Z}} \|m_j^\alpha\|_{W^s} 2^{-ks} \ell(Q)^{\alpha - \frac{nN}{\bar{r}}}, \tag{18}
$$

because $\frac{nN}{\bar{r}} - s - \alpha < 0$.

We can also compute

$$
\begin{aligned}
J_{j,k} &= \left(\int_{(2^{k+1}\widetilde{Q})^N \setminus (2^k \widetilde{Q})^N} |\check{m}_j(h + y_1, \ldots, h + y_N) - \check{m}_j(y_1, \ldots, y_N)|^{\bar{r}'} d\mathbf{y} \right)^{\frac{1}{\bar{r}'}} \\
&\leq \left(\int_{(2^{k+1}\widetilde{Q})^N \setminus (2^k \widetilde{Q})^N} \left(\int_0^1 |\bar{\mathbf{h}} \cdot \nabla \check{m}_j(y_1 + \theta h, \ldots, y_N + \theta h)| d\theta \right)^{\bar{r}'} d\mathbf{y} \right)^{\frac{1}{\bar{r}'}} \\
&\leq \left(\int_0^1 \left(\int_{(2^{k+1}\widetilde{Q})^N \setminus (2^k \widetilde{Q})^N} |\bar{\mathbf{h}} \cdot \nabla \check{m}_j(y_1 + \theta h, \ldots, y_N + \theta h)|^{\bar{r}'} d\mathbf{y} \right)^{\frac{1}{\bar{r}'}} d\theta \right) \\
&\lesssim \left(\int_{|\mathbf{y}| \sim 2^k \ell(Q)} |\bar{\mathbf{h}} \cdot \nabla \check{m}_j(y_1, \ldots, y_N)|^{\bar{r}'} d\mathbf{y} \right)^{\frac{1}{\bar{r}'}}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
J_{j,k} &\lesssim \sum_{i=1}^{nN} \ell(Q)\Big(\int_{|y|\sim 2^k\ell(Q)} |\partial_i \check{m}_j(y_1,\dots,y_N)|^{\bar r'}\,d\mathbf{y}\Big)^{\frac{1}{\bar r'}} \\
&\lesssim \sum_{i=1}^{nN} \ell(Q)(2^k\ell(Q))^{-s}\Big(\int_{\mathbb{R}^{nN}} (1+|y|^2)^{\frac{s\bar r'}{2}} \\
&\qquad \times |2^{j\alpha}2^{-jnN}\partial_i\check{m}_j(2^{-j}y_1,\dots,2^{-j}y_N)|^{\bar r'}\,d\mathbf{y}\Big)^{\frac{1}{\bar r'}} 2^{j(\frac{nN}{\bar r}-s-\alpha)} \qquad (19)\\
&\lesssim \sum_{i=1}^{nN} \ell(Q)(2^k\ell(Q))^{-s}2^{j(\frac{nN}{\bar r}-s-\alpha)}2^j\|2^{j\alpha}m(2^j\xi)\xi_i\psi(\xi)\|_{W^s} \\
&\lesssim \sum_{i=1}^{nN} \ell(Q)(2^k\ell(Q))^{-s}2^{j(\frac{nN}{\bar r}-s-\alpha+1)}\sup_{j\in\mathbb{Z}}\|m_j^\alpha\|_{W^s}.
\end{aligned}
$$

Since $\bar r$ can be selected so that $\frac{nN}{\bar r} - s - \alpha + 1 > 0$, we have

$$
\sum_{j<l} J_{j,k} \lesssim \sup_{j\in\mathbb{Z}}\|m_j^\alpha\|_{W^s}2^{-ks}\ell(Q)^{\alpha-\frac{nN}{\bar r}}. \qquad (20)
$$

Finally, from (18) with (20) we obtain

$$
\begin{aligned}
&\sum_{k=0}^{\infty}|2^kQ|^{\frac{N}{\bar r}-\frac{\alpha}{n}}\left(\int_{(2^{k+1}Q)^N\setminus(2^kQ)^N}|K(x,\mathbf{y})-K(z,\mathbf{y})|^{\bar r'}\,d\mathbf{y}\right)^{1/\bar r'} \\
&\quad\lesssim \sum_{k=0}^{\infty}|2^kQ|^{\frac{N}{\bar r}-\frac{\alpha}{n}}\Big(\sup_{j\in\mathbb{Z}}\|m_j^\alpha\|_{W^s}\Big)2^{-ks}\ell(Q)^{\alpha-\frac{nN}{\bar r}} \\
&\quad= \sup_{j\in\mathbb{Z}}\|m_j^\alpha\|_{W^s}\sum_{k=0}^{\infty}2^{-k(s+\alpha-\frac{nN}{\bar r})} \\
&\quad\lesssim \sup_{j\in\mathbb{Z}}\|m_j^\alpha\|_{W^s},
\end{aligned}
$$

which concludes the proof. □

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Weighted Norm Inequalities of $(1, q)$-Type for Integral and Fractional Maximal Operators

Stephen Quinn and Igor E. Verbitsky

Dedicated to Richard L. Wheeden

Abstract We study weighted norm inequalities of $(1, q)$- type for $0 < q < 1$,

$$\|\mathbf{G}\nu\|_{L^q(\Omega, d\sigma)} \le C\,\|\nu\|, \quad \text{for all positive measures } \nu \text{ in } \Omega,$$

along with their weak-type counterparts, where $\|\nu\| = \nu(\Omega)$, and G is an integral operator with nonnegative kernel,

$$\mathbf{G}\nu(x) = \int_\Omega G(x, y)d\nu(y).$$

These problems are motivated by sublinear elliptic equations in a domain $\Omega \subset \mathbb{R}^n$ with non-trivial Green's function $G(x, y)$ associated with the Laplacian, fractional Laplacian, or more general elliptic operator.

We also treat fractional maximal operators M_α $(0 \le \alpha < n)$ on $\mathbb{R}^n$, and characterize strong- and weak-type $(1, q)$-inequalities for M_α and more general maximal operators, as well as $(1, q)$-Carleson measure inequalities for Poisson integrals.

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S. Quinn • I.E. Verbitsky (✉)
Department of Mathematics, University of Missouri, Columbia, MO 65211, USA
e-mail: stephen.quinn@mail.missouri.edu; verbitskyi@missouri.edu

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1 Introduction

In this paper, we discuss recent results on weighted norm inequalities of $(1, q)$- type in the case $0 < q < 1$,

$$\|\mathbf{G}\nu\|_{L^q(\Omega, d\sigma)} \leq C\,\|\nu\|, \tag{1}$$

for all positive measures ν in Ω, where $\|\nu\| = \nu(\Omega)$, and $\mathbf{G}$ is an integral operator with nonnegative kernel,

$$\mathbf{G}\nu(x) = \int_\Omega G(x, y) d\nu(y).$$

Such problems are motivated by sublinear elliptic equations of the type

$$\begin{cases} -\Delta u = \sigma u^q & \text{in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases}$$

in the case $0 < q < 1$, where Ω is an open set in $\mathbb{R}^n$ with non-trivial Green's function $G(x, y)$, and $\sigma \geq 0$ is an arbitrary locally integrable function, or locally finite measure in Ω.

The only restrictions imposed on the kernel G are that it is quasi-symmetric and satisfies a weak maximum principle. In particular, $\mathbf{G}$ can be a Green operator associated with the Laplacian, a more general elliptic operator (including the fractional Laplacian), or a convolution operator on $\mathbb{R}^n$ with radially symmetric decreasing kernel $G(x, y) = k(|x - y|)$ (see [1, 12]).

As an example, we consider in detail the one-dimensional case where $\Omega = \mathbb{R}_+$ and $G(x, y) = \min(x, y)$. We deduce explicit characterizations of the corresponding $(1, q)$-weighted norm inequalities, give explicit necessary and sufficient conditions for the existence of weak solutions, and obtain sharp two-sided pointwise estimates of solutions.

We also characterize weak-type counterparts of (1), namely,

$$\|\mathbf{G}\nu\|_{L^{q,\infty}(\Omega, d\sigma)} \leq C\,\|\nu\|. \tag{2}$$

Along with integral operators, we treat fractional maximal operators M_α with $0 \leq \alpha < n$ on $\mathbb{R}^n$, and characterize both strong- and weak-type $(1, q)$-inequalities for M_α, and more general maximal operators. Similar problems for Riesz potentials were studied earlier in [6–8]. Finally, we apply our results to the Poisson kernel to characterize $(1, q)$-Carleson measure inequalities.

2 Integral Operators

2.1 Strong-Type $(1, q)$-Inequality for Integral Operators

Let $\Omega \subseteq \mathbb{R}^n$ be a connected open set. By $\mathcal{M}^+(\Omega)$ we denote the class of all nonnegative locally finite Borel measures in Ω. Let $G\colon \Omega \times \Omega \to [0, +\infty]$ be a nonnegative lower-semicontinuous kernel. We will assume throughout this paper that G is quasi-symmetric, i.e., there exists a constant $a > 0$ such that

$$a^{-1}\, G(x, y) \le G(y, x) \le a\, G(x, y), \quad x, y \in \Omega. \tag{3}$$

If $\nu \in \mathcal{M}^+(\Omega)$, then by $\mathbf{G}\nu$ and $\mathbf{G}^*\nu$ we denote the integral operators (potentials) defined respectively by

$$\mathbf{G}\nu(x) = \int_\Omega G(x, y)\, d\nu(y), \quad \mathbf{G}^*\nu(x) = \int_\Omega G(y, x)\, d\nu(y), \quad x \in \Omega. \tag{4}$$

We say that the kernel G satisfies the *weak maximum principle* if, for any constant $M > 0$, the inequality

$$\mathbf{G}\nu(x) \le M \quad \text{for all } x \in S(\nu)$$

implies

$$\mathbf{G}\nu(x) \le hM \quad \text{for all } x \in \Omega,$$

where $h \ge 1$ is a constant, and $S(\nu) := \operatorname{supp} \nu$. When $h = 1$, we say that $\mathbf{G}\nu$ satisfies the *strong maximum principle*.

It is well-known that Green's kernels associated with many partial differential operators are quasi-symmetric, and satisfy the weak maximum principle (see, e.g., [2, 3, 12]).

The kernel G is said to be *degenerate* with respect to $\sigma \in \mathcal{M}^+(\Omega)$ provided there exists a set $A \subset \Omega$ with $\sigma(A) > 0$ and

$$G(\cdot, y) = 0 \quad d\sigma - \text{a.e. for } y \in A.$$

Otherwise, we will say that G is *non-degenerate* with respect to σ. (This notion was introduced in [19] in the context of (p, q)-inequalities for positive operators $T\colon L^p \to L^q$ in the case $1 < q < p$.)

Let $0 < q < 1$, and let G be a kernel on $\Omega \times \Omega$. For $\sigma \in \mathcal{M}^+(\Omega)$, we consider the problem of the existence of a *positive solution* u to the integral equation

$$u = \mathbf{G}(u^q d\sigma) \quad \text{in } \Omega, \quad 0 < u < +\infty \ d\sigma\text{–a.e.}, \quad u \in L^q_{\text{loc}}(\Omega). \tag{5}$$

We call u a positive *supersolution* if

$$u \geq \mathbf{G}(u^q d\sigma) \quad \text{in } \Omega, \quad 0 < u < +\infty \ \ d\sigma\text{–a.e.}, \quad u \in L^q_{\text{loc}}(\Omega). \tag{6}$$

This is a generalization of the sublinear elliptic problem (see, e.g., [4, 5], and the literature cited there):

$$\begin{cases} -\Delta u = \sigma u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where σ is a nonnegative locally integrable function, or measure, in Ω.

If Ω is a bounded C^2-domain then solutions to (7) can be understood in the "very weak" sense (see, e.g., [13]). For general domains Ω with a nontrivial Green function G associated with the Dirichlet Laplacian Δ in Ω, solutions u are understood as in (5).

Remark 2.1 In this paper, for the sake of simplicity, we sometimes consider positive solutions and supersolutions $u \in L^q(\Omega, d\sigma)$. In other words, we replace the natural local condition $u \in L^q_{\text{loc}}(\Omega, d\sigma)$ with its global counterpart. Notice that the local condition is necessary for solutions (or supersolutions) to be properly defined.

To pass from solutions u which are globally in $L^q(\Omega, d\sigma)$ to all solutions $u \in L^q_{\text{loc}}(\Omega, d\sigma)$ (for instance, very weak solutions to (7)), one can use either a localization method developed in [8] (in the case of Riesz kernels on $\mathbb{R}^n$), or *modified* kernels $\tilde{G}(x, y) = \frac{G(x,y)}{m(x)\, m(y)}$, where the modifier $m(x) = \min\Big(1, G(x, x_0)\Big)$ (with a fixed pole $x_0 \in \Omega$) plays the role of a regularized distance to the boundary $\partial\Omega$. One also needs to consider the corresponding $(1, q)$-inequalities with a weight m (see [16]). See the next section in the one-dimensional case where $\Omega = (0, +\infty)$.

Remark 2.2 Finite energy solutions, for instance, solutions $u \in W^{1,2}_0(\Omega)$ to (7), require the global condition $u \in L^{1+q}(\Omega, d\sigma)$, and are easier to characterize (see [6]).

The following theorem is proved in [16]. (The case where $\Omega = \mathbb{R}^n$ and $\mathbf{G} = (-\Delta)^{-\frac{\alpha}{2}}$ is the Riesz potential of order $\alpha \in (0, n)$ was considered earlier in [8].)

Theorem 2.3 *Let $\sigma \in \mathscr{M}^+(\Omega)$, and $0 < q < 1$. Suppose G is a quasi-symmetric kernel which satisfies the weak maximum principle. Then the following statements are equivalent:*

(1) *There exists a positive constant $\varkappa = \varkappa(\sigma)$ such that*

$$\|\mathbf{G}\nu\|_{L^q(\sigma)} \leq \varkappa \|\nu\| \quad \textit{for all } \nu \in \mathscr{M}^+(\Omega).$$

(2) *There exists a positive supersolution $u \in L^q(\Omega, d\sigma)$ to (6).*

(3) *There exists a positive solution $u \in L^q(\Omega, d\sigma)$ to (5), provided additionally that G is non-degenerate with respect to σ.*

Remark 2.4 The implication (1) $\Rightarrow$ (2) in Theorem 2.3 holds for any nonnegative kernel G, without assuming that it is either quasi-symmetric, or satisfies the weak maximum principle. This is a consequence of Gagliardo's lemma [10, 21]; see details in [16].

Remark 2.5 The implication (3) $\Rightarrow$ (1) generally fails for kernels G which do not satisfy the weak maximum principle (see examples in [16]).

The following corollary of Theorem 2.3 is obtained in [16].

Corollary 2.6 *Under the assumptions of Theorem 2.3, if there exists a positive supersolution* $u \in L^q(\Omega, \sigma)$ *to* (6)*, then* $\mathbf{G}\sigma \in L^{\frac{q}{1-q}}(\Omega, d\sigma)$.

Conversely, if $\mathbf{G}\sigma \in L^{\frac{q}{1-q},1}(\Omega, d\sigma)$*, then there exists a non-trivial supersolution* $u \in L^q(\Omega, \sigma)$ *to* (6) *(respectively, a solution u, provided G is non-degenerate with respect to* σ*).*

2.2 The One-Dimensional Case

In this section, we consider positive weak solutions to sublinear ODEs of the type (7) on the semi-axis $\mathbb{R}_+ = (0, +\infty)$. It is instructive to consider the one-dimensional case where elementary characterizations of $(1, q)$-weighed norm inequalities, along with the corresponding existence theorems and explicit global pointwise estimates of solutions are available. Similar results hold for sublinear equations on any interval $(a, b) \subset \mathbb{R}$.

Let $0 < q < 1$, and let $\sigma \in \mathscr{M}^+(\mathbb{R}_+)$. Suppose u is a positive weak solution to the equation

$$- u'' = \sigma u^q \quad \text{on } \mathbb{R}_+, \quad u(0) = 0, \tag{8}$$

such that $\lim_{x\to+\infty} \frac{u(x)}{x} = 0$. This condition at infinity ensures that u does not contain a linear component. Notice that we assume that u is concave and increasing on $[0, +\infty)$, and $\lim_{x\to 0+} u(x) = 0$.

In terms of integral equations, we have $\Omega = \mathbb{R}_+$, and $G(x, y) = \min(x, y)$ is the Green function associated with the Sturm-Liouville operator $\Delta u = u''$ with zero boundary condition at $x = 0$. Thus, (8) is equivalent to the equation

$$u(x) = \mathbf{G}(u^q d\sigma)(x) := \int_0^{+\infty} \min(x, y)u(y)^q d\sigma(y), \quad x > 0, \tag{9}$$

where σ is a locally finite measure on $\mathbb{R}_+$, and

$$\int_0^a y\, u(y)^q d\sigma(y) < +\infty, \quad \int_a^{+\infty} u(y)^q d\sigma(y) < +\infty, \quad \text{for every } a > 0. \tag{10}$$

This "local integrability" condition ensures that the right-hand side of (9) is well defined. Here intervals $(a, +\infty)$ are used in place of balls $B(x, r)$ in $\mathbb{R}^n$.

Notice that

$$u'(x) = \int_x^{+\infty} u(y)^q d\sigma(y), \quad x > 0. \tag{11}$$

Hence, u satisfies the global integrability condition

$$\int_0^{+\infty} u(y)^q d\sigma(y) < +\infty \tag{12}$$

if and only if $u'(0) < +\infty$.

The corresponding $(1, q)$-weighted norm inequality is given by

$$\|\mathbf{G}\nu\|_{L^q(\sigma)} \le \varkappa \|\nu\|, \tag{13}$$

where $\varkappa = \varkappa(\sigma)$ is a positive constant which does not depend on $\nu \in \mathscr{M}^+(\mathbb{R}_+)$. Obviously, (13) is equivalent to

$$\|H_+\nu + H_-\nu\|_{L^q(\sigma)} \le \varkappa \|\nu\| \quad \text{for all } \nu \in \mathscr{M}^+(\mathbb{R}_+), \tag{14}$$

where $H_\pm$ is a pair of Hardy operators,

$$H_+\nu(x) = \int_0^x y\, d\nu(y), \quad H_-\nu(x) = x \int_x^{+\infty} d\nu(y).$$

The following proposition can be deduced from the known results on two-weight Hardy inequalities in the case $p = 1$ and $0 < q < 1$ (see, e.g., [20]). We give here a simple independent proof.

Proposition 2.7 *Let* $\sigma \in \mathscr{M}^+(\mathbb{R}_+)$*, and let* $0 < q < 1$*. Then* (13) *holds if and only if*

$$\varkappa(\sigma)^q = \int_0^{+\infty} x^q d\sigma(x) < +\infty, \tag{15}$$

where $\varkappa(\sigma)$ *is the best constant in* (13)*.*

Proof Clearly,

$$H_+\nu(x) + H_-\nu(x) \le x \|\nu\|, \quad x > 0.$$

Hence,

$$\|H_+\nu + H_-\nu\|_{L^q(\sigma)} \le \Big(\int_0^{+\infty} x^q d\sigma(x)\Big)^{\frac{1}{q}} \|\nu\|,$$

which proves (14), and hence (13), with $\varkappa = \Big(\int_0^{+\infty} x^q d\sigma(x)\Big)^{\frac{1}{q}}$.

Conversely, suppose that (14) holds. Then, for every $a > 0$, and $\nu \in \mathscr{M}^+(\mathbb{R}_+)$,

$$\begin{aligned}
&\Big(\int_0^a x^q d\sigma(x)\Big)\Big(\int_a^{+\infty} d\nu(y)\Big)^q \\
&\le \int_0^a \Big(x \int_x^{+\infty} d\nu(y)\Big)^q d\sigma(x) \\
&\le \int_0^{+\infty} (H_-\nu)^q d\sigma \le \varkappa^q \|\nu\|^q.
\end{aligned}$$

For $\nu = \delta_{x_0}$ with $x_0 > a$, we get

$$\int_0^a x^q d\sigma(x) \le \varkappa^q.$$

Letting $a \to +\infty$, we deduce (15). □

Clearly, the Green kernel $G(x, y) = \min(x, y)$ is symmetric, and satisfies the strong maximum principle. Hence, by Theorem 2.3, Eqs. (8) and (9) have a nontrivial (super)solution $u \in L^q(\mathbb{R}_+, \sigma)$ if and only if (15) holds.

From Proposition 2.7, we deduce that, for "localized" measures $d\sigma_a = \chi_{(a,+\infty)} d\sigma$ $(a > 0)$, we have

$$\varkappa(\sigma_a) = \Big(\int_a^{+\infty} x^q d\sigma(x)\Big)^{\frac{1}{q}}. \tag{16}$$

Using this observation and the localization method developed in [8], we obtain the following existence theorem for general weak solutions to (7), along with sharp pointwise estimates of solutions.

We introduce a new potential

$$\mathbf{K}\sigma(x) := x\Big(\int_x^{+\infty} y^q d\sigma(y)\Big)^{\frac{1}{1-q}}, \quad x > 0. \tag{17}$$

We observe that $\mathbf{K}\sigma$ is a one-dimensional analogue of the potential introduced recently in [8] in the framework of intrinsic Wolff potentials in $\mathbb{R}^n$ (see also [7] in the radial case). Matching upper and lower pointwise bounds of solutions are obtained below by combining $\mathbf{G}\sigma$ with $\mathbf{K}\sigma$.

Theorem 2.8 *Let $\sigma \in \mathscr{M}^+(\mathbb{R}_+)$, and let $0 < q < 1$. Then Eq.* (7)*, or equivalently* (8) *has a nontrivial solution if and only if, for every $a > 0$,*

$$\int_0^a x\, d\sigma(x) + \int_a^{+\infty} x^q d\sigma(x) < +\infty. \tag{18}$$

Moreover, if (18) *holds, then there exists a positive solution u to* (7) *such that*

$$C^{-1}\Big[\Big(\int_0^x y\, d\sigma(y)\Big)^{\frac{1}{1-q}} + \mathbf{K}\sigma(x)\Big] \tag{19}$$

$$\le u(x) \le C\Big[\Big(\int_0^x y\, d\sigma(y)\Big)^{\frac{1}{1-q}} + \mathbf{K}\sigma(x)\Big]. \tag{20}$$

The lower bound in (19) *holds for any non-trivial supersolution u.*

Remark 2.9 The lower bound

$$u(x) \ge (1-q)^{\frac{1}{1-q}}\Big[\mathbf{G}\sigma(x)\Big]^{\frac{1}{1-q}}, \quad x > 0, \tag{21}$$

is known for a general kernel G which satisfies the strong maximum principle (see [11], Theorem 3.3; [16]), and the constant $(1-q)^{\frac{1}{1-q}}$ here is sharp. However, the second term on the left-hand side of (19) makes the lower estimate stronger, so that it matches the upper estimate.

Proof The lower bound

$$u(x) \ge (1-q)^{\frac{1}{1-q}}\Big[\int_0^x y\, d\sigma(y)\Big]^{\frac{1}{1-q}}, \quad x > 0, \tag{22}$$

is immediate from (21).

Applying Lemma 4.2 in [8], with the interval $(a, +\infty)$ in place of a ball B, and combining it with (16), for any $a > 0$ we have

$$\int_a^{+\infty} u(y)^q d\sigma(y) \ge c(q)\varkappa(\sigma_a)^{\frac{q}{1-q}} = c(q)\Big[\int_a^{+\infty} y^q d\sigma(y)\Big]^{\frac{1}{1-q}}.$$

Hence,

$$u(x) \ge \mathbf{G}(u^q d\sigma) \ge x\int_x^{+\infty} u(y)^q d\sigma(y) \ge c(q)\, x\Big[\int_x^{+\infty} y^q d\sigma(y)\Big]^{\frac{1}{1-q}}.$$

Combining the preceding estimate with (22), we obtain the lower bound in (19) for any non-trivial supersolution u. This also proves that (18) is necessary for the existence of a non-trivial positive supersolution.

Conversely, suppose that (18) holds. Let

$$v(x) := c\left[\left(\int_0^x y\, d\sigma(y)\right)^{\frac{1}{1-q}} + \mathbf{K}\sigma(x)\right], \quad x > 0, \tag{23}$$

where c is a positive constant. It is not difficult to see that v is a supersolution, so that $v \geq \mathbf{G}(v^q d\sigma)$, if $c = c(q)$ is picked large enough. (See a similar argument in the proof of Theorem 5.1 in [7].)

Also, it is easy to see that $v_0 = c_0(\mathbf{G}\sigma)^{\frac{1}{1-q}}$ is a subsolution, i.e., $v_0 \leq \mathbf{G}(v_0^q d\sigma)$, provided $c_0 > 0$ is a small enough constant. Moreover, we can ensure that $v_0 \leq v$ if $c_0 = c_0(q)$ is picked sufficiently small. (See details in [7] in the case of radially symmetric solutions in $\mathbb{R}^n$.) Hence, there exists a solution which can be constructed by iterations, starting from $u_0 = v_0$, and letting

$$u_{j+1} = \mathbf{G}(u_j^q d\sigma), \quad j = 0, 1, \ldots .$$

Then by induction $u_j \leq u_{j+1} \leq v$, and consequently $u = \lim_{j\to+\infty} u_j$ is a solution to (9) by the Monotone Convergence Theorem. Clearly, $u \leq v$, which proves the upper bound in (19). □

2.3 Weak-Type $(1, q)$-Inequality for Integral Operators

In this section, we characterize weak-type analogues of $(1, q)$-weighted norm inequalities considered above. We will use some elements of potential theory for general positive kernels G, including the notion of *inner capacity*, cap$(\cdot)$, and the associated *equilibrium* (extremal) measure (see [9]).

Theorem 2.10 *Let $\sigma \in \mathscr{M}^+(\Omega)$, and $0 < q < 1$. Suppose G satisfies the weak maximum principle. Then the following statements are equivalent:*

(1) *There exists a positive constant $\varkappa_w$ such that*

$$\|\mathbf{G}\nu\|_{L^{q,\infty}(\sigma)} \leq \varkappa_w \|\nu\| \quad \textit{for all } \nu \in \mathscr{M}^+(\Omega).$$

(2) *There exists a positive constant c such that*

$$\sigma(K) \leq c\Big(\operatorname{cap}(K)\Big)^q \quad \textit{for all compact sets } K \subset \Omega.$$

(3) $\mathbf{G}\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$.

Proof (1) ⇒ (2) Without loss of generality we may assume that the kernel G is *strictly positive*, that is, $G(x, x) > 0$ for all $x \in \Omega$. Otherwise, we can consider the kernel G on the set $\Omega \setminus A$, where $A := \{x \in \Omega : G(x, x) \neq 0\}$, since A is negligible for the corresponding $(1, q)$-inequality in statement (1). (See details in [16] in the case of the corresponding strong-type inequalities.)

We remark that the kernel G is known to be strictly positive if and only if, for any compact set $K \subset \Omega$, the inner capacity cap(K) is finite [9]. In this case there exists an equilibrium measure λ on K such that

$$\mathbf{G}\lambda \geq 1 \text{ n.e. on } K, \quad \mathbf{G}\lambda \leq 1 \text{ on } S(\lambda), \quad \|\lambda\| = \text{cap}(K). \tag{24}$$

Here n.e. stands for *nearly everywhere*, which means that the inequality holds on a given set except for a subset of zero capacity [9].

Next, we remark that condition (1) yields that σ is absolutely continuous with respect to capacity, i.e., $\sigma(K) = 0$ if cap$(K) = 0$. (See a similar argument in [16] in the case of strong-type inequalities.) Consequently, $\mathbf{G}\lambda \geq 1$ $d\sigma$-a.e. on K. Hence, by applying condition (1) with $\nu = \lambda$, we obtain (2).

(2) $\Rightarrow$ (3) We denote by σ_E the restriction of σ to a Borel set $E \subset \Omega$. Without loss of generality we may assume that σ is a finite measure on Ω. Otherwise we can replace σ with σ_F where F is a compact subset of Ω. We then deduce the estimate

$$\|\mathbf{G}\sigma_F\|_{L^{\frac{q}{1-q},\infty}(\sigma_F)} \leq C < \infty,$$

where C does not depend on F, and use the exhaustion of Ω by an increasing sequence of compact subsets $F_n \uparrow \Omega$ to conclude that $\mathbf{G}\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$ by the Monotone Convergence Theorem.

Set $E_t := \{x \in \Omega \colon \mathbf{G}\sigma(x) > t\}$, where $t > 0$. Notice that, for all $x \in (E_t)^c$,

$$\mathbf{G}\sigma_{(E_t)^c}(x) \leq \mathbf{G}\sigma(x) \leq t.$$

The set $(E_t)^c$ is closed, and hence the preceding inequality holds on $S(\sigma_{(E_t)^c})$. It follows by the weak maximum principle that, for all $x \in \Omega$,

$$\mathbf{G}\sigma_{(E_t)^c}(x) \leq \mathbf{G}\sigma(x) \leq h\, t.$$

Hence,

$$\{x \in \Omega \colon \mathbf{G}\sigma(x) > (h+1)t\} \subset \{x \in \Omega \colon \mathbf{G}\sigma_{E_t}(x) > t\}. \tag{25}$$

Denote by $K \subset \Omega$ a compact subset of $\{x \in \Omega \colon \mathbf{G}\sigma_{E_t}(x) > t\}$. By (2), we have

$$\sigma(K) \leq c \left(\text{cap}(K) \right)^q$$

If λ is the equilibrium measure on K, then $\mathbf{G}\lambda \leq 1$ on $S(\lambda)$, and $\lambda(K) = \text{cap}(K)$ by (24). Hence by the weak maximum principle $\mathbf{G}\lambda \leq h$ on Ω. Using quasi-symmetry of the kernel G and Fubini's theorem, we have

$$\begin{aligned}\text{cap}(K) &= \int_K d\lambda \\ &\leq \frac{1}{t}\int_K \mathbf{G}\sigma_{E_t}\, d\lambda \\ &\leq \frac{a}{t}\int_{E_t} \mathbf{G}\lambda d\sigma \\ &\leq \frac{ah}{t}\sigma(E_t).\end{aligned}$$

This shows that

$$\sigma(K) \leq \frac{c(ah)^q}{t^q}\left(\sigma(E_t)\right)^q.$$

Taking the supremum over all $K \subset E_t$, we deduce

$$\left(\sigma(E_t)\right)^{1-q} \leq \frac{c(ah)^q}{t^q}.$$

It follows from the preceding estimate and (25) that, for all $t > 0$,

$$t^{\frac{q}{1-q}}\sigma\Big(\{x \in \Omega\colon \mathbf{G}\sigma(x) > (h+1)t\}\Big) \leq t^{\frac{q}{1-q}}\sigma(E_t) \leq c^{\frac{1}{1-q}}(ah)^{\frac{q}{1-q}}.$$

Thus, (3) holds.

(3) $\Rightarrow$ (2) By Hölder's inequality for weak L^q spaces, we have

$$\begin{aligned}\|\mathbf{G}\nu\|_{L^{q,\infty}(\sigma)} &= \left\|\frac{\mathbf{G}\nu}{\mathbf{G}\sigma}\mathbf{G}\sigma\right\|_{L^{q,\infty}(\sigma)} \\ &\leq \left\|\frac{\mathbf{G}\nu}{\mathbf{G}\sigma}\right\|_{L^{1,\infty}(\sigma)} \|\mathbf{G}\sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)} \\ &\leq C\,\|\mathbf{G}\sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)}\,\|\nu\|,\end{aligned}$$

where the final inequality,

$$\left\|\frac{\mathbf{G}\nu}{\mathbf{G}\sigma}\right\|_{L^{1,\infty}(\sigma)} \leq C\,\|\nu\|,$$

with a constant $C = C(h, a)$, was obtained in [16], for quasi-symmetric kernels G satisfying the weak maximum principle. □

3 Fractional Maximal Operators

Let $0 \leq \alpha < n$, and let $\nu \in \mathscr{M}^+(\mathbb{R}^n)$. The fractional maximal function $M_\alpha \nu$ is defined by

$$M_\alpha \nu(x) := \sup_{Q \ni x} \frac{|Q|_\nu}{|Q|^{1-\frac{\alpha}{n}}}, \quad x \in \mathbb{R}^n, \tag{26}$$

where Q is a cube, $|Q|_\nu := \nu(Q)$, and $|Q|$ is the Lebesgue measure of Q. If $f \in L^1_{\text{loc}}(\mathbb{R}^n, d\mu)$ where $\mu \in \mathscr{M}^+(\mathbb{R}^n)$, we set $M_\alpha(fd\mu) = M_\alpha \nu$ where $d\nu = |f| d\mu$, i.e.,

$$M_\alpha(fd\mu)(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f| \, d\mu, \quad x \in \mathbb{R}^n. \tag{27}$$

For $\sigma \in \mathscr{M}^+(\mathbb{R}^n)$, it was shown in [22] that in the case $0 < q < p$,

$$M_\alpha : L^p(dx) \to L^q(d\sigma) \iff M_\alpha \sigma \in L^{\frac{q}{p-q}}(d\sigma), \tag{28}$$

$$M_\alpha : L^p(dx) \to L^{q,\infty}(d\sigma) \iff M_\alpha \sigma \in L^{\frac{q}{p-q},\infty}(d\sigma), \tag{29}$$

provided $p > 1$.

More general two-weight maximal inequalities

$$\|M_\alpha(fd\mu)\|_{L^q(\sigma)} \leq \varkappa \, \|f\|_{L^p(\mu)}, \quad \text{for all } f \in L^p(\mu), \tag{30}$$

where characterized by E.T. Sawyer [18] in the case $p = q > 1$, R.L. Wheeden [24] in the case $q > p > 1$, and the second author [22] in the case $0 < q < p$ and $p > 1$, along with their weak-type counterparts,

$$\|M_\alpha(fd\mu)\|_{L^{q,\infty}(\sigma)} \leq \varkappa_w \, \|f\|_{L^p(\mu)}, \quad \text{for all } f \in L^p(\mu), \tag{31}$$

where $\sigma, \mu \in \mathscr{M}^+(\mathbb{R}^n)$, and $\varkappa, \varkappa_w$ are positive constants which do not depend on f.

However, some of the methods used in [22] for $0 < q < p$ and $p > 1$ are not directly applicable in the case $p = 1$, although there are analogues of these results for real Hardy spaces, i.e., when the norm $\|f\|_{L^p(\mu)}$ on the right-hand side of (30) or (31) is replaced with $\|M_\mu f\|_{L^p(\mu)}$, where

$$M_\mu f(x) := \sup_{Q \ni x} \frac{1}{|Q|_\mu} \int_Q |f| d\mu. \tag{32}$$

We would like to understand similar problems in the case $0 < q < 1$ and $p = 1$, in particular, when $M_\alpha : \mathscr{M}^+(\mathbb{R}^n) \to L^q(d\sigma)$, or equivalently, there exists a constant

$\varkappa > 0$ such that the inequality

$$\|M_\alpha \nu\|_{L^q(\sigma)} \leq \varkappa \, \|\nu\| \tag{33}$$

holds for all $\nu \in \mathscr{M}^+(\mathbb{R}^n)$.

In the case $\alpha = 0$, Rozin [17] showed that the condition

$$\sigma \in L^{\frac{1}{1-q},1}(\mathbb{R}^n, dx)$$

is sufficient for the Hardy-Littlewood operator $M = M_0 : L^1(dx) \to L^q(\sigma)$ to be bounded; moreover, when σ is radially symmetric and decreasing, this is also a necessary condition. We will generalize this result and provide necessary and sufficient conditions for the range $0 \leq \alpha < n$. We also obtain analogous results for the weak-type inequality

$$\|M_\alpha \nu\|_{L^{q,\infty}(\sigma)} \leq \varkappa_w \, \|\nu\|, \quad \text{for all } \nu \in \mathscr{M}^+(\mathbb{R}^n), \tag{34}$$

where $\varkappa_w$ is a positive constant which does not depend on ν.

We treat more general maximal operators as well, in particular, dyadic maximal operators

$$M_\rho \nu(x) := \sup_{Q \in \mathscr{Q} : Q \ni x} \rho_Q \, |Q|_\nu, \tag{35}$$

where $\mathscr{Q}$ is the family of all dyadic cubes in $\mathbb{R}^n$, and $\{\rho_Q\}_{Q \in \mathscr{Q}}$ is a fixed sequence of nonnegative reals associated with $Q \in \mathscr{Q}$. The corresponding weak-type maximal inequality is given by

$$\|M_\rho \nu\|_{L^{q,\infty}(\sigma)} \leq \varkappa_w \, \|\nu\|, \quad \text{for all } \nu \in \mathscr{M}^+(\mathbb{R}^n). \tag{36}$$

3.1 Strong-Type Inequality

Theorem 3.1 *Let $\sigma \in M^+(\mathbb{R}^n)$, $0 < q < 1$, and $0 \leq \alpha < n$. The inequality (33) holds if and only if there exists a function $u \not\equiv 0$ such that*

$$u \in L^q(\sigma), \quad \textit{and} \quad u \geq M_\alpha(u^q \sigma).$$

Moreover, u can be constructed as follows: $u = \lim_{j\to\infty} u_j$, where $u_0 := (M_\alpha \sigma)^{\frac{1}{1-q}}$, $u_{j+1} \geq u_j$, and

$$u_{j+1} := M_\alpha(u_j^q \sigma), \quad j = 0, 1, \ldots. \tag{37}$$

In particular, $u \geq (M_\alpha \sigma)^{\frac{1}{1-q}}$.

Proof ($\Rightarrow$) We let $u_0 := (M_\alpha \sigma)^{\frac{1}{1-q}}$. Notice that, for all $x \in Q$, we have $u_0(x) \geq \left(\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\right)^{\frac{1}{1-q}}$. Hence,

$$u_1(x) := M_\alpha(u_0^q d\sigma)(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q u_0^q d\sigma \geq \sup_{Q \ni x} \left(\frac{|Q|_\sigma}{|Q|^{1-\frac{\alpha}{n}}}\right)^{\frac{1}{1-q}} = u_0(x).$$

By induction, we see that

$$u_{j+1} := M_\alpha(u_j^q d\sigma) \geq M_\alpha(u_{j-1}^q d\sigma) = u_j, \quad j = 1, 2, \ldots.$$

Let $u = \lim u_j$. By (33), we have

$$\begin{aligned} \|u_{j+1}\|_{L^q(\sigma)} &= \|M_\alpha(u_j^q \sigma)\|_{L^q(\sigma)} \\ &\leq \varkappa \|u_j\|_{L^q(\sigma)}^q \\ &\leq \varkappa \|u_{j+1}\|_{L^q(\sigma)}^q. \end{aligned}$$

From this we deduce that $\|u_{j+1}\|_{L^q(\sigma)} \leq \varkappa^{\frac{1}{1-q}}$ for $j = 0, 1, \ldots$. Since the norms $\|u_j\|_{L^q(\sigma)}^q$ are uniformly bounded, by the Monotone Convergence Theorem, we have for $u := \lim_{j\to\infty} u_j$ that $u \in L^q(\sigma)$. Note that by construction $u = M_\alpha(u^q d\sigma)$.

($\Leftarrow$) We can assume here that $M_\alpha \nu$ is defined, for $\nu \in \mathscr{M}^+(\mathbb{R}^n)$, as the centered fractional maximal function,

$$M_\alpha \nu(x) := \sup_{r>0} \frac{\nu(B(x,r))}{|B(x,r)|^{1-\frac{\alpha}{n}}},$$

since it is equivalent to its uncentered analogue used above. Suppose that there exists $u \in L^q(\sigma)$ ($u \not\equiv 0$) such that $u \geq M_\alpha(u^q d\sigma)$. Set $\omega := u^q d\sigma$. Let $\nu \in \mathscr{M}^+(\mathbb{R}^n)$.

We note that we have

$$\begin{aligned} \frac{M_\alpha \nu(x)}{M_\alpha \omega(x)} &= \frac{\sup_{r>0} \frac{|B(x,r)|_\nu}{|B(x,r)|^{1-\frac{\alpha}{n}}}}{\sup_{\rho>0} \frac{|B(x,\rho)|_\omega}{|B(x,\rho)|^{1-\frac{\alpha}{n}}}} \\ &\leq \sup_{r>0} \frac{|B(x,r)|_\nu}{|B(x,r)|_\omega} \\ &=: M_\omega \nu(x). \end{aligned}$$

Thus,

$$
\begin{aligned}
\|M_\alpha \nu\|_{L^q(\sigma)} &= \left\| \frac{M_\alpha \nu}{M_\alpha \omega} \right\|_{L^q((M_\alpha \omega)^q d\sigma)} \\
&\le \left\| \frac{M_\alpha \nu}{M_\alpha \omega} \right\|_{L^q(d\omega)} \\
&\le \|M_\omega \nu\|_{L^q(d\omega)} \\
&\le C \, \|M_\omega \nu\|_{L^{1,\infty}(\omega)} \le C\|\nu\|,
\end{aligned}
$$

by Jensen's inequality and the $(1, 1)$-weak-type maximal function inequality for $M_\omega \nu$. This establishes (33). □

3.2 Weak-Type Inequality

For $0 \le \alpha < n$, we define the *Hausdorff content* on a set $E \subset \mathbb{R}^n$ to be

$$
H^{n-\alpha}(E) := \inf \left\{ \sum_{i=1}^{\infty} r_i^{n-\alpha} : E \subset \bigcup_{i=1}^{\infty} B(x_i, r_i), \right\} \tag{38}
$$

where the collection of balls $\{B(x_i, r_i)\}$ forms a countable covering of E (see [1, 15]).

Theorem 3.2 *Let $\sigma \in M^+(\mathbb{R}^n)$, $0 < q < 1$, and $0 \le \alpha < n$. Then the following conditions are equivalent:*

(1) *There exists a positive constant $\varkappa_w$ such that*

$$
\|M_\alpha \nu\|_{L^{q,\infty}(\sigma)} \le \varkappa_w \, \|\nu\| \quad \textit{for all } \nu \in \mathscr{M}(\mathbb{R}^n).
$$

(2) *There exists a positive constant $C > 0$ such that*

$$
\sigma(E) \le C \, (H^{n-\alpha}(E))^q \quad \textit{for all Borel sets } E \subset \mathbb{R}^n.
$$

(3) $M_\alpha \sigma \in L^{\frac{q}{1-q}, \infty}(\sigma)$.

Remark 3.3 In the case $\alpha = 0$ each of the conditions (1)–(3) is equivalent to $\sigma \in L^{\frac{1}{1-q}, \infty}(dx)$.

Proof (1) ⇒ (2) Let $K \subset E$ be a compact set in $\mathbb{R}^n$ such that $H^{n-\alpha}(K) > 0$. It follows from Frostman's theorem (see the proof of Theorem 5.1.12 in [1]) that there exists a measure ν supported on K such that $\nu(K) \le H^{n-\alpha}(K)$, and, for every $x \in K$ there exists a cube Q such that $x \in Q$ and $|Q|_\nu \ge c \, |Q|^{1-\frac{\alpha}{n}}$, where c depends only on n and α. Hence,

$$M_\alpha \nu(x) \geq \sup_{Q \ni x} \frac{|Q|_\nu}{|Q|^{1-\frac{\alpha}{n}}} \geq c \quad \text{for all } x \in K,$$

where c depends only on n and α. Consequently,

$$c^q\, \sigma(K) \leq \|M_\alpha \nu\|^q_{L^{q,\infty}(\sigma)} \leq \varkappa_w^q \Big(H^{n-\alpha}(K)\Big)^q.$$

If $H^{n-\alpha}(E) = 0$, then $H^{n-\alpha}(K) = 0$ for every compact set $K \subset E$, and consequently $\sigma(E) = 0$. Otherwise,

$$\sigma(K) \leq \varkappa_w^q \Big(H^{n-\alpha}(K)\Big)^q \leq \varkappa_w^q \Big(H^{n-\alpha}(K)\Big)^q,$$

for every compact set $K \subset E$, which proves (2) with $C = c^{-q}\varkappa_w^q$.

(2) $\Rightarrow$ (3) Let $E_t := \{x : M_\alpha \sigma(x) > t\}$, where $t > 0$. Let $K \subset E_t$ be a compact set. Then for each $x \in K$ there exists $Q_x \ni x$ such that

$$\frac{\sigma(Q_x)}{|Q_x|^{1-\frac{\alpha}{n}}} > t.$$

Now consider the collection $\{Q_x\}_{x\in K}$, which forms a cover of K. By the Besicovitch covering lemma, we can find a subcover $\{Q_i\}_{i\in I}$, where I is a countable index set, such that $K \subset \bigcup_{i\in I} Q_i$ and $x \in K$ is contained in at most b_n sets in $\{Q_i\}$. By (2), we have

$$\sigma(K) \leq [H^{n-\alpha}(K)]^q,$$

and by the definition of the Hausdorff content we have

$$H^{n-\alpha}(K) \leq \sum |Q_i|^{1-\alpha/n}.$$

Since $\{Q_i\}$ have bounded overlap, we have

$$\sum_{i\in I} \sigma(Q_i) \leq b_n \sigma(K).$$

Thus,

$$\sigma(K) \leq \left(b_n \frac{\sigma(K)}{t}\right)^q,$$

which shows that

$$t^{\frac{q}{1-q}}\sigma(K) \le (b_n)^{\frac{1}{1-q}} < +\infty.$$

Taking the supremum over all $K \subset E_t$ in the preceding inequality, we deduce $M_\alpha\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$.

(3) $\Rightarrow$ (1). We can assume again that M_α is the centered fractional maximal function, since it is equivalent to the uncentered version. Suppose that $M_\alpha\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$. Let $\nu \in \mathscr{M}(\mathbb{R}^n)$. Then, as in the case of the strong-type inequality,

$$\frac{M_\alpha\nu(x)}{M_\alpha\sigma(x)} = \frac{\sup_{r>0}\frac{|B(x,r)|_\nu}{|B(x,r)|^{1-\frac{\alpha}{n}}}}{\sup_{\rho>0}\frac{|B(x,\rho)|_\sigma}{|B(x,\rho)|^{1-\frac{\alpha}{n}}}} \le \sup_{r>0}\frac{|B(x,r)|_\nu}{|B(x,r)|_\sigma} =: M_\sigma\nu(x).$$

Thus, by Hölder's inequality for weak L^p-spaces,

$$\begin{aligned}\|M_\alpha\nu\|_{L^{q,\infty}(\sigma)} &\le \|(M_\alpha\sigma)\,(M_\sigma\nu)\|_{L^{q,\infty}(\sigma)}\\ &\le \|M_\alpha\sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)}\,\|M_\sigma\nu\|_{L^{1,\infty}(\sigma)}\\ &\le c\|M_\alpha\sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)}\,\|\nu\|,\end{aligned}$$

where in the last line we have used the $(1, 1)$-weak-type maximal function inequality for the centered maximal function $M_\sigma\nu$. □

We now characterize weak-type $(1, q)$-inequalities (36) for the generalized dyadic maximal operator M_ρ defined by (35). The corresponding (p, q)-inequalities in the case $0 < q < p$ and $p > 1$ were characterized in [22]. The results obtained in [22] for weak-type inequalities remain valid in the case $p = 1$, but some elements of the proofs must be modified as indicated below.

Theorem 3.4 *Let $\sigma \in \mathscr{M}^+(\mathbb{R}^n)$, $0 < q < 1$, and $0 \le \alpha < n$. Then the following conditions are equivalent:*

(1) *There exists a positive constant $\varkappa_w$ such that (36) holds.*
(2) $M_\rho\sigma \in L^{\frac{q}{1-q},\infty}(\sigma)$.

Proof (2) $\Rightarrow$ (1) The proof of this implication is similar to the case of fractional maximal operators. Let $\nu \in \mathscr{M}(\mathbb{R}^n)$. Denoting by $Q, P \in \mathscr{Q}$ dyadic cubes in $\mathbb{R}^n$, we estimate

$$\frac{M_\rho \nu(x)}{M_\rho \sigma(x)} = \frac{\sup_{Q\ni x}(\rho_Q\, |Q|_\nu)}{\sup_{P\ni x}(\rho_P\, |P|_\sigma)} \leq \sup_{Q\ni x} \frac{|Q|_\nu}{|Q|_\sigma} =: M_\sigma \nu(x).$$

By Hölder's inequality for weak L^p-spaces,

$$\begin{aligned}\|M_\rho \nu\|_{L^{q,\infty}(\sigma)} &\leq \|(M_\rho \sigma)\,(M_\sigma \nu)\|_{L^{q,\infty}(\sigma)} \\ &\leq \|M_\rho \sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)}\, \|M_\sigma \nu\|_{L^{1,\infty}(\sigma)} \\ &\leq c\|M_\rho \sigma\|_{L^{\frac{q}{1-q},\infty}(\sigma)}\, \|\nu\|,\end{aligned}$$

by the $(1,1)$-weak-type maximal function inequality for the dyadic maximal function M_σ.

(1) $\Rightarrow$ (2) We set $f = \sup_Q(\lambda_Q \chi_Q)$ and $d\nu = f\, d\sigma$, where $\{\lambda_Q\}_{Q\in\mathcal{Q}}$ is a finite sequence of non-negative reals. Then obviously

$$M_\rho \nu(x) \geq \sup_Q(\lambda_Q \rho_Q \chi_Q), \quad \text{and} \quad \|\nu\| \leq \sum_Q \lambda_Q\, |Q|_\sigma.$$

By (1), for all $\{\lambda_Q\}_{Q\in\mathcal{Q}}$,

$$\|\sup_Q(\lambda_Q \rho_Q \chi_Q)\|_{L^{q,\infty}(\sigma)} \leq \varkappa_\upsilon \sum_Q \lambda_Q\, |Q|_\sigma.$$

Hence, by Theorem 1.1 and Remark 1.2 in [22], it follows that (2) holds. □

4 Carleson Measures for Poisson Integrals

In this section we treat $(1,q)$-Carleson measure inequalities for Poisson integrals with respect to Carleson measures $\sigma \in \mathcal{M}^+(\mathbb{R}^{n+1}_+)$ in the upper half-space $\mathbb{R}^{n+1}_+ = \{(x,y): x \in \mathbb{R}^n, y > 0\}$. The corresponding weak-type (p,q)-inequalities for all $0 < q < p$ as well as strong-type (p,q)-inequalities for $0 < q < p$ and $p > 1$, were characterized in [23]. Here we consider strong-type inequalities of the type

$$\|\mathbf{P}\nu\|_{L^q(\mathbb{R}^{n+1}_+,\sigma)} \leq \varkappa\, \|\nu\|_{\mathcal{M}^+(\mathbb{R}^n)}, \quad \text{for all } \nu \in \mathcal{M}^+(\mathbb{R}^n), \tag{39}$$

for some constant $\varkappa > 0$, where $\mathbf{P}\nu$ is the Poisson integral of $\nu \in \mathscr{M}^+(\mathbb{R}^n)$ defined by

$$\mathbf{P}\nu(x, y) := \int_{\mathbb{R}^n} P(x - t, y) d\nu(t), \quad (x, y) \in \mathbb{R}^{n+1}_+.$$

Here $P(x, y)$ denotes the Poisson kernel associated with $\mathbb{R}^{n+1}_+$.

By $\mathbf{P}^*\mu$ we denote the formal adjoint (balayage) operator defined, for $\mu \in \mathscr{M}^+(\mathbb{R}^{n+1}_+)$, by

$$\mathbf{P}^*\mu(t) := \int_{\mathbb{R}^{n+1}_+} P(x - t, y) d\mu(x, y), \quad t \in \mathbb{R}^n.$$

We will also need the symmetrized potential defined, for $\mu \in \mathscr{M}^+(\mathbb{R}^{n+1}_+)$, by

$$\mathbf{PP}^*\mu(x, y) := \mathbf{P}\Big[(\mathbf{P}^*\mu)dt\Big] = \int_{\mathbb{R}^{n+1}_+} P(x - \tilde{x}, y + \tilde{y}) d\mu(\tilde{x}, \tilde{y}), \quad (x, y) \in \mathbb{R}^{n+1}_+.$$

As we will demonstrate below, the kernel of $\mathbf{PP}^*\mu$ satisfies the weak maximum principle with constant $h = 2^{n+1}$.

Theorem 4.1 *Let $\sigma \in \mathscr{M}^+(\mathbb{R}^{n+1}_+)$, and let $0 < q < 1$. Then inequality (39) holds if and only if there exists a function $u > 0$ such that*

$$u \in L^q(\mathbb{R}^{n+1}_+, \sigma), \quad \textit{and} \quad u \geq \mathbf{PP}^*(u^q\sigma) \quad \text{in } \mathbb{R}^{n+1}_+.$$

Moreover, if (39) holds, then a positive solution $u = \mathbf{PP}^*(u^q\sigma)$ such that $u \in L^q(\mathbb{R}^{n+1}_+, \sigma)$ can be constructed as follows: $u = \lim_{j\to\infty} u_j$, where

$$u_{j+1} := \mathbf{PP}^*(u_j^q\sigma), \quad j = 0, 1, \dots, \quad u_0 := c_0(\mathbf{PP}^*\sigma)^{\frac{1}{1-q}}, \tag{40}$$

for a small enough constant $c_0 > 0$ (depending only on q and n), which ensures that $u_{j+1} \geq u_j$. In particular, $u \geq c_0\, (\mathbf{PP}^*\sigma)^{\frac{1}{1-q}}$.

Proof We first prove that (39) holds if and only if

$$\|\mathbf{PP}^*\mu\|_{L^q(\mathbb{R}^{n+1}_+,\sigma)} \leq \varkappa\, \|\mu\|_{\mathscr{M}^+(\mathbb{R}^{n+1}_+)}, \quad \text{for all } \mu \in \mathscr{M}^+(\mathbb{R}^{n+1}_+). \tag{41}$$

Indeed, letting $\nu = \mathbf{P}^*\mu$ in (39) yields (41) with the same embedding constant $\varkappa$.

Conversely, suppose that (41) holds. Then by Maurey's factorization theorem (see [14]), there exists $F \in L^1(\mathbb{R}^{n+1}_+, \sigma)$ such that $F > 0$ $d\sigma$-a.e., and

$$\|F\|_{L^1(\mathbb{R}^{n+1}_+,\sigma)} \leq 1, \qquad \sup_{(x,y)\in\mathbb{R}^{n+1}_+} \mathbf{PP}^*(F^{1-\frac{1}{q}} d\sigma)(x, y) \leq \varkappa. \tag{42}$$

By letting $y \downarrow 0$ in (42) and using the Monotone Convergence Theorem, we deduce

$$\sup_{x \in \mathbb{R}^n} \mathbf{P}^*(F^{1-\frac{1}{q}} d\sigma)(x) \le \varkappa. \tag{43}$$

Hence, by Jensen's inequality and (43), for any $\nu \in \mathscr{M}^+(\mathbb{R}^n)$, we have

$$\|\mathbf{P}\nu\|_{L^q(\mathbb{R}^{n+1}_+,\sigma)} \le \|\mathbf{P}\nu\|_{L^1(\mathbb{R}^{n+1}_+, F^{1-\frac{1}{q}} d\sigma)} = \|\mathbf{P}^*(F^{1-\frac{1}{q}} d\sigma)\|_{L^1(\mathbb{R}^n, d\nu)} \le \varkappa \, \|\nu\|_{\mathscr{M}^+(\mathbb{R}^n)}.$$

We next show that the kernel of $\mathbf{PP}^*$ satisfies the weak maximum principle with constant $h = 2^{n+1}$. Indeed, suppose $\mu \in \mathscr{M}^+(\mathbb{R}^{n+1}_+)$, and

$$\mathbf{PP}^*\mu(\tilde{x}, \tilde{y}) \le M, \quad \text{for all } (\tilde{x}, \tilde{y}) \in S(\mu).$$

Without loss of generality we may assume that $S(\mu) \Subset \mathbb{R}^{n+1}_+$ is a compact set. For $t \in \mathbb{R}^n$, let $(x_0, y_0) \in S(\mu)$ be a point such that

$$|(t, 0) - (x_0, y_0)| = \operatorname{dist}\Big((t, 0), S(\mu)\Big).$$

Then by the triangle inequality, for any $(\tilde{x}, \tilde{y}) \in S(\mu)$,

$$|(x_0, y_0) - (\tilde{x}, -\tilde{y})| \le |(x_0, y_0) - (t, 0)| + |(t, 0) - (\tilde{x}, -\tilde{y})| \le 2|(t, 0) - (\tilde{x}, \tilde{y})|.$$

Hence,

$$\sqrt{|t - \tilde{x}|^2 + \tilde{y}^2} \ge \frac{1}{2} \sqrt{\Big[|x_0 - \tilde{x}|^2 + (y_0 + \tilde{y})^2\Big]}.$$

It follows that, for all $t \in \mathbb{R}^n$ and $(\tilde{x}, \tilde{y}) \in S(\mu)$, we have

$$P(t - \tilde{x}, \tilde{y}) \le 2^{n+1} P(x_0 - \tilde{x}, y_0 + \tilde{y}).$$

Consequently, for all $t \in \mathbb{R}^n$,

$$\mathbf{P}^*\mu(t) \le 2^{n+1}\mathbf{PP}^*\mu(x_0, y_0) \le 2^{n+1}M.$$

Applying the Poisson integral $\mathbf{P}[dt]$ to both sides of the preceding inequality, we obtain

$$\mathbf{PP}^*\mu(x, y) \le 2^{n+1}M \quad \text{for all } (x, y) \in \mathbb{R}^{n+1}_+.$$

This proves that the weak maximum principle holds for $\mathbf{PP}^*$ with $h = 2^{n+1}$. It follows from Theorem 2.3 that (39) holds if and only if there exists a non-trivial $u \in$

$L^q(\mathbb{R}^{n+1}_+, \sigma)$ such that $u \geq \mathbf{PP}^*(u^q d\sigma)$. Moreover, a positive solution $u = \mathbf{PP}^*(u^q \sigma)$ can be constructed as in the statement of Theorem 4.1 (see details in [16]). □

Corollary 4.2 *Under the assumptions of Theorem 4.1, inequality (39) holds if and only if there exists a function $\phi \in L^1(\mathbb{R}^n)$, $\phi > 0$ a.e., such that*

$$\phi \geq \mathbf{P}^*\Big[(\mathbf{P}\phi)^q d\sigma\Big] \quad a.e. \quad in\ \mathbb{R}^n.$$

Moreover, if (39) holds, then there exists a positive solution $\phi \in L^1(\mathbb{R}^n)$ to the equation $\phi = \mathbf{P}^*\Big[(\mathbf{P}\phi)^q d\sigma\Big]$.

Proof If (39) holds then by Theorem 4.1 there exists $u = \mathbf{PP}^*(u^q d\sigma)$ such that $u > 0$ and $u \in L^q(\mathbb{R}^{n+1}_+, \sigma)$. Setting $\phi = \mathbf{P}^*(u^q d\sigma)$, we see that

$$\mathbf{P}\phi = \mathbf{PP}^*(u^q d\sigma) = u,$$

so that $\phi = \mathbf{P}^*[(\mathbf{P}\phi)^q d\sigma]$, and consequently

$$\|\phi\|_{L^1(\mathbb{R}^n)} = \|u\|^q_{L^q(\mathbb{R}^{n+1}_+, \sigma)} = \int_{\mathbb{R}^n} u(x, y) dx < \infty.$$

Conversely, if there exists $\phi > 0$, $\phi \in L^1(\mathbb{R}^n)$ such that $\phi \geq \mathbf{P}^*\Big[(\mathbf{P}\phi)^q d\sigma\Big]$, then letting $u = \mathbf{P}\phi$, we see that u is a positive harmonic function in $\mathbb{R}^{n+1}_+$ so that

$$u(x, y) = \mathbf{P}\phi(x, y) \geq \mathbf{PP}^*(u^q d\sigma)(x, y), \quad (x, y) \in \mathbb{R}^{n+1}_+.$$

Notice that the kernel $P(x - \tilde{x}, y + \tilde{y})$ of the operator $\mathbf{PP}^*$ has the property

$$\int_{\mathbb{R}^n} P(x - \tilde{x}, y + \tilde{y}) dx = 1, \quad y > 0, \ (\tilde{x}, \tilde{y}) \in \mathbb{R}^{n+1}_+,$$

and consequently, for all $y > 0$,

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^{n+1}_+} P(x - \tilde{x}, y + \tilde{y}) u(\tilde{x}, \tilde{y})^q d\sigma(\tilde{x}, \tilde{y}) \, dx = \int_{\mathbb{R}^{n+1}_+} u(\tilde{x}, \tilde{y})^q d\sigma(\tilde{x}, \tilde{y}),$$

Hence,

$$\|u\|^q_{L^q(\mathbb{R}^{n+1}_+, \sigma)} = \int_{\mathbb{R}^n} \Big[\mathbf{PP}^*(u^q d\sigma)\Big](x, y) \, dx \leq \int_{\mathbb{R}^n} u(x, y) dx = \|\phi\|_{L^1(\mathbb{R}^n)} < \infty.$$

Thus, inequality (39) holds by Theorem 4.1. □

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New Bellman Functions and Subordination by Orthogonal Martingales in L^p, $1 < p \le 2$

Prabhu Janakiraman and Alexander Volberg

Abstract We present a new Bellman function that gives estimates for L^p norm, $1 < p < 2$, of differentially subordinated martingales if one of them has extra symmetries. Our Bellman function is obtained by explicitly solving a corresponding Monge–Ampère equation. In one particular case this Bellman function can be given by an explicit and simple formula. This corresponds to $p = 3/2$.

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1 Introduction: Orthogonal Martingales and the Beurling-Ahlfors Transform

The main result of this note is Theorem 7 below. Of main interest is the array of new Bellman functions, which are very different from the Burkholder's function.

A complex-valued martingale $Y = Y_1 + iY_2$ is said to be *orthogonal* if the quadratic variations of the coordinate martingales are equal and their mutual covariation is 0:

$$\langle Y_1 \rangle = \langle Y_2 \rangle \,, \quad \langle Y_1, Y_2 \rangle = 0.$$

In [2], Bañuelos and Janakiraman make the observation that the martingale associated with the Beurling-Ahlfors transform is, in fact, an orthogonal martingale. They show that Burkholder's proof in [9] naturally accommodates for this property and leads to an improvement in the estimate of the Ahlfors–Beurling transform $\|B\|_p$,

P. Janakiraman • A. Volberg (✉)
Department of Mathematics, Michigan State University, East Lansing, MI, USA
e-mail: pjanakir1978@gmail.com; volberg@math.msu.edu

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which is given by the formula

$$Bf(z) := \frac{1}{\pi} \int \frac{f(\zeta)}{\zeta - z} dm_2(\zeta)\,.$$

Theorem 1 (One-Sided Orthogonality)

(i) *(Left-side orthogonality) Suppose* $2 \le p < \infty$. *If* Y *is an orthogonal martingale and* X *is any martingale such that* $\langle Y \rangle \le \langle X \rangle$, *then*

$$\|Y\|_p \le \sqrt{\frac{p^2 - p}{2}} \|X\|_p. \tag{1}$$

(ii) *(Right-side orthogonality) Suppose* $1 < p < 2$. *If* X *is an orthogonal martingale and* Y *is any martingale such that* $\langle Y \rangle \le \langle X \rangle$, *then*

$$\|Y\|_p \le \sqrt{\frac{2}{p^2 - p}} \|X\|_p. \tag{2}$$

It is not known whether these estimates are the best possible.

Remark The result for left-side orthogonality was proved in [2]. The result for right-side orthogonality was stated in [20]. In [20] we emulate [2] to provide in a rather simple way an estimate on right-side orthogonality and in the regime $1 < p \le 2$. In the present work we tried to come up with a better constant for this regime, as the sharpness of these constants in [2] and [20] is somewhat dubious. For that purpose we build some family of new (funny and interesting) Bellman functions, very different from the original Burkholder's function. Even though the approach is quite different from the one in [2] and [20], the constants we obtain here are *the same*! So, maybe they are sharp after all [1, 3, 4, 6–8, 10–13, 15–19, 21]. The Bellman function approach to harmonic analysis problems was used in [22–29]. Implicitly it was used in [30] as well. It was extended in [33–37].

If X and Y are the martingales associated with f and Bf respectively, then Y is orthogonal, $\langle Y \rangle \le 4 \langle X \rangle$, see [2] (and Theorem 5 below), and hence by (1), one obtains

$$\|Bf\|_p \le \sqrt{2(p^2 - p)} \|f\|_p \text{ for } p \ge 2. \tag{3}$$

By interpolating this estimate $\sqrt{2(p^2 - p)}$ with the known $\|B\|_2 = 1$, Bañuelos and Janakiraman establish the present best estimate in the conjecture by Iwaniec:

$$\|B\|_p \le 1.575(p^* - 1), \tag{4}$$

where $p^* = \max(p, \frac{p}{p-1})$. This is the best to date estimate known for all p. For large p, however, a better estimate is contained in [5]:

$$\|B\|_p \le 1.39(p - 1),\ \ p \ge 1000\,. \tag{5}$$

The conjecture of Iwaniec states that

$$\|B\|_p \leq (p^* - 1). \tag{6}$$

The reader who wants to see the operator theory origins of the problems in this article may consult [27, 32].

2 New Questions and Results

Since B is associated with left-side orthogonality and since we know $\|B\|_p = \|B\|_{p'}$, two important questions arise:

(i) If $2 \leq p < \infty$, what is the best constant C_p in the left-side orthogonality problem: $\|Y\|_p \leq C_p\|X\|_p$, where Y is orthogonal and $\langle Y\rangle \leq \langle X\rangle$?
(ii) Similarly, if $1 < p' < 2$, what is the best constant $C_{p'}$ in the left-side orthogonality problem?

We have separated the two questions, since Burkholder's proof (and his function) already gives a good answer, when $p \geq 2$. This was the main observation of [2].

However, no estimate (better than $p - 1$) follows from analyzing Burkholder's function, when $1 < p' < 2$. Perhaps, we may hope that $C_{p'} < \sqrt{\frac{p^2-p}{2}}$, when $1 < p' = \frac{p}{p-1} < 2$, which would then imply a better estimate for $\|B\|_p$. This paper destroys this hope by finding $C_{p'}$; see Theorem 2. We also ask and answer an analogous question of right-side orthogonality when $2 < p < \infty$. In the spirit of Burkholder [14], we believe these questions are of independent interest in martingale theory and may have deeper connections with other areas of mathematics.

Remark The following sharp estimates are proved in [5], they cover the left-side orthogonality for the regime $1 < p \leq 2$ and the right-side orthogonality for the regime $2 \leq p < \infty$. Notice that these two complementary regimes have some non-trivial estimates: 1) for $2 \leq p < \infty$ and left-side orthogonality in [2], 2) for $1 < p \leq 2$, and the right-side orthogonality in this note and in [20], but their sharpness is somewhat dubious.

Theorem 2 *Let $Y = (Y_1, Y_2)$ be an orthogonal martingale and let $X = (X_1, X_2)$ be an arbitrary martingale.*

(i) *Let $1 < p' \leq 2$. Suppose $\langle Y\rangle \leq \langle X\rangle$. Then the least constant that always works in the inequality $\|Y\|_{p'} \leq C_{p'}\|X\|_{p'}$ is*

$$C_{p'} = \frac{1}{\sqrt{2}}\frac{z_{p'}}{1 - z_{p'}} \tag{7}$$

where $z_{p'}$ is the smallest root in $(0, 1)$ of the bounded Laguerre function $L_{p'}$.

(ii) *Let $2 \le p < \infty$. Suppose $\langle X\rangle \le \langle Y\rangle$. Then the least constant that always works in the inequality $\|X\|_p \le C_p\|Y\|_p$ is*

$$C_p = \sqrt{2}\frac{1-z_p}{z_p} \tag{8}$$

where z_p is the smallest root in $(0,1)$ of the bounded Laguerre function L_p.

Bounded Laguerre function L_p is a bounded function that solves the ODE

$$sL_p''(s) + (1-s)L_p'(s) + pL_p(s) = 0.$$

3 Orthogonality

Let $Z = (X, Y), W = (U, V)$ be two $\mathbb{R}^2$-valued martingales on the filtration of 2-dimensional Brownian motion $B_s = (B_{1s}, B_{2s})$. Let $A = \begin{bmatrix} -1, & i \\ i, & 1 \end{bmatrix}$. We want W to be the martingale transform of Z (defined by matrix A). Let

$$X(t) = \int_0^t \overrightarrow{x}(s)\cdot dB_s\,,$$

$$Y(t) = \int_0^t \overrightarrow{y}(s)\cdot dB_s\,,$$

where X, Y are *real-valued* processes, and $\overrightarrow{x}(s), \overrightarrow{y}(s)$ are $\mathbb{R}^2$-valued "martingale differences".

Put

$$Z(t) = X(t) + iY(t)\,, Z(t) = \int_0^t (\overrightarrow{x}(s) + i\overrightarrow{y}(s))\cdot dB_s\,, \tag{9}$$

and

$$W(t) = U(t) + iV(t)\,, W(t) = \int_0^t (A(\overrightarrow{x}(s) + i\overrightarrow{y}(s)))\cdot dB_s\,. \tag{10}$$

We will denote

$$W = A \star Z\,.$$

As before

$$U(t) = \int_0^t \overrightarrow{u}(s) \cdot dB_s\,,$$

$$V(t) = \int_0^t \overrightarrow{v}(s) \cdot dB_s\,,$$

$$W(t) = \int_0^t (\overrightarrow{u}(s) + i\overrightarrow{v}(s)) \cdot dB_s\,.$$

We can easily write components of $\overrightarrow{u}(s), \overrightarrow{v}(s)$:

$$u_1(s) = -x_1(s) - y_2(s)\,,\ v_1(s) = x_2(s) - y_1(s)\,, i = 1, 2\,,$$
$$u_2(s) = x_2(s) - y_1(s)\,,\ v_2(s) = x_1(s) + y_2(s)\,, i = 1, 2\,.$$

Notice that

$$\overrightarrow{u} \cdot \overrightarrow{v} = u_1 v_1 + u_2 v_2 = -(x_1 + y_2)(x_2 - y_1) + (x_2 - y_1)(x_1 + y_2) = 0\,. \qquad (11)$$

3.1 Local Orthogonality

The processes

$$\langle X, U\rangle(t) := \int_0^t \overrightarrow{x} \cdot \overrightarrow{u}\, ds\,,\ \langle X, V\rangle(t) := \int_0^t \overrightarrow{x} \cdot \overrightarrow{v}\, ds\,,$$

$$\langle Y, U\rangle(t) := \int_0^t \overrightarrow{y} \cdot \overrightarrow{u}\, ds\,,\ \langle Y, V\rangle(t) := \int_0^t \overrightarrow{y} \cdot \overrightarrow{v}\, ds\,,$$

$$\langle X, X\rangle(t) := \int_0^t \overrightarrow{x} \cdot \overrightarrow{x}\, ds\,,\ \langle Y, Y\rangle(t) := \int_0^t \overrightarrow{y} \cdot \overrightarrow{y}\, ds\,,$$

$$\langle X, Y\rangle(t) := \int_0^t \overrightarrow{x} \cdot \overrightarrow{y}\, ds\,,\ \langle U, U\rangle(t) := \int_0^t \overrightarrow{u} \cdot \overrightarrow{u}\, ds\,,$$

$$\langle V, V\rangle(t) := \int_0^t \overrightarrow{v} \cdot \overrightarrow{v}\, ds\,,\ \langle U, V\rangle(t) := \int_0^t \overrightarrow{u} \cdot \overrightarrow{v}\, ds\,.$$

are called the covariance processes. We can denote

$$d\langle X, U\rangle(t) := \overrightarrow{x}(t) \cdot \overrightarrow{u}(t)\,,\ d\langle X, V\rangle(t) := \overrightarrow{x}(t) \cdot \overrightarrow{v}(t)\,,$$
$$d\langle Y, U\rangle(t) := \overrightarrow{y}(t) \cdot \overrightarrow{u}(t)\,,\ d\langle Y, V\rangle(t) := \overrightarrow{y}(t) \cdot \overrightarrow{v}(t)\,,$$
$$d\langle X, X\rangle(t) := \overrightarrow{x}(t) \cdot \overrightarrow{x}(t)\,,\ d\langle Y, Y\rangle(t) := \overrightarrow{y}(t) \cdot \overrightarrow{y}(t)\,,$$

$$d\langle X, Y\rangle(t) := \overrightarrow{x}(t)\cdot\overrightarrow{y}(t)\,,\ d\langle U, U\rangle(t) := \overrightarrow{u}(t)\cdot\overrightarrow{u}(t)\,,$$
$$d\langle V, V\rangle(t) := \overrightarrow{v}(t)\cdot\overrightarrow{v}(t)\,,\ d\langle U, V\rangle(t) := \overrightarrow{u}(t)\cdot\overrightarrow{v}(t)\,,$$
$$d\langle Z, Z\rangle(t) := (\overrightarrow{x}(t)\cdot\overrightarrow{x}(t) + \overrightarrow{y}(t)\cdot\overrightarrow{y}(t))\,,\ d\langle W, W\rangle(t) := (\overrightarrow{u}(t)\cdot\overrightarrow{u}(t) + \overrightarrow{v}(t)\cdot\overrightarrow{v}(t))\,.$$

Of importance is the following observation.

Lemma 3 *Let* $A = \begin{bmatrix} -1, & i \\ i, & 1 \end{bmatrix}$. *Then*

$$d\langle U, V\rangle(t) = 0\,. \tag{12}$$

Or

$$\overrightarrow{u}(t)\cdot\overrightarrow{v}(t) = 0\,.$$

Also we have the following statement.

Lemma 4 *With the same A*

$$d\langle U, U\rangle(t) \le 2\, d\langle Z, Z\rangle(t)\,.$$
$$d\langle V, V\rangle(t) \le 2\, d\langle Z, Z\rangle(t)\,.$$

Or

$$\overrightarrow{u}(t)\cdot\overrightarrow{u}(t) \le 2\,(\overrightarrow{x}(t)\cdot\overrightarrow{x}(t) + \overrightarrow{y}(t)\cdot\overrightarrow{y}(t))\,,$$
$$\overrightarrow{v}(t)\cdot\overrightarrow{v}(t) \le 2\,(\overrightarrow{x}(t)\cdot\overrightarrow{x}(t) + \overrightarrow{y}(t)\cdot\overrightarrow{y}(t))\,.$$

Or

$$d\langle W, W\rangle(t) \le 4\, d\langle Z, Z\rangle(t)\,. \tag{13}$$

Proof

$$\overrightarrow{u}(t)\cdot\overrightarrow{u}(t) = (x_1 + y_2)^2 + (x_2 - y_1)^2 = 2\,(x_1\, y_2 - x_2\, y_1) +$$
$$(x_1)^2 + (y_2)^2 + (x_2)^2 + (y_1)^2 \le 2\,((x_1)^2 + (y_2)^2 + (x_2)^2 + (y_1)^2) = 2\, d\langle Z, Z\rangle\,.$$

The same can be shown for v. □

Definition The complex martingale $W = A \star Z$ is called the Ahlfors-Beurling transform of martingale Z.

Now let us quote again the theorem of Bañuelos–Janakiraman from [2]:

Theorem 5 *Let Z, W be any two martingales on the filtration of Brownian motion, such that W is an orthogonal martingale in the sense of (12): $d\langle U, V\rangle = 0$, and such that there is a subordination property*

$$d\langle W, W\rangle \le d\langle Z, Z\rangle \tag{14}$$

Let $p \ge 2$. Then

$$(\mathbf{E}\,|W|^p)^{1/p} \le \sqrt{\frac{p^2-p}{2}}(\mathbf{E}\,|Z|^p)^{1/p}\,. \tag{15}$$

Further we will use the notation

$$\|Z\|_p := (\mathbf{E}\,|Z|^p)^{1/p}\,.$$

Applied to our case (with the help of Lemmas 3, 4) we get the following theorem from Theorem 5.

Theorem 6 $\|W\|_p = \|A \star Z\|_p \le \sqrt{2(p^2-p)}\|Z\|_p\,,\ \ \forall p \ge 2\,.$

4 Subordination by Orthogonal Martingales in $L^{3/2}$

For $1 < p \le 2$ one has the following

Theorem 7 *Let Z, W be any two $\mathbb{R}^2$ martingales as above, and let W be an orthogonal martingale in the sense that :*

$$d\langle U, V\rangle = 0\,.$$

Let us also assume

$$d\langle U, U\rangle = d\langle V, V\rangle\,. \tag{16}$$

Let Z be subordinated to the orthogonal martingale W:

$$d\langle Z, Z\rangle \le \langle W, W\rangle \tag{17}$$

Then for $1 < q \le 2$

$$\|Z\|_q \le \sqrt{\frac{2}{q^2-q}}\|W\|_q\,. \tag{18}$$

Below we will give the proof for all $q \in (1, 2]$, but first we will give the proof only for $q = 3/2$. Moreover, our general-case proof may indicate that the constant $\sqrt{\frac{2}{p^2-p}}$ is sharp after all. (Note that a completely different proof, but with the same constant, is given in [20].)

Proof We assume that $F = (\Phi, \Psi)$ (or $F = \Phi + i\Psi$) is a martingale on the filtration of Brownian motion

$$\Phi(t) = \int_0^t \overrightarrow{\phi}(s) \cdot dB_s\,, \ \ \Psi(t) = \int_0^t \overrightarrow{\psi}(s) \cdot dB_s\,,$$
$$X(t) = \int_0^t \overrightarrow{x}(s) \cdot dB_s\,, \ \ Y(t) = \int_0^t \overrightarrow{y}(s) \cdot dB_s\,,$$
$$U(t) = \int_0^t \overrightarrow{u}(s) \cdot dB_s\,, \ \ V(t) = \int_0^t \overrightarrow{v}(s) \cdot dB_s\,,$$

and that these vector processes and their components satisfy Lemmas 3 and 4, namely:

$$u_1 v_1 + u_2 v_2 = 0\,, \tag{19}$$

$$(u_1)^2 + (u_2)^2 = (v_1)^2 + (v_2)^2\,, \tag{20}$$

$$\Im \mathbf{E}\,(F \cdot Z) = \int_0^t (d\langle \Phi, X\rangle + d\langle \Psi, Y\rangle)\, ds = \int_0^t (\phi_1 x_1 + \phi_2 x_2 + \psi_1 y_1 + \psi_2 y_2)\, ds\,.$$

Hence,

$$|\Im \mathbf{E}\,(Z \cdot F)| \le \int_0^t ((\phi_1)^2 + (\phi_2)^2 + (\psi_1)^2 + (\psi_2)^2)^{1/2}((x_1)^2 + (x_2)^2 + (y_1)^2 + (y_2)^2)^{1/2}\, ds\,. \tag{21}$$

By subordination assumption (17) we have

$$|\Im \mathbf{E}\,(Z \cdot F)| \le \int_0^t ((u_1)^2 + (u_2)^2 + (v_1)^2 + (v_2)^2)^{1/2}((\phi_1)^2 + (\phi_2)^2 + (\psi_1)^2 + (\psi_2)^2)^{1/2}\, ds\,. \tag{22}$$

Our next goal is to prove that

$$\sqrt{\frac{3}{2}} \int_0^t ((u_1)^2 + (u_2)^2 + (v_1)^2 + (v_2)^2)^{1/2}((\phi_1)^2 + (\phi_2)^2 + (\psi_1)^2 + (\psi_2)^2)^{1/2} \le 2\left(\frac{\|W\|_{3/2}^{3/2}}{3/2} + \frac{\|F\|_3^3}{3}\right). \tag{23}$$

Let us polarize the last equation to convert its RHS to $2\|W\|_{3/2}\|F\|_3$. Then let us use the combination of (22) and (4). Then we obtain the desired estimate

$$\|Z\|_{3/2} \le \frac{2\sqrt{2}}{\sqrt{3}}\|W\|_{3/2}\,, \tag{24}$$

which is equivalent to the claim of Theorem 7 for $q = 3/2$.

We are left to prove (4). For that we will need next several sections. □

5 Bellman Functions and Martingales, the Proof of (4)

Suppose we have the function of 4 real variables such that

$$B(y_{11}, y_{12}, y_{21}, y_{22}) \le \frac{2}{3}(y_{11}^2 + y_{12}^2)^{3/2} + \frac{4}{3}(y_{21}^2 + y_{22}^2)^{1/2}\,, \tag{25}$$

$$\langle d^2B(y_{11}, y_{12}, y_{21}, y_{22}) \begin{bmatrix} dy_{11} \\ dy_{12} \\ dy_{21} \\ dy_{22} \end{bmatrix}, \begin{bmatrix} dy_{11} \\ dy_{12} \\ dy_{21} \\ dy_{22} \end{bmatrix} \rangle \ge \tag{26}$$

$$\tau(dy_{11}^2 + dy_{12}^2) + \frac{1}{\tau}(dy_{21}^2 + dy_{22}^2) + \frac{3\tau}{4x_2}\Big(\frac{y_{22}dy_{21} - y_{21}dy_{22}}{x_2}\Big)^2$$
$$+ \frac{\tau x_1}{\sqrt{x_1^2 + 3x_2}}\Big[\frac{y_{11}dy_{11} + y_{12}dy_{12}}{x_1} + \frac{1}{\tau}\frac{y_{21}dy_{21} + y_{22}dy_{22}}{x_2}\Big]^2,$$

where

$$\frac{3}{4}\frac{\tau}{(y_{21}^2 + y_{22}^2)^{1/2}} + \frac{2}{\tau} \ge \frac{3}{\tau}\,. \tag{27}$$

Then we can prove (4). Let us start by writing Itô's formula for the process $b(t) := B(\Phi(t), \Psi(t), U(t), V(t))$:

$$db = \langle \nabla B(\Phi, .., V), (d\Phi(t), \ldots, dV(t)\rangle + \frac{1}{2}(d^2B(\phi_1, \psi_1, u_1, v_1) + d^2B(\phi_2, \psi_2, u_2, v_2))\,.$$

Here d^2B stands for the Hessian bilinear form. It is applied to vector $(\phi_1, \psi_1, u_1, v_1)$ and then to vector $(\phi_2, \psi_2, u_2, v_2)$. Of course, the second derivatives of B constituting this form are calculated at point (Φ, Ψ, U, V). All this is at time t. The first term is a martingale with zero average, and it disappears after taking the expectation.

Therefore,

$$\mathbf{E}\,(b(t)-b(0)) = \mathbf{E}\int_0^t db(s)\,ds =$$

$$\frac{1}{2}\int_0^t ((d^2B(\phi_1,\psi_1,u_1,v_1) + d^2B(\phi_2,\psi_2,u_2,v_2))\,ds =: \frac{1}{2}\int_0^t dI\,. \tag{28}$$

The sum in (28) is the Hessian bilinear form on vector (ϕ_1,ψ_1,u_1,v_1) plus the Hessian bilinear form on vector (ϕ_2,ψ_2,u_2,v_2). Using (26) we can add these two forms with a definite cancellation:

$$\begin{aligned}
&dI = \tau((\phi_1)^2 + (\psi_1)^2) + 1/\tau((u_1)^2 + (v_1)^2)+\\
&\frac{3}{4}\frac{\tau}{(U^2+V^2)^{1/2}}\frac{V^2(u_1)^2 + U^2(v_1)^2 - 2UVu_1v_1}{U^2+V^2} + \text{ Positive term}+\\
&\tau((\phi_2)^2 + (\psi_2)^2) + 1/\tau((u_2)^2 + (v_2)^2)+\\
&\frac{3}{4}\frac{\tau}{(U^2+V^2)^{1/2}}\frac{V^2(u_2)^2 + U^2(v_2)^2 - 2UVu_2v_2}{U^2+V^2} + \text{ Positive term}\,.
\end{aligned}$$

Notice that orthogonality (19) and equality of norms (20):

$$d\langle U,V\rangle = 0\,, \tag{29}$$

$$d\langle U,U\rangle = d\langle V,V\rangle\,, \tag{30}$$

imply pointwise equalities $u_1v_1 + u_2v_2 = 0$ and thus

$$V^2(u_1)^2+U^2(v_1)^2+V^2(u_2)^2+U^2(v_2)^2 = \frac{1}{2}(U^2+V^2)((u_1)^2+(u_2)^2+(v_1)^2+(v_2)^2)\,.$$

Therefore, UV–term above will disappear, and we will get

$$\begin{aligned}
&dI = \tau((\phi_1)^2 + (\psi_1)^2 + (\phi_2)^2 + (\psi_2)^2) + 1/\tau((u_1)^2 + (v_1)^2 + (u_2)^2 + (v_2)^2))+\\
&\frac{3}{4}\frac{\tau}{(U^2+V^2)^{3/2}}\cdot\frac{1}{2}(U^2+V^2)((u_1)^2 + (u_2)^2 + (v_1)^2 + (v_2)^2) + \text{ Positive} =\\
&\tau((\phi_1)^2 + (\psi_1)^2 + (\phi_2)^2 + (\psi_2)^2)+\\
&\frac{1}{2}\left(\frac{3}{4}\frac{\tau}{(U^2+V^2)^{1/2}} + \frac{2}{\tau}\right)((u_1)^2 + (v_1)^2 + (u_2)^2 + (v_2)^2)) + \text{ Positive}\,.
\end{aligned}$$

Hence, by using (27) we get

$$dI \geq \tau(\|\overrightarrow{\phi}\|^2 + \|\overrightarrow{\psi}\|^2) + \frac{3}{2}\cdot\frac{1}{\tau}(\|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2)\,.$$
$$\geq 2\sqrt{\frac{3}{2}}(\|\overrightarrow{\phi}\|^2 + \|\overrightarrow{\psi}\|^2)^{1/2}(\|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2)^{1/2}\,. \tag{31}$$

Let us combine now (28) and (31). We get

$$\sqrt{\frac{3}{2}}\int_0^t (\|\overrightarrow{\phi}\|^2 + \|\overrightarrow{\psi}\|^2)^{1/2}(\|\overrightarrow{u}\|^2 + \|\overrightarrow{v}\|^2)^{1/2}\,ds \leq \frac{1}{2}dI \leq \mathbf{E}\,(b(t))\,. \tag{32}$$

We used (25) that claims $b \geq 0$. But it also claims that

$$b(t) = B(\Phi(t), \Psi(t), U(t), V(t)) \leq 2\left(\frac{|(U,V)|^{3/2}}{3/2} + \frac{|(\Phi,\Psi)|^3}{3}\right). \tag{33}$$

Combine (32) and (33). We obtain (4).

To find the function with (25) and (26) we need the next section.

6 Special Function $B = \frac{2}{9}(y_{11}^2 + y_{12}^2) + 3(y_{21}^2 + y_{22}^2)^{1/2})^{3/2} + \frac{2}{9}((y_{11}^2 + y_{12}^2))^{3/2}$

It is useful if the reader thinks that $y_{11}, y_{12}, y_{21}, y_{22}$ are correspondingly Φ, Ψ, U, V.

Also in what follows $dy_{11}, dy_{12}, dy_{21}, dy_{22}$ can be viewed as ϕ_1, ψ_1, u_1, v_1 and ϕ_2, ψ_2, u_2, v_2.

Let $\mathbf{B}_{n+m}(x)$ be a real-valued function of $n+m$ variables $x = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$. Define a function $\mathbf{B}_{nk+m}(y)$ of n vector-valued variables $y_i = (y_{i1}, \ldots, y_{ik})$, $1 \leq i \leq n$, and m scalar variables y_i, $n+1 \leq i \leq n+m$, as follows:

$$\mathbf{B}_{nk+m}(y) = \mathbf{B}_{n+m}(x),$$

where

$$x_i = \|y_i\| := \Big(\sum_{j=1}^k y_{ij}^2\Big)^{\frac{1}{2}} \qquad \text{for } i \leq n,$$
$$x_i = y_i \qquad \text{for } i > n.$$

Omitting indices we shall denote by $\frac{d^2\mathbf{B}}{dx^2}$ and $\frac{d^2\mathbf{B}}{dy^2}$ the Hessian matrices of $\mathbf{B}_{n+m}(x)$ and $\mathbf{B}_{nk+m}(x)$, respectively.

7 Hessian of a Vector-Valued Function

Lemma 8 *Let P_j be the following operator from $\mathbb{R}^k$ to $\mathbb{R}$:*

$$P_j h = \frac{(h, y_j)}{x_j},$$

i.e., it gives the projection to the direction y_j. Let P be the block-diagonal operator from $\mathbb{R}^{kn+m} = \mathbb{R}^k \oplus \mathbb{R}^k \oplus \cdots \oplus \mathbb{R}^k \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ to $\mathbb{R}^{n+m} = \mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R}$ whose first n diagonal elements are P_j and the rest is identity. Then

$$\frac{d^2\mathbf{B}}{dy^2} = P^* \frac{d^2\mathbf{B}}{dx^2} P + diag\left\{ (I - P_i^* P_i) \frac{1}{x_i} \frac{\partial \mathbf{B}}{\partial x_i} \right\},$$

or

$$\begin{aligned} d^2\mathbf{B} = & \sum_{i,j=1}^{n} \frac{\partial^2 \mathbf{B}}{\partial x_i \partial x_j} \cdot \frac{\sum_{s=1}^k y_{is} dy_{is}}{x_i} \cdot \frac{\sum_{r=1}^k y_{jr} dy_{jr}}{x_j} \\ & + 2 \sum_{i=1}^{n} \sum_{j=n+1}^{n+m} \frac{\partial^2 \mathbf{B}}{\partial x_i \partial x_j} \cdot \frac{\sum_{s=1}^k y_{is} dy_{is}}{x_i} \cdot dy_j \\ & + \sum_{i=n+1}^{n+m} \sum_{j=n+1}^{n+m} \frac{\partial^2 \mathbf{B}}{\partial x_i \partial x_j} \cdot dy_i \cdot dy_j \\ & + \sum_{i=1}^{n} \frac{1}{x_i} \frac{\partial \mathbf{B}}{\partial x_i} \cdot \left(\sum_{j=i}^{k} dy_{ij}^2 - \left(\frac{\sum_{j=1}^k y_{ij} dy_{ij}}{x_i} \right)^2 \right). \end{aligned}$$

7.1 Positive Definite Quadratic Forms

Let

$$Q = Ax^2 + 2Bxy + Cy^2$$

be a positive definite quadratic form. We are interested in the best possible constant D such that

$$Q \geq 2D|x|\,|y| \qquad \text{for all } x, y \in \mathbb{R}\,.$$

After dividing this inequality over $|x|\,|y|$ we get

$$At \pm 2B + \frac{C}{t} \geq 2D \qquad \text{for all } t \in \mathbb{R} \setminus \{0\}\,.$$

The left-hand side has its minimum at the point $t = \sqrt{\frac{C}{A}}$. Therefore the best D is $\sqrt{AC} - |B|$.

Now we would like to present Q as a sum of three squares:

$$Q = D(\tau x^2 + \frac{1}{\tau}y^2) + (\alpha x + \beta y)^2\,,$$

which would immediately imply the required estimate. We think that

$$(A - D\tau)x^2 + 2Bxy + (C - \frac{D}{\tau})y^2$$

is a complete square, whence

$$(A - D\tau)(C - \frac{D}{\tau}) = B^2$$

or

$$CD\tau^2 - (AC - B^2 + D^2)\tau + AD = 0\,,$$
$$C\tau^2 - 2\sqrt{AC}\tau + A = 0\,.$$

Therefore, $\tau = \sqrt{\frac{A}{C}}$ and

$$Q = (\sqrt{AC} - |B|)\Big(\sqrt{\frac{A}{C}}x^2 + \sqrt{\frac{C}{A}}y^2\Big) + |B|\sqrt{\frac{A}{C}}\Big(x + \operatorname{sign} B\sqrt{\frac{C}{A}}y\Big)^2 \qquad (34)$$

7.2 Example

Let

$$\mathbf{B}_2(x) = \frac{2}{9}(x_1^2 + 3x_2)^{3/2} + \frac{2}{9}x_1^3\,, \qquad (35)$$

$$\mathbf{B}_4(y) = B_2(x); \qquad x_i = \sqrt{y_{i1}^2 + y_{i2}^2}\,.$$

Calculate the derivatives:

$$\frac{\partial \mathbf{B}_2}{\partial x_1} = \frac{2}{3} x_1 (\sqrt{x_1^2 + 3x_2} + x_1)\,, \frac{\partial \mathbf{B}_2}{\partial x_2} = \sqrt{x_1^2 + 3x_2}\,,$$

$$A = \frac{\partial^2 \mathbf{B}_2}{\partial x_1^2} = \frac{2(\sqrt{x_1^2 + 3x_2} + x_1)^2}{3\sqrt{x_1^2 + 3x_2}}\,,$$

$$B = \frac{\partial^2 \mathbf{B}_2}{\partial x_1 \partial x_2} = \frac{x_1}{\sqrt{x_1^2 + 3x_2}}\,, C = \frac{\partial^2 \mathbf{B}_2}{\partial x_2^2} = \frac{3}{2\sqrt{x_1^2 + 3x_2}}\,,$$

$$D = \sqrt{AC} - |B| = 1\,,$$

Also

$$\tau = \sqrt{\frac{A}{C}} = \frac{2}{3}(\sqrt{x_1^2 + 3x_2} + x_1)\,, \tag{36}$$

$$\frac{1}{\tau} = \frac{\sqrt{x_1^2 + 3x_2} - x_1}{2x_2}\,. \tag{37}$$

After substitution in the expressions of the preceding sections we get

$$\begin{aligned} d^2\mathbf{B}_4 = {} & \tau\Big(\frac{y_{11}dy_{11} + y_{12}dy_{12}}{x_1}\Big)^2 + \frac{1}{\tau}\Big(\frac{y_{21}dy_{21} + y_{22}dy_{22}}{x_2}\Big) \\ & + \frac{\tau x_1}{\sqrt{x_1^2 + 3x_2}}\Big[\frac{y_{11}dy_{11} + y_{12}dy_{12}}{x_1} + \frac{1}{\tau}\frac{y_{21}dy_{21} + y_{22}dy_{22}}{x_2}\Big]^2 \\ & + \frac{2}{3}\Big(\sqrt{x_1^2 + 3x_2} + x_1\Big)\Big(\frac{y_{12}dy_{11} - y_{11}dy_{12}}{x_1}\Big)^2 \\ & + \frac{\sqrt{x_1^2 + 3x_2}}{x_2}\Big(\frac{y_{22}dy_{21} - y_{21}dy_{22}}{x_2}\Big)^2 \\ = {} & \tau(dy_{11}^2 + dy_{12}^2) + \frac{1}{\tau}(dy_{21}^2 + dy_{22}^2) + \frac{3\tau}{4x_2}\Big(\frac{y_{22}dy_{21} - y_{21}dy_{22}}{x_2}\Big)^2 \\ & + \frac{\tau x_1}{\sqrt{x_1^2 + 3x_2}}\Big[\frac{y_{11}dy_{11} + y_{12}dy_{12}}{x_1} + \frac{1}{\tau}\frac{y_{21}dy_{21} + y_{22}dy_{22}}{x_2}\Big]^2. \end{aligned}$$

7.3 Verifying (27)

Here, using (36), (37) we get

$$\tau = \frac{2}{3}((y_{11}^2 + y_{12}^2) + 3(y_{21}^2 + y_{22}^2)^{1/2})^{1/2} + (y_{11}^2 + y_{12}^2)^{1/2}) . \tag{38}$$

And henceforth

$$\frac{1}{\tau} = \frac{((y_{11}^2 + y_{12}^2) + 3(y_{21}^2 + y_{22}^2)^{1/2})^{1/2} - (y_{11}^2 + y_{12}^2)^{1/2}}{2(y_{21}^2 + y_{22}^2)^{1/2}} . \tag{39}$$

Let us now (when we know τ) check the condition (27):

$$\frac{3}{4}\frac{\tau}{(y_{21}^2 + y_{22}^2)^{1/2}} + \frac{2}{\tau} = \frac{1}{2}\frac{(y_{11}^2 + y_{12}^2) + 3(y_{21}^2 + y_{22}^2)^{1/2})^{1/2} + (y_{11}^2 + y_{12}^2)^{1/2}}{(y_{21}^2 + y_{22}^2)^{1/2}} +$$
$$\frac{(y_{11}^2 + y_{12}^2) + 3(y_{21}^2 + y_{22}^2)^{1/2})^{1/2} - (y_{11}^2 + y_{12}^2)^{1/2}}{(y_{21}^2 + y_{22}^2)^{1/2}} =$$
$$\frac{3(y_{11}^2 + y_{12}^2) + 3(y_{21}^2 + y_{22}^2)^{1/2})^{1/2} - (y_{11}^2 + y_{12}^2)^{1/2}}{2(y_{21}^2 + y_{22}^2)^{1/2}} \geq \frac{3}{\tau} .$$

So, yes, we finished the proof of the fact that function

$$B = \frac{2}{9}(y_{11}^2 + y_{12}^2) + 3(y_{21}^2 + y_{22}^2)^{1/2})^{3/2} + \frac{2}{9}((y_{11}^2 + y_{12}^2))^{3/2}$$

satisfies all differential properties we wished, and thus it is proving our main result for $q = \frac{3}{2}$. In fact, we saw that it proves (4). In its turn we saw that (4) implies (24), which is the same as proving Theorem 7 for $q = 3/2$.

We are very lucky that B is found in the explicit form. There are only two such exponents, for which explicit form exists: $q = \frac{3}{2}$ and $q = 2$.

8 Explanation of How We Found This Special Function *B*: Pogorelov's Theorem

We owe the reader an explanation of where we got this function B, which played such a prominent part above.

Let $p \geq 2$. We want to find a function satisfying the following properties:

- 1) B is defined in the whole plane $\mathbb{R}^2$ and $B(u, v) = B(-u, v) = B(u, -v)$;
- 2) $0 \leq B(u, v) \leq (p - 1)(\frac{1}{p}|u|^p + \frac{1}{q}|v|^q)$;

- 3) Everywhere we have inequality for Hessian quadratic form $d^2B(u,v) \ge 2|du||dv|$;
- 4) Homogeneity: $B(c^{1/p}u, c^{1/q}v) = c\,B(u,v)$, $c > 0$;
- 5) Function B should be the "best" one satisfying 1), 2), 3).

We understand the last statement as follows: B must saturate inequalities to make them equalities on a natural subset of $\mathbb{R}^2$ in 2) and on a natural subset of the tangent bundle of $\mathbb{R}^2$ in 3).

Let us start with 3). This inequality just means that $d^2B(u,v) \ge 2dudv$, $d^2B(u,v) \ge -2dudv$ for any $(u,v) \in \mathbb{R}^2$ and for any $(du,dv) \in \mathbb{R}^2$. In other words, this is just positivity of matrices

$$\begin{bmatrix} B_{uu}, & B_{uv}-1 \\ B_{vu}-1, & B_{vv} \end{bmatrix} \ge 0\,, \begin{bmatrix} B_{uu}, & B_{uv}+1 \\ B_{vu}+1, & B_{vv} \end{bmatrix} \ge 0\,. \qquad (40)$$

Now we want (40) to barely occur. In other words, we want one of the matrices in (40) to have a zero determinant for every (u,v).

Notice that symmetry 1) allows us to consider B only in the first quadrant. Here we will assume the second matrix in (40) to have zero determinant in the first quadrant.

So let us assume for $u > 0, v > 0$

$$5) \qquad \det \begin{bmatrix} B_{uu}, & B_{uv}+1 \\ B_{vu}+1, & B_{vv} \end{bmatrix} = 0\,. \qquad (41)$$

Let us introduce

$$A(u,v) := B(u,v) + uv\,.$$

So, we require that

$$\det \begin{bmatrix} A_{uu}, & A_{uv} \\ A_{vu}, & A_{vv} \end{bmatrix} = 0\,. \qquad (42)$$

Returning to saturation of 2): we require that $B(u,v) = \phi(u,v) := (p-1)(\frac{1}{p}u^p + \frac{1}{q}v^q)$ at a non-zero point. By homogeneity 4) we have this equality on the whole curve Γ, where Γ is invariant under transformations $u \to c^{1/p}u$, $v \to c^{1/q}v$.

$$B(u,v) = \phi(u,v) := (p-1)(\frac{1}{p}u^p + \frac{1}{q}v^q) \text{ on the curve } v^q = \gamma u^p\,. \qquad (43)$$

Notice that γ is unknown at this moment. We are going to solve (42) and (43), so that our solution satisfies (40), 1), 2), 3), 4).

Remark We strongly suspect that such solution is still non-unique. On the other hand, one cannot "improve" 1), 2), 3), 4) by, say, changing 2 in 3) to a bigger constant, or making a constant $p-1$ in 2) smaller.

Recall that we have also the symmetry conditions on $A(u,v)+uv =: B(u,v)$. They are

$$B(-u,v)=B(u,v)\,,\ B(u,-v)=B(u,v)\,.$$

We assume the smoothness of B. It is a little bit ad hoc assumption, and we will be using it as such, namely, we will assume it when it is convenient and we will be on guard not to come to a contradiction. Anyway, assuming now the smoothness of B on the v-axis we get that the symmetry implies the Neumann boundary condition on B on v-axis: $\frac{\partial}{\partial u}B(0,v)=0$, that is

$$\frac{\partial}{\partial u}A(0,v)=v\,. \tag{44}$$

Solving the homogeneous Monge-Ampère equation is the same as building a surface of zero gaussian curvature. We base the following on a Theorem of Pogorelov [31]. The reader can see the algorithm in [36]. So we will be brief. Solution A must have the form

$$A(u,v)=t_1\cdot u+t_2\cdot v-t\,, \tag{45}$$

where $t_1:=A_u(u,v), t_2:=A_v(u,v), t(u,v)$ are unknown function of u,v, but, say, t_1,t_2 are certain functions of t. Moreover, Pogorelov's theorem says that

$$u\cdot dt_1+v\cdot dt_2-dt=0\,,\ \text{meaning}\ u\cdot\frac{dt_1}{dt}+v\cdot\frac{dt_2}{dt}-1=0\,. \tag{46}$$

We write homogeneity condition 4) as follows $A(c^{1/p}u,c^{1/q}v) = cA(u,v)$, differentiate in c and plug $c=1$. Then we obtain

$$A(u,v)=\frac{1}{p}t_1\cdot u+\frac{1}{q}t_2\cdot v\,, \tag{47}$$

which being combined with (45) gives

$$\frac{1}{q}t_1\cdot u+\frac{1}{p}t_2\cdot v-t=0\,. \tag{48}$$

Notice a simple thing, when t is fixed (46) gives us the equation of a line in (u,v) plane. Call this line L_t. Functions t_1,t_2 are certain (unknown at this moment) functions of t, so again, for a fixed t equation (48) also gives us a line. Of course this must be L_t. Comparing the coefficients we obtain differential equations on t_1, t_2:

$$q\frac{dt_1}{t_1} = \frac{dt}{t}\,,\; p\frac{dt_2}{t_2} = \frac{dt}{t}\,. \tag{49}$$

We write immediately the solutions in the following form:

$$t_1(t) = pC_1|t|^{\frac{1}{q}}\,,\; t_2(t) = qC_2|t|^{\frac{1}{p}}\,. \tag{50}$$

Plugging this into (47) one gets

$$A(u,v) = C_1 t^{\frac{1}{q}} u + C_2 t^{\frac{1}{p}} v\,,\;\; B(u,v) = C_1 t^{\frac{1}{q}} u + C_2 t^{\frac{1}{p}} v - uv\,, \tag{51}$$

where $t(u, v)$ (see (48)) is defined from the following implicit formula

$$t = \frac{p}{q} C_1 t^{\frac{1}{q}} u + \frac{q}{p} C_2 t^{\frac{1}{p}} v\,. \tag{52}$$

To define unknown constants C_1, C_2 we have only one boundary condition (44). However we have one more condition. It is a free boundary condition (we think that $p \geq 2 \geq q$)

$$B(u,v) = \phi(u,v) := (p-1)(\frac{1}{p}u^p + \frac{1}{q}v^q) \text{ on the curve } \Gamma := \{v^q = \gamma^q u^p\}\,. \tag{53}$$

This seems to be not saving us because we have three unknowns C_1, C_2, γ and two conditions: (44) and (53). But we will require in addition that $B(u, v)$ and $\phi(u, v)$ have the same tangent plane on the curve Γ:

$$\frac{B_u(u,v)}{B_v(u,v)} = \frac{\phi_u(u,v)}{\phi_v(u,v)} \text{ on the curve } \Gamma = \{v^q = \gamma^q u^p\}\,. \tag{54}$$

Now we are going to solve (44), (53), (54), to find C_1, C_2, γ and plug them into (51) and (52).

First of all

$$v = A_u(0,v) = t_1(0,v)\,.$$

So $v/pC_1 = t(0,v)^{\frac{1}{q}}$ from (50). Plug $u = 0$ into (52) to get $t(0,v)^{\frac{1}{q}} = \frac{q}{p}C_2 v$. Combining we get

$$C_1 C_2 = \frac{1}{q}\,.$$

Now we use (54).

$$\frac{t_1 - v}{t_2 - u} = \frac{u^{p-1}}{v^{q-1}} = \frac{u^p}{v^q}\frac{v}{u} = \frac{1}{\gamma^q}\frac{v}{u}\,.$$

Using (50) we get

$$\frac{pC_1 t^{\frac{1}{q}} - v}{qC_2 t^{\frac{1}{p}} - u} = \frac{1}{\gamma^q}\frac{v}{u}\,. \tag{55}$$

Let us write Γ as $u^p = \frac{1}{\gamma}uv$ or $v^q = \gamma^{q-1}uv$, and let us write on Γ

$$\begin{cases} t^{\frac{1}{q}} &= av \\ t^{\frac{1}{p}} &= bu \end{cases} \tag{56}$$

The reader will easily see from what follows that a, b are constants. From (55)

$$(pC_1 a - 1)\gamma^q uv = (qC_2 b - 1)uv\,. \tag{57}$$

Also from (56)

$$\frac{a^q}{b^p} = \frac{1}{\gamma^q}\,, \tag{58}$$

and from (56) and (52)

$$ab = \frac{p}{q}C_1 a + \frac{q}{p}C_2 b\,. \tag{59}$$

From (56), (53) it follows

$$C_1 a + C_2 b - 1 = (p-1)(\frac{1}{p}\cdot\frac{1}{\gamma} + \frac{1}{q}\cdot\gamma^{q-1})\,. \tag{60}$$

We already proved

$$C_1 C_2 = \frac{1}{q}\,. \tag{61}$$

We have five equations (55)–(61) on five unknowns C_1, C_2, a, b, γ.

One solution is obvious:

$$\gamma = 1\,, a = qC_2\,, b = pC_1\,, p^p C_1^p = q^q C_2^q\,,$$

from where one finds

$$C_1 = \frac{1}{p} p^{\frac{1}{p}}\,,\ C_2 = \frac{1}{q} p^{\frac{1}{q}}\,. \tag{62}$$

Therefore,

$$B(u, v) = \frac{1}{p} p^{\frac{1}{p}} t^{\frac{1}{q}} u + \frac{1}{q} p^{\frac{1}{q}} t^{\frac{1}{p}} v - uv\,, \tag{63}$$

where t is defined from

$$t = \frac{1}{q} p^{\frac{1}{p}} t^{\frac{1}{q}} u + \frac{1}{p} p^{\frac{1}{q}} t^{\frac{1}{p}} v\,. \tag{64}$$

If we specify $p = 3, q = \frac{3}{2}$ we get

$$C_1 = \frac{1}{3} 3^{\frac{1}{3}}\,,\ C_2 = \frac{2}{3} 3^{\frac{2}{3}}\,. \tag{65}$$

$$t^{\frac{2}{3}} = \frac{2}{3} 3^{\frac{1}{3}} t^{\frac{1}{3}} u + \frac{1}{3} 3^{\frac{2}{3}} v\,, \tag{66}$$

and solving the quadratic equation on $s := t^{\frac{1}{3}}$: $s^2 - 2C_1 us - \frac{C_2}{2} v = 0$, we get (the right root will be with $+$ sign)

$$t^{\frac{1}{3}}(u, v) = s = C_1 u + \sqrt{C_1^2 u^2 + \frac{C_2}{2} v}\,. \tag{67}$$

Therefore, $B(u, v)$ being equal to $C_1 s^2 u + C_2 sv - uv$ is ($C_1 C_2 = \frac{2}{3}$, see (61))

$$B(u, v) = C_1 u(2C_1 us + \frac{C_2}{2} v) + C_2 vs - uv = (2C_1^2 u^2 + C_2 v)s + \frac{1}{2} C_1 C_2 uv - uv\,,$$

and so

$$B(u, v) = (2C_1^2 u^2 + C_2 v)(C_1 u + \sqrt{C_1^2 u^2 + \frac{C_2}{2} v}) - \frac{2}{3} uv\,,$$

$$= (2C_1^2 u^2 + C_2 v)\sqrt{C_1^2 u^2 + \frac{C_2}{2} v} + 2C_1^3 u^3 + (C_1 C_2 - \frac{2}{3}) uv\,.$$

The last term disappears (see (61)), and we get

$$B(u, v) = 2(C_1^2 u^2 + \frac{C_2}{2} v)\sqrt{C_1^2 u^2 + \frac{C_2}{2} v} + 2C_1 u^3 = 2C_1^3 (u^2 + \frac{C_2}{2C_1} v)^{\frac{3}{2}} + 2C_1^3 u^3\,.$$

Finally from (65)

$$B(u,v) = \frac{2}{9}((u^2+3v)^{\frac{3}{2}} + u^3)\,. \tag{68}$$

This is exactly the function in (35). This function gave us our main theorem for $p = 3$. We have just explained how we got it.

By the way, in this particular case the transcendental equation on γ becomes the usual cubic equation on $\sqrt{\gamma}$: $2\sqrt{\gamma} + 1 = 4 - \frac{1}{\gamma}$, which has only one real solutions $\gamma = 1$.

9 Explanation: Pogorelov's Theorem Again

We owe the reader the explanation, why we chose the function $A(u,v) = B(u,v) + uv$ rather than $A(u,v) = B(u,v) - uv$ to have the degenerate Hessian form.

We want to find a function satisfying the following properties (in what follows $p \geq 2$):

- 1) B is defined in the whole plane $\mathbb{R}^2$ and $B(u,v) = B(-u,v) = B(u,-v)$;
- 2) $0 \leq B(u,v) \leq \phi(u,v) = (p-1)(\frac{1}{p}|u|^p + \frac{1}{q}|v|^q)$;
- 3) Everywhere we have inequality for Hessian quadratic form $d^2B(u,v) \geq 2|du||dv|$;
- 4) Homogeneity: $B(c^{1/p}u, c^{1/q}v) = c\,B(u,v)$, $c > 0$;
- 5) Function B should be the "best" one satisfying 1), 2), 3).

(i) What do we mean by *best* function? We would like B to be the 'largest' function below $\phi(u,v)$ such that the convexity condition in 3) holds. We expect that such a function should equal the upper bound $\phi(u,v)$ at some point(s) and the inequality in 3) should be equality where possible.

(ii) Due to the symmetry in 1), we can restrict our attention to $\{u > 0, v > 0\}$.

(iii) If we have at some (u,v), $B(u,v) = \phi(u,v)$, then condition 4) implies that $B(c^{1/p}u, c^{1/q}v) = cB(u,v) = c\phi(u,v) = \phi(c^{1/p}u, c^{1/q}v)$. Hence they remain equal on a curve $\{(u,v) : v^q = \gamma^q u^p\}$ for some γ.

(iv) The condition $< d^2B \cdot (u,v), (u,v) > \geq 2|u||v|$ means that the 'directional convexity' in direction (u,v) stays above the value $2|u||v|$. This means that the directional convexity of B is above that of both the functions uv and $-uv$. Equivalently we are asserting the positive definiteness of the matrices:

$$\begin{pmatrix} B_{uu} & B_{uv}-1 \\ B_{vu}-1 & B_{vv} \end{pmatrix} \geq 0, \begin{pmatrix} B_{uu} & B_{uv}+1 \\ B_{vu}+1 & B_{vv} \end{pmatrix} \geq 0. \tag{69}$$

(v) In order to optimize (69), we require that one of the matrices is degenerate (with "$=0$"). Suppose that the first matrix is degenerate. This means that the function $A(u,v) = B(u,v) - uv$ has a degenerate Hessian. At every point, one of its two non-negative eigenvalues is 0, and the function has 0 convexity in the direction of corresponding eigenvector. Since the matrix is positive definite, it follows that 0 is the minimal eigenvalue, hence the graph of this function is a surface with gaussian curvature 0.

Moreover the directional convexity of $B - uv$ is greater than that of $B + uv$ in directions of negative slope and less than in directions of positive slope. If we want $B + uv$ to have non-degenerate positive Hessian, then the degeneracy of $B - uv$ must occur in the positive slope direction.

Let us analyze the function $A(u, v) = B(u, v) - uv$. A theorem of Pogorolev tells us that A will be a linear function on lines of degeneracy. That is, it will have the form:

$$A(u, v) = t_1 u + t_2 v - t \tag{70}$$

where $t_1(u, v)$, $t_2(u, v)$ and $t(u, v)$ are constant on the lines given by

$$\frac{dt_1}{dt} u + \frac{dt_2}{dt} v - 1 = 0. \tag{71}$$

We can say two things about the coefficient functions, that the eigen-lines that intersect the positive y axis must also have $\frac{dt_1}{dt_2} \leq 0$ and $\frac{dt_2}{dt} \geq 0$ - this information comes from (71) and the fact that the eigen-lines have positive slope. At the moment we know nothing else about the coefficient functions. We will use the various boundary conditions on B, hence on A to determine them.

(i) First observe that since $B(u, v) = B(-u, v) = B(u, -v)$, we may expect that B is smooth on at least one of the two axes, assume on the y axis, and hence the corresponding derivative $\partial_u B(0, v) = 0$. This means:

$$\partial_u A(0, v) = -v. \tag{72}$$

(ii) We already assumed that

$$B(u, v) = \phi(u, v) = (p - 1)(\frac{u^p}{p} + \frac{v^q}{q}) \tag{73}$$

on some curve $\Gamma = \{v^q = \gamma^q u^p\}$.

(iii) Let us also assume that the tangent planes of B and ϕ agree on Γ. This means that the gradients of the two functions $B(u, v) - z$ and $\phi(u, v) - z$ should be parallel at the points $(u, v, \phi(u, v))$ where $(u, v) \in \Gamma$. Therefore

$$(\partial_u \phi, \partial_v \phi, -1) = \lambda(\partial_u B, \partial_v B, -1),$$

which implies $\lambda = 1$ and

$$B_u(u,v) = (p-1)u^{p-1}, B_v(u,v) = (p-1)v^{q-1} \tag{74}$$

on the curve Γ. Similarly on Γ,

$$A_u(u,v) = (p-1)u^{p-1} - v, A_v(u,v) = (p-1)v^{q-1} - u. \tag{75}$$

Recall:

$$A(u,v) = t_1 u + t_2 v - t \tag{76}$$

where $t_1(u,v) = A_u(u,v)$, $t_2(u,v) = A_v(u,v)$ and $t(u,v)$ are constant on the lines given by

$$\frac{dt_1}{dt}u + \frac{dt_2}{dt}v - 1 = 0. \tag{77}$$

We also have the homogeneity condition: $A(c^{1/p}u, c^{1/q}v) = cA(u,v)$. Differentiating this with respect to c and setting $c = 1$ gives:

$$A(u,v) = \frac{1}{p}A_u(u,v)u + \frac{1}{q}A_v(u,v)v \tag{78}$$

$$= \frac{1}{p}t_1 u + \frac{1}{q}t_2 v. \tag{79}$$

Comparing (76) and (79), we have

$$\frac{1}{q}t_1 u + \frac{1}{p}t_2 v - t = 0. \tag{80}$$

Now comparing (77) and (80) gives

$$\frac{dt_1}{dt} = \frac{1}{q}\frac{t_1}{t}, \frac{dt_2}{dt} = \frac{1}{p}\frac{t_2}{t}. \tag{81}$$

Solving these differential equations, we have

$$t_1(t) = C_1|t|^{1/q}, t_2(t) = C_2|t|^{1/p}. \tag{82}$$

Putting this into (80) gives:

$$t = \frac{1}{q}C_1|t|^{1/q}u + \frac{1}{p}C_2|t|^{1/p}v \tag{83}$$

Let us make two observations: Recall that if our eigen-line intercepts the positive y axis and has positive slope, then $\frac{dt_1}{dt_2} = \frac{q}{p}\frac{C_1}{C_2}|t|^{\frac{1}{q}-\frac{1}{p}} \le 0$ and $\frac{dt_2}{dt} \ge 0$. If $t > 0$, then $\frac{dt_2}{dt} = \frac{1}{p}C_2|t|^{-1/q}$, and if $t < 0$, then $\frac{dt_2}{dt} = -\frac{1}{p}C_2|t|^{-1/q}$. We conclude from this:

(i) If $t > 0$, then $C_1C_2 \le 0$ and $C_2 \ge 0$, hence $C_1 \le 0$,
(ii) If $t < 0$, then $C_1C_2 \le 0$ and $C_2 \le 0$, hence $C_1 \ge 0$.

Let us bring in the following: $t_1 = A_u(0, v) = -v$. The first equality is from Pogorolev and the second is the boundary condition (72). Then (82) implies that

$$-v = C_1|t(0, v)|^{1/q} \tag{84}$$

and (83) implies that

$$t(0, v) = \frac{1}{p}C_2|t(0, v)|^{1/p}v. \tag{85}$$

Conclude:

(i) If $v > 0$, then $C_1 < 0$. The previous observations imply $t > 0$ and $C_2 \ge 0$. We are concerned at present with this case of positive y intercept.
(ii) From (84) and (85), we conclude

$$C_1C_2 = -p. \tag{86}$$

Next from (75), we know that on Γ,

$$t_1 = (p-1)u^{p-1} - v = (\frac{p-1}{\gamma} - 1)v, t_2 = (p-1)v^{q-1} - u = ((p-1)\gamma^{q-1} - 1)u. \tag{87}$$

In terms of t, this says

$$C_1t^{1/q} = (\frac{p-1}{\gamma} - 1)v, C_2t^{1/p} = ((p-1)\gamma^{q-1} - 1)u. \tag{88}$$

Write on Γ

$$\begin{cases} t^{\frac{1}{q}} = aC_2v \\ t^{\frac{1}{p}} = bC_1u \end{cases} \tag{89}$$

Note that $a \ge 0$ and $b \le 0$ due to the signs of C_1 and C_2. Substituting in (88) and using (86) gives

$$a = \frac{1}{p} - \frac{1}{q\gamma}, b = \frac{1}{p} - \frac{1}{q}\gamma^{q-1}. \tag{90}$$

Note that (89) also implies that

$$\frac{a^q C_2^q}{|b|^p |C_1|^p} = \frac{1}{\gamma^q} \tag{91}$$

Hence (90) and (91) imply

$$(\frac{\gamma}{p} - \frac{1}{q})^q C_2^q = (\frac{1}{q}\gamma^{q-1} - \frac{1}{p})^p |C_1|^p. \tag{92}$$

(92), (86) and the fact $pq = p + q$ imply that

$$C_2 = \left(\frac{p((p-1)\gamma^{\frac{1}{p-1}} - 1)^{p-1}}{\gamma - (p-1)} \right)^{\frac{1}{p}}. \tag{93}$$

Next observe that (83), (86) and (89) imply

$$ab = \frac{1}{q}a + \frac{1}{p}b \tag{94}$$

and hence by (90)

$$(\frac{1}{p} - \frac{1}{q\gamma})(\frac{1}{p} - \frac{1}{q}\gamma^{q-1}) = \frac{1}{pq} + \frac{1}{p^2} - \frac{1}{q^2\gamma} - \frac{1}{pq}\gamma^{q-1} \tag{95}$$

The equation that follows from making substitutions into the boundary condition (73) $B = \phi$ on Γ and $A = B - uv$ gives no new relationship. So we can avoid its consideration.

Simplifying (95) shows that γ is solution to the equation

$$\gamma^{q-1} - (q-1)\gamma + 2 - q = 0. \tag{96}$$

The rest of the analysis is yet to be done. However, note that $\frac{B_u}{u} = \frac{\phi_u}{u}$ on Γ, and on the corresponding eigen-line, we can understand it by using the fact that $A_u = B_u - v$ is constant. This may help later.

10 The Case When $p = 3$ and $q = \frac{3}{2}$

Observe that by setting $\delta = \gamma^{q-1}$, we can rewrite (96) as

$$\delta^{p-1} - (p-1)\delta + 2 - p = 0. \tag{97}$$

Let us analyze the case when $p = 3$. Then this equation becomes

$$\delta^2 - 2\delta - 1 = 0 \tag{98}$$

whose unique positive solution is $\delta = 1 + \sqrt{2}$. Therefore

$$\gamma = (1 + \sqrt{2})^2 = 3 + 2\sqrt{2}. \tag{99}$$

Then using (86), (90) and (93), we obtain

$$a = -\frac{5}{3} + \frac{4\sqrt{2}}{3}, b = \frac{1}{3} - \frac{2}{3}(3 + 2\sqrt{2})^{1/2} \tag{100}$$

and

$$C_1 = \frac{-3^{\frac{2}{3}}(1 + 2\sqrt{2})^{1/3}}{(2\sqrt{3 + 2\sqrt{2}} - 1)^{2/3}}, C_2 = \frac{3^{1/3}(2\sqrt{3 + 2\sqrt{2}} - 1)^{2/3}}{(1 + 2\sqrt{2})^{1/3}} \tag{101}$$

Now we will explicitly find $B(u, v)$. Recall

$$\begin{aligned} B(u, v) &= \frac{1}{p}t_1 u + \frac{1}{q}t_2 v + uv \\ &= \frac{C_1}{p}t^{1/q}u + \frac{C_2}{q}t^{1/p}v + uv \\ &= \frac{C_1}{3}t^{2/3}u + \frac{2C_2}{3}t^{1/3}v + uv. \end{aligned}$$

$$\begin{aligned} t &= \frac{1}{q}C_1 t^{1/q}u + \frac{1}{p}C_2 t^{1/p}v \\ &= \frac{2}{3}C_1 t^{2/3}u + \frac{1}{3}C_2 t^{1/3}v. \end{aligned}$$

Let $s = t^{1/3}$. Then we have $s^2 - \frac{2}{3}C_1 us - \frac{1}{3}C_2 v = 0$ and

$$s = \frac{C_1}{3}u + \frac{1}{3}\sqrt{C_1^2 u^2 + 3C_2 v}.$$

$$\begin{aligned} B(u, v) &= \frac{C_1}{3}s^2 u + \frac{2}{3}C_2 s v + uv \\ &= \frac{C_1}{3}\left(\frac{2C_1^2}{9}u^2 + \frac{3C_2 v}{9} + \frac{2C_1 u}{9}\sqrt{C_1^2 u^2 + 3C_2 v}\right)u \\ &\quad + \frac{2}{9}C_1 C_2 uv + \frac{2}{9}C_1 C_2 uv + \frac{2}{9}C_2 v\sqrt{C_1^2 u^2 + 3C_2 v} + uv. \end{aligned}$$

Use the fact that $C_1C_2 = -3$ to simplify and obtain:

$$B(u,v) = \frac{2}{27}(C_1^2u^2 + 3C_2v)^{3/2} + \frac{2}{27}C_1^3u^3. \tag{102}$$

$$\begin{aligned}
B_u &= \frac{2}{9}C_1^3u^2 + \frac{1}{9}(C_1^2u^2 + 3C_2v)^{1/2}2C_1^2u \\
B_v &= \frac{C_2}{3}(C_1^2u^2 + 3C_2v)^{1/2} \\
B_{uu} &= \frac{2}{9}C_1^2\left[\frac{(\sqrt{C_1^2u^2 + 3C_2v} + C_1u)^2}{\sqrt{C_1^2u^2 + 3C_2v}}\right] \\
B_{uv} &= \frac{|C_1|uv}{\sqrt{C_1^2u^2 + 3C_2v}} \\
B_{vv} &= \frac{C_2^2}{2\sqrt{C_1^2u^2 + 3C_2v}}
\end{aligned}$$

$$\tau := \sqrt{\frac{B_{uu}}{B_{vv}}} = \frac{2}{3}\frac{|C_1|}{C_2}(\sqrt{C_1^2u^2 + 3C_2v} + C_1u) \tag{103}$$

$$\frac{1}{\tau} = \frac{\sqrt{C_1^2u^2 + 3C_2v} - C_1u}{2|C_1|v} \tag{104}$$

$$\begin{aligned}
\frac{B_u}{u} &= \frac{2}{9}C_1^3u + \frac{1}{9}(C_1^2u^2 + 3C_2v)^{1/2}2C_1^2 \\
\frac{B_v}{v} &= \frac{C_2(C_1^2u^2 + 3C_2v)^{1/2}}{3v}.
\end{aligned}$$

We can use $|C_1|C_2 = 3$ to deduce $\frac{B_u}{u} = \tau$. Next we compute the quadratic form associated with B by using the formulation before:

$$\begin{aligned}
&Q(dx,dy) = B_{uu}dx^2 + 2B_{uv}dxdy + B_{vv}dy^2 \\
&= (\sqrt{B_{uu}B_{vv}} - |B_{uv}|)\left(\sqrt{\frac{B_{uu}}{B_{vv}}}dx^2 + \sqrt{\frac{B_{vv}}{B_{uu}}}dy^2\right) + |B_{uv}|\sqrt{\frac{B_{uu}}{B_{vv}}}\left(dx + \operatorname{sign}(B_{uv})\sqrt{\frac{B_{vv}}{B_{uu}}}dy\right)^2 \\
&= (\sqrt{B_{uu}B_{vv}} - |B_{uv}|)\left(\tau dx^2 + \frac{1}{\tau}dy^2\right) + |B_{uv}|\tau\left(dx + \operatorname{sign}(B_{uv})\frac{1}{\tau}dy\right)^2.
\end{aligned}$$

Now let $\mathbf{B}(y_{11}, y_{12}, y_{21}, y_{22}) := B(\sqrt{y_{11}^2 + y_{12}^2}, \sqrt{y_{11}^2 + y_{12}^2}) = B(x_1, x_2)$. Then the associated quadratic form becomes

$$
\begin{aligned}
d^2\mathbf{B} = \tau\Big(\frac{y_{11}dy_{11} + y_{12}dy_{12}}{x_1}\Big)^2 + \frac{1}{\tau}\Big(\frac{y_{21}dy_{21} + y_{22}dy_{22}}{x_2}\Big) \\
+ \frac{\tau|C_1|x_1}{\sqrt{C_1^2x_1^2 + 3C_2x_2}}\Big[\frac{y_{11}dy_{11} + y_{12}dy_{12}}{x_1} + \frac{1}{\tau}\frac{y_{21}dy_{21} + y_{22}dy_{22}}{x_2}\Big]^2 \\
+ \Big(\frac{B_u}{u} = \tau\Big)\Big(\frac{y_{12}dy_{11} - y_{11}dy_{12}}{x_1}\Big)^2 \\
+ \Big(\frac{B_v}{v} = \frac{C_2(C_1^2x_1^2 + 3C_2x_2)^{1/2}}{3x_2}\Big)\Big(\frac{y_{22}dy_{21} - y_{21}dy_{22}}{x_2}\Big)^2 \\
= \tau(dy_{11}^2 + dy_{12}^2) + \frac{1}{\tau}(dy_{21}^2 + dy_{22}^2) \\
+ \Big(\frac{C_2(C_1^2x_1^2 + 3C_2x_2)^{1/2}}{3x_2} - \frac{1}{\tau}\Big)\Big(\frac{y_{22}dy_{21} - y_{21}dy_{22}}{x_2}\Big)^2 \\
+ \frac{\tau|C_1|x_1}{\sqrt{C_1^2x_1^2 + 3C_2x_2}}\Big[\frac{y_{11}dy_{11} + y_{12}dy_{12}}{x_1} + \frac{1}{\tau}\frac{y_{21}dy_{21} + y_{22}dy_{22}}{x_2}\Big]^2 \\
= \tau(dy_{11}^2 + dy_{12}^2) + \frac{1}{\tau}(dy_{21}^2 + dy_{22}^2) \\
+ \Big(\frac{3C_2\tau}{4C_1^2x_2}\Big)\Big(\frac{y_{22}dy_{21} - y_{21}dy_{22}}{x_2}\Big)^2 \\
+ \frac{\tau|C_1|x_1}{\sqrt{C_1^2x_1^2 + 3C_2x_2}}\Big[\frac{y_{11}dy_{11} + y_{12}dy_{12}}{x_1} + \frac{1}{\tau}\frac{y_{21}dy_{21} + y_{22}dy_{22}}{x_2}\Big]^2.
\end{aligned}
$$

In order for the quadratic form to have the self-improving property, we need

$$\frac{3C_2\tau}{4C_1^2x_2} + \frac{2}{\tau} \geq \frac{c}{\tau} \tag{105}$$

for suitable constant c. In fact if $\frac{C_2}{C_1^2} = 1$, we know that $c = 3$. This suggests that the right constant is $2 + \frac{C_2}{C_1^2} \approx 3.276142375$. (Calculation gives $|C_1| \approx 1.329660319$ and $C_2 \approx 2.256215334$, hence $\frac{C_2}{C_1^2} \approx 1.276142375$.)

If the rest of the process is the same as with the previous estimate, then the over all constant estimate would be approximately

$$\frac{2\sqrt{2}}{\sqrt{3.276142375}} \approx 1.562656814.$$

11 The Proof of Theorem 7 for General $q \in (1, 2]$

Recall that we found for $1 < q \le 2 \le p < \infty$, $1/p + 1/q = 1$ the following function

$$B(u, v) = B_q(u, v) = \frac{p^{\frac{1}{p}}}{p} t^{\frac{1}{q}} u + \frac{p^{\frac{1}{q}}}{q} t^{\frac{1}{p}} v - uv\,, \text{ where} \tag{106}$$

$$t = t(u, v) \text{ is the solution of } t = \frac{p^{\frac{1}{p}}}{q} t^{\frac{1}{q}} u + \frac{p^{\frac{1}{q}}}{p} t^{\frac{1}{p}} v\,. \tag{107}$$

Our goal is to represent the Hessian form of this implicitly given B as a sum of squares. This requires some calculations.

$$B_u = \frac{p^{\frac{1}{p}}}{p} t^{\frac{1}{q}} - v + \frac{1}{pq} S \frac{t'_u}{t}\,, \tag{108}$$

$$B_v = \frac{p^{\frac{1}{q}}}{q} t^{\frac{1}{p}} - u + \frac{1}{pq} S \frac{t'_v}{t}\,, \tag{109}$$

where

$$S := p^{\frac{1}{p}} t^{\frac{1}{q}} u + p^{\frac{1}{q}} t^{\frac{1}{p}} v\,. \tag{110}$$

Also

$$t'_u = \frac{p^{\frac{1}{p}}}{p} t^{\frac{1}{q}} \cdot \frac{t}{t - \frac{p^{\frac{1}{p}}}{q^2} t^{\frac{1}{q}} u - \frac{p^{\frac{1}{q}}}{p^2} t^{\frac{1}{p}} v}\,,$$

which, after using (107), (110) gives

$$\frac{t'_u}{t} = p \cdot p^{\frac{1}{p}} \frac{t^{\frac{1}{q}}}{S}\,. \tag{111}$$

Similarly,

$$\frac{t'_v}{t} = q \cdot p^{\frac{1}{q}} \frac{t^{\frac{1}{p}}}{S}\,. \tag{112}$$

Recall also that we had

$$A = A(u, v) = \frac{p^{\frac{1}{p}}}{p} t^{\frac{1}{q}} u + \frac{p^{\frac{1}{q}}}{q} t^{\frac{1}{p}} v\,. \tag{113}$$

Using the notations (110) and (113) we can compute the Hessian of $B = B_q$. Namely,

$$B_{uu} = \frac{2p^{\frac{1}{p}}}{pq} t^{\frac{1}{q}} \frac{t'_u}{t} - \frac{1}{pq} A (\frac{t'_u}{t})^2 + \frac{1}{pq} S \frac{t''_{uu}}{t} .$$

$$B_{vv} = \frac{2p^{\frac{1}{q}}}{pq} t^{\frac{1}{p}} \frac{t'_v}{t} - \frac{1}{pq} A (\frac{t'_v}{t})^2 + + \frac{1}{pq} S \frac{t''_{vv}}{t} .$$

$$B_{uv} = \frac{p\, t}{S} - \frac{1}{pq} A \frac{t'_u t'_v}{t^2} + + \frac{1}{pq} S \frac{t''_{uv}}{t}$$

Plugging

$$\frac{t''_{uu}}{t} = (\frac{1}{q} + 1 - \frac{1}{p})(\frac{t'_u}{t})^2 - \frac{t}{S}(\frac{t'_u}{t})^2$$

and using (111) we get the following concise formulas:

$$B_{uu} = \frac{1}{pq} S (\frac{t'_u}{t})^2 . \tag{114}$$

$$B_{vv} = \frac{1}{pq} S (\frac{t'_v}{t})^2 . \tag{115}$$

$$B_{uv} + 1 = \frac{1}{pq} S \frac{t'_u t'_v}{t^2} . \tag{116}$$

Let us introduce the notations:

$$\alpha = \frac{t'_u}{t}, \; \beta = \frac{t'_v}{t}, \; m = \frac{1}{pq} S, \; \tau = \frac{\alpha}{\beta} .$$

Then we saw in the previous sections that the Hessian quadratic form of B

$$Q(dx_1, dx_2) = B_{uu} dx_1^2 + 2(B_{uv} + 1) dx_1 dx_2 + B_{vv} dx_2^2$$

will have the form

$$Q = \frac{\alpha}{\beta} dx_1^2 + \frac{\beta}{\alpha} dx_2^2 + \frac{\alpha}{\beta}(m\alpha\beta - 1)(dx_1 + \frac{\beta}{\alpha} dx_2)^2 . \tag{117}$$

It is useful if the reader thinks that in what follows $y_{11}, y_{12}, y_{21}, y_{22}$ are, correspondingly, Φ, Ψ, U, V.

Also in what follows $dy_{11}, dy_{12}, dy_{21}, dy_{22}$ can be viewed as ϕ_1, ψ_1, u_1, v_1 and ϕ_2, ψ_2, u_2, v_2.

Our goal now is to "tensorize" the form Q. This operation means in our particular case to consider the new function, now of 4 real variables (or 2 complex variables if one prefers), given by

$$\mathcal{B} := \mathcal{B}(y_{11}, y_{12}, y_{21}, y_{22}) := B(x_1, x_2)\,, \text{ where } x_1 := \sqrt{y_{11}^2 + y_{12}^2}\,, x_2 := \sqrt{y_{21}^2 + y_{22}^2}$$

and to write its Hessian quadratic form. In the previous section we saw the formula for doing that:

$$\mathbb{Q} = \frac{\alpha}{\beta}\left(\frac{y_{11}dy_{11} + y_{12}dy_{12}}{x_1}\right)^2 + \frac{\beta}{\alpha}\left(\frac{y_{21}dy_{21} + y_{22}dy_{22}}{x_2}\right)^2 +$$
$$\frac{\alpha}{\beta}(m\alpha\beta - 1)\left(\frac{y_{11}dy_{11} + y_{12}dy_{12}}{x_1} + \frac{\beta}{\alpha}\frac{y_{21}dy_{21} + y_{22}dy_{22}}{x_2}\right)^2 +$$
$$\frac{B_u}{u}\left(\frac{y_{12}dy_{11} - y_{11}dy_{12}}{x_1}\right)^2 + \frac{B_v}{v}\left(\frac{y_{22}dy_{21} - y_{21}dy_{22}}{x_2}\right)^2.$$

To show that this quadratic form has an interesting self-improving property we are going to make some calculations. First of all notice that

$$\tau = \frac{\alpha}{\beta} = \frac{p \cdot p^{\frac{1}{p}} \cdot t^{\frac{1}{q}}}{q \cdot p^{\frac{1}{q}} \cdot t^{\frac{1}{p}}} \tag{118}$$

Now we start with combining (108) with (111)

$$B_u = \frac{p^{\frac{1}{p}}}{p} t^{\frac{1}{q}} - v + \frac{1}{pq} S \frac{p \cdot p^{\frac{1}{p}} \cdot t^{\frac{1}{q}}}{S} = p^{\frac{1}{p}} t^{\frac{1}{q}} - v\,. \tag{119}$$

Let us see that

$$\frac{p}{q} \frac{p^{\frac{1}{p}}}{p^{\frac{1}{q}}} \frac{t^{\frac{1}{q}}}{t^{\frac{1}{p}}} = \frac{p^{\frac{1}{p}} t^{\frac{1}{q}}}{u} - \frac{v}{u}\,. \tag{120}$$

This is the same as

$$p \cdot p^{\frac{1}{p}} \cdot t^{\frac{1}{q}} u = qpt - q \cdot p^{\frac{1}{q}} \cdot t^{\frac{1}{p}} v\,.$$

But the last claim is correct, it is just the implicit equation (107) for t. So (120) is correct. So, combining (118) and (119) we obtain

$$\frac{B_u}{u} = \frac{\alpha}{\beta}\,. \tag{121}$$

We would expect that $\frac{B_v}{v} = \frac{\beta}{\alpha} = \frac{1}{\tau}$ by symmetry, but actually $\frac{B_v}{v} > \frac{\beta}{\alpha}$ for $p > 2$ and this allows us to have an improved inequality for $\mathbb{Q}$. Let us see how.

Using (107) we get

$$\frac{B_v}{v} - \frac{\beta}{\alpha} = \frac{p^{\frac{1}{q}} t^{\frac{1}{p}}}{v} - \frac{u}{v} - \frac{q \cdot p^{\frac{1}{q}} \cdot t^{\frac{1}{p}}}{p \cdot p^{\frac{1}{p}} \cdot t^{\frac{1}{q}}} =$$

$$\frac{p^2 t - p \cdot p^{\frac{1}{p}} \cdot t^{\frac{1}{q}} u - q \cdot p^{\frac{1}{q}} \cdot t^{\frac{1}{p}} v}{p \cdot p^{\frac{1}{p}} \cdot t^{\frac{1}{q}} v} =$$

$$\frac{(p^2 - pq)t}{p \cdot p^{\frac{1}{p}} \cdot t^{\frac{1}{q}} v} = \frac{p-q}{p^{\frac{1}{p}}} \frac{t^{\frac{1}{p}}}{v} =$$

$$\frac{p-q}{p^{\frac{1}{p}}} \frac{\frac{p^{\frac{1}{p}}}{q} t^{\frac{1}{q}} u + \frac{p^{\frac{1}{q}}}{p} t^{\frac{1}{p}} v}{t^{\frac{1}{q}} v} =$$

$$\frac{(1-q/p)}{p^{\frac{1}{p}-\frac{1}{q}}} t^{\frac{1}{p}-\frac{1}{q}} + \left(\frac{p}{q} - 1\right).$$

In particular, using (118)

$$\frac{B_v}{v} - \frac{\beta}{\alpha} + \frac{2}{\tau} \geq \frac{(1-q/p)}{p^{\frac{1}{p}-\frac{1}{q}}} t^{\frac{1}{p}-\frac{1}{q}} + 2 \frac{q \cdot p^{\frac{1}{q}} \cdot t^{\frac{1}{p}}}{p \cdot p^{\frac{1}{p}} \cdot t^{\frac{1}{q}}} =$$

$$\frac{(1-q/p)}{p^{\frac{1}{p}-\frac{1}{q}}} t^{\frac{1}{p}-\frac{1}{q}} + \frac{2q/p}{p^{\frac{1}{p}-\frac{1}{q}}} t^{\frac{1}{p}-\frac{1}{q}} =$$

$$\frac{q}{p^{\frac{1}{p}-\frac{1}{q}}} t^{\frac{1}{p}-\frac{1}{q}} = p \cdot \frac{1}{\tau}.$$

This is what we need

$$\frac{B_v}{v} - \frac{\beta}{\alpha} + \frac{2}{\tau} = p \cdot \frac{1}{\tau} + (p/q - 1)\frac{u}{v} \geq p \cdot \frac{1}{\tau}. \tag{122}$$

Now let us take a look at $\mathbb{Q}$ and let us plug (121) and (122) into it. Then

$$\mathbb{Q} \geq \tau(dy_{11}^2 + dy_{12}^2) + \frac{1}{\tau}(dy_{21}^2 + dy_{22}^2) + (\frac{B_v}{v} - \frac{\beta}{\alpha})\left(\frac{y_{22}dy_{21} - y_{21}dy_{22}}{x_2}\right)^2. \tag{123}$$

Now imagine that we apply this estimate to **two** different collection of vectors $(dy_{11}, dy_{12}, dy_{21}, dy_{22})$, $(dy'_{11}, dy'_{12}, dy'_{21}, dy'_{22})$. Moreover, suppose that we have orthonormality condition

$$dy_{21} \cdot dy_{22} + dy'_{21} \cdot dy'_{22} = 0\,, dy_{21}^2 + (dy'_{21})^2 = dy_{22}^2 + (dy'_{22})^2\,. \tag{124}$$

Then we get from (123), (124)

$$\mathbb{Q}(dy) + \mathbb{Q}(dy') \geq \tau(dy_{11}^2 + dy_{12}^2 + (dy'_{11})^2 + (dy'_{12})^2) + 1/\tau(dy_{21}^2 + dy_{22}^2 + (dy'_{21})^2 + (dy'_{22})^2) +$$
$$(\frac{B_v}{v} - \frac{\beta}{\alpha})\frac{y_{22}^2 + y_{21}^2}{x_2^2}\frac{(dy_{21}^2 + dy_{22}^2 + (dy'_{21})^2 + (dy'_{22})^2)}{2}\,.$$

We denote $\xi_1^2 := dy_{11}^2 + dy_{12}^2 + (dy'_{11})^2 + (dy'_{12})^2, \xi_2^2 := dy_{21}^2 + dy_{22}^2 + (dy'_{21})^2 + (dy'_{22})^2$. Using that $\frac{y_{22}^2+y_{21}^2}{x_2^2} = 1$ and (122) we rewrite the RHS and get

$$\mathbb{Q}(dy) + \mathbb{Q}(dy') \geq \tau \cdot \xi_1^2 + \frac{1}{2}(\frac{B_v}{v} - \frac{\beta}{\alpha} + \frac{2}{\tau})\xi_2^2 \geq \tau \cdot \xi_1^2 + 1/\tau \cdot \frac{p}{2}\xi_2^2 \geq$$
$$2\sqrt{\frac{p}{2}}(dy_{11}^2 + dy_{12}^2 + (dy'_{11})^2 + (dy'_{12})^2)^{\frac{1}{2}}(dy_{21}^2 + dy_{22}^2 + (dy'_{21})^2 + (dy'_{22})^2)^{\frac{1}{2}}\,. \tag{125}$$

So we won $\sqrt{2/p} = \sqrt{\frac{2(q-1)}{q}}$ in comparison with the usual Burkholder estimate, which would be $\leq \frac{1}{q-1}$. So the estimate for the orthogonal martingale will be $\leq \sqrt{\frac{2(q-1)}{q}} \cdot \frac{1}{q-1} = \sqrt{\frac{2}{q(q-1)}}$.

And we get Theorem 7.

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Bounded Variation, Convexity, and Almost-Orthogonality

Michael Wilson

Abstract Let $N \geq 2$ be fixed. Suppose that, for every dyadic cube Q in $\mathbf{R}^d$, we have: N convex regions $\{R_i(Q)\}_1^N$, subsets of Q; and N complex numbers $\{c_i(Q)\}_1^N$ such that $|c_i(Q)| \leq 1$ and $\sum_1^N c_i(Q)|R_i(Q)| = 0$. Define $\tilde{h}_{(Q)}(x) \equiv |Q|^{-1/2}(\sum_1^N c_i(Q)\chi_{R_i(Q)}(x))$. We prove a technical theorem which implies that, for all such collections $\{\tilde{h}_{(Q)}\}_{Q\in\mathcal{D}}$ and all finite linear combinations $\sum \lambda_Q \tilde{h}_{(Q)}(x)$,

$$\left\| \sum \lambda_Q \tilde{h}_{(Q)} \right\|_2 \leq (2+\sqrt{2})Nd \left(\sum |\lambda_Q|^2 \right)^{1/2}.$$

We show that, if $\{\tilde{h}1_{(Q)}\}_{Q\in\mathcal{D}}$ and $\{\tilde{h}2_{(Q)}\}_{Q\in\mathcal{D}}$ are two such families, the L^2 bounded linear operator T defined by

$$T(f) \equiv \sum_Q \langle f, \tilde{h}1_{(Q)} \rangle h2_{(Q)}$$

is, in a natural sense, stable with respect to small dilation and translation errors in the kernel functions $\{\tilde{h}1_{(Q)}\}_{Q\in\mathcal{D}}$ and $\{\tilde{h}2_{(Q)}\}_{Q\in\mathcal{D}}$.

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1 Introduction

A family $\{\psi_\gamma\}_{\gamma\in\Gamma} \subset L^2(\mathbf{R}^d)$ is called *almost-orthogonal* if there is finite R so that, for all finite subsets $\mathcal{F} \subset \Gamma$ and all linear sums $\sum_{\gamma\in\mathcal{F}} \lambda_\gamma \psi_\gamma$,

$$\left\| \sum_{\gamma\in\mathcal{F}} \lambda_\gamma \psi_\gamma \right\|_2 \leq R \left(\sum_{\mathcal{F}} |\lambda_\gamma|^2 \right)^{1/2}. \tag{1}$$

M. Wilson (✉)
Department of Mathematics, University of Vermont, Burlington, VT 05405, USA
e-mail: jmwilson@uvm.edu

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If R is the least such constant for which (1) holds, we say the family is almost-orthogonal with constant R.

"Almost-orthogonal" is a mild misnomer: "almost-orthonormal" may be more accurate. We recall that a family $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is orthonormal if, for all γ and γ' in Γ,

$$\langle\psi_\gamma,\psi_{\gamma'}\rangle \equiv \int_{\mathbf{R}^d}\psi_\gamma(x)\,\overline{\psi_{\gamma'}(x)}\,dx = \begin{cases}1 & \text{if } \gamma=\gamma';\\ 0 & \text{otherwise.}\end{cases}$$

The family $\{\psi_\gamma\}_{\gamma\in\Gamma}$ is orthonormal if and only if, for all finite sums as in (1), we have equality, with $R=1$.

A duality argument shows that $\{\psi_\gamma\}_{\gamma\in\Gamma}\subset L^2(\mathbf{R}^d)$ satisfies (1) if and only if, for all $f\in L^2(\mathbf{R}^d)$,

$$\left(\sum_\Gamma |\langle f,\psi_\gamma\rangle|^2\right)^{1/2} \le R\|f\|_{L^2}. \tag{2}$$

Combining (1) and (2), we see that, if $\{\psi_\gamma^{(1)}\}_{\gamma\in\Gamma}$ and $\{\psi_\gamma^{(2)}\}_{\gamma\in\Gamma}$ are two almost-orthogonal families in $L^2(\mathbf{R}^d)$, with respective constants R_1 and R_2, then, for all $f\in L^2(\mathbf{R}^d)$,

$$\sum_\Gamma \langle f,\psi_\gamma^{(1)}\rangle\psi_\gamma^{(2)}$$

converges unconditionally[1] to define a linear operator $T: L^2\to L^2$ with bound $\le R_1R_2$. The canonical example of such an operator is the identity, where $\{\psi_\gamma^{(1)}\}_{\gamma\in\Gamma}$ and $\{\psi_\gamma^{(2)}\}_{\gamma\in\Gamma}$ are both the same complete orthonormal family, such as the classical Haar functions [3]. Recall that an interval I is dyadic if $I=[j2^k,(j+1)2^k)$ for some integers j and k. For each such I we set

$$h^{(I)}(x)\equiv\chi_{I_l}(x)-\chi_{I_r}(x),$$

where I_l is I's left half and I_r is I's right half. (We also use this notation for non-dyadic intervals.) The *Haar function* associated to I is $h^{(I)}(x)/|I|^{1/2}$, where, here and henceforth, $|E|$ is a set E's Lebesgue measure (of varying dimension!).

One can define "Haar functions" adapted to dyadic cubes in $\mathbf{R}^d$. A *cube* $Q\subset\mathbf{R}^d$ is a cartesian product of d intervals $I_i(Q)$ of equal length: $Q=\prod_1^d I_i(Q)$. We call their common length Q's *sidelength*, denoted $\ell(Q)$. The cube is *dyadic* if each $I_i(Q)$ is a dyadic interval. The set of all dyadic cubes in $\mathbf{R}^d$ is $\mathcal{D}$. The dimension d will vary but be clear from the context. We get d-dimensional Haar functions for the Qs

[1] We state our precise meaning of "unconditional convergence" in Definition 1.

in $\mathcal{D}$ by taking products

$$\mu_1(x_1) \times \mu_2(x_2) \times \cdots \times \mu_d(x_d)$$

where $x = (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d$ and each μ_j is $h^{(I_j(Q))}$ or $\chi_{I_j(Q)}$. We run over all such products *except* the one for which every μ_j equals $\chi_{I_j(Q)}$. This yields, for each $Q \in \mathcal{D}$, an orthogonal set of $2^d - 1$ functions, $\{h_{(i)}^{(Q)}\}_1^{2^d-1}$. Each $h_{(i)}^{(Q)}$ is supported on Q (where it only takes on the values ± 1), has integral equal to 0, and is constant on Q's immediate dyadic subcubes. We normalize the set by dividing each $h_{(i)}^{(Q)}$ by $|Q|^{1/2}$. The resulting "Haar functions",

$$\left\{ \frac{h_{(i)}^{(Q)}}{|Q|^{1/2}} \right\}_{Q \in \mathcal{D},\, 1 \le i < 2^d} \tag{3}$$

make up a complete orthonormal family for $L^2(\mathbf{R}^d)$, letting us write

$$f = \sum_{Q,i} \frac{\langle f, h_{(i)}^{(Q)} \rangle}{|Q|} h_{(i)}^{(Q)}, \tag{4}$$

for any $f \in L^2(\mathbf{R}^d)$.

Formula (4) is true, but is it *stable*? If we want to use (4) to investigate f, we have to estimate integrals

$$\langle f, h_{(i)}^{(Q)} \rangle = \int_{\mathbf{R}^d} f(x)\, h_{(i)}^{(Q)}(x)\, dx,$$

which are likely to have small errors. We might make translation errors: instead of $f(x)$ we have $f(x + \vec{\tau}_1^{\,i}(Q))$, where (we hope) $|\vec{\tau}_1^{\,i}(Q)| < \ell(Q)$, the computed inner product is

$$\int_{\mathbf{R}^d} f(x)\, h_{(i)}^{(Q)}(x - \vec{\tau}_1^{\,i}(Q))\, dx \equiv \langle f, h1_{(i)}^{(Q)} \rangle.$$

We can expect similar translation errors—call them $\vec{\tau}_2^{\,i}(Q)$—in the other $h_{(i)}^{(Q)}$s occurring in (4), resulting in "perturbed" Haar functions $h2_{(i)}^{(Q)}$. If we try to add up part of (4), we face

$$\sum_{Q,i} \frac{\langle f, h1_{(i)}^{(Q)} \rangle}{|Q|} h2_{(i)}^{(Q)}. \tag{5}$$

If the $\vec{\tau}_k^i(Q)$s have norms $\leq \eta\ell(Q)$, where η is small, then we hope that

$$\left\| f - \sum_{Q,i} \frac{\langle f, h1_{(i)}^{(Q)} \rangle}{|Q|} h2_{(i)}^{(Q)} \right\|_2 \leq C(\eta)\|f\|_2 \tag{6}$$

for some function $C(\eta)$ going to 0 as $\eta \to 0$.

But it is not clear that the families $\{hk_{(i)}^{(Q)}/|Q|^{1/2}\}_{Q,i}$ $(k = 1, 2)$ are even almost-orthogonal. The problem comes from the Haar functions' jumps. We can fix this by working with a smoother family. Let $0 < \alpha \leq 1$. Suppose that, for each $Q \in \mathcal{D}$, we have a function $\phi^{(Q)} : \mathbf{R}^d \to \mathbf{C}$ such that:

(a) $\operatorname{supp} \phi^{(Q)} \subset Q$;
(b) $|\phi^{(Q)}(x) - \phi_{(Q)}(x')| \leq (|x - x'|/\ell(Q))^\alpha$ for all x and x';
(c) $\int \phi^{(Q)}\, dx = 0$.

It is well known that $\{\phi^{(Q)}/|Q|^{1/2}\}_{Q\in\mathcal{D}}$ is almost-orthogonal in $L^2(\mathbf{R}^d)$ [3, 4]. If $\{\phi_{(1)}^{(Q)}/|Q|^{1/2}\}_{Q\in\mathcal{D}}$ and $\{\phi_{(2)}^{(Q)}/|Q|^{1/2}\}_{Q\in\mathcal{D}}$ are two such families then the unconditionally convergent sum

$$\sum_{Q\in\mathcal{D}} \frac{\langle f, \phi_{(1)}^{(Q)} \rangle}{|Q|} \phi_{(2)}^{(Q)}(x), \tag{7}$$

defines bounded linear operator $T : L^2 \to L^2$. This sum is also *stable*. Let $0 < \eta < 1/2$ and let $\{\vec{\tau}_i(Q)\}_{Q\in\mathcal{D}}$ $(i = 1, 2)$ be two families of vectors in $\mathbf{R}^d$ such that $|\vec{\tau}_i(Q)| \leq \eta\ell(Q)$. Define $\widetilde{\phi_{(i)}^{(Q)}}(x) = \phi_{(i)}^{(Q)}(x - \vec{\tau}_1(Q))$ $(i = 1, 2)$. The families $\{\widetilde{\phi_{(i)}^{(Q)}}/|Q|^{1/2}\}_{Q\in\mathcal{D}}$ are almost-orthogonal, with constants $\leq C(\alpha, d)$ [3, 4], implying that

$$\widetilde{T}(f) \equiv \sum_{Q\in\mathcal{D}} \frac{\langle f, \widetilde{\phi_{(1)}^{(Q)}} \rangle}{|Q|} \widetilde{\phi_{(2)}^{(Q)}}$$

defines a bounded linear operator on L^2. Moreover, for every $0 < r < \alpha$, there is a constant $C = C(\alpha, r, d)$ so that, for all $f \in L^2(\mathbf{R}^d)$ [4],

$$\left\| T(f) - \widetilde{T}(f) \right\|_2 \leq C\eta^r \|f\|_2; \tag{8}$$

and analogous results hold in $L^p(\mathbf{R}^d)$ if $1 < p < \infty$ [4]. The $\phi_{(i)}^{(Q)}$s' smoothness is crucial here. But with the $hk_{(i)}^{(Q)}$s, "α is 0", and the Hölder smooth $\phi^{(Q)}$s seem better for working with wavelet representations of operators. This superiority is somewhat specious. In the real world, (7) is discretized: the $\phi^{(Q)}$s are replaced by discontinuous, piecewise constant functions. Sums like (4) provide a model to understand their sensitivity to errors.

It turns out that the perturbed Haar systems *are* almost-orthogonal in $L^2(\mathbf{R}^d)$ (Theorems 1 and 2) and series like (4) are stable: they satisfy (6) with $C(\eta)$ equal to a dimensional constant times $\eta^{1/2}$ (Theorem 3). The almost-orthogonality and stability results hold for much more general systems, perturbations, and operators than those discussed above, and the exponent on η is sharp.

Our proofs of these facts start from a familiar concept. A function $f : [a, b] \to \mathbf{C}$ is said to be *of bounded variation on* $[a, b]$ (written $f \in BV[a, b]$) [1] if there is a finite M so that, for all partitions $P = \{a = x_0 < \cdots < x_n = b\}$ of $[a, b]$,

$$\sum_{1}^{n} |f(x_k) - f(x_{k-1})| \leq M.$$

The supremum over all such sums is called f's total variation over $[a, b]$ and is denoted $V_f[a, b]$. (When we write $V_f(I)$ and $I = [a, b]$, we mean $V_f[a, b]$.) If $f \in BV[a, b]$ then $f \in BV[c, d]$ for every $[c, d] \subset [a, b]$, and, for every partition P as above,

$$\sum_{1}^{n} V_f[x_{k-1}, x_k] = V_f(\cup_1^n [x_{k-1}, x_k]) = V_f[a, b].$$

We say that a function is of bounded variation on $\mathbf{R}$ if the supremum of the preceding expression, over all closed bounded intervals, is finite; and we call that supremum the function's total variation on $\mathbf{R}$.

For every cube $Q \subset \mathbf{R}^d$, let $NBV(Q)$ be the set of $f : \mathbf{R}^d \to \mathbf{C}$ such that: (a) f is measurable; (b) f's support is a subset of $\overline{Q}$ (the closure of Q); (c) for each $1 \leq i \leq d$, f is of bounded variation with respect to x_i on $\mathbf{R}$, with total variation on $\mathbf{R}$ being ≤ 1; (d) $\int f\, dx = 0$.

Condition (c) means: If we fix the $x_2, x_3, \ldots, x_d$ components of $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$, then the function $f(\cdot, x_2, x_3, \ldots, x_d)$ has total variation ≤ 1 on $\mathbf{R}$, with the analogous statements for x_2, x_3, etc.

Our fundamental result is:

Theorem 1 *If $f^{(Q)} \in NBV(Q)$ for every $Q \in \mathcal{D}$ then*

$$\left\{ \frac{f^{(Q)}}{|Q|^{1/2}} \right\}_{Q \in \mathcal{D}}$$

is almost-orthogonal in $L^2(\mathbf{R}^d)$, with constant $\leq \left(1 + \frac{1}{\sqrt{2}}\right) d$.

Theorem 1 immediately implies the fact stated in the abstract:

Corollary 1 *Let $N \geq 2$. Suppose that, for every dyadic cube $Q \subset \mathbf{R}^d$, we have N convex regions $\{R_i(Q)\}_1^N$, subsets of Q, and N complex numbers $\{c_i(Q)\}_1^N$ such that*

$|c_i(Q)| \leq 1$ *and* $\sum_1^N c_i(Q)|R_i(Q)| = 0$. *Define, for every* $Q \in \mathcal{D}$,

$$\tilde{h}_{(Q)}(x) \equiv |Q|^{-1/2} \left(\sum_1^N c_i(Q) \chi_{R_i(Q)}(x) \right).$$

Then, for every finite linear combination $\sum_{Q \in \mathcal{D}} \lambda_Q \tilde{h}_{(Q)}$,

$$\left\| \sum_{Q \in \mathcal{D}} \lambda_Q \tilde{h}_{(Q)} \right\|_2 \leq \left(2 + \sqrt{2}\right) Nd \left(\sum_{Q \in \mathcal{D}} |\lambda_Q|^2 \right)^{1/2}.$$

Proof Each function $\sum_1^N c_i(Q) \chi_{R_i(Q)}$ equals $2N$ times some $f^{(Q)} \in NBV(Q)$.

Corollary 1 holds no matter what the convex bodies are (cones, spheres, parallelepipeds, cylinders, etc.) or how they are placed (overlapping, disjoint, etc.). Careful placement gives a better constant.

Corollary 2 *Suppose that, for every dyadic cube* $Q \subset \mathbf{R}^d$, *we have* 2^d *convex regions* $\{R_i(Q)\}_1^{2^d}$, *where each* $R_i(Q)$ *is a subset of a* ***unique*** *immediate dyadic subcube of* Q, *and that we have complex numbers* $\{c_i(Q)\}_1^{2^d}$ *such that* $|c_i(Q)| \leq 1$ *and* $\sum_1^{2^d} c_i(Q)|R_i(Q)| = 0$. *Define, for every* $Q \in \mathcal{D}$,

$$\tilde{h}_{(Q)}(x) \equiv |Q|^{-1/2} \left(\sum_1^{2^d} c_i(Q) \chi_{R_i(Q)}(x) \right).$$

For every finite linear combination $\sum_{Q \in \mathcal{D}} \lambda_Q \tilde{h}_{(Q)}$,

$$\left\| \sum_{Q \in \mathcal{D}} \lambda_Q \tilde{h}_{(Q)} \right\|_2 \leq \left(4 + 2\sqrt{2}\right) d \left(\sum_{Q \in \mathcal{D}} |\lambda_Q|^2 \right)^{1/2}.$$

Again, it's simple: because of how we placed the $R_i(Q)$s, each function $\sum_1^{2^d} c_i(Q) \chi_{R_i(Q)}$ equals 4 times some $f^{(Q)} \in NBV(Q)$.

After proving Theorem 1 we look at the stability of almost-orthogonal expansions of the form

$$T(g) \equiv \sum_{Q \in \mathcal{D}} \frac{\langle g, f_1^{(Q)} \rangle}{|Q|} f_2^{(Q)}, \tag{9}$$

where each $f_i^{(Q)} \in NBV(Q)$. Corollary 3 shows that, for any $g \in L^2$, the series in (9) converges unconditionally to define T as a bounded linear operator on L^2. In Theorem 3 we show that the operator defined by (9) is L^2-stable with respect to small dilation and translation errors in the functions $f_i^{(Q)}$. We now say precisely what those small errors are.

Given a family of functions $\{f^{(Q)}\}_{Q\in\mathcal{D}}$, where each $f^{(Q)} \in NBV(Q)$, we suppose we have two sequences of vectors $\{\vec{\delta}(Q)\}_{Q\in\mathcal{D}}$ and $\{\vec{\tau}(Q)\}_{Q\in\mathcal{D}}$ in $\mathbf{R}^d$. The vectors $\vec{\tau}(Q)$ are assumed to be small and the vectors $\vec{\delta}(Q)$ are assumed to be close to $\vec{1} \equiv (1, 1, 1, \ldots, 1)$. Precisely, for some $0 < \eta < 1/2$, $|\vec{1} - \vec{\delta}(Q)| + |\vec{\tau}(Q)| < \eta$ for all $Q \in \mathcal{D}$. If $\vec{\delta}(Q) = (\delta_1, \delta_2, \ldots, \delta_d)$ and $x = (x_1, x_2, \ldots, x_d) \in \mathbf{R}^d$ we shall set $\vec{\delta}(Q)x \equiv (\delta_1 x_1, \delta_2 x_2, \delta_3 x_3, \ldots, \delta_d x_d)$.

We define the *perturbed* form of $f^{(Q)}$ by

$$\widetilde{f^{(Q)}}(x) \equiv f^{(Q)}\left(\vec{\delta}(Q)(x - x_Q + \ell(Q)\vec{\tau}(Q)) + x_Q\right). \tag{10}$$

The effect of replacing x with $\vec{\delta}(Q)(x - x_Q + \ell(Q)\vec{\tau}(Q)) + x_Q$ is to shift $f^{(Q)}$'s "center" a bit and dilate it slightly "relative to x_Q". For example, if

$$g(x) = \chi_{B(x_Q;\ell(Q))}(x),$$

the characteristic function of a ball roughly comparable to Q, and $\vec{\delta}(Q) = (\delta, \delta, \delta, \ldots, \delta)$, then

$$g(\vec{\delta}(Q)(x - x_Q + \ell(Q)\vec{\tau}(Q)) + x_Q) = \chi_{B(x_Q-\ell(Q)\vec{\tau}(Q);\ell(Q)/\delta)}(x):$$

the center shifts by a small multiple of $\ell(Q)$ and the radius gets multiplied by δ^{-1}. (For a general $\delta(Q)$, the ball becomes an ellipsoid.) For an operator T like (9) built from two families $\{f_j^{(Q)}\}_{Q\in\mathcal{D}}$ ($j = 1, 2$), we assume we have sequences of vectors $\{\vec{\delta}_j(Q)\}_{Q\in\mathcal{D}}$ and vectors $\{\vec{\tau}_j(Q)\}_{Q\in\mathcal{D}}$ such that $|\vec{1} - \vec{\delta}_j(Q)| + |\vec{\tau}_j(Q)| < \eta$, from which we define the analogous $\widetilde{f_j^{(Q)}}$s as given by formula (10). We define a *perturbation* of T in the obvious way:

$$\widetilde{T}(g) \equiv \sum_{Q\in\mathcal{D}} \frac{\langle g, \widetilde{f_1^{(Q)}}\rangle}{|Q|} \widetilde{f_2^{(Q)}}. \tag{11}$$

In Sect. 3 we prove:

Theorem 2 *The operator defined by (11) is L^2 bounded, with norm $\leq C(d)$.*

Theorem 3 *There is a constant $C = C(d)$, independent of η, so that, for all operators T and $\widetilde{T}$ (as defined by (9) and (11), respectively), and all $g \in L^2(\mathbf{R}^d)$,*

$$\|T(g) - \widetilde{T}(g)\|_2 \leq C(d)\eta^{1/2}\|g\|_2.$$

The exponent $1/2$ is the best possible. Let $\{f_j^{(Q)}/|Q|^{1/2}\}_{Q\in\mathcal{D}}$ ($j = 1, 2$) be the Haar functions on $\mathbf{R}$ and let $g = h_{[0,1)}$. Leave the $f_1^{(Q)}$s alone but shift $h_{[0,1)}$ in the

$f_2^{(Q)}$ system to the right by $0 < \eta < 1/10$. Then $T(g)(x) = h_{[0,1)}(x)$, $\widetilde{T}(g)(x) = h_{[0,1)}(x-\eta)$, and $\|T(g) - \widetilde{T}(g)\|_2 \sim \eta^{1/2}$.

At two places the reader may wonder why we are doing things certain ways when others seem simpler. Remarks there (labeled "Point 1" and "Point 2") direct the reader to an appendix (Sect. 3) for motivations. Originally we tried to put these in the introduction, but attempts to motivate the motivations (before stating the proofs) made the paper too long and confusing. We removed them, thinking nobody would care about them anyway, but the referee asked about precisely those issues. We then had the idea of addressing them in an appendix. We are grateful to the referee for getting us to explain ourselves, and helping to make the paper not too long and just confusing enough. The "points" remarks occur, respectively, after the proofs of Lemma 1 and Theorem 1.

We write $A \sim B$—where A and B are positive quantities depending on some parameters—to mean that there are positive numbers c_1 and c_2 ("comparability constants") so that

$$c_1 A \leq B \leq c_2 A; \tag{12}$$

and, if c_1 and c_2 do happen to depend on parameters, (12) does not become trivial. We will often use 'C' to denote a constant which might change to occurrence. We will not always state the parameters C depends on. If E and F are sets, we write $E \subset F$ to express $E \subseteq F$.

We indicate the end of the proof with the symbol ♠.

2 The Proof of Theorem 1

We begin with two lemmas.

Lemma 1 *Let I be a closed, bounded interval. Suppose that $f : I \to \mathbf{C}$ is of bounded variation, with $V_f(I) \leq 1$, $b : I \to \mathbf{R}$ is integrable, and $\int b\,dx = 0$. Then:*

$$\left| \int_I f\, b\, dx \right| \leq (1/2)\|b\|_1. \tag{13}$$

Proof of Lemma 1 Take $\|b\|_1 = 1$. Assume first that f is real. If b^+ and b^- are b's positive and negative parts then $\int b^+\,dx = \int b^-\,dx = 1/2$, implying

$$\int f(x)\, b^+(x)\, dx = (1/2)s_1$$

and

$$\int f(x)\, b^-(x)\, dx = (1/2)s_2,$$

where s_1 and s_2 are two numbers lying in $[\inf_I f, \sup_I f]$. Therefore

$$\left|\int f(x)\,(b^+(x) - b^-(x))\,dx\right| = (1/2)|s_1 - s_2| \leq (1/2) \sup_{x,y\in I} |f(x) - f(y)| \leq 1/2.$$

If f is not real, let α be a complex number with modulus equal to 1 such that

$$\left|\int_I f\,b\,dx\right| = \int (\alpha f(x))\,b(x)\,dx = \int (\Re(\alpha f(x)))\,b(x)\,dx,$$

and apply the same argument to $\Re(\alpha f)$. ♠

Point 1. Using bounded variation seems like overkill. For f defined on I we can set

$$\Omega_f(I) \equiv \sup\{|f(x) - f(y)| : x, y \in I\}.$$

If $\Omega_f(I) \leq 1$ we'll get

$$\left|\int_I f\,b\,dx\right| \leq (1/2)\|b\|_1.$$

Why use $V_f(I)$? See the appendix.

The second lemma lets us prove Theorem 1 by induction on d.

Lemma 2 *Suppose that $d \geq 2$, $Q \subset \mathbf{R}^d$ is a cube, and $f : \mathbf{R}^d \to \mathbf{C}$ lies in $NBV(Q)$. Write $Q \equiv I_1(Q) \times K(Q)$, where $K(Q) = \prod_2^d I_i(Q)$. For $y \in \mathbf{R}^{d-1}$ define*

$$\phi(y) \equiv \ell(Q)^{-1} \int_{I_1(Q)} f(t, y)\,dt.$$

Then $\phi \in NBV(K(Q))$.

Proof of Lemma 2 It is trivial that supp $\phi \subset \overline{K(Q)}$ and $\int \phi\,dy = 0$. For $2 \leq j \leq d$, let $\{y_k\}_0^n$ be points in $\mathbf{R}^{d-1}$ differing only in their x_j coordinates, where these increase with k. Then:

$$\sum_1^n |\phi(y_k) - \phi(y_{k-1})| \leq \ell(Q)^{-1} \int_{I_1(Q)} \left(\sum_1^n |f(t, y_k) - f(t, y_{k-1})|\right) dt \leq 1,$$

because $f \in NBV(Q)$. ♠

We now prove Theorem 1.

Let $d = 1$. Take Q and I, dyadic intervals. Consider the inner product

$$\left\langle \frac{f^{(Q)}}{|Q|^{1/2}}, \frac{h^{(I)}}{|I|^{1/2}} \right\rangle, \tag{14}$$

where $f^{(Q)} \in NBV(Q)$ and $h^{(I)}/|I|^{1/2}$ is the classical Haar function associated to I. If $Q \cap I = \emptyset$ or Q is properly contained in I then (14) is 0. If $I \subset Q$ then, by Lemma 1,

$$\left| \left\langle \frac{f^{(Q)}}{|Q|^{1/2}}, \frac{h^{(I)}}{|I|^{1/2}} \right\rangle \right| \leq (1/2) V_{f^{(Q)}}(\bar{I}) \left(\frac{|I|}{|Q|} \right)^{1/2}.$$

Therefore, for each $j \geq 0$,

$$\begin{aligned}
\sum_{\substack{I \subset Q \\ \ell(I) = 2^{-j}\ell(Q)}} \left| \left\langle \frac{f^{(Q)}}{|Q|^{1/2}}, \frac{h^{(I)}}{|I|^{1/2}} \right\rangle \right| &\leq (1/2) 2^{-j/2} \sum_{\substack{I \subset Q \\ \ell(I) = 2^{-j}\ell(Q)}} V_{f^{(Q)}}(\bar{I}) \\
&= (1/2) 2^{-j/2} V_{f^{(Q)}}(\overline{Q}) \qquad (15) \\
&\leq (1/2) 2^{-j/2}.
\end{aligned}$$

For each $Q \in \mathcal{D}$,

$$\begin{aligned}
\sum_{I \in \mathcal{D}} \left| \left\langle \frac{f^{(Q)}}{|Q|^{1/2}}, \frac{h^{(I)}}{|I|^{1/2}} \right\rangle \right| &= \sum_{\substack{I \in \mathcal{D} \\ I \subset Q}} \left| \left\langle \frac{f^{(Q)}}{|Q|^{1/2}}, \frac{h^{(I)}}{|I|^{1/2}} \right\rangle \right| \\
&\leq (1/2) \sum_{0}^{\infty} 2^{-j/2} \\
&= 1 + \frac{1}{\sqrt{2}}.
\end{aligned}$$

For every $I \in \mathcal{D}$,

$$\begin{aligned}
\sum_{Q \in \mathcal{D}} \left| \left\langle \frac{f^{(Q)}}{|Q|^{1/2}}, \frac{h^{(I)}}{|I|^{1/2}} \right\rangle \right| &= \sum_{\substack{Q \in \mathcal{D} \\ I \subset Q}} \left| \left\langle \frac{f^{(Q)}}{|Q|^{1/2}}, \frac{h^{(I)}}{|I|^{1/2}} \right\rangle \right| \\
&\leq (1/2) \sum_{0}^{\infty} 2^{-j/2} \\
&= 1 + \frac{1}{\sqrt{2}}.
\end{aligned}$$

By the Schur Test, the linear mapping $L : \ell^2(\mathcal{D}) \to \ell^2(\mathcal{D})$ defined by

$$L\left(\{\lambda_Q\}_{Q\in\mathcal{D}}\right) \equiv \left\{\sum_{Q\in\mathcal{D}} \lambda_Q \left\langle \frac{f^{(Q)}}{|Q|^{1/2}}, \frac{h^{(I)}}{|I|^{1/2}} \right\rangle\right\}_{I\in\mathcal{D}}$$

has a bound less than or equal to $1 + \frac{1}{\sqrt{2}}$. Let $g = \sum_{Q\in\mathcal{D}} \lambda_Q \frac{f^{(Q)}}{|Q|^{1/2}}$ be a finite linear sum. The classical Haar functions form a complete orthonormal set in $L^2(\mathbf{R})$. Therefore,

$$\begin{aligned}\int |g|^2\, dx &= \sum_I \left|\left\langle g, \frac{h^{(I)}}{|I|^{1/2}}\right\rangle\right|^2 \\ &= \sum_I \left|\sum_{Q\in\mathcal{D}} \lambda_Q \left\langle \frac{f^{(Q)}}{|Q|^{1/2}}, \frac{h^{(I)}}{|I|^{1/2}}\right\rangle\right|^2 \\ &\le \left(1 + \frac{1}{\sqrt{2}}\right)^2 \sum_{Q\in\mathcal{D}} |\lambda_Q|^2,\end{aligned}$$

proving the Theorem 1 when $d = 1$.

Assume the result for $d-1 \ge 1$, with constant $C(d-1)$; i.e., assume that if $f^{(Q)} \in NBV(Q)$ for every $(d-1)$-dimensional dyadic cube Q, and $\sum_{Q\in\mathcal{D}} \lambda_Q \frac{f^{(Q)}}{|Q|^{1/2}}$ is any finite linear combination, then

$$\left(\int_{\mathbf{R}^{d-1}} \left|\sum_{Q\in\mathcal{D}} \lambda_Q \frac{f^{(Q)}}{|Q|^{1/2}}\right|^2 dx\right)^{1/2} \le C(d-1)\left(\sum_{Q\in\mathcal{D}} |\lambda_Q|^2\right)^{1/2}.$$

Consider the family

$$\left\{\frac{f^{(Q)}}{|Q|^{1/2}}\right\}_{Q\in\mathcal{D}},$$

where every Q is a d-dimensional dyadic cube and each $f^{(Q)} \in NBV(Q)$. Put $f^{(Q)}(x) = f^{(Q)}(x', y)$, where $x' \in \mathbf{R}$ and $y \in \mathbf{R}^{d-1}$. Write

$$f^{(Q)}(x', y) = f_1^{(Q)}(x', y) + f_2^{(Q)}(x', y),$$

where

$$f_1^{(Q)}(x',y) = \left(f^{(Q)}(x',y) - \ell(Q)^{-1}\int_{I_1(Q)} f^{(Q)}(t,y)\,dt\right)\chi_{\overline{I_1(Q)}}(x')\chi_{\overline{K(Q)}}(y)$$

$$f_2^{(Q)}(x',y) = \left(\ell(Q)^{-1}\int_{I_1(Q)} f^{(Q)}(t,y)\,dt\right)\chi_{\overline{I_1(Q)}}(x')\chi_{\overline{K(Q)}}(y),$$

and $I_1(Q)$ and $K(Q)$ are as in the statement of Lemma 2, so that $Q = I_1(Q)\times K(Q)$.

By our $d = 1$ result, for each fixed $y \in \mathbf{R}^{d-1}$, the family $\{\ell(Q)^{-1/2}f_1^{(Q)}(x',y)\}_{Q\in\mathcal{D}}$ is almost-orthogonal in $L^2(\mathbf{R})$, with constant $\le C(1)$. This is because, for each fixed y, the function $\ell(Q)^{-1/2}f_1^{(Q)}(x',y)$ is either identically 0 (with respect to x') or it's a suitably scaled, uniformly bounded-variation function, with integral 0, adapted to a unique dyadic interval $I_1(Q)$. Note that subtracting a term of the form $c\chi_{\overline{I_1(Q)}}(x')$ does not change $f^{(Q)}$'s total variation in x' on $\overline{I_1(Q)}$, and so does not affect the relevant Schur Test estimates. (See the proof of Lemma 1.)

For each fixed $y \in \mathbf{R}^{d-1}$,

$$\int_{\mathbf{R}}\left|\sum \lambda_Q|Q|^{-1/2}f_1^{(Q)}(x',y)\right|^2 dx' \le C(1)^2\sum|\lambda_Q|^2\ell(Q)^{-(d-1)}\chi_{\overline{K(Q)}}(y).$$

Since $|\overline{K(Q)}| = \ell(Q)^{d-1}$, integrating in y yields

$$\int_{\mathbf{R}^d}\left|\sum \lambda_Q|Q|^{-1/2}f_1^{(Q)}(x',y)\right|^2 dx'\,dy \le C(1)^2\sum|\lambda_Q|^2. \tag{16}$$

By induction (and because of Lemma 2), for each fixed $x' \in \mathbf{R}$, the family $\{\ell(Q)^{-(d-1)/2}f_2^{(Q)}(x',y)\}_{Q\in\mathcal{D}}$ is almost-orthogonal in $L^2(\mathbf{R}^{d-1})$, with constant $\le C(d-1)$. (As with the $f_1^{(Q)}$s, for some x', $f_2^{(Q)}(x',y)$ is identically 0 in y—which is fine.) Hence, for each fixed $x' \in \mathbf{R}$,

$$\int_{\mathbf{R}^{d-1}}\left|\sum \lambda_Q|Q|^{-1/2}f_2^{(Q)}(x',y)\right|^2 dy \le C(d-1)^2\sum|\lambda_Q|^2\ell(Q)^{-1}\chi_{\overline{I_1(Q)}}(x').$$

Now integrating in x' yields:

$$\int_{\mathbf{R}^d}\left|\sum \lambda_Q|Q|^{-1/2}f_2^{(Q)}(x',y)\right|^2 dx'\,dy \le C(d-1)^2\sum|\lambda_Q|^2. \tag{17}$$

Combining (16) and (17) yields

$$\left\|\sum \lambda_Q\frac{f^{(Q)}}{|Q|^{1/2}}\right\|_2 \le (C(1)+C(d-1))\left(\sum|\lambda_Q|^2\right)^{1/2},$$

which implies

$$C(d) \leq d\, C(1).$$

The Schur Test gives $C(1) \leq 1 + \frac{1}{\sqrt{2}}$. We have Theorem 1. ♠

Point 2. Why do induction? We have d-dimensional Haar functions (3). Why not get Schur test estimates directly from inner products between them and the functions in

$$\left\{ \frac{f^{(Q)}}{|Q|^{1/2}} \right\}_{Q \in \mathcal{D}} ?$$

See the appendix.

Theorem 1 implies the L^2 boundedness of a certain "rough" operator (see the introduction), defined as a limit of finite sums. We need to specify in what way this limit is taken.

Definition 1 We say that a sequence $\{\mathcal{E}_k\}_1^\infty$ of *finite* subsets of $\mathcal{D}$ *fills up* $\mathcal{D}$ if every $Q \in \mathcal{D}$ is in all but finitely many $\mathcal{E}_k$s. (This holds if the $\mathcal{E}_k$s are increasing and $\cup_k \mathcal{E}_k = \mathcal{D}$.) Let $\{\lambda_Q\}_{Q \in \mathcal{D}}$ be a sequence of complex numbers, and $\{g_{(Q)}\}_{Q \in \mathcal{D}}$ a sequence of functions in $L^2(\mathbf{R}^d)$, each indexed over the family of dyadic cubes $\mathcal{D}$. We say that

$$\sum_{Q \in \mathcal{D}} \lambda_Q g_{(Q)}$$

converges unconditionally to $h \in L^2(\mathbf{R}^d)$ if, for every sequence of finite subsets $\{\mathcal{E}_k\}_1^\infty$ that fills up $\mathcal{D}$,

$$\lim_{k \to \infty} \left\| h - \sum_{Q \in \mathcal{E}_k} \lambda_Q g_{(Q)} \right\|_2 = 0.$$

Corollary 3 *Let* $\{f_1^{(Q)}\}_{Q \in \mathcal{D}}$ *and* $\{f_2^{(Q)}\}_{Q \in \mathcal{D}}$ *be two families such that* $f_i^{(Q)} \in NBV(Q)$ *for all* $Q \in \mathcal{D}$ *and* $i = 1, 2$. *If* $g \in L^2(\mathbf{R}^d)$ *then the series*

$$\sum_{Q \in \mathcal{D}} \frac{\langle g, f_1^{(Q)} \rangle}{|Q|} f_2^{(Q)} \tag{18}$$

converges unconditionally to some h in L^2. *Moreover,*

$$\|h\|_2 \leq \left(\left(1 + \frac{1}{\sqrt{2}} \right) d \right)^2 \|g\|_2.$$

In other words, (18) defines a linear operator $T : L^2 \to L^2$ with norm $\leq ((1+\frac{1}{\sqrt{2}})d)^2$.

Proof of Corollary 3 Let $g \in L^2(\mathbf{R}^d)$, and suppose that $\mathcal{E} \subset \mathcal{D}$ is a finite subset. Define

$$T_{\mathcal{E}}(g) \equiv \sum_{\mathcal{E}} \frac{\langle g, f_1^{(Q)}\rangle}{|Q|} f_2^{(Q)}.$$

By Theorem 1,

$$\|T_{\mathcal{E}}(g)\|_2 \leq \left(1+\frac{1}{\sqrt{2}}\right) d \left(\sum_{\mathcal{E}} \frac{|\langle g, f_1^{(Q)}\rangle|^2}{|Q|}\right)^{1/2} \leq \left(\left(1+\frac{1}{\sqrt{2}}\right) d\right)^2 \|g\|_2 < \infty. \tag{19}$$

If $\{\mathcal{E}_k\}_1^\infty$ is a sequence of finite subsets that fills up $\mathcal{D}$ then, for any m and n,

$$\|T_{\mathcal{E}_m}(g) - T_{\mathcal{E}_n}(g)\|_2 \leq \left(1+\frac{1}{\sqrt{2}}\right) d \left(\sum_{\mathcal{E}_m \Delta \mathcal{E}_n} \frac{|\langle g, f_1^{(Q)}\rangle|^2}{|Q|}\right)^{1/2};$$

which, because of (19), goes to 0 as m and n go to infinity. (Apply dominated convergence to the sums over the symmetric differences $\mathcal{E}_m \Delta \mathcal{E}_n$.) Therefore $\{T_{\mathcal{E}_k}(g)\}_k$ is Cauchy in $L^2(\mathbf{R}^d)$ and converges to an h with norm $\leq ((1+\frac{1}{\sqrt{2}})d)^2\|g\|_2$. The function h is unique because, if $\{\mathcal{E}_k\}_1^\infty$ and $\{\mathcal{E}'_k\}_1^\infty$ fill up $\mathcal{D}$, so does $\{\mathcal{E}_1, \mathcal{E}'_1, \mathcal{E}_2, \mathcal{E}'_2, \ldots\}$. ♠

3 The Proofs of Theorems 2 and 3

As with Theorem 1, we will first work in one dimension, where we will sometimes call dyadic intervals I or J, and sometimes Q.

Both proofs make use of a simple fact whose proof can be found in [2] and [3].

Lemma 3 *If $\tilde{\mathcal{D}}$ denotes the family of concentric triples of dyadic intervals in $\mathbf{R}$ then $\tilde{\mathcal{D}}$ can be decomposed into 3 disjoint families,*

$$\tilde{\mathcal{D}} = \cup_1^3 \mathcal{G}_i,$$

such that, for each $1 \leq i \leq 3$: a) $\forall I, J \in \mathcal{G}_i$, either $I \cap J = \emptyset$ or one is a subset of the other; b) every $I \in \mathcal{G}_i$ is the right or left half of a $J \in \mathcal{G}_i$; c) $\forall I \in \mathcal{G}_i$, I's right and left halves belong to $\mathcal{G}_i$; d) $\mathbf{R}$ is covered by the set of $I \in \mathcal{G}_i$ of length 3; and therefore, for any k, $\mathbf{R}$ is covered by the set of $I \in \mathcal{G}_i$ of length $3 \cdot 2^k$.

As an immediate corollary of Lemma 3, the set of concentric triples of dyadic *cubes* in $\mathbf{R}^d$ (also denoted $\tilde{\mathcal{D}}$) can be split into 3^d disjoint families, each one having the analogous inclusion/exclusion and relative size properties as the set of dyadic cubes. The proof is trivial: for every $\vec{a} = (a_1, \ldots, a_d) \in \{1, 2, 3\}^d$, let $\mathcal{G}_{\vec{a}}$ be the set of *cubes* $Q = \prod_1^d I_i(Q)$ such that each $I_i(Q) \in \mathcal{G}_{a_i}$.

The Proof of Theorem 2 If I is a dyadic interval, we use $\tilde{I}$ to denote I's concentric triple, and we define $h^{(\tilde{I})}$ by

$$h^{(\tilde{I})}(x) = \chi_{\tilde{I}_l}(x) - \chi_{\tilde{I}_r}(x).$$

Then $h^{(\tilde{I})}/|\tilde{I}|^{1/2}$ is the "Haar function" associated to $\tilde{I}$. Because of Lemma 3, for each $1 \leq i \leq 3$, $\{h^{(\tilde{I})}/|\tilde{I}|^{1/2} : \tilde{I} \in \mathcal{G}_i\}$ forms a complete orthonormal basis for $L^2(\mathbf{R})$.

For each $1 \leq i \leq 3$ we let $\mathcal{F}_i$ be the set of dyadic intervals Q such that $\tilde{Q} \in \mathcal{G}_i$. We note that if $Q \in \mathcal{F}_i$ and $f^{(Q)} \in NBV(Q)$ then $\widetilde{f^{(Q)}} \in NBV(\tilde{Q})$, where $\tilde{Q} \in \mathcal{G}_i$. We claim that if $1 \leq i \leq 3$ and $\{f^{(Q)}\}_{Q \in \mathcal{F}_i}$ is any family such that each $f^{(Q)} \in NBV(Q)$ then

$$\left\{ \frac{\widetilde{f^{(Q)}}}{|Q|^{1/2}} \right\}_{Q \in \mathcal{F}_i}$$

is almost-orthogonal in $L^2(\mathbf{R})$, with a constant less than or equal to an absolute C. The proof is easy. We only need to bound

$$\left| \left\langle \frac{\widetilde{f^{(Q)}}}{|Q|^{1/2}}, \frac{h^{(\tilde{I})}}{|\tilde{I}|^{1/2}} \right\rangle \right| \tag{20}$$

for Q and I both lying in $\mathcal{F}_i$. But we have already seen this sort of thing. If $\tilde{Q} \cap \tilde{I} = \emptyset$ or $\tilde{Q}$ is strictly contained in $\tilde{I}$ then the inner product is 0. Otherwise $\tilde{I} \subset \tilde{Q}$, with $|\tilde{I}| = 2^{-j}|\tilde{Q}|$ for some $j \geq 0$, and (20) is less than or equal to

$$\left(\frac{|\tilde{I}|}{|Q|} \right)^{1/2} V_{\widetilde{f^{(Q)}}}(\tilde{I}) = 3^{1/2} 2^{-j/2} V_{\widetilde{f^{(Q)}}}(\tilde{I}).$$

For every $Q \in \mathcal{F}_i$ and $j \geq 0$,

$$\sum_{\substack{I \in \mathcal{F}_i:\ \tilde{I} \subset \tilde{Q} \\ |\tilde{I}| = 2^{-j}|\tilde{Q}|}} \left| \left\langle \frac{\widetilde{f^{(Q)}}}{|Q|^{1/2}}, \frac{h^{(\tilde{I})}}{|\tilde{I}|^{1/2}} \right\rangle \right|$$

is less than or equal to a constant times

$$2^{-j/2} V_{\widetilde{f^{(Q)}}}(\tilde{Q}) \leq 2^{-j/2},$$

implying that, for every $Q \in \mathcal{F}_i$,

$$\sum_{I \in \mathcal{F}_i} \left| \left\langle \frac{\widetilde{f^{(Q)}}}{|Q|^{1/2}}, \frac{h^{(\tilde{I})}}{|\tilde{I}|^{1/2}} \right\rangle \right| \leq C(1 + \frac{1}{\sqrt{2}}) \leq C.$$

Similarly, for every $I \in \mathcal{F}_i$,

$$\sum_{Q \in \mathcal{F}_i} \left| \left\langle \frac{\widetilde{f^{(Q)}}}{|Q|^{1/2}}, \frac{h^{(\tilde{I})}}{|\tilde{I}|^{1/2}} \right\rangle \right| \leq C(1 + \frac{1}{\sqrt{2}}) \leq C.$$

Combining the two inequalities proves our claim.

For every $\vec{a} \in \{1, 2, 3\}^d$, let $\mathcal{F}_{\vec{a}}$ be the family of dyadic cubes Q such that $\tilde{Q} \in \mathcal{G}_{\vec{a}}$. Fix an $\vec{a} \in \{1, 2, 3\}^d$. If $Q \in \mathcal{F}_{\vec{a}}$ then $\widetilde{f^{(Q)}} \in NBV(\tilde{Q})$. We can now repeat the inductive argument from the proof of Theorem 1 to get that

$$\left\{ \frac{\widetilde{f^{(Q)}}}{|Q|^{1/2}} \right\}_{Q \in \mathcal{F}_{\vec{a}}}$$

is almost-orthogonal in $L^2(\mathbf{R}^d)$, with constant $\leq Cd$, where C is the constant we get for $d = 1$. We get the same estimate for every $\vec{a} \in \{1, 2, 3\}^d$, implying that

$$\left\{ \frac{\widetilde{f^{(Q)}}}{|Q|^{1/2}} \right\}_{Q \in \mathcal{D}}$$

is almost-orthogonal in $L^2(\mathbf{R}^d)$, with constant $\leq C3^d d \equiv C(d)$. A repetition of the argument in the proof of Corollary 3 shows that, for any $g \in L^2(\mathbf{R}^d)$,

$$\sum_{Q \in \mathcal{D}} \frac{\langle g, \widetilde{f_1^{(Q)}} \rangle}{|Q|} \widetilde{f_2^{(Q)}}$$

converges unconditionally in L^2 to define a bounded linear operator $\widetilde{T} : L^2 \to L^2$ with norm $\leq C(d)^2$. ♠

The Proof of Theorem 3 Write $T(g)-\widetilde{T}(g)$ as $S_1(g)+S_2(g)$, where

$$S_1(g) = \sum_{Q\in\mathcal{D}} \frac{\langle g, f_1^{(Q)} - \widetilde{f_1^{(Q)}}\rangle}{|Q|} f_2^{(Q)}$$

$$S_2(g) = \sum_{Q\in\mathcal{D}} \frac{\langle g, \widetilde{f_1^{(Q)}}\rangle}{|Q|} \left(f_2^{(Q)} - \widetilde{f_2^{(Q)}}\right).$$

Because of (2) and Theorem 2, Theorem 3 will follow once we show that, for all finite linear sums

$$\sum_{Q\in\mathcal{D}} \lambda_Q \left(\frac{f_i^{(Q)} - \widetilde{f_i^{(Q)}}}{|Q|^{1/2}}\right)$$

$(i=1,2)$, we have

$$\left\|\sum_{Q\in\mathcal{D}} \lambda_Q \left(\frac{f_i^{(Q)} - \widetilde{f_i^{(Q)}}}{|Q|^{1/2}}\right)\right\|_2 \le C\eta^{1/2}\left(\sum_{Q\in\mathcal{D}} |\lambda_Q|^2\right)^{1/2} \tag{21}$$

for a constant C only depending on d. Inequality (21) will follow from Theorem 2 and a technical, one-dimensional lemma (Lemma 4). We prove Lemma 4 first. We warn the reader that its proof requires an additional (fortunately very easy) lemma (Lemma 5).

Since the $f_i^{(Q)}$s' subscripts are now irrelevant, we no longer write them.

Until otherwise stated, $\mathcal{D}$, $\tilde{\mathcal{D}}$, $\mathcal{F}_i$, and $\mathcal{G}_i$ refer to families of *intervals*.

Lemma 4 *For each $Q\in\mathcal{F}_i$, let $g^{(Q)}:\mathbf{R}\to\mathbf{R}$ have support contained in $\overline{Q}$ and be of bounded variation, with total variation ≤ 1. (Note: we do* not *require that $\int g^{(Q)}\,dx = 0$.) Let $\{\delta(Q)\}_{Q\in\mathcal{D}}$ and $\{\tau(Q)\}_{Q\in\mathcal{D}}$ be two sequences of real numbers indexed over $\mathcal{D}$, such that $|1-\delta(Q)|+|\tau(Q)|<\eta<1/2$ for all $Q\in\mathcal{D}$. Define $\widetilde{g^{(Q)}}(x)\equiv g^{(Q)}(\delta(Q)(x-x_Q+\ell(Q)\tau(Q))+x_Q)$ and, for each $\tilde{I}\in\mathcal{G}_i$, set*

$$a(Q,\tilde{I}) \equiv \left\langle \frac{g^{(Q)} - \delta(Q)\widetilde{g^{(Q)}}}{|Q|^{1/2}}, \frac{h^{(\tilde{I})}}{|\tilde{I}|^{1/2}}\right\rangle.$$

There is an absolute C such that, for all $Q\in\mathcal{F}_i$ and $\tilde{I}\in\mathcal{G}_i$,

$$\sum_{\tilde{I}\in\mathcal{G}_i} |a(Q,\tilde{I})| \tag{22}$$

and

$$\sum_{Q\in\mathcal{F}_i} |a(Q,\tilde{I})| \tag{23}$$

are both bounded by $C\eta^{1/2}$.

Proof of Lemma 4 If $Q \in \mathcal{F}_i$ and $\tilde{I} \in \mathcal{G}_i$ then

$$\begin{aligned}&\delta(Q)\int \widetilde{g^{(Q)}}(x)\, h^{(\tilde{I})}(x)\,dx\\ &= \delta(Q)\int g^{(Q)}\left(\delta(Q)(x - x_Q + \ell(Q)\tau(Q)) + x_Q\right)\, h^{(\tilde{I})}(x)\,dx\\ &= \int g^{(Q)}(u)\, h^{(\tilde{I})}\left(\delta(Q)^{-1}(u - x_Q - \delta(Q)\ell(Q)\tau(Q)) + x_Q\right)\,du,\end{aligned}$$

after substituting $u = \delta(Q)(x - x_Q + \ell(Q)\tau(Q)) + x_Q$. Therefore,

$$\int \left(g^{(Q)}(x) - \delta(Q)\widetilde{g^{(Q)}}(x)\right)\, h^{(\tilde{I})}(x)\,dx = \int g^{(Q)}(x)\,\gamma^{(\tilde{I})}(x)\,dx, \tag{24}$$

where $\gamma^{(\tilde{I})}(x)$ equals

$$h^{(\tilde{I})}(x) - h^{(\tilde{I})}\left(\delta(Q)^{-1}(x - x_Q - \delta(Q)\ell(Q)\tau(Q)) + x_Q\right). \tag{25}$$

We note a fact which will be important soon. Although we do not assume that $\int g^{(Q)}\,dx = 0$, we do have $\int \left(g^{(Q)}(x) - \delta(Q)\widetilde{g^{(Q)}}(x)\right)\,dx = 0$, ensuring that (24) equals 0 if $\tilde{I} \not\subset \tilde{Q}$: if $\tilde{I} \not\subset \tilde{Q}$ and $\tilde{I} \cap \tilde{Q} \neq \emptyset$, the support of $g^{(Q)}(x) - \delta(Q)\widetilde{g^{(Q)}}$ is entirely contained in either the right or the left half of $\tilde{I}$, across which $h^{(\tilde{I})}$ is constant.

The key to the proof of Lemma 4 is a good estimate for the right-hand side of (24), which follows from Lemma 1 and a bound on $\|\gamma^{(\tilde{I})}\|_1$. For the latter we need the simple lemma mentioned above.

Lemma 5 *If I is a bounded interval, with endpoints $a < b$, and I' is another bounded interval, with endpoints $a' < b'$, then*

$$\int |\chi_I(x) - \chi_{I'}(x)|\,dx \le |a - a'| + |b - b'|. \tag{26}$$

Proof of Lemma 5 Assume that $b-a \le b'-a'$. If $b \le a'$ the left-hand side of (26) is $b-a+b'-a'$, while $|a-a'| \ge b-a$ and $|b-b'| \ge b'-a'$. If $a \le a' < b$ the left-hand side of (26) is exactly $a' - a + b' - b$ (because $b \le b'$), and if $a' < a < b \le b'$ it is $a - a' + b' - b$. The other cases follow from symmetry. ♠

We continue the proof of Lemma 4. Recall that $h^{(\tilde{I})}$ has the form $\chi_{[a,b)} - \chi_{[a',b')}$, where $[a,b) = \tilde{I}_l$ and $[a',b') = \tilde{I}_r$. If $\phi : \mathbf{R} \to \mathbf{R}$ is a strictly increasing bijection then

$$\chi_{[\alpha,\beta)}(\phi(x)) = \chi_{[\phi^{-1}(\alpha),\phi^{-1}(\beta))}(x).$$

Set $\phi(x) = \delta(Q)^{-1}(x - x_Q - \delta(Q)\ell(Q)\tau(Q)) + x_Q$. Then $\phi^{-1}(x) = \delta(Q)(x - x_Q + \ell(Q)\tau(Q)) + x_Q$. For ease of reading we will refer to ϕ^{-1} as ψ. We can write

$$\begin{aligned} &h^{(\tilde{I})}\left(\delta(Q)^{-1}(x - x_Q - \delta(Q)\ell(Q)\tau(Q)) + x_Q\right) \\ &\quad = \chi_{[\psi(a),\psi(b))}(x) - \chi_{[\psi(a'),\psi(b'))}(x), \end{aligned}$$

and therefore the L^1 norm of

$$\gamma^{(\tilde{I})}(x) \equiv h^{(\tilde{I})}(x) - h^{(\tilde{I})}\left(\delta(Q)^{-1}(x - x_Q - \delta(Q)\ell(Q)\tau(Q)) + x_Q\right)$$

is less than or equal to

$$|a - \psi(a)| + |b - \psi(b)| + |a' - \psi(a')| + |b' - \psi(b')|. \tag{27}$$

A quick calculation yields

$$a - \psi(a) = (a - x_Q)(1 - \delta(Q)) - \delta(Q)\ell(Q)\tau(Q), \tag{28}$$

with similar expressions for the other terms.

We recall that $Q \in \mathcal{F}_i$, $\tilde{I} \in \mathcal{G}_i$, and that the inner product (24) is zero unless $\tilde{I} \subset \tilde{Q}$; thus, for the only cases of interest, $\ell(\tilde{I}) = 2^{-j}\ell(\tilde{Q})$ for some $j \geq 0$. Given $0 < \eta < 1/2$, let N be the unique natural number such that $\eta \in [2^{-N-1}, 2^{-N})$. For such $\tilde{I}$, the absolute value of (28)—*and thus* $\|\gamma^{(\tilde{I})}\|_1$—is less than or equal to a constant times $2^{-N}\ell(Q)$.

We will give two bounds on the absolute value of (24), depending on whether $j \leq N$ or $j > N$. We only use (28) for the $j \leq N$ estimate.

If $j \leq N$ (so that $\tilde{I}$ is not too small compared to $\tilde{Q}$), then the absolute value of (24) is less than or equal to a constant times $2^{-N}\ell(\tilde{Q})V_{g^{(Q)}}(\bar{\tilde{I}})$.

If $j > N$ (meaning that $\tilde{I}$ is very small compared to $\tilde{Q}$) then the absolute value of (24) is less than or equal to

$$\left(V_{g^{(Q)}}(\bar{\tilde{I}}) + V_{\widetilde{g^{(Q)}}}(\bar{\tilde{I}})\right)\|h^{(\tilde{I})}\|_1,$$

which is the same as

$$2^{-j}\ell(\tilde{Q})\left(V_{g^{(Q)}}(\bar{\tilde{I}}) + V_{\widetilde{g^{(Q)}}}(\bar{\tilde{I}})\right). \tag{29}$$

Of course, what we need to bound is not the absolute value of (24), but the same divided by $|Q|^{1/2}|\tilde{I}|^{1/2} \sim 2^{-j/2}\ell(Q)$. (Recall that we are still working in $d = 1$.) If

$j \le N$, the quotient is less than or equal to a constant times $2^{j/2}2^{-N}V_{g(Q)}(\tilde{\bar{I}})$. If $j > N$ the corresponding estimate is $2^{-j/2}\left(V_{g(Q)}(\tilde{\bar{I}}) + V_{\delta(Q)\widetilde{g^{(Q)}}}(\tilde{\bar{I}})\right)$. Therefore, if $Q \in \mathcal{F}_i$, $\tilde{I} \in \mathcal{G}_i$, $\tilde{I} \subset \tilde{Q}$, and $\ell(\tilde{I}) = 2^{-j}\ell(\tilde{Q})$, then

$$|a(Q,\tilde{I})| \le \begin{cases} C2^{j/2}2^{-N}V_{g(Q)}(\tilde{\bar{I}}) & \text{if } j \le N; \\ C2^{-j/2}\left(V_{g(Q)}(\tilde{\bar{I}}) + V_{\delta(Q)\widetilde{g^{(Q)}}}(\tilde{\bar{I}})\right) & \text{if } j > N; \end{cases}$$

while $a(Q,\tilde{I}) = 0$ if $\tilde{I} \not\subset \tilde{Q}$.

We now estimate (22)

$$\sum_{\tilde{I}} |a(Q,\tilde{I})| = \sum_{\tilde{I}:\, \tilde{I}\subset\tilde{Q}} |a(Q,\tilde{I})|$$

and (23)

$$\sum_{Q} |a(Q,\tilde{I})| = \sum_{Q:\, \tilde{I}\subset\tilde{Q}} |a(Q,\tilde{I})|.$$

Estimate of (22):

$$\begin{aligned} \sum_{\tilde{I}:\, \tilde{I}\subset\tilde{Q}} |a(Q,\tilde{I})| &\le C\sum_{j\ge 0}\sum_{\tilde{I}:\ell(\tilde{I})=2^{-j}\ell(\tilde{Q})} |a(Q,\tilde{I})| \\ &= C2^{-N}\sum_{0\le j\le N} 2^{j/2}\sum_{\tilde{I}:\ell(\tilde{I})=2^{-j}\ell(\tilde{Q})} V_{g(Q)}(\tilde{\bar{I}}) \\ &+ C\sum_{j>N}2^{-j/2}\sum_{\tilde{I}:\ell(\tilde{I})=2^{-j}\ell(\tilde{Q})}\left(V_{g(Q)}(\tilde{\bar{I}}) + V_{\delta(Q)\widetilde{g^{(Q)}}}(\tilde{\bar{I}})\right) \\ &= (I) + (II), \end{aligned}$$

where

$$\begin{aligned} (I) &= C2^{-N}\sum_{0\le j\le N} 2^{j/2}\sum_{\tilde{I}:\ell(\tilde{I})=2^{-j}\ell(\tilde{Q})} V_{g(Q)}(\tilde{\bar{I}}) \\ (II) &= C\sum_{j>N}2^{-j/2}\sum_{\tilde{I}:\ell(\tilde{I})=2^{-j}\ell(\tilde{Q})}\left(V_{g(Q)}(\tilde{\bar{I}}) + V_{\delta(Q)\widetilde{g^{(Q)}}}(\tilde{\bar{I}})\right). \end{aligned}$$

For each Q and $j \ge 0$,

$$\sum_{\tilde{I}:\ell(\tilde{I})=2^{-j}\ell(\tilde{Q})} V_{g(Q)}(\tilde{\bar{I}}) \le V_{g(Q)}(\tilde{\bar{Q}}) \le 1$$

and

$$\sum_{\tilde{I}:\ell(\tilde{I})=2^{-j}\ell(\tilde{Q})} \left(V_{g^{(Q)}}(\bar{\tilde{I}}) + V_{\delta(Q)\widetilde{g^{(Q)}}}(\bar{\tilde{I}}) \right) \leq V_{g^{(Q)}}(\bar{\tilde{Q}}) + V_{\delta(Q)\widetilde{g^{(Q)}}}(\bar{\tilde{Q}}) \leq 5/2,$$

because the change of variable does not affect the total variation and $|\delta(Q)| \leq 3/2$. Therefore

$$(I) \leq C2^{-N} \sum_{0 \leq j \leq N} 2^{j/2} \leq C2^{-N/2}$$

$$(II) \leq C \sum_{j>N} 2^{-j/2} \leq C2^{-N/2},$$

implying that $\sum_{\tilde{I}:\, \tilde{I} \subset \tilde{Q}} |a(Q,\tilde{I})| \leq C2^{-N/2}$.

Estimate of (23): This is like case a), but simpler, because, for each $j \geq 0$ and $\tilde{I}$, there is only one $\tilde{Q}$ such that $\tilde{I} \subset \tilde{Q}$ and $\ell(\tilde{I}) = 2^{-j}\ell(\tilde{Q})$. We get the same estimate: $\leq C2^{-N/2}$.

That proves Lemma 4, since $\eta \sim 2^{-N}$. ♠

The Schur Test and Lemma 4 imply that if $\{g^{(Q)}\}_{Q\in\mathcal{D}}$ and $\{\widetilde{g^{(Q)}}\}_{Q\in\mathcal{D}}$ are two families as given in Lemma 4's hypotheses, then, for any finite linear sum

$$\sum_{Q\in\mathcal{D}} \lambda_Q \left(\frac{g^{(Q)} - \delta(Q)\widetilde{g^{(Q)}}}{|Q|^{1/2}} \right),$$

we have

$$\left\| \sum_{Q\in\mathcal{D}} \lambda_Q \left(\frac{g^{(Q)} - \delta(Q)\widetilde{g^{(Q)}}}{|Q|^{1/2}} \right) \right\|_2 \leq C\eta^{1/2} \left(\sum_{Q\in\mathcal{D}} |\lambda_Q|^2 \right)^{1/2}, \tag{30}$$

with C an absolute constant. This implies Theorem 3 when $d = 1$. We can write

$$f^{(Q)} - \widetilde{f^{(Q)}} = \left(f^{(Q)} - \delta(Q)\widetilde{f^{(Q)}} \right) + (\delta(Q) - 1)\widetilde{f^{(Q)}}.$$

Lemma 4 implies that

$$\left\| \sum_{Q\in\mathcal{D}} \lambda_Q \left(\frac{f^{(Q)} - \delta(Q)\widetilde{f^{(Q)}}}{|Q|^{1/2}} \right) \right\|_2 \leq C\eta^{1/2} \left(\sum_{Q\in\mathcal{D}} |\lambda_Q|^2 \right)^{1/2}; \tag{31}$$

while, by Theorem 2,

$$\left\| \sum_{Q\in\mathcal{D}} \lambda_Q \left(\frac{\widetilde{f^{(Q)}}}{|Q|^{1/2}} \right) \right\|_2 \le C \left(\sum_{Q\in\mathcal{D}} |\lambda_Q|^2 \right)^{1/2}. \tag{32}$$

Since $|1-\delta(Q)| \le \eta \le \eta^{1/2}$ for all Q, we get (21) and thus Theorem 3 *in one dimension*.

We now prove (21) for general d. From here on we work in $\mathbf{R}^d$: $\mathcal{F}_{\vec{a}}$, $\mathcal{G}_{\vec{a}}$, $\mathcal{D}$, and $\tilde{\mathcal{D}}$ are families of *cubes*; $\vec{\delta}(Q)$ and $\vec{\tau}(Q)$ are *vectors*.

Fix $\vec{a} \in \{1,2,3\}^d$. For $Q \in \mathcal{F}_{\vec{a}}$ we write $\vec{\delta}(Q)$ as $(\delta_1(Q), \delta_2(Q), \ldots, \delta_d(Q))$ and $\vec{\tau}(Q)$ as $(\tau_1(Q), \tau_2(Q), \ldots, \tau_d(Q))$. Associated to each $\vec{\delta}(Q)$ and $\vec{\tau}(Q)$ will be two finite sequences of vectors $\{\tilde{\delta}_j(Q)\}_0^d$ and $\{\tilde{\tau}_j(Q)\}_0^d$, defined by

$$\begin{aligned}
\tilde{\delta}_0(Q) &\equiv \vec{1} \\
\tilde{\delta}_1(Q) &\equiv (\delta_1(Q), 1, 1, \ldots, 1) \\
\tilde{\delta}_2(Q) &\equiv (\delta_1(Q), \delta_2(Q), 1, 1, \ldots, 1) \\
\tilde{\delta}_3(Q) &\equiv (\delta_1(Q), \delta_2(Q), \delta_3(Q), 1, 1, \ldots, 1) \\
&\ldots \\
\tilde{\delta}_d(Q) &= \delta(Q)
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\tau}_0(Q) &= 0 \\
\tilde{\tau}_1(Q) &= (\tau_1(Q), 0, 0, \ldots, 0) \\
\tilde{\tau}_2(Q) &= (\tau_1(Q), \tau_2(Q), 0, 0, \ldots, 0) \\
\tilde{\tau}_3(Q) &= (\tau_1(Q), \tau_2(Q), \tau_3(Q), 0, 0, \ldots, 0) \\
&\ldots \\
\tilde{\tau}_d(Q) &= \tau(Q).
\end{aligned}$$

In other words, considered as a dilation operator, $\tilde{\delta}_0(Q)$ starts as the identity, and then, as j advances, morphs—one variable at a time—into $\vec{\delta}(Q)$; while $\tilde{\tau}_j(Q)$ similarly morphs from the identity into $\vec{\tau}(Q)$, but now considered as a sequence of *translation* operators. *Keep in mind that $\delta_j(Q)$ and $\tau_j(Q)$ are numbers (components of the vectors $\vec{\delta}(Q)$ and $\vec{\tau}(Q)$) while $\tilde{\delta}_j(Q)$ and $\tilde{\tau}_j(Q)$ are vectors.*

Define $\zeta_0^{(Q)}(x) \equiv f^{(Q)}(x)$ and, for $1 \leq k \leq d$,

$$\zeta_k^{(Q)}(x) = \left(\prod_1^k \delta_j(Q)\right) f^{(Q)}(\tilde{\delta}_k(Q)(x - x_Q + \ell(Q)\tilde{\tau}_k(Q)) + x_Q).$$

After noticing that $\zeta_d^{(Q)}(x) = \left(\prod_1^d \delta_k(Q)\right)\widetilde{f^{(Q)}}(x)$, we write

$$\begin{aligned} f^{(Q)}(x) - \widetilde{f^{(Q)}}(x) &= f^{(Q)}(x) - \left(\prod_1^d \delta_k(Q)\right)\widetilde{f^{(Q)}}(x) + \left(\left(\prod_1^d \delta_k(Q)\right) - 1\right)\widetilde{f^{(Q)}}(x) \\ &= \zeta_0^{(Q)}(x) - \zeta_d^{(Q)}(x) + \left(\left(\prod_1^d \delta_k(Q)\right) - 1\right)\widetilde{f^{(Q)}}(x) \\ &= \left[\sum_{k=1}^d \left(\zeta_{k-1}^{(Q)}(x) - \zeta_k^{(Q)}(x)\right)\right] + \left[\left(\left(\prod_1^d \delta_k(Q)\right) - 1\right)\widetilde{f^{(Q)}}(x)\right] \\ &\equiv [I] + [II]\,. \end{aligned}$$

The term $[II]$ is no problem, because

$$\left|\left(\left(\prod_1^d \delta_k(Q)\right) - 1\right)\right| \leq C(d)\eta$$

and Theorem 2 controls the almost-orthogonal "norm" of $\{\widetilde{f^{(Q)}}/|Q|^{1/2}\}_{\mathcal{F}_{\vec{a}}}$.

To see what is going on with $[I]$, we look at the first term in the sum,

$$\zeta_0^{(Q)}(x) - \zeta_1^{(Q)}(x) = f^{(Q)}(x) - \delta_1(Q) f^{(Q)}(\tilde{\delta}_1(Q)(x - x_Q + \ell(Q)\tilde{\tau}_1(Q)) + x_Q). \tag{33}$$

Write $x = (x_1, x_2, \ldots, x_d)$ as (x_1, x^*), where $x_1 \in \mathbf{R}$ and $x^* \in \mathbf{R}^{d-1}$. For fixed x^*, (33) is

$$f^{(Q)}(x_1, x^*) - \delta_1(Q) f^{(Q)}(\delta_1(Q)(x_1 - (x_Q)_1 + \ell(Q)\tau_1(Q)) + (x_Q)_1, x^*) \tag{34}$$

(note the absence of tildes), because the (respective) dilation and translation operators $\tilde{\delta}_1(Q)$ and $\tilde{\tau}_1(Q)$ do not affect the x^* components at all.

To ease reading we refer to (34) as $\omega^{(Q)}(x)$.

For $\tilde{Q} \in \mathcal{G}_{\vec{a}}$, write $\tilde{Q} = I_1(\tilde{Q}) \times K(\tilde{Q})$, as in the statement of Lemma 2. Then

$$\omega^{(Q)}(x_1, x^*) = \omega^{(Q)}(x_1, x^*)\chi_{I_1(\tilde{Q})}(x_1)\chi_{K(\tilde{Q})}(x^*)$$

and, for every fixed $x^* \in \mathbf{R}^{d-1}$ and every finite linear sum

$$\sum_{Q\in\mathcal{F}_{\vec{a}}} \lambda_Q \left(\frac{\omega^{(Q)}}{|Q|^{1/2}}\right) = \sum_{Q\in\mathcal{F}_{\vec{a}}} \lambda_Q \left(\frac{\omega^{(Q)}(x_1, x^*)\chi_{I_1(\tilde{Q})}(x_1)\chi_{K(\tilde{Q})}(x^*)}{|Q|^{1/2}}\right),$$

we have, by the one-dimensional version of Theorem 3,

$$\int_{\mathbf{R}} \left|\sum_{Q\in\mathcal{F}_{\vec{a}}} \lambda_Q \left(\frac{\omega^{(Q)}(x_1, x^*)}{|Q|^{1/2}}\right)\right|^2 dx_1 \le C\eta \sum_{Q\in\mathcal{F}_{\vec{a}}} |\lambda_Q|^2 |K(\tilde{Q})|^{-1} \chi_{K(\tilde{Q})}(x^*). \tag{35}$$

Here we are arguing just as we did in estimating (16), but incorporating the '$\le C\eta$' bound we have from the one-dimensional Theorem 3 (see (30)–(32)). We get η this time, and not $\eta^{1/2}$, because we are not taking the square root of the integral. When we integrate (35) in x^* we get, for every $\vec{a} \in \{1, 2, 3\}^d$,

$$\begin{aligned}\int_{\mathbf{R}^d} \left|\sum_{Q\in\mathcal{F}_{\vec{a}}} \lambda_Q \left(\frac{\omega^{(Q)}(x)}{|Q|^{1/2}}\right)\right|^2 dx &= \int_{\mathbf{R}\times\mathbf{R}^{d-1}} \left|\sum_{Q\in\mathcal{F}_{\vec{a}}} \lambda_Q \left(\frac{\omega^{(Q)}(x_1, x^*)}{|Q|^{1/2}}\right)\right|^2 dx_1\, dx^* \\ &\le C\eta \sum_{Q\in\mathcal{F}_{\vec{a}}} |\lambda_Q|^2.\end{aligned}$$

The other summands in $[I]$ are handled in a similar fashion, successively treating the variables $x_2, \ldots, x_d$ as we did x_1. For example, $\zeta_1^{(Q)}(x) - \zeta_2^{(Q)}(x)$ equals $\delta_1(Q)$ times

$$f^{(Q)}(\tilde{\delta}_1(Q)(x - x_Q + \ell(Q)\tilde{\tau}_1(Q)) + x_Q) - \delta_2(Q) f^{(Q)}(\tilde{\delta}_2(Q)(x - x_Q + \ell(Q)\tilde{\tau}_2(Q)) + x_Q),$$

where the functions' two arguments, respectively

$$(\tilde{\delta}_1(Q)(x - x_Q + \ell(Q)\tilde{\tau}_1(Q)) + x_Q \tag{36}$$

and

$$\tilde{\delta}_2(Q)(x - x_Q + \ell(Q)\tilde{\tau}_2(Q)) + x_Q, \tag{37}$$

differ only in their second components. The second component of (36) is x_2, and that of (37) is

$$\delta_2(Q)(x_2 - (x_Q)_2 + \ell(Q)\tau_2(Q)) + (x_Q)_2.$$

But their first components both equal

$$\delta_1(Q)(x_1 - (x_Q)_1 + \ell(Q)\tau_1(Q)) + (x_Q)_1;$$

and, for $3 \le k \le d$, each kth component for both functions equals x_k.

If we now define, more or less as before,

$$\omega^{(Q)}(x) \equiv \zeta_1^{(Q)}(x) - \zeta_2^{(Q)}(x),$$

then the preceding argument applies virtually verbatim to yield

$$\int_{\mathbf{R}^d} \left| \sum_{Q \in \mathcal{F}_{\vec{a}}} \lambda_Q \left(\frac{\omega^{(Q)}(x)}{|Q|^{1/2}} \right) \right|^2 dx \le C\eta \sum_{Q \in \mathcal{F}_{\vec{a}}} |\lambda_Q|^2$$

for every $\vec{a} \in \{1, 2, 3\}^d$. (Recall that $\delta_1(Q)$ is essentially 1.) The same argument applies to the other summands $\zeta_{k-1}^{(Q)} - \zeta_k^{(Q)}$ for $3 \le k \le d$ to yield the same estimates. When we add up over all k and all $\vec{a} \in \{1, 2, 3\}^d$, and include the term $[II]$, we get

$$\left\| \sum_{Q \in \mathcal{D}} \lambda_Q \left(\frac{f^{(Q)} - \widetilde{f^{(Q)}}}{|Q|^{1/2}} \right) \right\|_2 \le C\eta^{1/2} \left(\sum_{Q \in \mathcal{D}} |\lambda_Q|^2 \right)^{1/2}$$

for all finite linear sums,

$$\sum_{Q \in \mathcal{D}} \lambda_Q \left(\frac{f^{(Q)} - \widetilde{f^{(Q)}}}{|Q|^{1/2}} \right),$$

where C depends on d. That's (21). Theorem 3 is proved. ♠

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Appendix

Point 1. The total variation $V_f(I)$ adds like a measure; $\Omega_f(I)$ doesn't. In particular, if $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ then

$$\sum_1^n V_f[x_{k-1}, x_k] \le V_f[a, b],$$

which we need (see (15)); but

$$\sum_1^n \Omega_f[x_{k-1}, x_k] \le \Omega_f[a, b]$$

fails. (As with V_f, we use $\Omega_f[x, y]$ to mean $\Omega_f([x, y])$.)

Point 2. We give two answers; the second makes the first redundant.

a) Theorem 1 yields

$$\left\| \sum \lambda_Q \frac{f^{(Q)}}{|Q|^{1/2}} \right\|_2 \le C(d) \left(\sum |\lambda_Q|^2 \right)^{1/2},$$

with $C(d)$ growing at worst linearly in d. If we apply Schur's test to

$$\left\{ \frac{f^{(Q)}}{|Q|^{1/2}} \right\}_{Q \in \mathcal{D}}$$

and the orthonormal family (3) then, for each J and Q in $\mathcal{D}$ such that $J \subset Q$, we have to consider the inner products

$$|Q|^{-1/2} |J|^{-1/2} \langle f^{(Q)}, h_i^{(J)} \rangle$$

for **every** $1 \le i < 2^d$, and the Schur bound grows exponentially in d.

But it's worse than that.

b) For fixed $d > 1$ set $Q_0 = [0, 1)^d$ and let B be the ball (open or closed) of radius $1/3$ centered at $(1/2, 1/2, 1/2, \ldots, 1/2)$ (the center of Q_0). Define $f(x) = \chi_B(x) - |B| \chi_{Q_0}(x)$. Then f is a bounded multiple of a function in $NBV(Q_0)$. We look at the terms appearing in the Schur test,

$$|J|^{-1/2} |Q_0|^{-1/2} \langle f, h_i^{(J)} \rangle = |J|^{-1/2} \langle f, h_i^{(J)} \rangle, \tag{38}$$

for $J \subset Q_0$ and a fixed i (the value of i doesn't matter: say it's 1). Let 2^{-k} be J's sidelength, where $k > 0$. The inner product (38) equals 0 if $J \subset B$ or $J \cap B = \emptyset$. It's possibly non-zero if J straddles B's boundary, and when that happens our best estimate for the absolute value of (38) is roughly

$$|J|^{-1/2} |J| = |J|^{1/2} = 2^{-kd/2}.$$

The $(d-1)$-dimensional measure of B's boundary is ~ 1 (it depends on d but not on k). The cubes J have diameters $\sim 2^{-k}$. For large k, $\sim 2^{k(d-1)}$ of them can straddle B's boundary. (The reader might want to sketch this.) Therefore, when we add up the estimated absolute values of (38) for these J's, we get $\sim 2^{-kd/2}2^{k(d-1)} = 2^{k(d/2-1)}$, which sums (over k) to infinity.

Philosophical Remark We think Theorem 1 holds because of subtle cancelation in the sums

$$\sum \lambda_Q \frac{f^{(Q)}}{|Q|^{1/2}},$$

which the bounded-variation-plus-induction argument lets us exploit without really understanding.

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