Chapter 6
Adaptive Fault Tolerant Backstepping Control for High-Order Nonlinear Systems

6.1 Introduction

It is well known that system physical components may become faulty which may cause system performance deterioration or worth, may lead to instability that can further produce catastrophic accidents. The fault effects require to be compensated to enhance the reliability and safety of the system. Accommodating faults to maintain acceptable system performances is particularly important for life-critical systems. In order to improve system reliability and to guarantee system stability in all situations, many effective FTC approaches have been proposed the literature.

Fuzzy logic systems (FLSs), as universal function approximators, have been widely used to model the nonlinearities with arbitrary preciseness. Due to the capability, fuzzy logic systems are also adopted to solve identification and control problems in nonlinear systems [1–6]. Various adaptive fuzzy control approaches, based on the feedback linearization, were developed for controlling uncertain nonlinear systems. Robust adaptive backstepping control [1, 5–10] and observer-based backstepping control [11–13] attracted much attention from many researchers, and many excellent results were obtained during the past decades.

Recently, stable control problems of high-order systems attracted the interest of many researchers [14–19]. In [14], the authors presented a continuous feedback solution to the problem of global strong stabilization, for genuine nonlinear systems that may not be stabilized, even locally, by a smooth feedback. The same authors extended their results in [15], where they investigated the reference tracking problem in nonlinear systems with disturbances. However, the control schemes in [14, 15] do not guarantee the closed-loop systems' stability or better tracking performance under faulty conditions.

In this chapter, we investigate the problem of active FTC for a class of high-order nonlinear uncertain systems with actuator gain faults. Compared with some existing works, the following main contributions are worth to be emphasized:

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(1) In literature, results concerning FTC in the literature like [20–31] consider the 1-order systems. This chapter extends the results to the more general systems, i.e., so-called high-order systems as [32–37], and an observer-based active fault-tolerant backstepping control scheme is proposed.

(2) Differing from the classical backstepping technology, our fault-tolerant control scheme does not need computing the high order derivatives of virtual control signal at each step of backstepping design procedure, which thus reduces the computation complexity.

(3) In general, the denominator of the fault-tolerant control law contains the estimate of the gain fault. If the denominator equals zero, a singularity occurs. In the proposed FTC scheme, the controller singularity is avoided without using a projection algorithm.

(4) In contrast with [20–25], the proposed FTC scheme does not require the a priori knowledge of the signs of control gain terms.

The rest of this chapter is organized as follows. In Sect. 6.2, the problem formulation, Nussbaum-type function and mathematical description of FLS, are introduced. Actuator faults are described and the FTC objectives are formulated. In Sect. 6.3, the main technical results of this chapter are given, which include fault detection, isolation, estimation and fault-tolerant control scheme design. The aircraft control application is presented in Sect. 6.4 and simulation results are given and demonstrate the effectiveness of the proposed technique. Finally, Sect. 6.5 draws the conclusion.

6.2 Problem Formulation and Mathematical Description of FLSs

In this section, we will formulate control problem. Then, the FLS description is introduced.

6.2.1 Problem Statement

Considers the following nonlinear systems:

$$\begin{cases} \dot{x}_i = x_{i+1}^p, \ i = 1, \ldots, n-1 \\ \dot{x}_n = f(x) + \sum_{j=1}^{m} g_j(x) u_j^p \\ y = x_1 \end{cases} \tag{6.1}$$

where $x = [x_1, x_2, \ldots, x_n]^T \in R^n$ denotes the state vector, $y = x_1$ denotes the system output, $u_j \in R$, $j = 1, 2, \ldots, m$ denote control inputs, $p \geq 1$ is a known positive odd number, $f(x) \in R$ denotes an unknown continuous smooth function,

$g_j(x) \in R$, $j = 1, \ldots, m$ are complete unknown control gain functions, i.e., the value and sign of $g_j(x)$ are both unknown.

Remark 6.1 System (6.1) is more general than the considered system in [18] which was described as $\dot{x}_i = x_{i+1}^p$, $i = 1, \ldots, n-1$ and $\dot{x}_n = u^p$. In addition, since actuator faults were not considered in [18], only one actuator was used. In this chapter, the FTC problem will be considered. In order to ensure the dependability of the controlled system, redundant actuators are added which leads to an over-actuated system.

In practical application, actuators may become faulty. In this chapter, actuator loss-of-effectiveness failures are considered, which can be modeled as follows.

$$u_j^f = k_j(x)u_j, \quad j = 1, \ldots, m, \quad t \geq t_j \tag{6.2}$$

where unknown function $k_j(x)$ denotes the remaining control rate, t_j is unknown fault occurrence time.

The control objectives, which are valid in normal (no fault) and faulty conditions, are to design the proper control inputs $u = [u_1, \ldots, u_m]^T$ which ensure that the system output can track asymptotically the reference model signal y_d with the tracking error converging to a small neighborhood of the origin and the closed-loop system is uniformly ultimately bounded (SGUUB). Under normal condition (no fault), u is designed to ensure boundedness of the closed-loop signals and asymptotic stability. Meanwhile, the FDI algorithm is working. As soon as actuator faults are detected and isolated, the fault accommodation algorithm is activated and a proper FTC input u is used such that the tracking performance is still maintained stable under faulty situation.

In order to design an appropriate controller, the following lemmas are introduced.

Lemma 6.1 ([38]) $\forall q > 1$, *being an odd integer,* $a, b \in R$, *the following inequality holds:*

$$|a + b|^q \leq (|a| + |b|)^q \leq 2^{q-1}|a^q + b^q| \tag{6.3}$$

Lemma 6.2 ([38]) $\forall m > 0 \in R, \forall n > 0 \in R$ *and* $r(x, y) > 0 \in R$, *the following inequality holds:*

$$|x|^m|y|^n \leq \frac{m}{m+n}r(x, y)|x|^{m+n} + \frac{n}{m+n}r^{-\frac{m}{n}}(x, y)|y|^{m+n} \tag{6.4}$$

Lemma 6.3 ([11]) *For* $\alpha \in R^{n_a}$, $\beta \in R^{n_b}$, $M \in R^{n_a \times n_b}$, *and arbitrary matrices* $X \in R^{n_a \times n_a}$, $Y \in R^{n_a \times n_b}$, $Z \in R^{n_b \times n_b}$, *if* $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$, *then*

$$-2\alpha^T M\beta \leq \begin{bmatrix} \alpha \\ \beta \end{bmatrix}^T \begin{bmatrix} X & Y - M \\ Y^T - M^T & Z \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \tag{6.5}$$

6.2.2 Nussbaum Type Gain

Any continuous function $N(s) : R \to R$ is a function of Nussbaum type if it has the following properties:

$$\lim_{s\to+\infty} \sup \frac{1}{s}\int_0^s N(\varsigma)d\varsigma = +\infty \tag{6.6}$$

$$\lim_{s\to-\infty} \inf \frac{1}{s}\int_0^s N(\varsigma)d\varsigma = -\infty \tag{6.7}$$

For example, the continuous functions $\varsigma^2\cos(\varsigma)$, $\varsigma^2\sin(\varsigma)$, and $e^{\varsigma^2}\cos((\pi/2)\varsigma)$ verify the above properties and are thus Nussbaum-type functions [39]. The even Nussbaum function $e^{\varsigma^2}\cos((\pi/2)\varsigma)$ is used throughout this chapter.

Lemma 6.4 ([40, 41]) *Let $V(\cdot)$ and $\varsigma(\cdot)$ be smooth functions defined on $[0, t_f)$ with $V(t) \geq 0, \forall t \in [0, t_f)$, and $N(\cdot)$ be an even smooth Nussbaum-type function. If the following inequality holds:*

$$V(t) \leq c_0 + \int_0^t (\underline{g}N(\varsigma) + 1)\dot{\varsigma}d\tau, \forall t \in [0, t_f) \tag{6.8}$$

where $\underline{g} \neq 0$ is a constant, and c_0 represents a suitable constant, then $V(t)$, $\varsigma(t)$ and $\int_0^t \underline{g}N(\varsigma)\dot{\varsigma}d\tau$ must be bounded on $[0, t_f)$.

Lemma 6.5 ([41]) *Let $V(\cdot)$ and $\varsigma(\cdot)$ be smooth functions defined on $[0, t_f)$ with $V(t) \geq 0, \forall t \in [0, t_f)$, and $N(\cdot)$ be an even smooth Nussbaum-type function. For $\forall t \in [0, t_f)$, if the following inequality holds,*

$$V(t) \leq c_0 + e^{-c_1 t}\int_0^t \underline{g}(\tau)N(\varsigma)\dot{\varsigma}e^{c_1\tau}d\tau + e^{-c_1 t}\int_0^t \dot{\varsigma}e^{c_1\tau}d\tau \tag{6.9}$$

where constant $c_1 > 0$, $\underline{g}(\cdot)$ is a time-varying parameter which takes values in the unknown closed intervals $I := [l^{-1}, l^{+1}]$ with $0 \notin I$, and c_0 represents some suitable constant, then $V(t)$, $\varsigma(t)$ and $\int_0^t \underline{g}(\tau)N(\varsigma)\dot{\varsigma}d\tau$ must be bounded on $[0, t_f)$.

6.2.3 Mathematical Description of FLSs

A fuzzy logic system consists of four parts: the knowledge base, the fuzzifier, the fuzzy inference engine working on fuzzy rules, and the defuzzifier. The knowledge base for FLS comprises a collection of fuzzy if-then rules of the following form:

$$\begin{aligned} R^l :\ & if\ x_1\ is\ A_1^l\ and\ x_2\ \ is\ A_2^l\ \cdots\ and\ x_n\ \ is\ A_n^l, \\ & then\ \ y\ is\ B^l,\ \ l = 1, 2, \ldots, M \end{aligned} \tag{6.10}$$

where $\underline{x} = [x_1, \ldots, x_n]^T \subset R^n$ and y are the FLS input and output, respectively. Fuzzy sets A_i^l and B^l are associated with the fuzzy functions $\mu_{A_i^l}(x_i) = \exp(-(\frac{x_i - a_i^l}{b_i^l})^2)$ and $\mu_{B^l}(y^l) = 1$, respectively. M is the rules number. Through singleton function, center average defuzzification and product inference [42], the FLS can be expressed as:

$$y(x) = \sum_{l=1}^{M} \bar{y}^l \left(\prod_{i=1}^{n} \mu_{A_i^l}(x_i) \right) \Big/ \sum_{l=1}^{M} \left(\prod_{i=1}^{n} \mu_{A_i^l}(x_i) \right) \tag{6.11}$$

where $\bar{y}^l = \max_{y \in R} \mu_{B^l}$. Define the fuzzy basis functions as:

$$\xi_l(x) = \prod_{i=1}^{n} \mu_{A_i^l}(x_i) \sum_{l=1}^{M} \left(\prod_{i=1}^{n} / \mu_{A_i^l}(x_i) \right)$$

and define $\theta^T = [\bar{y}^1, \bar{y}^2, \ldots, \bar{y}^M] = [\theta_1, \theta_2, \ldots, \theta_M]$ and $\xi(x) = [\xi_1(x), \ldots, \xi_M(x)]^T$, then the above FLS can be rewritten as:

$$y(x) = \theta^T \xi(x) \tag{6.12}$$

The stability results obtained in FLS control literature are semi-global in the sense that, as long as the input variable of the FLS remains within some pre-fixed compact set, where the compact set can be made as large as desired, there exist controllers with sufficiently large number of FLS rules such that all the signals in the closed-loop remain bounded.

Lemma 6.6 ([5, 6]) *Let $f(x)$ be a continuous function defined on a compact set Ω. Then for any constant $\varepsilon > 0$, there exists a FLS such as*

$$\sup_{x \in \Omega} |f(x) - \theta^T \xi(x)| \leq \varepsilon$$

In this chapter, using FLS, the unknown functions $f(x)$, $g_j(x)$ and $g_{kj}(x)$, $j = 1, 2, \ldots, m$, are approximated as

$$\hat{f}(x) = \hat{\theta}_f^T \xi_f(x), \quad \hat{f}(\hat{x}) = \hat{\theta}_f^T \xi_f(\hat{x})$$

$$\hat{g}_j(x) = \hat{\theta}_{gj}^T \xi_{gj}(x), \quad \hat{g}_j(\hat{x}) = \hat{\theta}_{gj}^T \xi_{gj}(\hat{x})$$

$$\hat{g}_{kj}(x) = \hat{\theta}_{gkj}^T \xi_{gkj}(x), \quad \hat{g}_{kj}(\hat{x}) = \hat{\theta}_{gkj}^T \xi_{gkj}(\hat{x})$$

Let define the optimal parameter vector θ_f^*, θ_{gj}^* and θ_{gkj}^* as

$$\theta_f^* = \arg\min_{\theta \in \Omega_f} [\sup_{x \in U, \hat{x} \in \hat{U}} |f(x) - \hat{f}(\hat{x})|]$$

$$\theta_{gj}^* = \arg\min_{\theta_{gj} \in \Omega_{gj}} [\sup_{x \in U, \hat{x} \in \hat{U}} |g_j(x) - \hat{g}_{gj}(\hat{x},)|]$$

$$\theta_{gkj}^* = \arg\min_{\theta_{gkj} \in \Omega_{gkj}} [\sup_{x \in U, \hat{x} \in \hat{U}} |g_{kj}(x) - \hat{g}_{kj}(\hat{x})|]$$

where Ω_f, Ω_{gj}, Ω_{gkj}, U and $\hat{U}$ are compact regions for $\hat{\theta}_f$, $\hat{\theta}_{gj}$, $\hat{\theta}_{gkj}$, x and $\hat{x}$, respectively; $\hat{\theta}_f$, $\hat{\theta}_{gj}$, $\hat{\theta}_{gkj}$ and $\hat{x}$ are the estimates of θ_f^*, θ_{gj}^*, θ_{gkj}^* and x, respectively. Similar to [11–13], The FLS minimum approximation errors and actual approximation errors are defined as

$$\varepsilon_f = f(x) - \theta_f^{*T} \xi_f(\hat{x}), \quad \delta_f = f(x) - \hat{\theta}_f^T \xi_f(\hat{x})$$

$$\varepsilon_{gj} = g_j(x) - \theta_{gj}^{*T} \xi_{gj}(\hat{x}), \quad \delta_{gj} = g_j(x) - \hat{\theta}_{gj}^T \xi_{gj}(\hat{x})$$

$$\varepsilon_{gkj} = g_{kj}(x) - \theta_{gkj}^{*T} \xi_{gkj}(\hat{x}), \quad \delta_{gkj} = g_{kj}(x) - \hat{\theta}_{gkj}^T \xi_{gkj}(\hat{x})$$

Now, the following assumptions are made.

Assumption 6.1 There exist unknown positive real constants ε_f^*, δ_f^*, ε_{gj}^*, δ_{gj}^*, ε_{gkj}^*, δ_{gkj}^* and known positive real constants $\bar{M}_{\varepsilon f}$, $\bar{M}_{\delta f}$, $\bar{M}_{\varepsilon gj}$, $\bar{M}_{\varepsilon gkj}$, such that $|\varepsilon_f| \le \varepsilon_f^*$, $\varepsilon_f^* \le \bar{M}_{\varepsilon f}$, $|\delta_f| \le \delta_f^*$, $\delta_f^* \le \bar{M}_{\delta f}$, $|\varepsilon_{gj}| \le \varepsilon_{gj}^*$, $\varepsilon_{gj}^* \le \bar{M}_{\varepsilon gj}$, $|\varepsilon_{gkj}| \le \varepsilon_{gkj}^*$, $\varepsilon_{gkj}^* \le \bar{M}_{gkj}$.

Assumption 6.2 There exist known positive real constants $M_{\theta f}$, $M_{\theta gj}$ and M_{gkj} such that $||\theta_f^*|| \le M_{\theta f}$, $||\theta_{gj}^*|| \le M_{\theta gj}$ and $||\theta_{gkj}^*|| \le M_{\theta gkj}$.

In order to facilitate the descriptions, in the following, $f(x)$, $g(x)$, $g_{kj}(x)$, $\hat{f}(\hat{x})$, $\hat{g}(\hat{x})$, $\hat{g}_{kj}(\hat{x})$, $\xi_f(\hat{x})$, $\xi_{gj}(\hat{x})$ and $\xi_{gkj}(\hat{x})$ are abbreviated to f, g, g_{kj}, $\hat{f}$, $\hat{g}$, $\hat{g}_{kj}$, ξ_f, ξ_{gj} and ξ_{gkj}, respectively.

6.3 Main Results

In this section, the main technical results of this chapter are given. We will first consider the stability control problem of system (6.1) under normal conditions, design a bank of observers to generate residuals, investigate the FDI algorithm based on the observers, and propose a FTC scheme to tolerate the fault using estimated fault information.

6.3.1 Fault Detection

In order to detect the fault, the following observer is constructed.

$$\begin{cases} \dot{\hat{x}}_i = \hat{x}_{i+1}^p + l_i(y - \hat{y}),\ i = 1, \ldots, n-1 \\ \dot{\hat{x}}_n = \hat{f} + \sum_{j=1}^{m} [\hat{g}_j + \hat{\varepsilon}_{gj}] u_j^p + l_n(y - \hat{y}) \\ \hat{y} = \hat{x}_1 = C\hat{x} \end{cases} \tag{6.13}$$

where $l_i, i = 1, \ldots, n$ are constant parameters that will be designed later.

Let $\hat{x} = [\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_n]^T$ and define observer errors $e_i = x_i - \hat{x}_i, i = 1, \ldots, n$, then observer error dynamics can be described as follows:

$$\begin{cases} \dot{e}_i = x_{i+1}^p - \hat{x}_{i+1}^p \\ \quad = (e_{i+1} + \hat{x}_{i+1})^p - \hat{x}_{i+1}^p - l_i(y - \hat{y}) \\ \quad = e_{i+1}^p - l_i(y - \hat{y}) + \sum_{l=1}^{p} C_p^l e_{i+1}^l \hat{x}_{i+1}^{p-l} \\ \dot{e}_n = f - \hat{f} + \sum_{j=1}^{m} (g_j - \hat{g}_j - \hat{\varepsilon}_{gj}) u_j^p - l_n(y - \hat{y}) \end{cases} \tag{6.14}$$

Using the notation $e = x - \hat{x}$, the above error dynamics can be re-written as:

$$\dot{e} = Ae_p + Re_p - L(y - \hat{y}) + d + B(d_f + d_g) \tag{6.15}$$

where $e_p = [e_1^p, \ldots, e_n^p]^T, d_i = \sum_{l=1}^{p} C_p^l e_{i+1}^l \hat{x}_{i+1}^{p-l}, i = 1, \ldots n-1, d_f = f - \hat{f} = \delta_f, d_g = \sum_{j=1}^{m} (g_j - \hat{g}_j - \hat{\varepsilon}_{gj}) u_i^p$, and

$$A = \begin{bmatrix} -r_1 & & \\ \vdots & I & \\ -r_n & 0 \cdots & 0 \end{bmatrix}, R = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}, L = \begin{bmatrix} l_1 \\ \vdots \\ l_n \end{bmatrix}, C = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}^T, d = \begin{bmatrix} d_1 \\ \vdots \\ 0 \end{bmatrix}, B = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

In the following we will use the backstepping technique to design the fault-tolerant controller.

Define

$$z_1 = \hat{x}_1 - y_d, z_i = \hat{x}_i - \alpha_{i-1}(\hat{x}_1, \ldots, \hat{x}_{i-1}), i = 2, 3, \ldots, n \tag{6.16}$$

where $\alpha_0 = 0$, $z_{n+1} = 0$, and $\alpha_{i-1}, i = 1, \ldots, n-1$ are virtual controls which will be designed at each step, $\alpha_n = u$ is the actual control input. The recursive design procedure contains n steps. From Step 1 to Step $n-1$, virtual control α_{i-1} is designed at each step. Finally an overall control law u is constructed at step n.

Step 1:
From $z_1 = \hat{x}_1 - y_d$, one has

$$\begin{aligned}\dot{z}_1 = \dot{\hat{x}}_1 - \dot{y}_d = \hat{x}_2^p = (z_2 + \alpha_1)^p + l_1(y - \hat{y}) - \dot{y}_d \\ = \alpha_1^p + \sum\nolimits_{j=1}^{p} C_p^j z_2^j \alpha_1^{p-j} + l_1(y - \hat{y}) - \dot{y}_d\end{aligned} \tag{6.17}$$

Define

$$V_1 = V_{11} + V_e, \quad V_{11} = \frac{1}{2} z_1^2, V_e = e^T P e$$

where $P = P^T > 0$ denotes a matrix with appropriate dimensions. Differentiating V_{11} with respect to time t leads to

$$\dot{V}_{11} = z_1 \dot{z}_1 = z_1 \alpha_1^p + z_1 \sum\nolimits_{j=1}^{p} C_p^j z_2^j \alpha_1^{p-j} + z_1 l_1(y - \hat{y}) - z_1 \dot{y}_d \tag{6.18}$$

Notice that, $p + 1 \geq 2$ is an even number. Differentiating V_e with respect to time t, from Lemma 6.3, it leads to

$$\begin{aligned}\dot{V}_e = 2e^T[P(A+K) + (A+K)^T P]e_p + 2e^T Pd + 2e^T PBd_f - e^T(PLC + C^T L^T P)e \\ \leq \begin{bmatrix} e \\ e_p \end{bmatrix}^T \begin{bmatrix} X - PLC - C^T L^T P & Y + P(A+R) \\ Y^T + (A+R)^T P & Z \end{bmatrix} \begin{bmatrix} e \\ e_p \end{bmatrix} + 2e^T P(d + Bd_f + Bd_g)\end{aligned} \tag{6.19}$$

where X, Y, Z denote matrices with appropriate dimensions, and $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$.

From Lemma 6.2, one has

$$\begin{aligned}\sum\nolimits_{k=1}^{p} C_p^k e_2^k \hat{x}_2^{p-k} \leq \sum\nolimits_{k=1}^{p} C_p^k \frac{k}{p} |e_2|^p \cdot \sigma + \sum\nolimits_{k=1}^{p} C_p^k \frac{p-k}{p} |\hat{x}_2|^p \cdot \sigma^{-\left(\frac{k}{p-k}\right)} \\ = \left[\sum\nolimits_{k=1}^{p} C_p^k \frac{k}{p} \sigma\right] \cdot |e_2|^p + \left[\sum\nolimits_{k=1}^{p} C_p^k \frac{p-k}{p} \sigma^{-\left(\frac{k}{p-k}\right)}\right] \cdot |\hat{x}_2|^p \\ = w_{e1}|e_2|^p + w_{e2}|\hat{x}_2|^p\end{aligned} \tag{6.20}$$

where $w_{e1} = \left[\sum_{k=1}^{p} C_p^k \frac{k}{p} \sigma\right]$, $w_{e2} = \left[\sum_{k=1}^{p} C_p^k \frac{p-k}{p} \sigma^{-\left(\frac{k}{p-k}\right)}\right]$.

Define

$$\sigma = \frac{p}{\lambda \sum_{k=1}^{p} C_p^k k}$$

where $\lambda > 1$ is a design parameter. Since $0 < \sigma \leq 1$, one has $w_{e1}|e_2|^p \leq \frac{1}{\lambda}|e_2|^p$. Therefore,

$$\sum\nolimits_{k=1}^{p} C_p^k e_2^k \hat{x}_2^{p-k} \leq \frac{1}{\lambda}|e_2|^p + w_{e2}|\hat{x}_2|^p.$$

Further one has

$$(\sum_{k=1}^{p} C_p^k e_2^k \hat{x}_2^{p-k})^2 \leq \frac{2}{\lambda^2}(|e_2|^p)^2 + 2(w_{e2})^2(|\hat{x}_2|^p)^2.$$

Similarly, one has

$$(\sum_{k=1}^{p} C_p^k e_i^k \hat{x}_i^{p-k})^2 \leq \frac{2}{\lambda^2}|e_i|^p + 2(w_{e2})^2(|\hat{x}_i|^p)^2, i = 2, \ldots, n$$

Hence,

$$d^T d \leq [\frac{1}{\lambda}|e_2|^p + w_{e2}|\hat{x}_2|^p, \ldots, \frac{1}{\lambda}|e_n|^p + w_{e2}|\hat{x}_n|^p, 0] \begin{bmatrix} \frac{1}{\lambda}|e_2|^p + w_{e2}|\hat{x}_2|^p \\ \vdots \\ \frac{1}{\lambda}|e_n|^p + w_{e2}|\hat{x}_n|^p \\ 0 \end{bmatrix}$$

$$= \sum_{i=2}^{n} \frac{2}{\lambda^2}(|e_i|^p)^2 + 2(w_{e2})^2(|\hat{x}_i|^p)^2$$

$$= \frac{2}{\lambda^2}\sum_{i=2}^{n}(|e_i|^p)^2 + \sum_{i=2}^{n} 2(w_{e2})^2(|\hat{x}_i|^p)^2$$

$$= [|e_2|^p, \ldots, |e_n|^p, 0] \begin{bmatrix} |e_2|^p \\ \vdots \\ |e_n|^p \\ 0 \end{bmatrix} + \sum_{i=2}^{n} 2(w_{e2})^2(|\hat{x}_i|^p)^2$$

$$= [|e_1|^p, |e_2|^p, \ldots, |e_n|^p] \begin{bmatrix} |e_1|^p \\ |e_2|^p \\ \vdots \\ |e_n|^p \end{bmatrix} - (e_1{}^p)^2 + \sum_{i=2}^{n} 2(w_{e2})^2(|\hat{x}_i|^p)^2$$

$$= e_p^T e_p - (e_1{}^p)^2 + \sum_{i=2}^{n} 2(w_{e2})^2(|\hat{x}_i|^p)^2$$

From Young's inequality, one has

$$e^T P d \leq e^T P P^T e + d^T d \leq e^T P P e + e_p^T e_p - (e_1{}^p)^2 + \sum_{i=2}^{n} 2(w_{e2})^2(|\hat{x}_i|^p)^2$$

$$2e^T B P d_f = e^T P B \delta_f \leq e^T P P e + \delta_f^2 \leq e^T P P^T e + (\delta_f^*)^2 \leq e^T P P e + (\bar{M}_{\delta f})^2.$$

Further, one has

$$\dot{V}_e \leq \begin{bmatrix} e \\ e_p \end{bmatrix}^T \begin{bmatrix} X - PLC - C^T L^T P + 2PP & Y + P(A+R) \\ Y^T + (A+R)^T P & Z + I \end{bmatrix} \begin{bmatrix} e \\ e_p \end{bmatrix} + \bar{\Delta}_0 + 2e^T P B d_g$$

where $\bar{\Delta}_0 = -(e_1{}^p)^2 + \sum_{i=2}^{n} 2(w_{e2})^2(|\hat{x}_i|^p)^2 + (\bar{M}_{\delta f})^2$, I denotes identity matrix with appropriate dimensions.

Hence, one has

$$\dot{V}_1 \leq z_1\alpha_1^p + z_1 \sum_{j=1}^{p} C_p^j z_2^j \alpha_1^{p-j} + z_1 l_1(y - \hat{y}) - z_1\dot{y}_d + \Delta_0 + 2e^T PBd_g + \begin{bmatrix} e \\ e_p \end{bmatrix}^T \begin{bmatrix} X - PLC - C^T L^T P + 2PP & Y + P(A+R) \\ Y^T + (A+R)^T P & Z + I \end{bmatrix} \begin{bmatrix} e \\ e_p \end{bmatrix}$$

Obviously, if matrices $X, Y, Z, Q > 0$ and $P = P^T > 0$ are chosen appropriately such that $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$ and

$$\begin{bmatrix} X - PLC - C^T L^T P + 2PP & Y + P(A+R) \\ Y^T + (A+R)^T P^T & Z + I \end{bmatrix} \leq -Q$$

where I denotes identity matrix with appropriate dimensions, then,

$$\begin{aligned} \dot{V}_1 \leq\ & z_1\alpha_1^p + z_1 \sum_{j=1}^{p} C_p^j z_2^j \alpha_1^{p-j} + z_1 l_1(y - \hat{y}) - z_1\dot{y}_d + \bar{\Delta}_0 + \\ & 2e^T PBd_g - \begin{bmatrix} e \\ e_p \end{bmatrix}^T Q \begin{bmatrix} e \\ e_p \end{bmatrix} \\ \leq & -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} e^T Pe + z_1\alpha_1^p + z_1 \sum_{j=1}^{p} C_p^j z_2^j \alpha_1^{p-j} + \\ & z_1 l_1(y - \hat{y}) - z_1\dot{y}_d + \bar{\Delta}_0 + 2e^T PBd_g \end{aligned} \tag{6.21}$$

Let $\Delta_0 = z_1 l_1(y - \hat{y}) - z_1\dot{y}_d + \bar{\Delta}_0$, one has

$$\dot{V}_1 \leq -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} e^T Pe + z_1\alpha_1^p + z_1 \sum_{j=1}^{p} C_p^j z_2^j \alpha_1^{p-j} + \Delta_0 + 2e^T PBd_g \tag{6.22}$$

Thus, virtual control α_1 can be modified as

$$\alpha_1 = \begin{cases} \sqrt[p]{(-\frac{1}{2}z_1 - \frac{\Delta_0}{z_1}}, & z_1 \in \Omega_{c_{z_1}}^0 \\ 0, & z_1 \in \Omega_{c_{z_1}} \end{cases} \tag{6.23}$$

Remark 6.2 In general, virtual control α_1 can be chosen as follows

$$\alpha_1 = \sqrt[p]{(-\frac{1}{2}z_1 - \frac{\Delta_0}{z_1}} \tag{6.24}$$

Just as pointed out in [41], for the above virtual control (6.23), controller singularity may occur since $\frac{\Delta_0}{z_1}$ is not well defined at $z_1 = 0$. Therefore, care must be taken to guarantee the boundedness of the control. It is noted that the controller singularity takes place at the point $z_1 = 0$, where the control objective is supposed to be achieved. From a practical point of view, once the system reaches its origin, no control action should be taken for less power consumption. As $z_1 = 0$ is hard to detect owing to the existence of measurement noise, it is more practical to relax our control objective of convergence to a “ball” rather than to the origin.

Similar to [41], let define $\Omega_{c_{z_i}} \subset \Omega$ and $\Omega^0_{c_{z_i}}$ s.t.

$$\Omega_{c_{z_i}} := \{z_i \,|\, |z_i| < c_{z_i}\} \Omega^0_{c_{z_i}} := \Omega - \Omega_{c_{z_i}},\ i = 1, \ldots, m$$

where $c_{z_i} > 0$ is a constant that can be chosen arbitrarily small and “-" is used to denote the complement of set B in set A as $A - B := \{x | x \in A \text{ and } x \notin B\}$. Thus, virtual control α_1 can be modified as (6.23).

Step 2.

Since $z_2 = \hat{x}_2 - \alpha_1$, one has

$$\begin{aligned} \dot{z}_2 &= \dot{\hat{x}}_2 - \frac{\partial \alpha_1}{\partial \hat{x}_1}(\hat{x}_2^p + l_i(y-\hat{y})) = \hat{x}_3^p - \frac{\partial \alpha_1}{\partial \hat{x}_1}\hat{x}_2^p - \frac{\partial \alpha_1}{\partial \hat{x}_1} l_1(y-\hat{y}) \\ &= (z_3 + \alpha_2)^p - \frac{\partial \alpha_1}{\partial \hat{x}_1}(z_2 + \alpha_1)^p - \frac{\partial \alpha_1}{\partial \hat{x}_1} l_i(y-\hat{y}) \\ &= \alpha_2^p + \sum\nolimits_{j=1}^{p} C_p^j z_3^j \alpha_2^{p-j} - \frac{\partial \alpha_1}{\partial \hat{x}_1} \sum\nolimits_{j=0}^{p} C_p^j z_2^j \alpha_1^{p-j} - \frac{\partial \alpha_1}{\partial \hat{x}_1} l_1(y-\hat{y}) \end{aligned} \tag{6.25}$$

Define

$$V_2 = V_1 + \frac{1}{2} z_2^2$$

Differentiating V_2 with respect to time t, leads to

$$\begin{aligned} \dot{V}_2 \le & \dot{V}_1 + \dot{z}_2 z_2 = -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} e^T P e - \frac{1}{2} z_1^2 + z_1 \sum\nolimits_{j=1}^{p} \left[C_p^j \alpha_1^{p-j} z_2^j \right] + z_2 \alpha_2^p + \\ & z_2 \sum\nolimits_{j=1}^{p} C_p^j z_3^j \alpha_2^{p-j} - z_2 \frac{\partial \alpha_1}{\partial \hat{x}_1} \sum\nolimits_{j=0}^{p} C_p^j z_2^j \alpha_1^{p-j} - z_2 \frac{\partial \alpha_1}{\partial \hat{x}_1} l_1(y-\hat{y}) + 2e^T P B d_g \end{aligned}$$

Let

$$\Delta_1 = \left\{ z_1 \sum\nolimits_{j=1}^{p} \left[C_p^j |\alpha_1^{p-j} z_2^{j-1}| \right] + \frac{\partial \alpha_1}{\partial \hat{x}_1} \sum\nolimits_{j=0}^{p} C_p^j |z_2^j \alpha_1^{p-j}| + |z_2| \frac{\partial \alpha_1}{\partial \hat{x}_1} l_1(y-\hat{y})| \right\} \tag{6.26}$$

$$\dot{V}_2 \le -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} e^T P e - \frac{1}{2} z_1^2 + \Delta_1 + z_2 \alpha_2^p + z_2 \sum\nolimits_{j=1}^{p} C_p^j z_3^j \alpha_2^{p-j} + 2e^T P B d_g \tag{6.27}$$

Similarly, choose a virtual control as follows

$$\alpha_2 = \begin{cases} \sqrt[p]{(-\frac{1}{2}z_2 - \frac{\Delta_1}{z_2}}, \ z_2 \in \Omega^0_{c_{z_2}} \\ 0, \ z_2 \in \Omega_{c_{z_2}} \end{cases} \tag{6.28}$$

Substituting α_2 into (6.27), it yields

$$\dot{V}_2 \leq -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}e^T Pe - \frac{1}{2}z_1^2 - \frac{1}{2}z_2^2 + z_2 \sum_{j=1}^{p} C_p^j z_3^j \alpha_2^{p-j} + 2e^T PBd_g \tag{6.29}$$

Step k:
Since $z_k = \hat{x}_k - \alpha_{k-1}$, one has

$$\begin{aligned} \dot{z}_k &= \dot{\hat{x}}_k - \sum_{l=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial \hat{x}_l}(\hat{x}_{l+1}^p + l_{l+1}(y - \hat{y})) \\ &= \hat{x}_{k+1}^p - \sum_{l=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial \hat{x}_l}(\hat{x}_{l+1}^p + l_{l+1}(y - \hat{y})) \\ &= (z_{k+1} + \alpha_k)^p - \sum_{l=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial \hat{x}_l}(\hat{x}_{l+1}^p + l_{l+1}(y - \hat{y})) \\ &= \alpha_k^p + \sum_{j=1}^{p} C_p^j z_{k+1}^j \alpha_k^{p-j} - \sum_{l=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial \hat{x}_l}(\hat{x}_{l+1}^p + l_{l+1}(y - \hat{y})) \end{aligned} \tag{6.30}$$

Define

$$V_k = V_{k-1} + \frac{1}{2}z_k^2$$

Differentiating V_k with respect to time t, leads to

$$\begin{aligned} \dot{V}_k \leqslant & -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}e^T Pe - \frac{1}{2}\sum_{i=1}^{k-1} z_i^2 + \Delta_{k-1} + z_k \alpha_k^p + \\ & z_k \sum_{j=1}^{p} C_p^j z_{k+1}^j \alpha_k^{p-j} + 2e^T PBd_g \end{aligned} \tag{6.31}$$

where

$$\Delta_{k-1} = \left\{ \begin{aligned} & z_{k-1} \sum_{j=1}^{p} \left[C_p^j |\alpha_{k-1}^{p-j} z_k^{j-1}| \right] + \\ & \sum_{i=1}^{k-1} \frac{\partial \alpha_{k-1}}{\partial \hat{x}_i} [\sum_{l=0}^{p} C_p^l |z_k^l \alpha_{k-1}^{p-l}| + |l_i(y - \hat{y})|] \end{aligned} \right\}.$$

Just as α_{k-1}, virtual control α_k is chosen as follows

$$\alpha_k = \begin{cases} \sqrt[p]{(-\frac{1}{2}z_k - \frac{\Delta_{k-1}}{z_k}}, \ z_k \in \Omega^0_{c_{z_k}} \\ 0, \ z_k \in \Omega_{c_{z_k}} \end{cases} \tag{6.32}$$

Substituting α_k into (6.28), yields

$$\dot{V}_k \leqslant -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} e^T P e - \frac{1}{2}\sum_{i=1}^{k} z_i^2 + \sum_{j=1}^{p} C_p^j z_{k+1}^j \alpha_k^{p-j} + 2e^T P B d_g \quad (6.33)$$

Step n:
Since $z_n = \hat{x}_n - \alpha_{n-1}$, one has

$$\begin{aligned}
\dot{z}_n &= \dot{\hat{x}}_n - \sum_{l=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}(\hat{x}_{l+1}^p + l_{l+1}(y-\hat{y})) \\
&= \hat{f} + \sum_{j=1}^{m} (\hat{g}_j + \varepsilon_j) u_j^p + l_n(y-\hat{y}) - \sum_{l=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}(\hat{x}_{l+1}^p + l_{l+1}(y-\hat{y})) \\
&= f - \delta_f + \sum_{j=1}^{m} (\hat{g}_j + \varepsilon_j) u_j^p + l_n(y-\hat{y}) - \sum_{l=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}(\hat{x}_{l+1}^p + l_{l+1}(y-\hat{y})) \\
&= \tilde{\theta}_f^T \xi_f + \hat{\theta}_f^T \xi_f + \gamma_f + \sum_{j=1}^{m} (\hat{g}_j + \varepsilon_j) u_j^p + l_n(y-\hat{y}) - \\
&\quad \sum_{l=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_l}(\hat{x}_{l+1}^p + l_{l+1}(y-\hat{y}))
\end{aligned} \quad (6.34)$$

Define

$$V_n = V_{n-1} + \frac{1}{2} z_n^2 + \frac{1}{2\eta_1} \tilde{\theta}_f^T \tilde{\theta}_f + \frac{1}{2\eta_2} \tilde{\gamma}_f^2 + \frac{1}{2\eta_3} \sum_{j=1}^{m} (\tilde{\theta}_{gj}^T \tilde{\theta}_{gj} + \tilde{\varepsilon}_{gj}^2) \quad (6.35)$$

where $\gamma_f^* = \varepsilon_f^* + \delta_f^*$, $\tilde{\gamma}_f = \gamma_f^* - \hat{\gamma}_f$, $\tilde{\theta}_f = \theta_f^* - \hat{\theta}_f$, $\tilde{\gamma}_f = \gamma_f^* - \hat{\gamma}_f$, $\tilde{\theta}_{gj} = \theta_{gj}^* - \hat{\theta}_{gj}$, $\tilde{\varepsilon}_{gj} = \varepsilon_{gj}^* - \hat{\varepsilon}_{gj}$, $\hat{\theta}_f$, $\hat{\gamma}_f$, $\hat{\theta}_{gj}$, $\hat{\varepsilon}_{gj}$ are the estimates of θ_f^*, γ_f^*, θ_{gj}^*, ε_{gj}^*, and $\eta_1 > 0$, $\eta_2 > 0$, $\eta_3 > 0$ are adaptive rates.

Differentiating V_n with respect to time t, leads to

$$\begin{aligned}
\dot{V}_n \leqslant& -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} e^T P e - \frac{1}{2}\sum_{i=1}^{n-1} z_i^2 + z_n \tilde{\theta}_f^T \xi_f + |z_n| \tilde{\gamma}_f + z_n \sum_{j=1}^{m} (\hat{g}_j + \hat{\varepsilon}_j) u_j^p + \\
& \Delta_{n-1} - \frac{1}{\eta_1} \tilde{\theta}_f^T \dot{\hat{\theta}}_f - \frac{1}{\eta_2} \tilde{\gamma}_f \dot{\hat{\gamma}}_f + 2e^T P d_g - \frac{1}{\eta_3} \sum_{j=1}^{m} (\tilde{\theta}_{gj}^T \dot{\hat{\theta}}_{gj} + \tilde{\varepsilon}_{gj} \dot{\hat{\varepsilon}}_{gj}) \\
\leqslant& -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} e^T P e - \frac{1}{2}\sum_{i=1}^{n-1} z_i^2 + z_n \sum_{j=1}^{m} (\hat{g}_j + \hat{\varepsilon}_j) u_j^p + \Delta_{n-1} + 2e^T P d_g - \\
& \frac{1}{\eta_3} \sum_{j=1}^{m} (\tilde{\theta}_{gj}^T \dot{\hat{\theta}}_{gj} + \tilde{\varepsilon}_{gj} \dot{\hat{\varepsilon}}_{gj}) + \tilde{\theta}_f^T (z_n \xi_f - \frac{1}{\eta_1} \dot{\hat{\theta}}_f) + \tilde{\gamma}_f (|z_n| - \frac{1}{\eta_2} \dot{\hat{\gamma}}_f)
\end{aligned}$$

where

$$\Delta_{n-1} = \left\{ \begin{array}{l} z_{n-1} \sum_{j=1}^{p} \left[C_p^j |\alpha_{n-1}^{p-j} z_n^{j-1}| \right] + z_n(\hat{\theta}_f^T \xi_f(\hat{x}, v) + l_n(y - \hat{y}) + \\ \sum_{i=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_i} [\sum_{j=0}^{p} C_p^j |z_k^j \alpha_{k-1}^{p-j}| + |l_i(y - \hat{y})|] - \\ \sum_{j=1}^{n-1} \frac{\partial \alpha_{n-1}}{\partial \hat{x}_j} (\hat{x}_{j+1}^p + l_{j+1}(y - \hat{y})) + |z_n| \hat{\gamma}_f \end{array} \right\}.$$

Since

$$\begin{aligned} 2e^T PBd_g &= \sum_{j=1}^{m} 2e^T P_n(g_j - \hat{g}_j - \hat{\varepsilon}_{gj}) u_j^p \\ &= \sum_{j=1}^{m} 2e^T P_n(\theta_{gj}^{*T} \xi_{gj} + \varepsilon_{gj} - \hat{\theta}_{gj}^T \xi_{gj} - \hat{\varepsilon}_{gj})) u_j^p) \\ &= \sum_{j=1}^{m} 2e^T P_n \tilde{\theta}_{gj}^T \xi_{gj} u_j^p + \\ &\quad \sum_{j=1}^{m} 2e^T P_n(\varepsilon_{gj}^* - \hat{\varepsilon}_{gj}) u_j^p + \sum_{j=1}^{m} 2e^T P_n(\varepsilon_{gj} - \varepsilon_{gj}^*) u_j^p \\ &= \sum_{j=1}^{m} 2e^T P_n \tilde{\theta}_{gj}^T \xi_{gj} u_j^p + \sum_{j=1}^{m} 2e^T P_n \tilde{\varepsilon}_{gj} u_j^p + \\ &\quad \sum_{i=1}^{m} 2e^T P_n(\varepsilon_{gj} - \varepsilon_{gj}^*) u_j^p \end{aligned}$$

from the above inequality, one has

$$\begin{aligned} \dot{V}_n \leqslant & -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} e^T Pe - \frac{1}{2} \sum_{i=1}^{n-1} z_i^2 + z_n \tilde{\theta}_f^T \xi_f + |z_n| \tilde{\gamma}_f + z_n \sum_{j=1}^{m} (\hat{g}_j + \hat{\varepsilon}_j) u_j^p + \\ & \Delta_{n-1} + \sum_{j=1}^{m} 2e^T P_n(\tilde{\theta}_{gj}^T \xi_{gj} + \tilde{\varepsilon}_{gj}) u_j^p + \sum_{i=1}^{m} 2e^T P_n(\varepsilon_{gj} - \varepsilon_{gj}^*) u_j^p - \\ & \frac{1}{\eta_1} \tilde{\theta}_f^T \dot{\hat{\theta}}_f - \frac{1}{\eta_2} \tilde{\gamma}_f \dot{\hat{\gamma}}_f - \frac{1}{\eta_3} \sum_{j=1}^{m} (\tilde{\theta}_{gj}^T \dot{\hat{\theta}}_{gj} + \tilde{\varepsilon}_{gj} \dot{\hat{\varepsilon}}_{gj}) \\ = & -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)} e^T Pe + \sum_{j=1}^{m} [z_n(\hat{g}_j + \hat{\varepsilon}_{gj}) + 2e^T P_n(\varepsilon_{gj} - \varepsilon_{gj}^*)] u_j^p - \\ & \sum_{j=1}^{m} [\tilde{\theta}_{gj}^T (2e^T P_n \xi_{gj} u_j^p - \frac{\dot{\hat{\theta}}_{gj}}{\eta_3}) + \tilde{\varepsilon}_{gj} (2e^T P_n u_j^p - \frac{\dot{\hat{\varepsilon}}_{gj}}{\eta_3})] + \\ & \tilde{\theta}_f^T (z_n \xi_f - \frac{1}{\eta_1} \dot{\hat{\theta}}_f) + \tilde{\gamma}_f (|z_n| - \frac{1}{\eta_2} \dot{\hat{\gamma}}_f) - \frac{\sum_{i=1}^{n-1} z_i^2}{2} + \Delta_{n-1} \end{aligned} \tag{6.36}$$

Choose control law $\alpha_{n,i}$, $i = 1, 2, \ldots, m$ and adaptation functions $\dot{\hat{\theta}}_f, \dot{\hat{\gamma}}_f, \dot{\hat{\theta}}_{gj}, \dot{\hat{\varepsilon}}_{gj}$ as follows:

$$\alpha_{n,i} = u_i = \alpha_k = \begin{cases} \sqrt[p]{\dfrac{N(\varsigma)(-\frac{1}{2} z_n - \frac{\Delta_{n-1}}{z_n})}{m}}, \ z_k \in \Omega_{c_{z_n}}^0 \\ 0, \ z_k \in \Omega_{c_{z_n}} \end{cases} \tag{6.37}$$

where $\dot{\varsigma} = -\frac{1}{2}z_n^2 - \Delta_{n-1}$,

$$\dot{\hat{\theta}}_f = \eta_1 z_n \xi_f - \eta_f \hat{\theta}_f \tag{6.38}$$

$$\dot{\hat{\gamma}}_f = \eta_2 |z_n| - \eta_\gamma \hat{\gamma}_f \tag{6.39}$$

$$\dot{\hat{\theta}}_{gj}^T = 2\eta_3 e^T P_n \xi_{gj} u_j^p + \eta_{gj} \hat{\theta}_{gj} \tag{6.40}$$

$$\dot{\hat{\varepsilon}}_{gj} = 2\eta_3 e^T P_n u_j^p + \eta_{gj} \hat{\varepsilon}_{gj} \tag{6.41}$$

and $\eta_f > 0, \eta_\gamma > 0, \eta_{gj} > 0$ are design parameters, u_j is a bounded control input which is applied simultaneously to the ith actuator in the system (6.1) and the observer (6.13).

Applying Young's inequality, one has

$$\frac{\eta_f}{\eta_1}\tilde{\theta}_f^T\hat{\theta}_f = \frac{\eta_f}{\eta_1}\tilde{\theta}_f^T(\theta_f^* - \tilde{\theta}_f) = -\frac{\eta_f}{\eta_1}\tilde{\theta}_f^T\tilde{\theta}_f + \frac{\eta_f}{\eta_1}\tilde{\theta}_f^T\theta_f^* \leqslant -\frac{\eta_f}{2\eta_1}\tilde{\theta}_f^T\tilde{\theta}_f + \frac{\eta_f}{2\eta_1}\theta_f^{*T}\theta_f^*,$$

$$\frac{\eta_\gamma}{\eta_2}\tilde{\gamma}_f\hat{\gamma}_f = \frac{\eta_\gamma}{\eta_2}\tilde{\gamma}_f(\gamma_f^* - \tilde{\gamma}_f) = -\frac{\eta_\gamma}{\eta_2}\tilde{\gamma}_f^2 + \frac{\eta_\gamma}{\eta_2}\tilde{\gamma}_f\gamma_f^* \leqslant -\frac{\eta_\gamma}{2\eta_2}\tilde{\gamma}_f^2 + (\frac{\eta_\gamma}{2\eta_2}\gamma_f^*)^2,$$

$$\frac{\eta_{gj}}{\eta_1}\tilde{\theta}_{gj}^T\hat{\theta}_{gj} = \frac{\eta_{gj}}{\eta_1}\tilde{\theta}_{gj}^T(\theta_{gj}^* - \tilde{\theta}_{gj}) = -\frac{\eta_{gj}}{\eta_1}\tilde{\theta}_{gj}^T\tilde{\theta}_{gj} + \frac{\eta_{gj}}{\eta_1}\tilde{\theta}_{gj}^T\theta_{gj}^* \leqslant -\frac{\eta_{gj}}{2\eta_1}\tilde{\theta}_{gj}^T\tilde{\theta}_{gj} + \frac{\eta_{gj}}{2\eta_1}\theta_{gj}^{*T}\theta_{gj}^*$$

$$\frac{\eta_{gj}}{\eta_3}\tilde{\varepsilon}_{gj}\hat{\varepsilon}_{gj} = \frac{\eta_{gj}}{\eta_3}\tilde{\varepsilon}_{gj}(\varepsilon_{gj}^* - \tilde{\varepsilon}_{gj}) = -\frac{\eta_{gj}}{\eta_3}\tilde{\varepsilon}_{gj}^2 + \frac{\eta_{gj}}{\eta_3}\tilde{\varepsilon}_{gj}\varepsilon_{gj}^* \leqslant -\frac{\eta_{g,i}}{2\eta_3}\tilde{\varepsilon}_{gj}^2 + \frac{\eta_{gj}}{2\eta_3}(\varepsilon_{gj}^*)^2$$

Substituting the above inequalities into (6.36), it yields

$$\begin{aligned}
\dot{V}_n \leqslant & -\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}e^T P e - \frac{1}{2}\sum_{i=1}^n z_i^2 + \\
& \sum_{i=1}^m \{z_n(\hat{g}_j + \hat{\varepsilon}_{gj}) + 2e^T P_n(\varepsilon_{gj} - \varepsilon_{gj}^*)]N(\varsigma)\dot{\varsigma} + \dot{\varsigma}\} \\
& \leqslant -gV_n + \frac{\eta_f}{2\eta_1}\theta_f^{*T}\theta_f^* + (\frac{\eta_\gamma}{2\eta_2}\gamma_f^*)^2 + \sum_{j=1}^m (\frac{\eta_{gj}}{2\eta_1}\theta_{gj}^{*T}\theta_{gj}^* + \frac{\eta_{gj}}{2\eta_3}(\varepsilon_{gj}^*)^2) + \\
& \sum_{i=1}^m \{z_n(\hat{g}_j + \hat{\varepsilon}_{gj}) + 2e^T P_n(\varepsilon_{gj} - \varepsilon_{gj}^*)]N(\varsigma)\dot{\varsigma} + \dot{\varsigma}\} \\
& \leqslant -gV_n + \frac{\eta_f}{2\eta_1}M_{\theta f}^2 + \frac{\eta_\gamma}{2\eta_2}(\bar{M}_{\varepsilon f} + \bar{M}_{\delta f})^2 + \sum_{j=1}^m (\frac{\eta_{gj}}{2\eta_1}M_{\theta gj}^2 + \frac{\eta_{gj}}{2\eta_3}\bar{M}_{\varepsilon gj}^2) \\
& \sum_{j=1}^m \{z_n(\hat{g}_j + \hat{\varepsilon}_{gj}) + 2e^T P_n(\varepsilon_{gj} - \varepsilon_{gj}^*)]N(\varsigma)\dot{\varsigma} + \dot{\varsigma}\} \\
& \leqslant -gV_n + \mu + hN(\varsigma) + 1)\dot{\varsigma}
\end{aligned} \tag{6.42}$$

where

$$g = \min\{\frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}, \frac{1}{2}, \frac{\eta_f}{2\eta_1}, \frac{\eta_\gamma}{2\eta_2}, \frac{\eta_{g1}}{2\eta_3}, \ldots, \frac{\eta_{gm}}{2\eta_3}\},$$

$$\mu = \frac{\eta_f}{2\eta_1} M_{\theta f}^2 + \frac{\eta_\gamma}{2\eta_2}(\bar{M}_{\varepsilon f} + \bar{M}_{\delta f})^2 + \sum\nolimits_{j=1}^{m} \frac{\eta_{gj}}{2\eta_3}(\bar{M}_{\theta gj}^2 + \bar{M}_{\varepsilon gj}^2),$$

$$h = \sum\nolimits_{j=1}^{m} [z_n(\hat{g}_j + \hat{\varepsilon}_{gj}) + 2e^T P_n(\varepsilon_{gj} - \varepsilon_{gj}^*)].$$

The above control design procedures and analysis are summarized in the following theorem.

Theorem 6.1 *Consider nonlinear system (6.1) under Assumptions 6.1 and 6.2, control law (6.37) and adaptive laws (6.38–6.41). If matrices* $X, Y, Z, Q > 0$ *and* $P = P^T > 0$ *are such that* $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$ *and*

$$\begin{bmatrix} X - PLC - C^T L^T P + 2PP & Y + P(A+R) \\ Y^T + (A+R)^T P & Z + I \end{bmatrix} \leqslant -Q \tag{6.43}$$

we can guarantee the following properties under bounded initial conditions

(1) all signals in the closed-loop system are semi-globally uniformly ultimately bounded;

(2) the vectors z_i *remain in the compact set* $\Omega_{z_i}^0$, $i = 1, 2, \ldots, n$ *specified as*

$$\Omega_{z_i}^0 := \left\{ \begin{array}{l} (z_i, \tilde{\theta}_f, \tilde{\gamma}_f, \tilde{\theta}_{gj}, \tilde{\varepsilon}_{gj}, e) \Big| \, |z_i| \leqslant \sqrt{2\bar{\mu}}, ||\tilde{\theta}_f|| \leqslant \sqrt{2\eta_1\bar{\mu}}, \\ |\tilde{\gamma}_f| \leqslant \sqrt{2\eta_2\bar{\mu}}, ||\tilde{\theta}_{gj}|| \leqslant \sqrt{2\eta_3\bar{\mu}}, \\ |\tilde{\varepsilon}_{gj}| \leqslant \sqrt{2\eta_3\bar{\mu}}, ||e|| \leqslant \sqrt{\dfrac{\bar{\mu}}{\lambda_{\min}(P)}} \end{array} \right\}$$

whose size is $\bar{\mu} = \frac{\mu}{g} + c_g + V_n(0) > 0$, *which can be adjusted by appropriately choosing the design parameters* $\eta_1, \eta_2, \eta_3, \eta_f, \eta_\gamma, \eta_{g,1}, \ldots, \eta_{g,m}$.

Proof Since $\dot{V}_n \leqslant -gV_n + \mu + hN(\varsigma) + 1)\dot{\varsigma}$, one has

$$\begin{aligned} V_n(t) &\leqslant \frac{\mu}{g} + [V_n(0) - \frac{\mu}{g}]e^{-gt} + e^{-gt}\int_0^t (hN(\varsigma) + 1)e^{-g\tau}\dot{\varsigma}d\tau \\ &\leqslant \frac{\mu}{g} + V_n(0)e^{-gt} + e^{-gt}\int_0^t (hN(\varsigma) + 1)e^{-g\tau}\dot{\varsigma}d\tau \end{aligned} \tag{6.44}$$

Applying Lemma 6.5, we can conclude that, $V_n(t)$, $\int_0^t (hN(\varsigma)+1)e^{-g\tau}\dot{\varsigma}d\tau$ and $\varsigma(t)$ are SGUUB on $[0, t_f)$. According to Proposition 2 in [39], if the solution of the closed-loop system is bounded, then $t_f = +\infty$. Let c_g be the upper bound of $\int_0^t (hN(\varsigma)+1)e^{-g\tau}\dot{\varsigma}d\tau$, we have the following inequalities:

$$e^{-gt}\int_0^t (hN(\varsigma)+1)e^{-g\tau}\dot{\varsigma}d\tau \leqslant \int_0^t (hN(\varsigma)+1)e^{-g\tau}\dot{\varsigma}d\tau \leqslant c_g$$

Thus, (6.44) becomes

$$V_n(t) \leqslant \frac{\mu}{g} + c_g + V_n(0) = \bar{\mu} \tag{6.45}$$

Hence, if matrices X, Y, Z, Q and positive definite symmetric matrices P are chosen appropriately such that $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$ and (6.38) holds, then, the proposed control input (6.37) can ensure that $V_n(t)$ is bounded, namely, the closed-loop system is semi-globally uniformly ultimately bounded. Noting the definitions of $V_n(t)$ and $z_i, i = 1, 2, \ldots, n$, we have $\frac{1}{2}z_i^2 \leqslant V_n(t) \leqslant \bar{\mu}$ and $\frac{1}{2\eta_1}\tilde{\theta}_f^T\tilde{\theta}_f \leqslant \bar{\mu}$. Furthermore, we have $|z_i| \leqslant \sqrt{2\bar{\mu}}$, $||\tilde{\theta}_f|| \leqslant \sqrt{2\eta_1\bar{\mu}}$. Similarly, we have $|\tilde{\gamma}_f| \leqslant \sqrt{2\eta_2\bar{\mu}}$, $||\tilde{\theta}_{g,i}|| \leqslant \sqrt{2\eta_3\bar{\mu}}$, $|\tilde{\varepsilon}_{g,i}| \leqslant \sqrt{2\eta_3\bar{\mu}}$, $||e|| \leqslant \sqrt{\frac{\bar{\mu}}{\lambda_{\min}(P)}}$. From the above analysis, we can conclude that there do exist compact sets $\Omega_{z_i}^0$ such that $z_i \in \Omega_{z_i}^0, \forall t \geqslant 0$. The proof is completed.

From Theorem 6.1, one has

$$||e|| \leqslant \sqrt{\frac{2\bar{\mu}}{\lambda_{\min}(P)}} \tag{6.46}$$

Furthermore, the detection residual can be defined as

$$J = ||y(t) - \hat{y}(t)|| \tag{6.47}$$

From (6.46), it can be seen that the following inequality holds in the healthy case:

$$J \leqslant ||C||\sqrt{\frac{2\bar{\mu}}{\lambda_{\min}(P)}} \tag{6.48}$$

Then, the fault detection can be performed using the following mechanism:

$$\begin{cases} J \leqslant T_d \text{ no fault occurred,} \\ J > T_d \text{ fault has occurred} \end{cases} \tag{6.49}$$

where threshold T_d is defined as follows:

$$T_d = ||C||\sqrt{\frac{2\bar{\mu}}{\lambda_{\min}(P)}} \tag{6.50}$$

6.3.2 Fault Isolation and Estimation

Since the system has m actuators and it is assumed that only one actuator becomes faulty at one time, we have m possible faulty cases in total. When the sth ($1 \leqslant s \leqslant m$) actuator is faulty, the faulty model can be described as:

$$u_s^f = \rho_s(x)u_s \tag{6.51}$$

The faulty system (6.1) can be described as follows:

$$\begin{cases} \dot{x}_{s,i} = x^p_{s,i+1}, \ i = 1, \ldots, n-1 \\ \dot{x}_{s,n} = f + \sum_{\substack{j=1 \\ j\neq s}}^{m} g_j u_j^p - g_s \rho_s^p u_s^p \\ y_s = x_{s,1} \end{cases} \tag{6.52}$$

After a fault has been detected, the isolation scheme is activated. Now, the following m nonlinear fault isolation observers are designed as follows:

$$\begin{cases} \dot{\hat{x}}_{s,i} = \hat{x}^p_{s,i+1} + l_{si}(y_s - \hat{y}_s), \ i = 1, \ldots, n-1 \\ \dot{\hat{x}}_{s,n} = \hat{\theta}_f^T \xi_f + \sum_{j=1, j\neq r}^{m} [\hat{g}_g + \hat{\varepsilon}_{gj}]u_j^p + l_{sn}(y - \hat{y}) + (\hat{\theta}^T_{gkr}\xi + \hat{\varepsilon}_{gkr})u_r^p \\ \hat{y}_s = \hat{x}_{s,1} = C\hat{x}_s \end{cases} \tag{6.53}$$

where $l_{si}, i = 1, 2, \ldots, n, s = 1, 2 \cdots .m$ are constants, which will be designed later, $\hat{\theta}^T_{g\rho,r}\xi_{g\rho,r}(\hat{x}_s, v)$ is the estimate of $g_r(x, v)\rho_r^p(x_r)$, $r = 1, \ldots, m$.

Let $\hat{x}_s = [\hat{x}_{s,1}, \hat{x}_{s,2}, \ldots, \hat{x}_{s,n}]^T$, the error terms $e_s = x_s - \hat{x}_s$ and $e_{ys} = y_s - \hat{y}_s$ are respectively the state error and output error between the faulty plant and the sth observer. The above error dynamics can be re-written as:

$$\dot{e}_s = A_s e_s^p + R_s e_s^p - L_s(y_s - \hat{y}_s) + d_s + B_s(d_f + d_g + \rho_s) \tag{6.54}$$

where $e_{sp} = [e_{s,1}^p, \ldots, e_{s,n}^p]^T$, $d_f = f - \hat{\theta}_f^T \xi_f$, $\rho_s = g_s k_s^p u_s^p - [\hat{\theta}_{gkr}^T \xi_{gkr} + \hat{\varepsilon}_{gkr}] u_r^p$, $d_g = \sum_{\substack{i=1 \\ i \neq s, i \neq r}}^m (g_j - \hat{g}_j - \hat{\varepsilon}_{gj}) u_j^p$ and

$$A = \begin{bmatrix} -r_1 & & \\ \vdots & I & \\ -r_n & 0 \cdots 0 \end{bmatrix}, R_s = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}, L_s = \begin{bmatrix} l_{s1} \\ \vdots \\ l_{sn} \end{bmatrix}, C_s = \begin{bmatrix} 1 \\ \vdots \\ 0 \end{bmatrix}^T,$$

$$d_s = \begin{bmatrix} \sum_{k=1}^p C_p^k e_2^k \hat{x}_{s,2}^{p-k} \\ \vdots \\ 0 \end{bmatrix}, B_s = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$$

Similar to the previous subsection, differentiating $V_{se} = e_s^T P_s e_s$ with respect to time t and using (6.20) and (6.54), it leads to

$$\begin{aligned} \dot{V}_{se} &= e_s^T P_s \dot{e}_s + \dot{e}_s^T P_s e_s \\ &= 2e_s^T [P_s(A_s + R_s) + (A_s + R_s)^T P_s] e_s^p + \\ &\quad 2e_s^T P_s (d + B_s d_f + B_s d_g + B_s \rho_s) - e_s^T (P_s L_s C_s + C_s^T L_s^T P_s) e_s \end{aligned}$$

From Young's inequality, one has

$$\begin{aligned} e_s^T P_s d &\le e_s^T P_s P_s e_s + d^T d \\ &\le e_s^T P_s P_s e_s + e_{sp}^T e_{sp} - (e_{s,1}{}^p)^2 + \sum_{i=2}^n 2(w_{e2})^2 (|\hat{x}_{s,i}|^p)^2 \end{aligned}$$

$$\begin{aligned} 2e_s^T P_s B_s d_f &= e_s^T P_s B_s d_f \le e_s^T P_s P_s e_s + d_f^2 \le e_s^T P_s P_s e_s + (\delta_f^*)^2 \\ &\le e_s^T P_s P_s e_s + (\bar{M}_{\delta f})^2 \end{aligned}$$

Furthermore, one has

$$\begin{aligned} \dot{V}_{se} \le &\begin{bmatrix} e_s \\ e_s^p \end{bmatrix}^T \begin{bmatrix} X_s - P_s L_s C_s - C_s^T L_s^T P_s + P_s P_s & Y_s + P_s(A_s + R_s) \\ Y_s^T + (A_s + R_s)^T P_s & Z_s + I \end{bmatrix} \begin{bmatrix} e_s \\ e_s^p \end{bmatrix} + \\ &\Delta_0 + 2e_s^T P_s B_s (d_g + \rho_s) \end{aligned} \tag{6.55}$$

where $\Delta_0 = -(e_{s,1}^p)^2 + \sum_{i=2}^n 2(w_{e2})^2 (|\hat{x}_{s,i}|^p)^2 + (\bar{M}_{\delta f})^2$.

In the following, stability analysis will be given at two cases, i.e., $s = r$ or $s \neq r$.

Case 1: $s = r$

Since

$$
\begin{aligned}
2e_s^T P_s B_s(\rho_s + d_g) &= 2e_s^T P_{sn}[\sum\nolimits_{j=1, j\neq s}^{m} (g_j - \hat{g}_j - \hat{\varepsilon}_{gj})u_j^p + (g_{ks} - \hat{g}_{ks} - \hat{\varepsilon}_{gks})u_s^p] \\
&= 2e_s^T P_{sn}(\sum\nolimits_{j=1, j\neq s}^{m} (\theta_{gj}^{*T}\xi_{gj} + \varepsilon_{gj} - \hat{\theta}_{gj}^T\xi_{gj} - \hat{\varepsilon}_{gj})u_j^p + \\
&\quad (\theta_{gks}^{*T}\xi_{gks} + \varepsilon_{gks} - \hat{\theta}_{gks}^T\xi_{gks} - \hat{\varepsilon}_{gks})u_s^p) \\
&= \sum\nolimits_{j=1, j\neq s}^{m} 2e_s^T P_{sn}\tilde{\theta}_{gj}^T\xi_{gj}u_j^p + \sum\nolimits_{j=1, j\neq s}^{m} 2e_s^T P_{sn}\tilde{\varepsilon}_{gj}u_j^p + \\
&\quad \sum\nolimits_{j=1, j\neq s}^{m} 2e_s^T P_{sn}(\varepsilon_{gj} - \hat{\varepsilon}_{gj}^*)u_j^p + \\
&\quad 2e_s^T P_{sn}\tilde{\theta}_{gks}^T\xi_{gks}u_s^p + 2e_s^T P_{sn}\tilde{\varepsilon}_{gks}u_s^p + 2e_s^T P_{sn}(\varepsilon_{gks} - \hat{\varepsilon}_{gks}^*)u_s^p
\end{aligned} \tag{6.56}
$$

$$
\begin{aligned}
\dot{V}_{se} \leq &\begin{bmatrix} e_s \\ e_s^p \end{bmatrix}^T \begin{bmatrix} X_s - P_sL_sC_s - C_s^TL_s^TP_s + 2P_sP_s^T & Y_s + P_s(A_s + R_s) \\ Y_s^T + (A_s + R_s)^TP_s & Z_s + I \end{bmatrix} \begin{bmatrix} e_s \\ e_s^p \end{bmatrix} + \\
&\Delta_0 + \sum\nolimits_{j=1, j\neq s}^{m} 2e_s^T P_{sn}\tilde{\theta}_{gj}^T\xi_{gj}u_j^p + \sum\nolimits_{j=1, j\neq s}^{m} 2e_s^T P_{sn}\tilde{\varepsilon}_{gj}u_j^p + \\
&\sum\nolimits_{j=1, j\neq s}^{m} 2e_s^T P_{sn}(\varepsilon_{gj} - \hat{\varepsilon}_{gj}^*)u_j^p + \\
&2e_s^T P_{sn}\tilde{\theta}_{gks}^T\xi_{gks}u_s^p + 2e_s^T P_{sn}\tilde{\varepsilon}_{gks}u_s^p + 2e_s^T P_{sn}(\varepsilon_{gks} - \hat{\varepsilon}_{gks}^*)u_s^p
\end{aligned} \tag{6.57}
$$

where P_{sn} is the nth column of P_s.

Similar to the above subsection, define

$$z_{s,1} = x_{s,1} = y_s$$

$$z_{s,i} = \hat{x}_{s,i} - \alpha_{s,i-1}(\hat{x}_{s,1}, \ldots, \hat{x}_{s,i-1}), i = 2, 3, \ldots, n$$

$$V_{s,1} = V_{s,11} + V_{se}, \quad V_{s,11} = \frac{1}{2}z_{s,1}^2$$

$$V_{s,i} = V_{s,i-1} + \frac{1}{2}z_{s,i-1}^2, i = 2, 3, \ldots, n$$

and choose a virtual control $\alpha_{s,i}$, $i = 1, 2, \ldots, n-1$ and practical control $\alpha_{s,nj}$, $j = 1, \ldots, m$ as follows

$$
\alpha_{s,1} = \begin{cases} \sqrt[p]{(-\frac{1}{2}z_{s,1} - \frac{\Delta_0}{z_{s,1}}}, & z_{s,1} \in \Omega_{c_{s,z_{s1}}}^0 \\ 0, & z_{s,1} \in \Omega_{c_{s,zs_1}} \end{cases} \tag{6.58}
$$

$$
\alpha_{s,2} = \begin{cases} \sqrt[p]{(-\frac{1}{2}z_{s,2} - \frac{\Delta_1}{z_{s,2}}}, & z_{s,2} \in \Omega_{c_{s,zs_2}}^0 \\ 0, & z_{s,2} \in \Omega_{c_{s,zs_2}} \end{cases} \tag{6.59}
$$

$$\alpha_{s,k} = \begin{cases} \sqrt[p]{(-\frac{1}{2}z_{s,k} - \frac{\Delta_{k-1}}{z_{s,k}}}, \ z_{s,k} \in \Omega^0_{c_{s,z_k}} \\ 0, \ z_{s,k} \in \Omega_{c_{s,z_k}} \end{cases} \tag{6.60}$$

$$\alpha_{s,nj} = u_j = \begin{cases} \sqrt[p]{\frac{N(\varsigma)(-\frac{1}{2}z_{s,n} - \frac{\Delta_{n-1}}{z_n})}{m}}, \ z_{s,k} \in \Omega^0_{c_{z_{s,zn}}} \\ 0, \ z_{s,k} \in \Omega_{c_{s,zn}} \end{cases} \tag{6.61}$$

where $\dot{\varsigma} = -\frac{1}{2}z^2_{s,n} - \Delta_{n-1}$, $\Omega_{c_{s,z_i}}, i = 1, \ldots, n$ are defined as $\Omega_{c_{z_k}}$ in the previous subsection. The adaptive laws are designed as follows:

$$\dot{\hat{\theta}}_f = 2\eta_1 e_s^T P_n \xi_f + \eta_f \hat{\theta}_f \tag{6.62}$$

$$\dot{\hat{\gamma}}_f = \eta_2 |z_n| + \eta_\gamma \hat{\gamma}_f \tag{6.63}$$

$$\dot{\hat{\theta}}_{gj} = 2\eta_3 e_s^T P_n \xi_{gj} u_j^p + \eta_{gj} \hat{\theta}_{gj} \tag{6.64}$$

$$\dot{\hat{\varepsilon}}_{gj} = 2\eta_3 e_s^T P_n u_j^p + \eta_{gj} \hat{\varepsilon}_{gj} \tag{6.65}$$

$$\dot{\hat{\theta}}_{gks} = 2\eta_4 e_s^T P_n \xi_{gks} u_s^p + \eta_{gks} \hat{\theta}_{gks} \tag{6.66}$$

$$\dot{\hat{\varepsilon}}_{gks} = 2\eta_4 e_s^T P_n u_s^p + \eta_{gks} \hat{\varepsilon}_{gks} \tag{6.67}$$

where u_j is a bounded control input which is applied simultaneously to the jth actuator in the system (6.1) and the observer (6.53), and $\eta_1 > 0$, $\eta_2 > 0$, $\eta_3 > 0$, $\eta_4 > 0$, $\eta_f > 0$, $\eta_\gamma > 0$, $\eta_{gks} > 0$, $\eta_{gj} > 0$, $\eta_{gks} > 0$ are design parameters.

Define

$$\begin{aligned} V_s =& V_{s,n} + \frac{1}{2\eta_1}\tilde{\theta}_f^T \tilde{\theta}_f + \frac{1}{2\eta_2}\tilde{\gamma}_f^2 + \\ & \frac{1}{2\eta_3}\sum\nolimits_{j=1, j\neq s}^m (\tilde{\theta}_{gj}^T \tilde{\theta}_{gj} + \tilde{\varepsilon}_{gj}^2) + \frac{1}{2\eta_4}(\tilde{\theta}_{gks}^T \tilde{\theta}_{gks} + \varepsilon_{gks}^2) \end{aligned} \tag{6.68}$$

Similar to the previous subsection, differentiating V_s with respect to time t, one has

$$\begin{aligned} \dot{V}_s \leq& \dot{V}_{s,n} + \frac{1}{\eta_1}\tilde{\theta}_f^T \dot{\hat{\theta}}_f + \frac{1}{\eta_2}\tilde{\gamma}_f \dot{\hat{\gamma}}_f + \frac{1}{\eta_4}(\tilde{\theta}_{gks}^T \dot{\hat{\theta}}_{gks} + \tilde{\varepsilon}_{gks}\dot{\hat{\varepsilon}}_{gks}) + \\ & \frac{1}{\eta_3}\sum\nolimits_{j=1, j\neq s}^m [\tilde{\theta}_{gj}^T \dot{\hat{\theta}}_{gj} + \tilde{\varepsilon}_{gj}\dot{\hat{\varepsilon}}_{gj}] \end{aligned} \tag{6.69}$$

It is obvious that if

$$\begin{bmatrix} X_s - P_s L_s C_s - C_s^T L_s^T P_s + 2P_s P_s & Y_s + P_s(A_s + R_s) \\ Y_s^T + (A_s + R_s)^T P_s & Z_s + I \end{bmatrix} < -Q_s \tag{6.70}$$

where X, Y, Z denote matrices with appropriate dimensions, respectively, and $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$, matrix $Q_s > 0$, then from (6.69), one has

$$\begin{aligned} \dot{V}_s \leq & \dot{V}_{s,n} - \begin{bmatrix} e_s \\ e_s^p \end{bmatrix}^T Q_s \begin{bmatrix} e_s \\ e_s^p \end{bmatrix} + \Delta_0 + \sum\nolimits_{j=1, j\neq s}^{m} 2e_s^T P_{sn} \tilde{\theta}_{gj}^T \xi_{gj} u_j^p + \\ & \sum\nolimits_{j=1, j\neq s}^{m} 2e_s^T P_{sn} \tilde{\varepsilon}_{gj} u_j^p + \sum\nolimits_{j=1, j\neq s}^{m} 2e_s^T P_{sn} (\varepsilon_{gj} - \hat{\varepsilon}_{gj}^*) u_j^p + \\ & 2e_s^T P_{sn} \tilde{\theta}_{gks}^T \xi_{gks} u_s^p + 2e_s^T P_{sn} \tilde{\varepsilon}_{gks} u_s^p + 2e_s^T P_{sn} (\varepsilon_{gks} - \hat{\varepsilon}_{gks}^*) u_s^p + \\ & \frac{\tilde{\theta}_f^T \dot{\hat{\theta}}_f}{\eta_1} + \frac{\tilde{\gamma}_f \dot{\hat{\gamma}}_f}{\eta_2} + \frac{\sum_{j=1, j\neq s}^{m} [\tilde{\theta}_{gj}^T \dot{\hat{\theta}}_{gj} + \tilde{\varepsilon}_{gj} \dot{\hat{\varepsilon}}_{gj}]}{\eta_3} + \frac{\tilde{\theta}_{gks}^T \dot{\hat{\theta}}_{gks} + \tilde{\varepsilon}_{gks} \dot{\hat{\varepsilon}}_{gks}}{\eta_4} \end{aligned} \tag{6.71}$$

Similar to (6.42) in the above subsection, considering (6.62–6.67), from (6.71), one has

$$\dot{V}_s \leq -g_s V_s + \bar{\mu}_s + (\bar{h}(\hat{x}) N(\varsigma)\dot{\varsigma} + \dot{\varsigma}) \tag{6.72}$$

where

$$\begin{aligned} \mu_s = & \frac{\eta_f}{2\eta_1} M_{\theta f}^2 + \frac{\eta_f}{2\eta_1} (\bar{M}_{\varepsilon f} + \bar{M}_{\delta f})^2 + \sum\nolimits_{j=1, j\neq s}^{m} \frac{\eta_{gj}}{2\eta_3} (\bar{M}_{\theta gj}^2 + \bar{M}_{\varepsilon gj}^2) + \\ & \frac{\eta_{gks}}{2\eta_4} (\bar{M}_{\theta gks}^2 + \bar{M}_{\varepsilon gks}^2) \end{aligned}$$

$$g_s = \min\{\frac{1}{2}, \frac{\eta_f}{2\eta_1}, \frac{\eta_\gamma}{2\eta_2}, \frac{\eta_{g1}}{2\eta_3}, \ldots, \frac{\eta_{gm}}{2\eta_3}, \frac{\eta_{gks}}{2\eta_4}, \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}\}$$

$$\begin{aligned} \bar{h}(\hat{x}) = & \sum\nolimits_{j=1, j\neq s}^{m} [z_n(\hat{g}_j + \hat{\varepsilon}_{gj}) + 2e^T P_n (\varepsilon_{gj} - \varepsilon_{gj}^*)] + \\ & z_n(\hat{g}_{ks} + \hat{\varepsilon}_{gks}) + 2e^T P_n (\varepsilon_{gks} - \varepsilon_{gks}^*) \end{aligned}$$

Since $\dot{V}_s \leq -g_s V_s + \bar{\mu}_s + (\bar{h}(\hat{x}) N(\varsigma)\dot{\varsigma} + \dot{\varsigma})$, one has

$$\begin{aligned} V_s(t) \leq & \frac{\bar{\mu}_s}{g_s} + [V_s(0) - \frac{\mu_{\cdot s}}{g_s}] e^{-g_s t} + e^{-gt} \int_0^t (\bar{h}(\hat{x}) N(\varsigma) + 1)\dot{\varsigma} e^{-g_s \tau} \dot{\varsigma} d\tau \\ \leq & \frac{\bar{\mu}_s}{g_s} + V_s(0) e^{-g_s t} + e^{-g_s t} \int_0^t (\bar{h}(\hat{x}) N(\varsigma) + 1)\dot{\varsigma} e^{-g_s \tau} \dot{\varsigma} d\tau \end{aligned} \tag{6.73}$$

Applying Lemma 6.5, we can conclude that, $V_n(t)$, $\int_0^t (\bar{h}(\hat{x})N(\varsigma)+1)\dot{\varsigma}e^{-g\tau}\dot{\varsigma}d\tau$ and $\varsigma(t)$ are SGUUB on $[0, t_f)$. According to Proposition 2 in [39], if the solution of the closed-loop system is bounded, then $t_f = +\infty$. Let c_g be the upper bound of $\int_0^t \bar{h}(\hat{x})(N(\varsigma)+1)\dot{\varsigma}e^{-g_s\tau}\dot{\varsigma}d\tau$, we have the following inequalities:

$$e^{-g_s t}\int_0^t (\bar{h}(\hat{x})N(\varsigma)+1)\dot{\varsigma}e^{-g_s\tau}\dot{\varsigma}d\tau \le c_g$$

Thus, (6.73) becomes

$$V_s(t) \le \frac{\bar{\mu}_s}{g_s} + c_g + V_s(0) = \mu_s \tag{6.74}$$

Hence, if matrices X_s, Y_s, Z_s, Q_s and the positive definite symmetric matrix P_s are chosen appropriately such that $\begin{bmatrix} X_s & Y_s \\ Y_s^T & Z_s \end{bmatrix} > 0$ and (6.74) holds, then, the proposed control input (6.61) and adaptive laws (6.62–6.67) can ensure that $V_s(t)$ is bounded, namely, the closed-loop system is semi-globally uniformly ultimately bounded. That is to say, all signals of the closed-loop system remain the following compact set Ω_1,

$$\Omega_1 := \left\{ \begin{array}{l} (z_i, \tilde{\theta}_f, \tilde{\gamma}_f, \tilde{\theta}_{gj}, \tilde{\varepsilon}_{gj}, \tilde{\theta}_{gks}, \tilde{\varepsilon}_{gks}, e) \Big| \, |z_i| \le \sqrt{2\mu_s}, ||\tilde{\theta}_f|| \le \sqrt{2\eta_1\mu_s}, \\ |\tilde{\gamma}_f| \le \sqrt{2\eta_2\mu_s}, ||\tilde{\theta}_{gj}|| \le \sqrt{2\eta_3\mu_s}, |\tilde{\varepsilon}_{gj}| \le \sqrt{2\eta_3\mu_s}, \\ ||\tilde{\theta}_{gks}|| \le \sqrt{2\eta_4\mu_s}, |\tilde{\varepsilon}_{gks}| \le \sqrt{2\eta_4\mu_s}, ||e|| \le \sqrt{\dfrac{\mu_s}{\lambda_{\min}(P_s)}} \end{array} \right\}$$

Case 2: $s \ne r$

Since $s \ne r$, from the faulty (6.52) and the observer (6.53), one has

$$2e_s^T P_s B_s \rho_s = 2e_s^T P_s B_s[(g_{ks} - \hat{g}_s - \hat{\varepsilon}_{gs})u_s^p + (g_r - \hat{g}_{kr} - \hat{\varepsilon}_{gkr})u_r^p] \tag{6.75}$$

From the adaptive laws (6.64–6.67), one has

$$\dot{\hat{\theta}}_{gs} \ne \dot{\hat{\theta}}_{gks}, \dot{\hat{\varepsilon}}_{gs} \ne \dot{\hat{\varepsilon}}_{gks}, \dot{\hat{\theta}}_{gr} \ne \dot{\hat{\theta}}_{gkr}, \dot{\hat{\varepsilon}}_{gr} \ne \dot{\hat{\varepsilon}}_{gkr}$$

It is noted that $2e_s^T P_s B_s[(g_{ks} - \hat{g}_s - \hat{\varepsilon}_{gs})u_s^p + (g_r - \hat{g}_{kr} - \hat{\varepsilon}_{gkr})u_r^p]$ varies infinitely since $\dot{\hat{\theta}}_{gs} \ne \dot{\hat{\theta}}_{gks}$, $\dot{\hat{\theta}}_{gr} \ne \dot{\hat{\theta}}_{gkr}$, $\dot{\hat{\varepsilon}}_{gs} \ne \dot{\hat{\varepsilon}}_{gks}$ and $\dot{\hat{\varepsilon}}_{gr} \ne \dot{\hat{\varepsilon}}_{gkr}$, which further cause that $V_s(t)$ varies infinitely. As a result, basically, all signals of the closed-loop systems such as e_{si} do not remain Ω_1 using the above control law and adaptive laws.

The above design procedure and analysis are summarized in the following theorem.

Theorem 6.2 *Consider the faulty system (6.52) under Assumptions 6.1 and 6.2, with virtual controls (6.58–6.60), control law (61) and adaptive laws (6.62–6.67). If matrices X_s, Y_s, Z_s, $Q_s > 0$ and $P_s = P_s^T > 0$ are such that* $\begin{bmatrix} X_s & Y_s \\ Y_s^T & Z_s \end{bmatrix} > 0$ *and*

$$\begin{bmatrix} X_s - P_s L_s C_s - C_s^T L_s^T P_s + 2P_s P_s & Y_s + P_s(A_s + R_s) \\ Y_s^T + (A_s + R_s)^T P_s & Z_s + I \end{bmatrix} < -Q_s \tag{6.76}$$

then, we can guarantee the following properties under bounded initial conditions, when the rth actuator is faulty,

(1) for $s = r$, the closed-loop system is semi-globally uniformly ultimately stable, and all signals involved in the closed-loop systems remain a small neighborhood of the origin, i.e., Ω_1 specified as

$$\Omega_1 := \left\{ \begin{array}{l} (z_i, \tilde{\theta}_f, \tilde{\gamma}_f, \tilde{\theta}_{gj}, \tilde{\varepsilon}_{gj}, \tilde{\theta}_{gks}, \tilde{\varepsilon}_{gks}, e) \Big| \, |z_i| \le \sqrt{2\mu_s}, ||\tilde{\theta}_f|| \le \sqrt{2\eta_1\mu_s}, \\ |\tilde{\gamma}_f| \le \sqrt{2\eta_2\mu_s}, ||\tilde{\theta}_{gj}|| \le \sqrt{2\eta_3\mu_s}, |\tilde{\varepsilon}_{gj}| \le \sqrt{2\eta_3\mu_s}, \\ ||\tilde{\theta}_{gks}|| \le \sqrt{2\eta_4\mu_s}, |\tilde{\varepsilon}_{gks}| \le \sqrt{2\eta_4\mu_s}, ||e|| \le \sqrt{\dfrac{\mu_s}{\lambda_{\min}(P_s)}} \end{array} \right\}$$

(2) $s \neq r$, all signals of the closed-loop systems do not remain the compact set Ω_1.

Remark 6.3 It is valuable to point out that, if the design parameters such as $\eta_i, i = 1, \ldots, 4$, $\eta_f, \eta_\gamma, \eta_{gks}, \eta_{gj}$, $j = 1, \ldots, m$ are appropriately chosen, μ_s is small enough, and all signals of the closed-loop system converge to a smaller neighborhood of the origin, which means that better control performance is obtained.

Now, we denote the residuals between the real system and isolation estimators as follows:

$$J_s(t) = \left\| \hat{y}_s(t) - y(t) \right\| = ||Ce(t)||, \quad 1 \le s \le m \tag{6.77}$$

According to Theorem 6.2, when the rth actuator is faulty, i.e., $s = r$, the residual $e_s(t)$ must tend to Ω_1; while for any $s \neq r$, basically, $e_s(t)$ does not belong to Ω_1. Hence, the isolation law for actuator fault can be designed as

$$\begin{cases} J_s(t) \le T_I, s = r \Rightarrow \text{the } r\text{th actuator is faulty} \\ J_s(t) > T_I, s \neq r \end{cases} \tag{6.78}$$

where threshold T_I is defined as follows.

$$T_I = ||C|| \sqrt{\frac{\mu_s}{\lambda_{\min}(P_s)}}$$

6.3.3 Fault Accommodation

After that the fault information is obtained, we will consider the fault-tolerant control problem of system (6.1), and design a fault-tolerant control law to recover the control system's dynamics performance when an actuator fault occurs. Firstly, we consider the fuzzy control problem for the following nominal system without actuator faults:

$$\begin{cases} \dot{x}_i = x_{i+1}^p, i = 1, \ldots, n-1 \\ \dot{x}_n = f(x) + \sum_{j=1}^{m} g_j(x) u_j^p \\ y = x_1 \end{cases}$$

Consider matrices $X, Y, Z, Q > 0$ and $P = P^T > 0$ such that $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$ and

$$\begin{bmatrix} X - PLC - C^T L^T P + 2PP & Y + P(A+R) \\ Y^T + (A+R)^T P & Z + I \end{bmatrix} \leq -Q$$

virtual control laws (6.58–6.60), control input (6.61) and adaptive laws (6.62–6.67).

From Theorem 6.1, under Assumptions 6.1 and 6.2, the closed-loop system is semi-globally uniformly ultimately stable, and all signals involved in the closed-loop systems converge to a small neighborhood of the origin.

On the basis of the estimated actuator fault, the fault tolerant controller is constructed as

$$u_s = \frac{\hat{\rho}_s u_s^N}{\hat{\rho}_s^2 + \varepsilon_u} \tag{6.79}$$

where $\varepsilon_u > 0$ is a design parameter, u_s^N is the sth desired control input under healthy condition, $\hat{\rho}_s$ is the estimate of $g_s k_s$, which is used to compensate for the gain fault k_s.

Theorem 6.3 *Consider the high-order system (6.1) under Assumptions 6.1 and 6.2, fault model (6.2), virtual and practical control laws (6.58–6.61) and adaptive laws (6.62–6.67). If there exist matrices $X, Y, Z, Q > 0$ and $P = P^T > 0$ with appropriate dimensions, such that $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} > 0$ and*

$$\begin{bmatrix} X - PLC - C^T L^T P + 2PP & Y + P(A+R) \\ Y^T + (A+R)^T P & Z + I \end{bmatrix} \leq -Q \tag{6.80}$$

then, the faulty system (6.1) is asymptotically stable under the feedback FTC (6.79) and all signals involved in the closed-loop system are semi-globally uniformly ultimately bounded, converging asymptotically to a small neighborhood of zero, i.e. $||\tilde{\theta}_f|| \leq \sqrt{2\eta_{sf}\mu_s}$, $||\tilde{\theta}_{gj}|| \leq \sqrt{2\eta_{gj}\mu_s}$, $||\tilde{\theta}_{g\rho,s}|| \leq \sqrt{2\eta_{gks}\mu_s}$, $||e|| \leq \sqrt{\frac{2\mu_s}{\lambda_{\min}(P_s)}}$, where

$$\mu_s = \frac{\bar{\mu}_s}{g_s} + c_g + V_s(0),\ g_s = \min\{\frac{1}{2}, \frac{\eta_f}{2\eta_1}, \frac{\eta_\gamma}{2\eta_2}, \frac{\eta_{g1}}{2\eta_3}, \ldots, \frac{\eta_{gm}}{2\eta_3}, \frac{\eta_{gks}}{2\eta_4}, \frac{\eta_{gks}}{2\eta_4}, \frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}\},$$

$$\begin{aligned}\bar{\mu}_s =& \frac{\eta_f}{2\eta_1} M_{\theta f}^2 + \frac{\eta_\gamma}{2\eta_2}(\bar{M}_{\varepsilon f}^2 + \bar{M}_{\delta f}^2) + \\ & \sum_{j=1, j\neq s}^{m} \frac{\eta_{gj}}{\eta_3}[\bar{M}_{\theta gj}^2 + \bar{M}_{\varepsilon gj}^2] + \frac{\eta_{gks}}{2\eta_4} M_{\theta gks}^2 + \frac{\eta_{gks}}{2\eta_4}\bar{M}_{\varepsilon gks}^2\end{aligned}$$

Proof Similar to the proof of Theorem 6.1, it is easy to obtain the conclusions of Theorem 3. The detailed proof is thus omitted here.

6.4 Simulation Results

In this section, a practical aircraft longitudinal motion dynamics, which can be described as a 1-order nonlinear system, namely $p = 1$, and a high-order numerical example where $p = 3$, are taken to show the effectiveness of the proposed fault tolerant control scheme.

6.4.1 *An Application to Aircraft Longitudinal Motion Dynamics*

In this subsection, we apply the proposed FTC scheme to diagnose and accommodate failures in an aircraft longitudinal motion dynamics. The aircraft longitudinal motion dynamics of the twin otter [43] can be described as 1-order nonlinear system as follows:

$$\begin{cases} \dot{V} = \dfrac{F_x \cos(\alpha) + F_z \sin(\alpha)}{m} \\ \dot{\alpha} = q + \dfrac{-F_x \sin(\alpha) + F_z \cos(\alpha)}{mV} \\ \dot{\theta} = q \\ \dot{q} = \dfrac{M}{I_y} \end{cases} \tag{6.81}$$

where V is the velocity, α is the angle of attack, θ is the angle of pitch and q is the pitch rate, m is the mass, I_y is the moment of inertia, and $F_x = \bar{q}SC_x(\alpha, q, \delta_{e1}, \delta_{e2}) + T_1\cos\gamma_1 + T_2\cos\gamma_2 - mg\sin(\theta)$, $F_z = \bar{q}SC_z(\alpha, q, \delta_{e1}, \delta_{e2}) + T_1 \cdot \sin\gamma_1 + T_2\sin\gamma_2 - mg\cos(\theta)$, $M = \bar{q}cSC_m(\alpha, q, \delta_{e1}, \delta_{e2})$, where $\bar{q} = \frac{1}{2}\rho V^2$ is the dynamic pressure, ρ is the air density, S is the wing area, c is the mean chord, T_1 and T_2 are independent thrusts with corresponding thrust misalignments γ_1 and γ_2. The functions C_x, C_z, C_m are of the polynomial form: $C_x = C_{x1}\alpha + C_{x2}\alpha^2 + C_{x3} + C_{x4}(d_1\delta_{e1} + d_2\delta_{e2})$, $C_z = C_{z1}\alpha + C_{x2}\alpha^2 + C_{z3} + C_{z4}(d_1\delta_{e1} + d_2\delta_{e2}) + C_{x5}q$, $C_m = C_{m1}\alpha + C_{m2}\alpha^2 + C_{m3} +$

$C_{m4}\,(d_1\delta_{e1}+d_2\delta_{e2})+C_{m5}q$, where δ_{e1} and δ_{e2} are the elevator angles of an augmented two-pieces elevators used as two actuators u_1 and u_2 for failure compensation study. Choosing V, α, θ and q as the states x_1, x_2, x_3 and x_4, and δ_{e1}, δ_{e2}, T_1, T_2 as the inputs u_1, u_2, u_3, u_4, (6.81) will be put into the state form:

$$\begin{cases} \dot{x}_1 = (c_1^T\phi_0(x_2)x_1^2+\phi_1(x))\cos(x_2)+ \\ \quad +(c_2^T\phi_0(x_2)x_1^2+\phi_2(x))\sin(x_2)+ \\ \quad d_1g_1(x)u_1+d_2g_1(x)u_2+g_{31}(x)u_3+g_{41}(x)u_4 \\ \dot{x}_2 = x_4-(c_1^T\phi_0(x_2)x_1+\phi_1(x)\frac{1}{x_1})\sin(x_2)+ \\ \quad (c_2^T\phi_0(x_2)x_1+\phi_2(x)\frac{1}{x_1})\cos(x_2)+ \\ \quad d_1g_2(x)u_1+d_2g_2(x)u_2+g_{32}(x)u_3+g_{42}(x)u_4 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = \theta^T\varphi(x)+b_1x_1^2u_1+b_2x_1^2u_2 \end{cases} \tag{6.82}$$

where

$$\phi_0(x_2)=[x_2, x_2^2, 1]^T, \quad \phi_1(x)=p_0\sin(x_3)$$

$$\phi_2(x)=p_1x_4x_1^2+p_0\sin(x_3),$$

$$g_1(x)=a_1x_1^2\cos(x_2)+a_2x_1^2\sin(x_2)$$

$$g_2(x)=-a_1x_1\sin(x_2)+a_2x_1\sin(x_2)$$

$$g_{31}(x)=\cos(\gamma_1)\cos(x_2)+\sin(\gamma_1)\sin(x_2)$$
$$g_{41}(x)=\cos(\gamma_2)\cos(x_2)+\sin(\gamma_2)\sin(x_2)$$

$$g_{32}(x)=-\cos(\gamma_1)\frac{\sin(x_2)}{x_1}+\sin(\gamma_1)\frac{\cos(x_2)}{x_1}$$
$$g_{42}(x)=-\cos(\gamma_2)\frac{\sin(x_2)}{x_1}+\sin(\gamma_2)\frac{\cos(x_2)}{x_1}$$
$$\varphi(x)=[x_1^2x_2, x_1^2x_2^2, x_1^2, x_1^2x_4]^T$$

and $\theta, p_1, a_1, a_2, , b_1, b_2, c_1, c_2, d_1, d_2, \gamma_1, \gamma_2$ are unknown constant parameters while p_0 is the gravity constant which is known. There exists a diffeomorphism $[\xi, x]^T = T(\chi) = [T_1(\chi), T_2(\chi), x_3, x_4]^T$ such that (6.82) can be transform into the parameter-strict-feedback form, where the positive odd number $p = 1$

$$\begin{cases} \dot{x}_3 = x_4 \\ \dot{x}_4 = \vartheta^T\phi(x)+\sum_{i=1}^{2} b_ix_1^2u_i \end{cases} \tag{6.83}$$

and the zero dynamics $\dot{\xi} = \phi(\xi, \chi) + \Phi(\xi, \chi)\vartheta$, where $\vartheta \in R^4$ is an unknown constant vector. The relative degree o equals 2. The aircraft parameters in the simulation study are chosen based on the data sheet in [44]: $m = 4600\,kg$, $I_y = 31027\,kg\,m^2$, $S = 39.2\,m^2$, $c = 1.98\,m$, $T_x = 4864\,N$, $T_z = 212\,N$, $\rho = 0.7377\,kg/m^3$ at the altitude of 5000 m, and for the 0° flap setting. In addition, $C_{x1} = 0.39$, $C_{x2} = 2.9099$, $C_{x3} = -0.0758$, $C_{x4} = 0.0961$, $C_{z1} = -7.0186$, $C_{z2} = 4.1109$, $C_{z3} = -0.3112$, $C_{z4} = -0.2340$, $C_{z5} = -0.1023$, $C_{m1} = -0.8789$, $C_{m2} = -3.852$, $C_{m3} =-0.0108$, $C_{m4} = -1.8987$, $C_{m5} = -0.6266$ are unknown constants. Reference signal $y_d$is set as $y_d = e^{-0.05t} \cdot \sin(0.2t)$. The initial states and estimates are set as $\chi(0) = [75, 0, 0, 15, 0]^T = e^{-0.05t}\sin(0.2t)$, $\hat{\vartheta}(0) = [0, 0, -0.004, 0]$. It is assumed that the zero dynamics $\dot{\xi} = \phi(\xi, \chi) + \Phi(\xi, \chi)\vartheta$ is input-to-state stable with respect to$x$ taken as the input. In addition, b_i, $i = 1, \ldots, m$ are assumed to be complete unknown, i.e., these values and signs are both unknown.

The fault case considered in this example is modeled as

$$u_1^f(t) = \begin{cases} u_1(t), \ t < 10 \\ (1 - \rho_1(x))u_1(t), \ t \geq 10 \end{cases}, \quad u_2^f(t) = u_2(t)$$

where $\rho_1(x) = 0.4\cos(x_3)$.

Firstly, the matrices inequality (6.43) are transformed to LMI, then by using Matlab toolbox to solve the matrices inequalities, one can obtain symmetric matrix $X, Y, Z, P, Q, X_s, Y_s, Z_s, P_s, Q_s$ and the nominal controller gains K_i. Therefore, one can design the desired control (6.37). Using this desired control, we can design fault-tolerant controller (6.79). In this example, we assume that the system state is not fully measured and thus the observer (6.53) is used to estimate the system state. Consequently, the observer-based fault-tolerant control (6.79) is used to control the faulty system. The simulation results are presented in Figs. 6.1, 6.2, 6.3, 6.4, 6.5 and 6.6. From Figs. 6.1 and 6.2, it is seen that, under normal operating condition, the system states globally asymptotically converge to a small neighborhood of the origin. Figures 6.3 and 6.4 show that, when an actuator fault occurs, when keeping the normal controller, the system states deviate significantly from the neighborhood. However, as shown in Figs. 6.5 and 6.6, using the proposed FTC (6.79), better tracking performance is obtained, again.

6.4.2 A High-Order Numerical Example

Consider the following high-order nonlinear system

$$\begin{cases} \dot{x}_1 = x_2^3 \\ \dot{x}_2 = u_1^3 + u_2^3 \end{cases} \tag{6.84}$$

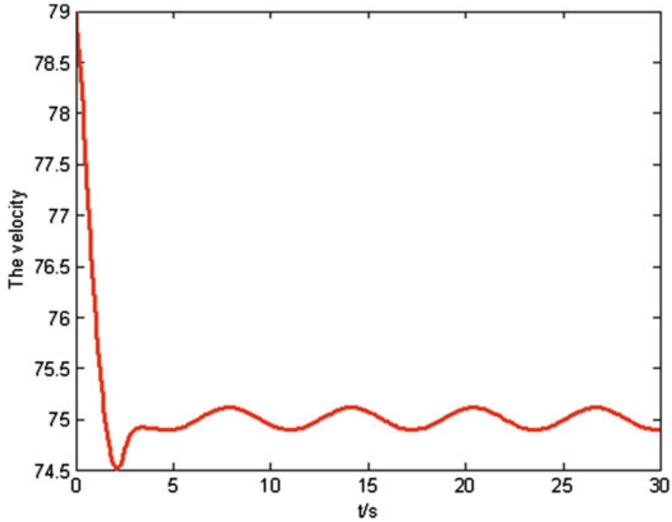

Fig. 6.1 Time response of the velocity without fault

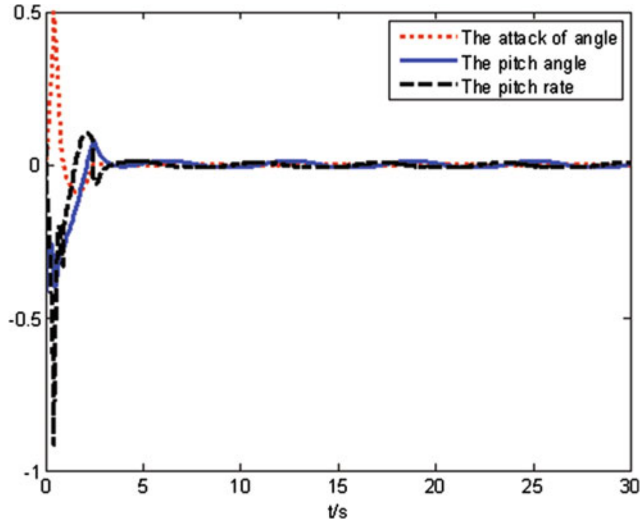

Fig. 6.2 Time response of the attack of angle, the pitch angle and the pitch rate without fault

The fault case considered in this example is modeled as

$$u_1^f(t) = \begin{cases} u_1(t), \ t < 10 \\ (1 - \rho_1(x))u_1(t), \ t \geq 10 \end{cases} \quad u_2^f(t) = u_2(t)$$

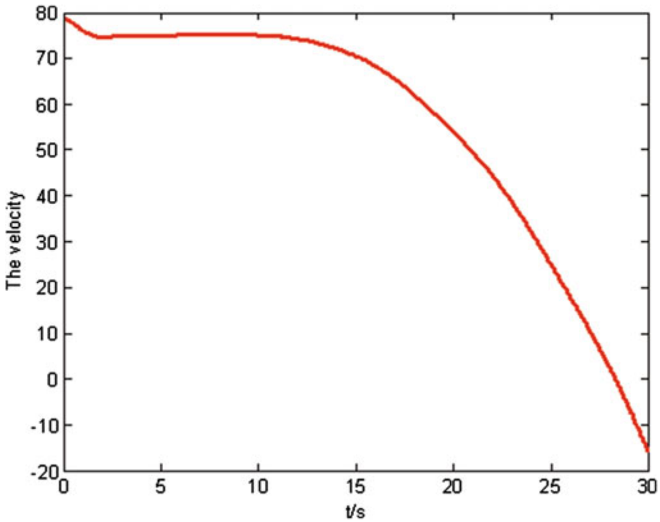

Fig. 6.3 Time response of the velocity without FTC

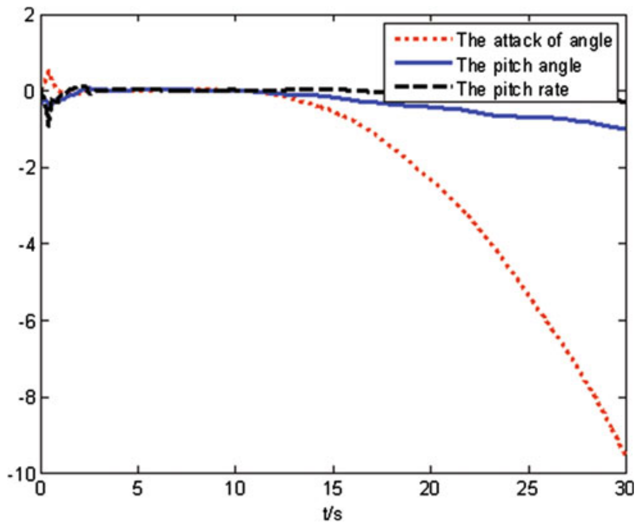

Fig. 6.4 Time response of the attack of angle, the pitch angle and the pitch rate without FTC

where $\rho_1(x) = 0.8\cos(2+x_1+x_2)$, the fault occurs at time $t = 10s$. As expected, we can find that system output y follows well $y_d = 0$ as shown in Fig. 6.7. Meanwhile, Figs. 6.8 and 6.9 illustrate that, under the faulty condition, the system output y does not converge to the desired reference signal without FTC, however, using FTC, the system has better tracking performance.

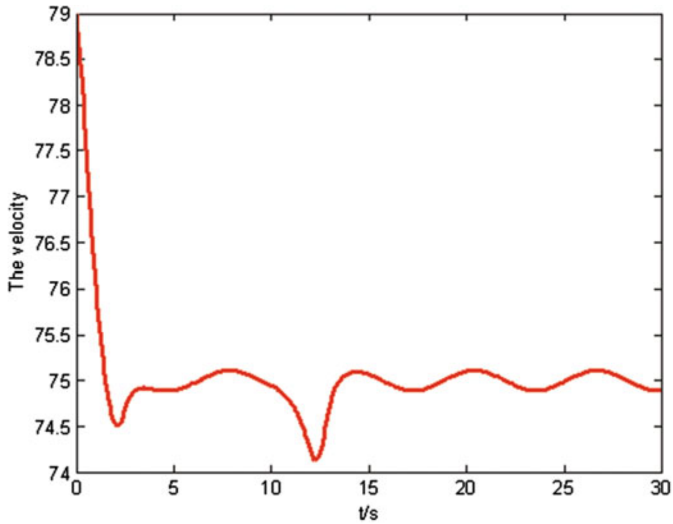

Fig. 6.5 Time response of the velocity with FTC

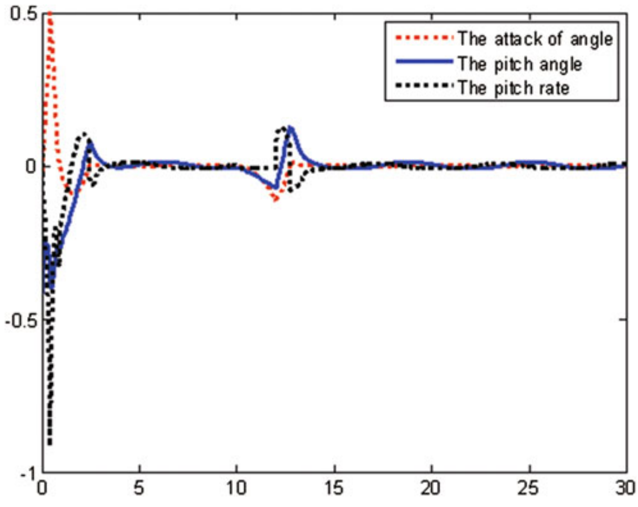

Fig. 6.6 Time response of the attack of angle, the pitch angle and the pitch rate with FTC

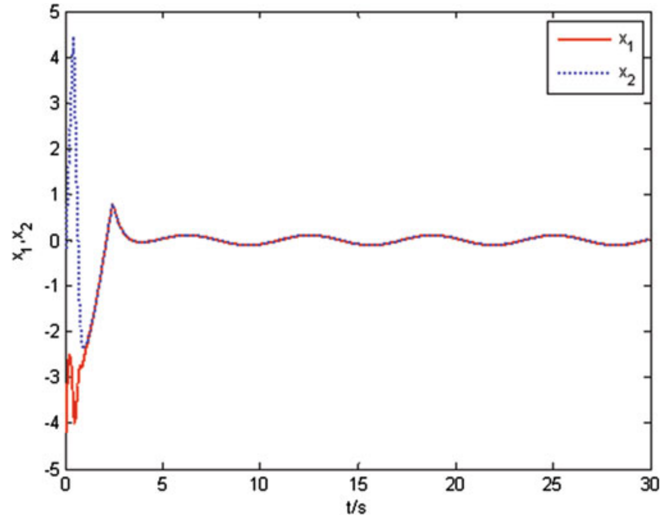

Fig. 6.7 State response under normal condition

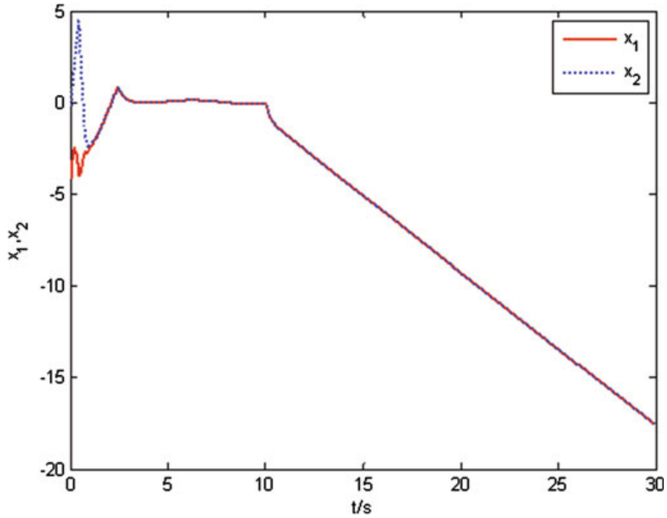

Fig. 6.8 State response under faulty condition without FTC

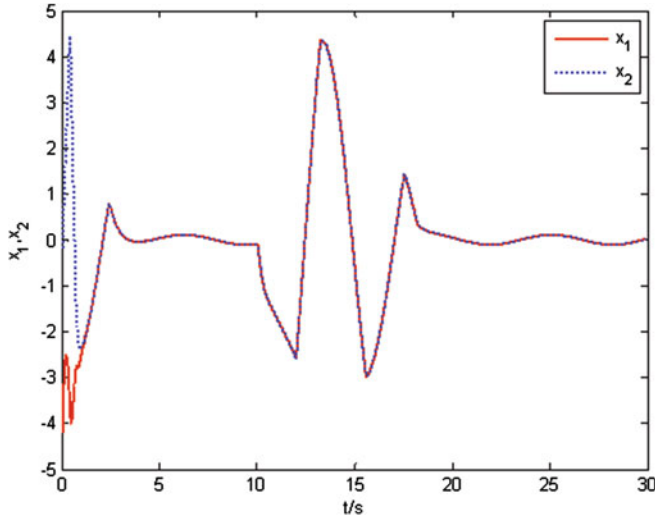

Fig. 6.9 State response under faulty condition with FTC

6.5 Conclusions

In this chapter, the fault-tolerant control problem for a class of uncertain nonlinear systems in presence of actuator faults is discussed. We first design a bank of observers to detect, isolate and estimate the fault. Then a sufficient condition for the existence of an FDI observer is derived. Simulation show that the designed fault detection, isolation and estimation algorithms and fault-tolerant control scheme have better dynamic performances in the presence of actuator faults.

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